Sensitivity Study and Stability Analysis of the Marotzke and Stone
Ocean-Atmosphere Model under Buoyancy and Energy Constraints

by

Christie Wood

Submitted to the Department of Earth, Atmospheric and Planetary Sciences in partial fulfillment of the Requirements for the Degree of Bachelor of Science in Earth, Atmospheric and Planetary Sciences at the Massachusetts Institute of Technology

May 16, 2005

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Abstract

The oceanic circulation plays a significant role in earth’s climate by transporting heat polewards. Here the ocean’s thermohaline circulation is studied using the Marotzke and Stone four box ocean-atmosphere model under both the buoyancy constraint proposed by Stommel and the energy constraint proposed by Huang. Both a sensitivity study and stability analysis is performed. In the sensitivity study it is evident that the model under the buoyancy constraint reacted to variations in flow between ocean boxes, net incoming radiation gradient and longwave radiation reflection coefficient in the same manner regardless of whether it was in thermal mode or haline mode. Under the energy constraint the study yielded similar results in both modes, with the exception of the equilibrium temperature and salinity having opposite relationships to the energy parameter in the thermal and haline modes. The model under the buoyancy constraint under both limits becomes more sensitive to changes in both the net incoming radiation gradient and the longwave radiation reflection coefficient as the rate of flow between the two ocean boxes decreases. The model under the energy constraint, in contrast, becomes more sensitive to changes in the longwave radiation reflection coefficient as the energy parameter decreases but the model’s sensitivity to changes in the net incoming radiation gradient is unaltered by changes in the energy parameter. This suggests that the model under the energy constraint is less sensitive to global climate changes. The stability analysis shows that the model under both the buoyancy and energy constraint is stable to realistic perturbations in temperature and salinity.

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1. Introduction

Earth’s complex ocean circulation plays a significant role in the planet’s climate system. The ocean covers approximately 71% of the Earth’s surface and has a volume totaling $3.2 \times 10^{17} m^3$. As a result, it has a heat capacity that is roughly 1000 times that of the atmosphere. It is therefore expected that the oceanic circulation plays a significant role in earth’s climate system by transporting heat. When studying the ocean circulation one realizes that it is forced by both buoyancy and wind. Since the ocean has an albedo of approximately 10% (and therefore readily absorbs solar and IR radiation), it is able to exchange heat and moisture with the atmosphere at its upper surface. The buoyancy driven circulation is therefore driven largely by cooling from above. In addition, winds blowing over the ocean exert a stress on it. The comparatively fast wind driven circulation plays a particularly important role in the upper kilometer or so of the ocean whereas the buoyancy driven circulation affects the ocean circulation at much greater depths. Due to the very long timescales and very weak currents involved, the buoyancy driven circulation is much less observed and understood. This study addresses the buoyancy driven circulation using a four-box ocean-atmosphere model.

There is also a humanistic incentive to study the overturning circulation. Most scientists accept that global warming is a reality. While many consequences of global warming have been envisioned, the impact of global warming on the overturning circulation is unknown. One of the more serious consequences of global warming could be a decrease in salinity of North Atlantic waters due to melting ice caps which would inhibit the water from sinking and therefore shut down the thermohaline circulation. This would have a global effect on temperature and the most significant impact would be on
Northwestern Europe which is currently kept warm compared to its North American counterpart by the heat transported in the North Atlantic Current. Paleoclimate and proxy data suggest that the overturning circulation has undergone significant changes in the past due to abrupt climate changes, so in the present situation it is important to try to understand the possible effects of global warming.

In this study the four-box ocean-atmosphere model used by Marotzke and Stone (1995) is studied. The ocean component is Stommel's (1961) two-box ocean model and the atmospheric component gives the surface heat and freshwater fluxes as residuals of the atmospheric energy and moisture budgets which are assumed to be in balance. In section 2 a discussion of the thermohaline circulation is given. In section 3, we review the buoyancy driven two-box ocean model designed by Henry Stommel. The fourth section of the paper discusses the possible need of adding a source of mechanical mixing to Stommel's model as proposed by Huang (1999). In section 5 we address Marotzke and Stone's model under the "buoyancy" constraint originally proposed by Stommel and in section 6 we look at the model under the "energy" constraint proposed by Huang. Preliminary results indicate that the two constraints lead to quite different thermal and haline equilibrium modes. In section 7 a sensitivity study of the resulting equilibria is carried out, varying important parameters such as vertical diffusion of heat and forcing by incoming radiation and strength of the mechanical energy required to drive the circulation. Section 8 is devoted to analyzing the stability of the equilibrium solutions.
2. Thermohaline Circulation

The thermohaline circulation is the deep-reaching meridional overturning cell of the ocean circulation (Fig. 2.1). Two properties of water that play a significant role in the meridional overturning cell are: cold water is denser than warm water and saline water is denser than fresh water. It is also important to note that in the narrow range of temperatures and salinities found in the ocean, temperature typically influences density more than salinity. When modeling the thermohaline circulation it is convenient to use the thermal expansion and haline contraction coefficients. The thermal expansion coefficient of water is

$$\alpha = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial T} \tag{2.1}$$

(with salinity and pressure kept constant) and has a typical value of

$$2 \times 10^{-4} \, ^{o}C^{-1} \text{ (Marshall and Plumb).}$$

The haline contraction coefficient is

$$\beta = \frac{1}{\rho_0} \frac{\partial \rho}{\partial S} \tag{2.2}$$
(with temperature and pressure kept constant). This coefficient typically has a value of about $8 \times 10^{-4} \text{ psu}^{-1}$ (Marshall and Plumb).

Based on the properties of water mentioned above we would expect ocean convection to occur in cold regions where interior stratification is small. This is likely to occur at high latitudes during winter where surface density can increase by direct cooling, which will therefore reduce temperature and increase density. Although the density of water in the high latitudes is increased by ice formation which raises the salinity of the water directly below the ice, water in the low latitudes is actually more saline than high latitude water. However, as previously mentioned, temperature is typically has more influence over water density.

![Zonally averaged overturning circulation in the North Atlantic](image)

**Figure 2.2** Zonally averaged overturning circulation in the North Atlantic (Marshall and Plumb)

The zonally averaged overturning circulation in the North Atlantic is schematically drawn in figure 2.2. Cold, salty water sinks in the North Atlantic and travels to the abyss. It then upwells throughout the mid- and low-latitudes (in order to maintain the thermocline at a relatively constant depth) and returns to the source of deep
convection. Meridional overturning cells occur in the southern hemisphere as well.

Figure 2.3 is an illustration of the ocean global overturning circulation proposed by David Webb.

Figure 2.3 Cartoon of ocean global circulation proposed by David Webb. (Marshall and Plumb)
3. Stommel’s Two-Box Model

Henry Stommel studied the thermohaline circulation using a highly idealized model consisting of two well mixed boxes of different temperature and salinity (Stommel, 1961). These boxes were connected by pipes at the surface of the water and at depth, and each box was surrounded by a reservoir of water separated from the boxes by porous walls (Fig 3.1).

![Figure 3.1 A schematic drawing of Stommel’s two-box experiment. (Stommel, 1961)](image)

In the drawing, $S$ and $T$ are the constant salinities and temperatures of the two reservoirs, $S_1$ and $T_1$ are the salinity and temperature of vessel 1, $S_2$ and $T_2$ are the salinity and temperature of vessel 2, and $q$ is the flow rate (Stommel, 1961). There are two stable equilibrium solutions to the differential equations of the two box model (these equations will be presented in section 5) (Stommel, 1961). The “thermal mode” solution for which temperature differences drive the circulation, is characterized by deep convective cells formed by cooling and subsequent sinking of surface waters in the high latitudes. The “haline mode” solution for which salinity differences drive the
circulation, is characterized by deep convective cells formed primarily due to strong net evaporation and sinking of surface water in low latitudes. The present day ocean is in the thermal mode. The principal assumption made in Stommel’s model is that the strength of the thermohaline circulation is proportional to the density difference between the two boxes which will be referred to as the buoyancy constraint.

4. Sandstrom’s Theorem

The ocean plays an important role in the climate system by transporting a significant amount of heat polewards. The total poleward heat flux diagnosed by satellite measurements is shown in figure 4.1. The total amount of poleward heat flux associated with the oceanic current is about 2 PW in the subtropical north Atlantic and the total oceanic contribution to poleward heat flux may consist roughly of fifty percent of the total heat flux (Huang, 2004).

Although there is a huge heat flux through the upper surface of the ocean, the ocean is not a heat engine. Sandstrom’s theorem states that a closed steady circulation
can be maintained in the ocean only if the heating source is situated at a level lower than the cooling source. Sandstrom's theorem applies to both the atmosphere and the oceans. The atmosphere is heated from below and cooled from above and therefore acts as a heat engine. The ocean is heated and cooled at the surface and according to Sandstrom's theorem there can be no circulation driven by thermal forcing. Therefore, to maintain the observed quasi-steady circulation in the ocean, external sources of mechanical energy are required to balance the loss of mechanical energy caused by frictional processes.

There are many sources (sinks) of mechanical energy in the ocean (Fig. 4.2). Wind stress applied at the ocean surface drives both currents and waves. Energy input through the surface current can be approximated as the scalar product of wind stress and surface geostrophic velocity. There is also an input of energy through the Ekman spiral, which occurs in the frictional boundary layer (Ekman layer) in roughly the upper 100 m. of the ocean, where wind stress is balanced by the frictional force. The convergence of the Ekman flux in the subtropical gyre gives rise to a pumping velocity at the base of the Ekman layer which pushes warm water into the ocean interior, thus forming the main thermocline in the subtropical ocean. This process increases gravitational potential energy. Another important process occurring in the surface mixed layer is the convective adjustment due to cooling and salinification. During the cooling process, water at the sea surface becomes heavier than at depth, producing a gravitational instability which leads to rapid convection. During this process, gravitational potential energy is converted into kinetic energy for turbulence and internal waves. Another sink of gravitational potential energy is baroclinic instability. The steep isopycnal surfaces in the oceans, such as those present at the meridional boundaries of the wind-driven gyre, are unstable due to
baroclinic instability. Because of this a large amount of gravitational potential energy can be converted into the eddy kinetic energy. A further source of energy due to wind stress is the one injected into surface waves. This input can be treated as a form drag for the atmospheric boundary layer. When discussing mechanical energy of the ocean, one also needs to include diapycnal and isopycnal mixing. Isopycnal mixing involves the least amount of gravitational potential energy so it is the dominating form of mixing on large scales. Diapycnal mixing in the ocean is supported primarily by wind stress and tidal dissipation. Finally, other sources/sinks of mechanical energy include: geothermal heat, bottom drag and atmospheric loading. An understanding of ocean energetics is crucially important to determining which mechanisms drive the meridional Atlantic overturning cell originating the thermohaline circulation. (Huang, 2004)

Figure 4.2: Mechanical energy diagram for the ocean circulation. (Huang, 2004)
Stommel’s assumption that the strength of the overturning circulation is proportional to the differential buoyancy forcing at the ocean surface has been challenged in several papers. Huang (1999) references Sandstrom’s theorem stated above. As previously mentioned, in the real ocean, the heating and cooling sources are located at the sea surface, which obviously violates the theorem. Huang suggests that the mechanical energy needed to overcome the stratification and drive the observed overturning circulation is created in the interior through vertical mixing by tides, internal waves, eddies, and to a limited extent, geothermal energy.

5. The Marotzke and Stone Model

In 1995 Marotzke and Stone published a paper in the Journal of Physical Oceanography entitled “Atmospheric Transports, the Thermohaline Circulation and Flux Adjustments in a Simple Coupled Model” which addresses a simple four box ocean-atmosphere model of the North Atlantic from 10N to 75N. The ocean model is Stommel’s and the atmospheric model gives the surface heat and freshwater fluxes as residuals of the atmospheric energy and moisture budgets which are assumed to be in balance. Figure 5.1 shows a schematic drawing of the four box model constructed by Marotzke and Stone. The model consists of two ocean boxes and two atmospheric boxes.
Figure 5.1. Schematic drawing of the four box ocean-atmosphere model constructed by Marotzke and Stone (1995).

The ocean boxes are well mixed and have a depth $D$. The righthand boxes represent the higher latitudes and the left hand boxes represent the lower latitudes. $H_1$ and $H_2$ are the ocean heat gain through the surface in the higher latitudes and the lower latitudes respectively. $H_{01}$ and $H_{02}$ are atmospheric energy gain at the top and $H_d$ is the meridional energy transport in the atmosphere. $E$ is the net evaporation in the low latitudes and the net precipitation at higher latitudes and $F_w$ is the meridional atmospheric moisture transport.

The conservation equations for the ocean are (Stommel, 1961; Marotzke, 1990)

$$\frac{dT_1}{dt} = H_1 + |q|T$$

(5.1)

$$\frac{dT_2}{dt} = H_2 - |q|T$$

(5.2)

$$\frac{dS_1}{dt} = -H_s + |q|S$$

(5.3)
\[
\frac{dS_2}{dt} = H_s - |q|S \tag{5.4}
\]

where \( T \) is the temperature difference between the lower and higher latitudes

\[
T = T_2 - T_1 \tag{5.5}
\]

\( S \) is the salinity difference

\[
S = S_2 - S_1 \tag{5.6}
\]

and the virtual surface salinity flux is related to the surface freshwater flux \( E \) through

\[
H_s = S_o \frac{E}{D} \tag{5.7}
\]

where \( S_o \) is the constant reference salinity. Knowing that the atmospheric moisture flux of approximately .44Sv, must be balanced by net evaporation at low latitudes and net precipitation at high latitudes, Marotzke and Stone (1995) derived the following equation for \( E \)

\[
E = \frac{1}{\varepsilon_w} \gamma T \tag{5.8}
\]

where \( \varepsilon_w \) is the ratio of the ocean area to the catchment area of the ocean basin and \( \gamma \) is the freshwater flux coefficient.

\[
H_s = \frac{1}{\varepsilon_w} \frac{S_o}{D} \gamma T \tag{5.9}
\]

The flow strength \( q \) is related to the meridional density gradient by the linear law

\[
q = k[\alpha T - \beta S] \tag{5.10}
\]

where \( \alpha \) and \( \beta \) are the thermal and haline expansion coefficients mentioned in section 2 and \( k \) is a proportionality constant. Assuming that the circulation is proportional to the density difference between the ocean boxes is the fundamental assumption made by
Stommel about the thermohaline circulation. In Marotzke and Stone (1995) they show that the meridional heat flux gradient can be written as

$$H_2 - H_1 = \frac{(A - (2\chi + B)T)}{\epsilon \rho_o D}$$

(5.11)

where $A$ is the net incoming radiation at low latitudes minus the net incoming radiation at high latitudes, $B$ is the longwave radiation coefficient, $\epsilon$ is the fractional area of the ocean compared to the surface area of the earth, $\chi$ is the heat flux coefficient and $\epsilon \rho_o D$ is the heat capacity of a unit water column. Subtracting Eq. 5.1 form Eq. 5.2 and applying Eq. 5.10 and Eq. 5.11 we arrive at the following equation

$$\frac{dT}{dt} = \frac{(A - (2\chi + B)T)}{\epsilon \rho D} - 2k[aT - \beta S]$$

(5.12)

Subtracting Eq. 5.3 from 5.4 and applying Eq. 5.9 and Eq. 5.10 we arrive at

$$\frac{dS}{dt} = \frac{2S_o yT}{\epsilon \rho D} - 2k[aT - \beta S]S$$

(5.13)

Setting the time derivatives equal to zero, we can solve for the equilibrium temperature and salinity differences. Doing so, we find the following equation for the salinity difference as a function of the equilibrium temperature.

$$S = \frac{\alpha T}{\beta} \mp \frac{(A - (2\chi + B)T)}{2k\beta \epsilon \rho D}$$

(5.14)

$$\left(\left(\frac{2\chi + B}{\epsilon \rho \beta D} + \frac{2S_o}{\epsilon \rho D}\right)\right) T^3 \mp \left(\left(\frac{(2\chi + B)/\epsilon \rho D)^2}{2k\beta}\right) + \frac{\alpha A}{\epsilon \rho \beta D}\right) T^2 \mp \left(\frac{A(2\chi + B)}{2\epsilon \rho D} - \frac{A^2}{2(\epsilon \rho D)^2 k\beta} \right) = 0$$

(5.15)

In these equations the signs depend on the sense of circulation. The upper signs correspond to the thermal mode (where $\alpha T - \beta S > 0$) and the lower signs to the haline
mode (where $\alpha T - \beta S > 0$). The values given for these parameters in the original buoyancy driven model are as follows:

\[
\begin{align*}
A_2 &= 90 W m^{-2} \\
A_1 &= -40 W m^{-2} \\
A &= 130 W m^{-2} \\
B &= 1.7 W m^{-2} K^{-1} \\
\chi &= 1.3 W m^{-2} K^{-1} \\
c &= 4000 J kg^{-1} K^{-1} \\
\alpha &= 2 \times 10^{-4} K^{-1} \\
\beta &= 8 \times 10^{-4} psu^{-1} \\
\gamma &= 2 \times 10^{-10} ms^{-1} K^{-1} \\
k &= 2 \times 10^{-8} s^{-1} \\
\varepsilon &= 0.1666666667 \\
\varepsilon_w &= 0.3 \\
D &= 5000 m \\
S_0 &= 35 psu \\
\rho_0 &= 1000
\end{align*}
\]

If we solve Eq. 5.14 and Eq. 5.15 with these values, we obtain a thermal mode equilibrium $(T_T, S_T) = (26.7 K, 1.5 psu)$ and a haline mode equilibrium of

$(T_H, S_H) = (29.5 K, 8.4 psu)$.

6. The Marotzke and Stone Model under the Energy Constraint

In the climate model, the tube is a closed square loop. Using the density equation, including diffusion, the density distribution within the tube is governed by the balance between advection and diffusion.

\[
\frac{w}{\partial z} = k \frac{\partial^2 \rho}{\partial z^2}
\]  

(6.1)
where \( w \) is the velocity in the \( z \) direction which is the direction of the tube, \( \rho \) is the density and \( k \) is the diffusivity. The scale for the vertical velocity is therefore

\[
w = \frac{k}{D}
\]  

(6.2)

Huang (1999) claims that \( k \) can be related to the energy dissipation through

\[
kN^2 = \alpha \epsilon
\]  

(6.3)

where \( N^2 \) is the buoyancy frequency, \( \alpha \) is a proportionality constant and \( \epsilon \) is the energy dissipation rate per unit mass. We are going to consider density variations which result from both thermal and salinity forcing and we rewrite equation (6.2)

\[
w = \frac{E}{g'}
\]  

(6.4)

where \( g' = g(\alpha T - \beta S) \). By continuity

\[
q = \frac{w}{h} = \frac{E}{g'h}
\]  

(6.5)

where \( h \) is the depth of the water column. Introducing the following energy parameter

\[
e = \frac{\rho_0 E}{gh}
\]  

(6.6)

and substituting Eq. 6.6 and \( g' \) into Eq. 6.5 we obtain

\[
q = \frac{w}{h} = \frac{e}{\rho_0(\alpha T - \beta S)}
\]  

(6.7)

This result is quite different from the flow strength equation used in Stommel's model.

If we now apply (6.7) to our ocean conservation equations (Eq. 5.1-5.4) we find

\[
\frac{dT}{dt} = \frac{(A - (2\chi + B)T)}{\alpha \rho_0 D} - \frac{2eT}{\rho_0(\alpha T - \beta S)}
\]  

(6.8)
If we set the time derivatives equal to zero we obtain the following equilibrium expressions for the temperature and salinity differences.

\[
\frac{dS}{dt} = \frac{2S_0 \gamma T}{\varepsilon_w D} - \frac{2eS}{\rho_0 |\alpha T - \beta S|}
\]

(6.9)

\[
S = \frac{\alpha T}{\beta} \mp \frac{2e\varepsilon D\varepsilon}{\beta (A - (2\gamma + B)T)}
\]

(6.10)

\[
T = \left( \frac{\alpha A \mp 2e\varepsilon D\varepsilon}{\alpha (2\gamma + B) + \frac{2\gamma S_0 c\beta \rho_0 \varepsilon}{\varepsilon_w}} \right)
\]

(6.11)

Again, the upper signs correspond to the thermal mode and the lower signs to the haline mode.

7. Sensitivity study of the Model Under both Buoyancy and Energy Constraints

7.1 The model under the buoyancy constraint

In this section, a sensitivity study on the equilibrium equations under the buoyancy constraint is presented. We look at how variations in \(k\), a parameter which is proportional to the flow strength, affect the equilibrium temperature and salinity gradients. We then look at how variations in \(A\) and \(B\), the net incoming radiation at low latitudes minus the net incoming radiation at high latitudes and the longwave radiation coefficient respectively, affect the equilibrium temperature gradient (\(A\) and \(B\) are absent from the equations for the equilibrium salinity gradient under the buoyancy constraint).

In addition, we will examine how variations in \(A\) and \(B\) affect the relationship between the equilibrium temperature gradient and \(k\). Since the equation for the equilibrium temperature gradient is quite complex we perform our sensitivity study on the
equilibrium equations for the temperature and salinity gradients under the limits $\alpha T \gg \beta S$ and $\alpha T \ll \beta S$. The equilibrium equation for the temperature gradient under the first limit is

$$T_{eq} = -\frac{\left(\frac{2\alpha + B}{\alpha \beta D}\right)^2 + \sqrt{\left(\frac{2\alpha + B}{\alpha \beta D}\right)^2 + \frac{8\alpha A}{\alpha \beta D}}}{4\alpha}$$

(7.1.1)

There is a hyperbolic relationship between the equilibrium temperature gradient and $k$ (fig. 7.1.1). When $k$ increases, the flow between the two ocean boxes increases, which results in a lower equilibrium temperature gradient.

![Graph of equilibrium temperature gradient](image)

**Figure 7.1.1** Graph of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \gg \beta S$) as it varies with $k$.

The equilibrium equation for the salinity gradient under the same limit is

$$S_{eq} = \frac{S_0}{k \alpha \epsilon_w D}$$

(7.1.2)

Figure 7.1.2 shows that there is also a hyperbolic relationship between the equilibrium salinity gradient and $k$. When $k$ increases the flow between the two ocean
boxes increases. This results in a lower equilibrium salinity gradient. In addition, there is a linear relationship between the equilibrium temperature gradient and the equilibrium salinity gradient as they vary with $k$. (fig. 7.1.3).

**Figure 7.1.2.** Graph of equilibrium salinity gradient (from the model under the buoyancy constraint and the limit $\alpha T \gg \beta S$) as it varies with $k$.

**Figure 7.1.3** Graph of equilibrium temperature gradient as it relates to the equilibrium salinity with the same $k$ value (from the model under the buoyancy constraint and the limit $\alpha T \gg \beta S$).
Now we look at the effects of increasing the parameters $A$ and $B$ (the net incoming radiation at low latitudes minus the net incoming radiation at high latitudes and the longwave radiation reflection coefficient respectively). In figures 7.1.4 and 7.1.5 we vary the parameter $A$. From figure 7.1.4 it is evident that the equilibrium temperature gradient increases when $A$ increases. In other words, an increase in the net incoming radiation in the lower latitudes leads to a higher temperature in the same region. Therefore there will be simultaneous increases in the net incoming radiation gradient and the temperature gradient between the two ocean boxes. Figure 7.1.5 illustrates the effect of $A$ on the relationship between equilibrium temperature gradient and $k$ and shows that increasing $A$ increases the equilibrium temperature gradient and the steepness of the curve. In figure 7.1.5 the arrow points in the direction of graphs with increasing $A$. Therefore, as the gradient in net incoming radiation increases, the equilibrium temperature gradient decreases more as the flow increases.

![Graph of equilibrium temperature gradient](image)

**Figure 7.1.4** Graph of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T >> \beta S$) as it varies with $A$. 

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Figure 7.1.5 The relationship of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \gg \beta S$) with $k$ for different values of $A$.

From figure 7.1.6 it is evident that the equilibrium temperature gradient decreases when $B$ increases. Therefore, if the ocean boxes are reflecting more of the incoming radiation the equilibrium temperature gradient will be lower. Then figure 7.1.5, which illustrates the effect of $B$ on the relationship between equilibrium temperature and $k$, shows that increasing $B$ decreases the equilibrium temperature gradient and the steepness of the curve. In figure 7.1.7 the arrow indicates the direction in which the graphs vary with increasing $B$. From figures 7.1.4 through 7.1.7 we can see that increasing $A$, the net incoming radiation gradient, has the opposite effect as increasing $B$, the longwave radiation reflection coefficient, on the equilibrium temperature gradient.
Figure 7.1.6 Graph of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $aT >> \beta S$) as it varies with $B$.

Figure 7.1.7 The relationship of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $aT >> \beta S$) with $k$ for different values of $B$. 
In the limit $\alpha T \ll \beta S$ the equation for the equilibrium temperature gradient is

$$T_{eq} = \frac{A}{3k\beta S \alpha \rho D + 2\chi + B}$$

(7.1.3)

Although the relationship between $k$ and the equilibrium temperature gradient seems to be linear for realistic values of $k$ under this limit (fig. 7.1.8), if we look at the graph as $k$ becomes infinitely large we realize that there is a hyperbolic relationship between the equilibrium temperature and $k$(fig. 7.1.9) as was seen under the first limit. This is expected, since an increase in flow between the two ocean boxes should decrease the equilibrium temperature gradient in the model.

![Graph of equilibrium temperature gradient](image)

Figure 7.1.8 Graph of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \ll \beta S$) as it varies with $k$. 

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Figure 7.1.9 Graph of the relationship between the equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \ll \beta S$) with $k$ as $k$ gets very large.

The equation for the equilibrium salinity gradient under the second limit is

$$S_{eq} = \frac{S_0 \gamma T}{\epsilon \omega D k \beta}$$

(7.1.4)

Figure 7.1.10 shows that there is a hyperbolic relationship between the equilibrium salinity gradient and $k$. This again is expected because increasing the flow between boxes will lower the salinity gradient.
Figure 7.1.10 Graph of equilibrium salinity gradient (from the model under the buoyancy constraint and the limit $\alpha T << \beta S$) as it varies with k.

Figure 7.1.11 shows that when the net incoming radiation gradient is increased the equilibrium temperature increases. Figure 7.1.12 illustrates that increasing the reflected longwave radiation coefficient has the opposite effect on the equilibrium temperature gradient. Figure 7.1.13, shows that increasing the net incoming radiation gradient increases the equilibrium temperature and the steepness of the slope. Figure 7.1.14 shows that increasing the longwave radiation reflection coefficient has the opposite effect, it decreases the equilibrium temperature and the steepness of the curve. Overall, the figures in this section show that variations in the selected parameters have the same effect on the temperature gradient under both limits.
Figure 7.1.11 Graph of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \ll \beta S$) as it varies with $A$.

Figure 7.1.12 Graph of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \ll \beta S$) as it varies with $B$. 
Figure 7.1.13 The relationship of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \ll \beta S$) with $k$ for different values of $A$.

Figure 7.1.14 The relationship of equilibrium temperature gradient (from the model under the buoyancy constraint and the limit $\alpha T \ll \beta S$) with $k$ for different values of $B$. 
7.2 The model under the energy constraint

We now look at the model under the energy constraint and perform a similar sensitivity study. We are going to look at how variations in \( e \), the rate of mixing, affect the equilibrium temperature and salinity gradients. We then look at how variations in \( A \) and \( B \), the net incoming radiation at low latitudes minus the net incoming radiation at high latitudes and the longwave radiation coefficient respectively, affect the equilibrium temperature and salinity gradients. In addition, we examine how variations in \( A \) and \( B \) affect the relationship between the equilibrium temperature gradient and \( e \). The equation for the equilibrium temperature and salinity were derived in section 8 (eq. 6.11 and eq. 6.10) where the upper signs correspond to the thermal mode and the lower to the haline mode.

\[
T_{eq} = \left( \frac{\alpha A \mp 2e \epsilon D_{\epsilon}}{\alpha(2X + B) + \frac{2\gamma S_0 c \beta \rho_0 \epsilon}{\epsilon_w}} \right) \\
S_{eq} = \frac{\alpha T}{\beta} \mp \frac{2\epsilon \epsilon D T}{\beta(A - (2X + B)T)}
\]

We first look at the energy constrained model under the thermal mode. Since we are looking at the energy constrained equations we first look at variations in equilibrium temperature and salinity gradients with variations in the parameter \( e \). As is clearly evident in figures 7.2.1 and 7.2.2, the equilibrium temperature and salinity gradients decrease linearly with increasing \( e \) values. In other words, an increase in density mixing lowers the temperature and salinity gradients between the ocean boxes.
Figure 7.2.1 Graph of equilibrium temperature gradient (from the model under the energy constraint and in the thermal mode) as it varies with e.

Figure 7.2.2 Graph of equilibrium salinity gradient (from the model under the energy constraint and in the thermal mode) as it varies with e.

The slope is maintained when $A$ values are increased but the equilibrium temperatures decreases (fig. 7.2.3). When $B$ is increased the equilibrium temperature decreases and the slope decreases (fig. 7.2.4).
Figure 7.2.3 The relationship between the equilibrium temperature (from the model under the energy constraint and in the thermal mode) and e for different values of A.

Figure 7.2.4 The relationship between the equilibrium temperature (from the model under the energy constraint and in the thermal mode) and e for different values of B.

In figures 7.2.5 to 7.2.8 the results from the sensitivity study done on the haline mode equations are shown. We see similar linear relationships between e and the equilibrium temperature and salinity gradients. However, both the salinity and temperature gradients increase with increasing e. In addition variations in A and B have a
similar effect on these relationships as they did in the thermal mode. This was expected since the haline and thermal mode equations under the energy constraint are almost identical.

**Figure 7.2.5** Graph of equilibrium temperature (from the model under the energy constraint and in the haline mode) as it varies with $e$.

**Figure 7.2.6** Graph of equilibrium salinity (from the model under the energy constraint and in the haline mode) as it varies with $e$. 

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Figure 7.2.7 The relationship between the equilibrium temperature (from the model under the energy constraint and in the haline mode) and e for different values of A.

Figure 7.2.8 The relationship between the equilibrium temperature (from the model under the energy constraint and in the haline mode) and e for different values of B.
8. Stability Analysis

Now we look at the stability of our differential equations for temperature and salinity under both the buoyancy and energy constraints. First we will look at the system under the buoyancy constraint. The differential equation for temperature is

\[ \frac{dT}{dt} = \frac{(A - (2 \chi + B)T)}{\alpha \rho D} + 2k(\alpha T - \beta S)T \]  

(8.1)

The sign of the second term on the right hand side of the equation depends on the mode of circulation with the upper sign applying to the thermal case and the lower sign to the haline case. To analyze the stability of the equilibrium to infinitesimal perturbations we write

\[ T = T_{eq} + T' \]  

(8.2)

where \( T_{eq} \) is the equilibrium value and \( T' \) is the infinitesimal perturbation. Substituting (8.2) into (8.1) we have

\[ \frac{dT_{eq}}{dt} + \frac{dT'}{dt} = \frac{A}{\alpha \rho D} - \frac{(2 \chi + B)T_{eq}}{\alpha \rho D} = \frac{(2 \chi + B)T_{eq}}{\alpha \rho D} + 2kT_{eq}(\alpha T_{eq} - \beta S_{eq}) + 2kT'(\alpha T_{eq} - \beta S_{eq} + \alpha T' - \beta S') \]

(8.3)

Since, by definition, the derivative of the equilibrium temperature is zero we can neglect the first term on the left hand side of the equation and the first, second and fourth terms from the right hand side. The equation therefore simplifies to

\[ \frac{dT}{dt} = -\frac{(2 \chi + B)}{\alpha \rho D} T + 2kT_{eq}(\alpha T_{eq} - \beta S_{eq}) + 2kT'(\alpha T_{eq} - \beta S_{eq} + \alpha T' - \beta S') \]

(8.4)

The product of prime terms is extremely small so we can linearize the equation and arrive at the following

\[ \frac{dT'}{dt} = -\frac{(2 \chi + B)}{\alpha \rho D} T' + 2k(T_{eq}(\alpha T' - \beta S') + T'(\alpha T_{eq} - \beta S_{eq})) \]

(8.5)
The same method can be applied to the differential equation for salinity and we arrive at the following linearized equation

\[
\frac{dS'}{dt} = \frac{2S_0 T'}{\varepsilon w D} + 2k\left(S_{eq}(\alpha T' - \beta S') + S'(\alpha T_{eq} - \beta S_{eq})\right)
\]  \hspace{1cm} (8.6)

Figures 8.1 and 8.2 illustrate the behavior of these linearized equations under the thermal and haline modes respectively as the perturbations in temperature and salinity are increased.

The same method can be applied to the differential equation for salinity and temperature under the energy constraint and we arrive at the following linearized equations

\[
\frac{dT'}{dt} = \frac{(2\chi + B)T'}{\alpha \rho D} - 2e\left(\frac{T'}{\rho_0 (\alpha T_{eq} - \beta S_{eq})} - \frac{T_{eq}(\alpha T' - \beta S')}{\rho_0 (\alpha T_{eq} - \beta S_{eq})^2}\right)
\]  \hspace{1cm} (8.7)

\[
\frac{dS'}{dt} = \frac{2S_0 T'}{\varepsilon w D} - 2e\left(\frac{S'}{\rho_0 (\alpha T_{eq} - \beta S_{eq})} - \frac{S_{eq}(\alpha T' - \beta S')}{\rho_0 (\alpha T_{eq} - \beta S_{eq})^2}\right)
\]  \hspace{1cm} (8.8)

Figures 8.3 and 8.4 illustrate the behavior of these linearized equations under the thermal and haline modes respectively as the perturbations in temperature and salinity are increased.

Although a quick glance at figures 8.1 -8.4 would lead one to believe that there are both significant stabilities and instabilities to increased perturbations in temperature and salinity, the values on the y axes of the graphs are actually quite small and therefore the slopes are approximately zero and both models under both thermal and haline modes are stable to increasing perturbations in salinity and temperature.
Figure 8.1 Results of stability analysis on the linearized differential equations for temperature and salinity (in the model under the buoyancy constraint and under the thermal mode) to increasing temperature and salinity perturbations.
Figure 8.2 Results of stability analysis on the linearized differential equations for temperature and salinity (in the model under the buoyancy constraint and under the haline mode) to increasing temperature and salinity perturbations.
Figure 8.3. Results of stability analysis on the linearized differential equations for temperature and salinity (in the model under the energy constraint and under the thermal mode) to increasing temperature and salinity perturbations.
A stability analysis was then performed on the full (non-linearized) equations to check the validity of our approximation. It was discovered that almost all of the graphs remained the same except the differential equations for temperature subject to variations in the temperature perturbation under the buoyancy constraint in the haline mode and under the energy constraint in the thermal mode. These two graphs illustrated a quadratic and unstable behavior (fig. 8.5 and fig. 8.6) as the perturbations became very large. However, for realistic perturbations these equations remained linear and decreasing and thus our results found using the linearized equations were correct for realistic perturbations.
Figure 8.5 A graph of the non-linearized differential equation for temperature (under the buoyancy constraint in the haline mode) as it varies with increasing perturbations in temperature.

Figure 8.6 A graph of the non-linearized differential equation for temperature (under the energy constraint in the thermal mode) as it varies with increasing perturbations in temperature.
9. Conclusions

Through this study many characteristics of the Marotzke and Stone model were observed. In the sensitivity study it was evident that the model under the buoyancy constraint reacted to variations in flow, net incoming radiation gradient and longwave radiation reflection coefficient in the same manner regardless of whether it was in thermal mode or haline mode. However, under the energy constraint the equilibrium temperature and salinity had opposite relationships to the energy parameter in the thermal and haline modes. It is interesting to compare the relative sensitivities of the two models to variations in the net incoming radiation gradient and the longwave radiation reflection coefficient. The model under the buoyancy constraint under both limits becomes more sensitive to changes in both of these parameters as the rate of flow between the two ocean boxes decreases. The model under the energy constraint, in contrast, becomes more sensitive to changes in the longwave radiation reflection coefficient as the energy parameter decreases but the model’s sensitivity to changes in the net incoming radiation gradient is unaltered by changes in the energy parameter. This suggests that the model under the energy constraint is less sensitive to global climate changes. Overall, the stability analysis shows that the linearized model under both the buoyancy and energy constraint is stable to perturbations in temperature and salinity. The non-linearized model was stable as long as the perturbations in temperature and salinity were realistic, thus justifying our simplifying the equations by linearization.

Some considerations can be made regarding possible scenarios of future global warming. The parameter B, the long-wave radiation reflection coefficient, can be considered to depict the long wave radiation exiting from the top of the atmosphere. B
decreases with increasing global mean temperature, as global warming causes enhanced evaporation, thereby deepening clouds and consequently enhancing the absorption of back-scattered long-wave radiation. The parameter $A$ represents the difference between low and high latitudes radiative forcing (incoming radiation). Global warming would enhance $A$, as the subtropics dry out and the high latitudes experience heavier precipitation.

Finally, the energy dissipation rate is proportional to the vertical density distribution (eq. 6.3). Although ice melting is not included in the model, most climate models do predict (and observations from Greenland confirm) significant high-latitude glacial melt as a consequence of global warming. This large scale introduction of cold but very fresh water may lighten the sinking waters of the high latitudes thereby reducing the vertical stratification. This density reduction could weaken the energy dissipation rate and the haline mode might become the dominant one. While the simplicity of this model must be considered carefully in its interpretation, these implications are worthy of being explored with more sophisticated multi-box models.
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11. Bibliography


