Vector Calculus of Periodic Non-Sinusoidal Signals for Decomposition of Power Components in Single and Multiphase Circuits

by

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Abstract

A vector space is used to represent periodic voltage and current signals in electrical networks. This representation is used to derive a periodic steady state impedance calculus, using a matrix impedance analogous to the complex impedance of sinusoidal steady state circuit analysis. The calculus provides simple matrix equations to calculate the total non-sinusoidal periodic response of linear time-invariant circuits and certain non-linear circuits.

The vector representation also provides convenient equations for time average power quantities, such as average power, apparent power, and inactive power. The vector expressions for power components are used to derive a reactive power vector, which has properties analogous to the reactive power in sinusoidal systems. The vector not only obeys conservation, but also has a magnitude equal to the total inactive power. These two properties are desirable in any non-sinusoidal generalization of reactive power but both do not hold for any of the scalar decompositions of reactive power found in the literature. The vector formulation provides mathematical insight into the nature of the long-standing confusion over definitions of non-sinusoidal reactive power. While the dimensionality of the reactive power vector is high, projections of the vector are shown to obey conservation and can be used as signed, scalar measures of reactive power for specific applications.

The vector space representation is extended to multiport circuits, and specifically, multiphase circuits. Several waveform decompositions are shown, including a symmetrical components decomposition for non-sinusoidal three-phase waveforms. The impedance matrix calculus is extended to include mixed, three and four wire, delta and wye-connected, three-phase circuits. The multiphase impedance calculus provides convenient matrix equations that can be used to solve for the periodic steady state response of multiphase LTI circuits. The multiphase extension includes a definition for the multiphase non-sinusoidal reactive power vector, which, using the symmetrical components decomposition, can provide a balanced/unbalanced decomposition of inactive power.

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Contents

Chapter 1

Introduction

The work in this thesis is motivated by the need for power systems analysis tools that can be used when harmonics are present. In particular, the appropriate definition for reactive power in the presence of harmonics has been, and continues to be, the subject of much debate and confusion in engineering journals. At present, the standard IEEE definitions used in the industry are confusing and problematic, yet no better definitions have been widely adopted.[1] Increasing levels of harmonics, injected by non-linear and power electronic switching devices, are making the traditional sinusoidal approximations less valid and the need for new definitions more vivid.

The theory presented in this thesis revisits the subject using vector and matrix methods of linear algebra. The analysis provides some insight into the nature of the reactive power confusion. With further refinement, the vector approach could yield new analysis tools and intuition, which would have widespread application to metering, control and power factor optimization in periodic systems with harmonics.

1.1 History of Nonsinusoidal Reactive Power

Confusion over reactive power for periodic signals with harmonics dates to 1927, when Budeanu introduced an orthogonal decomposition of apparent power into active, reactive, and distortion power components.[2] While these components were observed to add in quadrature to equal the apparent power, neither reactive nor distortion power components could be assigned any physical significance. Furthermore, unlike the active and reactive components, distortion power disobeys conservation, making it counterintuitive as a time-

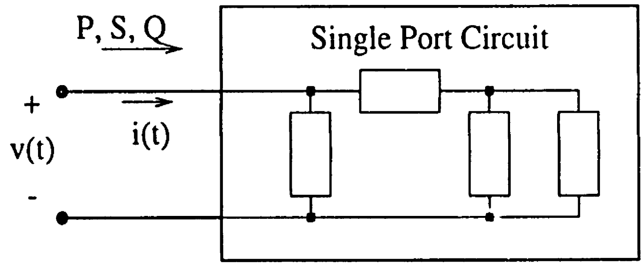

Figure 1-1: Single Port Circuit in Periodic Steady State

average power component.

Noting the limitations of Budeanu's two-component decomposition, Fryze introduced a single component definition of reactive power.[3] Fryze's reactive power is a signed orthogonal component accounting for the difference between apparent and average power. The definition is useful because if its magnitude is zero, apparent and average power are equal, corresponding to unity power factor. The sign portion of Fryze's definition, however, is not consistent and can not be uniquely defined. Furthermore, the definition does not obey conservation, meaning that a circuit with positive reactive power does not, in general, provide compensation for a circuit with negative reactive power.

Many other authors [4]-[12] have proposed alternative definitions of time-average reactive power quantities, but none have succeeded in defining a decomposition that not only accounts for the difference between apparent and average power, but also satisfies conservation. The reason for the difficulty becomes apparent when the problem is mapped to a vector space. As will be shown in this thesis, the inactive component of apparent power consists of many orthogonal elements, each of which is conserved and has an arbitrary sign convention. It is not possible to combine these orthogonal components into a single signed measure of reactive power that both reflects the inactive power and also obeys conservation.

1.2 Problem Statement

The work in this thesis focuses on the periodic, band-limited case, where the voltage and current waveforms throughout the circuit repeat, exactly, with period T, and contain only a finite number of harmonics. First single phase, single port circuits are considered, and then the notation is extended to include multiphase circuits in Chapter 5.

A single port circuit is shown in Figure 1-1. The port voltage and current waveforms are periodic such that

$$v(t) = v(t+T) \qquad \text{and} \qquad i(t) = i(t+T) \tag{1.1}$$

With the current defined as positive into the circuit port, the instantaneous power, $p(t)$, into the port is the product of voltage and current. The average, or active power, P, is the average of the product.

$$p(t) = v(t)i(t) \qquad \text{and} \qquad P = \overline{v(t)i(t)} \tag{1.2}$$

The unit used for power is the *Watt* or $J\ s^{-1}$, reflecting a flow of energy. P is a signed quantity, with positive corresponding to power *into* the port.

The apparent power, S, is the product of rms values, or the geometric mean of averages.

$$S = v_{rms} i_{rms} = \sqrt{\overline{v(t)^2}\ \overline{i(t)^2}} \tag{1.3}$$

Although S has the same physical units as the *Watt*, the unit VA is typically used, because S does not measure the flow of energy. S is not a signed quantity but a geometric mean, so one does not refer to the apparent power *into* the port, but to the apparent power *of* the port.

Power factor is a measure of how much of the apparent power results in a net transfer of energy into the port. Power factor can take the units $Watts/VA$, although it has no physical units.

$$PF = \frac{P}{S} \qquad \text{where} \qquad PF \leq 1 \tag{1.4}$$

The component of S that does not reflect active power will be called the total inactive power, Q. Q can be considered orthogonal to P, such that $Q = \sqrt{S^2 - P^2}$. Like S, Q is strictly non-negative and has the unit VA, as it does not correspond to a flow of energy. The decomposition of S into orthogonal components P and Q is depicted in the power triangle in Figure 1-2.

Power factor and inactive power are useful quantities for solving efficiency related optimization problems. Unity power factor corresponds to zero inactive power, where all of the apparent power is associated with useful energy transfer. In general, an increase in power factor results in improved efficiency, because losses are typically reduced. Such op-

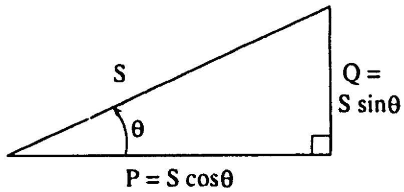

Figure 1-2: Power Component Decomposition

timization problems are called power factor optimization problems and usually correspond to solving for the most efficient operating conditions, given a set of constraints. Instead of a maximization of PF, the optimization could be stated as a minimization of $\frac{Q}{|P|}$. If P is constrained to be constant, the equivalent optimization is a minimization of Q only.

1.3 Review of Single Frequency Reactive Power

The definition of reactive power in purely sinusoidal systems is widely accepted. When the voltage and current waveforms throughout a circuit are sinusoidal, share a common frequency, ω, and have no D.C. component, reactive power, R, is a signed scalar quantity with a magnitude equal to the inactive power, Q. R has both physical significance and intuitive properties. In fact, the definition of R is based more on the intuition behind reactive power than on any particular mathematical decomposition of instantaneous power components.

Viewed in the time domain, for example, the appropriate definition for R is not clear. Using trigonometric identities, the equation for instantaneous power, $p(t)$, can be simplified to three orthogonal terms.

$$\begin{aligned} v(t) &= V cos(\omega t + \theta_V) \qquad \text{and} \qquad i(t) = I\, cos(\omega t + \theta_I) \\ p(t) &= v(t)i(t) = VI cos(\omega t + \theta_V) cos(\omega t + \theta_I) \\ &= \tfrac{VI}{2} cos(\theta_V - \theta_I) + \tfrac{VI}{2} cos(2\omega t + \theta_V + \theta_I) \\ &= \tfrac{VI}{2} cos(\theta_V - \theta_I) + \tfrac{VI}{2} cos(\theta_V - \theta_I) cos(2(\omega t + \theta_I)) + \tfrac{VI}{2} sin(\theta_V - \theta_I) sin(2(\omega t + \theta_I)) \end{aligned} \tag{1.5}$$

Using an alternate trigonometric simplification yields

$$p(t) = \frac{VI}{2}cos(\theta_I - \theta_V) + \frac{VI}{2}cos(\theta_I - \theta_V)cos(2(\omega t + \theta_V)) + \frac{VI}{2}sin(\theta_I - \theta_V)sin(2(\omega t + \theta_V)) \tag{1.6}$$

The first term is the same in both (1.5) and (1.6). This term is a constant equal to the average power, P. The second and third terms are orthogonal oscillatory components, with frequency 2ω and zero average value. The amplitude of the second term is P in both equations, but the amplitude of the third term has opposite signs in (1.5) and (1.6). As only the third term is associated with reactive power, reactive power is not *the* oscillatory component of instantaneous power, but *one of the* oscillatory components of instantaneous power. The exact expression for reactive power is not clear from the time-domain expressions.

The definition of inactive power, however, can be seen in Equation (1.6). The amplitude of the total 2ω oscillation is $\frac{VI}{2}$, which equals the apparent power, S. Because the amplitude of the first orthogonal 2ω component is P, then the amplitude of the second orthogonal component must be the inactive power, $Q = \sqrt{S^2 - P^2}$.

Viewed in the frequency domain, the definition of R is similarly unclear. The Fourier transform of the product of two sinusoids is the convolution of two complex-conjugate impulse pairs. As shown in Figure 1-3, the Fourier transform of $p(t)$ consists of a real impulse at zero frequency and a complex conjugate impulse pair at $\pm 2\omega$. The total area of the $\pm 2\omega$ impulse pair is S, but the phase angle of the impulse depends on the absolute time reference of the waveforms. Thus reactive power is not the *imaginary* power.

The most elegant expression for reactive power comes from phasor notation. If the voltage and current sinusoids are represented using the complex phasors, $\hat{V}$ and $\hat{I}$, then the time-average power components can be expressed using complex algebra.

$$\begin{aligned}
&v(t) = Re\{\hat{V}e^{j\omega t}\} \qquad \text{and} \qquad i(t) = Re\{\hat{I}e^{j\omega t}\} \\
&p(t) = Re\left\{\frac{\hat{V}\hat{I}^*}{2}\right\} + Re\left\{\frac{\hat{V}\hat{I}^*}{2}e^{2j\omega t}\right\} \\
&S = \left|\frac{\hat{V}\hat{I}^*}{2}\right| = \left|\frac{\hat{I}\hat{V}^*}{2}\right| \\
&P = Re\left\{\frac{\hat{V}\hat{I}^*}{2}\right\} = Re\left\{\frac{\hat{I}\hat{V}^*}{2}\right\} \\
&R = Im\left\{\frac{\hat{V}\hat{I}^*}{2}\right\} \neq Re\left\{\frac{\hat{I}\hat{V}^*}{2}\right\}
\end{aligned} \tag{1.7}$$

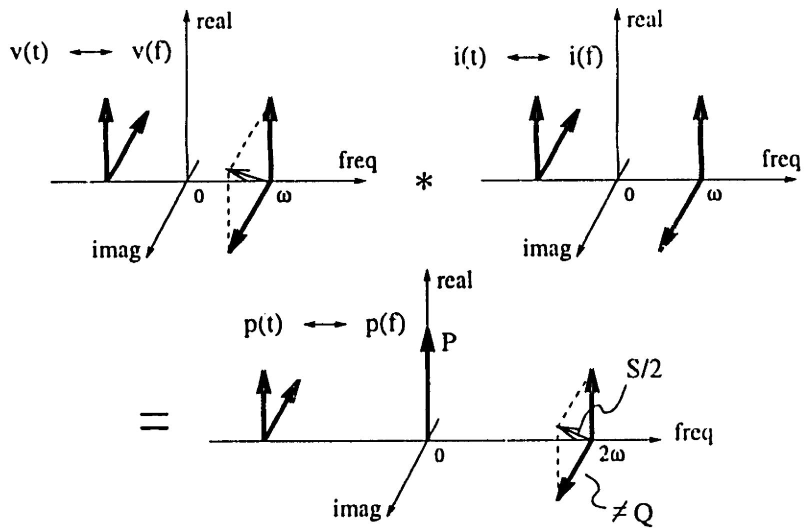

Figure 1-3: Frequency-Domain Power Representation

The phasor expressions for orthogonal time-average power components are widely used, leading to the common use of the terms *real* and *imaginary power* for *time-average* and *reactive power*. The $\neq$ in (1.7) reflects the sign ambiguity inherent in the definition of reactive power. The choice of $\frac{\hat{V}\hat{I}^*}{2}$ over $\frac{\hat{I}\hat{V}^*}{2}$ is only a matter of convention.

Intuitive Properties of Reactive Power

While the exact mathematical definition is somewhat hidden in abstract notation, the physical properties of sinusoidal reactive power have led to an intuitive understanding that is widely held. The intuition is based on the fact that the magnitude of R is equal to the inactive power, Q. As Q is associated with useless, or inefficient, oscillatory power transfer, minimizing $|R|$ maximizes efficiency. While this statement is not strictly true in all cases, it is a very good approximation in most power related applications.

A second property that is important in the intuitive understanding of reactive power is the conservation property. The total reactive power into a circuit port is the sum of the reactive powers into all the elements of the circuit. Therefore, in order to reduce the effective $|R|$ of a circuit, a compensation circuit can be added, either in series or parallel, which cancels the reactive power flow and improves efficiency.

The strong intuition behind reactive power conservation has led to the rating of capacitors directly in units of reactive power, the VAR. According to the most common sign convention, inductive loads *consume* reactive power, while capacitors *generate* reactive power. Technicians can provide compensation circuits simply by looking up the reactive consumption of a particular load and purchasing a similarly rated VAR capacitor. Large uncompensated reactive power loads are often charged according to reactive power *consumption*, and the conservation property is important to the valuation of the reactive energy unit.

The two most important properties of sinusoidal reactive power are conservation and the relationship to inactive power. The proper extension of the definition to include harmonics should account for these properties. This extension is the subject of Chapter 3.

Chapter 2

Vector Representation of Periodic Signals

2.1 Orthonormal Waveform Decomposition

We begin by considering periodic, real-valued voltage and current waveforms of an arbitrary single port circuit element. If the voltage and current waveforms are periodic with period T, such that $v(t+T) = v(t)$ and $i(t+T) = i(t)$, the waveforms can be considered members of the L_2 periodic function space with a scalar product and induced norm given in (2.1).

$$
\begin{aligned}
< x(t), y(t) > &= \frac{1}{T}\int_0^T x(t)y(t)dt \\
\|x(t)\| = < x(t), x(t) >^{\frac{1}{2}} &= \left[\frac{1}{T}\int_0^T x^2(t)dt\right]^{\frac{1}{2}}
\end{aligned} \tag{2.1}
$$

This function space is convenient for expressing rms and time-average power components of the circuit port in terms of the voltage and current member functions, $v(t)$ and $i(t)$. The rms value is the induced norm, while the average power, P, is the scalar product, and the apparent power, S, is a product of norms.

$$
\begin{aligned}
v_{rms} &= \|v(t)\| \\
P &= \overline{v(t)\, i(t)} = < v(t), i(t) > \\
S &= v_{rms}\, i_{rms} = \|v(t)\|\ \|i(t)\|
\end{aligned} \tag{2.2}
$$

The function space is then mapped to an ℓ_2 vector space, $\Re^n$, by expressing the periodic signals as a linear combination of n orthonormal basis functions, $\phi_j(t)$.

$$x(t) = \sum_{j=1}^{n} x_j \, \phi_j(t) \tag{2.3}$$

This equation has the form of a linear *synthesis* transform, where the inverse, or *decomposition* transform would be

$$x_j = \frac{1}{T} \int_0^T \phi_j(t) \, x(t) \, dt \tag{2.4}$$

This transform pair can be expressed in vector form by stacking the coefficients, x_j into a constant vector X, and the basis functions, $\phi_j(t)$ into a vector of time functions $\Phi(t)$.

$$x(t) = X^T \, \Phi(t) \tag{2.5}$$

where X^T denotes the transpose of X, and

$$X = \frac{1}{T} \int_0^T \Phi(t) \, x(t) \, dt \tag{2.6}$$

The constant vector X, then, fully represents the signal waveform, $x(t)$, in the ℓ_2 vector space associated with a particular orthonormal basis $\Phi(t)$.

The orthonormality of $\Phi(t)$ is expressed in the L_2 function space. The basis functions are orthogonal in the time-average sense, and normal in the *rms* sense.

$$< \phi_i(t), \phi_j(t) >= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.7}$$

or in vector notation

$$< \Phi(t), \Phi^T(t) >= \frac{1}{T} \int_0^T \Phi(t) \Phi^T(t) dt = \mathbb{I} \tag{2.8}$$

where $\mathbb{I}$ is the identity matrix.

The orthonormal property of $\Phi(t)$ provides a norm and scalar product equivalence between the L_2 function space and the ℓ_2 vector space. Substituting the vector expression

(2.5) into (2.1) gives

$$\begin{aligned} < x(t), y(t) >&= \frac{1}{T}\int_0^T X^T\Phi(t)Y^T\Phi(t)dt \\ &= X^T\left(\frac{1}{T}\int_0^T \Phi(t)\Phi^T(t)dt\right) Y \\ &= X^T Y = < X , Y > \end{aligned} \tag{2.9}$$

The norm equivalence makes makes the vector space similarly convenient for expressing rms and time-average power quantities, now in terms of the constant vectors for voltage and current.

$$\begin{aligned} v_{rms} &= \|V\| = \sqrt{V^TV} \\ P &=< V, I >= V^TI \\ S &= \|V\| \, \|I\| = \sqrt{V^TVI^TI} \end{aligned} \tag{2.10}$$

The vector expressions hold for any choice of orthonormal basis, provided that the same basis is used for both voltage and current, and provided that the basis is sufficiently rich to decompose, to some desired level of accuracy, the actual waveforms.

2.1.1 Frequency-Domain Representation

One common orthonormal basis for representing periodic signals is the Fourier basis. Here the basis consists of a constant, or D.C. component, and cosine and sine pairs at multiples of the fundamental frequency, $\omega = \frac{2\pi}{T}$.

$$\Phi_f(t) = \begin{bmatrix} 1 \\ \sqrt{2}cos(\omega t) \\ \sqrt{2}sin(\omega t) \\ \sqrt{2}cos(2\omega t) \\ \sqrt{2}sin(2\omega t) \\ \sqrt{2}cos(3\omega t) \\ \sqrt{2}sin(3\omega t) \\ \vdots \end{bmatrix} \tag{2.11}$$

While the length of this basis is infinite, $\Phi_f(t)$ can be truncated to length, $n \geq 2Tf_m + 1$, where f_m is the largest frequency in the band-limited signals. n is constrained to be an odd number, due to the single D.C. term plus the even number of harmonic components.

In the Fourier basis, a vector, X_f, fully represents the periodic waveform, $x(t)$. The elements of X_f are the *rms* values of the orthogonal frequency components in $x(t)$. Because the decomposition is orthonormal, the *rms* of $x(t)$ is the two-norm of the vector, $\|X_f\|$.

2.1.2 Time-Domain Representation

Another useful vector representation decomposes the waveform into n time samples, evenly spaced over the period, T. The decomposition represents the periodic waveform not with frequency components, but with orthogonal *rms* time components. As with the Fourier basis, n must be odd.

The basis functions for this vector space are time-delayed periodic sync functions. [1]

$$\Phi_t(t) = \begin{bmatrix} \psi(t) \\ \psi(t - T/n) \\ \psi(t - 2T/n) \\ \vdots \\ \psi(t - (n-1)T/n) \end{bmatrix} \tag{2.12}$$

where

$$\psi = \frac{sin(\frac{n\pi}{T}t)}{\sqrt{n}\, sin(\frac{\pi}{T}t)} \tag{2.13}$$

This basis is orthogonal in the time-average sense, and normalized in the *rms* sense. For large n, the basis approximates a set of delayed periodic impulses, with period T, height $\sqrt{n}$, approximate width $1/n$, and unity *rms* value.

Using this basis, a time waveform $x(t)$ is represented by the n-vector, X_t, where the k^{th} element of X_t is equal to $\frac{x(t)}{\sqrt{n}}$ sampled at $t = \frac{(k-1)T}{n}$. The vector, then, is a discrete representation of the periodic waveform normalized by $\sqrt{n}$. The normalization preserves the *rms* character of the vector, such that each element reflects the *rms* of each sample, not the *value* of each sample. Because the samples are orthogonal, the *rms* of the waveform is again the norm of the vector, $\|X_t\|$.

The time-domain basis is closely related to the Fourier basis, through an orthonormal transformation equivalent to the Discrete Fourier Transform. The transformation to discrete

[1] The periodic sync function is the basis obtained by combining the Discrete Fourier Transform *analysis* equation with the Discrete Fourier Series *synthesis* equation. See [13] for more information on these transforms.

time from the frequency domain can be expressed as multiplication by an $n \times n$ orthonormal transformation matrix, M_{tf}, where $M_{tf}M_{tf}^T = \mathbb{I}$. The rows of M_{tf} consist of discrete samples of the Fourier basis functions with sampling frequency n/T.

$$M_{tf} = \frac{1}{\sqrt{n}} \begin{bmatrix} \Phi_f^T(0) \\ \Phi_f^T(\frac{T}{n}) \\ \Phi_f^T(2\frac{T}{n}) \\ \vdots \\ \Phi_f^T(T - \frac{T}{n}) \end{bmatrix} \tag{2.14}$$

This transformation provides a convenient notation for transforming between the time domain and the frequency domain.

$$\begin{aligned} \Phi_t(t) &= M_{tf}\Phi_f(t) \qquad & X_t(t) &= M_{tf}X_f \\ \Phi_f(t) &= M_{ft}\Phi_t(t) \qquad & X_f(t) &= M_{ft}X_t \end{aligned} \tag{2.15}$$

where $M_{ft} = M_{tf}^T$ provides the transformation to frequency from the time domain.

The constraint that n is odd applies to both the time and frequency domain representations and reflects the fact that there is an odd number of degrees of freedom in a band-limited periodic waveform. The truncation of the Fourier basis to length n is equivalent to setting the time-domain sampling frequency to $\frac{n}{T}$. If the frequency basis can be truncated to length n without loss of information, then the waveform can be sampled at $\frac{n}{T}$ without aliasing. This statement is equivalent to the Nyquist Sampling Theorem, which states that there will be no aliasing if $f_m < \frac{1}{2}f_s$, where f_m is the maximum frequency in the continuous waveform, and f_s is the sampling frequency. The maximum frequency in an n-length Fourier basis is $\frac{n-1}{2T}$, so $\frac{n}{T}$ is the lowest synchronous sampling rate above the Nyquist rate.

2.2 Periodic Steady State Circuit Analysis

Using vector representation of signals with a specific basis, it is possible to extend sinusoidal steady (SSS) analysis techniques for linear time invariant (LTI) circuits to the more general periodic steady state case. In SSS analysis the complex impedance, $\hat{Z}$, is used to relate the complex steady state voltage and current phasors of a branch of the circuit, $\hat{V} = \hat{Z}\hat{I}$ and

$\hat{I} = \frac{\hat{V}}{\hat{Z}}$. Series connections add impedances, $\hat{Z} = \hat{Z}_1 + \hat{Z}_2$, while parallel connections add inverse impedance, $\frac{1}{\hat{Z}} = \frac{1}{\hat{Z}_1} + \frac{1}{\hat{Z}_2}$.

Using the impedance relationships, the SSS network solution is found using Kirchhoff's voltage and current laws (KVL and KCL), which apply not only to the instantaneous voltage and current, but also to the complex steady state phasors, $\hat{V}$ and $\hat{I}$. Short hand relationships such as Thevenin equivalent sources and the voltage and current divider laws make hand calculation of the steady state solution an elementary problem.

2.2.1 Analizing LTI Circuits Using the Fourier Basis

In band-limited periodic steady state, an LTI network solution contains a finite number of sinusoidal frequency components. Each frequency satisfies the SSS relationships, and the total response can be obtained by superposition. Instead of solving each frequency separately, however, the total periodic steady state solution can be found by stacking the SSS constraints for each frequency into a matrix equation that relates the voltage and current vectors in the Fourier basis.

Analogous to the complex impedance, the vector relationship between the periodic steady state voltage and current of a circuit branch is an $n \times n$ impedance matrix, Z, such that $V = ZI$ and $I = Z^{-1}V$. Here it is assumed that V and I are Fourier basis vectors, although the $_f$ subscript has been dropped for convenience. As each frequency in an LTI network satisfies the network constraints independently, it is not possible for different frequencies to cross-couple in the steady state response of a linear circuit. The Z matrix, then, is block diagonal, where each 2×2 block represents the voltage-current relationship of one harmonic. The single top-left entry in Z relates the zero-frequency, or D.C. voltage and current.

The form of the Z matrix for passive single port LTI elements, resistors, inductors, and capacitors, is found by transforming the time-domain constituent relation to the Fourier vector space, using the transform given in Equation (2.6). For LTI resistors,

$$v(t) = Ri(t) \quad \longleftrightarrow \quad \frac{1}{T}\int_0^T \Phi_f(t)v(t)dt = \frac{1}{T}\int_0^T \Phi_f(t)Ri(t)dt \tag{2.16}$$

Substituting V and I, the voltage and current vectors in the Fourier basis, the equation becomes

$$V_f = RI_f = Z_R I \quad \text{where} \quad Z_R = R\mathbb{I} \tag{2.17}$$

The impedance matrix for linear resistors, then, is proportional to the identity matrix, $\mathbb{I}$.

The impedance matrix for an inductor is similarly found by using the Fourier transform.

$$v(t) = L\frac{d}{dt}i(t) \quad \longleftrightarrow \quad \frac{1}{T}\int_0^T \Phi_f(t)v(t)dt = \frac{1}{T}\int_0^T \Phi_f(t)L\frac{d}{dt}i(t)dt \tag{2.18}$$

Substituting V and I gives

$$V = L\left[\frac{1}{T}\int_0^T \Phi_f(t)\frac{d}{dt}\Phi_f^T(t)dt\right] I \tag{2.19}$$

The time derivative of the Fourier basis may be expressed as the basis itself multiplied by the transpose of a special matrix called the J_ω matrix.

$$\frac{d}{dt}\Phi(t) = J_\omega^T \Phi(t) \tag{2.20}$$

J_ω is the $n \times n$ block diagonal, skew-symmetric matrix.

$$J_\omega = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & \omega & 0 & 0 & 0 & 0 & . \\ 0 & -\omega & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 2\omega & 0 & 0 & . \\ 0 & 0 & 0 & -2\omega & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 3\omega & . \\ 0 & 0 & 0 & 0 & 0 & -3\omega & 0 & . \\ . & . & . & . & . & . & . & . \end{bmatrix} \tag{2.21}$$

where ω is the fundamental frequency, $\frac{2\pi}{T}$.

Substituting (2.20) into (2.19) gives the impedance relationship for an ideal inductor, which is proportional to J_ω.

$$V = LJ_\omega I = Z_L I \quad \text{where} \quad Z_L = LJ_\omega \tag{2.22}$$

The current-voltage relationship for a capacitor, $i(t) = C\frac{d}{dt}v(t)$, is the dual of the inductor relationship, so the impedance matrix for a capacitor is proportional to the inverse of J_ω.

$$I = CJ_\omega V = Z_C^{-1} V \quad \text{where} \quad Z_C = (CJ_\omega)^{-1} \tag{2.23}$$

The problem with (2.22) and (2.23) is that the J_ω matrix is singular due to the zero frequency entry in the top left. The singularity exists because it is not possible to determine the steady state D.C. voltage across an ideal capacitor from the D.C. current. In fact, there is no steady state capacitor voltage if the D.C. current is non-zero. Likewise, there is no steady state inductor current if the D.C. inductor voltage is non-zero. The singularity reflects actual physical properties of the inductor and capacitor, which are important to include in the periodic steady state analysis.

Numerical problems due to the singularity in J_ω can be solved by defining J_ω in the limit.

$$J_\omega = \lim_{\rho \to 0} \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & . \\ 0 & 0 & \omega & 0 & 0 & . \\ 0 & -\omega & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & 2\omega & . \\ 0 & 0 & 0 & -2\omega & 0 & . \\ . & . & . & . & . & . \end{bmatrix} \tag{2.24}$$

Using the limit, J_ω^{-1} exists and becomes,

$$J_\omega^{-1} = \lim_{\rho \to 0} \begin{bmatrix} \frac{1}{\rho} & 0 & 0 & 0 & 0 & . \\ 0 & 0 & \frac{1}{\omega} & 0 & 0 & . \\ 0 & -\frac{1}{\omega} & 0 & 0 & 0 & . \\ 0 & 0 & 0 & 0 & \frac{1}{2\omega} & . \\ 0 & 0 & 0 & \frac{1}{2\omega} & 0 & . \\ . & . & . & . & . & . \end{bmatrix} \tag{2.25}$$

With J_ω defined in the limit, the basic impedance relationships relating the periodic steady state voltage and current vectors for a particular circuit branch are

$$V = ZI \quad \text{and} \quad I = Z^{-1} V \tag{2.26}$$

Impedance matrices for LTI resistors, inductors, and capacitors are shown in Figure 2-1

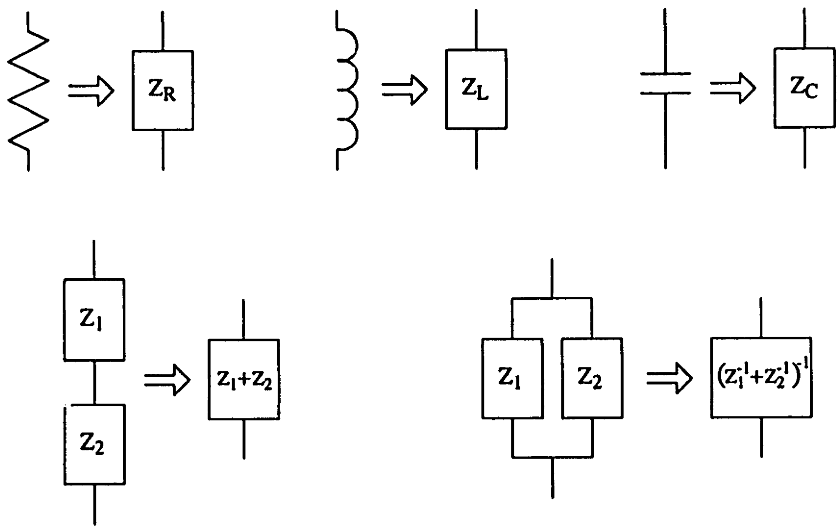

Figure 2-1: Periodic Steady State Impedance Representation

and are expressed in terms of the identity matrix, $\mathbb{I}$, and the J_ω matrix defined in the limit.

$$Z_R = R\mathbb{I} \qquad Z_L = \lim_{\rho \to 0} LJ_\omega \qquad Z_C = \lim_{\rho \to 0} (CJ_\omega)^{-1} \tag{2.27}$$

As in the SSS, the impedance matrix of a series connection is the sum of the individual impedance matrices, while the impedance matrix of a parallel connection is the inverse of the sum of inverses.

$$Z_{ser} = \lim_{\rho \to 0} Z_1 + Z_2 \qquad Z_{par} = \lim_{\rho \to 0} \left(Z_1^{-1} + Z_2^{-1}\right)^{-1} \tag{2.28}$$

Using (2.28), the equivalent steady state impedance matrix of a complicated LTI single port network is easily computed. The limit serves only to determine the top left element of Z, which gives the steady state D.C. voltage across the port as a linear function of D.C. current into the port. If the D.C. operating point is well determined, the D.C. element of the Z matrix will approach a finite value in the limit. If the network has a series capacitor or parallel inductor, the limit will approach ∞ or 0, as expected.

In any LTI network, the Z matrix has a block diagonal structure, where each 2×2 block corresponds to a particular harmonic frequency. Each block has skew-symmetric off-diagonal entries and identical diagonal entries; these two degrees of freedom are the SSS

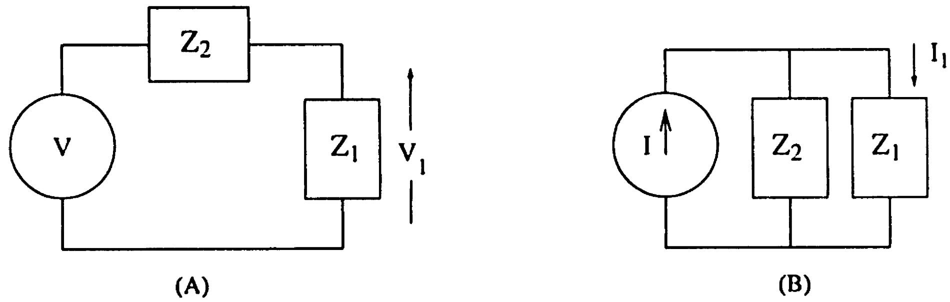

Figure 2-2: Voltage and Current Divider Laws

reactance and resistance of the circuit evaluated at the harmonic frequency. If the circuit happens to have a high Q resonance at one of the harmonic frequencies, the corresponding block would approach zero or infinity, depending on the nature of the resonance. As with the D.C. singularity, the singularity caused by resonance properly reflects the indeterminate nature of the periodic steady state response.

Short hand constituent relations for periodic steady state can be derived from LTI SSS methods. For example, the Thevenin equivalent voltage source of a port is the periodic open circuit voltage of the port. The Thevenin equivalent impedance is the effective impedance matrix seen looking into the port with all independent sources set to zero. As the special structure of LTI Z matrix has only n degrees of freedom, the Thevenin impedance matrix can be determined by measurements of open circuit voltage and short circuit current, provided there is excitation at every harmonic frequency.

For the circuit shown in Figure 2-2(A), an analogous voltage divider law can be used to determine the voltage, V_1, across Z_1 when the series connection of Z_1 and Z_2 is subject to the excitation voltage V. Here $I = (Z_1 + Z_2)^{-1}V$, and $V_1 = Z_1 I$, so the voltage divider law for the periodic steady state can be written.

$$V_1 = \lim_{\rho \to 0} Z_1 \left(Z_1 + Z_2\right)^{-1} V \qquad (2.29)$$

Similarly, for Figure 2-2(B), the current divider law specifies that if a parallel connection of Z_1 and Z_2 is subject to the total current, I, then I_1, the current through Z_1 is

$$I_1 = \lim_{\rho \to 0} Z_1^{-1} \left(Z_1^{-1} + Z_2^{-1}\right)^{-1} I = \lim_{\rho \to 0} \left(Z_1 + Z_2\right)^{-1} Z_2 I \qquad (2.30)$$

The $\lim_{\rho \to 0}$ is subsequently dropped for notational simplicity. The limit is implied in the Z matrix definition, and is only needed to solve the matrix inversion.

2.2.2 Analyzing LTI Circuits in the Time Domain

The preceding development of LTI periodic steady state analysis techniques used the Fourier basis, which decomposes the periodic voltage and current waveforms into vectors of orthogonal frequency components. In fact, any orthonormal decomposition could be used, although the simple block diagonal structure of the impedance matrix is unique to the Fourier basis. The time-domain basis, described in Section 2.1.2, provides another useful decomposition, where the Z matrix has a different simple structure.

To map the periodic steady state impedance relationships to the time domain, we use the orthonormal transformation matrix, M_{tf}, and its inverse, M_{ft}. We can re-write the basic periodic steady state impedance relationship in (2.26) as follows.

$$\begin{aligned} V_f &= Z_f I_f \\ M_{tf} V_f &= M_{tf} Z_f I_f = M_{tf} Z_f M_{ft} M_{tf} I_f \\ V_t &= M_{tf} Z_f M_{ft} I_t = Z_t I_t \end{aligned} \tag{2.31}$$

The $Z_t = M_{tf} Z_f M_{ft}$ transformation gives the time-domain impedance matrix from the block diagonal frequency-domain impedance. The inverse impedance matrix is transformed the same way. $Z_t^{-1} = (M_{tf} Z_f M_{ft})^{-1} = M_{tf} Z_f^{-1} M_{ft}$

Bearing in mind that V_t and I_t contain discrete samples of the waveforms over one period, and that $V_t = Z_t I_t$, the first column in Z_t must be the discrete periodic voltage resulting from a discrete periodic current impulse at $t = 0$. The special structure of the time-domain Z matrix, then, is that the columns are the discrete periodic impulse response relating current to voltage in the time domain. As the circuit is time invariant, the second column in Z_t must be the same, but shifted down circularly by one. Thus the time-domain LTI impedance matrix also has only n degrees of freedom.

Examples of periodic steady state circuit analysis can be found in Chapter 4, with applications of the reactive power definitions of Chapter 3. The circuit analysis techniques are extended to include multiphase circuits in Chapter 5. The next section provides an extension to include certain switching circuits.

2.2.3 Steady State Analysis of Periodic Switching Circuits

The method of periodic steady state analysis using the impedance matrix can be extended to include certain classes of non-linear and time varying circuits. While detailed study of non-linear circuit analysis is beyond the scope of this thesis, the extension to include certain types of switching circuits is straightforward. These circuits consist of only LTI elements and ideal switches, where the switching intervals are known and periodic with period T.

The ideal switches change the circuit topology between the different intervals of the period. During each interval, however, the circuit topology is linear and time invariant. When the switching intervals are known, the constituent relationships for the different intervals can be written in matrix form and then added to get the total periodic response. This method does not reflect the transient behavior of the circuit but solves for the periodic steady state response. This analysis technique is not possible using traditional SSS methods, because the switching elements provide a cross-coupling between harmonics and superposition no longer holds.

The analysis method can be applied to many different switching inverter topologies, and each topology takes a different matrix form. In this thesis, the method will be illustrated in a single example. Figure 2-3 shows the example switching circuit, which has a sinusoidal source intermittently connected to a parallel R-L-C load through an ideal switch. The switch is *on* during the intervals $[t_1, t_2]$ and $[t_3, t_4]$, and *off* for the rest of the period. The switching pattern repeats with the same period, T, as the sinusoidal source. The switching pattern can be represented by a binary signal, $sw(t)$, where

$$sw(t) = \begin{cases} 1 & ; \quad t_1 \leq t \leq t_2 \quad \text{or} \quad t_3 \leq t \leq t_4 \\ 0 & ; \quad otherwise \end{cases} \tag{2.32}$$

During the *on* interval, the load voltage is equal to the source voltage less the $R_s i(t)$ drop in the source resistance. During the *off* interval, the current is zero, and the load voltage rings at the resonant frequency of the R-L-C circuit. Even though the ring frequency does not necessarily coincide with a Fourier basis harmonic frequency, the periodic steady state load voltage repeats, exactly, with period T, and can therefore be represented with a Fourier basis vector.

The periodic steady state solution is found by defining a switching matrix, S_{ON}, which

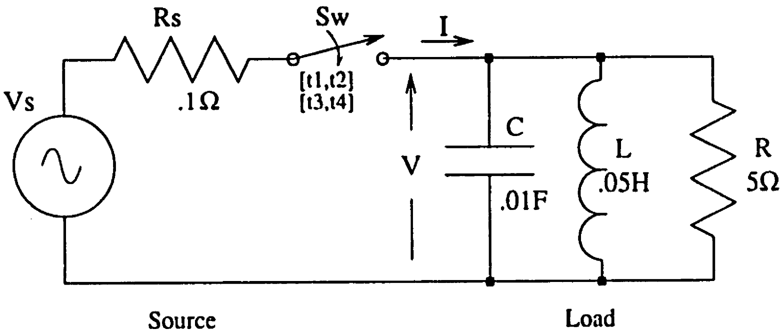

Figure 2-3: Example Switching Circuit

picks out the intervals when the switch is on. Using the time-domain basis, $S_{ON_t}\ V_t$ gives the time-domain vector representation of $sw(t)\ v(t)$. In the Fourier basis, $S_{ON_f}\ V_f$ is the Fourier vector representation of $sw(t)\ v(t)$. Because a time-domain vector consists of normalized samples of the corresponding waveform, S_{ON_t} is simply a diagonal matrix with diagonal entries equal to one for samples that fall in the *on* interval and zero for samples that fall in the *off* interval. For example,

$$S_{ON_t} V_t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ \vdots \end{bmatrix} \tag{2.33}$$

The frequency-domain switching matrix is found by Fourier transforming $sw(t)\ v(t)$.

$$\begin{aligned} S_{ON_f} V_f &= \frac{1}{T}\int_0^T \Phi(t) sw(t) v(t) dt = S_{ON_f} V_f = \frac{1}{T}\int_{\{[t1,t2][t3,t4]\}} \Phi(t)\Phi^T(t) dt\, V \\ S_{ON_f} &= \frac{1}{T}\int_{\{[t1,t2][t3,t4]\}} \Phi(t)\Phi^T(t) dt \end{aligned} \tag{2.34}$$

Except for the effects of aliasing, S_{ON_t} and S_{ON_f} are related by the orthonormal time-frequency transformation, $S_{ON_t} = M_{tf} S_{ON_f} M_{ft}$. Aliasing, however, can be significant, because the switching waveform is a square wave and contains high frequency components.

The effects of aliasing can be reduced by using a large n and low-pass filtering the switching waveform below the Nyquist frequency. For the purposes of this example, S_{ONf} is used without anti-alias filtering.

S_{ONt} and S_{ONf} are singular matrices, because it is not possible to reconstruct the entire unswitched waveform from the switched waveform. The complementary switching matrix, S_{OFF}, *picks out* the intervals when the switch is off, $(1 - sw(t))v(t) \leftrightarrow S_{OFF}V$, in either basis. The complementary switching matrix is simply the identity matrix minus the switching matrix.

$$S_{OFF} = \mathbb{I} - S_{ON} \qquad \text{or} \qquad S_{ON} + S_{OFF} = \mathbb{I} \tag{2.35}$$

Given S_{ON} and S_{OFF}, it is possible to write the circuit analysis equations for the *on* and *off* intervals separately, and then add the two equations to get the total response. First the load voltage, V, is expressed in terms of the source voltage, V_s, and the current, I.

$$\begin{array}{ll} ON: & S_{ON}V = S_{ON}(V_s - Z_{Rs}I) \\ OFF: & S_{OFF}V = S_{OFF}Z_L I \\ \hline TOTAL: & V = S_{ON}V_s - S_{ON}Z_{Rs}I + S_{OFF}Z_L I \end{array} \tag{2.36}$$

where Z_{Rs} and Z_L are the equivalent impedance matrices for the series source resistance and the parallel connected load. Next, the current vector is expressed in terms of V and substituted into (2.36).

$$\begin{array}{ll} ON: & S_{ON} = S_{ON}Z_L^{-1}V \\ OFF: & S_{OFF}I = 0 \\ \hline TOTAL: & I = S_{ON}Z_L^{-1}V \end{array} \tag{2.37}$$

$$V = S_{ON}V_s - S_{ON}Z_{Rs}S_{ON}Z_L^{-1}V + S_{OFF}Z_L S_{ON}Z_L^{-1}V \tag{2.38}$$

Finally, solving for V yields a single matrix equation for the periodic steady state load voltage of the circuit.

$$V = \left(\mathbb{I} + S_{ON}Z_{Rs}S_{ON}Z_L^{-1} - S_{OFF}Z_L S_{ON}Z_L^{-1}\right)^{-1} S_{ON}V_s \tag{2.39}$$

The equation can be simplified to a form resembling the matrix form of the voltage divider

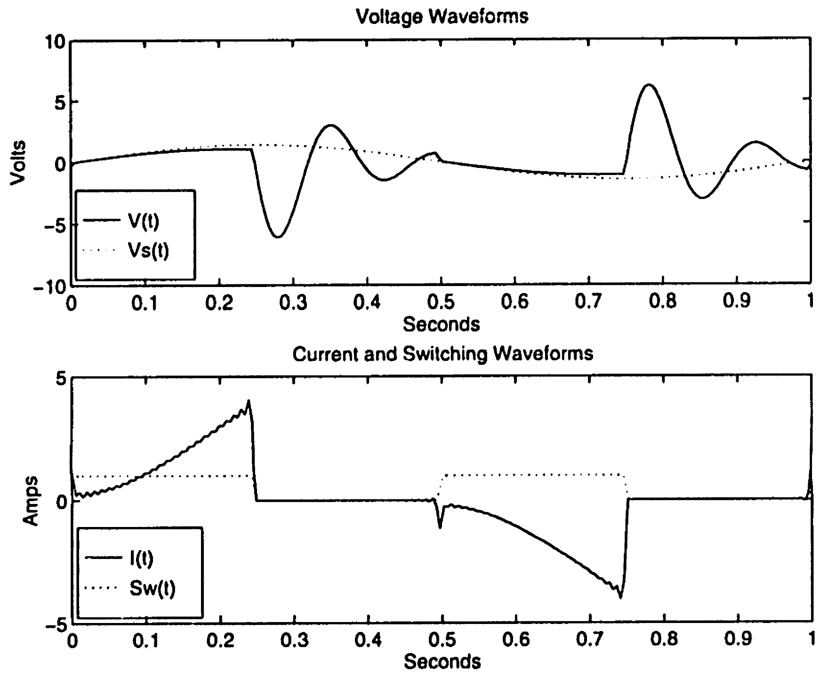

Figure 2-4: Switching Circuit Waveforms

law.

$$V = Z_L \left(Z_L + S_{ON} Z_{Rs} S_{ON} - S_{OFF} Z_L S_{ON}\right)^{-1} S_{ON} V_s \tag{2.40}$$

Using the component values shown in the circuit and a $1Hz.$ sinusoidal source, the circuit response is calculated using the Fourier basis. The resulting waveforms are then transformed to the time-domain, using the M_{tf} transformation matrix, and plotted in Figure 2-4. The top plot is the load voltage and the source voltage. The bottom plot is the switched current waveform and the switching waveform, $sw(t)$. The high-frequency ripple in the current waveform is due to the aliasing of the high frequency components of the switching waveform.

The form of (2.40) applies to the specific circuit topology of this example. Similar matrix equations can be found for other switching topologies, such as boost or buck converters. Given a particular circuit topology, however, the matrix equation provides a simple means of calculating the periodic steady state circuit response in a single computation. Further research on this method is needed to determine the effects of aliasing and investigate methods of reducing aliasing by low-pass filtering the switching waveform.

Chapter 3

Vector Decomposition of Power Components

The vector representation in any orthonormal basis provides a convenient notation for expressing time-average power components in terms of the constant vectors V and I.

$$P = < V, I > = V^T\, I$$
$$S = \|V\|\ \|I\| = \sqrt{V^T V I^T I} \tag{3.1}$$

Here the average power, P, is a signed scalar quantity, not a vector, representing the average energy flow in the waveforms. The apparent power, S, is also a scalar but is non-negative and does not reflect any particular physical phenomenon.

The power factor is the ratio of time-average power to apparent power, which is equal to the cosine of the small angle between the voltage and current vectors.

$$PF = \frac{P}{S} = \frac{V^T I}{\sqrt{V^T V I^T I}} = cos(\theta_{VI}) \tag{3.2}$$

The expression identifies active power, $P = S\,cos(\theta_{VI})$, as the base of a power triangle that is identical to the single frequency power triangle shown in Figure 1-2. The obvious orthogonal, inactive component of apparent power is $Q = S\,sin(\theta_{VI})$, just as in the single frequency case. However, the sign of Q is not defined for the multifrequency case, because θ_{VI} is an angle in n-space, which can not have a sign convention. The cosine is not sensitive to the sign of its argument, but the sine is. Q, then, is defined as the positive absolute value, which is unique but only reflects the magnitude of the inactive power component.

$$
\begin{aligned}
P &= S\,cos(\theta_{VI}) \\
Q &= S\,|sin(\theta_{VI})|
\end{aligned}
\tag{3.3}
$$

3.1 Vector Expressions for Reactive Power

With P and S defined in (3.1), we can write an algebraic expression for inactive power using the Pythagorean theorem.

$$
Q = \sqrt{S^2 - P^2} = \sqrt{V^T\,V\,I^T\,I - (V^T\,I)^2} \tag{3.4}
$$

Q is a non-negative scalar, equal to the length of the apparent power component that is orthogonal to time-average power. As in the definition of single frequency reactive power, Q can be viewed as a measure of the magnitude of reactive power.

In the Fourier basis, (3.4) expresses Q in terms of the rms values of cosine and sine frequency components of $v(t)$ and $i(t)$. The expression is general, however, and holds for any orthonormal decomposition. Further simplification can help illustrate the nature of the terms that contribute to Q.

$$
\begin{aligned}
Q &= \sqrt{V^T\,(V\,I^T - I\,V^T)\,I} \\
&= \sqrt{V^T\,(V\,I^T - (V\,I^T)^T)\,I}
\end{aligned}
\tag{3.5}
$$

Expanding (3.5) gives a scalar equation for Q in terms of the elements of V and I. After grouping terms, this scalar equation becomes

$$
Q^2 = \sum_{j=1}^{n} \sum_{k=j+1}^{n} (V_j I_k - V_k I_j)^2 \tag{3.6}
$$

where V_i and I_i are the i^{th} elements of the vectors V and I.

While it is not immediately obvious, (3.6) suggests that Q is the norm of a vector of orthogonal components that contribute to the total inactive power. The equation shows that Q^2 is the sum of squared terms, and that each term has the form of a cross product resembling the classical definition of reactive power in the sinusoidal case. As there are $m = \frac{1}{2}n(n-1)$ terms in (3.6), there is a fundamental dimensionality of the reactive power

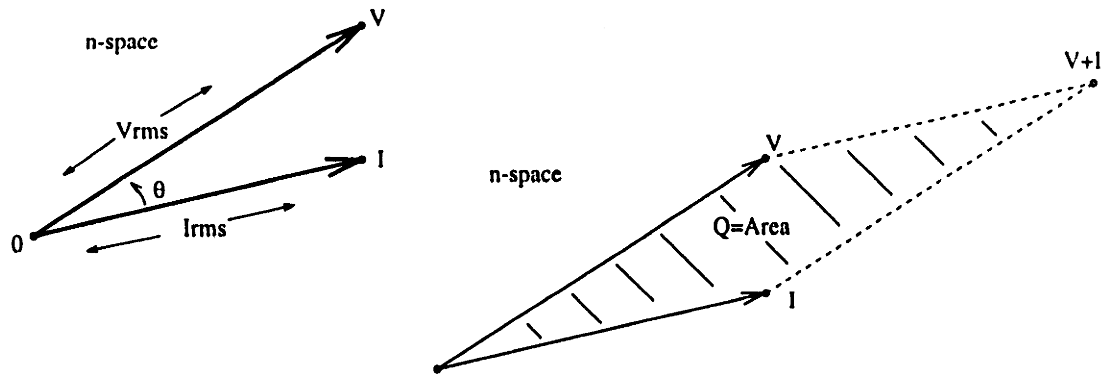

Figure 3-1: Voltage and Current Vector Parallelogram

decomposition that is related to, but not equal to, the dimensionality of the vector basis.

As the notion that reactive power is composed of m orthogonal components is a central result of this thesis, the formation of the reactive power vector, denoted R, will be viewed from several perspectives. First, R is shown to be the vector product, or cross product, of the voltage and current vectors. Next, the the grouping of frequency components into the m squared terms in (3.6) will be shown as a polynomial. Finally, R is shown to have a matrix equivalent, **R**, with a convenient mathematical form.

3.1.1 Reactive Power as a Vector Product

$Q = |V|\ |I|\ |sin(\theta_{VI})|$ is the area of the parallelogram shown in Figure 3-1. In three dimensions, the parallelogram area would be the magnitude of the vector cross-product, $V \times I$, but in n dimensions, the cross-product operator is not clearly defined.

The class of such n-dimensional vector products is called a 2-form, the exterior product of two 1-forms, as discussed in detail in [14], but an in-depth analysis of such theory is beyond the scope of this thesis. For the purposes of this thesis, the cross product can be defined as the m-vector with elements given by $(V_j I_k - V_k I_j)$, which equals the projected area of the parallelogram (V,I) on the plane of the coordinate axis pair, $(\vec{j}, \vec{k})$. There are $m = \frac{1}{2}n(n-1)$ elements in the cross-product vector, corresponding to the permutations of $j \neq k$.

As shown in Figure 3-2, the vector cross product is formed by directing each projected area in a unique orthogonal direction, $\vec{l}$, and taking the vector sum of the m directed areas. There are two important things to see from this picture. First, because $m \neq n$, the cross-

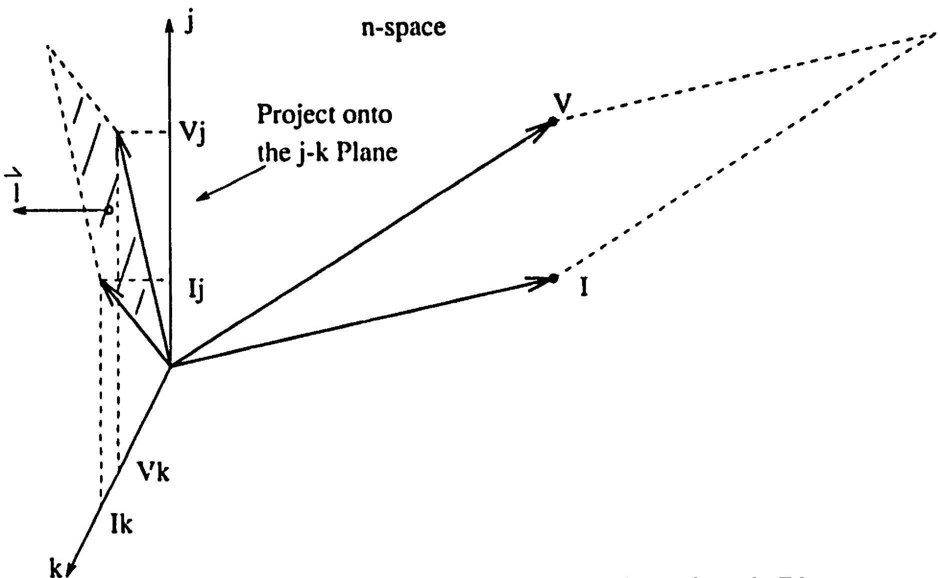

Figure 3-2: Parallelogram Projected to the j-k Plane

product vector does not exist in the same space as the voltage and current vectors. Instead, the principal directions of the cross product are associated with a pair of directions in V and I, and the pairing is arbitrary.

Second, when directing a projected area in the direction $\vec{l}$, the sign is chosen according to an arbitrary convention[1].

$$\vec{l} = \vec{j} \times \vec{k} = -\vec{k} \times \vec{j} \tag{3.7}$$

As there are essentially m orthogonal sign conventions implied in the vector product, there can not be one overall consistent sign definition associated with Q.

With the cross product in n dimensions defined, we can now write an equation for the reactive power vector using the $\times$ operator. However, the operator is defined in principle only. There is no simple linear algebraic expression that implements the cross product, due to the arbitrary ordering and sign convention. Therefore, the cross product is left as an operator, which takes two n-vectors and yields an m-vector related to the exterior product. Thus the reactive power vector is expressed as

$$R = V \times I \qquad \text{and} \qquad Q = \|R\| \tag{3.8}$$

[1]The equivalent convention commonly used for three dimensions is the right-hand rule.

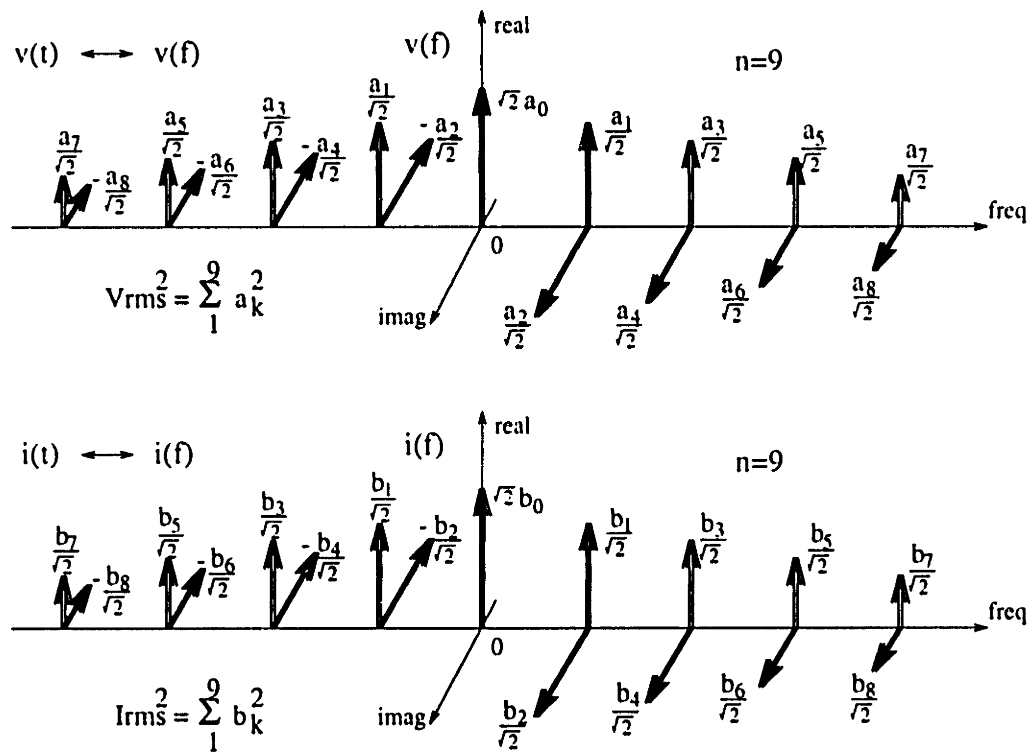

Figure 3-3: Frequency Components of the Voltage and Current Waveforms

3.1.2 Grouping of Reactive Power Terms

The formation of the m orthogonal reactive power components can also be shown as a polynomial. Equation (3.6) indicates that Q^2 is the sum of *squared* terms, so how do these squares arise from the difference $S^2 - P^2$? Figure 3-3 shows a frequency-domain representation of a voltage and current waveform consisting of a D.C. component and four harmonics. The D.C. component yields a real impulse in the frequency domain, and each harmonic gives a complex conjugate impulse pair, for a total of $n = 9$ degrees of freedom. For this example, we would expect $m = \frac{1}{2}n(n-1) = 36$ squared terms in Q.

With the rms-normalized coefficients for the voltage and current impulses denoted by $(a_0...a_8)$ and $(b_0...b_8)$, Parseval's relation implies that the the total mean-squared energy in the waveforms is given by

$$V_{rms}^2 = (a_0^2 + a_1^2 + ...a_8^2) \qquad \text{and} \qquad I_{rms}^2 = (b_0^2 + b_1^2 + ...b_8^2) \tag{3.9}$$

S^2 is the product of the two polynomials in (3.9), while P^2 is the overlap integral of $v(f)$

and $i(f)$, squared.

$$S^2 = (a_0^2 + a_1^2 + ...a_8^2)(b_0^2 + b_1^2 + ...b_8^2)$$
$$P^2 = (a_0 b_0 + a_1 b_1 + ...a_8 b_8)^2 \tag{3.10}$$

Expanding (3.10) shows that S has 81 terms, while P has 45 terms, and $81 - 45 = 36 = m$, but how do the the $81 + 45 = 126$ terms in the difference combine? As shown in Equation (3.11), the $2n$ terms of the form $\pm a_i^2 b_i^2$ cancel by subtraction, leaving 108 terms. These terms, in groups of three, form perfect squares, which are the 36 terms in Q^2.

$$Q^2 = S^2 - P^2 = a_0^2 b_0^2 + a_0^2 b_1^2 + a_1^2 b_0^2 + a_1^2 b_1^2 + ...$$
$$-(a_0 b_0)^2 - 2a_0 b_0 a_1 b_1 - (a_1 b_1)^2 - ... \tag{3.11}$$
$$= (a_0 b_1 - a_1 b_0)^2 + ()^2 + ... \qquad m = 36 \text{ Terms.}$$

The m elements of Q do not reflect frequency components of the instantaneous power waveform, $p(t) \leftrightarrow p(f)$, which would be given by the convolution, $v(t)i(t) \leftrightarrow v(f) * i(f)$. Convolving the impulses in Figure 3-3 shows that $p(f)$ consists of 17 impulses, one real, and 16 in complex conjugate pairs. $p(f)$, then, has only $2n - 1$ orthogonal components, while Q has $\frac{1}{2}n(n-1)$.

The elements in Q simply correspond to cross terms between orthogonal V and I components, which contribute to S but not to P. These cross terms do not reflect physical power quantities or a decomposition of the flow of energy. While each component of Q is signed, according to an arbitrary convention, Q cannot have a single sign convention, because there are m orthogonal conventions.

3.1.3 Matrix Form of the Reactive Power Vector

The vector expression for Q^2 given in Section 3.1 is the product of the V and I vectors with a matrix, which will be called the reactive power matrix, denoted in bold-face, $\mathbf{R}$.

$$Q^2 = V^T(VI^T - IV^T)I = V^T\mathbf{R}I$$
$$\mathbf{R} = VI^T - IV^T \tag{3.12}$$

$\mathbf{R}$ is the skew-symmetric part of the outer product of V and I, with zeros on the main diagonal, and the negative of the upper triangle appearing in the lower triangle. The ij^{th}

element of $\mathbf{R}$ is given by

$$[\mathbf{R}]_{jk} = V_j I_k - V_k I_j \tag{3.13}$$

which indicates that the upper triangle contains the m elements of the reactive power vector, R. While the vector $R = V \times I$ has no convenient algebraic form, the matrix $\mathbf{R} = VI^T - IV^T$ is easily expressed in terms of outer products. Therefore, it is sometimes more convenient to use the matrix form in algebraic expressions involving reactive power. Both R and $\mathbf{R}$ will be used subsequently, depending on which is more convenient.

Q is the root mean square of the m orthogonal elements of R, and can, therefore, be expressed as the two-norm of R. Q can also be expressed as a norm of the matrix $\mathbf{R}$. Specifically, the Frobenius[2] norm normalized by $\frac{1}{\sqrt{2}}$.

$$Q = \|R\| = \frac{1}{\sqrt{2}}\|\mathbf{R}\|_f \tag{3.14}$$

3.2 Properties of the Reactive Power Vector

The theoretical development of a reactive power vector has been shown in the previous section, but the theory provides nothing more than a definition. For the definition to be of practical use, the properties must be shown. For the vector to be considered a measure of reactive power, the properties of the vector should resemble those of classical reactive power defined in the sinusoidal steady state.

3.2.1 Relationship to Sinusoidal Reactive Power

For the special case of sinusoidal steady state signals, the voltage and current sinusoids are typically represented as complex phasors.

$$\hat{V} = V_r + jV_i \qquad \text{such that} \qquad v(t) = Re\{\hat{V}e^{jwt}\} = V_r cos(wt) - V_i sin(wt) \tag{3.15}$$

The complex number is a convenient mathematical form for expressing two-dimensional quantities as scalars; there are equivalent expressions wherein voltage and current are left as two-dimensional vectors of *rms* frequency components. With the ordering convention of the Fourier basis defined in Equation (2.11), the vectors for voltage and current would be

[2]The Frobenius norm of a matrix, A, is the root mean square of the elements of A. $\|A\|_f = \sum_j \sum_k A_{jk}^2$.

given by

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} V_r \\ -V_i \end{bmatrix} \qquad \text{and} \qquad I = \frac{1}{\sqrt{2}} \begin{bmatrix} I_r \\ -I_i \end{bmatrix} \tag{3.16}$$

The vector space has $n = 2$, so $m = \frac{1}{2}n(n-1) = 1$, and the reactive power vector reduces to a single, signed scalar. This scalar is equal to the sinusoidal reactive power as classically defined.

$$R = V_1 I_2 - V_2 I_1 = \frac{1}{2}(V_i I_r - V_r I_i) = \frac{1}{2} Im\{\hat{V}\hat{I}^*\} \tag{3.17}$$

Note that the sign convention of the scalar R matches $\frac{1}{2}Im\{\hat{V}\hat{I}^*\}$, because the arbitrary order of the Fourier basis was chosen accordingly. Such a single sign convention is only possible when $m = 1$ and is not possible when harmonics or a D.C. component are present.

3.2.2 Conservation of the Reactive Power Vector

One of the strongest or most intuitive properties of sinusoidal reactive power is that it is conserved. The total reactive power *entering* the port of a circuit is the sum of the reactive power *dissipated* in all the elements of the circuit. To compensate a circuit that *consumes* reactive power, one adds an element, in series or in parallel, that *generates* reactive power. Even though reactive power doesn't correspond to a net flow of energy, the conservation property allows us to form an intuition similar to that of conservation of energy.

For any general measure of power, reactive, time-average, or otherwise, conservation property implies that the sum of the power into all elements and ports of a network is zero.

$$\sum_k \rho^k = 0 \tag{3.18}$$

where ρ^k is some generalized measure power at element or port k.

Conservation of energy guarantees that instantaneous power is conserved at each instant of time, so (3.18) holds for $\rho \stackrel{\Delta}{=} p(t)$. As the time-average operator is linear, it commutes with the summation in (3.18), so average power is also conserved.

$$\sum_k P^k = \sum_k \overline{p^k(t)} = \overline{\sum_k p^k(t)} = 0 \tag{3.19}$$

Other power definitions, however, may not obey conservation. S and Q, for example, are non-negative by definition and cannot, therefore, sum to zero unless all are zero.

Tellegen's theorem can be used to prove conservation for a set of power definitions called generalized powers. As shown in [15, 16], a particular power is a generalized power and will obey conservation if it can be expressed in the following form.

$$\rho^k = \Lambda_v(v^k)\, \Lambda_i(i^k) \tag{3.20}$$

where v^k, and i^k, are the voltage and current at the k^{th} circuit element, and Λ_v and Λ_i are linear operators, or more generally, Kirchhoff voltage and current operators. The proof of (3.20) requires proving conservation for only two types of circuit connection, series and parallel. As any circuit can be constructed as a number of nested series and parallel connections, proof of overall conservation requires only proving series conservation and parallel conservation.

In a series connection, the current is identical in each series element, and the series voltage is the sum of the element voltages. The Kirchhoff voltage operator is defined as an operator that preserves KVL, such that $\sum \Lambda_v(v^k)$ for the series elements equals $\Lambda_v(\sum v^k)$. As the current, and therefore $\Lambda_i(i)$, is identical for all elements, the total power for the series connection is $\rho = \Lambda_v(\sum v^k)\Lambda_i(i)$, which equals the sum of the power for each series element, $\sum \Lambda_v(v^k)\Lambda_i(i)$, so ρ is conserved.

Similarly, in a parallel connection, all elements see the same voltage, while the total parallel current is the sum of the element currents. As Kirchhoff current operator preserves KCL, $\sum \Lambda_i(i^k) = \Lambda_i(\sum i^k)$ for all parallel elements. The total power is $\rho = \Lambda_v(v)\Lambda_i(\sum i^k) = \sum \Lambda_v(v)\Lambda_i(i^k)$, so ρ is again conserved.

The form of conserved generalized powers in (3.20) can be further extended by applying a linear operator, β, as shown in [17]. As did the time-average operator in (3.19), β commutes with the summation, preserving the conservation property. The more general form for generalized powers is

$$\rho_k = \beta\left(\Lambda_v(v^k)\, \Lambda_i(i^k)\right) \tag{3.21}$$

The definition for vector definition for Q cannot be written in the form (3.21) and, in fact, does not obey conservation. Each of the elements in the reactive power vector, however, do have the required form, because the elements of the vectors V and I are linear transforms of the voltage and current waveforms, as shown in Equation (2.6). The elements of R have the form $V_jI_k - V_kI_j$, which is a linear combination of products of transformed voltage and current, satisfying (3.21).

As the components of R are generalized powers and are conserved, the vector itself is conserved, and (3.18) can then be written in vector form or matrix form.

$$\sum_k R^k = \underline{0} \qquad \text{and} \qquad \sum_k \mathbf{R}^k = \underline{\mathbf{0}} \tag{3.22}$$

where $\underline{0}$ and $\underline{\mathbf{0}}$ are the zero m-vector and $n \times n$ zero matrix. This classification is independent of the specific decomposition of voltage and current, and will hold for any orthonormal basis, $\Phi(t)$.

3.2.3 The Local Property

A defined electrical quantity such as a power can be said to be local if its definition only requires local information.[18] Calculating the average power in a circuit branch, for example, requires only the voltage and current measurements of the circuit branch. No other information about the circuit is necessary. The local property is desirable for any measured quantity, because a detailed analysis of the whole circuit is not always possible.

All of the power quantities defined so far are local. S, P, Q, and R, as well as the generalized powers in (3.21), are all expressed in terms of the local voltage and current. Certain power measures yet to be defined are, in fact, not local, but are useful when information about the whole circuit is known. It is important to consider the local property when assessing the merits of a particular power definition.

The reactive power vector has been shown to have the desirable properties for a measure of reactive power. It is conserved, local, and has a magnitude equal to the total inactive power, $\sqrt{S^2 - P^2}$. While a formal proof is beyond the scope of this thesis, a conjecture of this work is that the reactive power vector is the only such definition for reactive power. The R vector is not unique – it depends not only on the orthonormal basis used to decompose voltage and current, but also on the arbitrary ordering convention in the definition of the cross product – but the $m = \frac{1}{2}n(n-1)$ dimensions in R are fundamental. Any conserved, local measurement of total inactive power must reflect these m orthogonal components and, therefore, have a form equivalent to an R vector.

3.3 Scalar Measures of Reactive Power

The reactive power vector has been defined and shown to have useful properties as a measure of reactive power in periodic signals with harmonics. However, the dimension, $m = \frac{1}{2}n(n-1)$, grows as n^2 for highly distorted signals. Such a quantity is of little practical use except in symbolic form. The following section shows how the information in the reactive power vector can be reduced to a scalar, with the subsequent loss of desirable properties.

3.3.1 Norms of the Reactive Power Vector

The dimensionality of reactive power can be reduced by considering the norm of R, or the norm of certain elements of R, as a measure of reactive power. The conservation property is lost when elements are combined with a norm, because the orthogonal sign conventions of each element can not be reflected in the combined quantity. Norms can be useful, however, because the total magnitude contribution from several dimensions is reflected in a single quantity.

$Q = \|R\|$ is such a norm, and does reflect the total magnitude of all the m components of R. Other norms could be used for certain applications; for example, an infinity norm would reflect the largest component in R, although the infinity norm would no longer reflect the total inactive power, $\sqrt{S^2 - P^2}$.

Sub-norms, or norms of certain elements of R, might be used to subdivide and measure different types of reactive power. In the Fourier basis, for example, the norm of the even elements of the first upper diagonal in the $\mathbf{R}$ matrix would reflect the magnitude contribution from the sinusoidal reactive powers taken at each harmonic frequency. The norm of all other upper triangle elements would reflect the remaining the reactive power. The two norms would add, in quadrature, to equal the total inactive power. Neither Q_1 nor Q_2 would be conserved, but such a two-component quadrature decomposition could still be useful for particular metering or optimization problems.

3.3.2 Projections of the Reactive Power Vector

Scalar measures of reactive power can also be defined as projections of the reactive power vector. Projections are formed as linear combinations of the elements of R, and therefore hold the conservation property. Projections, however, only reflect one of the m dimensions in R, and do not reduce the dimensionality.

Projections are formed as the dot product of R and a unit vector. If the unit vector is to point in the direction of some reference reactive power vector R_{ref}, then the normalized unit vector is $\frac{R_{ref}}{\|R_{ref}\|} = \frac{R_{ref}}{Q_{ref}}$, and the projection, q, is

$$q = R^T \frac{R_{ref}^T}{Q_{ref}} \tag{3.23}$$

As the vector R has no simple algebraic form in terms of the vectors V and I, the matrix form for $\mathbf{R}$ and $\mathbf{R}_{ref}$ will be used. In matrix form, the projection is expressed using the linear trace operator.

$$q = \frac{1}{2} tr \left\{ \mathbf{R}^T \frac{\mathbf{R}_{ref}}{Q_{ref}} \right\} = \frac{1}{2Q_{ref}} tr \left\{ \mathbf{R}^T \mathbf{R}_{ref} \right\} \tag{3.24}$$

Substituting $\mathbf{R} = VI^T - IV^T$ and using the fact that $\mathbf{R}_{ref} = -\mathbf{R}_{ref}^T$ leads to the following simplification.

$$\begin{aligned} q &= \tfrac{1}{2Q_{ref}} tr \left\{ (VI^T - IV^T)^T \mathbf{R}_{ref} \right\} \\ &= \tfrac{1}{2Q_{ref}} tr \left\{ IV^T \mathbf{R}_{ref} \right\} - \tfrac{1}{2Q_{ref}} tr \left\{ VI^T \mathbf{R}_{ref} \right\} \\ &= \tfrac{1}{2Q_{ref}} tr \left\{ V^T \mathbf{R}_{ref} I \right\} - \tfrac{1}{2Q_{ref}} tr \left\{ I^T \mathbf{R}_{ref} V \right\} \\ &= \tfrac{1}{2Q_{ref}} tr \left\{ V^T \mathbf{R}_{ref} I \right\} - \tfrac{1}{2Q_{ref}} tr \left\{ V^T \mathbf{R}_{ref}^T I \right\} \\ &= \tfrac{1}{2Q_{ref}} V^T \left(\mathbf{R}_{ref} - \mathbf{R}_{ref}^T \right) I \\ q &= V^T \tfrac{\mathbf{R}_{ref}}{Q_{ref}} I \end{aligned} \tag{3.25}$$

Equation (3.25) is the general form of a projection in the direction of the reference reactive power matrix. The equation applies whether $\mathbf{R}_{ref}$ specifies a fixed direction or a direction that depends on a voltage and current pair in a circuit. One useful fixed projection is the projection of R in the direction of the sinusoidal reactive power of the fundamental. In the Fourier basis, the corresponding $\frac{\mathbf{R}_{ref}}{Q_{ref}}$ is a zero matrix, except for the elements $(2,3)$ and $(3,2)$, which are ± 1.

One potential variable projection is the projection of R in the direction of R. Here $\frac{\mathbf{R}_{ref}}{Q_{ref}}$

is equal to $\frac{\mathbf{R}}{Q}$, and the projection reduces to $\|R\| = Q$, as expected.

$$q = V^T \frac{\mathbf{R}}{Q} I = \frac{V^T(VI^T - IV^T)I}{\sqrt{V^T(VI^T - IV^T)I}} = \sqrt{V^T(VI^T - IV^T)I} = Q \tag{3.26}$$

Q, however, is not conserved, as the $\mathbf{R}$ reference direction is different at each element of a circuit. Only projections in a particular direction obey conservation.

A more useful variable projection, which does obey conservation and can be used for single port circuit analysis, is the projection, q^k, of circuit element k, in the direction of the reactive power vector of the circuit port. The projection can be stated both ways, q^k is both the projection of R^k in the direction of R^{port} and the projection of R^{port} in the direction of R^k.

$$q^k = V^{kT} \frac{\mathbf{R}^{port}}{Q^{port}} I^k = V^{portT} \frac{\mathbf{R}^k}{Q^k} I^{port} \tag{3.27}$$

The conservation property guarantees that $\sum_k q^k$ over all circuit elements is equal to q^{port}, defined *into* the port. q^{port} is equal to the the total port inactive power, Q^{port}. q^k, then, is a signed scalar measure of the contribution of element k to the port reactive power. If q^k is positive, then element k is increasing Q^{port}, or *consuming* reactive power. If q^k is negative, then element k is decreasing Q^{port} by *compensating* or *generating* reactive power.

The q^k measure of reactive power does not have the local property. In order to calculate q^k, the voltage and current of both element k and the port must be known. In general, q^k could be useful in assessing the main contributors to the reactive power at an interconnect, but the non-local nature of q^k imposes practical considerations.

3.4 Coordinate Rotation

While the form of the reactive power vector applies to any orthonormal basis, the values of the elements of R do depend on the specific basis used to decompose the voltage and current waveforms. The transformation from one basis to another can be seen as an orthonormal rotation of the vector space. The transformation takes the form of multiplication by an orthonormal matrix, M, as shown in Section 2.1.2. Using the matrix expression $\mathbf{R} = VI^T - IV^T$, the reactive power in one basis, $\mathbf{R}_a$, can be expressed in terms of the reactive power in another basis, $\mathbf{R}_b$, and the transformation matrix, M_{ab}.

$$\begin{aligned} \mathbf{R}_a &= V_a I_a^T - I_a V_a^T = M_{ab} V_b (M_{ab} I_b)^T - M_{ab} I_b (M_{ab} V_b)^T \\ \mathbf{R}_a &= M_{ab} \mathbf{R}_b M_{ab}^T = M_{ab} \mathbf{R}_b M_{ba} \end{aligned} \tag{3.28}$$

3.4.1 Time-Frequency Transformation

As was shown in Section 2.1.2, the transformation from n frequency components to n evenly spaced, *rms* normalized, discrete samples is orthonormal for odd n. The transformation matrix, M_{tf} defined in Equation (2.14) is orthonormal, such that $M_{tf}^{-1} = M_{tf}^T = M_{ft}$, so no information is lost during the transformation.

The vector expressions for the reactive power vector are independent of basis, so R vector can be calculated using either time or frequency samples, and transformation between the domains is a simple matrix operation. Similarly, the vector expressions for projections of the reactive power matrix can be implemented using time samples. For example, q^k in (3.27) can be calculated using a time domain algorithm.

$$\begin{aligned} q^k &= V^{k^T} \frac{\mathbf{R}^{port}}{Q^{port}} I^k \\ &= \frac{V^{k^T} V^{port} I^{port^T} I^k - V^{k^T} I^{port} V^{port^T} I^k}{\sqrt{V^{port^T} V^{port} I^{port^T} I^{port} - (V^{port^T} I^{port})^2}} \\ &= \frac{\left(\sum_{s=1}^{n} V_{[s]}^{k} V_{[s]}^{port}\right)\left(\sum_{s=1}^{n} I_{[s]}^{k} I_{[s]}^{port}\right) - \left(\sum_{s=1}^{n} V_{[s]}^{k} I_{[s]}^{port}\right)\left(\sum_{s=1}^{n} I_{[s]}^{k} V_{[s]}^{port}\right)}{\sqrt{\left(\sum_{s=1}^{n} V_{[s]}^{port^2}\right)\left(\sum_{s=1}^{n} I_{[s]}^{port^2}\right) - \left(\sum_{s=1}^{n} V_{[s]}^{port} I_{[s]}^{port}\right)^2}} \end{aligned} \tag{3.29}$$

If the sampling rate, $\frac{n}{T}$, is too low, aliasing occurs. The aliasing in time is equivalent to the truncation to n components in the frequency domain. The aliased samples can be transformed to the frequency domain, however, the resulting frequency components are erroneous. Similarly, a truncated frequency representation can be transformed to the time domain, but the time-domain representation will not match the sampled waveform. Therefore, accurately determining the R vector from time-domain samples requires the use of anti-aliasing filters.

3.4.2 Shift Invariance

The elements of the voltage and current vectors depend on the choice of orthonormal basis, which requires the choice of a time origin. A shift of time origin is a coordinate rotation, which causes the elements of V and I to circulate.[3] The norms $\|V\|$, $\|I\|$ and $\|R\|$, and therefore S, P, and Q, are not affected by the rotation and are *shift invariant.* The elements of R, however, circulate as V and I rotate. As the frequency-domain rotation is proportional to harmonic frequency, the frequency-domain reactive power vector for large n is very sensitive to time origin.

The calculation of R from samples of the voltage and current waveforms requires a precise sampling frequency, which must be integer multiple of the $1/T$. A slight error in the sampling frequency will cause the elements of R to circulate slowly over time. As precise synchronous sampling is often not practical, the shift invariance property must be considered when designing a particular measure of reactive power. Appropriately designed norms, for example, could be shift invariant, as could a projection where the reference direction rotates synchronously with R. ς^k, defined in (3.27), is an example of a shift invariant projection.

3.5 Comparison with Other Reactive Power Definitions

The reactive power vector described in this thesis is a decomposition of total inactive power, $Q = \sqrt{S^2 - P^2}$, into m orthogonal elements, where $m = \frac{1}{2}n(n-1)$. The reactive power vector is fundamentally a time-average quantity as it is derived from the time-average quantities S and P. As does the vector decomposition of voltage and current, the reactive power vector depends on the choice of basis functions. The dimensionality, m, however, is fundamental to the cross product.

Many time-average reactive power decompositions have been suggested, some of which will be reviewed here. Instantaneous reactive power quantities, such as Akagi's generalized instantaneous reactive power for multiphase systems, [19], are not reviewed. This instantaneous definition is based on a two-dimensional vector representation of the instantaneous voltage and current of a multiphase system. The vectors themselves are instantaneous quantities and are not specifically oriented to periodic signals. As the instantaneous definition is not defined for single phase waveforms, comparison to the time-average reactive power vector is difficult.

In most cases, proposed time-average definitions can be interpreted as norms or projec-

[3] The elements of the time-domain vectors shift circularly, while the frequency-domain cosine/sine pairs rotate proportional to the harmonic frequency

tions of the R vector. Often the definitions suffer the loss of some useful properties, because they attempt to represent m orthogonal conserved power components with fewer than m scalar quantities. Those definitions that are projections consist of linear combinations of the elements of R and obey conservation, but each projection can only account for only one of the m dimensions in R. The other $m-1$ dimensions, if non-zero, must be accounted for, if the decomposition is to reflect the total inactive power. Those definitions that are norms combine orthogonal projections of R in quadrature, thus representing the magnitude contribution from several dimensions. The sign convention, however, is lost when components are combined, and the resulting definition is not conserved.

The original Budeanu definition of reactive power is given in [2].

$$Q_b = \sum_i E_i I_i sin(\theta_i) \tag{3.30}$$

where E_i, I_i, and θ_i are the *rms* voltage and current and the phase angle difference of the i^{th} harmonic. In the Fourier basis, each harmonic is represented with a cosine/sine pair, and each term in (3.30) represents the interaction of the cosine and sine components at a particular frequency. These terms appear as every other entry in the first upper diagonal in the reactive power matrix of the Fourier basis, $\mathbf{R}_f$. In Equation (3.30) these terms are added linearly, while all other components are ignored. Budeanu's definition, then, is a projection of R in a fixed direction.

Budeanu's definition is conserved, local, and shift invariant, but the direction of the projection has no particularly useful interpretation. As Q_b only reflects one of the m dimensions of reactive power, Q_b is not useful as a measure of power factor or inactive power.

In order to account for the other reactive components, Budeanu introduced Distortion Power.

$$D_b = \sqrt{S^2 - P^2 - Q_b^2} \tag{3.31}$$

D_b, then, is the norm of $m-1$ orthogonal projections of R, which are also orthogonal to the Q_b projection. As D_b adds components in quadrature, the definition does not obey conservation. Because the direction of the Q_b projection is arbitrary, the D_b component has no generally useful interpretation.

Perhaps Budeanu intended that Q_b reflect the individual interactions of the $n/2$ frequencies, and that D_b represent the cross terms between different frequencies. If so, Q_b would have been more properly defined as the norm of the $n/2$ components in the first upper diagonal of $\mathbf{R}_f$, and D_b would then have been the norm of all the other elements. The resulting two-element orthogonal decomposition would have properly decomposed the total inactive power, but neither of the norm quantities would have been conserved.

Fryze, in [3], defined a single signed measure of reactive power.

$$Q_f = \pm\sqrt{S^2 - P^2} \tag{3.32}$$

The magnitude of Q_f is the norm of all the elements of R, which is potentially useful because minimizing $\|Q_f\|$ minimizes S, maximizing power factor. The sign definition, however, is misleading in that the definition is a norm and is not conserved. Presumably, the sign was added to reflect the sign of fundamental reactive power component, which would be dominant in the nearly sinusoidal case. If the intention was to obtain a single conserved quantity to represent $\sqrt{S^2 - P^2}$ at a particular circuit port, the definition would more properly have been the projection of R in the direction of R_{port}, which is the definition of q^k given in this thesis.

Many authors have provided discussion on specific drawbacks and advantages of the Budeanu and Fryze definitions. While the debate is very much ongoing, most current research focuses on the magnitude portion of Q_f, referred to in this thesis as inactive power, as the more useful quantity for optimization of power transfer. Several authors provide orthogonal decompositions of Q_f into components which have a specific interpretation in optimization of power factor. Ofori, for example, combining the work of Czarnecki and Emanuel, defines the magnitude component that could be compensated by a parallel connected ladder network of lossless LTI elements.[12] Such components are norms of projections of R, which do not obey conservation but can provide a useful summary of reactive power information for metering purposes. In active compensation of loads, however, such quantities are of limited use, because important sign information is lost in the norm.

Emanuel, in [7], encourages the decomposition of reactive power into two components, Q_1 and Q_H. Q_1 is the signed sinusoidal reactive power of the fundamental component of voltage and current, and Q_H is norm of the remaining components such that $Q_f^2 = Q_1^2 + Q_H^2$. Q_1 is $V_1 I_2 - I_1 V_2$, which is one of the elements of R in the Fourier basis. For nearly sinusoidal signals, this element is the dominant orthogonal term in inactive power, and maintaining the sign information for this term allows the dominant portion of reactive power to be conserved and have an associated direction of *flow*. Q_H, however, retains no sign information and is not conserved.

Page, in [8], defined capacitive reactive power and inductive reactive power, Q_C and Q_L, as the two non-orthogonal components of Q that could be compensated by a parallel capacitor and inductor. These powers are, in fact, projections of the R vector in the directions of R_C and R_L, the reactive power vectors of a pure capacitance and pure inductance subject to the same voltage, V. Page correctly noted that these quantities can have a sign and are conserved in a parallel circuit where the voltage is common. Therefore, a negative Q_C can be compensated by adding a capacitor in parallel. Page also noticed that the Q_C and Q_L are not generally orthogonal or parallel, and introduced a refinement to account for the cross terms.

Page's projections did not account for the other $m-2$ orthogonal components of R, which must be considered if the decomposition is to account for the total inactive power. His projections can help solve for the best parallel compensation capacitor or inductor but do not help with other compensation topologies, such as a series compensator or a more complicated compensation network. As shown in an example in Section 4.3, single element parallel compensation does not always provide substantial power factor improvement.

Wyatt and Ilić, in [11], discusses the merits of an instantaneous reactive power.

$$p_{react,2}(t) = v(t)\frac{d}{dt}i(t) - i(t)\frac{d}{dt}v(t) \tag{3.33}$$

The time average of this quantity can also be interpreted as a measure of reactive power, and can be expressed in vector notation. In the Fourier basis, $\frac{d}{dt}\Phi(t) = J_\omega^T\Phi(t)$, where J_ω is the block diagonal, anti-symmetric matrix given in Section 2.2.1. The time average in (3.33) is simplified by taking the constants outside the integral.

$$\begin{aligned} \frac{1}{T}\int_0^T p_{react,2}(t)dt &= V^T J_\omega I - V^T J_\omega^T I \\ &= 2V^T J_\omega I \end{aligned} \tag{3.34}$$

While (3.34) is not normalized, it has the form of a fixed projection of R. Like Budeanu's Q_b, the projection contains only interactions between the sine and cosine components at the same frequency. Unlike Q_b, each component is scaled by its frequency.

In linear sinusoidal systems, reactive power can also be interpreted in terms of average stored electric and magnetic energy.

$$Q = 2\omega \left(\overline{W_M(t)} - \overline{W_E(t)} \right) \tag{3.35}$$

Because $W_M(t)$ and $W_E(t)$ both oscillate as $cos(2\omega t)$ but are 180° out of phase, (3.35) can also be written in terms of the total stored energy, $W(t) = W_M(t) + W_E(t)$.

$$Q = rms \left\{ \frac{d}{dt} (W(t)) \right\} \tag{3.36}$$

This expression has the nice interpretation that Q, in the sinusoidal case, is an rms measure of the oscillatory transfer of stored energy. This fact, however, is a result of the 180° phase difference between $W_M(t)$ and $W_E(t)$, and no such relationship holds when harmonics are present or when nonlinear circuits are considered.

Chapter 4

Single Port Circuit Examples

The following Chapter combines the periodic steady state analysis techniques of Section 2.2 with the vector expressions for reactive power in to several circuit examples. The examples are drawn from power electronics and power systems, with examples of both metering and power factor optimization.

4.1 Power Electronics Metering Example

Figure 4-1 shows a one-port circuit example consisting of a linear R-L-C circuit in parallel with a switched resistive load. The triac switch is self commutating with a 90° firing angle, so it is on for the latter half of each half-cycle. The voltage source at the circuit port contains a 1 Hertz fundamental component as well as small components of the third and fifth harmonics. The port current contains these harmonics as well as additional harmonics introduced by the switching discontinuity.

The port voltage is shown in Figure 4-2, along with the total port current, and the current in the R-L-C and switched resistor loads. The current is found using the periodic steady state analysis methods of Section 2.2.1. The R-L-C load current is found using the impedance matrix of R_1, L_1, and C_1.

$$Z_{RLC} = \left[Z_{C1}^{-1} + (Z_{R1} + Z_{L1})^{-1}\right]^{-1} \qquad \text{and} \qquad I_{RLC} = Z_{RLC}^{-1} V \tag{4.1}$$

To calculate the current in the switched resistive load, the vector for the $R2$ voltage is calculated using the Fourier transform of Equation (2.6). As the triac is switched on at

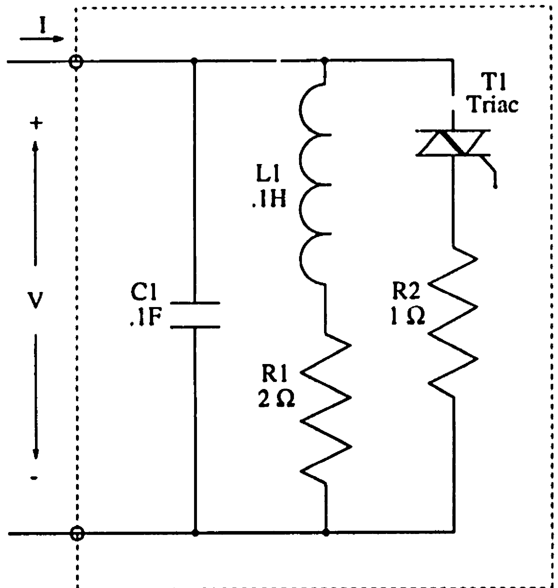

Figure 4-1: Example Circuit

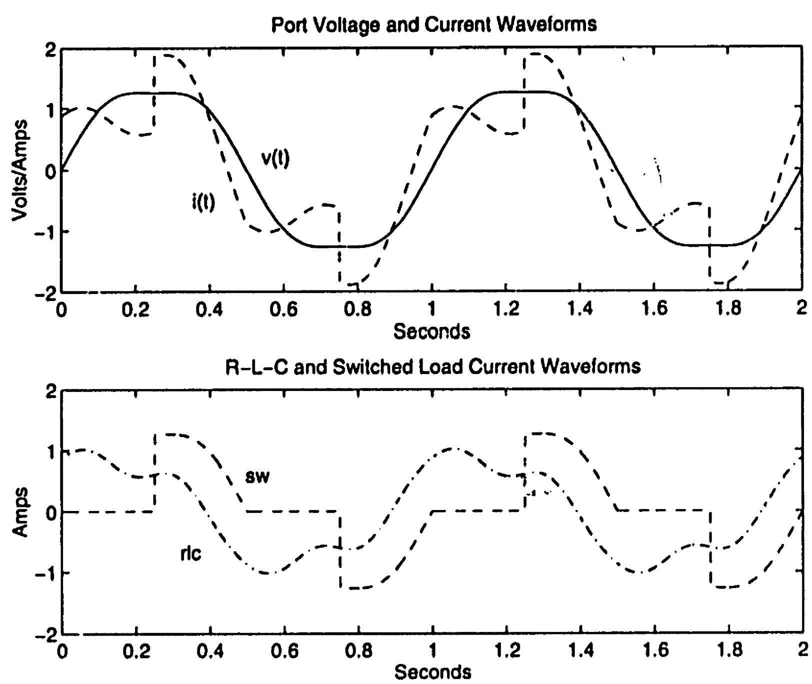

Figure 4-2: Example Circuit Waveforms

$t = \frac{T}{4}$ and $t = \frac{3T}{4}$ and turns off at $t = \frac{T}{2}$ and $t = T$, the Fourier transform can be calculated in terms of the port voltage vector, V.

$$\begin{aligned} V_{R2} &= \frac{1}{T}\int_0^T \Phi(t)v_{R2}(t)dt \\ &= \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} \Phi(t)v(t)dt \ + \ \frac{1}{T}\int_{\frac{3T}{4}}^{T} \Phi(t)v(t)dt \\ &= \left(\frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} \Phi(t)\Phi^T(t)dt\frac{1}{T}\int_{\frac{3T}{4}}^{T} \Phi(t)\Phi^T(t)dt\right) V \end{aligned} \tag{4.2}$$

The vector for the switch resistor current is then calculated from

$$I_{sw} = Z_{R2}^{-1}V_{R2} \tag{4.3}$$

Because the LTI Z matrix is block diagonal, the linear load current contains only those frequencies that appear in the port voltage. The switched load current, however, contains additional frequencies generated by the switching discontinuity. These harmonics are reflected in Equation (4.2) by the non-block-diagonal matrix inside the parentheses. The vector representation of the voltage and current waveforms are given in Table 4.1. As there are no even harmonics or D.C. components in any of the waveforms, only the odd harmonics are given. For this example, the Fourier basis was truncated after the 19^{th} harmonic.

For the $n = 39$ basis used here, the reactive power vector has length $m = 190$ corresponding to all possible cross product terms between the voltage and current. Although many of the elements in the port reactive power vector are zero due to the sparseness in V, there are far to many elements to interpret individually. Instead, the projection of R_k in the direction of R_{port} is used to measure the contribution of the k^{th} element to the total port inactive power. This projection is the defined as q in Section 3.3.2.

Table 4.2 summarizes the power components for the circuit example, listing P, S, Q, and q, for each element in the circuit and for the port. Average power is dissipated in the resistive elements, with a small dissipation in the switch introduced by aliasing associated with the truncation of the Fourier basis. Unlike S and Q, P is conserved, so the average power into the port equals the total dissipation in the circuit. As shown in the data, q is also conserved, because it is a projection of the reactive power vector.

q provides a measure of the contribution of the circuit element to the Q seen at the

$\phi(t)$	V	I_{RLC}	I_{SW}	I
$\sqrt{2}cos(\omega t)$	0.0000	0.4853	-0.3491	0.1363
$\sqrt{2}sin(\omega t)$	1.0000	0.4551	0.4982	0.9533
$\sqrt{2}cos(3\omega t)$	0.0000	0.1635	0.3109	0.4744
$\sqrt{2}sin(3\omega t)$	0.1000	0.0265	0.0518	0.0783
$\sqrt{2}cos(5\omega t)$	0.0000	-0.0292	-0.0736	-0.1028
$\sqrt{2}sin(5\omega t)$	-0.0100	-0.0014	-0.0068	-0.0082
$\sqrt{2}cos(7\omega t)$	0.0000	0.0000	0.0965	0.0965
$\sqrt{2}sin(7\omega t)$	0.0000	0.0000	0.0018	0.0018
$\sqrt{2}cos(9\omega t)$	0.0000	0.0000	-0.0526	-0.0526
$\sqrt{2}sin(9\omega t)$	0.0000	0.0000	-0.0018	-0.0018
$\sqrt{2}cos(11\omega t)$	0.0000	0.0000	0.0580	0.0580
$\sqrt{2}sin(11\omega t)$	0.0000	0.0000	0.0018	0.0018
$\sqrt{2}cos(13\omega t)$	0.0000	0.0000	-0.0387	-0.0387
$\sqrt{2}sin(13\omega t)$	0.0000	0.0000	-0.0018	-0.0018
$\sqrt{2}cos(15\omega t)$	0.0000	0.0000	0.0413	0.0413
$\sqrt{2}sin(15\omega t)$	0.0000	0.0000	0.0018	0.0018
$\sqrt{2}cos(17\omega t)$	0.0000	0.0000	-0.0305	-0.0305
$\sqrt{2}sin(17\omega t)$	0.0000	0.0000	-0.0018	-0.0018
$\sqrt{2}cos(19\omega t)$	0.0000	0.0000	0.0320	0.0320
$\sqrt{2}sin(19\omega t)$	0.0000	0.0000	0.0018	0.0018

Table 4.1: Vector Representation of Voltage and Current.

	P	S	Q	q
C1	0.0000	0.6600	0.6600	0.3416
L1	0.0000	0.1471	0.1471	-0.0598
R1	0.4577	0.4577	0.0000	0.0000
T1	0.0080	0.4971	0.4970	0.2454
R2	0.4955	0.4955	0.0000	0.0000
Total	0.9612	2.2575	1.3042	0.5272
Port	0.9612	1.0963	0.5272	0.5272

Table 4.2: Average, Apparent and Inactive Power, and the Projection, q.

port. For this example, the capacitor is contributing to the port reactive power, while the inductor is providing compensation. While the resistive elements do not contribute reactive power, the triac switch does. Even though the switch is not an energy storage device, it generates reactive power, because the nonlinear switching characteristic generates current harmonics.

4.2 Power System Metering Example

The projection q can also be used as a measure in power systems. Figure 4-3 shows a sinusoidal source, V, supplying power to an intermediate bus, V_s, through an effective impedance Z_s. Two loads draw power from the bus through the transmission line impedances Z_1 and Z_2. The first load draws a non-sinusoidal current, I_1, which induces harmonic distortion in the circuit. The second load is an LTI impedance, Z_3, which would draw sinusoidal current if the voltage waveform were not distorted.

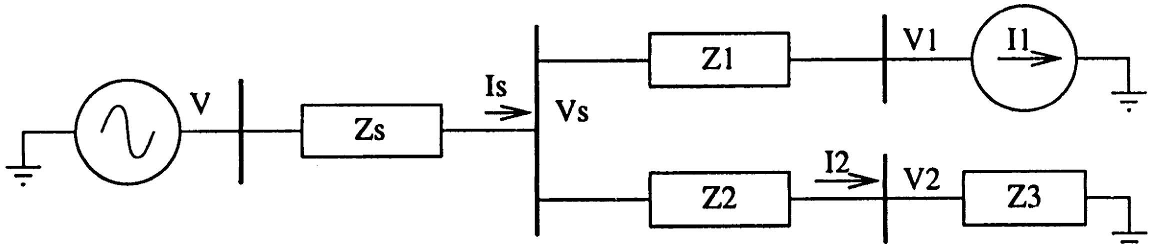

Figure 4-3: Example Power System

As the source voltage is sinusoidal and the transmission line impedances are linear, the distorted current load is the only source of harmonics in the system. The metering problem of this example is to design a scalar measure of reactive power that could be used to penalize the two loads according to the waveforms seen at the intermediate bus. With knowledge of the transmission line impedances, Z_1 and Z_2, and measurements taken at the intermediate bus, V_s, I_1, and I_2, the contribution of each load to the total inactive power, Q_s, can be calculated. The contribution of element k is q_k, the projection of R_k in the direction of R_s.

$$q_k = V_k^T \left(\frac{V_s I_s^T - I_s V_s^T}{Q_s} \right) I_k \tag{4.4}$$

For this example, V is a $1V_{rms}$, $1Hz$. source, and the distorted current, I_1, contains

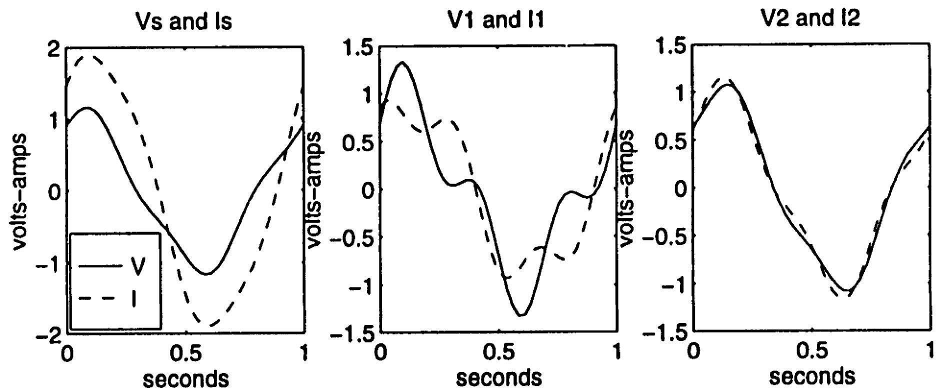

Figure 4-4: Bus Voltage and Current Waveforms

both a lagging fundamental and a strong 3^{rd} harmonic. Using the Fourier basis defined in (2.11), the vectors for these sources are

$$V_s = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{and} \quad I_s = \begin{bmatrix} 0 \\ .4 \\ .5 \\ 0 \\ 0 \\ .2 \\ 0 \\ \vdots \end{bmatrix} \tag{4.5}$$

The three transmission line impedances are identical, consisting of a 0.01Ω resistor in series with a $.05H$ inductor. The load impedance, Z_3, is a parallel R-L-C circuit with $R = 1\Omega$, $L = .5H$, and $C = 0.05F$.

$$\begin{aligned} Z_s \;=\; Z_1 \;=\; Z_2 \;&=\; 0.01\,\mathbb{I} + 0.05\,J_\omega \\ Z_3 \;&=\; ((1.0\,\mathbb{I})^{-1} + (1.0\,J_\omega)^{-1} + (0.05\,J_\omega))^{-1} \end{aligned} \tag{4.6}$$

Given the measurements V_s, I_1, and I_2, the periodic steady state voltages, V_1 and V_2 are calculated from the impedance relationships, $V_1 = Vs - Z_1 I_1$ and $V_2 = Vs - Z_2 I_2$. The waveforms are shown in Figure 4-4. (4.4) is then used to calculate the reactive power

	P	q
Load 1	0.406	0.177
Load 2	0.508	-0.012
Line Z1	0.005	0.102
Line Z2	0.005	0.150
Total	0.924	0.417
Port	0.924	0.417

Table 4.3: Average Power and the Projection, q^k.

projections of the two loads and the two transmission lines, q_1,q_1, q_{Z1} and q_{Z1}. Due to the conservation property, the total inactive power at the intermediate bus is equal to the sum of these reactive power projections.

As shown in the table, q_k provides a conserved measure of the port inactive power. q_1 is large and positive, as Load 1 is the primary source of both distortion and lagging current. Load 2 is credited with providing compensation, as Z_3 is nearly perfectly compensated at the fundamental frequency, and the impedance helps to smooth the harmonics injected by Load 1. Both the transmission lines are a significant source of reactive power as they contain large inductances without compensating capacitors.

4.3 Power Factor Optimization Example

The next example shows how a projection of the reactive power vector can be used to solve a simple power factor optimization problem in *closed form*. The circuit example of Section 4.1 is used as a load, and a parallel compensation element is added to optimize the power factor at the circuit port. The circuit is shown in Figure 4-5.

As shown by the positive q_{C1} and negative q_{L1} in Table 4.2, the parallel capacitor is contributing to the port reactive power, while the inductor is providing compensation. Therefore, to provide additional compensation, a parallel inductance, L_C, is used in this example. The problem is to find the best value for L_C such that the power factor of the port is maximized. This example is a constrained optimization in that compensation with a single inductor can improve the power factor but cannot achieve unity power factor.

The power factor is the average power divided by the apparent power, $PF = \frac{P}{S} = \frac{P}{\sqrt{P^2+Q^2}}$. When P is positive, maximizing power factor is equivalent to minimizing $\frac{S^2}{P^2} = 1 + \frac{Q^2}{P^2}$. Thus the value of L_C that maximizes power factor also minimizes $\frac{Q}{P}$. Because the

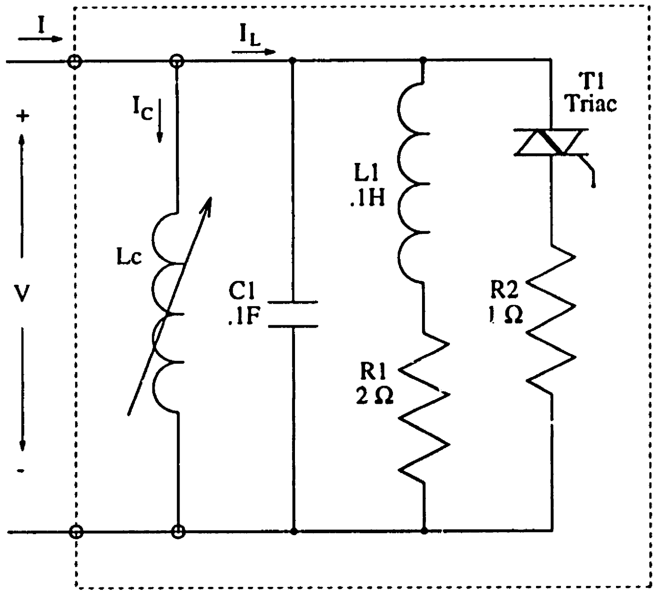

Figure 4-5: Example Compensation Circuit

compensator is lossless and the voltage source is constant, P is not affected by compensation and is a constant. Maximizing the power factor, then, corresponds to minimizing Q at the port.

$$L_C^* = argmax\left\{PF_{(L_C)}\right\} = argmin\left\{Q_{(L_C)}\right\} \tag{4.7}$$

Q is not a generalized power and does not obey conservation, so the port Q is not simply the sum of load reactive power, Q_L, and compensator reactive power, Q_C. The reactive power vector, however, is a generalized power, so the port R vector is the sum of R_L and R_C, and the port Q is the norm of this vector sum.

$$Q = \|R_L + R_C\| \tag{4.8}$$

Because Q is a non-negative quantity, the minimum Q coincides with the minimum Q^2 as well, and Q^2 can be expressed as an inner product.

$$\begin{aligned} Q^2 &= (R_L + R_C)^T(R_L + R_C) \\ &= R_L^T R_L + 2R_L^T R_C + R_C^T R_C \\ &= Q_L^2 + 2R_L^T R_C + Q_C^2 \end{aligned} \tag{4.9}$$

where Q_L is the inactive power of the load, Q_C is the inactive power of the compensator.

The inner product of the middle term can be expressed as a projection of R_C in the direction of R_L times Q_L. Using the matrix notation for the reactive power projection, (4.9) becomes

$$Q^2 = Q_L^2 + 2V^T \mathbf{R_L} I_C + Q_C^2 \tag{4.10}$$

As the compensator is lossless, $P_C = 0$, so $Q_C^2 = S_C^2 = V^T V I_C^T I_C$, yielding

$$Q^2 = Q_L^2 + 2V^T \mathbf{R_L} I_C + V^T V I_C^T I_C \tag{4.11}$$

Both Q_L and R_L are constant, because neither V nor I_L are affected by compensation. Their values can be calculated using the voltage and current vectors derived in the example in Section 4.1. (4.11) can be used to find the optimum value of L_C in closed form. Using the impedance matrix for the compensator, I_C can be expressed in terms of V, parameterized by L_C. As the impedance matrix is proportional to L_C, I_C is proportional to the inverse of L_C.

$$I_C = Z_{L_C}^{-1} V = \frac{1}{L_C} J_\omega^{-1} V \tag{4.12}$$

Substituting this expression into (4.11) yields a scalar quadratic equation in $\frac{1}{L_C}$.

$$\begin{aligned} Q^2 = & \, Q_L^2 + 2V^T \mathbf{R_L} J_\omega^{-1} V \tfrac{1}{L_C} \\ & + V^T V V^T J_\omega^{-1^T} J_\omega^{-1} V \tfrac{1}{L_C^2} \end{aligned} \tag{4.13}$$

Inserting the values for the example circuit gives.

$$Q^2 = 0.0256 \frac{1}{L_C^2} - 0.0490 \frac{1}{L_C} + 0.2779 \tag{4.14}$$

This equation has a single minimum, at $L_C = 1.05H$, where $Q = 0.505$. Figure 4-6 shows the port reactive power and power factor as a function of L_C in the vicinity of this minimum.

The compensated power factor is 0.885, improved from 0.877 by L_C. Due to the extreme harmonic content of the waveforms, the power factor can not be improved beyond 0.885 without a more complicated compensator designed to better match harmonics in current to those in voltage. The third and fifth harmonics, which exist in the voltage, could be adjusted by a linear network of tuned circuits. Nulling the higher harmonics in current would require

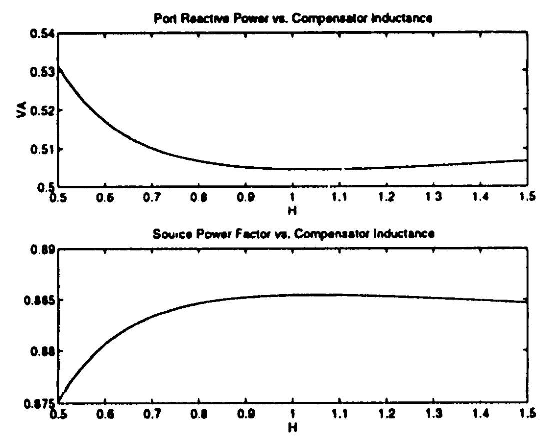

Figure 4-6: Optimum Compensation

a nonlinear or active compensator capable of generating current harmonics where none exist in voltage. In the more complicated optimization, Equation (4.13) would no longer be quadratic, and finding the global minimum in closed form would be difficult. Still, for a small number of control parameters, the global minimum could be found numerically using the techniques presented in this example.

Also, in this example the circuit port is connected to an ideal, zero-impedance voltage source. In minimizing power factor, the source voltage does not change, and the load current remains constant. In a more realistic example, a small source impedance would make the port voltage and load current slightly sensitive to the compensator inductance. The closed form minimization equation would then lose its simple quadratic form, again making a numerical solution more practical. The next example shows such a minimization.

4.4 Numerical Power Factor Optimization Example

In the sample circuit in Figure 4-7, an approximate steady state induction motor model has harmonic current injection from saturation modeled as a constant current source. The matrix equation for power factor has a more complicated form than in the previous example, because the source impedance is non-zero. Still, the optimum compensation can be found numerically by plotting the power factor for a range of compensation capacitance and looking for the global maximum.

The source voltage, V_s, is purely sinusoidal, with a $1Hz$. fundamental, while the current

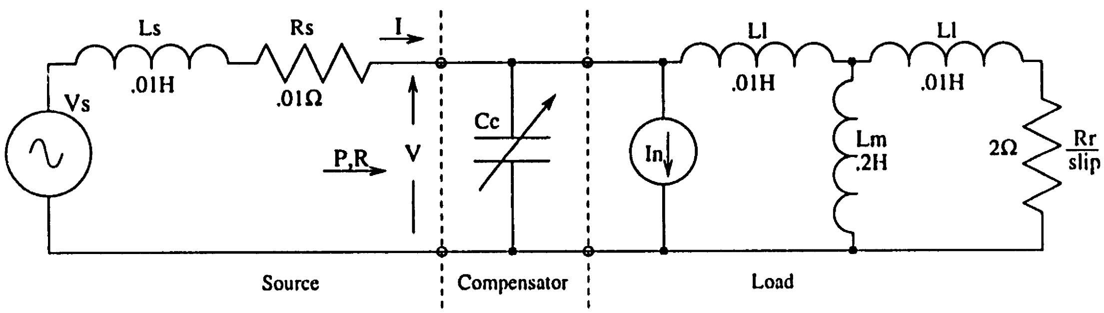

Figure 4-7: Compensation Circuit with Non-Zero Source Impedance

injection, I_n, introduces a small portion of 1^{st} and 3^{rd} harmonics to reflect core saturation. In the Fourier basis, the vectors for these sources are

$$V_s = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{and} \quad I_s = \begin{bmatrix} 0 \\ -.1 \\ .1 \\ 0 \\ 0 \\ .05 \\ .05 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{4.15}$$

The distorted load voltage is found using matrix impedance relationships.

$$\begin{aligned} V &= \left(Z_l^{-1} + Z_c^{-1}\right)^{-1} \left(Z_s + \left(Z_l^{-1} + Z_c^{-1}\right)^{-1}\right) V_s + \left(Z_s^{-1} + Z_s^{-1} + Z_s^{-1}\right)^{-1} I_s \\ I &= \left(Z_s^{-1} + Z_s^{-1} + Z_s^{-1}\right)\left(Vs - V\right) \end{aligned} \tag{4.16}$$

where Z_l, Z_c and Z_s, are the equivalent impedance matrices for the load, compensator, and source.

$$\begin{aligned} Z_s &= R_s\,\mathbf{II} + L_s\,J_\omega \\ Z_l &= L_l\,J\omega + \left((L_m\,J\omega)^{-1} + \left(\tfrac{R_r}{slip}\,\mathbf{II} + L_l\,J_\omega\right)^{-1}\right)^{-1} \\ Z_c &= (C_c\,J_\omega)^{-1} \end{aligned} \tag{4.17}$$

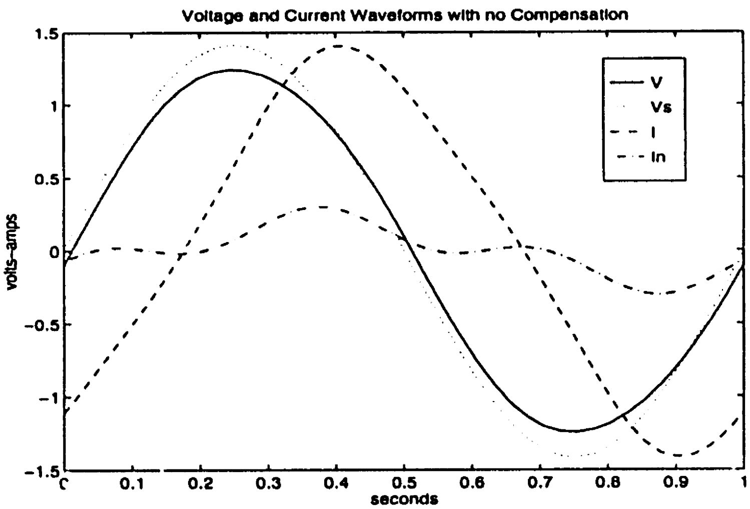

Figure 4-8: Uncompensated Voltage and Current Waveforms

With no compensation, $C_c = 0$, and the load voltage and current waveforms are shown in Figure 4-8. The small injection of 3^{rd} harmonic produces a slight distortion in the voltage waveform, but the distortion is slight, because the source impedance is small.

The power factor, $\frac{P}{S}$, can be calculated either for the load-plus-compensator, or for the internal voltage source, V_s. Due to the source impedance, the load and source voltages are different, and the power factor of the load is not the same as the power factor seen by V_s.

$$PF_l = \frac{P_l}{S_l} = \frac{V^T I}{\sqrt{V^T V I^T I}} \qquad \text{and} \qquad PF_s = \frac{P_s}{S_s} = \frac{V_s^T I}{\sqrt{V_s^T V_s I^T I}} \tag{4.18}$$

Figure 4-9 shows the compensated load power factor for a range of compensation capacitance. The global maximum occurs at $C_c = 0.12F$, while there is a local maximum $C_c = .18F$ due to the harmonic distortion. Figure 4-10 shows the load plus compensator current with optimal compensation, $C_c = .12F$. The harmonic distortion is more noticeable in the compensated voltage waveform, and the maximum power factor is approximately .9. No further power factor improvement is possible with a single parallel capacitor.

The power factor seen at the source is not the same as the power factor of the compensated load. Furthermore, as shown in Figure 4-11, the maximum source power factor does not occur at the same value of compensation capacitance. As for which maximum is *optimal*, it depends on the exact definition of optimal. If percentage resistive losses in the source impedance are to be minimized, then PF_s should be maximized. Because V_s is

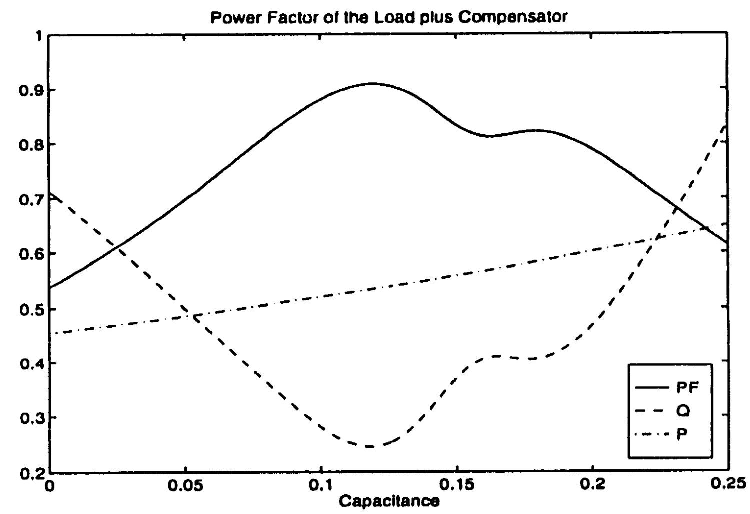

Figure 4-9: Power Factor vs. Compensation Capacitance

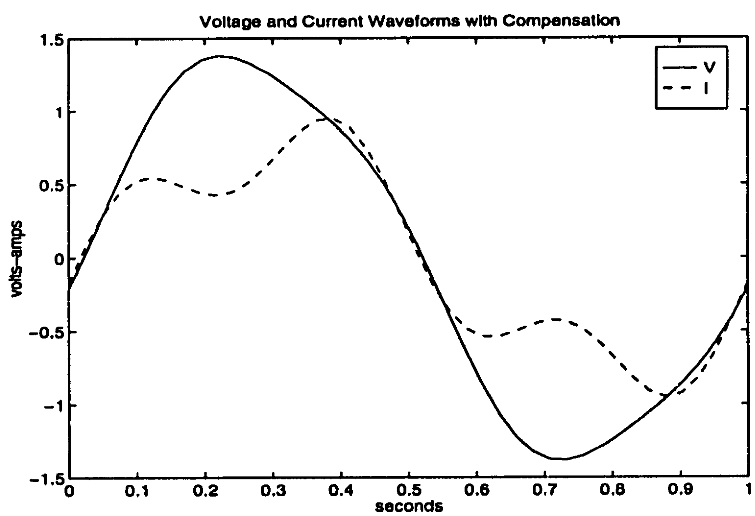

Figure 4-10: Compensated Voltage and Current Waveforms

constant, minimizing $\frac{S_s}{P}$ minimizes $\frac{I_{rms}}{P}$. In general, care must be taken to specify precisely what is to be optimized.

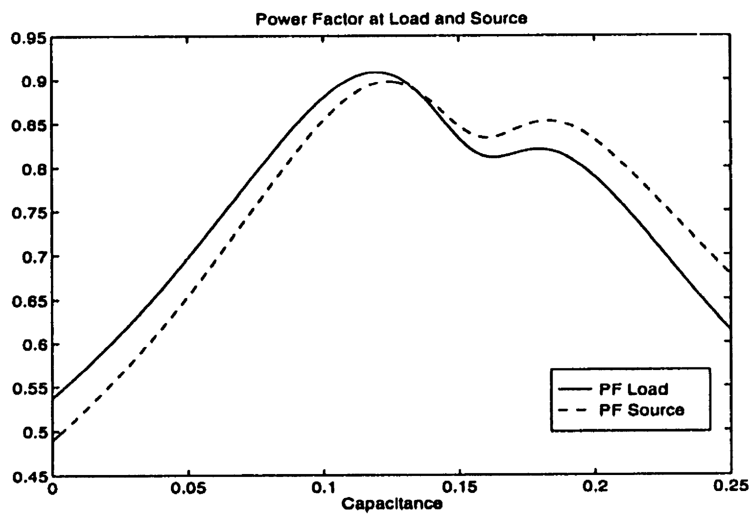

Figure 4-11: Source and Load Power Factor Comparison

Chapter 5

Multiphase Systems

Multiport waveforms can also be represented as vectors given an appropriate set of basis functions. By combining the vectors representing voltage and current in each circuit port into collective vectors, the entire set of periodic waveforms can be represented as two constant vectors, V and I. These vectors can then be used to express time-average power components just as in the single phase case. While the results could be generalized to multiport and polyphase circuits, this analysis will focus on the three-phase case.

5.1 Vector Representation of Three-Phase Signals

Figure 5-1 shows a three-phase one-port, which has either three or four wires depending on whether a neutral wire exists. The set of voltage waveforms refers to the voltage of each phase $v_1(t)$, $v_2(t)$ and $v_3(t)$, measured relative to a common reference. The set of current

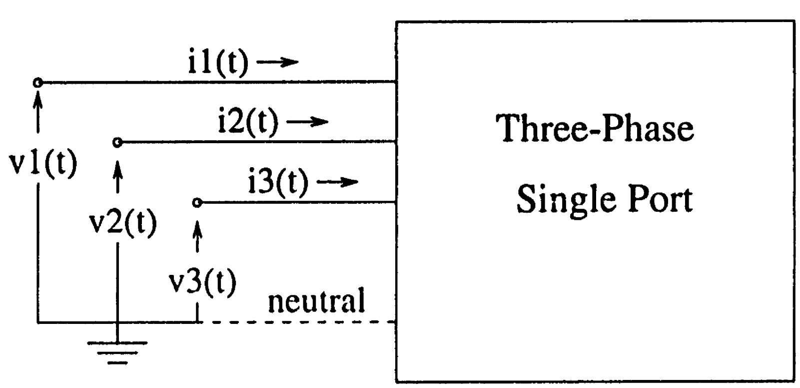

Figure 5-1: Three-Phase Single Port Connection

waveforms refers to the currents flowing in each phase, $i_1(t)$, $i_2(t)$ and $i_3(t)$. These waveform sets can be assembled into row vectors, and expressed in terms of a constant vector times an orthonormal basis, $\underline{\Phi}(t)$.

$$\begin{aligned}
[v_1(t)\ \ v_2(t)\ \ v_3(t)] &= \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}^T \underline{\Phi}(t) = V^T\underline{\Phi}(t) \\
[i_1(t)\ \ i_2(t)\ \ i_3(t)] &= \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}^T \underline{\Phi}(t) = I^T\underline{\Phi}(t)
\end{aligned} \tag{5.1}$$

Here V and I are constant vectors of length $3n$, composed of the sub-vectors V_1, V_2, V_3, and I_1, I_2, I_3, which are the vector representation of the single phase waveforms. $\underline{\Phi}(t)$ is a $3n \times 3$ matrix of time functions, forming an orthonormal basis such that

$$\frac{1}{T}\int_0^T \underline{\Phi}(t)\underline{\Phi}^T(t)dt = \mathbf{I} \tag{5.2}$$

A special note is needed regarding three-wire versus four-wire systems. In a three-wire system, there is no neutral connection, and the three currents waveforms must sum to zero. With this constraint, the current is overspecified and could be represented with only two of the waveforms. We chose to keep the extra current waveform to maintain consistency with the four-wire representation, as if the neutral exists but happens to be carrying no current.

Similarly, the voltage waveforms in a three-wire system are often taken as line-to-line. Each phase voltage is taken relative to that of the previous phase. As the sum of these waveforms must always equal zero, the three-phase voltage is also overspecified. To maintain consistency with the four-wire case, the voltage is instead represented relative to a common reference, which is equal to the average of the instantaneous voltages of each phase. Thus voltages are measured line-to-neutral, or in the absence of a neutral, line-to-average.

The set of line-to-average voltages, v_{la}, can be seen as a transformation of the line-to-line voltages, v_{ll}, although the transformation is not orthonormal.

$$[v_{1la}(t)\ v_{2la}(t)\ v_{3la}(t)] = [v_{1ll}(t)\ v_{2ll}(t)] \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix} \tag{5.3}$$

5.1.1 The Uniform Basis

There are many choices for the orthonormal matrix of basis functions, $\underline{\mathbf{\Phi}}(t)$. In general, each different basis provides a different decomposition of the multiphase waveforms, and certain decompositions can be more convenient for different applications. One obvious choice decomposes each individual phase waveform using an identical vector basis, $\Phi(t)$, as in the single phase case.

$$\underline{\mathbf{\Phi}}_u(t) = \begin{bmatrix} \Phi(t) & \vec{0} & \vec{0} \\ \vec{0} & \Phi(t) & \vec{0} \\ \vec{0} & \vec{0} & \Phi(t) \end{bmatrix} \tag{5.4}$$

Where $\Phi(t)$ is an n-vector single phase basis, and $\vec{0}$ is an n-vector of zeros. Although $\underline{\mathbf{\Phi}}_u(t)$ is a rectangular matrix, the basis is still orthonormal.

$$\frac{1}{T}\int_0^T \underline{\mathbf{\Phi}}_u(t)\underline{\mathbf{\Phi}}_u^T(t)dt = \mathbb{I} \tag{5.5}$$

This basis is called a uniform basis, because the sub-vectors represent each phase waveform in a common vector basis. For sufficiently rich vector basis, such as the Fourier basis, the uniform basis is sufficient to represent any multiphase waveform set. However, the vector representation does not reflect the specific inter-phase relationship inherent in multiphase systems. For example, three-phase systems are designed so that each phase waveform resembles that of the previous phase delayed in time by $\pm T/3$. The $\pm$ corresponds to positive and negative sequence waveforms. This time-relationship can be reflected in the basis as shown in the next section.

5.1.2 The Balanced Basis

To reflect the time delayed relationship between multiphase waveforms, the columns of $\underline{\mathbf{\Phi}}(t)$ are delayed according to the appropriate phase sequence (positive or negative.) This basis is called a balanced basis, because a balanced set of waveforms results in identical sub-vectors, V_1, V_2 and V_3.

$$\underline{\mathbf{\Phi}}_b(t) = \begin{bmatrix} \Phi(t) & \vec{0} & \vec{0} \\ \vec{0} & \Phi(t-\frac{T}{3}) & \vec{0} \\ \vec{0} & \vec{0} & \Phi(t-\frac{2T}{3}) \end{bmatrix} \tag{5.6}$$

For a sufficiently rich vector basis, Φ, such as the Fourier basis, the time delay can be represented as an orthonormal coordinate rotation, or multiplication by an $n \times n$ matrix D, which is orthonormal, such that $D^T D = \mathbb{I}$, $D^2 = D^T$, and $D^3 = D^0 = \mathbb{I}$.

$$\begin{aligned} \Phi(t - \tfrac{T}{3}) &= D\Phi(t) \\ \Phi(t - \tfrac{2T}{3}) &= D^2\Phi(t) \\ \Phi(t - \tfrac{3T}{3}) &= D^3\Phi(t) = \Phi(t) \end{aligned} \tag{5.7}$$

For the Fourier vector basis, the rotation matrix, D, preserves the D.C. component and rotates the cosine-sine pair of the k^{th} harmonic by $\frac{2\pi k}{3} radians$. If the 2×2 rotation matrix for each frequency pair is δ^k, then

$$\delta = \begin{bmatrix} cos(\frac{2\pi}{3}) & -sin(\frac{2\pi}{3}) \\ sin(\frac{2\pi}{3}) & cos(\frac{2\pi}{3}) \end{bmatrix} \quad \delta^2 = \begin{bmatrix} cos(\frac{4\pi}{3}) & -sin(\frac{4\pi}{3}) \\ sin(\frac{4\pi}{3}) & cos(\frac{4\pi}{3}) \end{bmatrix} \quad \delta^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.8}$$

$$D = \begin{bmatrix} 1 & 0\ 0 & 0\ 0 & 0\ 0 & \\ 0 & \multirow{2}{*}{δ^1} & 0\ 0 & 0\ 0 & \\ 0 & & 0\ 0 & 0\ 0 & \\ 0 & 0\ 0 & \multirow{2}{*}{δ^2} & 0\ 0 & \\ 0 & 0\ 0 & & 0\ 0 & \\ 0 & 0\ 0 & 0\ 0 & \multirow{2}{*}{δ^3} & \\ 0 & 0\ 0 & 0\ 0 & & \\ & & & & \ddots \end{bmatrix} \tag{5.9}$$

Using the delay rotation matrix, D, the balanced matrix basis, $\underline{\Phi}_b(t)$, can be represented as an orthonormal rotation of the uniform basis $\underline{\Phi}_u(t)$.

$$\begin{bmatrix} \Phi(t) & \vec{0} & \vec{0} \\ \vec{0} & \Phi(t - \frac{T}{3}) & \vec{0} \\ \vec{0} & \vec{0} & \Phi(t - \frac{2T}{3}) \end{bmatrix} = \begin{bmatrix} D^0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D^1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D^2 \end{bmatrix} \begin{bmatrix} \Phi(t) & \vec{0} & \vec{0} \\ \vec{0} & \Phi(t) & \vec{0} \\ \vec{0} & \vec{0} & \Phi(t) \end{bmatrix} \tag{5.10}$$

$$\underline{\Phi}_b(t) = \mathbf{M_{bu}}\, \underline{\Phi}_u(t)$$

where $\mathbf{0}$ is an $n \times n$ matrix of zeros, and D^0 is the $n \times n$ identity matrix. Here the $3n \times 3n$ block-diagonal rotation matrix, $\mathbf{M_{bu}}$, is also orthonormal, and the resulting rotated basis is also orthonormal.

This rotated basis is convenient for representing a set of multiphase waveforms, because in the resulting vector space, the sub-vectors V_1, V_2, and V_3 are equal under balanced conditions. Other orthonormal transformations can be used to decompose waveforms into balanced and unbalanced components, as shown in the next section.

5.1.3 The Basis of Symmetrical Rotating Components

The symmetrical components decomposition widely used for single frequency multiphase systems decomposes a three-phase set of waveforms into three balanced waveforms, one rotating with positive sequence, one with negative sequence, and one non-rotating, zero sequence.[20] For the Fourier basis, such a decomposition could be constructed for the multifrequency case by considering the non-block-diagonal rotation matrix M.

$$\underline{\Phi}_s(t) = \mathbf{M_{su}}\underline{\Phi}_u(t) \tag{5.11}$$

where

$$\mathbf{M_{su}} = \frac{1}{\sqrt{3}}\begin{bmatrix} D^0 & D^1 & D^2 \\ D^0 & D^2 & D^1 \\ D^0 & D^0 & D^0 \end{bmatrix} \tag{5.12}$$

In the resulting vector space, the vector V would be composed of sub-vectors V_p, V_n, and V_z, for the positive, negative and zero sequence waveforms.

$$[v_1(t)\ \ v_2(t)\ \ v_3(t)] = \begin{bmatrix} V_p \\ V_n \\ V_z \end{bmatrix}^T \frac{1}{\sqrt{3}}\begin{bmatrix} D^0 & D^1 & D^2 \\ D^0 & D^2 & D^1 \\ D^0 & D^0 & D^0 \end{bmatrix}\begin{bmatrix} \Phi(t) & \vec{0} & \vec{0} \\ \vec{0} & \Phi(t) & \vec{0} \\ \vec{0} & \vec{0} & \Phi(t) \end{bmatrix} \tag{5.13}$$

However, the matrix $\mathbf{M_{su}}$ is not orthonormal due to the presence of the D.C. term and harmonics at multiples of three times the fundamental frequency. These terms are not affected by the rotation matrix D and lead to a rank deficiency in $\mathbf{M_{su}}$. Simply stated, the positive, negative, and zero sequence components at D.C. and multiples of 3 harmonics are identical, in-phase, and balanced. The singularity reflects the fact that it is not possible to decompose an unbalanced waveform into three balanced time-delayed waveforms. Thus, strictly speaking, it is not possible to represent multifrequency waveforms using the single frequency definition of symmetrical components.

One approach to defining a more general symmetrical components decomposition for the multifrequency case is to specifically address the frequency components causing the rank deficiency in the rotation matrix. To allow for imbalances at D.C. and at multiples of three harmonics, $\mathbf{M_{su}}$ can be orthogonalized such that imbalances are represented in V_p and V_n as either positive or negative sequence imbalances, while the balanced component shows up in the zero sequence vector V_z. The corresponding rotation matrix can be written as

$$\mathbf{M_{su}} = \frac{1}{\sqrt{3}} \begin{bmatrix} D^0 & A_1 D^1 & A_2 D^2 \\ D^0 & A_2 D^2 & A_1 D^1 \\ D^0 & D^0 & D^0 \end{bmatrix} \tag{5.14}$$

where D is defined as in (5.9) and

$$A_i = \begin{bmatrix} a_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & a_i & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & a_i & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ & & & & & & & & & \ddots \end{bmatrix} \tag{5.15}$$

where $\quad a_1 = -\frac{\sqrt{3}+1}{2} \qquad a_2 = \frac{\sqrt{3}-1}{2}$

With this correction, $\mathbf{M_{su}}$ becomes orthonormal, so the basis $\underline{\Phi}_s(t) = \mathbf{M_{su}}\underline{\Phi}_u(t)$ is orthonormal. The sub-vectors V_p, V_n and V_z, then, are a decomposition of the multiphase voltage waveforms into positive, negative and zero sequence parts. The vector of symmetrical components can be expressed in terms of the sub-vectors, V_1, V_2, and V_3, representing the three individual waveforms in the uniform basis.

$$\begin{bmatrix} V_p \\ V_n \\ V_z \end{bmatrix} = \mathbf{M_{su}} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \mathbf{M_{us}} \begin{bmatrix} V_p \\ V_n \\ V_z \end{bmatrix} \tag{5.16}$$

where $\mathbf{M_{us}}$ is the inverse rotation matrix, $\mathbf{M_{su}}^T$.

The positive sequence part of the decomposition includes a balanced, positively rotating component for harmonics 1, 2, 4, 5, 7, etc., and a purely unbalanced, non-rotating component for harmonics 0, 3, 6, etc. The negative sequence part includes a balanced, negatively rotating component for harmonics 1, 2, 4, 5, 7, etc., and a purely unbalanced, orthogonal, non-rotating component for harmonics 0, 3, 6, etc. The zero sequence part includes a balanced non-rotating component for all frequencies.

This decomposition provides a convenient representation for multiphase waveforms in that a balanced multiphase set of waveforms with either a positive or negative sequence relationship yields only V_p or V_n non-zero. However, the presence of balanced harmonics 0, 3, 6, etc., are reflected in V_z, while imbalances are reflected in V_p and V_n depending on the orientation of the imbalance.

In three wire systems, zero sequence components can not exist, so the symmetrical components formulation is sound. However, in four wire systems, the decomposition can be confusing. For the single frequency case, there is no D.C. component and only two basis functions, and this multifrequency formulation is identical to the single frequency definition of symmetrical components. In the next section, an alternate decomposition is developed to provide a more consistent definition for waveforms with a specific phase sequence.

5.1.4 The Basis of Phase Symmetrical Components

When the phase sequence of the multiphase waveforms is defined as positive, the balanced, or in-phase component can be defined as the component which appears identically in each phase with the appropriate time delay. This component will be represented by the n-vector V_b. The unbalanced part can be decomposed into two balanced components, one where the phase angle of each frequency in $v_1(t)$ leads those of $v_2(t)$, which lead those of $v_3(t)$, by 120°. The second unbalanced component has a lagging 120° phase relationship. These unbalanced components are represented by the n-vectors V_+ and V_-.

$$[v_1(t)\ \ v_2(t)\ \ v_3(t)] = \begin{bmatrix} V_b \\ V_+ \\ V_- \end{bmatrix}^T \underline{\Phi}_p(t) \quad \text{where} \quad \underline{\Phi}_p(t) = \mathbf{M_{pu}}\underline{\Phi}_u(t) \tag{5.17}$$

The unbalanced decomposition again bears the exception of the D.C. component, which is a scalar and has no phase angle. i.e. it is not possible to decompose a D.C. imbalance into multiple balanced D.C. components. The inconsistency is solved by making the D.C. components of V_b, V_+ and V_- represent a specific orthonormal decomposition of the D.C. values of the three waveforms.

$$\begin{bmatrix} V_{bDC} \\ V_{+DC} \\ V_{-DC} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ 1 & \frac{\sqrt{3}-1}{2} & -\frac{\sqrt{3}+1}{2} \end{bmatrix} \begin{bmatrix} V_{1DC} \\ V_{2DC} \\ V_{3DC} \end{bmatrix} \tag{5.18}$$

Here V_b properly reflects the balanced component, while V_+ and V_- contain a somewhat arbitrary decomposition of the unbalanced D.C. part. Unlike the symmetrical components formulation, the harmonics at multiples of 3 times the fundamental frequency do not require an unbalanced decomposition. The orthonormal rotation matrix for the phase symmetrical components basis then becomes

$$\mathbf{M_{pu}} = \begin{bmatrix} D^0 & D^1 & D^2 \\ D^0 & A_1 D^1 & A_2 D^2 \\ D^0 & A_2 D^1 & A_1 D^2 \end{bmatrix} \tag{5.19}$$

where

$$A_1 = \begin{bmatrix} -\frac{\sqrt{3}+1}{2} & 0 \;\; 0 & 0 \;\; 0 & 0 \;\; 0 & \\ 0 & & 0 \;\; 0 & 0 \;\; 0 & \\ & \delta & & & \\ 0 & & 0 \;\; 0 & 0 \;\; 0 & \\ 0 & 0 \;\; 0 & & 0 \;\; 0 & \\ & & \delta & & \\ 0 & 0 \;\; 0 & & 0 \;\; 0 & \\ 0 & 0 \;\; 0 & 0 \;\; 0 & & \\ & & & \delta & \\ 0 & 0 \;\; 0 & 0 \;\; 0 & & \\ & & & & \ddots \end{bmatrix} \tag{5.20}$$

and

$$A_2 = \begin{bmatrix} \frac{\sqrt{3}-1}{2} & 0\ 0 & 0\ 0 & 0\ 0 & \\ 0 & \delta^{-1} & 0\ 0 & 0\ 0 & \\ 0 & & 0\ 0 & 0\ 0 & \\ 0 & 0\ 0 & \delta^{-1} & 0\ 0 & \\ 0 & 0\ 0 & & 0\ 0 & \\ 0 & 0\ 0 & 0\ 0 & \delta^{-1} & \\ 0 & 0\ 0 & 0\ 0 & & \\ & & & & \ddots \end{bmatrix} \tag{5.21}$$

$$\delta = \begin{bmatrix} cos(\frac{2\pi}{3}) & -sin(\frac{2\pi}{3}) \\ sin(\frac{2\pi}{3}) & cos(\frac{2\pi}{3}) \end{bmatrix} \quad \text{and} \quad \delta^{-1} = \delta^2 = \begin{bmatrix} cos(\frac{4\pi}{3}) & -sin(\frac{4\pi}{3}) \\ sin(\frac{4\pi}{3}) & cos(\frac{4\pi}{3}) \end{bmatrix} \tag{5.22}$$

With the correction for the D.C. components, the rotation is orthonormal, so the decomposition can fully represent any positive sequence multiphase waveform. The sub-vectors of symmetrical components, V_b, V_+ and V_-, can be expressed in terms of the sub-vectors for the three individual waveforms in the uniform basis, V_1, V_2, and V_3.

$$\begin{bmatrix} V_b \\ V_+ \\ V_- \end{bmatrix} = \mathbf{M_{pu}} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \mathbf{M_{up}} \begin{bmatrix} V_b \\ V_+ \\ V_- \end{bmatrix} \tag{5.23}$$

where $\mathbf{M_{up}}$ is the inverse rotation matrix, $\mathbf{M_{pu}}^T$.

In the phase symmetrical components decomposition, the sub-vector V_b represents the balanced positive sequence component at all frequencies, while the vectors V_+ and V_- represent a leading and lagging phase angle decomposition of the unbalanced component. For the single frequency case, where there is no D.C. component and only two basis functions, this decomposition is equivalent to that of symmetrical components, with V_b, V_+ and V_- corresponding to the positive, negative and zero sequence components, respectively.

5.2 Multiphase Periodic Steady State Circuit Analysis

The periodic steady state circuit analysis tools outlined in Section 2.2.1 can be extended to include multiphase systems by stacking the impedance relationships of the individual phases into a multiphase impedance relationship. For four-wire, wye-connected circuits, the extension is simple, because the phases are completely decoupled. In a delta-connected or three-wire system, however, the sum of the phase currents must always be zero, and this constraint leads to a singularity in the conductance matrix. Problems due to the singularity can be solved in the limit, leading to a notationally compact LTI circuit analysis method for multiphase systems.

5.2.1 Wye-Connected Four-Wire Systems

A four wire, wye-connected, multiphase electrical network can be characterized by three basic types of circuit connection; a single element termination to neutral, a series connection of elements, and a parallel connection of elements. As shown in Figure 5-2, there are no coupling impedances between phases, only to neutral. The neutral wire is considered an integral part of the element, because the existence of the neutral is important in setting up the algebraic solution.

The periodic steady state impedance relationships for the three phases in the single element termination are completely decoupled. The three impedance equations can be stacked into a single block-diagonal matrix equation.

$$\begin{aligned} V_1 &= Z_1 I_1 \\ V_2 &= Z_2 I_2 \\ V_3 &= Z_3 I_3 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} Z_1 & 0 & 0 \\ 0 & Z_2 & 0 \\ 0 & 0 & Z_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \tag{5.24}$$

The impedance matrix relates the voltage and current vectors of the uniform basis, $\mathbf{\Phi}_u(t)$, and will be denoted in bold, $\mathbf{Z}$. The inverse impedance is the conductance matrix, $\mathbf{G} = \mathbf{Z}^{-1}$, which is also block diagonal.

$$V = \mathbf{Z}I \qquad \text{and} \qquad I = \mathbf{G}V \tag{5.25}$$

The impedance is easily transformed from the uniform basis to any other orthonormal basis using a rotation matrix. As derived for the time to frequency transformation in Section 2.2.2, the uniform basis impedance matrix is transformed to the phase symmetrical basis using $\mathbf{M_{pu}}$. The transformed matrix is no longer block diagonal but represents a

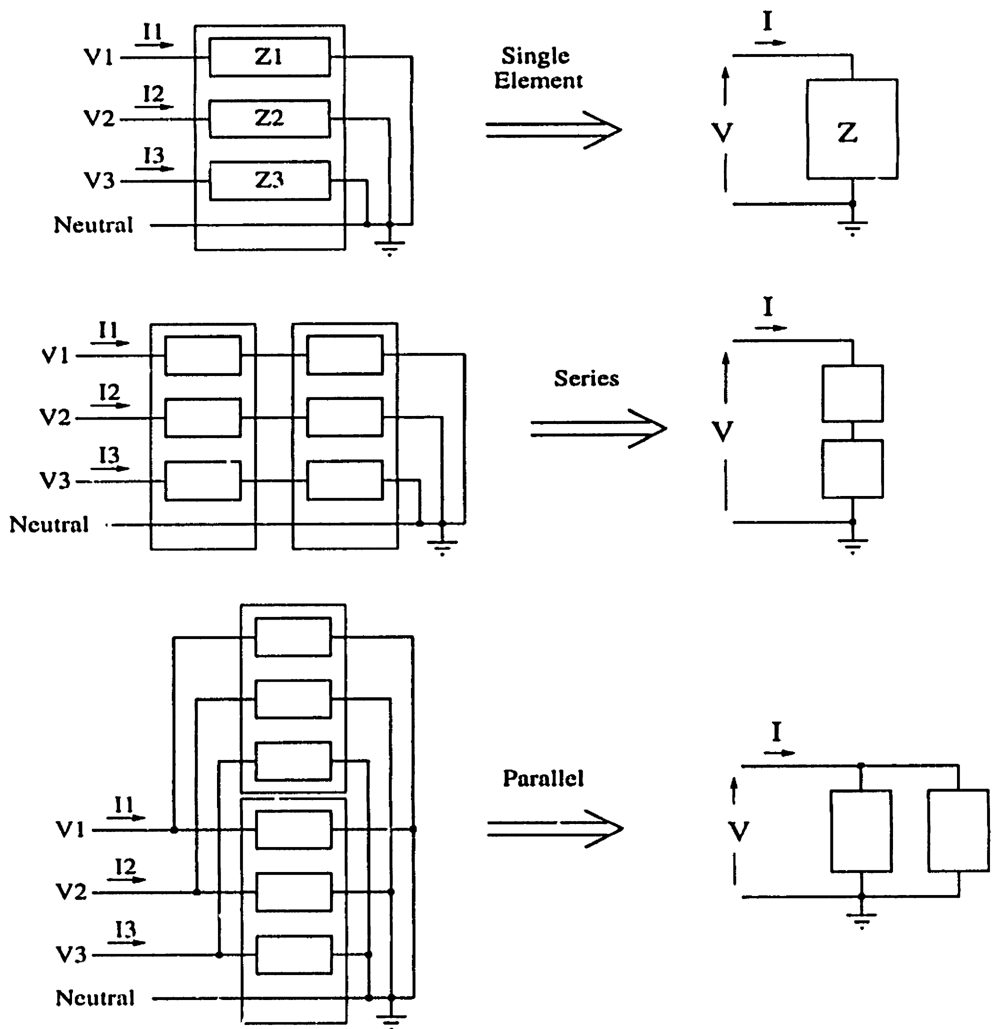

Figure 5-2: Wye-Connected Four-Wire Three-Phase Circuit Connections

balanced/unbalanced decomposition of the circuit impedance.

$$\mathbf{Z_p} = \mathbf{M_{pu}} \mathbf{Z_u} \mathbf{M_{up}} \tag{5.26}$$

The multiphase impedances of the series and parallel connections shown in Figure 5-2 combine in the same manner as the single phase impedances. Series connections add impedances, while parallel connections add conductances. Thus complicated wye-connected circuits can be modeled as a collection of nested series and parallel connections. One important difference between the single and multiphase analysis is that the multiphase voltage reference *must be* the neutral node, and there can be only one neutral node in a wye-connected circuit. A floating neutral node can be modeled as a three-wire system, as described in the next section.

5.2.2 Wye-Connected Three-Wire Elements

In a wye-connected, three-wire circuit, the neutral wire does not exist, so in the single element termination, the center node is floating. As there is no fourth current path, the current vectors for the three phases must always sum to zero. The three rows of blocks in the uniform basis conductance matrix must, therefore, sum to zero, leading to a singularity. The impedance matrix, which calculates the multiphase voltage vector from the current vector, is infinite in the direction of current imbalances.

The multiphase impedance matrix is developed starting from the per-phase impedance or conductance relationships.

$$\begin{aligned} V_1 &= Z_1 I_1 + V_n & \qquad & I_1 = G_1(V_1 - V_n) \\ V_2 &= Z_2 I_2 + V_n & \text{or} \qquad & I_2 = G_2(V_2 - V_n) \\ V_3 &= Z_3 I_3 + V_n & & I_3 = G_3(V_3 - V_n) \end{aligned} \tag{5.27}$$

where V_n is the voltage vector for the floating node. For the wye-connected element, it is more convenient to start with the impedance relationships. As shown in Figure 5-3, the floating neutral singularity is eliminated by connecting a resistance, R_n, between the floating node and the global neutral wire. The effects of the added resistance are made arbitrarily small by taking the limit $R_n \to \infty$.

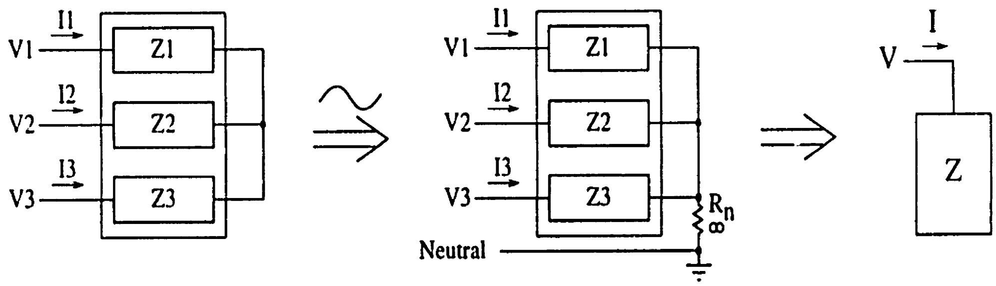

Figure 5-3: Three-Wire Wye-Connected Termination Approximation

The added resistance provides an additional equation for V_n in (5.27).

$$V_n = R_n(I_1 + I_2 + I_3) \tag{5.28}$$

Combining (5.27) and (5.28) into one matrix equation yields

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \left(\begin{bmatrix} Z_1 & 0 & 0 \\ 0 & Z_2 & 0 \\ 0 & 0 & Z_3 \end{bmatrix} + R_n \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} & \mathbb{1} \end{bmatrix} \right) \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \tag{5.29}$$

Thus the impedance matrix, **Z**, for the a three-wire wye-connected single element termination is the equivalent block-diagonal four-wire impedance of the element plus a non-block-diagonal termination impedance proportional to R_n. As **Z** is invertible, the conductance matrix also exists.

$$\mathbf{Z} = \begin{bmatrix} Z_1 & 0 & 0 \\ 0 & Z_2 & 0 \\ 0 & 0 & Z_3 \end{bmatrix} + R_n \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} & \mathbb{1} \end{bmatrix} \tag{5.30}$$

As shown in Figure 5-3, the three-wire termination is represented in schematic form as a single connection element, with no ground connection.

Three-Wire, Delta-Connected Terminations

The conductance matrix for a delta-connected load is similarly singular must be defined in the limit. As shown in Figure 5-4, the delta connection contains no floating node, but has only inter-phase impedances. The singular conductance relationship can be written in

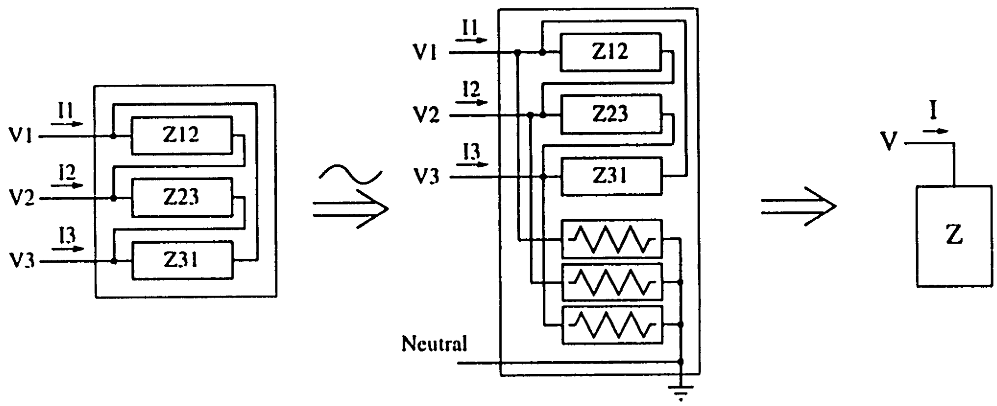

Figure 5-4: Three-Wire Delta-Connected Termination Approximation

matrix form starting with the per-phase conductance relationship.

$$\begin{aligned}
I_1 &= G_{12}(V_1 - V_2) + G_{13}(V_1 - V_3) \\
I_2 &= G_{21}(V_2 - V_1) + G_{23}(V_2 - V_3) \\
I_3 &= G_{31}(V_3 - V_1) + G_{32}(V_3 - V_2) \\
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} &= \begin{bmatrix} (G_{12}+G_{13}) & -G_{12} & -G_{13} \\ -G_{21} & (G_{21}+G_{23}) & -G_{23} \\ -G_{31} & -G_{32} & (G_{31}+G_{32}) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
\end{aligned} \tag{5.31}$$

where $G_{ij} = G_{ji}$ by reciprocity. The **G** matrix is singular because its columns sum to zero. The singularity is solved by adding a multiphase, wye-connected, four-wire resistive termination, R_n, in parallel, and taking the limit as $R_n \to \infty$. The parallel connection adds a diagonal conductance matrix, proportional to $\frac{1}{R_n}$. Thus for a single element delta-connected termination, the conductance matrix becomes

$$\mathbf{G} = \lim_{R_n \to \infty} \begin{bmatrix} (G_{12}+G_{13}) & -G_{12} & -G_{13} \\ -G_{21} & (G_{21}+G_{23}) & -G_{23} \\ -G_{31} & -G_{32} & (G_{31}+G_{32}) \end{bmatrix} + \frac{1}{R_n} \begin{bmatrix} \mathrm{II} & 0 & 0 \\ 0 & \mathrm{II} & 0 \\ 0 & 0 & \mathrm{II} \end{bmatrix} \tag{5.32}$$

5.2.3 Multiphase Circuit Analysis Example

With the modification for the three-wire single element delta and wye terminations, multiphase periodic steady state circuit models with mixed three and four wire elements can

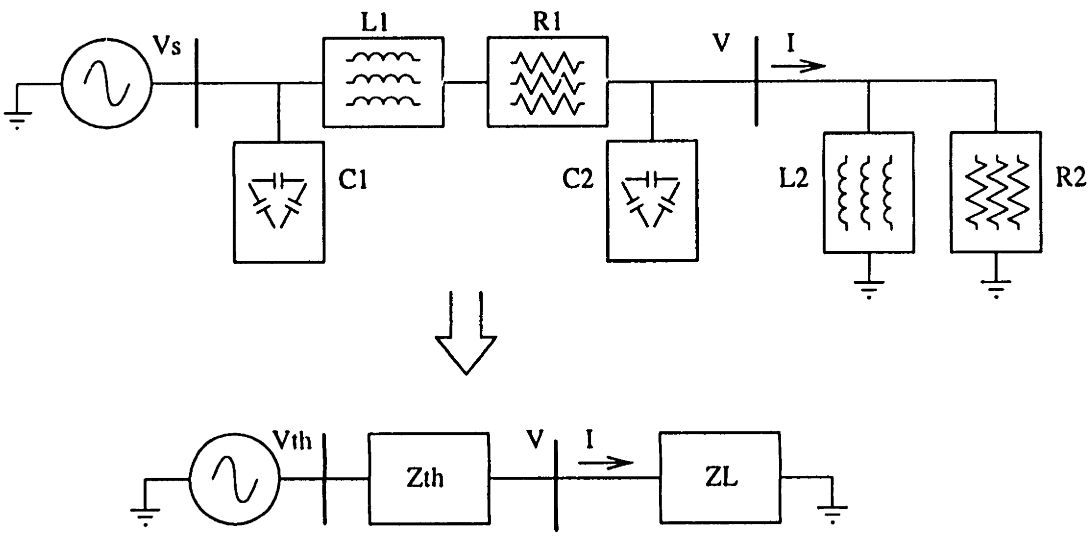

Figure 5-5: Multiphase Thevenin Equivalent Circuit

be constructed. The circuit is specified as a collection of four-wire series and parallel connections, and each termination is specified as either a four-wire wye, three-wire wye, or four-wire delta connection.

An example circuit is shown in Figure 5-5. A three-phase voltage source, or infinite bus, provides power to a parallel resistor-inductor load through a network resembling a transmission line. Transmission line capacitors C_1 and C_2 are delta connected, three-wire terminations, while the load terminations are four-wire, wye-connections. The transmission line inductor and resistor are series elements, not terminations, and so are modeled as four-wire elements.

As an example of multiphase circuit modeling, the multiphase thevenin equivalent for the infinite bus and transmission line can be easily calculated in terms of the uniform basis impedance matrices of the circuit elements. C_1 is in parallel with the voltage source and does not effect the thevenin equivalents, V_{th} and Z_{th}. C_2, however, loads the transmission line thereby changing the open circuit voltage according to the voltage divider law.

$$V_{th} = \mathbf{Z_{C2}} \left(\mathbf{Z_{L1}} + \mathbf{Z_{R1}} + \mathbf{Z_{C2}}\right)^{-1} V_s \tag{5.33}$$

The thevenin equivalent impedance is the parallel combination of C_2 and the series line impedance.

$$\mathbf{Z_{th}} = \left((\mathbf{Z_{L1}} + \mathbf{Z_{R1}})^{-1} + \mathbf{Z_{C2}}^{-1}\right)^{-1} \tag{5.34}$$

The equivalent impedance of the load, $\mathbf{Z_L}$, is the parallel combination of L_2 and R_2. The uniform basis voltage and current vectors, V and I, are then expressed in terms of $\mathbf{Z_{th}}$ and $\mathbf{Z_L}$.

$$\begin{aligned} \mathbf{Z_L} &= (\mathbf{Z_{L2}}^{-}1 + \mathbf{Z_{R2}}^{-}1)^{-1} \\ V &= \mathbf{Z_L}\,(\mathbf{Z_{th}} + \mathbf{Z_L})^{-1}\,V_{th} \\ I &= (\mathbf{Z_{th}} + \mathbf{Z_L})^{-1}\,V_{th} \end{aligned} \tag{5.35}$$

This example is analyzed for typical component values in Section 5.4.

5.3 Multiphase Power Components

5.3.1 Average Power

The total instantaneous power in a multiphase system is the sum of instantaneous power in each phase. With the voltages taken with respect to a common reference, i.e. line-to-neutral or line-to-average as defined in Section 5.1, the total power can be written as an inner product.

$$p(t) = [v_1(t)\ v_2(t)\ v_3(t)] \begin{bmatrix} i_1(t) \\ i_2(t) \\ i_3(t) \end{bmatrix} = v(t)i^T(t) \tag{5.36}$$

Substituting the vector expressions for voltage and current using any orthonormal basis,

$$p(t) = V^T\ \underline{\mathbf{\Phi}}(t)\ \underline{\mathbf{\Phi}}^T(t)\ I \tag{5.37}$$

Because $\underline{\mathbf{\Phi}}$ is orthonormal, the time-average total power simplifies to a single scalar product, as in the single phase case.

$$P = \frac{1}{T}\int_0^T V^T \underline{\mathbf{\Phi}}(t)\underline{\mathbf{\Phi}}^T(t) I dt = V^T I \tag{5.38}$$

Therefore, the expression for P is independent of the choice of basis. The only terms contributing to average power are the products of identical voltage and current components.

5.3.2 Apparent Power

In a multiphase system, the apparent power is the multiphase rms voltage times the multiphase *rms* current. Carefully defined, the multiphase *rms* is not the sum of the rms of each phase, but the square-root of the sum of the squares of rms values.

$$v_{trms} = \sqrt{\sum_i v_{irms}^2} \neq \sum_i v_{irms} \tag{5.39}$$

This definition reflects the fact that the waveforms are physically orthogonal, i.e. exist in different wires. Again, the definition assumes that the voltages are measured line-to-neutral, if the neutral wire exists, or line-to-average, in the absence of a neutral.

Substituting the vector expressions for voltage and current using any orthonormal basis, the multiphase *rms* becomes the square root of the inner product of the vector with itself. As in the single phase case, the multiphase *rms* is the vector two-norm, and S is the product of the norms of V and I. The expression for S, then, is also independent of the choice of basis.

$$S = \sqrt{V^T V}\,\sqrt{I^T I} = \|V\|\,\|I\| \tag{5.40}$$

5.3.3 Total Reactive Power and the Reactive Power Vector

Because the definitions for multiphase P and S are identical to the single phase case, the multiphase definition of Q has the same form and dimensionality as before. Both S and P are independent of basis, so the value of the scalar Q does not depend on the specific decomposition of the voltage and current waveforms.

$$Q = \sqrt{S^2 - P^2} = \sqrt{V^T V I^T I - V^T I V^T I} \tag{5.41}$$

Q can be viewed as the norm of a vector, R, but because the voltage and current vectors are now length $3n$, the reactive power vector is length $\frac{1}{2}3n(3n-1)$, or roughly nine times larger than in the single phase case. The elements of R do depend on the choice of basis but, for a particular basis, obey conservation and can be used to define scalar projections of reactive power.

The reactive power matrix, $\mathbf{R}$, is defined as before and is again skew-symmetric with zeros on the main diagonal.

$$\mathbf{R} = VI^T - IV^T \tag{5.42}$$

With V and I represented as sub-vectors for each phase, the $\mathbf{R}$ matrix has a partitioned structure representing the interactions of the different phases.

$$\mathbf{R} = \begin{bmatrix} (V_1 I_1^T - I_1 V_1^T) & (V_1 I_2^T - I_1 V_2^T) & (V_1 I_3^T - I_1 V_3^T) \\ (V_2 I_1^T - I_2 V_1^T) & (V_2 I_2^T - I_2 V_2^T) & (V_2 I_3^T - I_2 V_3^T) \\ (V_3 I_1^T - I_3 V_1^T) & (V_3 I_2^T - I_3 V_2^T) & (V_3 I_3^T - I_3 V_3^T) \end{bmatrix} \tag{5.43}$$

When the sub-vectors correspond to a symmetrical or balanced/unbalanced decomposition, the partitions of the reactive power matrix are $n \times n$ matrices representing the interactions of the different components. This component-wise partitioning allows the contributions to reactive power to be classified as balanced reactive power, unbalanced reactive power, and cross terms. If waveforms are nearly balanced, the unbalanced blocks may be unimportant, leaving only the $n \times n$ balanced reactive power matrix and perhaps the cross terms.

5.4 Multiphase Reactive Power Examples

Sinusoidal Reactive Power in a Balanced Network

The first example of multiphase reactive power involves a purely sinusoidal, balanced network. The circuit in Figure 5-5 will be used with the following component values.

$$\begin{aligned} V_s &= 1V\,rms\,, 60Hz, balanced. \\ C_1 &= C_2 = 100\mu F \\ L_1 &= 1mH \\ R_1 &= 0.01\Omega \\ L_2 &= 10mH \\ R_2 &= 2\Omega \end{aligned} \tag{5.44}$$

As there is only one frequency present, the underlying basis functions are the cosine/sine pair at $60Hz$. The rms source voltage is $1V\,rms$, so the source voltage vector, V_s, is unit length. As the impedance relationships in Section 5.2 use the uniform basis, the balanced voltage source is expressed using $\mathbf{M_{ub}}$, the balanced to unbalanced transformation given in Equation (5.10).

$$V_s = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \mathbf{M_{ub}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{5.45}$$

The uniform basis multiphase impedance matrices of the four-wire elements, L_1, R_1, L_2 and R_2, are found from

$$\mathbf{Z_L} = L \begin{bmatrix} J_\omega & 0 & 0 \\ 0 & J_\omega & 0 \\ 0 & 0 & J_\omega \end{bmatrix} \quad \text{and} \quad \mathbf{Z_R} = R \begin{bmatrix} \mathbb{I} & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \end{bmatrix} \tag{5.46}$$

The impedance of the three-wire delta-connected capacitors is found by first writing the conductance matrix as in Equation (5.32).

$$\mathbf{G_C} = \lim_{R_n \to \infty} C \begin{bmatrix} 2J_\omega & -J_\omega & -J_\omega \\ -J_\omega & 2J_\omega & -J_\omega \\ -J_\omega & -J_\omega & 2J_\omega \end{bmatrix} + \frac{1}{R_n} \begin{bmatrix} \mathbb{I} & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \end{bmatrix} \tag{5.47}$$

$$\mathbf{Z_C} = \mathbf{G_C}^{-1}$$

As there are only two basis functions for the sinusoidal case, $\mathbb{I}$ is the 2×2 identity matrix and

$$J_\omega = 2\pi \begin{bmatrix} 0 & -60 \\ 60 & 0 \end{bmatrix} \tag{5.48}$$

Using the thevenin equivalent and solution equations for this circuit given in (5.35), the voltage and current vectors for the load bus are calculated. The average, apparent, and inactive powers, and the reactive power matrix are then calculated from V and I.

$$V = \begin{bmatrix} 0.588 \\ -0.026 \\ -0.316 \\ -0.497 \\ -0.272 \\ 0.522 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 0.287 \\ -0.169 \\ -0.290 \\ -0.164 \\ 0.003 \\ 0.333 \end{bmatrix}$$

$$P = 0.520W \qquad S = 0.589VA \qquad Q = 0.276VAR \tag{5.49}$$

$$R = \begin{bmatrix} 0 & -0.092 & -0.080 & 0.046 & 0.080 & 0.046 \\ 0.092 & 0 & -0.046 & -0.080 & -0.046 & 0.080 \\ 0.080 & 0.046 & 0 & -0.092 & -0.080 & 0.046 \\ -0.046 & 0.080 & 0.092 & 0 & -0.046 & -0.080 \\ -0.080 & 0.046 & 0.080 & 0.046 & 0 & -0.092 \\ -0.046 & -0.080 & -0.046 & 0.080 & 0.092 & 0 \end{bmatrix}$$

Because the circuit is exactly balanced, there is a symmetrical structure to the V, I and R, but because the uniform basis does not properly reflect the multiphase symmetry, the structure of the vectors is not obvious. Mapping the vectors to the basis of phase symmetrical components, using $\mathbf{M_{pu}}$ defined in Equation (5.19), brings out the symmetry. The one remaining non-zero element in the upper triangle of R is the signed component corresponding to balanced sinusoidal reactive power.

$$\mathbf{M_{pu}}V = \begin{bmatrix} 1.019 \\ -0.044 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M_{pu}}I = \begin{bmatrix} 0.498 \\ -0.292 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.50}$$

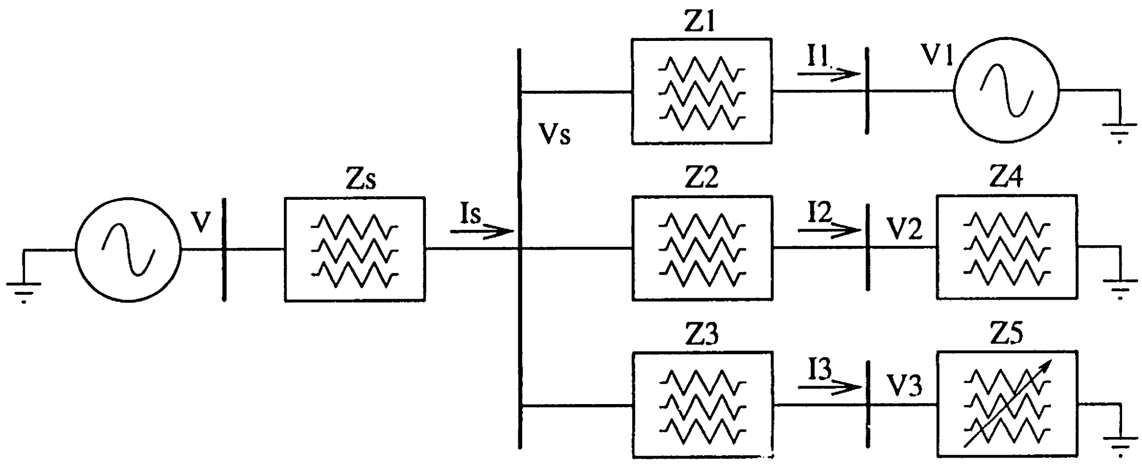

Figure 5-6: Unbalanced Three-Phase Resistive Network

$$\mathbf{M_{pu}}R\mathbf{M_{up}} = \begin{bmatrix} 0 & -0.276 & 0 & 0 & 0 & 0 \\ 0.276 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.51}$$

Unbalanced Reactive Power Example

The next example demonstrates how multiphase reactive power can arise from an phase imbalance even if each phase has no reactive power. The circuit is shown in Figure 5-6. The two voltage sources are sinusoidal, in phase, and balanced. All the impedances are resistive and, with the exception of Z_5, balanced. The bus voltages V_s, V_2 and V_3, have an imbalance introduced by Z_5, but are otherwise in phase with the two voltage sources. The following component values are used:

	P	S	Q	q
V_1	0.431	0.435	0.057	-0.057
Z_1	0.034	0.034	0	0
Z_2	0.054	0.054	0	0
Z_3	0.040	0.040	0	0
Z_4	0.539	0.539	0	0
Z_5	0.456	0.475	0.134	0.134
Total	1.554	1.577	0.191	0.077
V_s	1.554	1.556	0.076	0.077

Table 5.1: Power components for the Multiphase Example.

$$V_s = 1V\,rms\,, 60Hz, balanced$$

$$V_1 = 0.75\,V_s$$

$$\mathbf{Z_s} = \mathbf{Z_1} = \mathbf{Z_2} = \mathbf{Z_3} = \begin{bmatrix} 0.1\Omega\Pi & 0 & 0 \\ 0 & 0.1\Omega\Pi & 0 \\ 0 & 0 & 0.1\Omega\Pi \end{bmatrix}$$

$$\mathbf{Z_4} = \begin{bmatrix} 1\Omega\Pi & 0 & 0 \\ 0 & 1\Omega\Pi & 0 \\ 0 & 0 & 1\Omega\Pi \end{bmatrix} \qquad (5.52)$$

$$\mathbf{Z_5} = \begin{bmatrix} 3\Omega\Pi & 0 & 0 \\ 0 & 1\Omega\Pi & 0 \\ 0 & 0 & 1\Omega\Pi \end{bmatrix}$$

Using the same basis as the previous example, the voltage and current vectors for each bus can be calculated. The power components for the three loads are calculated from (V_1,I_1), (V_2,I_2), and (V_3,I_3) and are shown in Table 5.1. Also shown is q_k, the reactive power projection of element k in the direction of the reactive power of the common bus, V_s.

As shown in the data, the conserved projection q provides a measure of the contribution of each element to the inactive power at the common bus. q is zero for each balanced resistive element, because the voltage and current vectors are exactly proportional. q for the unbalanced resistance Z_5 is large and positive, because Z_5 is the only source of imbalance in the circuit. The voltage source V_1 is credited with providing compensating, or helping to balance the common bus voltage. The impedance load Z_4 does not provide compensation as the resistances of each phase are decoupled.

The common bus voltage and current imbalance can be easily seen by transforming the vectors to the basis of symmetrical phase components.

$$\mathbf{M_{pu}}Vs = \begin{bmatrix} V_b \\ V_+ \\ V_- \end{bmatrix} = \begin{bmatrix} 0.807 \\ 0 \\ 0.005 \\ 0 \\ 0.005 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M_{pu}}Is = \begin{bmatrix} I_b \\ I_+ \\ I_- \end{bmatrix} = \begin{bmatrix} 1.925 \\ 0 \\ -0.054 \\ 0 \\ -0.054 \\ 0 \end{bmatrix} \tag{5.53}$$

The directional information contained in the reactive power vector can also be seen by looking at the **R** matrix in the symmetrical basis.

$$\mathbf{M_{pu}}(V_s I_s^T - I_s V_s^T)\mathbf{M_{up}} = \begin{bmatrix} 0 & 0 & -0.054 & 0 & -0.054 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.054 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.054 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.54}$$

Chapter 6

Conclusions

There are three primary results in this thesis. These results will be summarized in the following sections.

6.1 Periodic Steady State Circuit Analysis

In exploring the use of orthonormal decompositions of periodic signals, the methods of sinusoidal steady-state (SSS) linear time invariant (LTI) circuit analysis are generalized to the periodic steady-state. The extension to include harmonics is based on the definition of an impedance matrix, which plays the role of the complex impedance used in SSS methods. The periodic steady state response is calculated in the frequency domain, using a block diagonal impedance matrix, or directly in the discrete sampled time domain, using a periodic impulse response matrix. The frequency and time-domain vector spaces are related by a simple orthonormal transformation, or matrix multiplication, which is derived from the Discrete Fourier Transform.

A matrix singularity results from an ill-defined D.C. operating point or the unstable excitation of a resonant mode in an LTI circuit. The singularity reflects the fact that there is no steady-state solution for these cases. Numerical problems due to the singularity are easily solved in practice by adding a very large parallel resistance, which then gives a unique, finite, steady state solution.

The relationships of Kirchhoff voltage and current laws, series and parallel effective impedance. Thevenin equivalents, and voltage and current divider laws, are all easily expressed in matrix form for the periodic steady state. The generalized methods provide a convenient means of finding the total periodic response of LTI circuits in one computational step, i.e. without solving for the SSS response at every frequency. The method can be extended to include certain time varying and non-linear circuits, such as the periodic switching circuit shown in Section 2.2.3. While the underlying computational complexity of the matrix expressions is large, the simple notation is easily implemented using matrix manipulation software.

Future Work

The special structure of the $n \times n$ impedance matrix for LTI circuits has only n degrees of freedom. Analysis of certain types of time-varying and non-linear systems can result in a matrix with a full structure. In the frequency domain, the full matrix can relate the coupling between frequencies, solving a more general problem than is possible with single frequency SSS methods. A detailed study is needed to classify the different types of circuits that can be analyzed in this manner.

Additional work is also required to quantify the effects of aliasing. For odd n, the DFT mapping from frequency to the discrete time domain is lossless. Choosing n both truncates the frequency basis and specifies the discrete sampling frequency. In general, n can be chosen sufficiently large that aliasing is negligible, but an error analysis is needed to provide a tolerance bound related to the amount of aliasing.

Finally, the periodic steady state analysis tools could be implemented as a set of routines implemented with a matrix analysis package, such as Matlab. The resulting circuit design environment would provide an intermediate level of detail, between hand calculation, and full simulation.

6.2 The Reactive Power Vector

The second result given given in this thesis is the definition of a reactive power vector. Using the expressions for time-average and apparent power in terms of the n-vectors for voltage and current, the total inactive power is shown to be the rms of $m = \frac{1}{2}n(n-1)$ terms. The form and properties of these terms strongly resemble the classical definition of reactive power for sinusoidal systems.

Interpreting the m terms as orthogonal components, an m vector for reactive power,

R, can be constructed. Inactive power is then the norm, $\|R\|$. R can be expressed as a cross-product of V and I, but there is no convenient linear algebra expression for the n-dimensional cross product. R can, however, be expressed in a matrix form, denoted $\mathbf{R}$, which is the anti-symmetric part of the exterior product of V and I.

While the theoretical definition of R is important, the dimensionality, m, grows with n^2, which can make R impractical as a measure of reactive power. However, sub-norms and projections of R can be designed for specific applications providing scalar measures to reflect the desired information. Such measures can then be used in solving metering and power factor compensation problems. One such useful measure is q^k, defined in Section 3.3.2, which is the projection of the reactive power of element k in a circuit, in the direction of the reactive power of the circuit port. q^k provides a simple conserved indicator of where in the circuit the port reactive power 'flows.'

Future Work

As harmonic effects in the power system are of increasing concern, there is a strong need for a consistent and useful definition of reactive power for non-sinusoidal systems. While the dimension of R is preventively large, the definition is consistent, and the method of analysis using norms and projections of R is useful. With refinement, the definitions herein could provide the needed framework for reactive power analysis.

The time and frequency-domain vector expressions for reactive power have many potential applications, including metering, control and optimization of power systems and power electronic devices. The design of specific norms and projections of R should be studied for specific applications. Combined with the periodic steady state analysis methods of Section 2.2, the new reactive power definitions could provide a new optimization methodology.

While only the time-domain and frequency-domain basis were used in this thesis, any decomposition can be used, including one that is designed specifically for a certain circuit topology or operating condition. For example, for a given excitation, the periodic voltage and current waveforms of a circuit with p ports and e energy storage and non-linear devices can be decomposed with a minimum of $n = p + e$ basis functions. The resulting reactive power vector, for small circuit models, has modest dimension, and might provide a practical measure. A network theoretic study of the reactive power vector could provide a better understanding of the degrees of freedom in the problem.

6.3 Symmetrical Decomposition of Multiphase Waveforms

The final result of this thesis is the definition of the symmetrical components definition for multiphase signals with harmonics. Section 5.1.3 shows how the single frequency three-phase symmetrical components can be extended, with exception of D.C. and multiples of three harmonics. A non-symmetrical orthogonal decomposition can be used to deal with the exceptions. Section 5.1.4 shows a more consistent decomposition into phase-leading and phase-lagging components, where only the D.C. components require a non-symmetrical decomposition.

The periodic steady state analysis methods are extended to multiport, and more specifically, multiphase systems. The method defines a collective impedance matrix for multiphase circuit elements. The matrix takes a different form for four-wire elements, three-wire wye-connected elements, and three-wire delta-connected elements. The general method allows the periodic steady state solution for balanced or unbalanced, mixed three and four-wire circuit to be solved using compact matrix expression that are directly analogous to the single phase expressions derived earlier.

Additional study is required to complete the definition. Analysis problems traditionally solved using single frequency symmetrical components should be attempted using the non-sinusoidal definition. If the decomposition proves useful, a more formal development of the theory would be beneficial.

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