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Output feedback concurrent learning model reference adaptive control

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Output Feedback Concurrent Learning Model Reference Adaptive Control

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Concurrent learning model reference adaptive control has recently been shown to guarantee simultaneous state tracking and parameter estimation error convergence to zero without requiring the restrictive persistency of excitation condition of other adaptive methods. This simultaneous convergence drastically improves the transient performance of the adaptive system since the true model is learned, but prior results were limited to systems with full state feedback. This paper presents an output feedback form of the concurrent learning controller for a novel extension to partial state feedback systems. The approach modifies a baseline LQG/LTR adaptive law with a recorded data stack of output and state estimate vectors. This maintains the guaranteed stability and boundedness of the baseline adaptive method, while improving output tracking error response. Simulations of flexible aircraft dynamics demonstrate the improvement of the concurrent learning system over a baseline output feedback adaptive method.

I. Introduction

Model reference adaptive control (MRAC) methods have been widely used to provably stabilize nonlinear systems in the presence of potentially destabilizing uncertainties.[1–4] These algorithms force the uncertain system to track reference model dynamics with desirable stability and performance characteristics in order to guarantee stability of the actual unstable system. The MRAC approach does so by estimating a parameterization of the uncertainties in the system using the differences between the reference model and actual dynamical response. These estimates are then used to select an appropriate control input to suppress the uncertainties and track the reference model. While MRAC methods do guarantee that the adaptive system will be stable and the response will converge towards that of the reference model, they do not guarantee that the estimates will converge to their true values. Only under a certain condition called persistency of excitation (PE)[5,6] will the estimates provably converge to their true values; however, PE is a restrictive condition and may not be practical in many applications.

The concurrent learning model reference adaptive control (CL-MRAC) approach has been shown to enable a relaxation of the persistency of excitation condition. This concurrent learning approach has been proven to guarantee that the weight estimates asymptotically converge to their actual values while maintaining guaranteed stability without relying upon persistency of excitation.[7] Where the standard baseline MRAC adaptive law solely relies upon instantaneous tracking errors to update the weight estimates, the concurrent learning approach appends recorded data to the baseline adaptive law. In effect, this history stack adds memory to the adaptive law and allows the weights to update even when the instantaneous tracking error is zero. In comparison to a baseline adaptive system, the CL-MRAC system demonstrates improved transient

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tracking error even before the weights have fully converged to their true parameterization as a result of the memory of the history stack.[8,9] The CL-MRAC method has been successfully demonstrated on numerous experimental and flight test systems with noisy full state measurements.[8,10]

One important subproblem for adaptive systems is the issue of output feedback, that is partial state measurements where fewer independent measurements than state variables are available. There are numerous output feedback approaches that extend MRAC to systems with partial state feedback. The lack of full state feedback prevents the standard MRAC adaptive law from being employed as the state tracking error is no longer available. Output tracking approaches[11–13] replace the state tracking error in the adaptive law with output tracking error from the measurements. In effect, the adaptive controller limits itself to the output response. Instead of neglecting unmeasurable states, state observer methods[14–16] add observer dynamics to estimate the state vector. This maintains a similar structure to the baseline state feedback method, but introduces state estimate error between the estimate and actual states that affects the adaptive system. Adaptive LQG/LTR[17,18] falls into the latter category, but attempts to address the problems with state estimate error using a high gain observer. This high gain observer aims to reduce the state estimate error and recover the desirable properties associated with a state feedback, LQR-controlled system. Just like in the state feedback case, the output feedback adaptive control methods focus on tracking error convergence.

While CL-MRAC has been extensively demonstrated on physical systems, it has been limited to systems with full state measurements. This paper presents an output feedback form of the concurrent learning adaptive control approach for systems with partial state feedback. While the lack of full state measurements prevents the same convergence properties as full state CL-MRAC, concurrent learning time history stacks improve the transient performance of the adaptive system. The adaptive LQG/LTR method from Lavretsky[17] is chosen as the baseline procedure for its desirable convergence properties and to minimize output tracking error. The concurrent learning controller is shown to maintain the same uniform ultimate boundedness of the state and weight estimate errors of the baseline adaptive LQG/LTR method, but improves upon the output tracking convergence. Despite the lack of full state measurements, the output feedback concurrent learning approach is shown to recover the guaranteed learning property of the state feedback method outside of a compact set defined by the state estimate and weight estimate errors. Simulations of flexible aircraft dynamics demonstrate this result.

II. Problem Formulation

Consider a class of uncertain, nonlinear, multiple-input multiple-output (MIMO) dynamical systems. Let $x(t) \in \mathbb{R}^n$ be the state vector, which is not fully measurable. Instead, partial state feedback is available with the measured output is $y(t) \in \mathbb{R}^p$, where $p < n$. The control input is given by $u(t) \in \mathbb{R}^m$. The class of systems considered has at least the same number of output as inputs, $p \geq m$. The uncertain dynamics are described by the following formulation

$$\dot{x}(t) = Ax(t) + B(u(t) + \Delta(x)) \tag{1}$$
$$y(t) = Cx(t)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ are known. Pairs (A, B) and (A, C) are controllable and observable.

Assumption 1 The system (A,B,C) is minimum-phase and output controllable with a nonsingular matrix (CB) that satisfies $\texttt{rank}(CB) = \texttt{rank}(B) = m$.

Assumption 2 The system uncertainties are represented by term $\Delta(x) \in \mathbb{R}^m$. This vector $\Delta(x)$ is the matched uncertainty in the span of the input matrix and is parameterized by a constant, unknown weighting matrix $W^* \in \mathbb{R}^{k \times m}$ and a regressor vector $\phi(x) \in \mathbb{R}^k$.

$$\Delta(x) = W^{*T}\phi(x) \tag{2}$$

Matrix W^* is unknown, but there exists a known bound on the uncertainty, W^*_{max}, such that $||W^*|| \leq W^*_{max}$.

This regressor vector can be either structured or unstructured, but a known structure is assumed throughout the remainder of the paper for ease of presentation. Unstructured regressor vectors can be directly taken from the results for the state feedback case.[5] The regressor vector is also assumed to be Lipschitz continuous.

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The uncertain system tracks a known reference model chosen for desirable performance criteria. With reference state $x_m(t) \in \mathbb{R}^n$, these reference dynamics are given by

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{3}$$
$$y_m(t) = C x_m(t)$$

where both x_m and y_m are known to the controller. State matrix $A_m \in \mathbb{R}^{n \times n}$ is chosen to be Hurwitz and matched with pair (A, B) from the system in Eq. (1) such that $A_m = A - BK$ with $K \in \mathbb{R}^{m \times n}$. Input matrix $B_m \in \mathbb{R}^{n \times m_2}$ is given by $B_m = BK_r$ with $K_r \in \mathbb{R}^{m \times m_2}$. The external reference command $r(t) \in \mathbb{R}^{m_2}$ is piecewise continuous and bounded so that the reference system in Eq. (3) is bounded.

While both x_m and y_m are available from the reference dynamics, only y is measurable from the actual system. Therefore, the state tracking error $e = x - x_m$ is unavailable, but output tracking error $e_y = Ce = y - y_m$ is measurable. Instead, a state observer $L \in \mathbb{R}^{n \times m}$ is used to estimate the states of the system. The observer dynamics are given as

$$\dot{\hat{x}} = A_m \hat{x} + B_m r + L(y - \hat{y}) \tag{4}$$
$$\hat{y} = C\hat{x}$$

with estimated state $\hat{x}(t) \in \mathbb{R}^n$ and output $\hat{y}(t) \in \mathbb{R}^p$. This introduces state estimate $\tilde{x} = \hat{x} - x$ and output estimate $\tilde{y} = C\tilde{x} = \hat{y} - y$ errors. Just like with tracking error, only the output estimate error $\tilde{y}$ will be available for use by the adaptive controller.

The control input $u(t)$ is segmented into three components:

$$u = u_{rm} + u_{pd} - u_{ad} \tag{5}$$

a feedforward reference term $u_{rm} = K_r r$, a feedback control term $u_{pd} = -K\hat{x}$, and an adaptive control input u_{ad}. The feedforward and feedback terms attempt to force the system to track the reference model, while the adaptive control input attempts to cancel the uncertainties.

III. Adaptive Output Feedback Control

Previous CL-MRAC implementations have relied upon systems with full state feedback to achieve stability and convergence guarantees. In these problems, the concurrent learning stack augments the baseline state feedback adaptive law. For the output feedback system, the same adaptive law is not possible; instead, the adaptive LQG/LTR method is used as the baseline adaptive law.[17, 19] This method applies a high gain observer to approximately recover Strictly Positive Real (SPR) properties for the system and demonstrates already desirable stability and convergence properties.

The baseline controller follows the same structure described by Lavretsky.[17] The observer L from the observer dynamics in Eq. (4) is the steady state Kalman gain

$$L = PC^T R_v^{-1} \tag{6}$$

that gives Hurwitz matrix $A_v = A - LC$. The symmetric positive definite matrix P satisfies the Algebraic Riccati Equation.

$$P(A + \eta I)^T + (A + \eta I)P - PC^T R_v^{-1} CP + Q_v = 0 \tag{7}$$

Positive constant scalar η is used as a robustifying term for the nominal state matrix A. This choice of η will affect the convergence properties of the system discussed in Section VII.A. Matrices Q_v and R_v are formed using scalar constant $v > 0$ and matrices $Q_0 \geq 0$ and $R_0 > 0$.

$$Q_v = Q_0 + \frac{v + 1}{v} \bar{B} \bar{B}^T \tag{8}$$

$$R_v = \frac{v}{v + 1} R_0 \tag{9}$$

Matrix $\bar{B} = \begin{bmatrix} B & B_2 \end{bmatrix}$ is the squared-up form of the input matrix B with ficticious input matrix B_2. This B_2 matrix is chosen from the squaring up procedure described by Lavretsky[17] to give a minimum phase system with triple $(A, \bar{B}, C)$.

With the inverse of P, $\tilde{P} = P^{-1}$, Eq. (7) can be rewritten as the Lyapunov equation.

$$A_v^T \tilde{P} + \tilde{P} A_v = -C^T R_v^{-1} C - \tilde{P} Q_v \tilde{P} - 2\eta \tilde{P} < 0 \tag{10}$$

The output feedback relation $C^T R_0^{-\frac{1}{2}} \omega$ approximates the matrix $\tilde{P} \bar{B}$ from the state feedback adaptive law

$$\tilde{P} \bar{B} = C^T R_0^{-\frac{1}{2}} \omega + O(v) \tag{11}$$

with error $O(v)$. Since there are more outputs that inputs, Eq. (11) can also be written with the actual input matrix B and matrix $S = \begin{bmatrix} I & 0 \end{bmatrix}$.

$$\tilde{P} B = C^T R_0^{-\frac{1}{2}} \omega S^T + O(v) \tag{12}$$

The matrix ω is created from unitary matrices U and V

$$\omega = (UV)^T \tag{13}$$

which are obtained from singular value decomposition with $\bar{B}$.

$$U\Sigma V = \texttt{svd}(\bar{B}^T C^T R_0^{-\frac{1}{2}}) \tag{14}$$

A. Concurrent Learning Adaptive Law

The baseline adaptive law is modified with a concurrent learning stack of uncertainty estimates. The uncertainty lies in the column space spanned by the input matrix B. In a state feedback approach, the uncertainty can be directly measured from the system response by

$$\Delta = B^\dagger (\dot{x} - Ax - Bu) \tag{15}$$

where $B^\dagger$ is the pseudo-inverse $B^\dagger = (B^T B)^{-1} B^T$.

For the output feedback problem, the state vector and its derivative are not available; therefore, the uncertainty measurement from Eq. (15) is not possible. Instead, only the output response and the state estimate vector are available. This fundamentally handicaps the convergence of estimates of uncertainties in the system since information will be lost due to the lack of full state measurements. While Eq. (15) and thus perfect measurement of the uncertainty is infeasible, the available measurements can be used to estimate the uncertainty. The state response of $\dot{x}$ is replaced by the output response $\dot{y}$, the closest measurement to $\dot{x}$ available. Because the output is a lower-dimensional projection of the state vector, the terms are multiplied by the output matrix C to reflect the reduction in dimension. This estimate of the uncertainty maintains a similar structure to Eq. (15) with

$$\hat{\Delta} = (CB)^\dagger (\dot{y} - CA\hat{x} - CBu) \tag{16}$$

where $(CB)^\dagger$ is the pseudo-inverse $(CB)^\dagger = (B^T C^T CB)^{-1} B^T C^T$. This pseudo-inverse is guaranteed to exist from Assumption 1. The uncertainty estimate also has to replace the state vector x with its estimate $\hat{x}$ since x isn't measurable. In effect, $\hat{\Delta}$ is the difference between the observed output response $\dot{y}$ and the estimated response. Ultimately, the partial state feedback leaves the uncertainty estimate susceptible to the state estimate error and Eq. (16) can be simplified down to the true uncertainty Δ corrupted by the state estimate error.

$$\hat{\Delta} = \Delta - (CB)^\dagger CA\tilde{x} \tag{17}$$

This estimation of the uncertainty from the output response assumes the derivative of the output vector $\dot{y}$ is measurable. For the state feedback case, it was shown in proofs, flight tests, and experimental demonstrations that a fixed point smoother can be used to obtain estimates of $\dot{x}$ without loss of generality.[5, 8, 10] These results will also apply to the output feedback problem and a fixed point smoother can be used to obtain $\dot{y}$, although that will not be further discussed in this paper.

The adaptive control input u_{ad} attempts to cancel the true uncertainty $\Delta(x)$ using estimates of the weight parameterization $\hat{W}$ and the structured regressor vector. Due to the lack of full state feedback, both the weighting matrix W^* and its corresponding regressor vector $\phi(x)$ are unknown. Instead, the regressor $\phi(x)$

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is replaced with a new vector $\phi(\hat{x})$ made up of state estimates in place of the actual state since the structure of the regressor vector is known even if $\phi(x)$ is not.

$$u_{ad} = \hat{W}^T \phi(\hat{x}) \tag{18}$$

The weight estimates are then updated according to the following adaptive laws. The baseline adaptive LQG/LTR law of Lavretsky is given by

$$\dot{\hat{W}} = \mathtt{proj}(\hat{W}, -\Gamma\phi(\hat{x})\tilde{y}^T R_0^{-\frac{1}{2}}\omega S^T, W_{max}^*) \tag{19}$$

while the modified concurrent learning adaptive law includes a recorded time history stack.

$$\dot{\hat{W}} = \mathtt{proj}(\hat{W}, -\Gamma\phi(\hat{x})\tilde{y}^T R_0^{-\frac{1}{2}}\omega S^T - W_c\Gamma_c \sum_{i=1}^{\overline{p}} \phi(\hat{x}_i)\epsilon_i^T, W_{max}^*) \tag{20}$$

Here $\Gamma > 0$ is the diagonal, baseline gain matrix and $\Gamma_c > 0$ is the diagonal, concurrent learning gain matrix. The projection operator in Eqs. (19) and (20) bounds the weight estimates within a ball of radius W_{max}^* centered at the origin. Recall that this upper limit W_{max}^* is known from Assumption 2. The concurrent learning time history stack improves the convergence of the system by storing regressor vectors and uncertainty estimates for $\overline{p}$ data points. The modeling error ϵ_i at each data point i gives the error between the adaptive control input and the estimated uncertainty.

$$\epsilon_i = \hat{W}^T \phi(\hat{x}_i) - \hat{\Delta}_i \tag{21}$$

Since the lack of full state feedback prevents the use of true uncertainty, the concurrent learning adaptive law has to rely upon these estimated modeling errors to improve performance. This further highlights the importance of minimizing state estimate error as large estimation errors will cause large deviations from the true uncertainty.

As time progresses, more data points are added to the time history stack; however, after some time this history stack will become unwieldy and impractical. Instead, the number of stored data points is capped at $\overline{p}_{max}$ so that after reaching $\overline{p}_{max}$ points, the size of the data point stack remains fixed. Rather than freezing the history stack when $\overline{p}_{max}$ has been reached, the time history stack is updated according to the singular value maximizing approach[20] given in Algorithm 1. The goal is to only replace older data points with new data points if these new points will increase the value of the information stored in the stack, where the value of the information stored in the stack is quantified by Condition 1.

Condition 1 The data stack at every time t will have at least as many linearly independent regressor vectors as the dimension of the vectors. The rank of the history stack $Z_t = \left[\phi(\hat{x}_1), \ldots, \phi(\hat{x}_{\overline{p}})\right]$ will then be k, so that,

$$\mathtt{rank}(Z_t) = \mathtt{dim}(\phi(\hat{x})) = k.$$

∎

The singular value maximizing approach of Algorithm 1 ensures Condition 1 is met and guarantees the matrix $(Z_t Z_t^T)$ is positive definite. As new data points replace the least-valued older points, the minimum singular value of $(Z_t Z_t^T)$ increases, which improves the convergence of the tracking and weight error.

Lastly, the term $W_c \in \mathbb{R}^1$ is a binary scalar used to enable or disable the concurrent learning portion of the adaptive law based upon output estimate error $\tilde{y}$. Since the uncertainty estimate $\hat{\Delta}$ is corrupted by state estimation error, this can adversely affect the adaptation of the weight estimates. Additionally, since the output measurement is a lower-dimensional projection of the state vector, state estimate errors can accumulate outside this projection without check. Instead, the W_c term prevents weight estimates from inadvertently diverging by using the available information, here $\tilde{y}$, to stop adapting the weight parameterization when the output estimate error has fallen below some user-specified bound, ϵ_c. This simple binary measure is given by Algorithm 2.

Even though the concurrent learning portion of the adaptive law is disabled after $\tilde{y} < \epsilon_c$, the improvement in the output estimate and tracking error is clearly shown in the following section. Stability and convergence analysis for both Lavretsky's baseline adaptive law and the concurrent learning portion when it's enabled is shown in the appendix. By using the strengths of both approaches, the concurrent learning adaptive system is able to improve transient tracking error while still maintaining stability.

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Algorithm 1 Singular value maximizing algorithm

1: **if** $\frac{||\phi(\hat{x}) - \phi(\hat{x}_i)||^2}{||\phi(\hat{x})||} \geq \epsilon$ or $\texttt{rank}([Z_t, \phi(\hat{x})]) > \texttt{rank}(Z_t)$ **then**
2:     **if** $p < p_{max}$ **then**
3:         $p = p + 1$
4:         $Z_t(:, p) = \phi(\hat{x})$ and $\overline{\Delta}_t(:, p) = \hat{\Delta}$
5:     **else**
6:         $T = Z_t$
7:         $S_{old} = \min \texttt{SVD}(Z_t^T)$
8:         **for** $j = 1$ to p_{max} **do**
9:             $Z_t(:, j) = \phi(\hat{x})$
10:           $S(j) = \min \texttt{SVD}(Z_t^T)$
11:           $Z_t = T$
12:         **end for**
13:         find max S and corresponding column index i
14:         **if** max $S > S_{old}$ **then**
15:             $Z_t(:, i) = \phi(\hat{x})$ and $\overline{\Delta}(:, i) = \hat{\Delta}$
16:         **end if**
17:     **end if**
18: **end if**

Algorithm 2 Output error threshold for W_c

1: **if** $||\tilde{y}|| \leq \epsilon_c$ **then**
2:     $W_c = 0$
3: **else**
4:     $W_c = 1$
5: **end if**

IV. Numerical Simulations

In this section, the output feedback concurrent learning adaptive control algorithm is demonstrated through a simulation of a transport aircraft with additional flexible dynamics. This model is a reduced form of the full longitudinal model used by McLean[21] and Lavretsky[17] limited to the short period dynamics and a single wing bending mode. The short period dynamics are also modified to be unstable. This leaves four state variables, where α is the angle of attack (radians), q is the pitch rate (radians/sec), λ_1 is the wing deflection (ft), and $\dot{\lambda}_1$ is the rate of deflection (ft/sec). Two inputs are present in the system with elevator deflection δ_e and canard deflection δ_c, both in radians. There are only three output measurements in the system: angle of attack α, pitch rate q, and a normalized acceleration measurement a_z.

$$x = \begin{bmatrix} \alpha & q & \lambda_1 & \dot{\lambda}_1 \end{bmatrix}^T \qquad u = \begin{bmatrix} \delta_e & \delta_c \end{bmatrix}^T \qquad y = \begin{bmatrix} \alpha & q & a_z \end{bmatrix}^T$$

Since there more outputs than inputs, the squaring-up procedure must be used.

The nominal model dynamics are given by triple (A, B, C). This linear model is stable and a controller (K, K_r) gives the desired reference model (A_{ref}, B_{ref}) with reference elevator commands r. Instead, the actual state matrix is given by A_p. This uncertainty destabilizes the short period dynamics of the system. These matrices are listed below.

$$A = \begin{bmatrix} -1.6 & 1 & -1.1811 & -0.1181 \\ 2 & -2.446 & -1.813 & 1.1805 \\ 0 & 0 & 0 & 1 \\ -7.196 & -0.445 & -56.82 & -5.53 \end{bmatrix} \qquad A_p = \begin{bmatrix} -1.6 & 1 & -1.1811 & -0.1181 \\ 6.57 & -2.446 & -1.813 & 1.1805 \\ 0 & 0 & 0 & 1 \\ -6.79 & -0.751 & -50.21 & -4.87 \end{bmatrix}$$

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$$B = \begin{bmatrix} -0.07 & -0.006 \\ 3.726 & -0.28 \\ 0 & 0 \\ 0.572 & 0.019 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.49499 & -0.4251 & 0.75696 & 0.03593 \end{bmatrix}$$

The uncertainty in the system is then represented by

$$W^* = \begin{bmatrix} 0.86135 & -4.8604 \\ -0.35917 & -4.7778 \\ 7.7586 & 103.2069 \\ 0.77469 & 10.3051 \end{bmatrix} \qquad \phi(x) = \begin{bmatrix} \alpha \\ q \\ \lambda_1 \\ \dot{\lambda}_1 \end{bmatrix}$$

The simulation is run over a period of 150 seconds with bounded $\pm 5°$, $\pm 10°$, $+15°$, and $+20°$ elevator step commands spaced throughout the time period. The adaptive gains are set to $\Gamma = 4I_{4\times4}$ and $\Gamma_c = 0.6I_{4\times4}$ with terms $\eta = 0.2$ and $v = 0.1$. For the squaring-up procedure, the new zero is set to $s = -0.75$ to make the system minimum phase.

The same simulation is performed for both the baseline adaptive law Eq. (19) and concurrent learning adaptive law Eq. (20). Figure 1 depicts the output response of system with both adaptive laws. In the figure, the adaptive controllers manage to successfully track the desired output response; however, the improvement in output tracking error using the concurrent learning adaptive law is readily visible. The actual and estimated outputs converge very rapidly to the desired response and stay there due to the recorded data stack. The state response of the also shows this improved convergence, even though it is unavailable in practice. The baseline system in Figure 2 has slightly worse state tracking error than the concurrent learning system. These state responses also illustrate the state estimate error between the actual and estimated dynamics resulting from the state observer and partial state measurements.

Figure 3 compares the adaptive control input to the estimated and actual uncertainty for the adaptive controllers. Because the baseline method only considers instantaneous data, there is no equivalent estimated uncertainty term in the baseline adaptive controller. From Figure 3, the two adaptive control inputs successfully track the system uncertainties, while the baseline controller has more difficulty. Ideally in a full state feedback system, the estimated and actual uncertainties would be the same and the adaptive control inputs would converge to them. Since only partial state feedback is available and therefore a state estimate error exists, there is a slight discrepancy between the estimated and actual uncertainty. Despite this, the improvement in tracking response with the concurrent learning controller over the baseline adaptive controller is readily apparent.

V. Conclusion

This paper presents an output feedback form of the concurrent learning model reference adaptive control law for uncertain systems with partial state feedback. This approach replaces the state feedback adaptive law with an adaptive LQG/LTR algorithm. This baseline adaptive law is then modified with recorded data from the output response and state estimates. Demonstrations of the approach on aircraft dynamics highlights the performance of the concurrent learning approach over the baseline method.

VI. Acknowledgments

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(a) Angle of attack

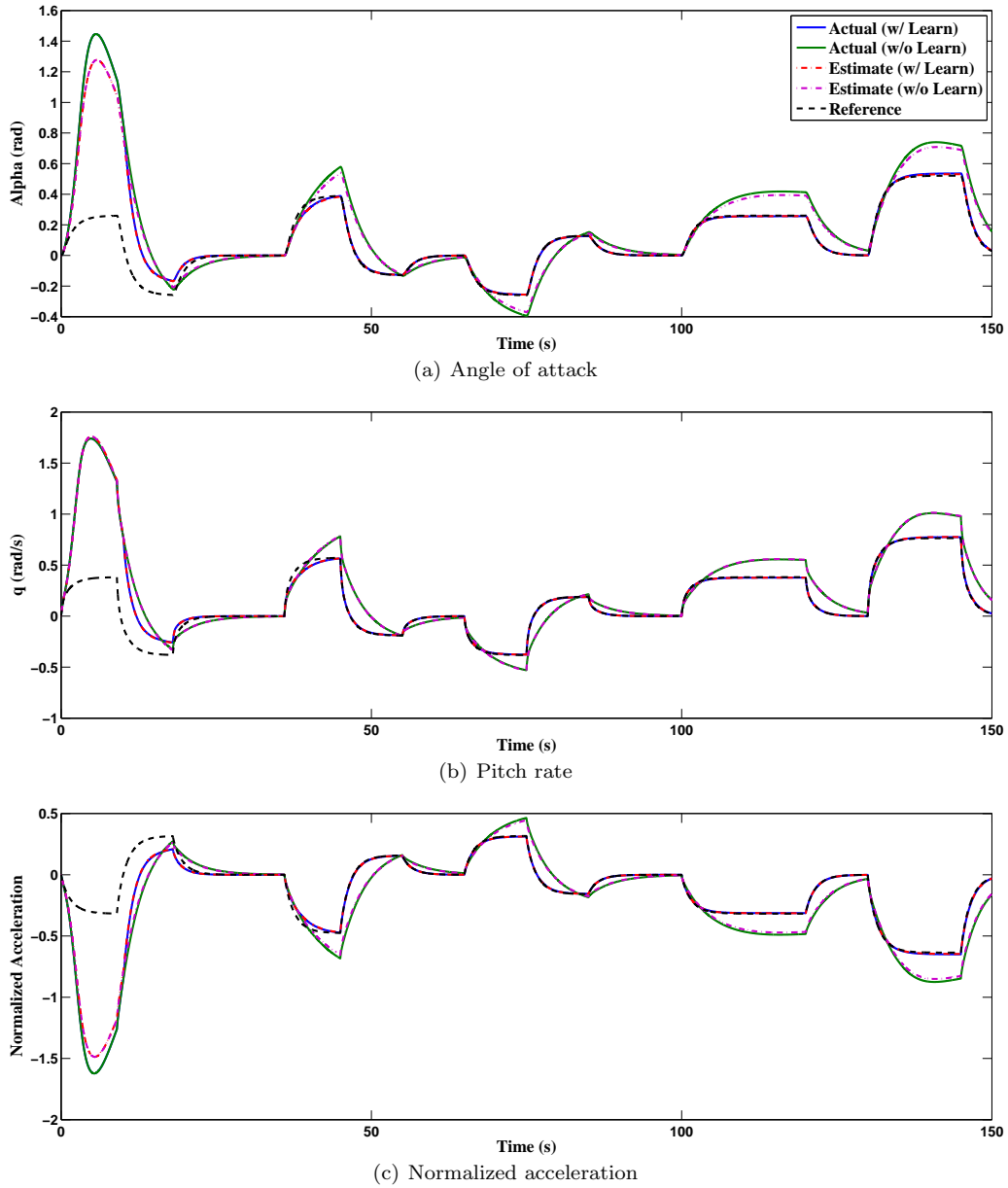

(b) Pitch rate

(c) Normalized acceleration

Figure 1. Output response of the adaptive system. The reference system is given by the black, dashed line. The green and magenta lines are taken from the system with the baseline adaptive law Eq. (19). The blue and red lines are taken from the system with the concurrent learning adaptive law Eq. (20). Notice the decrease in tracking error using the concurrent learning adaptive law.

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(a) Angle of attack

(b) Pitch rate

(c) Bending deflection
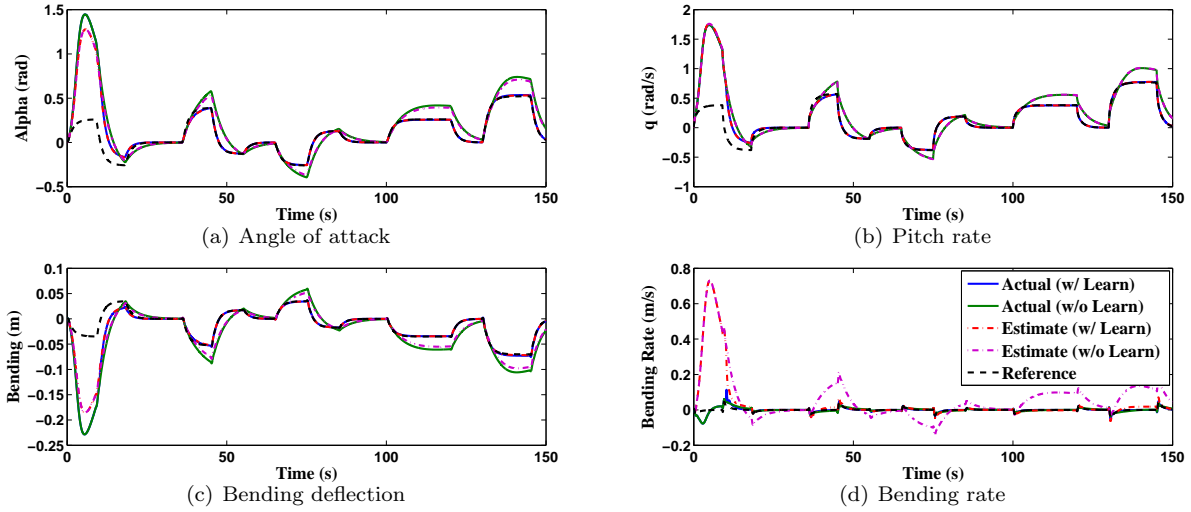
(d) Bending rate

Figure 2. State response of the adaptive system. The reference system is given by the black, dashed line. The green and magenta lines are taken from the system with the baseline adaptive law Eq. (19). The blue and red lines are taken from the system with the concurrent learning adaptive law Eq. (20). Not all the states are accessible, but are shown here to demonstrate the improvement in performance.

(a) Elevator deflection
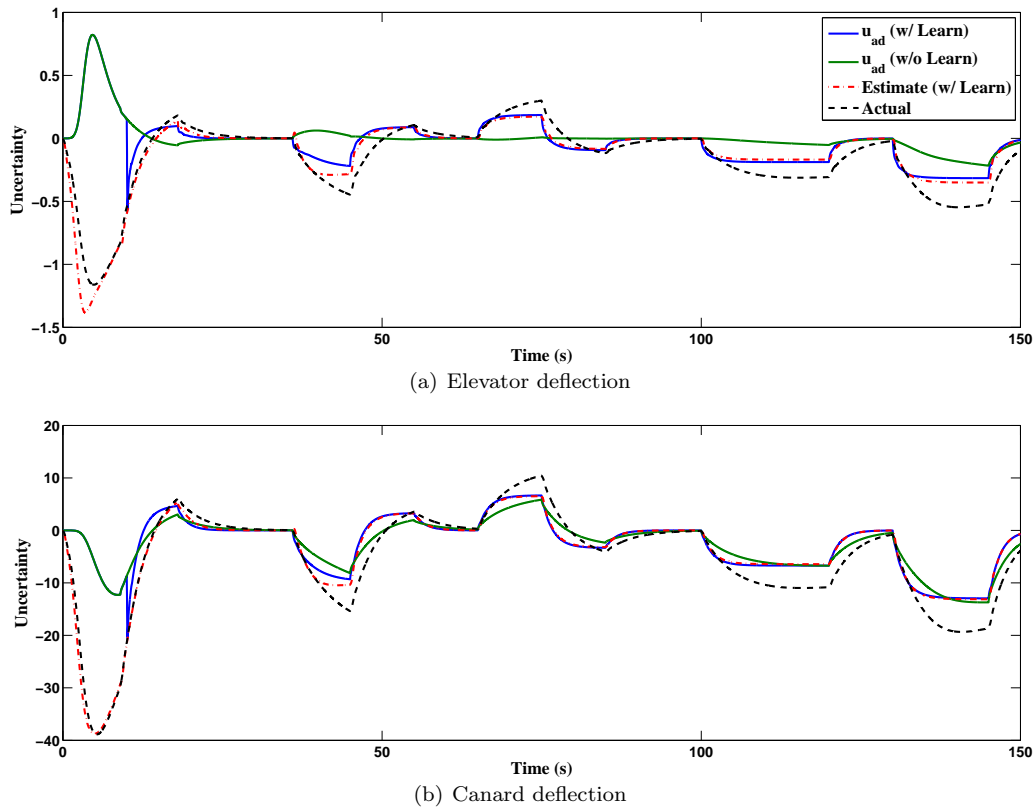
(b) Canard deflection

Figure 3. The adaptive control input Eq. (18) of the system with the standard adaptive law Eq. (19), the concurrent learning adaptive law Eq. (20), the corresponding estimated uncertainty Eq. (16), and actual uncertainty Eq. (15). The actual uncertainty is unavailable in practice due to the lack of full state feedback, but is shown here to demonstrate the accuracy of the estimated uncertainty $\hat{\Delta}$ in capturing the true uncertainty.

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VII. Appendix

A. Stability Analysis

The following section proves the stability and boundedness of the output feedback concurrent learning approach. Note that this stability analysis can be broken into two separate parts owing to Algorithm 2, when the learning portion is active ($W_c = 1$) and when it is not ($W_c = 0$). For the latter case when $W_c = 0$, the approach reverts base to the baseline adaptive control method in Eq. (19). This case was comprehensively proven by Lavretsky and Wise in their original work.[17] Using this adaptive law, the state estimate error $\tilde{x}$ and weight estimate error $\tilde{W}$ will converge to either the origin or a compact set. The stability of the state estimate error can then be used to show the subsequent state tracking error $e = x - x_m$ will also be uniformly ultimately bounded. This establishes the stability and boundedness of the system when concurrent learning is not enabled ($W_c = 0$).

The concurrent learning portion when $W_c = 1$ builds upon the results of the baseline method. The inclusion of the time history stack in Eq. (20) introduces additional terms that further complicate analysis of the adaptive system. The remainder of Appendix A will be devoted to proving the stability and boundedness of the system when concurrent learning is active ($W_c = 1$).

Before starting, consider the following definitions and relations:

$$\phi(\tilde{x}) = \phi(\hat{x}) - \phi(x)$$
$$||\phi(\tilde{x})|| \leq K_\phi ||\tilde{x}||$$
$$||\phi(\hat{x})|| \leq K_1 + K_2 ||\tilde{x}||$$

$$K_\Gamma = \Gamma^{-1} \Gamma_c > 0$$
$$A_{CB} = ||A^T C^T (CB)^\dagger||$$

where K_ϕ, K_1, and K_2 are positive, constant scalar terms. Γ and Γ_c were defined in Section III to be diagonal, positive definite matrices.

The uncertainty estimates $\hat{\Delta}$ stored in the time history stack are potentially adversely affected by state estimation error $\tilde{x}$. In Eq. (17), it was shown that each measurement $\hat{\Delta}$ estimates the true uncertainty Δ from the output response, but $\hat{\Delta}$ is also corrupted by the state estimation error $\tilde{x}$. Each data point stored in the history stack will therefore be affected by the estimation error at the time it was taken. However, nothing can be done about it since the state estimation error is unmeasurable. While the singular value maximizing algorithm ensures matrix $(Z_t Z_t^T)$ is positive definite, the sign and magnitude of $\tilde{x}$ are unknown and therefore it is unclear if matrix $(\phi(\hat{x})\phi(\tilde{x})^T)$ is positive semidefinite. Reducing the number of data points only slows down the rate of convergence and reduces the improvement over the baseline method. As the weight estimation errors improve, the state estimation error will decrease, although it is not guaranteed to converge to zero.

In order to quantify the effect of the station estimation error on the time history stack, the following assumption allows a bound to be placed on $\tilde{x}$. This is a very conservative estimate, but unfortunately it has not been relaxed. This is an ongoing aim of the research.

Assumption 3 The state estimation errors within the time history stack are upper bounded by a constant $\tilde{x}_{max}$: $||\tilde{x}_i|| \leq \tilde{x}_{max}$. With this upper bound on $\tilde{x}$, a conservative estimate of the effect the state estimate errors in the time history stack can be computed.

$$||\sum_{i=1}^{\bar{p}} \phi(\hat{x}_i)\phi(\tilde{x}_i)^T|| \leq K_1 \bar{p}^2 K_\phi \tilde{x}_{max} + K_2 \bar{p}^2 K_\phi \tilde{x}_{max}^2 \tag{22}$$

$$||\sum_{i=1}^{\bar{p}} \phi(\hat{x}_i)\tilde{x}_i^T|| \leq K_1 \bar{p}^2 \tilde{x}_{max} + K_2 \bar{p}^2 \tilde{x}_{max}^2 \tag{23}$$

This highlights the important of employing a method with good state estimation error. For this reason, Lavretsky and Wise's approach is particularly well suited.[17]

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Remark 1 A number of steps can be followed to ensure Assumption 3 is feasible. First, the time history stack does not need to be updated when the concurrent learning portion of Eq. (20) is activated ($W_c = 1$). In fact, when the adaptive law has reverted back to the baseline method ($W_c = 0$), the state estimation error $\tilde{x}$ was proven to be bounded. If the resulting time history stack is only updated when learning is disabled, then Assumption 3 is valid.

The following theorem considers the stability of the state estimate and weight estimate errors. Once the state estimation error $\tilde{x}$ is proven to be bounded, then the tracking error e can also be proven to be bounded.

Theorem 1 Consider the uncertain output feedback system in Eq. (1). When the history stack is updated according to the singular value maximizing algorithm, Algorithm 1, the state estimate error $\tilde{x}$ and weight estimate error $\tilde{W}$ are uniformly ultimately bounded around the origin $(\tilde{x}, \tilde{W}) = 0$.

Let V be the following Lyapunov function candidate for the stability of $\tilde{x}$ and $\tilde{W}$.

$$V = \frac{1}{2}\tilde{x}^T \tilde{P}\tilde{x} + \frac{1}{2}\texttt{trace}(\tilde{W}^T \Gamma^{-1}\tilde{W}) \tag{24}$$

Consider the time derivative of V

$$\dot{V} = \frac{1}{2}\tilde{x}^T(A_v^T\tilde{P} + \tilde{P}A_v)\tilde{x} + \tilde{x}^T\tilde{P}B\tilde{W}^T\phi(\hat{x}) \\ + \tilde{x}^T\tilde{P}BW^{*T}\phi(\tilde{x}) + \texttt{trace}(\tilde{W}^T\Gamma^{-1}\dot{\tilde{W}}) \tag{25}$$

After applying the concurrent learning adaptive law Eq. (20) and relation (12), the time derivative reduces to

$$\dot{V} = \frac{1}{2}\tilde{x}^T(A_v^T\tilde{P} + \tilde{P}A_v)\tilde{x} + \tilde{x}^T\tilde{P}BW^{*T}\phi(\tilde{x}) \\ + \tilde{x}^T O(v)\tilde{W}^T\phi(\hat{x}) - K_\Gamma \texttt{trace}(\tilde{W}^T\sum_{i=1}^{\bar{p}}\phi(\hat{x}_i)\epsilon_i^T) \tag{26}$$

From the Lyapunov equation for $A_v^T\tilde{P} + \tilde{P}A_v$ in Eq. (10), $\dot{V}$ is equivalent to the expression

$$\dot{V} = -\frac{1}{2}\tilde{x}^T(C^TR_v^{-1}C + \tilde{P}Q_v\tilde{P} + 2\eta\tilde{P})\tilde{x} + \tilde{x}^T\tilde{P}BW^{*T}\phi(\tilde{x}) \\ + \tilde{x}^T O(v)\tilde{W}^T\phi(\hat{x}) - K_\Gamma \texttt{trace}(\tilde{W}^T\sum_{i=1}^{\bar{p}}\phi(\hat{x}_i)\epsilon_i^T) \tag{27}$$

The concurrent learning data stack appears as the last term in the expression and $\dot{V}$ can be upper bounded according to

$$\dot{V} \leq -\frac{1}{2}(1 + \frac{1}{v})\lambda_{min}(R_0^{-1})||\tilde{y}||^2 - \frac{1}{2}\lambda_{min}(Q_0)\lambda_{min}^2(\tilde{P})||\tilde{x}||^2 \\ - \frac{1}{2}(1 + \frac{1}{v})||B^T\tilde{P}\tilde{x}||^2 - \eta\lambda_{min}(\tilde{P})||\tilde{x}||^2 \\ + W_{max}^*||B^T\tilde{P}\tilde{x}||||\phi(\tilde{x})|| + v||\tilde{x}||K_3||\tilde{W}||\,||\phi(\hat{x})|| \\ - K_\Gamma\texttt{trace}(\tilde{W}^T\sum_{i=1}^{\bar{p}}\phi(\hat{x}_i)\epsilon_i^T) \tag{28}$$

where K_3 is a positive, constant scalar.

The concurrent learning recorded data stack can be expanded to

$$\tilde{W}^T\sum_{i=1}^{\bar{p}}\phi(\hat{x}_i)\epsilon_i^T = (\tilde{W}^T\sum_{i=1}^{\bar{p}}\phi(\hat{x}_i)\phi(\hat{x}_i)^T\tilde{W}) \\ + (\tilde{W}^T\sum_{i=1}^{\bar{p}}\phi(\hat{x}_i)\phi(\tilde{x}_i)^TW^*) + (\tilde{W}^T\sum_{i=1}^{\bar{p}}\phi(\hat{x}_i)\tilde{x}_i^TA^TC^T(CB)^\dagger) \tag{29}$$

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From this expansion, the full term in Eq. (28) is then upper bounded according to the following inequality

$$-K_\Gamma \texttt{trace}(\tilde{W}^T \sum_{i=1}^{\overline{p}} \phi(\hat{x}_i)\epsilon_i^T) \le -K_\Gamma \lambda_{min}(Z_t Z_t^T)||\tilde{W}||^2$$

$$+ K_\Gamma ||\tilde{W}|| \; || \sum_{i=1}^{\overline{p}} \phi(\hat{x}_i)\phi(\tilde{x}_i)^T || \; ||W^*|| + K_\Gamma ||\tilde{W}|| \; || \sum_{i=1}^{\overline{p}} \phi(\hat{x}_i)\tilde{x}_i^T ||A_{CB} \tag{30}$$

The first term in Eq. (30) is negative definite for all $\tilde{W} \ne 0$ since matrix $(Z_t Z_t^T)$ is updated according to the singular value maximizing algorithm. The remaining two terms are dependent upon the unknown estimation error $\tilde{x}_i$ for each saved data point. Since the state estimation errors are bounded according to Assumption 3, the preceding statement can be written as

$$-K_\Gamma \texttt{trace}(\tilde{W}^T \sum_{i=1}^{\overline{p}} \phi(\hat{x}_i)\epsilon_i^T) \le -K_\Gamma \lambda_{min}(Z_t Z_t^T)||\tilde{W}||^2 + K_{\tilde{x}}||\tilde{W}|| \tag{31}$$

where $K_{\tilde{x}}$ and K_ϕ are positive, constant scalars.

$$K_{\tilde{x}} = K_\Gamma W_{max}^*(K_1 \overline{p}^2 K_\phi \tilde{x}_{max} + K_2 \overline{p}^2 K_\phi \tilde{x}_{max}^2) + K_\Gamma A_{CB}(K_1 \overline{p}^2 \tilde{x}_{max} + K_2 \overline{p}^2 \tilde{x}_{max}^2) \tag{32}$$

The derivative of the Lyapunov function can then be written as

$$\begin{aligned}
\dot{V} \le &-\frac{1}{2}(1 + \frac{1}{\upsilon})\lambda_{min}(R_0^{-1})||\tilde{y}||^2 - \frac{1}{2}\lambda_{min}(Q_0)\lambda_{min}^2(\tilde{P})||\tilde{x}||^2 \\
&-\frac{1}{2}(1 + \frac{1}{\upsilon})||B^T \tilde{P}\tilde{x}||^2 - \eta\lambda_{min}(\tilde{P})||\tilde{x}||^2 \\
&+ W_{max}^*||B^T \tilde{P}\tilde{x}|| \; ||\phi(\tilde{x})|| + \upsilon||\tilde{x}||K_3||\tilde{W}|| \; ||\phi(\hat{x})|| \\
&- K_\Gamma \lambda_{min}(\Omega)||\tilde{W}||^2 + K_{\tilde{x}}||\tilde{W}||
\end{aligned} \tag{33}$$

For ease of analysis, the derivative of the Lyapunov candidate in Eq. (33) is broken up into two cases: 1) weight error bounded by W_1 and 2) state estimate error bounded by x_2. Each case can then be examined independently. The system will then converge to the intersection of the two resulting compact sets for $\tilde{x}$ and $\tilde{W}$.

1. State estimate error convergence

First, for this section consider the case when the weight error is bounded: $||\tilde{W}|| \le W_1$. The derivative in Eq. (33) then becomes

$$\begin{aligned}
\dot{V} \le &-\frac{1}{2}(1 + \frac{1}{\upsilon})\lambda_{min}(R_0^{-1})||\tilde{y}||^2 - \frac{1}{2}\lambda_{min}(Q_0)\lambda_{min}^2(\tilde{P})||\tilde{x}||^2 \\
&-\frac{1}{2}(1 + \frac{1}{\upsilon})||B^T \tilde{P}\tilde{x}||^2 - \eta\lambda_{min}(\tilde{P})||\tilde{x}||^2 \\
&+ W_{max}^*||B^T \tilde{P}\tilde{x}|| \; ||\phi(\tilde{x})|| + \upsilon||\tilde{x}||K_3 W_1 \; ||\phi(\hat{x})|| \\
&- K_\Gamma \lambda_{min}(\Omega)W_1^2 + K_{\tilde{x}}W_1
\end{aligned} \tag{34}$$

As used before in,[17] this expression can be simplified to the combination of two negative definite terms of $\tilde{y}$ and $\tilde{x}$ and a quadratic function $\varphi'(\tilde{x}, w)$ with $w = ||B^T \tilde{P}\tilde{x}||$.

$$\dot{V} \le -\frac{1}{2}c_3 \lambda_{min}(R_0^{-1})||\tilde{y}||^2 - \eta\lambda_{min}(\tilde{P})||\tilde{x}||^2 \; -\frac{1}{2}\varphi'(\tilde{x}, w) \tag{35}$$

Here the quadratic function $\varphi'(\tilde{x}, w) = \zeta^T C\zeta - 2\zeta^T b + f$ is written as

$$\begin{aligned}
\varphi'(\tilde{x}, w) = &\begin{bmatrix} ||\tilde{x}|| & w \end{bmatrix} \begin{bmatrix} c_1 & -c_2 \\ -c_2 & c_3 \end{bmatrix} \begin{bmatrix} ||\tilde{x}|| \\ w \end{bmatrix} - 2 \begin{bmatrix} ||\tilde{x}|| & w \end{bmatrix} \begin{bmatrix} c_4 \\ 0 \end{bmatrix} \\
&+ 2K_\Gamma \lambda_{min}(Z_t Z_t^T)W_1^2 - 2K_{\tilde{x}}W_1
\end{aligned} \tag{36}$$

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$$C = \begin{bmatrix} c_1 & -c_2 \\ -c_2 & c_3 \end{bmatrix}$$

$$b = \begin{bmatrix} c_4 & 0 \end{bmatrix}^T$$

$$f = 2K_\Gamma \lambda_{min}(Z_t Z_t^T) W_1^2 - 2K_{\tilde{x}} W_1$$

The coefficients are defined as follows:

$$c_1 = \lambda_{min}(Q_0)\lambda_{min}^2(\tilde{P}_0) - 2\upsilon K_3 W_1 K_2$$
$$c_2 = W_{max}^* K_\phi$$
$$c_3 = 1 + \frac{1}{\upsilon}$$
$$c_4 = \upsilon K_3 \tilde{W}_1 \ K_1$$

The goal is to show Eq. (36) is negative definite and thus V converges towards the origin. If $\varphi'(\tilde{x}, w) \geq 0$, then $\dot{V} < 0$ and V converges to the origin. But if $\varphi'(\tilde{x}, w) < 0$, then $\dot{V} < 0$ only outside a compact set and thus the state estimation error $\tilde{x}$ provably converges to that set. To minimize this compact set, $|\varphi'(\tilde{x}, w)|$ must be minimized.

Two conditions must be met for $|\varphi'(\tilde{x}, w)|$ to have a minimum: First, matrix C must be positive definite. This means term $c_1 > 0$ and $c_1 c_3 - c_2^2 > 0$. Second, $\dot{\varphi}' = 0$ at the minimum, meaning $\zeta = C^{-1}b$ at the minimum.

An appropriate υ can be chosen to satisfy the first condition. For $c_1 > 0$, there exists a $\upsilon_{c_1} > \upsilon > 0$. Likewise, for $c_1 c_3 - c_2^2 > 0$, there also exists a $\upsilon_{c_3} > \upsilon > 0$ Selecting $\upsilon < \mathtt{min}(\upsilon_{c_1}, \upsilon_{c_3})$ will ensure $C > 0$. This result is the same as the baseline adaptive case from Lavretsky.[17]

The second condition highlights the change in convergence of the concurrent learning law over the baseline adaptive law. With $\zeta = C^{-1}b$ at the minimum,

$$\varphi'_{min} = -\frac{c_3 c_4^2}{c_1 c_3 - c_2^2} + f \tag{37}$$

where f is the same scalar term defined earlier. If $f > 0$, then φ'_{min} becomes less negative or even positive and the concurrent learning algorithm has improved the convergence of the state estimate errors. However, f can also be negative. While this will hurt state estimate error convergence, a number of factors restricts this. First, as the weights converge, $W_1 \to 0$ which causes $f \to 0$. Additionally, when the time history stack is updated according to the singular value maximizing algorithm, $\lambda_{min}(Z_t Z_t^T)$ will be increased, thus increasing f.

Substituting in for ϕ_{min}, the derivative of the Lyapunov function becomes

$$\dot{V} \leq -\frac{1}{2} c_3 \lambda_{min}(R_0^{-1}) ||\tilde{y}||^2 - \eta \lambda_{min}(\tilde{P}) ||\tilde{x}||^2 \\ - \frac{1}{2}\varphi'_{min} \tag{38}$$

If $\varphi'_{min} \geq 0$ then V converges to the origin. Since f will decrease as weight estimate error improves, this is less likely. When $\varphi'_{min} < 0$, $\dot{V} < 0$ outside the compact set given by

$$\Omega_{\tilde{x}} = \tilde{x} : ||\tilde{x}||^2 \leq \frac{|\varphi'_{min}(\upsilon_0)|}{2\eta \lambda_{min}(\tilde{P})} \tag{39}$$

2. Weight error convergence

Now consider the second case when state estimate error is bounded by $||\tilde{x}|| \leq x_2$ and therefore a second expression $w_2 = ||B^T \tilde{P} \tilde{x}||$. For this case, the projection operator in the adaptive law, Eq. (20), already ensured the weight error is bounded by $||\tilde{W}|| \leq 2W_{max}^*$ in the worst case. However, the weight estimate error

with the concurrent learning adaptive law can still tend to a smaller compact set than the bound proven by the projection operator alone, but will do no worse. The time derivative in Eq. (33) becomes

$$
\begin{aligned}
\dot{V} \leq & -\frac{1}{2}(1 + \frac{1}{\upsilon})\lambda_{min}(R_0^{-1})||\tilde{y}||^2 - \frac{1}{2}\lambda_{min}(Q_0)\lambda_{min}^2(\tilde{P})x_2^2 \\
& -\frac{1}{2}(1 + \frac{1}{\upsilon})w_2^2 - \eta\lambda_{min}(\tilde{P})x_2^2 \\
& + W_{max}^*w_2||\phi(\tilde{x})|| + \upsilon x_2 K_3||\tilde{W}|| \, ||\phi(\hat{x})|| \\
& - K_\Gamma\lambda_{min}(\Omega)||\tilde{W}||^2 + K_{\tilde{x}}||\tilde{W}||
\end{aligned}
\tag{40}
$$

which can be written compactly with quadratic function $\vartheta(\tilde{W})$

$$
\dot{V} \leq -\frac{1}{2}(1 + \frac{1}{\upsilon})\lambda_{min}(R_0^{-1})||\tilde{y}||^2 - \vartheta(\tilde{W})
\tag{41}
$$

This quadratic function is written as

$$
\vartheta(\tilde{W}) = a_w||\tilde{W}||^2 + b_w||\tilde{W}|| + c_w
\tag{42}
$$

with coefficients

$$
\begin{aligned}
a_w &= K_\Gamma\lambda_{min}(Z_t Z_t^T) \\
b_w &= -K_{\tilde{x}} - \upsilon K_3(K_1 + K_2 x_2)x_2 \\
c_w &= \frac{1}{2}\lambda_{min}(Q_0)\lambda_{min}^2(\tilde{P})x_2^2 + \frac{1}{2}(1 + \frac{1}{\upsilon})w_2^2 \\
&\quad + \eta\lambda_{min}(\tilde{P})x_2^2 - w_2 x_2 K_\phi W_{max}^*
\end{aligned}
$$

The derivative of the Lyapunov function will be negative definite for nonnegative values of $\vartheta(\tilde{W})$. From Condition 1, it is known that $a_w > 0$ and increases as new points are added to the data stack. While $a_w > 0$, $b_w < 0$, and the sign of c_w can be either positive or negative, $\dot{V}$ in Eq. (42) will be strictly negative for sufficiently large weight errors. That is the weight errors will converge as long as

$$
a_w||\tilde{W}||^2 > -b_w||\tilde{W}|| - c_w
\tag{43}
$$

Notice that as $\upsilon \to 0$, $b_w = -K_{\tilde{x}}$ and $c_w \approx \frac{1}{2}(1 + \frac{1}{\upsilon})w_2^2 > 0$. Additionally, as the state estimate errors improve and x_2 decreases, $c_w \to 0$ and $b_w = -K_{\tilde{x}}$. This means the recorded data stack terms in a_w and b_w will dominate as the error decreases. Ultimately, Eq. (43) defines a compact set $\Omega_{\tilde{W}_1}$ the weight error converges with $\vartheta(\tilde{W})$; however, the weight estimate errors are still bounded by the projection operator in the adaptive law Eq. (20). The compact set for the weight error convergence $\Omega_{\tilde{W}}$ will then be the smaller of $\Omega_{\tilde{W}_1}$ or the projection operator.

The Lyapunov function derivative in Eq. (33) describing the convergence of the system will be negative definite outside the compact set formed by the intersection of the compact sets: $\Omega = \Omega_{\tilde{x}} \times \Omega_{\tilde{W}}$. The system will then converge to this intersection.

Remark 2 Since the state estimate error $\tilde{x}$ is uniformly ultimately bounded (UUB), the tracking error e between the actual and reference systems is also ultimately bounded. The state tracking error is bounded according to the triangle inequality where

$$
||x - x_m|| \leq ||x - \hat{x}|| + ||\hat{x} - x_m||
\tag{44}
$$

The state estimate error $\tilde{x} = \hat{x} - x$ was shown to be UUB in Theorem 1 so its norm $||x - \hat{x}||$ is bounded. The reference state x_m is also bounded by definition. Lastly, the observer dynamics in (4) were a function of the reference input r and $\tilde{x}$, both of which are bounded. Therefore, $||\hat{x} - x_m||$ is bounded. The resulting tracking error $e = x_m - x$ is bounded since it is bounded below the summation of two bounded signals.

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