THE INHERENT VARIATION IN FATIGUE DAMAGE RESULTING FROM RANDOM VIBRATION

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SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August, 1961

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Department of Mechanical Engineering, August 4, 1961

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Thesis Supervisor

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ABSTRACT

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The stress-time history at a fatigue sensitive point in a resonant structure may be considered as the response of the structure to some excitation. If the excitation is a stochastic process, e.g., acoustic noise, then the stress-time response is also a stochastic process.

Inherent in the Palmgren-Miner cumulative damage criterion is a definition of fatigue damage. Using this definition we determine means of assigning fatigue damages to narrow band random stress histories. General formulas are derived for the mathematical expectation and variance of the fatigue damage in a fixed interval of time. Using a mathematical characterization of S-N fatigue curves, for narrow band Gaussian stress histories we then reduce the expectation of the damage to a simple form and the variance to a form involving the autocorrelation function of the stress histories. Next we obtain an expression for the variance of the fatigue damage for Gaussian stress histories generated by lightly damped, linear single degree-of-freedom systems with nominally white noise excitations. In the final section we discuss "equivalent" sinusoidal fatigue stress amplitudes in the light of the results obtained in the earlier sections.

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ACKNOWLEDGMENTS

First of all, the author wishes to express his gratitude to Professor Stephen H. Crandall for his patient assistance in supervising this thesis. I also wish to thank Professors Wallace E. Vander Velde and James A. Fay for their helpful comments as members of the thesis committee.

To M.I.T. and the Department of Mechanical Engineering I wish to express my appreciation for the opportunity of several years of graduate study.

Finally, I wish to thank Miss Joan Clarke for cheerfully doing the typing.

This research was partially supported by the USAF through the AFOSR of the ARDC, under Contract No. AF 49(638)-564 and partially by the Random Vibration Research Committee of the Applied Mechanics Division of the American Society of Mechanical Engineers.

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1. INTRODUCTION

1.1 Review of Fatigue Failure for Deterministic Stress-Time Histories

When an oscillating stress-time history occurs at a point in a structural member, fatigue damage to that member often is a result. Frequently, the structural member is part of a vibratory system and the oscillating stress-time history may be considered as the response of the appropriately defined system to some excitation. If the excitation is known, e.g. a sinusoid, and the system is well defined with prescribed initial conditions then the oscillating stress-time history can be uniquely determined. In such a case the problem of predicting the fatigue life of the member is reduced to determining the fatigue life while possessing a complete description of the stress-time history of the member. If the stress-time history is a sinusoid with zero mean stress, a large literature of S-N curves is available to predict the number of cycles until fatigue failure. If the mean stress is not zero additional data is required, e.g. Goodman and related diagrams [1] [1], to predict the fatigue life.

If a stress-time history is not a sinusoid then, without additional assumptions, its fatigue life cannot be predicted from an S-N curve or Goodman diagram. Often in practical situations stress-time histories are not sinusoids. In some of these cases it is possible to clearly identify cycles of stress, even though the amplitude and, or mean stress change as time progresses on the stress history. For these cases Palmgren [2], and independently Miner [3], have suggested a simple method of predicting fatigue lives from S-N curves or Goodman diagrams. Consider the stress history shown in Fig. 1.1. Individual cycles of stress are easily identifiable even though different amplitudes of stress are represented. For simplicity

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[1] Numbers in brackets refer to entries in the listing of references.

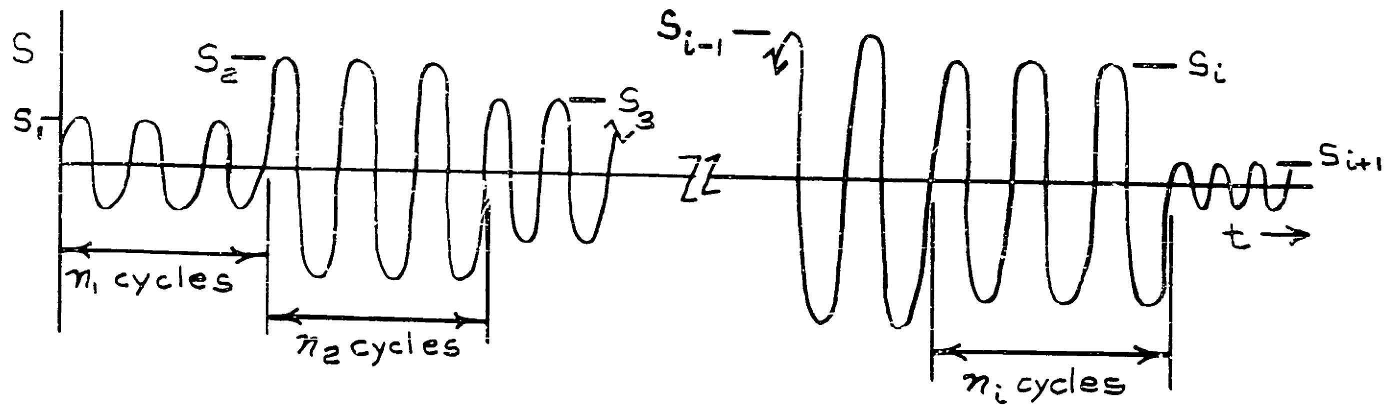

Figure 1.1 A stress history with identifiable stress cycles of different amplitudes.

we have shown the mean stresses to be zero. There occurs in the stress history n_1 cycles at stress amplitude S_1, n_2 cycles at stress amplitude S_2, . . . n_i cycles at stress amplitude S_i, ...

In Fig. 1.2 is shown the S-N curve for the material. N_1 is the number of cycles until failure at stress amplitude S_1, N_2 is the number of cycles until failure at stress amplitude S_2 etc. The method of

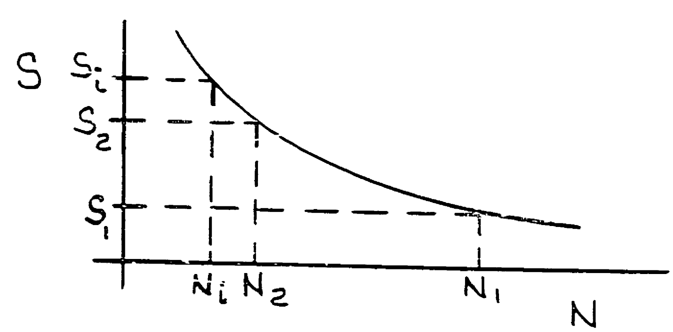

Figure 1.2 S-N curve for the material of the stress history of Figure 1.1.

Palmgren and Miner, sometimes called the Palmgren -Miner cumulative damage hypothesis, is that fatigue failure will occur when

$$\sum_i \frac{n_i}{N_i} = 1 \quad (1.1)$$

Hence, for each n_i cycles at stress amplitude S_i, the fraction n_i/N_i of the fatigue life of the material is "used up", where N_i is the number of cycles at stress amplitude S_i obtained from the S-N curve. The accumulation $\sum (n_i/N_i)$ which occurs during a fixed interval of time T can be thought of as a fatigue damage, D_T, incurred during T. Thus from Eq. 1.1 fatigue failure results when $D_T = 1$.

It has been implied so far that a precise knowledge of the stress-time history of a point in a structure, such as shown in Fig. 1.1, together with the S-N curve for the material of the structure enables one to calculate the fatigue life of the structure. In fact, however, an S-N curve is an "average" drawn through a scatterband of points plotted from the failure of individual fatigue specimens. The fatigue life of a specimen is sensitive to the surface conditions of the specimen, e.g. scratches, the distribution of grain sizes within the specimen, impurities in the metal, etc. Thus, the "width" of the scatterband of points through which an S-N curve is drawn can be reduced by carefully controlling the conditions of the fatigue tests but some scatter will always result.

Hence, the best that Eq. 1.1 can do is to predict an "average" fatigue life.

In addition to the unavoidable scatter in the fatigue lives of "identical" fatigue specimens another uncertainty occurs when one wishes to predict the fatigue life of a structural member of different geometrical shape than that of the test specimens from which the S-N curve for the material of the structure was obtained. Strictly speaking knowledge of an S-N curve allows us to predict with confidence only the fatigue life of a specimen "identical" to those used in obtaining the S-N curve. When using S-N data obtained from smooth specimens together with the Palmgren-Miner criterion, Eq. 1.1, to predict the fatigue lives of practical structures large errors can result [4].

Cumulative fatigue damage criteria more complex than the Palmgren-Miner criterion, Eq. 1.1, have been suggested [5]. However, none of these has been more widely accepted for practical calculations than the Palmgren-Miner criterion.

In the work to follow we rely heavily on the Palmgren-Miner criterion. We shall always assume that materials obey perfectly their S-N curves and the Palmgren-Miner criterion. We justify this assumption on the basis that our primary interest is to determine the variation in fatigue damage which results from the stochastic nature of the ensembles of stress histories which we study.

1.2 Extension to Stochastic Stress-Time Histories

If the excitation of a resonant structure is a stochastic process, e.g. acoustic noise, then the stress-time history at a point in the structure, considered as the response of the structure, is also a stochastic process. In particular the fatigue damage to the structure, considered as a function of time, is a stochastic process. In this thesis we study some of the statistical properties of fatigue damages associated with stochastic stress histories.

The mathematical tools for engineering studies of stochastic processes are primarily based on probability theory and generalized harmonic analysis. These tools have been organized into a systematic theory for the study of problems associated with communications engineering and automatic control theory, and are generally called statistical communication theory or random noise theory. A particularly important contribution to this theory has been made by Rice [6] . Those elements of the theory primarily concerned with the study of the responses of structures to stochastic excitations are usually called random vibration theory [7] . A brief summary of random vibration theory together with an extensive bibliography of the pertinent literature has been given by Crandall in [8] . The reader is assumed to be familiar with the fundamental notions and techniques of this theory as found for example in [9] .

Consider a stochastic stationary ensemble of stress histories with zero mean stresses. If examination of a typical sample function obtained from this ensemble, e.g. by using an oscilloscope, shows that the number of stationary points (maxima plus minima) is approximately equal to the number of zero crossings then we can speak of "cycles" of stress. We refer to a cycle of stress as that portion of a stress sample function between two adjacent zero crossings having the same slope sign. Assume such a stress random process to occur at a point in a structure. Further suppose that the S-N curve for the material of the structure is available. Then using the definition of fatigue damage , D_T, obtained in connection with the Palmgren-Miner damage hypothesis in Section 1.1, for any interval of time T of any given stress sample function we can determine a value for D_T. Since we are dealing with ensembles of such sample functions, for a given interval of time T, D_T is a random variable.

In Section 3 we derive a general formula, in the form of an integral, for the mathematical expectation of D_T, $E\left[D_T\right]$, for stochastic stationary ensembles of stress histories. In Section 6 we specialize the results of

Section 3 giving E $\left[D_T\right]$ in a simple form for Gaussian ensembles of stress histories. In Section 4 we derive a general formula, involving triple integrals, for the variance of D_T, $\sigma_{D_T}^{2}$, for stochastic stationary ensembles of stress histories. In Section 7 we specialize the results of Section 4 giving $\sigma_{D_T}^{2}$ in the form of single integrals for Gaussian ensembles of stress histories. In Section 8 we further specialize these results to ensembles of stress histories generated by linear single degree-of-freedom systems with white noise excitations, and perform the last integrations.

Using a different method Miles [10] has determined results which we show in Section 9 to be consistent with, and for all practical purposes equivalent to, our results of Section 6. One of Mile's principal results is an expression for an "equivalent fatigue stress." In Section 9 we discuss such "equivalent fatigue stresses" in the light of the expressions we have obtained for $\sigma_{D_T}^{2}$.

The appendices contain material which either deviates from the main line of thought of the text or is of a too detailed nature to be included in the text. The order we have used in presenting the material has been motivated primarily by a desire to present each potentially useful result with a minimum of assumptions.

2. ASSOCIATION OF FATIGUE DAMAGES WITH STRESS-TIME ENSEMBLE MEMBERS

Let Fig. 2.1 represent a typical sample function taken from a stochastic stationary ensemble of stress histories. The mean stress is assumed to

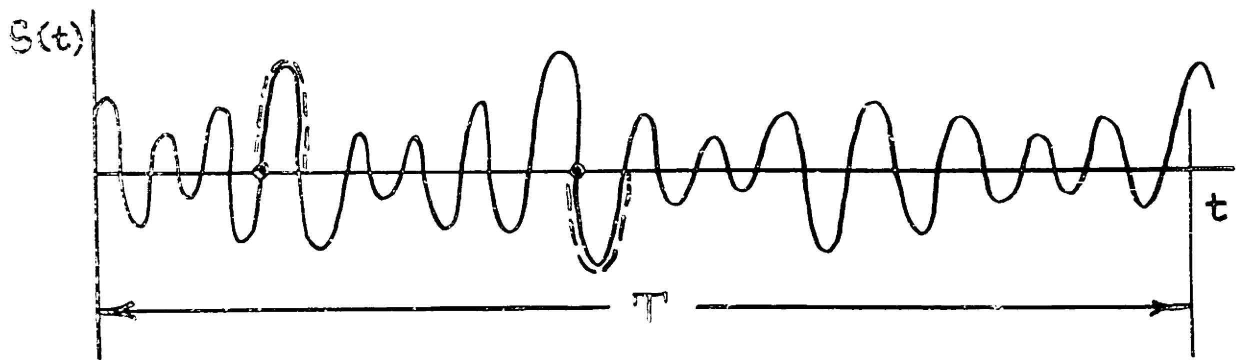

Figure 2.1 A typical narrow band stress sample function.

be zero. For each sample function for any period of time long in comparison to the average time between zero crossings we shall assume that the number of stationary points (maxima plus minima) is approximately equal to the number of zero crossings. (Any small wiggles superimposed on top of the relatively large oscillations are not included). We shall refer to a half cycle as that portion of a sample function between two adjacent zero crossings. With each half cycle we shall associate one zero crossing, the zero crossing at the beginning of the half cycle, and shall refer to this as the zero crossing associated with the half cycle. Two half cycles and their associated zero crossings are illustrated in Fig. 2.1.

With each half cycle of a sample function we associate an increment of fatigue damage, δD. Each increment of fatigue damage is a non-negative real number and is assumed to be incurred instantaneously at the time of the zero crossing associated with the half cycle. In a prescribed interval of time, T, the fatigue damage, D_T, incurred during T is the sum of the damage increments, δD, occurring during T. Thus D_T is a random variable possessing a probability distribution which depends on the value of T chosen. The increments of fatigue damage δD are scaled in such a way that fatigue failure is assumed to occur when $D_T = 1$.

Suppose the law for assigning the values of δD is known. Then for any sample function for which a time origin is prescribed there exists a unique time for which the value of D_T becomes unity, i.e., the time elapsed until fatigue failure for that sample function. This time until fatigue failure is a random variable. In Appendix A a relationship is derived giving the probability density function of the failure times in terms of the probability density functions of the damages, D_T, if these latter density functions are known for all values of T. Unfortunately, the probability density functions of D_T appear to be very difficult to obtain analytically so that at the present time the material of Appendix A is only of academic interest.

We shall now discuss the means we have used for associating the damage increments, δD, with the half cycles of the sample functions. Anticipating use of a modified form of the Palmgren-Miner cumulative damage criterion, ideally we would like to make the damage increments, δD, functions of the magnitudes of the stresses at the stationary points of the half cycles. Assuming that one stationary point occurs for each half cycle we could equivalently associate the damage increments directly with the stationary points, assuming the increments to be incurred instantaneously at the times of the stationary points. However, in practically important cases, mathematical difficulties can arise if this procedure is used. These difficulties are discussed in Appendix B.

The procedure we have used is to associate the damage increment for each half cycle with the zero crossing occurring at the beginning of the half cycle. The magnitude of the stress at the stationary point associated with the half cycle is then "predicted" from the magnitude of the slope at the zero crossing. Hence the damage increment δD is assumed to be a function of the magnitude of the slope at the beginning of the half cycle, $|\dot{S}|$, i.e. $\delta D = \delta D\ (|\dot{S}|)$. The general formulas for the expected fatigue damage and variance in the fatigue damage derived in Sections 3 and 4 (Eqs. 3.6 and 4.23 respectively) do not depend on any particular functional relation, $\delta D(|\dot{S}|)$.

Frequently the stress sample functions will have the appearance of sinusoids with slowly and randomly varying amplitude and phase where the time between zero crossings on the average is independent of the amplitudes of the half cycles. The response of a lightly damped linear single degree-of-freedom system to a wide band Gaussian excitation has this appearance. In such cases the magnitudes of the stresses at the stationary points can be "predicted" from the magnitudesof the slopes of the preceding zero crossings in the following manner. We assume that for the quarter cycle between each zero crossing and the following stationary point the sample functions are sinusoids of frequency, cycles/second, equal to one half of the expected number of zero crossings per second, $\bar{N}_0$. An expression for $\bar{N}_0$ for stationary Gaussian processes has been determined by Rice [11]. Thus if the frequency, rad./sec., of the sinusoid is ω_0, $\omega_0 = \pi \bar{N}_0$, then the assumption is that the magnitude of the stress at the stationary point, $|S_s|$ is given by

$$|S_s| = \frac{|\dot{S}|}{\omega_0} = \frac{|\dot{S}|}{\pi \bar{N}_0} \qquad (2.1)$$

where $|\dot{S}|$ is the magnitude of the slope of the stress at the zero crossing. This method has been used in the calculations of Sections 6,7 and 8.

In cases where the stress-time histories are responses of nonlinear systems the assumption Eq. 2.1 may be poor. In Appendix C an alternative procedure for predicting the magnitudes of the stationary points from the slopes of the preceding zero crossings is explained.

The procedure for using the Palmgren-Miner cumulative damage criterion to evaluate the damage increments from the "predicted" values of the stationary points, using Eq. 2.1, is explained in Section 5.

In order to be able to determine the standard deviation (or variance) of D_T it is necessary to have an analytical expression for D_T for an arbitrary sample function. This expression must allow us to take into account both the statistical variations in the number of zero crossings (i.e. half cycles) during the time interval, T, and the statistical variations in the slopes at the zero crossings (i.e. "predicted" values of stress at the stationary points.) This has been done in the following manner. We divide the time interval, $0 < t < T$, into a large number ,M, of equal intervals Δt, where

$$M \Delta t = T \tag{2.2}$$

Δt is taken so small that we may assume no more than one zero crossing will occur per interval. With an ensemble of stress sample functions we associate M random variables, ΔD_i, $i = 0, 1, 2, \ldots, M-1$, each ΔD_i being the damage increment incurred during the time interval $i \Delta t < t < (i+1) \Delta t$. If a zero crossing occurs within the interval $i \Delta t < t < (i+1) \Delta t$ then the random variable ΔD_i takes on the value $\delta D \; (\; |\dot{S}| \;)$ where $\dot{S}$ is the slope at the zero crossing. If no zero crossing occurs within the interval then ΔD_i takes a value of zero. Thus the total damage, D_T, accumulated by any sample function during the interval $0 < t < T$ is

$$D_T = \sum_{i=0}^{M-1} \Delta D_i \tag{2.3}$$

where the M values of ΔD_i refer , of course, to the same sample function. A technique similar to the one just explained has been suggested by Rice [12] in connection with the study of the mean square number of zero crossings of a random process in a fixed interval of time.

In order to determine if a zero crossing occurs within an arbitrary interval $i\Delta t < t < (i+1)\Delta t$ we make a further assumption restricting the size of Δt. Following a procedure used by Rice [13] , we choose Δt so small that the portions of all but a negligible fraction of the sample functions lying in the interval $i\Delta t < t < (i+1)\Delta t$ may be regarded as straight lines. Let $S(i\Delta t)$ and $\dot{S}(i\Delta t)$ be respectively the stress and slope at $t = i\Delta t$. Then a zero crossing will occur in the interval if and only if $S(i\Delta t)$ and $\dot{S}(i\Delta t)$ satisfy either conditions (a) or (b) below.

(a) $\dot{S}(i\Delta t) > 0$ and $\quad -\Delta t\, \dot{S}(i\Delta t) < S(i\Delta t) < 0$

(b) $\dot{S}(i\Delta t) < 0$ and $\quad 0 < S(i\Delta t) < -\Delta t\, \dot{S}(i\Delta t)$

Hence having chosen a sufficiently small Δt, $S(i\Delta t)$ and $\dot{S}(i\Delta t)$ determine if a zero crossing occurs in the interval. Furthermore, since the sample functions are assumed to be straight lines within an arbitrary interval, $i\Delta t < t < (i+1)\Delta t$, if a zero crossing occurs within the interval its slope at the point of crossing is $\dot{S}(i\Delta t)$. Then given that a zero crossing occurs within the interval, the incremental damage incurred in this interval,

$\delta D(|\dot{S}|)$, is equivalent to $\delta D(|\dot{S}(i\Delta t)|)$. Thus for arbitrary intervals $i\Delta t < t < (i+1)\Delta t$, each of the random variables ΔD_i may be considered as a function of the random variables $S(i\Delta t)$ and $\dot{S}(i\Delta t)$ as indicated in Eq. 2.4 below.

$$\Delta D_i\left(S(i\Delta t), \dot{S}(i\Delta t)\right) = \begin{cases} \delta D(|\dot{S}(i\Delta t)|) & \text{if either condition (a) or condition (b) is satisfied} \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

The random variables D_T given by Eq. 2.3 are thus functions of the 2M random variables $S(i\Delta t)$, $\dot{S}(i\Delta t)$, $i = 0, 1, 2 \ldots M-1$, $M\Delta t = T$.

3. DERIVATION OF GENERAL FORMULA FOR MATHEMATICAL EXPECTATION OF FATIGUE DAMAGE

In this section we derive a formula for the mathematical expectation of the fatigue damage, D_T, in terms of the damage incurred due to a single stress zero crossing (i.e. a single half cycle), $\delta D(|\dot{S}|)$, and the joint probability density function of the stress and its slope taken at the same time. The ensemble of stress histories is assumed to be a stationary random process.

In section 2 it was shown that D_T could be written as the sum of a set of random variables, ΔD_i, which are the increments of damage incurred in the time intervals $i\Delta t < t < (i+1)\Delta t$, i.e.,

$$D_T = \sum_{i=0}^{M-1} \Delta D_i \qquad \text{(2.3 repeated)}$$

Since the mathematical expectation of a sum of random variables is equal to the sum of the expectations we have

$$E\left[D_T\right] = E\left[\sum_{i=0}^{M-1} \Delta D_i\right]$$

$$= \sum_{i=0}^{M-1} E\left[\Delta D_i\right] \qquad (3.1)$$

As described in Section 2, Eq. 2.4, each ΔD_i is a function of the stress and slope at $t = i\Delta t$. Since the random process is assumed stationary the joint probability density function of $S(i\Delta t)$ and $\dot{S}(i\Delta t)$ is independent of the time $t = i \Delta t$. Hence the values of $E\left[\Delta D_i\right]$ are the same for all i, and using Eq. 2.2,

$$E\left[D_T\right] = M \quad E\left[\Delta D_i\right]$$
$$= \frac{T}{\Delta t} \quad E\left[\Delta D \quad (S(t), \ \dot{S} \ (t) \ )\right] \tag{3.2}$$

The arguments t in Eq. 3.2 have been retained to show that the random variables S and $\dot{S}$ are taken at the same instant of time.

The condition for evaluating $E\left[\Delta D \quad (S(t), \dot{S} \ (t) \ )\right]$ is given by Eq. 2.4. The region of the S(t) , $\dot{S}$ (t) plane for which S(t) and $\dot{S}(t)$ satisfy either conditions (a) or (b) of Section 2 is shown in Fig. 3.1. From Eq. 2.4 when the

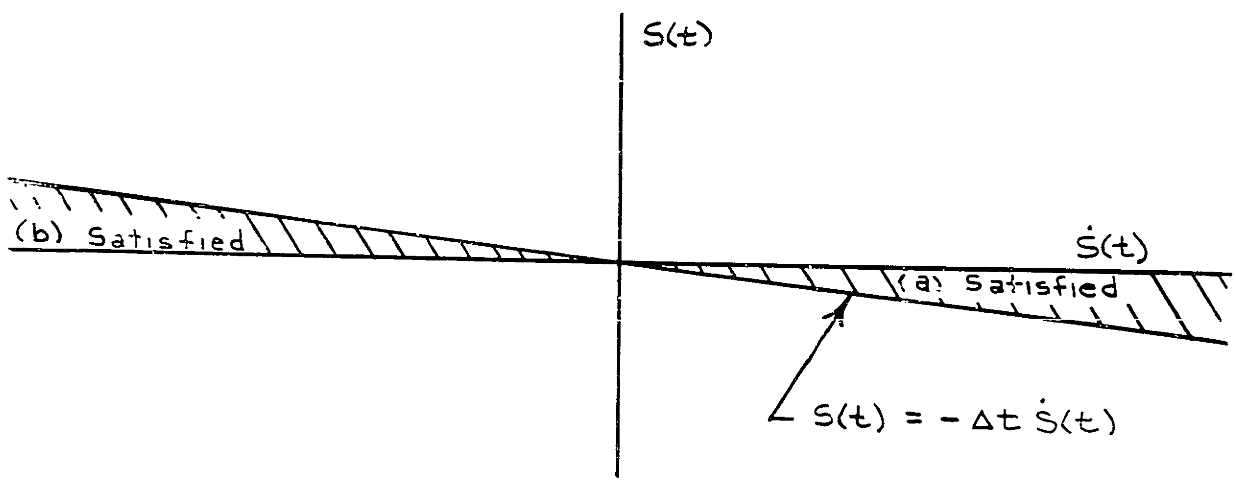

Figure 3.1 Illustrates region of the S(t), $\dot{S}(t)$ plane which satisfies conditions (a) or (b) of Section 2.

pair $\left\{ S \ (t), \ \dot{S} \ (t) \right\}$ fall outside the shaded area Δ D takes a value of zero. When $\left\{ S \ (t), \ \dot{S} \ (t) \right\}$ fall within the shaded area $\Delta D = \delta D (| \dot{S} \ (t) | \ )$.

Hence from the definition of mathematical expectation,

$$E\left[\Delta D \ (S(t), \dot{S}(t) \ )\right] = \int_{-\infty}^{0} d\dot{S} \ \delta D (| \dot{S} |) \int_{0}^{-\Delta t \dot{S}} ds \ f_2 \ (S, \dot{S})$$
$$+ \int_{0}^{\infty} d \dot{S} \ \delta D (| \dot{S} |) \int_{-\Delta t \dot{S}}^{0} d s \ f_2 \ (S, \dot{S}) \tag{3.3}$$

where $f_2(S, \dot{S})$ is the joint probability density function of $S(t)$ and $\dot{S}(t)$ and Δt is taken sufficiently small to satisfy the assumptions of Section 2. Substituting Eq. 3.3 into Eq. 3.2 and satisfying the (smallness) condition on Δt by letting $\Delta t \rightarrow 0$ the mathematical expectation of D_T can be written as

$$E\left[D_T\right] = T\left\{ \int_{-\infty}^{0} d\dot{S}\ \delta D\left(|\dot{S}|\right)\left[\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t}\int_{0}^{-\Delta t \dot{S}} dS\ f_2(S, \dot{S})\right] + \int_{0}^{\infty} d\dot{S}\ \delta D\left(|\dot{S}|\right)\left[\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t}\int_{-\Delta t \dot{S}}^{0} dS\ f_2(S, \dot{S})\right]\right\} \tag{3.4}$$

The operations within the brackets give

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t}\int_{0}^{-\Delta t \dot{S}} dS\ f_2(S, \dot{S}) = -\dot{S}\ f_2(0, \dot{S})$$

and

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t}\int_{-\Delta t \dot{S}}^{0} dS\ f_2(S, \dot{S}) = \dot{S}\ f_2(0, \dot{S}) \tag{3.5}$$

Hence,

$$E\left[D_T\right] = T\left[\int_{-\infty}^{0} d\dot{S}\ \delta D\left(|\dot{S}|\right)\left(-\dot{S}\right) f_2(0, \dot{S}) + \int_{0}^{\infty} d\dot{S}\ \delta D\left(|\dot{S}|\right)\dot{S}\ f_2(0, \dot{S})\right]$$

or

$$E\left[D_T\right] = T\int_{-\infty}^{\infty} \delta D\left(|\dot{S}|\right)|\dot{S}|\ f_2(0, \dot{S})\ d\dot{S} \tag{3.6}$$

This is the desired formula.

4. DERIVATION OF GENERAL FORMULA FOR VARIANCE OF FATIGUE DAMAGE

In this section we derive a formula for the variance of the fatigue damage D_T, in terms of the damage incurred due to a single stress zero crossing (i.e. a single half cycle), $\delta D(|\dot{S}|)$ and the four dimensional probability density function of the stress and its slope taken at two different times. The ensemble of stress histories is assumed to be a stationary random process.

The problem is similar to the determination of the mean square number of zero crossings of random signals as considered by Steinberg, et al. [14] and Miller and Freund [15]. The technique we use is considerably different, being similar to one suggested by Rice [16].

From the definition of variance, the variance $\sigma_{D_T}^2$ of D_T is given by

$$\begin{aligned} \sigma_{D_T}^2 &= E\left[(D_T - E[D_T])^2 \right] \\ &= E\left[D_T^2\right] - (E[D_T])^2 \end{aligned} \tag{4.1}$$

Since a general formula for $E[D_T]$ was derived in Section 3, Eq. 3.6, we need to consider here only $E\left[D_T^2\right]$. As pointed out in Sections 2 and 3, D_T can be expressed as

$$D_T = \sum_{i=0}^{M-1} \Delta D_i \qquad \text{(2.3 repeated)}$$

Hence,

$$\begin{aligned} E\left[D_T^2\right] &= E\left[\left(\sum_{i=0}^{M-1} \Delta D_i \right)^2 \right] \\ &= E\left[\sum_{i=0}^{M-1} \sum_{k=0}^{M-1} \Delta D_i \; \Delta D_k \right] \end{aligned}$$

$$E\left[D_T^2\right] = \sum_{i=0}^{M-1} \sum_{k=0}^{M-1} E\left[\Delta D_i \quad \Delta D_k\right] \tag{4.2}$$

The M^2 terms in the double summation are represented by the M^2 points in the i,k plane in Fig. 4.1. At each point (i, k) we associate the

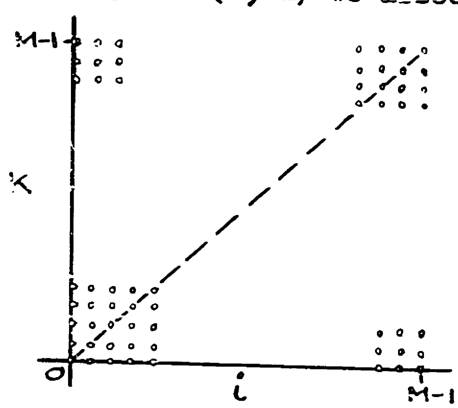

Figure 4.1 Each point represents one term in the double summation of Eq. 4.2.

contribution of the term $E\left[\Delta D_i \quad \Delta D_k\right]$ to the summation. However, since $E\left[\Delta D_k \quad \Delta D_i\right] = E\left[\Delta D_i \quad \Delta D_k\right]$ the contribution of an arbitrary point (k,i) is identical to the contribution of the point (i,k). Hence, the total contribution to the summation of the points above the dashed diagonal (not including the points on the diagonal) is identical to the contribution of the points below the diagonal. Eq. 4.2 can therefore be written as:

$$E\left[D_T^2\right] = 2 \sum_{i=1}^{M-1} \sum_{k=0}^{i-1} E\left[\Delta D_i \quad \Delta D_k\right] + \sum_{i=0}^{M-1} E\left[(\Delta D_i)^2\right] \tag{4.3}$$

The double summation in Eq. 4.3 gives the contribution of all points not on the dashed diagonal and the single summation gives the contribution of the points on the diagonal.

Since the ensemble of stress histories is assumed to be stationary and all of the time increments, Δt, corresponding to the various random variables ΔD_i, i= 0,1,2 . . . M-1, are of identical size, $E[\Delta D_i \ \Delta D_k]$ depends only in i-k. Hence we may write

$$E\left[\Delta D_i \ \Delta D_k\right] = E\left[\Delta D_o \ \Delta D_{i-k}\right]$$

This enables us to transform the double summation in Eq. 4.3 into a single summation over (i-k). On each dashed line in Fig. 4.2 lie all of the points i-k = constant, the constants being different for different dashed lines. Hence all points on the same dashed line give identical contributions to the double summation in Eq. 4.3.

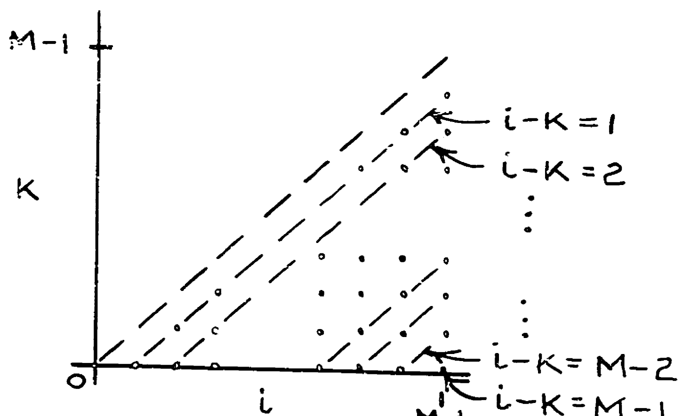

Figure 4.2 The contributions of the individual points on the same dashed line are identical for stationary stress histories.

From Fig. 4.2 there are M - (i-k) points on each dashed line. Hence from Eq. 4.3 $E\left[D_T^2\right]$ becomes

$$E\left[D_T^2\right] = 2\sum_{(i-k)=1}^{M-1}\left[M-(i-k)\right] E\left[\Delta D_o \ \Delta D_{i-k}\right] + \sum_{i=0}^{M-1} E\left[(\Delta D_i)^2\right]$$

$$= 2\sum_{\ell=1}^{M-1}(M-\ell)\, E\left[\Delta D_o \ \Delta D_\ell\right] + \sum_{i=0}^{M-1} E\left[(\Delta D_i)^2\right] \qquad (4.4)$$

We shall now determine an expression for $E\left[\Delta D_o \Delta D_\ell\right]$. The procedure is similar to that used in the determination of $E\left[\Delta D_i\right]$ in Section 3. For arbitrary i, ΔD_i is a function of $S(i\Delta t)$ and $\dot{S}(i\Delta t)$ as indicated in

Section 2, Eq. 2.4. Hence the product $\Delta D_o \, \Delta D_\ell$ is a function of the four random variables $S(0)$, $\dot{S}\,(0)$, $S\,(\ell \,\Delta t)$, $\dot{S}\,(\ell \,\Delta t)$. In order to shorten the equations to follow we define the new notation.

$$\tau = \ell \,\Delta t \tag{4.5}$$

$$S_o = S\,(0) \qquad \dot{S}_o = \dot{S}\,(0)$$

$$S_\tau = S\,(\ell \Delta t) \qquad \dot{S}_\tau = \dot{S}\,(\ell \Delta t) \tag{4.6}$$

From Eq. 2.4 the product $\Delta D_o \;\Delta D_\ell$ is zero unless either condition (a) or (b) of Section 2 is satisfied by both sets $\left\{ S_o \,,\, \dot{S}_o \right\}$ and $\left\{ S_\tau \,,\, \dot{S}_\tau \right\}$. Hence from the definition of mathematical expectation and Eq. 2.4 we have

$$E\left[\Delta D_o \, \Delta D_\ell \right] = \iiiint\limits_R \delta D\,(\,|\dot{S}_o|\,)\,\delta D(\,|\dot{S}_\tau|\,)\; f_4\,(S_o, \dot{S}_o, S_\tau, \dot{S}_\tau)\, dS_o \, d\dot{S}_\tau \, dS_o \, dS_\tau \tag{4.7}$$

where f_4 is the four dimensional probability density function of the indicated variables (with $\tau = \ell \,\Delta t$ entering f_4 as a parameter) and R is the region of the four dimensional space in S_o, $\dot{S}_o$, S_τ, $\dot{S}_\tau$ in which both pairs $\left\{ S_o, \dot{S}_o \right\}$ and $\left\{ S_\tau, \dot{S}_\tau \right\}$ satisfy either conditions (a) or (b) of Section 2. The shaded areas of Fig. 4.3 illustrate the satisfaction of these conditions. The integral in Eq. 4.7 can be treated as an i terated integral by integrating first over S_o and S_τ with

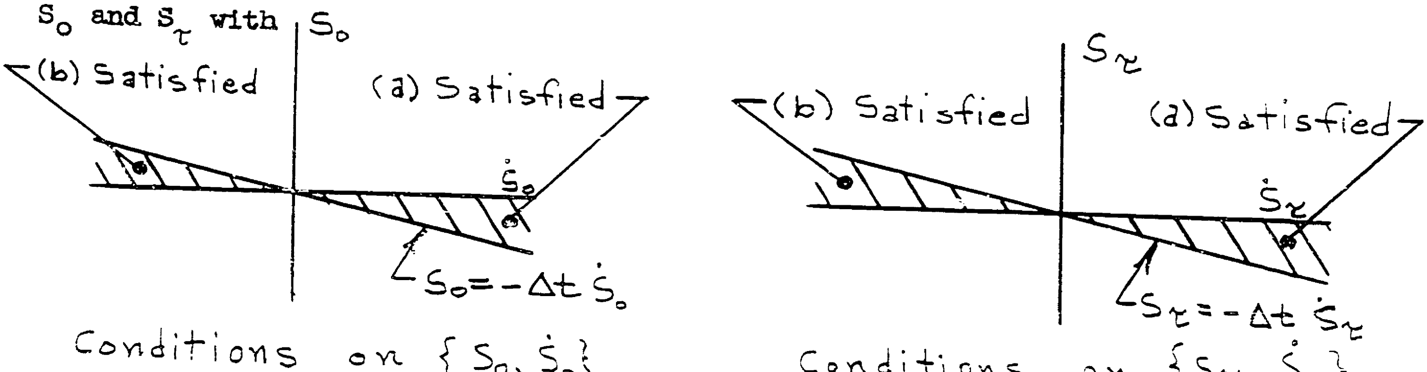

Figure 4.3 The region R in Eq. 4.7 is that region where both $\left\{ S_o, \dot{S}_o \right\}$ and $\left\{ S_\tau, \dot{S}_\tau \right\}$ lie in the shaded areas of their respective planes.

the limits prescribed by Fig. 4.3 and then over $\dot{S}_o$ and $\dot{S}_\tau$. The region R is separated into four parts corresponding to the four quadrants of the $\dot{S}_o$, $\dot{S}_\tau$ plane. Hence taking Δt sufficiently small to satisfy the assumptions of Section 2, we have

$$E\left[\Delta D_o \quad \Delta D_\ell\right] = \left\{\left[\int_{-\infty}^{0} d\dot{S}_o \int_{-\infty}^{0} d\dot{S}_\tau \int_{0}^{-\Delta t \dot{S}_o} dS_o \int_{0}^{-\Delta t \dot{S}_\tau} dS_\tau + \right.\right.$$
$$+ \int_{-\infty}^{0} d\dot{S}_o \int_{0}^{\infty} d\dot{S}_\tau \int_{0}^{-\Delta t \dot{S}_o} dS_o \int_{-\Delta t \dot{S}_\tau}^{0} dS_\tau + \int_{0}^{\infty} d\dot{S}_o \int_{-\infty}^{0} d\dot{S}_\tau \int_{-\Delta t \dot{S}_o}^{0} dS_o \int_{0}^{-\Delta t \dot{S}_\tau} dS_\tau +$$
$$\left.\left. + \int_{0}^{\infty} d\dot{S}_o \int_{0}^{\infty} d\dot{S}_\tau \int_{-\Delta t \dot{S}_o}^{0} dS_o \int_{-\Delta t \dot{S}_\tau}^{0} dS_\tau \right] \delta D\left(|\dot{S}_o|\right) \delta D\left(|\dot{S}_\tau|\right) f_4\left(S_o, \dot{S}_o, S_\tau, \dot{S}_\tau\right)\right\}$$

(4.8)

In order to satisfy the (smallness) condition on Δt we shall want to let $\Delta t \to 0$. In this operation the summation over ℓ in Eq. 4.4 passes to a Riemann integral over τ, and $E\left[\Delta D_o \Delta D_\ell\right]$ enters as $\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} \times E\left[\Delta D_o \quad \Delta D_\ell\right]$. Hence we shall now determine $\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} E\left[\Delta D_o \quad \Delta D_\ell\right]$. Consider the first of the four fold integrals in Eq. 4.8. The appropriate limiting operation on this integral can be written as

$$\int_{-\infty}^{0} d\dot{S}_o \int_{-\infty}^{0} d\dot{S}_\tau \; \delta D\left(|\dot{S}_o|\right) \delta D\left(|\dot{S}_\tau|\right) \left[\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} \int_{0}^{-\Delta t \dot{S}_o} dS_o \int_{0}^{-\Delta t \dot{S}_\tau} dS_\tau \; f_4\left(S_o, \dot{S}_o, S_\tau, \dot{S}_\tau\right)\right] \quad (4.9)$$

The operation within the brackets of Eq. 4.9 gives

$$\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} \int_{0}^{-\Delta t \dot{S}_o} dS_o \int_{0}^{-\Delta t \dot{S}_\tau} dS_\tau \; f_4\left(S_o, \dot{S}_o, S_\tau, \dot{S}_\tau\right) = (-\dot{S}_o)(-\dot{S}_\tau) f_4(0, \dot{S}_o, 0, \dot{S}_\tau)$$

(4.10)

Since both $\dot{S}_o$ and $\dot{S}_\tau$ are negative everywhere within their domain of integration in Eq. 4.9, $-\dot{S}_o$ and $-\dot{S}_\tau$ in Eq. 4.10 can be written as $|\dot{S}_o|$ and $|\dot{S}_\tau|$ respectively. Hence the result for the first of the four fold integrals in Eq. 4.8 can be written as

$$\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} \int_{-\infty}^{0} d\dot{S}_0 \int_{-\infty}^{0} d\dot{S}_\tau \int_{0}^{-\Delta t \dot{S}_0} dS_0 \int_{0}^{-\Delta t \dot{S}_\tau} dS_\tau \, \delta D(|\dot{S}_0|) \, \delta D(|\dot{S}_\tau|) \, f_4(S_0, \dot{S}_0, S_\tau, \dot{S}_\tau) =$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{0} \delta D(|\dot{S}_0|) \, \delta D(|\dot{S}_\tau|) \, |\dot{S}_0| \, |\dot{S}_\tau| \, f_4(0, \dot{S}_0, 0, \dot{S}_\tau) \, d\dot{S}_0 \, d\dot{S}_\tau \qquad (4.11)$$

The remaining three of the four fold integrals in Eq. 4.8 are handled in an analogous manner. The results can be gathered into a single double integral,

$$\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} E\left[\Delta D_0 \ \Delta D_\ell\right] =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta D(|\dot{S}_0|) \, \delta D(|\dot{S}_\tau|) \, |\dot{S}_0| \, |\dot{S}_\tau| \, f_4(0, \dot{S}_0, 0, \dot{S}_\tau) \, d\dot{S}_0 \, d\dot{S}_\tau \qquad (4.12)$$

where $\tau = \ell \Delta t$. To shorten the equations to follow we define

$$I_3(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta D(|\dot{S}_0|) \, \delta D(|\dot{S}_\tau|) \, |\dot{S}_0| \, |\dot{S}_\tau| \, f_4(0, \dot{S}_0, 0, \dot{S}_\tau) \, d\dot{S}_0 \, d\dot{S}_\tau \qquad (4.13)$$

We shall now let $\Delta t \to 0$ and hence $M \to \infty$ in the summation over ℓ in Eq. 4.4. For any size Δt the summation can be written as

$$\sum_{\ell=1}^{M-1} (M-\ell) \, E\left[\Delta D_0 \ \Delta D_\ell\right] =$$

$$= \sum_{\ell=1}^{M-1} (M \Delta t - \ell \Delta t) \frac{1}{(\Delta t)^2} E\left[\Delta D_0 \ \Delta D_\ell\right] \Delta t \qquad (4.14)$$

Recalling that T is the time interval during which the damages D_T are incurred and hence is independent of the size of Δt, and that $T = M \Delta t$ (Eq. 2.2), then as $\Delta t \to 0$, $M \to \infty$ in such a way that $M \Delta t$ remains constant. Hence assuming $I_3(\tau) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^2} E\left[\Delta D_0 \ \Delta D_\ell\right]$ to be continuous in the interval $0 \leqslant \tau \leqslant T$, and using $\tau = \ell \Delta t$ as the variable of integration, as $\Delta t \to 0$ the summation, Eq. 4.14 defines a Riemann integral between the limits $\tau = 0$ and $\tau = T - \Delta t$. The assumed continuity (and hence boundedness) of $I_3(\tau)$ allows us to replace the upper limit by $\tau = T$. Hence we have

$$\lim_{\Delta t \to 0} \sum_{\ell=1}^{M-1} (M-\ell)\, E\left[\Delta D_0 \, \Delta D_\ell\right] = \int_0^T (T-\tau) I_3(\tau)\, d\tau \tag{4.15}$$

where $I_3(\tau)$ is given by Eq. 4.13.

We shall now consider the summation over i in Eq. 4.4. It is desirable to express this summation as a function of $\delta D(|\dot{S}|)$ and the joint probability density function of the stress and its slope taken at the same instant of time, $f_2(S, \dot{S})$. The treatment is a straightforward extension of the arguments between Eqs. 3.1 and 3.6 in Section 3 and we shall not bother to repeat them. The result is

$$\sum_{i=0}^{M-1} E\left[(\Delta D_i)^2\right] = T \int_{-\infty}^{\infty} \delta D^2(|\dot{S}|)\, |\dot{S}|\, f_2(0,\dot{S})\, d\dot{S} \tag{4.16}$$

where

$$\delta D^2(|\dot{S}|) = \left[\delta D(|\dot{S}|)\right]^2 \tag{4.17}$$

We shall now combine the results obtained in Section 3 and this section into a single expression for the variance of D_T given by Eq. 4.1. First we make the definitions

$$I_1 = \int_{-\infty}^{\infty} \delta D^2(|\dot{S}|)\, |\dot{S}|\, f_2(0, \dot{S})\, d\dot{S} \tag{4.18}$$

$$I_2 = \int_{-\infty}^{\infty} \delta D(|\dot{S}|)\, |\dot{S}|\, f_2(0, \dot{S})\, d\dot{S} \tag{4.19}$$

$$I_3(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta D(|\dot{S}_o|)\,\delta D(|\dot{S}_\tau|)\,|\dot{S}_o|\,|\dot{S}_\tau|\,f_4(0,\dot{S}_o,0,\dot{S}_\tau)\,d\dot{S}_o\,d\dot{S}_\tau$$

(4.13 repeated)

Substituting Eq. 4.18 into Eq. 4.16, this result and Eq. 4.15 into Eq. 4.4 and this result together with Eqs. 4.19 and 3.6 into Eq. 4.1 gives the variance of D_T,

$$\sigma_{D_T}^2 = 2\int_0^T (T-\tau)\, I_3(\tau)\, d\tau + T I_1 - T^2 I_2^2 \tag{4.20}$$

However,

$$T^2 = 2\int_0^T (T-\tau)\, d\tau \tag{4.21}$$

Hence $\sigma_{D_T}^2$ can also be written as

$$\sigma_{D_T}^2 = TI_1 + 2\int_0^T (T-\tau)\left[I_3(\tau) - I_2^2\right] d\tau \tag{4.22}$$

or

$$\sigma_{D_T}^2 = T\left\{I_1 + 2\int_0^T \left[I_3(\tau) - I_2^2\right] d\tau\right\} - 2\int_0^T \tau\left[I_3(\tau) - I_2^2\right] d\tau \tag{4.23}$$

Eq. 4.23 is the principal result of this section.

The reason for grouping the terms $\left[I_3(\tau) - I_2^2\right]$ in Eqs. 4.22 and 4.23 is now given. For most ergodic ensembles as $\tau \to \infty$

$$f_4(0,\dot{S}_o,0,\dot{S}_\tau) \sim f_2(0,\dot{S}_o)\, f_2(0,\dot{S}_\tau) \tag{4.24}$$

that is, for large τ the process at $t=\tau$ is statistically independent of the process at $t=0$. In such cases as $\tau \to \infty$

$$\begin{aligned} I_3(\tau) &\sim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta D(|\dot{S}_o|)\,\delta D(|\dot{S}_\tau|)\,|\dot{S}_o|\,|\dot{S}_\tau|\,f_2(0,\dot{S}_o)\,f_2(0,\dot{S}_\tau)\,d\dot{S}_o\,d\dot{S} \\ &= \left[\int_{-\infty}^{\infty} \delta D(|\dot{S}|)\,|\dot{S}|\,f_2(0,\dot{S})\,d\dot{S}\right]^2 \\ &= I_2^2 \end{aligned} \tag{4.25}$$

Hence, for these cases as $\tau \to \infty$

$$\left[I_3 (\tau) - I_2^2 \right] \sim 0 \tag{4.26}$$

If $\left[I_3 (\tau) - I_2^2 \right]$ dies out sufficiently fast, then as $T \to \infty$,

$$\int_0^T \left[I_3 (\tau) - I_2^2 \right] d\tau \sim \text{a constant, say } C_1$$

and

$$\int_0^T \tau \left[I_3 (\tau) - I_2^2 \right] d\tau \sim \text{another constant, say } C_2 ,$$

where C_1 and C_2 are independent of T, and from Eq. 4.23, as $T \to \infty$

$$\sigma_{D_T}^2 \sim T \left\{ I_1 + 2 C_1 \right\} - 2 C_2 \tag{4.27}$$

Hence, for ergodic ensembles and large times, T, we might expect the variance of the damage incurred during $0 < t < T$ to be approximately proportional to T. In addition, from Eqs. 3.6 and 4.27 as $T \to \infty$ we expect the ratio $\sigma_{D_T} / E \left[D_T \right]$ to approach zero.

5. EVALUATION OF DAMAGE INCREMENTS $\delta D(|\dot{S}|)$ AND CHOICE OF MATHEMATICAL MODEL FOR S-N CURVES

5.1 Evaluation of Damage Increments $\delta D(|\dot{S}|)$ from S-N Curves

In Section 2 it was explained that with each half cycle and associated zero crossing of a stress sample function a damage increment $\delta D(|\dot{S}|)$ is assumed to be incurred, the damage increment being a function of the slope at the zero crossing. In the work contained in Sections 6, 7 and 8, we assume the stress histories to be sample functions obtained from Gaussian ensembles. Narrow band responses which are also Gaussian usually have the appearance of sinusoids with slowly and randomly varying amplitude and phase where the time between zero crossings on the average is independent of the amplitudes of the half cycles. Hence, in Sections 6, 7 and 8 we shall assume the amplitudes of the half cycles (the magnitudes of the stresses at the stationary points), $|S_s|$, to be well predicted from the slopes at their associated zero crossings by the assumption that the intervening quarter cycles are sinusoids as explained in Section 2. That is what we shall use

$$|S_s| = \frac{|\dot{S}|}{\pi \bar{N}_o} \qquad \text{(2.1 repeated)}$$

to "predict" values of $|S_s|$ from the magnitudes of the slopes at the zero crossings $|\dot{S}|$, where $\bar{N}_o$ is the expected number of zero crossings per unit time of the stress sample functions. An alternative procedure for "predicting" $|S_s|$ from $|\dot{S}|$ is outlined in Appendix C.

We shall now explain how we evaluate the damage increments $\delta D(|\dot{S}|)$ from S-N curves by using Eq. 2.1 and a modified form of the Palmgren-Miner criterion. As was mentioned in Section 2 the damage increments $\delta D(|\dot{S}|)$ are scaled so that fatigue failure will occur when the sum of these increments accumulates to unity. The Palmgren-Miner criterion, Eq. 1.1 predicts fatigue failure to occur when the total number of (whole) stress cycles, $\sum_i n_i$ is such that

$$\sum_i \frac{n_i}{N_i} = 1 \qquad \text{(1.1 repeated)}$$

where $N_i = N(S_i)$ is the number of (whole) sinusoidal stress cycles obtained from an S-N curve at the stress level S_i, which is the stress amplitude of the n_i cycles. Hence if a fatigue damage of $1/N(S_i)$ is associated with each (whole) stress cycle of amplitude S_i then Eq. 1.1 predicts fatigue failure to occur when the sum of these damages accumulates to unity. An obvious extension of this to half cycles of stress of amplitudes $|S_s|$ is to associate a damage increment of $1/2N(|S_s|)$ with each half cycle. This is the method we use. Using Eq. 2.1 to "predict" the amplitudes of the half cycles from their associated zero crossings gives us the damage increments,

$$\delta D(|\dot{S}|) = \frac{1}{2N(|\dot{S}|/\pi \bar{N}_0)} \qquad (5.1)$$

where N(S) is the number of (whole) cycles at stress amplitude S obtained from the appropriate S-N curve.

5.2 Choice of Mathematical Model for S-N Curves

Since N appears in Eq. 5.1, in order to be able to analytically evaluate the formulas derived in Sections 3 and 4 it is necessary to have a mathematical representation of S-N curves. S-N data are usually plotted on semi-log paper. However, when the S-N data for a large number of metals are plotted on full log coordinates the resultant curves are well approximated by straight lines over a large range of S. Fig. 5.1 represents such a plot for a high strength aluminum alloy. The straight portion of the curve

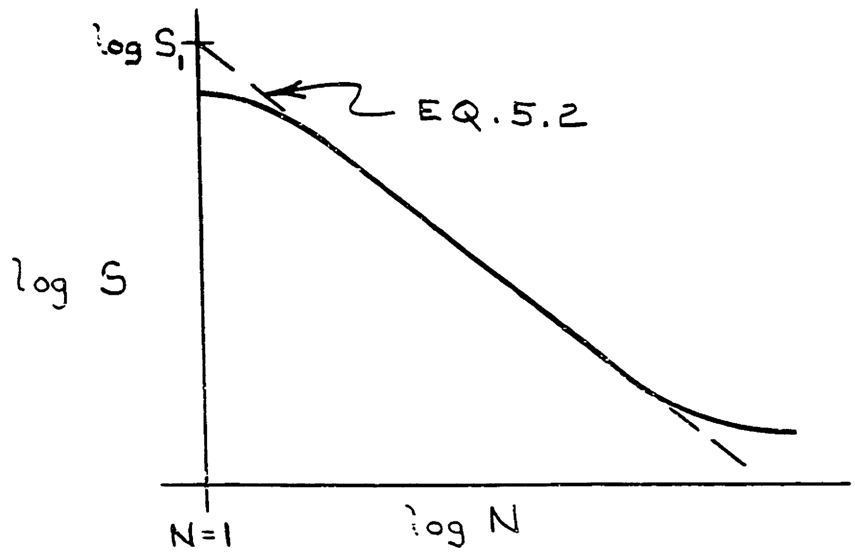

Figure 5.1 Logarithmic plot of S-N data for a high strength aluminum alloy and mathematical approximation, Eq. 5.2.

can be represented by the formula

$$N(S) = \left(\frac{S_1}{S}\right)^{\alpha} \qquad (5.2)$$

where S_1 and α are constants.

Taking the logarithms of both sides of Eq. 5.2 and rearranging gives

$$\log S = \log S_1 - \frac{1}{\alpha} \log N \qquad (5.3)$$

Thus the slope of the straight portion of the curve in Fig. 5.1 is $-\frac{1}{\alpha}$.

A typical value of α is 9. S_1 can be interpreted as the stress amplitude at which Eq. 5.2 predicts fatigue failure after one cycle. However, due to the curvature of (full log) S-N curves for large values of S , as shown in Fig. 5.1, values of S_1 are generally considerably larger than the ultimate strengths of their materials.

The representation given by Eq. 5.2 may be used if the amplitudes in the stress sample functions are such that most will fall on the straight portion of the logarithmic plot of S-N data. This representation has been used by Miles [17] , and we shall use it in the work of Sections 6,7 and 8. Substituting Eq. 5.2 into Eq. 5.1 we have

$$\delta D(|\dot{S}|) = \frac{1}{2} \left(\frac{|\dot{S}|}{S_1 \bar{N}_o \pi} \right)^{\alpha} \qquad (5$$

Eq. 5.4 gives the damage increments as a function of the slopes at the zero crossings, $\dot{S}$, where $\bar{N}_o$ is a property of the ensemble of stress historys and S_1 and α are constants which characterize the fatigue properties of the material.

6. EVALUATION OF EXPECTED FATIGUE DAMAGE FOR STATIONARY NARROW BAND GAUSSIAN STRESS HISTORIES

In this section we determine a formula for the mathematical expectation of the fatigue damage incurred during a fixed time T for stationary narrow band Gaussian stress histories. In Section 3 a general formula, Eq. 3.6, was derived for the expected fatigue damage.

$$E\left[D_T\right] = T\int_{-\infty}^{\infty} \delta D(|\dot{S}|)\ |\dot{S}|\ f_2(0,\dot{S})\ d\dot{S} \qquad \text{(3.6 repeated)}$$

For Gaussian ensembles of stress histories the joint probability density function of S(t) and $\dot{S}(t)$, $f_2(S,\dot{S})$ is a two dimensional normal distribution [18] . For stationary ensembles of stress histories $E\left[S(t)\ \dot{S}(t)\right] = 0$. Hence $f_2(0,\dot{S})$ in Eq. 3.6 is

$$f_2(0,\dot{S}) = \frac{1}{2\pi\sigma_s\ \sigma_{\dot{s}}}\ e^{-\frac{\dot{S}^2}{2\sigma_{\dot{S}}^2}} \qquad (6.1)$$

where

$$\sigma_s = \sqrt{E\left[(S-E[S])^2\right]} = \sqrt{E\left[S^2\right]}$$

and

$$\sigma_{\dot{s}} = \sqrt{E\left[(\dot{S}-E[\dot{S}])^2\right]} = \sqrt{E\left[\dot{S}^2\right]} \qquad (6.2)$$

since the mean stress is assumed to be zero. If ψ_τ is the autocorrelation function of S(t), that is,

$$\psi_\tau = \psi(\tau) = E\left[S(t)\ S(t+\tau)\right] \qquad (6.3)$$

then

$$\sigma_s = \psi_0^{\ 1/2} \qquad (6.4)$$

and from [19]

$$\sigma_{\dot{S}} = (-\psi_o'')^{1/2} \tag{6.5}$$

where

$$\psi_o'' = \left.\frac{d^2\psi_\tau}{d\tau^2}\right|_{\tau=0} \tag{6.6}$$

is a negative number.

An expression for $\delta D(|\dot{S}|)$ has been obtained in Section 5, Eq. 5.4,

$$\delta D(|\dot{S}|) = \frac{1}{2}\left(\frac{|\dot{S}|}{S_1 \bar{N}_o \pi}\right)^\alpha \qquad \text{(5.4 repeated)}$$

where the expected number of zero crossings per unit time for a stationary Gaussian process is given [20] by

$$\bar{N}_o = \frac{1}{\pi}\left[\frac{-\psi_o''}{\psi_o}\right]^{1/2} \tag{6.7}$$

Combining Eqs. 3.6, 6.1, 6.4, 6.5, 5.4, and 6.7 gives

$$E[D_T] = T\int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{|\dot{S}|}{S_1}\left[\frac{\psi_o}{-\psi_o''}\right]^{1/2}\right)^\alpha \frac{|\dot{S}|}{2\pi\psi_o^{1/2}(-\psi_o'')^{1/2}} e^{\frac{-\dot{S}^2}{2(-\psi_o'')}}\, d\dot{S}$$

$$= \frac{T\psi_o^{\frac{\alpha-1}{2}}}{2\pi S_1^\alpha\left[-\psi_o''\right]^{\frac{\alpha+1}{2}}}\ \frac{1}{2}\int_{-\infty}^{\infty}|\dot{S}|^{\alpha+1} e^{\frac{-\dot{S}^2}{2(-\psi_o'')}}\, d\dot{S}$$

$$= \frac{T\psi_o^{\frac{\alpha-1}{2}}}{2\pi S_1^\alpha\left[-\psi_o''\right]^{\frac{\alpha+1}{2}}}\int_{0}^{\infty}\dot{S}^{\alpha+1} e^{\frac{-\dot{S}^2}{2(-\psi_o'')}}\, d\dot{S} \tag{6.8}$$

Evaluating [21] the integral in Eq. 6.8 in terms of the gamma function, the expected damage becomes

$$E[D_T] = \frac{T}{2\pi}\left[\frac{-\psi_o''}{\psi_o}\right]^{1/2}\left(\frac{2\psi_o}{S_1^2}\right)^{\frac{\alpha}{2}}\Gamma\left(1+\frac{\alpha}{2}\right) \tag{6.9}$$

or using Eq. 6.7

$$E\left[D_T\right] = T \frac{\bar{N}_o}{2} \left(\frac{2\Psi_o}{S_1^{\,2}} \right)^{\frac{\alpha}{2}} \Gamma\left(1+\frac{\alpha}{2}\right) \qquad (6.10)$$

which is the principal result of this section. In Section 9 this result is shown to be consistent with the results of Miles [22] . Notice that since each "cycle" contains two zero crossings, the quantity $T \bar{N}_o/2$ in Eq. 6.10 is the mathematical expectation of the number of "cycles" in the interval $0 < t < T$. Hence the expected damage per "cycle" is $(2\Psi_o/S_1^{\,2})^{\alpha/2} \times \Gamma(1+\frac{\alpha}{2})$.

7. INTRODUCTION OF GAUSSIAN ASSUMPTION INTO FORMULA FOR VARIANCE OF FATIGUE DAMAGE

If we assume that the stationary ensembles of stress histories are Gaussian random processes then the integrals I_1, I_2, and $I_3(\tau)$, Eqs. 4.18, 4.19 and 4.13 respectively, can be evaluated in terms of the autocorrelation function of the stresses, ψ_τ, and the parameters characterizing the S-N curve, S_1 and α. The only remaining steps in the determination of $\sigma_{D_T}^{\ 2}$ are the integrations over τ indicated by Eq. 4.23. Assumption of a specific form of ψ_τ is necessary before these can be carried out.

Comparing Eqs. 3.6, 4.19 and 6.10 we find I_2 has already been evaluated in Section 6.

$$I_2 = \frac{1}{T} \; E\left[D_T\right]$$

$$= \frac{\bar{N}_o}{2}\left(\frac{2\psi_o}{S_1^{\ 2}}\right)^{\frac{\alpha}{2}} \Gamma\left(1+\frac{\alpha}{2}\right) \tag{7.1}$$

where $\bar{N}_o$ is given by Eq. 6.7.

The evaluation of I_1 is implicity contained in the evaluation of I_2. The parameter α is contained in the integrand of I_2 (Eq. 4.19) only as an exponent in $\delta D(|\dot{S}|)$ given by Eq. 5.4. Hence from Eq. 5.4 squaring $\delta D(|\dot{S}|)$ in Eq. 4.18 is equivalent to multiplying Eq. 4.19 by 1/2 and replacing α by 2α. We thus obtain I_1 immediately from I_2, Eq. 7.1 as

$$I_1 = \frac{\bar{N}_o}{4} \left(\frac{2\psi_o}{S_1^{\ 2}}\right)^{\alpha} \Gamma(1+\alpha) \tag{7.2}$$

We shall now evaluate $I_3(\tau)$, Eq. 4.13.

If an ensemble of stress histories is Gaussian then the probability density function, $f_4(S_o, \dot{S}_o, S_\tau, \dot{S}_\tau)$, is a four dimensional normal distribution

[23] . Hence $f_4\,(0, \dot{S}_0, 0, \dot{S}_\tau)$ can be written as

$$f_4\,(0, \dot{S}_0, 0, \dot{S}_\tau) =$$
$$= \frac{1}{4\pi^2 |\Lambda|^{1/2}}\; e^{-\frac{1}{2|\Lambda|}\left(\Lambda_{22}\dot{S}_0^{\,2} + 2\Lambda_{24}\dot{S}_0\dot{S}_\tau + \Lambda_{44}\dot{S}_\tau^{\,2}\right)} \tag{7.3}$$

The evaluation of the "Λ parameters" in terms of the autocorrelation function ψ_τ is in the readily available literature [24] and we shall list only the results here

$$
\begin{aligned}
|\Lambda| &= (\psi_0^{\,2} - \psi_\tau^{\,2})(\psi_0''^{\,2} - \psi_\tau''^{\,2}) + 2\psi_\tau'^{\,2}(\psi_0\psi_0'' - \psi_\tau\psi_\tau'') + \psi_\tau'^{\,4} \\
\Lambda_{22} &= -\psi_0''(\psi_0^{\,2} - \psi_\tau^{\,2}) - \psi_0\psi_\tau'^{\,2} \\
\Lambda_{24} &= \psi_\tau''(\psi_0^{\,2} - \psi_\tau^{\,2}) + \psi_\tau\psi_\tau'^{\,2} \\
\Lambda_{44} &= \Lambda_{22}
\end{aligned} \tag{7.4}
$$

where

$$
\begin{aligned}
\psi_\tau &= \psi(\tau) = E\,[\,S(t)\,S(t+\tau)\,] \\
\psi_0 &= \psi_\tau \Big|_{\tau=0} \\
\psi_\tau' &= \frac{d\psi_\tau}{d\tau} \\
\psi_\tau'' &= \frac{d\psi_\tau'}{d\tau} \\
\psi_0'' &= \psi_\tau'' \Big|_{\tau=0}
\end{aligned} \tag{7.5}
$$

Substituting Eq. 7.3 and the expression for the damage increments, Eq. 5.4, into $I_3(\tau)$, Eq. 4.13, gives

$$I_3(\tau) = \frac{1}{16\pi^2(s_1\pi \bar{N}_o)^{2\alpha} |\Lambda|^{1/2}} \times \tag{7.6}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{s}_o|^{\alpha+1} |\dot{s}_\tau|^{\alpha+1} e^{-\frac{1}{2|\Lambda|}(\Lambda_{22}\dot{s}_o^2 + 2\Lambda_{24}\dot{s}_o\dot{s}_\tau + \Lambda_{22}\dot{s}_\tau^2)} d\dot{s}_o \, d\dot{s}_\tau$$

We define the new variables

$$\chi_o = \left(\frac{\Lambda_{22}}{2|\Lambda|}\right)^{1/2} \dot{s}_o$$

$$\chi_\tau = \left(\frac{\Lambda_{22}}{2|\Lambda|}\right)^{1/2} \dot{s}_\tau \tag{7.7}$$

Taking the square root in the change of variables, Eq. 7.7 causes no trouble. This follows from the fact $|\Lambda| / \Lambda_{22}$ can be interpreted as a certain conditional variance [25] and hence is necessarily non-negative. Writing Eq. 7.6 in terms of χ_o and χ_τ gives

$$I_3(\tau) = \frac{1}{4\pi^2} \left(\frac{2}{s_1^2 \pi^2 \bar{N}_o^2}\right)^{\alpha} \frac{|\Lambda|^{\alpha+3/2}}{\Lambda_{22}^{\alpha+2}} \times$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\chi_o|^{\alpha+1} |\chi_\tau|^{\alpha+1} e^{-\chi_o^2 - 2\frac{\Lambda_{24}}{\Lambda_{22}}\chi_o\chi_\tau - \chi_\tau^2} d\chi_o \, d\chi_\tau \tag{7.8}$$

If we restrict $(\alpha + 1)$ to be an even integer then the absolute value signs can be removed from the integrand in Eq. 7.8. We shall make this assumption in all of the work that follows, i.e.,

$$\text{Assume } \alpha = \text{an odd integer} \tag{7.9}$$

The resulting integral has been evaluated by Rice [26] in terms of the hypergeometric function,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_o^{\alpha+1}\, \chi_\tau^{\alpha+1}\; e^{-\chi_o^2 - 2\chi_o\chi_\tau\cos\varphi - \chi_\tau^2}\, d\chi_o\, d\chi_\tau =$$

$$= \frac{\Gamma^2(1+\alpha/2)}{(\sin\varphi)^{2\alpha+3}}\; F\left(-\frac{\alpha+1}{2},\ -\frac{\alpha+1}{2};\ \frac{1}{2};\ \cos^2\varphi\right)$$

$$\alpha+1 = \text{even}, \quad 0 < \varphi < \pi \quad . \tag{7.10}$$

Since $-\Lambda_{24}/\Lambda_{22}$ can be considered as the correlation coefficient of a certain two dimensional normal distribution [27] it follows that $|\Lambda_{24}/\Lambda_{22}| \leq 1$. Hence the integral in Eq. 7.8 can be considered of the same form as Eq. 7.10.

Before using Eq. 7.10 we shall transform it into a more convenient form. Since $\sin\varphi \geq 0$ for $0 \leq \varphi \leq \pi$ the right hand side of Eq. 7.10 can also be written as

$$\Gamma^2\left(1+\frac{\alpha}{2}\right)\; \frac{F\left(-\frac{\alpha+1}{2},\ -\frac{\alpha+1}{2};\ \frac{1}{2};\ \cos^2\varphi\right)}{\left[1-\cos^2\varphi\right]^{\alpha+3/2}} \tag{7.11}$$

Using the transformation [28]

$$F(a,\ b;\ c;\ z) = (1-z)^{c-a-b}\; F(c-a,\ c-b;\ c;\ z) \tag{7.12}$$

(7.11) becomes

$$\Gamma^2\left(1+\frac{\alpha}{2}\right)\; F\left(1+\frac{\alpha}{2},\ 1+\frac{\alpha}{2};\ \frac{1}{2};\ \cos^2\varphi\right) \tag{7.13}$$

Hence, an alternative form for Eq. 7.10 is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_o^{\alpha+1}\ \chi_\tau^{\alpha+1}\ e^{-\chi_o^2 - 2\chi_o\chi_\tau \cos\varphi - \chi_\tau^2}\, d\chi_o\, d\chi_\tau = \Gamma^2\left(1+\frac{\alpha}{2}\right) F\left(1+\frac{\alpha}{2},\ 1+\frac{\alpha}{2};\ \frac{1}{2};\ \cos^2\varphi\right)$$

$$\alpha+1 = \text{even},\quad 0<\varphi<\pi. \qquad (7.14)$$

The form given by Eq. 7.10 has an advantage over that given by Eq. 7.14 since the hypergeometric series of Eq. 7.10 terminates after $\left(1+\frac{\alpha+1}{2}\right)$ terms.

Using the form given by Eq. 7.14 for the integral in $I_3(\tau)$, Eq. 7.8 we have

$$I_3(\tau) = \frac{1}{4\pi^2}\left(\frac{2}{S_1^2\pi^2\bar{N}_o^2}\right)^{\alpha}\Gamma^2\left(1+\frac{\alpha}{2}\right) \times$$

$$\frac{|\Lambda|^{\alpha+\frac{3}{2}}}{\Lambda_{22}^{\alpha+2}}\ F\left(1+\frac{\alpha}{2},\ 1+\frac{\alpha}{2};\ \frac{1}{2};\ \frac{\Lambda_{24}^2}{\Lambda_{22}^2}\right)$$

$$\alpha = \text{odd integer} \qquad (7.15)$$

where $|\Lambda|$, Λ_{22} and Λ_{24} are given in terms of the autocorrelation function by Eq. 7.4, and $\bar{N}_o$ by Eq. 6.7.

In section 4 in letting $\Delta\tau\to 0$ a summation was replaced by a Riemann integral, this operation being indicated by Eq. 4.15. In performing this operation $I_3(\tau)$ was assumed continuous in the interval $0 \leq \tau \leq T$. For stationary ensembles ψ_τ must be an even function of τ and hence

$$\psi_o' = \left.\frac{d\psi_\tau}{d\tau}\right|_{\tau=0} = 0 \qquad (7.16)$$

Using Eq. 7.16, examination of Eqs. 7.4 and 7.5 shows that for stationary ensembles

$$|\Lambda|\Big|_{\tau=0} = 0$$
$$\Lambda_{22}\Big|_{\tau=0} = 0$$
$$\Lambda_{24}\Big|_{\tau=0} = 0 \qquad (7.17)$$

Thus, the ratios $|\Lambda|^{\alpha+3/2} / \Lambda_{22}^{\alpha+2}$ and $\Lambda_{24}^{2} / \Lambda_{22}^{2}$ appearing in Eq. 7.15 are indeterminate at $\tau = 0$ and hence $I_3(0)$ is undefined. In order to satisfy the continuity assumption it is necessary that $\lim_{\tau \to 0+} I_3(\tau)$ exist, where the limit is the conventional right hand limit. In Appendix D we show that under fairly general conditions

$$\lim_{\tau \to 0+} I_3(\tau) = 0 \tag{7.18}$$

Hence we define

$$I_3(0) = 0 \tag{7.19}$$

thereby satisfying the continuity requirement at $\tau = 0$. For autocorrelation functions arising from physical problems we do not anticipate continuity problems in the remainder of the interval, $0 < \tau \leq T$.

8. EVALUATION OF VARIANCE OF FATIGUE DAMAGE FOR STRESS HISTORIES GENERATED BY SINGLE DEGREE-OF-FREEDOM SYSTEMS

In this section we evaluate $\sigma_{D_T}^2$ for stationary Gaussian ensembles of stress histories which may be considered as responses of lightly damped linear single degree-of-freedom systems to white noise excitations. That is, we assume the sample functions, S(t) are stationary Gaussian responses of systems described by the differential equation

$$\ddot{S} + 2\zeta\omega_n \dot{S} + \omega_n^2 S = F(t) \tag{8.1}$$

where ω_n is the natural frequency, rad./sec., of the undamped system, ζ is the (small) system damping and F(t) is a nominally white noise excitation. By a nominally white noise excitation we mean one whose spectral density is a constant over the range of frequencies which contribute significantly to the mean square response resulting from a true white noise excitation.

The evaluation of the autocorrelation function, $\psi_\tau = E\left[S(t)S(t+\tau)\right]$, of the response of the system described by Eq. 8.1 is straightforward and in the easily accessible literature [29].

$$\psi_\tau = \psi_0 e^{-\zeta\omega_n\tau}\left(\cos\sqrt{1-\zeta^2}\,\omega_n\tau + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n\tau\right), \quad \tau \geq 0 \tag{8.2}$$

$$= \frac{\psi_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n\tau} \cos\left(\sqrt{1-\zeta^2}\,\omega_n\tau - \varphi_\zeta\right), \quad \tau \geq 0 \tag{8.3}$$

where $\varphi_\zeta = \tan^{-1}\dfrac{\zeta}{\sqrt{1-\zeta^2}}$

Using Eqs. 6.6, 6.7 and 8.2 we find [1]

$$\bar{N}_o = \frac{\omega_n}{\pi} \tag{8.4}$$

The functions depending on τ in $I_3(\tau)$ Eq. 7.15, are evaluated using Eqs. 7.4, 7.5 and 8.2. After considerable manipulation we find [1]

$$|\Lambda| = \psi_o^4 \, \omega_n^4 \quad A_1(\zeta, \omega_n\tau) \tag{8.5}$$

$$\Lambda_{22} = \psi_o^3 \, \omega_n^2 \quad A_2(\zeta, \omega_n\tau) \tag{8.6}$$

$$\Lambda_{24} = -\psi_o^3 \, \omega_n^2 \quad A_4(\zeta, \omega_n\tau) \tag{8.7}$$

where for $\tau \geqslant 0$

$$A_1(\zeta, \omega_n\tau) = 1 - \frac{2}{1-\zeta^2} e^{-2\zeta\omega_n\tau} \left(1 - \zeta^2 \cos 2\sqrt{1-\zeta^2}\,\omega_n\tau\right) + e^{-4\zeta\omega_n\tau} \tag{8.8}$$

$$= \left(1 - e^{-2\zeta\omega_n\tau}\right)^2 - \frac{4\zeta^2}{1-\zeta^2} e^{-2\zeta\omega_n\tau} \sin^2 \sqrt{1-\zeta^2}\,\omega_n\tau \tag{8.9}$$

$$A_2(\zeta, \omega_n\tau) = 1 - \frac{e^{-2\zeta\omega_n\tau}}{1-\zeta^2}\left[1 + \zeta \sin\left(2\sqrt{1-\zeta^2}\,\omega_n\tau - \varphi_\zeta\right)\right] \tag{8.10}$$

$$A_2(\zeta, \omega_n\tau) = 1 - e^{-2\zeta\omega_n\tau} - \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-2\zeta\omega_n\tau}\left(\sin 2\sqrt{1-\zeta^2}\,\omega_n\tau + \frac{2\zeta}{\sqrt{1-\zeta^2}} \sin^2 \sqrt{1-\zeta^2}\,\omega_n\tau\right) \tag{8.11}$$

---

[1] From the facts that the existence of $E\left[\dot{S}^2\right]$ implies the existence of ψ_o'', that for any stationary process $\psi_o' = 0$, and that the operation $\left.\frac{d}{d\tau}\right|_{\tau=0}$ on the right hand side of Eq. 8.2 gives zero, it can be shown that if $E\left[\dot{S}^2\right]$ exists the operation $\left.\frac{d^2}{d\tau^2}\right|_{\tau=0}$ on the right hand side of Eq. 8.2 gives the true value of ψ_o'' even though Eq. 8.2 gives ψ_τ only for $\tau \geqslant 0$.

$$A_4(\xi, \omega_n\tau) = \frac{e^{-\xi\omega_n\tau}}{\sqrt{1-\xi^2}}\left[\cos(\sqrt{1-\xi^2}\,\omega_n\tau + \varphi_\xi) - e^{-2\xi\omega_n\tau}\cos(\sqrt{1-\xi^2}\,\omega_n\tau - \varphi_\xi)\right] \quad (8.12)$$

$$A_4(\xi, \omega_n\tau) = e^{-\xi\omega_n\tau}\left[(1-e^{-2\xi\omega_n\tau})\cos\sqrt{1-\xi^2}\,\omega_n\tau - \frac{\xi}{\sqrt{1-\xi^2}}(1+e^{-2\xi\omega_n\tau}) \times \sin\sqrt{1-\xi^2}\,\omega_n\tau\right] \quad (8.13)$$

$$\varphi_\xi = \tan^{-1}\frac{\xi}{\sqrt{1-\xi^2}}.$$

Furthermore, we shall use the abbreviation

$$\frac{\Lambda_{24}^{\;2}}{\Lambda_{22}^{\;2}} = \frac{A_4^2(\xi, \omega_n\tau)}{A_2^2(\xi, \omega_n\tau)} = A_3(\xi, \omega_n\tau) \quad (8.14)$$

Substituting Eq. 8.4 into Eqs. 7.1 and 7.2 gives

$$I_2 = \frac{\omega_n}{2\pi}\left(\frac{2\Psi_0}{S_1^{\;2}}\right)^{\alpha/2}\Gamma\left(1+\frac{\alpha}{2}\right) \quad (8.15)$$

$$I_1 = \frac{\omega_n}{4\pi}\left(\frac{2\Psi_0}{S_1^{\;2}}\right)^{\alpha}\Gamma(1+\alpha) \quad (8.16)$$

while substituting Eqs. 8.4, 8.5, 8.6 and 8.14 into Eq. 7.15 gives

$$I_3(\tau) = \frac{\omega_n^{\;2}}{4\pi^2}\left(\frac{2\Psi_0}{S_1^{\;2}}\right)^{\alpha}\Gamma^2\left(1+\frac{\alpha}{2}\right) \times \frac{A_2^{\alpha+\frac{3}{2}}(\xi, \omega_n\tau)}{A_2^{\alpha+2}(\xi, \omega_n\tau)} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi\omega_n\tau)\right)$$

$$\alpha = \text{odd integer}, \quad \tau \geq 0. \quad (8.17)$$

Substituting Eqs. 8.15, 8.16 and 8.17 into Eq. 4.23 and performing the integrations over τ will give an exact expression for $\sigma_{D_T}^{\;2}$ in terms of the time during which damage is incurred, T, the mean square stress, Ψ_0, the parameters characterizing the system, ω_n and ξ and the parameters characterizing the S-N curve, S_1 and α. Explicit dependence of the results on ω_n can be eliminated by defining a dimensionless time $t' = \omega_n t$ with the dimensionless variable of integration becoming

$$\tau' = \omega_n\tau \quad (8.18)$$

and the dimensionless period during which damage is incurred

$$T' = \omega_n T \tag{8.19}$$

Thus, substituting Eqs. 8.15, 8.16 and 8.17 into Eq. 4.23 and using τ' as the new variable of integration and T' as the dimensionless period during which damage is incurred we have for the variance of $D_{T'}$;

$$\sigma_{D_{T'}}^2 = \frac{1}{2\pi^2}\left(\frac{2\psi_0}{S_1^2}\right)^{\alpha}\Bigg(T'\Big\{\frac{\pi}{2}\Gamma(1+\alpha) + \Gamma^2\left(1+\frac{\alpha}{2}\right)\int_0^{T'}\left[\frac{A_1^{\alpha+\frac{3}{2}}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')}\,F\left(1+\frac{\alpha}{2},1+\frac{\alpha}{2};\frac{1}{2};A_3(\xi,\tau')\right)-1\right]d\tau'\Big\}$$
$$-\Gamma^2\left(1+\frac{\alpha}{2}\right)\int_0^{T'}\tau'\left[\frac{A_1^{\alpha+\frac{3}{2}}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')}\,F\left(1+\frac{\alpha}{2},1+\frac{\alpha}{2};\frac{1}{2};A_3(\xi,\tau')\right)-1\right]d\tau'\Bigg)$$

α = odd integer .

(8.20)

Furthermore, from Eqs. 7.1, 8.15 and 8.19

$$E\left[D_{T'}\right] = \frac{T'}{2\pi}\left(\frac{2\psi_0}{S_1^2}\right)^{\frac{\alpha}{2}}\Gamma\left(1+\frac{\alpha}{2}\right) \tag{8.21}$$

Hence the ratio $\sigma_{D_{T'}} / E\left[D_{T'}\right]$ is independent of ψ_0 and S_1, and depends explicitly only on T', α and ξ .

Examination of the expressions for A_1, A_2 and A_4, Eqs. 8.8 to 8.13, makes the possibility of performing exactly the integrations in Eq. 8.20 appear discouraging. However in order to be able to use S-N fatigue data to evaluate the fatigue damage increments it has been necessary to restrict the admissible ensembles of stress histories to those displaying the usual narrow and the dimensionless period during which damage is incurred

$$T' = \omega_n T \tag{8.19}$$

Thus, substituting Eqs. 8.15, 8.16 and 8.17 into Eq. 4.23 and using τ' as the new variable of integration and T' as the dimensionless period during which damage is incurred we have for the variance of D_T;

$$\sigma_{D_{T'}}^2 = \frac{1}{2\pi^2} \left(\frac{2\psi_0}{S_1^2} \right)^{\alpha} \Bigg(T' \Big\{ \frac{\pi}{2} \Gamma(1+\alpha) + + \Gamma^2 \left(1+\frac{\alpha}{2}\right) \int_0^{T'} \left[\frac{A_1^{\alpha+\frac{3}{2}}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')\right) - 1 \right] d\tau' \Big\}$$
$$- \Gamma^2 \left(1+\frac{\alpha}{2}\right) \int_0^{T'} \tau' \left[\frac{A_1^{\alpha+\frac{3}{2}}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')\right) - 1 \right] d\tau' \Bigg)$$

$$\alpha = \text{odd integer} \tag{8.20}$$

Furthermore, from Eqs. 7.1, 8.15 and 8.19

$$E\left[D_{T'}\right] = \frac{T'}{2\pi} \left(\frac{2\psi_0}{S_1^2} \right)^{\frac{\alpha}{2}} \Gamma\left(1+\frac{\alpha}{2}\right) \tag{8.21}$$

Hence the ratio $\sigma_{D_{T'}} / E\left[D_{T'}\right]$ is independent of ψ_0 and S_1, and depends explicitly only on T', α and ξ .

Examination of the expressions for A_1, A_2 and A_4, Eqs. 8.8 to 8.13, makes the possibility of performing exactly the integrations in Eq. 8.20 appear discouraging. However in order to be able to use S-N fatigue data to evaluate the fatigue damage increments it has been necessary to restrict the admissible ensembles of stress histories to those displaying the usual narrow band characteristics. For stress histories generated by single degree-of-freedom systems this restriction is satisfied by requiring the system dampings, ζ to be small, say $\zeta \leqslant .03$. Thus there is no loss in generality if by assuming $\zeta \ll 1$ we can obtain adequate approximations to the integrations. Assuming $\zeta \ll 1$ we have been able to approximate, analytically, the integrations in Eq. 8.20. In this section we merely state the small ζ approximations for A_1, A_2 and A_4, and then carry through with the integrations without much discussion of theerrors involved. A fairly detailed discussion of the approximations is given in Appendix E. In addition, for a typical set of parameters, $\alpha = 9$ and $\zeta = 1/60$, we carryout there a numerical integration using the exact expressions for A_1, A_2 and A_4, and compare the results with the analytical results obtained by assuming $\zeta \ll 1$. The agreement is very good.

From Eqs. 8.9, 8.11 and 8.13 obvious small ζ approximations exist for A_1, A_2 and A_4 except near $\omega_n \tau = \tau' = 0$. The approximations we make are

$$A_1(\zeta, \tau') \approx (1 - e^{-2\zeta\tau'})^2 \tag{8.22}$$

$$A_2(\zeta, \tau') \approx 1 - e^{-2\zeta\tau'} \tag{8.23}$$

$$A_4(\zeta, \tau') \approx e^{-\zeta\tau'}(1 - e^{-2\zeta\tau'}) \cos \tau' \tag{8.24}$$

all for $\zeta \ll 1$, $\tau' \geqslant \frac{\pi}{2}$.

We shall give separate consideration to the interval $0 \leqslant \tau' \leqslant \frac{\pi}{2}$.

Using Eq. 8.14 these approximations give

$$\frac{A_1^{\frac{\alpha+3}{2}}(\zeta, \tau')}{A_2^{\alpha-2}(\zeta, \tau')} F\left(1 + \frac{\alpha}{2}, 1 + \frac{\alpha}{2}; \frac{1}{2}; A_3(\zeta, \tau')\right) - 1 \approx$$

$$\approx (1 - e^{-2\zeta\tau'})^{\alpha+1} F\left(1 + \frac{\alpha}{2}, 1 + \frac{\alpha}{2}; \frac{1}{2}; e^{-2\zeta\tau'} \cos^2 \tau'\right) - 1 \tag{8.25}$$

for $\zeta \ll 1$, $\tau' \geqslant \frac{\pi}{2}$.

The hypergeometric function can be defined as

$$F(a\,;\,b\,;\,c\,;\,Z)=1+\frac{ab}{c}\;\frac{Z}{1!}\;+\;\frac{a(a+1)b(b+1)}{c(c+1)}\;\frac{Z^2}{2!}\;+\;\cdots \qquad (8.26)$$

Thus the hypergeometric function in the right hand side of Eq. 8.25 has the general appearance given in Fig. 8.1.

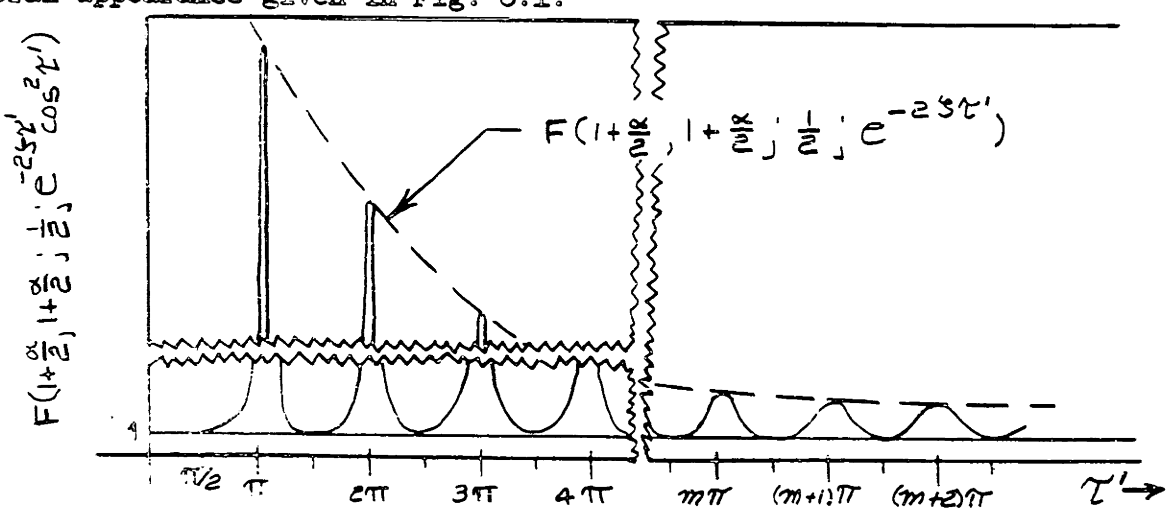

Figure 8.1 Sketch of the hypergeometric function appearing in the right hand side of Eq. 8.25.

We shall treat the first integral appearing in Eq. 8.20 first. The small ζ approximation of the integrand is given by the right hand side of Eq. 8.25 for $\tau' \geq \frac{\pi}{2}$. For $\zeta << 1$ the variations of $(1-e^{-2\zeta\tau'})^{\alpha+1}$ and the envelope of $F(1+\frac{\alpha}{2},\ 1+\frac{\alpha}{2}\,;\ \frac{1}{2}\,;\ e^{-2\zeta\tau'}\cos^2\tau')$ over any interval $\frac{n\pi}{2} \leq \tau' \leq \frac{(n+1)\pi}{2}$, $n=1,2,3\ldots$ are small in comparison to the variation of $F(1+\frac{\alpha}{2};\ 1+\frac{\alpha}{2}\,;\ \frac{1}{2}\,;\ e^{-2\zeta\tau'}\cos^2\tau')$ over the same interval (see fig. 8.1). The technique of approximate integration we use is the following. The integration is performed in two steps. In the first step we assume the variation of $e^{-2\zeta\tau'}$ in any interval $\frac{n\pi}{2} \leq \tau' \leq \frac{(n+1)\pi}{2}$, $n=1,2,3\ldots$, is negligible. With this assumption we perform the integration over τ between the limits $\tau'=\frac{n\pi}{2}$ and $\tau'=\frac{(n+1)\pi}{2}$, any $n=1,2,3\ldots$.

This would give us an approximation of the contribution to the integral for the interval $\frac{n\pi}{2} \leq \tau' \leq \frac{(n+1)\pi}{2}$ if $e^{-2\xi\tau'}$ (assumed constant) were evaluated by assigning to it some value of τ' within the same interval $\frac{n\pi}{2} \leq \tau' \leq \frac{(n+1)\pi}{2}$. Dividing this approximation of the contribution to the integral by thelength of the interval, $\pi/2$, would then give an approximation of the average value of the integrand over the interval $\frac{n\pi}{2} \leq \tau' \leq \frac{(n+1)\pi}{2}$. In this manner for each $n=1,2,3$ . . . an approximation to the average value of the integrand over the intervals $\frac{n\pi}{2} \leq \tau' \leq \frac{(n+1)\pi}{2}$ can be obtained. However, instead of assigning a value to $e^{-2\xi\tau'}$ for each interval corresponding to the values of $n=1,2,3$ . . . , we have left $e^{-2\xi\tau'}$ "free." The approximation to the average value of the integrand for each interval $\frac{n\pi}{2} \leq \tau' \leq \frac{(n+1)\pi}{2}$ then can be considered as a function of τ' (the τ' appearing in $e^{-2\xi\tau'}$) over this interval. For each of the intervals corresponding to $n=1,2,3$ . . . this function has the same form. Thus we obtain a single function which gives a continuous approximation to the average value of the integrand for all $\tau' \geq \pi/2$. Integrating this approximation between the limits $\tau'=\pi/2$ and $\tau' = T'$ then gives the small ξ approximation we have obtained for the first integral appearing in Eq. 8.20 over the range $\pi/2 \leq \tau' \leq T'$. Our motivation for this procedure is simple. It can be carried out analytically with reasonable effort and gives answers in a convenient form which agree very well (at least for the typical case $\alpha = 9$, $\xi = 1/60$) with numerical integrations using the exact expression for the integrand.

We now proceed with the integrations. Define

$$B = e^{-2\xi\tau'} \tag{8.27}$$

From Eqs. 8.20, 8.25, 8.27 and the above discussion the approximate average value of the first integrand in Eq. 8.20 is given by

$$I_4(B) = \frac{2}{\pi}\int_{\frac{n\pi}{2}}^{\frac{(n+1)\pi}{2}} \left[(1-B)^{\alpha+1} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau'\right) -1\right] d\tau' \qquad (8.28)$$

$$n=1,2,3 \ldots$$

where B is held constant in the integration. If we define

$$I_5(B) = \int_{\frac{n\pi}{2}}^{\frac{(n+1)\pi}{2}} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau'\right) d\tau' \qquad (8.29)$$

$$n= 1,2,3 \ldots$$

then $I_4(B)$ is given by

$$I_4(B) = \frac{2}{\pi}\,(1-B)^{\alpha+1}\, I_5(B) - 1 \qquad (8.30)$$

Since τ' appears in the integrand of Eq. 8.29 only in $\cos^2\tau'$ which is periodic with period π, it is necessary to consider $I_5(B)$ only over a single period, say $\frac{\pi}{2} < \tau' < \frac{3\pi}{2}$ i.e. n=1 and 2. Furthermore since $\cos^2\tau'$ is an even function about $\tau' = \pi$, the integrand of Eq. 8.29 is also even about $\tau' = \pi$ and $I_5(B)$ has the same form for the intervals $\frac{\pi}{2} \leq \tau' \leq \pi$ and $\pi \leq \tau' \leq \frac{3\pi}{2}$ i.e. n=1 and 2 respectively. Hence from the periodicity of $\cos^2\tau'$ it follows that $I_5(B)$ has the same form for all n=1,2,3 . . . . We evaluate $I_5(B)$ for n=1. Introduce the change of variable

$$x = \cos^2\tau' \qquad (8.31)$$

Then

$$dx = -2\cos\tau' \sin\tau' \; d\tau'$$

or

$$dx = 2x^{1/2}(1-x)^{1/2} d\tau' \qquad \frac{\pi}{2} \leq \tau' \leq \pi \qquad (8.32)$$

Hence, from Eq. 8.29 $I_5(B)$ becomes

$$I_5(B) = \int_0^1 \frac{F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; Bx)}{2x^{1/2}(1-x)^{1/2}} dx \tag{8.33}$$

This expression can be integrated [30] giving

$$I_5(B) = \frac{\pi}{2} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; 1; B) \tag{8.34}$$

Substituting this into Eq. 8.30 and now considering B as a function of τ', Eq. 8.27, gives us I_4, the approximate average value of the first integrand in Eq. 8.20 for $\tau' \geq \frac{\pi}{2}$.

$$I_4 = (1-e^{-2\xi\tau'})^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) - 1 \tag{8.35}$$

Using the transformation, Eq. 7.12, I_4 can be written as

$$I_4 = F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) - 1 \tag{8.36}$$

Hence from the comments preceding Eq. 8.27 the small ξ approximation of the first integral in Eq. 8.20 over the range $\frac{\pi}{2} \leq \tau' \leq T'$ is given by

$$\int_{\pi/2}^{T'} \left[\frac{A_1^{\alpha+3/2}(\xi, \tau')}{A_2^{\alpha+2}(\xi, \tau')} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi, \tau')) - 1 \right] d\tau' \approx$$

$$\approx \int_{\pi/2}^{T'} \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) - 1 \right] d\tau', \quad \xi \ll 1 \tag{8.37}$$

The integral on the right hand side of Eq. 8.37 could be evaluated in a more usable form if the lower limit were zero. For $\xi \ll 1$ and $0 \leqslant \tau' \leqslant \frac{\pi}{2}$ $e^{-2\xi\tau'}$ is approximately unity so that

$$\int_0^{\pi/2} \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}\right) - 1\right] d\tau' \approx$$

$$\approx \int_0^{\pi/2} \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right) - 1\right] d\tau' \qquad (8.38)$$

But

$$F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right) = \frac{\Gamma(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)} \qquad \text{if } \alpha > -1 \qquad (8.39)$$

so from Eq. 8.38

$$\int_0^{\pi/2} \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}\right) - 1\right] d\tau' \approx \frac{\pi}{2}\left[\frac{\Gamma(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)} - 1\right],$$

$$\xi \ll 1. \qquad (8.40)$$

Hence, using Eq. 8.40 to add and subtract the value of the integral for the interval $0 \leqslant \tau' \leqslant \frac{\pi}{2}$ to the right hand side of Eq. 8.37, we have

$$\int_{\pi/2}^{T'} \left[\frac{A_1^{\alpha+3/2}(\xi, \tau')}{A_2^{\alpha+2}(\xi, \tau')} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi, \tau')\right) - 1\right] d\tau' \approx$$

$$\approx \int_0^{T'} \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}\right) - 1\right] d\tau' - \frac{\pi}{2}\left[\frac{\Gamma(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)} - 1\right],$$

$$\xi \ll 1. \qquad (8.41)$$

We shall now consider the contribution to the first integral in Eq. 8.20 for the interval $0 \leq \tau' \leq \frac{\pi}{2}$. It is recalled that no satisfactory small ζ approximations for A_1, A_2 and A_4, Eqs. 8.8 to 8.13, were found for the region near $\tau' = 0$. From Eqs. 8.17 and 8.18 the first integral appearing in Eq. 8.20 can be written as

$$\int_0^{T'} \left[\frac{A_1^{\alpha + 3/2}(\zeta, \tau')}{A_2^{\alpha+2}(\zeta, \tau')} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\zeta, \tau')\right) - 1\right] d\tau' =$$

$$= \left[\frac{\omega_n^2}{4\pi^2}\left(\frac{2\Psi_o}{S_1^2}\right)^{\alpha} \Gamma^2\left(1+\frac{\alpha}{2}\right)\right]^{-1} \int_0^{T'} I_3 \, d\tau' - T' \tag{8.42}$$

and the contribution for the interval $0 \leq \tau' \leq \frac{\pi}{2}$ as

$$\int_0^{\pi/2} \left[\frac{A_1^{\alpha+3/2}(\zeta, \tau')}{A_2^{\alpha+2}(\zeta, \tau')} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\zeta, \tau')\right) - 1\right] d\tau' =$$

$$= \left[\frac{\omega_n^2}{4\pi^2}\left(\frac{2\Psi_o}{S_1^2}\right)^{\alpha} \Gamma^2\left(1+\frac{\alpha}{2}\right)\right]^{-1} \int_0^{\pi/2} I_3 \, d\tau' - \frac{\pi}{2} \tag{8.43}$$

Using physical arguments we show in Appendix F that

$$\int_0^{\pi/2} I_3 \, d\tau' \approx 0 \qquad \text{for } \zeta << 1 \tag{8.44}$$

which is meaningful in the sense that its contribution is neglibible in comparison with the corresponding integral over the remainder of the interval $\frac{\pi}{2} \leq \tau' \leq T'$ i.e.

$$\frac{\int_0^{\pi/2} I_3 \, d\tau'}{\int_{\pi/2}^{T'} I_3 \, d\tau'} <<< 1 \qquad \text{for } \zeta << 1, \quad T' \geq \frac{3\pi}{2}. \tag{8.45}$$

This result was verified by the numerical integration described in Appendix E for the case $\alpha = 9$, $\zeta = 1/60$. Substituting Eq. 8.44 into Eq. 8.43 and adding the resulting equation to Eq. 8.41 gives

$$\int_0^{T'} \left[\frac{A_1^{\alpha+3/2}(\zeta,\tau')}{A_2^{\alpha+2}(\zeta,\tau')} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\zeta,\tau')\right) - 1\right] d\tau' \approx \tag{8.46}$$

$$\approx \int_0^{T'} \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\zeta\tau'}\right) - 1\right] d\tau' - \frac{\pi}{2} \frac{\Gamma(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)}, \quad \zeta \ll 1 .$$

Using Eq. 8.26 we now integrate the right hand side of Eq. 8.46 term by term. The resulting series converges sufficiently fast for convenient numerical evaluation.

$$\int_0^{T'} \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\zeta\tau'}\right) - 1\right] d\tau' =$$

$$= \int_0^{T'} \left[\frac{\left(-\frac{\alpha}{2}\right)^2}{(1!)^2} e^{-2\zeta\tau'} + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2}{(2!)^2} e^{-4\zeta\tau'} + \dots \right.$$

$$\left. \dots + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2 \dots \left(-\frac{\alpha}{2}+n-1\right)^2}{(n!)^2} e^{-2n\zeta\tau'} + \dots \right] d\tau'$$

$$= \frac{1}{2\zeta} \left[\frac{\left(-\frac{\alpha}{2}\right)^2}{1(1!)^2} \left(1 - e^{-2\zeta T'}\right) + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2}{2(2!)^2} \left(1 - e^{-4\zeta T'}\right) + \dots \right.$$

$$\left. \dots + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2 \dots \left(-\frac{\alpha}{2}+n-1\right)^2}{n(n!)^2} \left(1 - e^{-2n\zeta T'}\right) + \dots \right] \tag{8.47}$$

For values of T' larger than say π/ζ the values of the exponentials in the above series are negligible ($e^{-2\pi} \approx .002$), and the effective value of T' in Eq. 8.47 can be taken as infinity. This result is in agreement with the comments at the end of Section 4. Since $T' = \omega_n T$ and from Eq. 8.4 the expected number of cycles per second is $\omega_n/2\pi$, the condition $T' > \pi/\zeta$ is equivalent to specifying that the expected number of cycles occurring during T be greater than $1/2\zeta$. This condition should easily be satisfied in all practical applications. Hence if we define

$$\Phi_1(\alpha) = \frac{(-\frac{\alpha}{2})^2}{1(1!)^2} + \frac{(-\frac{\alpha}{2})^2(-\frac{\alpha}{2}+1)^2}{2(2!)^2} + \ldots$$
$$\ldots + \frac{(-\frac{\alpha}{2})^2(-\frac{\alpha}{2}+1)^2 \ldots (-\frac{\alpha}{2}+n-1)^2}{n(n!)^2} + \ldots \qquad (8.48)$$

then

$$\int_0^{T'} \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\zeta\tau'})-1\right] d\tau' \approx$$
$$\approx \int_0^{\infty} \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}, 1; e^{-2\zeta\tau'})-1\right] d\tau' = \frac{1}{2\zeta}\,\Phi_1(\alpha) \qquad (8.49)$$

if the expected number of cycles during T $\geqslant \frac{1}{2\zeta}$. It is fortunate that this result appears as a product of a function of ζ times a function of α.

We now consider the second integral appearing in Eq. 8.20. The technique of integration we use is very similar to that used for the first integral. The integrands of the first and second integrals in Eq. 8.20 are identical except for the τ' multiplier in the second. We use the same small ζ approximations for A_1, A_2 and A_4, Eqs. 8.22 to 8.24 for the interval $\frac{\pi}{2} < \tau' < T'$. In obtaining the "approximate average value of the integrand," analogous to Eq. 8.28 for the first integral, we assume that the τ' multiplier is constant in addition to the assumption that $e^{-2\xi\tau'}$ is constant. Hence it follows immediately from Eqs. 8.20 and 8.36 or 8.37 that our small ξ approximation of the second integral appearing in Eq. 8.20 for the interval $\frac{\pi}{2} \leq \tau' \leq T'$ is given by

$$\int_{\pi/2}^{T'} \tau' \left[\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')) -1 \right] d\tau' \approx$$

$$\approx \int_{\pi/2}^{T'} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) -1 \right] d\tau' \quad , \xi << 1. \tag{8.50}$$

At first glance the assumption that the τ' multiplier $\approx$ constant in obtaining the integrand in the right hand side of the above equation would appear to lead to considerably more error than the $e^{-2\xi\tau'} \approx$ constant assumption. However, using the small ξ approximations for A_1, A_2 and A_4 we show in Appendix G that as $\xi \to 0$ the results given by our approximate method of integration approach the exact answers for each interval $(m-1/2)\pi \leq \tau' \leq (m+1/2)\pi$, m=1,2,3 . . .

Following the technique used for the first integral in Eq. 8.20, for $\xi << 1$ and $0 \leq \tau' \leq \frac{\pi}{2}$, $e^{-2\xi\tau'} \approx 1$, so that

$$\int_0^{\pi/2} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) -1 \right] d\tau' \approx$$

$$\approx \int_0^{\pi/2} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1, 1) -1 \right] d\tau' \quad , \xi << 1 \tag{8.51}$$

But $F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1) = \frac{\Gamma(1+\alpha)}{\Gamma^2(1+\frac{\alpha}{2})}$ if $\alpha > -1$

(8.39 repeated).

so from Eq. 8.51

$$\int_0^{\pi/2} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) - 1\right] d\tau' \approx \frac{\pi^2}{8}\left[\frac{\Gamma(1+\alpha)}{\Gamma^2(1+\frac{\alpha}{2})} - 1\right]$$

$\xi << 1.$ (8.

Using Eq. 8.52 to add and subtract the integral for the interval $0 \leq \tau' \leq \frac{\pi}{2}$ to the right hand side of Eq. 8.50, we have

$$\int_{\pi/2}^{T'} \tau' \left[\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')) - 1\right] d\tau' \approx$$

$$\approx \int_0^{T'} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) - 1\right] d\tau' - \frac{\pi^2}{8}\left[\frac{\Gamma(1+\alpha)}{\Gamma^2(1+\frac{\alpha}{2})} - 1\right],$$

$\xi << 1.$ (8.

From Eqs. 8.17, 8.44 and the fact that I_3 is always non-negative (see appendix F, Eq. F.1 and the discussion that follows) it follows that

$$\int_0^{\pi/2} \tau' \left[\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')) - 1\right] d\tau' \approx$$

$$\approx -\int_0^{\pi/2} \tau' d\tau' = -\frac{\pi^2}{8}, \quad \xi << 1.$$ (8.

Finally, adding Eqs. 8.53 and 8.54 gives

$$\int_0^{T'} \tau' \left[\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F\left(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')\right) -1 \right] d\tau' \approx$$

$$\approx \int_0^{T'} \tau' \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}\right) -1 \right] d\tau' - \frac{\pi^2}{8} \frac{\Gamma(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)},$$

$$\xi \ll 1. \qquad (8.55)$$

Using Eq. 8.47 we write the integrand appearing in the right hand side of Eq. 8.55 in series form and then integrate term by term.

$$\int_0^{T'} \tau' \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}\right) -1 \right] d\tau' =$$

$$= \int_0^{T'} \left[\frac{\left(-\frac{\alpha}{2}\right)^2}{(1!)^2} \tau' e^{-2\xi\tau'} + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2}{(2!)^2} \tau' e^{-4\xi\tau'} + \ldots \right.$$

$$\left. \ldots + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2 \ldots \left(-\frac{\alpha}{2}+n-1\right)^2}{(n!)^2} \tau' e^{-2n\xi\tau'} + \ldots \right] d\tau'$$

$$= \frac{1}{4\xi^2} \left\{ \frac{\left(-\frac{\alpha}{2}\right)^2}{1^2(1!)^2} \left[1 - e^{-2\xi T'}(1+2\xi T') \right] + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2}{2^2(2!)^2} \left[1 - e^{-4\xi T'} \right. \right.$$

$$\left. (1+4\xi T') \right] + \ldots$$

$$\left. \ldots + \frac{\left(-\frac{\alpha}{2}\right)^2 \left(-\frac{\alpha}{2}+1\right)^2 \ldots \left(-\frac{\alpha}{2}+n-1\right)^2}{n^2 (n!)^2} \left[1 - e^{-2n\xi T'}(1+2n\xi T') \right] + \ldots \right\} \qquad (8.56)$$

For values of T' larger than π / ξ the effective value of T' in Eq. 8.56 can again be taken as infinity although the approximation here is not quite as good as in Eq. 8 47 ($e^{-2\pi}[1+2\pi] \approx .01$). Hence if we define

$$\Phi_2(\alpha) = \frac{(-\frac{\alpha}{2})^2}{1^2\,(1!)^2} + \frac{(-\frac{\alpha}{2})^2(-\frac{\alpha}{2}+1)^2}{2^2\,(2!)^2} + \dots + \frac{(-\frac{\alpha}{2})^2(-\frac{\alpha}{2}+1)^2\dots(-\frac{\alpha}{2}+n-1)^2}{n^2\,(n!)^2} + \dots \tag{8.57}$$

then

$$\int_0^{T'} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\zeta\tau'}) - 1\right] d\tau' \approx$$

$$\approx \int_0^{\infty} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\zeta\tau'}) - 1\right] d\tau' = \frac{1}{4\zeta^2}\,\Phi_2(\alpha), \tag{8.58}$$

if the expected number of cycles during $T \geqslant \frac{1}{2\zeta}$.

We now combine these results to get our approximation of $\sigma_{D_{T'}}^2$. Substituting Eqs. 8.49 and 8.58 into Eqs. 8.46 and 8.55 respectively and the results into Eq. 8.20 gives us $\sigma_{D_{T'}}^2$.

$$\sigma_{D_{T'}}^2 \approx \frac{1}{4\pi^2} \left(\frac{2\psi_0}{s_1^2}\right)^{\alpha} \Gamma^2(1+\frac{\alpha}{2}) \times \tag{8.59}$$

$$\left[\frac{T'}{\zeta}\,\Phi_1(\alpha) - \frac{\Phi_2(\alpha)}{2\zeta^2} + \frac{\pi^2}{4}\,\frac{\Gamma(1+\alpha)}{\Gamma^2(1+\frac{\alpha}{2})}\right]$$

and from Eq. 8.21

$$\frac{\sigma_{D_{T'}}}{E\left[D_{T'}\right]} \approx \frac{1}{\sqrt{\zeta T'}} \left\{\Phi_1(\alpha) - \frac{1}{2\zeta T'}\left[\Phi_2(\alpha) - \zeta^2\frac{\pi^2}{2}\,\frac{\Gamma(1+\alpha)}{\Gamma^2(1+\frac{\alpha}{2})}\right]\right\}^{\frac{1}{2}} \tag{8.60}$$

Eqs. 8.59 and 8.60 applying if $\zeta \ll 1$, α = an odd integer and the expected number of cycles in $T \geqslant \frac{1}{2\zeta}$. If in addition $\zeta T' \gg 1$, then $\sigma_{D_{T'}}/E\left[D_{T'}\right]$

becomes

$$\frac{\sigma_{D_{T'}}}{E\left[D_{T'}\right]} \approx \left[\frac{\Phi_1(\alpha)}{\zeta T'}\right]^{1/2} \qquad (8.61)$$

if $\zeta \ll 1$, $\zeta T' \gg 1$, and α = an odd integer, where $T' = \omega_n T$. Eqs. 8.59, 8.60 and 8.61 are the principal results of Section 8 and may be considered the principal "practical" results of this thesis. Table 8.1 lists values of $\Phi_1(\alpha)$ and $\Phi_2(\alpha)$ which have been computed from the series Eq. 8.48 and 8.57 respectively, and values of $\pi\Gamma(1+\alpha)/\Gamma^2(1+\frac{\alpha}{2})$ computed from the relation $\Gamma(1+n) = n!$ and the duplication formula [31], $\Gamma(1/2)\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+1/2)$.

Table 8.1 Functions occurring in Eqs. 8.60 and 8.61.

α	$\Phi_1(\alpha)$	$\Phi_2(\alpha)$	$\pi\Gamma(1+\alpha)/\Gamma^2(1+\frac{\alpha}{2})$
1	.260	.255	4
3	2.32	2.29	10.67
5	8.04	7.14	34.13
7	23.4	17.6	117.0
9	67.1	40.9	416.1
11	198.1	97.1	1,513
13	608	242	5,587
15	1,940	640	20,860
17	6,350	1,790	78,520
19	21,300	5,220	297,600
21	72,800	15,800	1,134,000
23	253,000	49,400	4,337,000
25	889,000	158,000	16,660,000

Using the relation

$$T' = 2\pi \text{ (expected number of cycles occurring during T)} \qquad (8.62)$$

obtained from $T' = \omega_n T$ and Eq. 8.4 we have plotted in Fig. 8.2 the results given by Eq. 8.61 for $\sigma_{D_T} / E\left[D_T\right] = .20$. The expected number of cycles is used as ordinate rather than T because it is more convenient to think in terms of cycles when dealing with fatigue. For the ranges of the parameters used in plotting Fig. 8.2 the errors involved in using Eq. 8.61 as an approximation of Eq. 8.60 are negligible.

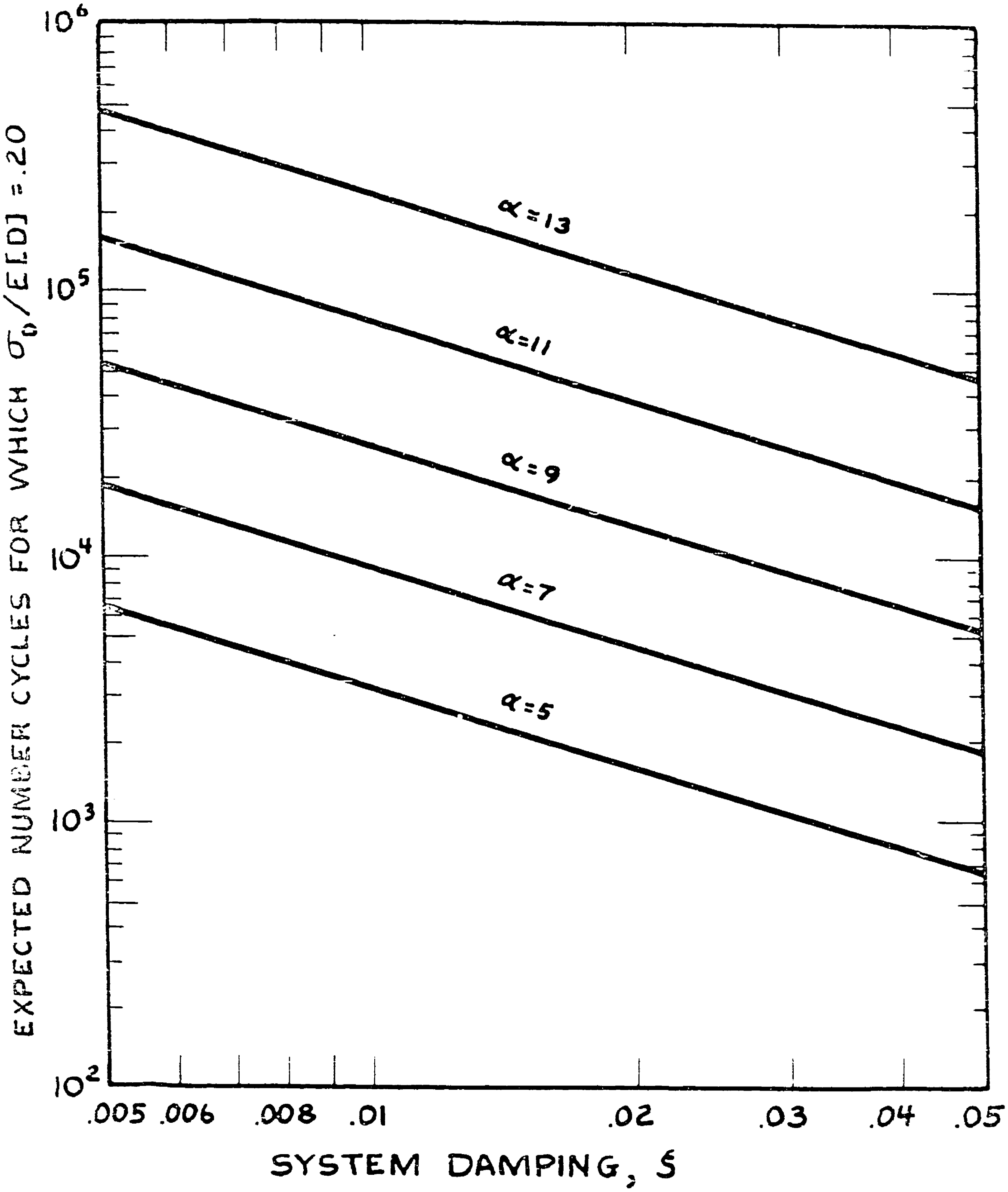

FIGURE 8.2 PLOT OF EQ. 8.61 FOR $\sigma_D/E[D] = .20$ USING EQ. 8.62 TO CONVERT DIMENSIONLESS TIME T' TO CYCLES.

9. ON "EQUIVALENT" SINUSOIDAL STRESS HISTORIES

In this section some aspects of "equivalent" sinusoidal stress amplitudes are discussed in connection with the "probable equivalent stress" introduced by Miles [32] . We begin by defining an "equivalent" sinusoidal stress amplitude, S_e, for a single stress sample function which is analogous to the "probable equivalent stress" of Miles. The "probable equivalent stress" S_T, may be considered as a definition of an "equivalent" sinusoidal stress amplitude for an ensemble of stress histories.

Since each "cycle" of a sample function contains two zero crossings, the mathematical expectation of the number of "cycles" in a time interval T, is $T\,\bar{N}_o/2$. Furthermore, with each stress sample function of length T there is associated a fatigue damage, D_T. Hence, for each stress sample function of length T of a stationary narrow band random process we define an "equivalent" sinusoidal stress amplitude, S_e, as that stress amplitude which will produce the same fatigue damage, D_T, after $T\,\bar{N}_o/2$ cycles. As described in Section 5.1 a damage increment of $1/N(S_e)$ is associated with each cycle of amplitude S_e (or a damage increment of $1/2N(S_e)$ with each half cycle). Thus using the model for S-N curves explained in Section 5.2 (Eq. 5.2), the relation defining S_e is

$$D_T = T\,\frac{\bar{N}_o}{2}\left(\frac{S_e}{S_1}\right)^{\alpha} \tag{9.1}$$

Hence the random variable S_e is a function of the random variable D_T. If the probability density function of D_T were available (we have determined expressions only for the variance of D_T) then the density functions of S_e

could be obtained, using Eq. 9.1, from the density function of D_T.

We now consider the "probable equivalent stress" S_r, introduced by Miles. The "equivalence" of S_r is based on the mathematical expectation of D_T, $E\left[D_T\right]$, which is a number characterizing the ensemble of stress sample functions. That is, using arguements analogous to those used in defning S_e, the "probable equivalent stress," S_r is defined by

$$E\left[D_T\right] = T\,\frac{\overline{N}_o}{2}\left(\frac{S_r}{S_1}\right)^{\alpha} \tag{9.2}$$

Thus $T\,\overline{N}_o/2$ sinusoidal stress cycles of amplitude S_r will produce the mathematical expectation of the fatigue damage incurred in a time interval of length T. Since $E\left[D_T\right]$ is proportional to T (Eq. 3.6), S_r is independent of T. Our method used in defining S_r is somewhat different than that of Miles. To show that the two definitions are equivalent (at least for stationary narrow band Gaussian ensembles of stress histories) we equate the right hand sides of Eq. 6.10 and 9.2 and solve for S_r

$$S_r = (\,2\,\psi_o\,)^{1/2}\left[\Gamma\left(1+\frac{\alpha}{2}\right)\right]^{1/\alpha} \tag{9.3}$$

which agrees with the results of Miles [33].

We shall now discuss some properties of S_e and S_r. From the definition of S_r Eq. 9.2 the damage produced during a time interval of length T by a sinusoidal stress of amplitude S_r and frequency $\overline{N}_o/2$ is $E\left[D_T\right]$. At the end of Section 4 and in particular in Eq. 8.59 we have shown that as $T \rightarrow \infty$ the variance of D_T increases without limit. From this it follows that the expected deviation between the damage produced by an arbitrarily chosen stress sample function and the damage produced by the sinusoidal stress of amplitude S_r and frequency $\bar{N}_0/2$ both during a time interval of length T, increases without limit as $T \rightarrow \infty$. By. Eq. 9.1 we have defined for each sample function an "equivalent" sinusoidal stress amplitude S_e which at a frequency of $\bar{N}_0/2$ will produce during a time interval of length T, the same fatigue damage as that produced by the sample function from which S_e is defined. It then follows that the expected deviation between the damage produced by a sinusoidal stress S_e (determined from an arbitrarily chosen stress sample function) and the damage produced by the sinusoidal stress S_r, both at the frequency $\bar{N}_0/2$ for a time interval of length T, increases without limit as $T \rightarrow \infty$.

It is also of interest to note that for finite time intervals T the mathematical expectation of S_e is in general not equal to S_r. Taking the mathematical expectation of both sides of Eq. 9.1 gives

$$E\left[D_T\right] = T\frac{\bar{N}_0}{2} \quad \frac{E\left[S_e^{\alpha}\right]}{S_1^{\alpha}} \tag{9.4}$$

and equating the right hand sides of Eqs. 9.2 and 9.4 we have

$$E\left[S_e^{\alpha}\right] = S_r^{\alpha} \tag{9.5}$$

However, in general $E\left[S_e^{\alpha}\right] \neq \left\{E\left[S_e\right]\right\}^{\alpha}$ and hence in general $E\left[S_e\right] \neq S_r$.

A special case of Liapounoff's inequality [34] gives

$$\left\{E\left[S_e\right]\right\}^{\alpha} \leqslant E\left[S_e^{\alpha}\right] \qquad \text{for } \alpha \geqslant 1 \tag{9.6}$$

so that from Eq. 9.5

$$E\left[S_e\right] \leqslant S_r \tag{9.7}$$

However, we shall next prove that as $T \rightarrow \infty$ if $\frac{\sigma D_T}{E[D_T]} \rightarrow 0$ then the probability density function of S_e approaches the Dirac delta function $\delta(S_e - S_r)$. We then have

$$\lim_{T\rightarrow\infty} E[S_e] = \int_0^\infty S_e \, \delta(S_e - S_r) \, dS_e = S_r \tag{9.8}$$

so that the equality in Eq. 9.7 is satisfied as $T \rightarrow \infty$.

We shall now show that as $T \rightarrow \infty$ if $\frac{\sigma D_T}{E[D_T]} \rightarrow 0$ then the probability density function of S_e approaches the Dirac delta function $\delta(S_e - S_r)$. The practical consequences of this are that for sufficiently large values of T, the value of the "equivalent" stress, S_e, obtained from a sample function chosen at random will be approximately equal to S_r. If this condition were not satisfied any meaningful practical application of the "probable equivalent stress" S_r, of Miles would be impossible. The plausibility of the condition $\frac{\sigma D_T}{E[D_T]} \rightarrow 0$ as $T \rightarrow \infty$ has been shown at the end of Section 4 for general ergodic narrow band ensembles of stress histories. In particular, we have shown this condition to be satisfied for the ensembles of stress histories of Section 8 which are generated by single degree-of-freedom systems (Eq. 8.61).

Equivalent to the statement that as $T \rightarrow \infty$ the probability density function of S_e approaches the Dirac delta function $\delta(S_e - S_r)$ is the condition

$$\lim_{T\rightarrow\infty} P_r \left[|S_e - S_r| > \epsilon \right] = 0 \tag{9.9}$$

for any $\epsilon > 0$. We now show Eq. 9.9 to be satisfied if $\lim_{T\rightarrow\infty} \frac{\sigma D_T}{E[D_T]} = 0$. The Bienayme-Tchebycheff inequality [35] applied to the random variable D_T gives

$$P_r \left[|D_T - \bar{D}_T| \geq K \, \sigma_{D_T} \right] \leq \frac{1}{K^2} \tag{9.10}$$

for $K > 0$ where

$$\bar{D}_T = E\left[D_T\right] \tag{9.11}$$

Eq. 9.10 implies

$$P_r\left[\left|\frac{D_T}{\bar{D}_T} - 1\right| \geq K \frac{\sigma_{D_T}}{\bar{D}_T}\right] \leq \frac{1}{K^2} \tag{9.12}$$

From Eqs. 9.1 and 9.2

$$\frac{D_T}{\bar{D}_T} = \left(\frac{S_e}{S_r}\right)^{\alpha} \tag{9.13}$$

and substituting this into Eq. 9.12 gives

$$P_r\left[\left|\left(\frac{S_e}{S_r}\right)^{\alpha} - 1\right| \geq K \frac{\sigma_{D_T}}{\bar{D}_T}\right] \leq \frac{1}{K^2} \tag{9.14}$$

But for $\alpha \geq 1$ and $\frac{S_e}{S_r} \geq 0$ (which are always satisfied)

$$\left|\frac{S_e}{S_r} - 1\right| \leq \left|\left(\frac{S_e}{S_r}\right)^{\alpha} - 1\right| \tag{9.15}$$

Hence Eq. 9.14 implies

$$P_r\left[\left|\frac{S_e}{S_r} - 1\right| \geq K \frac{\sigma_{D_T}}{\bar{D}_T}\right] \leq \frac{1}{K^2} \tag{9.16}$$

or

$$P_r\left[\left|S_e - S_r\right| \geq K\, S_r \frac{\sigma_{D_T}}{\bar{D}_T}\right] \leq \frac{1}{K^2} \tag{9.17}$$

Let us choose

$$K = \frac{\epsilon}{S_r} \frac{\bar{D}_T}{\sigma_{D_T}} \qquad ; \qquad \epsilon > 0 \tag{9.18}$$

then

$$\frac{1}{K^2} = \frac{S_r^{\,2}}{\epsilon^2} \left(\frac{\sigma D_T}{\bar{D}_T} \right)^2 \tag{9.19}$$

and Eq. 9.17 is equivalent to

$$P_r \left[|S_e - S_r| \geq \epsilon \right] \leq \frac{S_r^{\,2}}{\epsilon^2} \left(\frac{\sigma D_T}{\bar{D}_T} \right)^2 \tag{9.20}$$

for any $\epsilon > 0$. We recall that S_r is independent of T. Then the condition $\lim_{T \to \infty} \frac{\sigma D_T}{\bar{D}_T} = 0$ implies that subsequent to the choice of any $\epsilon > 0$ and any $\eta > 0$ there exists a number T_1 such that for all $T > T_1$

$$\frac{S_r^{\,2}}{\epsilon^2} \left(\frac{\sigma D_T}{\bar{D}_T} \right)^2 < \eta$$

which completes the proof of Eq. 9.9.

From Eq. 9.14 using the values of $\sigma_{D_T} / E\left[D_T\right]$ we have obtained, one could find very conservative bounds on the probabilities of $|S_e - S_r|$ exceeding specified values. However, these bounds would be much too conservative to be considered as estimates of the actual probabilities. In order to obtain such estimates one could guess a form for the probability density function of D_T depending only on the parameters $E\left[D_T\right]$ and σ_{D_T} values of which we have obtained. From this and Eqs. 9.1 and 9.2 it is then possible to obtain estimates of the probabilities of $|S_e - S_r|$ exceeding specified values.

APPENDIX

A. RELATIONSHIP BETWEEN THE PROBABILITY DENSITY FUNCTION OF THE TIME UNTIL FATIGUE FAILURE AND THE AND THE PROBABILITY DENSITY FUNCTIONS OF THE FATIGUE DAMAGE D_T

Let D_T be the fatigue damage accumulated in the time interval $0 < t < T$, the accumulation being stochastic. Fatigue failure occurs at the time when D_T accumulates to a value of unity. Let $F_D\ (D_T,\ T)$ designate the probability distribution function of D_T. T appears as a parameter in F_D. Then the probability density function of D_T, $f_D\ (D_T,\ T)$ is given by

$$f_D = \frac{\partial}{\partial D_T} F_D(D_T, T) \tag{A.1}$$

Let T_F designate the time of occurrence of fatigue failure, and $F_F(T_F)$ designate the probability distribution function of T_F. Then the probability density function of T_F is given by

$$f_F = \frac{d}{dT_F} F_F(T_F) \tag{A.2}$$

Then by the definition of the probability distribution function

$$F_D\ (1,\ T) = P_r\left[D_T < 1\right] = P_r\left[\text{fatigue failure occurs at } t > T\right]$$

$$= P_r\left[T_F > T\right]$$

$$= 1 - P_r\left[T_F < T\right]$$

But

$$P_r\left[T_F < T\right] = F_F\ (T)$$

hence

$$F_F\ (T_F) = 1 - F_D\ (1, T_F)$$

$$= 1 - \int_0^1 f_D\ (D_T, T_F)\ dD_T$$

and from Eq. A.2

$$f_F = -\frac{d}{dT_F}\int_0^1 f_D \quad (D_T, T_F) \quad dD_T \tag{A.3}$$

which determines the probability density function of the time until fatigue failure from the probability density functions of the fatigue damage D_T, given for all T.

B. DIFFICULTIES IN ASSOCIATING FATIGUE DAMAGE INCREMENTS WITH THE STATIONARY POINTS OF STOCHASTIC STRESS HISTORIES

It was mentioned in Section 2 that it would be desirable to associate the fatigue damage increment of each half cycle of a stochastic stationary narrow band stress history directly with the stationary point (i.e. the maxima or minima) of the half cycle. Implicit in this statement is the assumption that only one stationary point occurs per half cycle. When narrow band noise is viewed on an oscilloscope this indeed appears to be the case. In practically important cases however (e.g. the stress histories of Section 8 which are generated by single degree-of-freedom systems) mathematical analysis of the stress histories predicts the occurrence of a large number of stationary points for each half cycle. Physical arguments lead us to believe that at least for the narrow band stress histories generated by lightly damped linear single degree-of-freedom systems all of the stationary points associated with each half cycle are grouped in a small cluster about the apparent single stationary point per half cycle one views on an oscilloscope. Thus in order to be able to associate the fatigue damage increment of each half cycle with the apparent single stationary point per half cycle it would be necessary to find a means of mathematically characterizing the event which is the occurrence of such a cluster of stationary points. Characterizing this event would consist in determining a value of stress to associate with the cluster and in being able to count such a cluster as a single apparent stationary point for the purposes of fatigue damage. We have not been able to find a meaningful convenient way of mathematically characterizing such a cluster and thus we have resorted to associating the damage increments with the zero crossings, "predicting" the amplitudes of the apparent stationary points from the slopes of the zero crossings.

C. ON "PREDICTION" OF MAGNITUDES OF STATIONARY POINTS FROM SLOPES AT PRECEDING ZERO CROSSINGS

The results of Sections 6,7 and 8 depend on the assumption that the magnitudes of the stationary points are well "predicted" from the slopes at their preceding zero crossings by assuming that for the quarter cycle between each zero crossing and the following stationary point the sample functions are sinusoids of frequency, cycles/second, equal to one half of the expected number of zero crossings per second, $\bar{N}_o$. For the stress sample functions studied in Section 8 which are generated by lightly damped linear single degree-of-freedom systems it was found that $\bar{N}_o = \omega_n/\pi$ (Eq. 8.4) where ω_n is the natural frequency of the undamped system. Thus for these sample functions an assumption equivalent to the above is that for the quarter cycle between each zero crossing and the following stationary point the (stress) response of the system is assumed to be the response of the undamped system with no excitation, the initial conditions being specified by the state of the system at the zero crossings. This interpretation possesses the advantage of having an obvious extension to cases of stress sample functions which are generated by lightly damped nonlinear single degree-of-freedom systems. In the nonlinear case, however, the elapsed time between a zero crossing and the "predicted" stationary point will, in general, depend on the slope at the zero crossing.

D. EVALUATION OF $\lim_{\tau \to 0+} I_3(\tau)$ FOR GAUSSIAN ENSEMBLES OF STRESS HISTORIES (Eq. 7.15)

In this appendix we show that for stationary Gaussian ensembles of stress histories for which $E\left[\dot{S}^2\right] = -\psi_0''$ exists

$$\lim_{\tau \to 0+} I_3(\tau) = 0 \qquad \text{(7.18 repeated)}$$

if the autocorrelation function, $\psi(\tau)$ can be expressed as a power series in τ valid in some interval $0 \leq \tau \leq \epsilon$, $\epsilon > 0$, that is,

$$\psi(\tau) = \psi_0 + \frac{\psi_0''}{2}\tau^2 + \sum_{n=3}^{\infty} \frac{C_n}{n!}\tau^n \qquad \text{(D.1)}$$

and if $C_3 \neq 0$. These conditions are satisfied for the ensembles of stress histories considered in Section 8.

From Eq. 7.15, $I_3(\tau)$ is given by

$$I_3(\tau) = \frac{1}{4\pi^2}\left(\frac{2}{S_1^2\,\pi^2\,\bar{N}_0^2}\right)^{\alpha}\Gamma^2\left(1+\frac{\alpha}{2}\right) \times \frac{|\Lambda|^{\alpha+\frac{3}{2}}}{\Lambda_{22}^{\alpha+2}}\,F\left(1+\frac{\alpha}{2},\,1+\frac{\alpha}{2};\,\frac{1}{2};\,\frac{\Lambda_{24}^2}{\Lambda_{22}^2}\right) \qquad \text{(7.15 repeated)}$$

$$\alpha = \text{odd integer}$$

where $|\Lambda|$, Λ_{22} and Λ_{24} are given in terms of the autocorrelation function by Eq. 7.4 and $\bar{N}_0$ by Eq. 6.7. Hence if the limit

$$\lim_{\tau \to 0+} F\left(1+\frac{\alpha}{2},\,1+\frac{\alpha}{2};\,\frac{1}{2};\,\frac{\Lambda_{24}^2}{\Lambda_{22}^2}\right) \qquad \text{(D.2)}$$

is finite and if

$$\lim_{\tau \to 0+} \frac{|\Lambda|^{\alpha+3/2}}{\Lambda_{22}^{\alpha+2}} = 0 \qquad (D.3)$$

then Eq. 7.18 is satisfied.

We first show that the limit indicated by Eq. D.2 is finite. Since the hypergeometric function in Eq. D.2 converges for all $(\Lambda_{24}^2 / \Lambda_{22}^2) < 1$ in order that the limit, Eq. D.2, be finite it is sufficient that

$$\lim_{\tau \to 0+} \frac{\Lambda_{24}^2}{\Lambda_{22}^2} < 1 \qquad (D.4)$$

We shall now show that

$$\lim_{\tau \to 0+} \frac{\Lambda_{24}}{\Lambda_{22}} = \frac{1}{2} \qquad (D.5)$$

thereby satisfying the condition on the hypergeometric function. Expressions for Λ_{22} and Λ_{24} in terms of $\psi(\tau)$ and its derivatives are given in Eq. 7.4. Since $\psi_0' = 0$ for stationary processes (as indicated in Eq. D.1) repeated differentiation of Λ_{22} and Λ_{24} (as given by Eq. 7.4) yields

$$\Lambda_{22}\Big|_{\tau=0} = \frac{d\Lambda_{22}}{d\tau}\Big|_{\tau=0} = \frac{d^2\Lambda_{22}}{d\tau^2}\Big|_{\tau=0} = 0 \qquad (D.6)$$

$$\Lambda_{24}\Big|_{\tau=0} = \frac{d\Lambda_{24}}{d\tau}\Big|_{\tau=0} = \frac{d^2\Lambda_{24}}{d\tau^2}\Big|_{\tau=0} = 0 \qquad (D.7)$$

The derivatives $\frac{d^n\psi}{d\tau^n}\Big|_{\tau=0}$ which occur in the determination of Eqs. D.6 and D.7 are to be interpreted as the coefficients C_n in Eq. D.1. Differentiation of Λ_{22} and Λ_{24} once more gives

$$\left.\frac{d^3 \Lambda_{22}}{d\tau^3}\right|_{\tau=0} = -4\Psi_0 \Psi_0'' \Psi_0''' = -4\Psi_0 \Psi_0'' c_3 \tag{D.8}$$

and

$$\left.\frac{d^3 \Lambda_{24}}{d\tau^3}\right|_{\tau=0} = -2\Psi_0 \Psi_0'' \Psi_0''' = -2\Psi_0 \Psi_0'' c_3 \tag{D.9}$$

Hence repeated use of L'Hospital's rule gives

$$\lim_{\tau \to 0+} \frac{\Lambda_{24}}{\Lambda_{22}} = \frac{1}{2} \qquad \text{if } c_3 \neq 0 \tag{D.10}$$

The limit in Eq. D.10 is indicated as a right hand limit in view of the interpretation of the derivatives $\left.\frac{d^n \Psi}{d\tau^n}\right|_{\tau=0}$ as the coefficients c_n in the series representation of $\Psi(\tau)$ (Eq. D.1) which is assumed to represent $\Psi(\tau)$ only in the positive interval $0 \leq \tau \leq \epsilon$, $\epsilon > 0$.

We shall now show Eq. D.3 to be satisfied. From the expressions for Λ_{22} and $|\Lambda|$ in terms of $\Psi(\tau)$ and its derivatives given in Eq. 7.4 and the assumption of the existence of the series representation of $\Psi(\tau)$ given by Eq. D.1 it follows that Λ_{22} and $|\Lambda|$ may also be expressed as power series within the same interval $0 \leq \tau \leq \epsilon$ as Eq. D.1, say

$$\Lambda_{22} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \tau^n \tag{D.11}$$

and

$$|\Lambda| = \sum_{n=0}^{\infty} \frac{b_n}{n!} \tau^n \tag{D.12}$$

for $0 \leqslant \tau \leqslant \epsilon$. From Eqs. 7.4 and D.1 the coefficients a_n and b_n could be expressed in terms of ψ_0, ψ_0'' and the coefficients C_n however this will not be necessary here. From Eqs. D.6, D.8 and D.11 it follows that for small $\tau \geqslant 0, \Lambda_{22}$ is given by

$$\Lambda_{22} = -\frac{2}{3}\psi_0 \psi_0'' c_3 \tau^3 + 0(\tau^4) \qquad \tau \geqslant 0. \tag{D.13}$$

Using $\psi_0' = 0$, repeated differentiation of $|\Delta|$ (as given by Eq. 7.4) yields

$$|\Delta|\Big|_{\tau=0} = \frac{d|\Delta|}{d\tau}\Big|_{\tau=0} = \frac{d^2|\Delta|}{d\tau^2}\Big|_{\tau=0} = \frac{d^3|\Delta|}{d\tau^3}\Big|_{\tau=0} = 0 \tag{D.14}$$

and

$$\frac{d^4|\Delta|}{d\tau^4}\Big|_{\tau=0} = -8\psi_0 \psi_0'' \psi_0'''^2$$
$$= -8\psi_0 \psi_0'' c_3^2 \tag{D.15}$$

where as before the derivatives $\frac{d^n\psi}{d\tau^n}\Big|_{\tau=0}$ occurring in the determination of Eqs. D.14 and D.15 are to be interpreted as the coefficients C_n in Eq. D.1. Then from Eqs. D.12, D.14 and D.15 it follows that for small $\tau \geqslant 0, |\Delta|$ is given by

$$|\Delta| = -\frac{1}{3}\psi_0 \psi_0'' c_3^2 \tau^4 + 0(\tau^5) \qquad \tau \geqslant 0. \tag{D.16}$$

This result can also be obtained using the relation [36]

$$|\Delta| = \frac{\Lambda_{22}^2 - \Lambda_{24}^2}{\psi_0^2 - \psi_\tau^2} \tag{D.17}$$

Hence, from Eqs. D.13 and D.16 we have

$$\frac{|\Lambda|^{\alpha+3/2}}{\Lambda_{22}^{\alpha+2}} = \frac{\left[-\frac{1}{3}\psi_0\psi_0'' c_3^2\ \tau^4 + 0(\tau^5)\right]^{\alpha+3/2}}{\left[-\frac{2}{3}\psi_0\psi_0'' c_3\ \tau^3 + 0(\tau^4)\right]^{\alpha+2}}$$

$$= \frac{\left\{\tau^4\left[-\frac{1}{3}\psi_0\psi_0'' c_3^2 + 0(\tau)\right]\right\}^{\alpha+3/2}}{\left\{\tau^3\left[-\frac{2}{3}\psi_0\psi_0'' c_3 + 0(\tau)\right]\right\}^{\alpha+2}}$$

$$= \tau^{\alpha}\frac{\left[-\frac{1}{3}\psi_0\psi_0'' c_3^2 + 0(\tau)\right]^{\alpha+3/2}}{\left[-\frac{2}{3}\psi_0\psi_0'' c_3 + 0(\tau)\right]^{\alpha+2}} \tag{D.18}$$

and hence

$$\lim_{\tau\to 0+} \frac{|\Lambda|^{\alpha+3/2}}{\Lambda_{22}^{\alpha+2}} = 0$$

if $c_3 \neq 0$. Thus for the conditions specified Eq. 7.18 is satisfied.

E. DISCUSSION OF APPROXIMATIONS USED IN SECTION 8 AND COMPARISON OF A NUMERICAL INTEGRATION WITH THE CORRESPONDING APPROXIMATE ANALYTICAL INTEGRATION

First we shall briefly discuss the small ζ approximations, Eqs. 8.22, 8.23 and 8.24, of $A_1(\zeta, \tau')$ $A_2(\zeta, \tau')$ and $A_4(\zeta, \tau')$ Eqs. 8.8 to 8.13 where $\tau' = \omega_n \tau$.

The exact expression for $A_1(\zeta, \tau')$ can be written as

$$A_1(\zeta, \tau') = (1-e^{-2\zeta\tau'})^2 - \frac{4\zeta^2}{1-\zeta^2} e^{-2\zeta\tau'} \sin^2\sqrt{1-\zeta^2}\,\tau' \qquad \text{(8.9 repeated)}$$

while the approximation we have used in Section 8 is

$$A_1(\zeta, \tau') \approx (1-e^{-2\zeta\tau'})^2 \qquad \text{(8.22 repeated)}$$

for $\zeta \ll 1$ and $\tau' \gg \pi/2$. If we define the error ϵ_1 in the approximation, Eq. 8.22 as

$$\epsilon_1 = A_{1\,\text{approx.}} - A_{1\,\text{exact}}$$
$$= \frac{4\zeta^2}{1-\zeta^2} e^{-2\zeta\tau'} \sin^2\sqrt{1-\zeta^2}\,\tau' \qquad \text{(E.1)}$$

then the following observations can be made about the error ϵ_1. (a) From the sketch of the hypergeometric function, Fig. 8.1, substantial contributions to the integrals appearing in Eq. 8.20 occur only in the neighborhoods of $\tau' = \pi$, 2π, 3π, ... Since $\sin n\pi = 0$, $n=1,2,3$ . . ., for small ζ, ϵ_1 reduces periodically near the regions of τ' where there are substantial contributions to the integrals. (b) For small τ'

$$A_{1\,\text{approx.}} = (1-e^{-2\zeta\tau'})^2 = 4\zeta^2\tau'^2 + O(\tau'^3) \qquad \text{(E.2)}$$

and

$$\epsilon_1 = 4\,\zeta^2\,\tau'^2 + 0\,(\tau'^3) \tag{E.3}$$

hence

$$\lim_{\tau' \to 0} \frac{\epsilon_1}{A_1 \text{ approx.}} = 1 \tag{E.4}$$

and the approximation , Eq. 8.22, is poor near $\tau' = 0$. (c) For small ζ and large $(2\zeta\tau')$, A_1 approx. $\gg \epsilon_1$ and

$$\lim_{\tau' \to \infty} \frac{\epsilon_1}{A_1 \text{ approx.}} = 0 \tag{E.5}$$

Therefore, in addition to the periodic reduction of ϵ_1 near $\tau' = \pi, 2\pi, 3\pi, \ldots$, the approximation, Eq. 8.22, improves asymptotically with increasing τ'. Thus the approximation, Eq. 8.22, is poorest for small values of τ'.

The exact expression for $A_2(\zeta, \tau')$ can be written as

$$A_2(\zeta, \tau') = 1 - e^{-2\zeta\tau'} - \frac{\zeta}{\sqrt{1-\zeta^2}}\, e^{-2\zeta\tau'} \left(\sin 2\sqrt{1-\zeta^2}\,\tau' + \frac{2\zeta}{\sqrt{1-\zeta^2}} \sin^2\sqrt{1-\zeta^2}\,\tau'\right)$$

(8.11 repeated).

The approximation of $A_2(\zeta, \tau')$ we have used in Section 8 is

$$A_2(\zeta, \tau') \approx 1 - e^{-2\zeta\tau'} \qquad \text{(8.23 repeated)}$$

for $\zeta \ll 1$ and $\tau' \geq \pi/2$. If we define the error, ϵ_2, in the approximation, Eq. 8.23, as

$$\begin{aligned}\epsilon_2 &= A_{2 \text{ approx.}} - A_{2 \text{ exact}} \\ &= \frac{\zeta}{\sqrt{1-\zeta^2}}\, e^{-2\zeta\tau'} \left(\sin 2\sqrt{1-\zeta^2}\,\tau' + \frac{2\zeta}{\sqrt{1-\zeta^2}} \sin^2\sqrt{1-\zeta^2}\,\tau'\right)\end{aligned} \tag{E.6}$$

then the following observations can be made about ϵ_2. (a) In the neighborhoods of $\tau' = \pi, 2\pi, 3\pi, \ldots$ where there are substantial contributions to the integrals appearing in Eq. 8.20, ϵ_2 has a periodic reduction for $\zeta \ll 1$. (b) For small τ'

$$A_{2\text{ approx.}} = 1 - e^{-2\zeta\tau'} = 2\zeta\tau' + 0(\tau'^2)$$

and

$$\epsilon_2 = 2\zeta\tau' + 0(\tau'^2)$$

hence

$$\lim_{\tau' \to 0} \frac{\epsilon_2}{A_{2\text{ approx.}}} = 1$$

and the approximation, Eq. 8.23 is poor near $\tau' = 0$. (c) For small ζ and large $(2\zeta\tau')$, $A_{2\text{ approx.}} \gg \epsilon_2$, and

$$\lim_{\tau' \to \infty} \frac{\epsilon_2}{A_{2\text{ approx.}}} = 0$$

Therefore, in addition to the periodic reduction of ϵ_2 near $\tau' = \pi, 2\pi, 3\pi,$ the approximation, Eq. 8.23 improves asymptotically with increasing τ'. Thus the approximation, Eq. 8.23, is poorest for small values of τ'.

The exact expression for $A_4(\zeta, \tau')$ can be written as

$$A_4(\zeta,\tau') = e^{-\zeta\tau'}\left[(1 - e^{-2\zeta\tau'})\cos\sqrt{1-\zeta^2}\,\tau' - \frac{\zeta}{\sqrt{1-\zeta^2}}(1 + e^{-2\zeta\tau'})\sin\sqrt{1-\right.$$

(8.13 repeated)

while the approximation we have used in Section 8 is

$$A_4(\zeta,\tau') \approx e^{-\zeta\tau'}(1 - e^{-2\zeta\tau'})\cos\tau'$$

(8.24 repeated)

for $\zeta \ll 1$ and $\tau' > \pi/2$. Neglecting the $\sqrt{1-\zeta^2}$ in the argument of the cosine the error, ϵ_4 in the approximation, Eq. 8.24, is

$$\epsilon_4 = A_{4\ \text{approx.}} - A_{4\ \text{exact}}$$
$$\approx e^{-\zeta\tau'} \frac{\zeta}{\sqrt{1-\zeta^2}} (1+e^{-2\zeta\tau'}) \sin\sqrt{1-\zeta^2}\,\tau' \tag{E.11}$$

The following observations can be made about ϵ_4. (a) In the neighborhoods of $\tau' = \pi, 2\pi, 3\pi \ldots$, where there are substantial contributions to the integrals appearing in Eq. 8.20, ϵ_4 has a periodic reduction for $\zeta \ll 1$.
(b) For small τ'

$$A_{4\ \text{approx.}} = e^{-\zeta\tau'}(1- e^{-2\zeta\tau'}) \cos\tau' = 2\zeta\tau' + 0(\tau'^2) \tag{E.12}$$

and

$$\epsilon_4 \approx 2\zeta\tau' + 0(\tau'^2) \tag{E.13}$$

hence

$$\lim_{\tau' \to 0} \frac{\epsilon_4}{A_{4\ \text{approx.}}} \approx 1 \tag{E.14}$$

and the approximation, Eq. 8.24, is poor near $\tau' = 0$. (c) Both the approximation, Eq. 8.24, and ϵ_4 approach zero as $\tau' \to \infty$ but comparing $(1 - e^{-2\zeta\tau'})$ with $(1+ e^{-2\zeta\tau'})$ from Eqs. 8.24 and E.11 shows that ϵ_4 asymptotically becomes relatively less as τ' increases. Thus the approximation Eq. 8.24, is poorest near $\tau' = 0$. The above considerations have shown that in all three small ζ approximations, Eqs. 8.22, 8.23 and 8.24, the largest error occurs near $\tau' = 0$.

In obtaining the "approximate average value of the first integrand in Eq. 8.20", I_4, given by Eq. 8.28, $e^{-2\xi\tau'}$ was treated as a constant in the arbitrary interval $\frac{n\pi}{2} \leqslant \tau' \leqslant \frac{(n+1)\pi}{2}$, n=1,2,3 ... . The change in $e^{-2\xi\tau'}$ as τ' increases from $\frac{n\pi}{2}$ to $\frac{(n+1)\pi}{2}$ is $e^{-\xi n\pi}(1-e^{-\xi\pi})$. Hence for fixed ξ the largest variation in $e^{-2\xi\tau'}$ occurs in the interval corresponding to n=1. Here, as in the case of the small ξ approximations, it appears that the largest error occurs near $\tau' = 0$.

In order to obtain an estimate of the errors caused by the small approximations, Eqs. 8.22, 8.23 and 8.24, and the errors caused by treating $e^{-2\xi\tau'}$ as a constant in our approximate analytical evaluation of the first integral in Eq. 8.20, we have performed a numerical integration using exact expressions for A_1, A_2 and A_4 in the first integrand in Eq. 8.20. Since, as indicated in the above discussions, we anticipate the maximum errors in our approximations to occur near $\tau'=0$, we have performed the numerical integration only between $\tau'=0$ and $\tau'= 3\pi/2$ (see Fig. 8.1). The values, $\alpha= 9$ and $\xi= 1/60$ were chosen as typical values for which to perform the numerical integration. After preliminary calculations it was found that values of

$$\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')}\ F\left(1+\frac{\alpha}{2};\ 1+\frac{\alpha}{2};\ \frac{1}{2};\ A_3(\xi,\tau')\right)$$

(for $\alpha=9$, $\xi = 1/60$) were negligible except in the immediate neighborhood of $\tau' = \pi$. This is not surprising in view of Fig. 8.1. Thus using values of A_1, A_2 and A_4 obtained from the exact equations, Eqs. 8.8 to 8.13, we have numerically evaluated the integral

$$I_{exact} = \int_{2.8416}^{3.4416} \left[\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')) - 1 \right] d\tau' \tag{E.15}$$

for $\alpha = 9$ $\xi = 1/60$. The integrand of Eq. E.15 is shown by the solid line in Fig. E.1 . Using Simpson's rule it was found that

$$I_{exact} = 332 \tag{E.16}$$

Using the small ξ approximations of A_1, A_2 and A_4, Eqs. 8.22, 8.23 and 8.24, values of the integrand of Eq. E.15 were also computed for the same interval $2.8416 \leqslant \tau' \leqslant 3.4416$. These values are shown by the dashed line in Fig. E.1. For the region closest to $\tau' = \pi$ the computed difference between the approximate and exact integrands became of the same order as the anticipated error involved in the numerical calculations, this difference being too small to be conveniently shown in Fig. E.1. Using Simpson's rule and the values of the integrand of Eq. E.15 computed from the small ξ approximations of A_1, A_2 and A_4 it was found that

$$I_{approx.} = 329 \tag{E.17}$$

for the interval $2.8416 \leqslant \tau' \leqslant 3.4416$.

To check the method of integration used in Section 8, and in particular the assumpiton of treating $e^{-2\xi\tau}$ as a constant in obtaining the "approximate average value of the first integrand in Eq. 8.20," as mentioned above, we have also evaluated the integral using the method of Section 8.

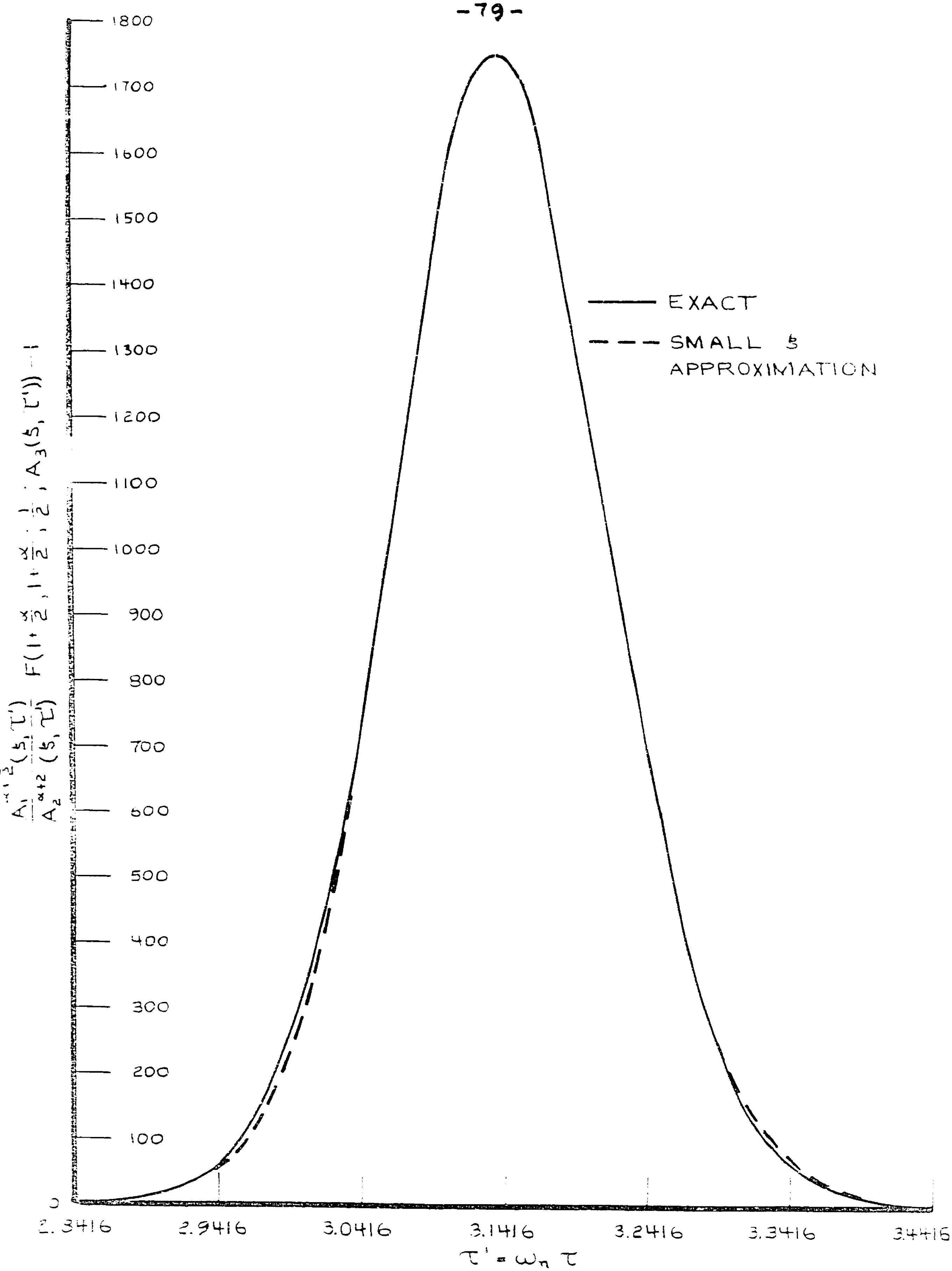

FIG. E.1 INTEGRAND OF EQ. E.15 FOR $\alpha = 9$ $\xi = 1/60$

Due to the "smoothing effect" of the method the limits here must be taken as $\pi/2$ and $3\pi/2$. The corresponding integral of Section 8 (see Eq. 8.47) is

$$\int_{\pi/2}^{3\pi/2}\left[F\left(-\frac{\alpha}{2},-\frac{\alpha}{2};1;e^{-2\xi\tau'}\right)-1\right]d\tau' =$$

$$=\frac{1}{2\xi}\left[\frac{\left(-\frac{\alpha}{2}\right)^2}{1(1!)^2}\left(e^{-2\xi\pi/2}-e^{-2\xi 3\pi/2}\right)+\frac{\left(-\frac{\alpha}{2}\right)^2\left(-\frac{\alpha}{2}+1\right)^2}{2(2!)^2}\left(e^{-4\xi\pi/2}-e^{-4\xi 3\pi/2}\right)+\ldots\right.$$

$$\left.\ldots+\frac{\left(-\frac{\alpha}{2}\right)^2\left(-\frac{\alpha}{2}+1\right)^2\ldots\left(-\frac{\alpha}{2}+n-1\right)^2}{n(n!)^2}\left(e^{-2n\xi\pi/2}-e^{-2n\xi 3\pi/2}\right)+\ldots\right] \tag{E.18}$$

and for $\alpha = 9$ and $\xi = 1/60$, evaluation of the above series gives

$$I_{Sec.\,8} = 328 \tag{E.19}$$

Since the exact values of

$$\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')}\;F\left(1+\frac{\alpha}{2},1+\frac{\alpha}{2};\frac{1}{2};A_3(\xi,\tau')\right)$$

for $\alpha = 9$ and $\xi = 1/60$, are effectively zero for the portion of the interval $\pi/2 < \tau' < 3\pi/2$ outside of $2.8416 < \tau' < 3.4416$, due to the minus one in the integrand of Eq. E.15 we should subtract $2.5416 \approx 3$ from the value of I_{exact}, Eq. E.16, to compare with $I_{Sec.\,8}$, Eq. E.19. Thus the approximate analytical method of Sec. 8 gives a value of 328 (Eq. E.19) to be compared with a value of 329 (Eq. E.16 with 3 subtracted) obtained numerically using the exact expressions for A_1, A_2 and A_4 both for the intervals, $\pi/2 \leq \tau' \leq 3\pi/2$. The approximations we have used in Section 8 are clearly adequate for engineering purposes.

In obtaining the numerical values of the integrand of Eq. E.15 using both the exact and approximate expressions for A_1, A_2 and A_4, the hypergeometric function of Eq. E.15 was transformed using Eq. 7.12, i.e.,

$$F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3) = \frac{F(-\frac{1+\alpha}{2}, -\frac{1+\alpha}{2}; \frac{1}{2}; A_3)}{(1-A_3)^{\alpha+3/2}} \tag{E.20}$$

For odd integral values of α the series representation of the hypergeometric function on the right hand side of Eq. E.20 converges after $\left(\frac{1+\alpha}{2}+1\right)$ terms. In order to facilitate the numerical evaluations this hypergeometric series was written in the following form.

$$F(-a,-a, \frac{1}{2}, X) = 1+\sum_{n=1}^{a} \frac{2^n\, n!}{1.3.5\ldots(2n-1)} \binom{a}{n}^2 X^n \tag{E.21}$$

where a=a positive integer and

$$\binom{a}{n} = \frac{a!}{(a-n)!\; n!}$$

are the binomial coefficients which have been tabulated.

F. DEMONSTRATION OF EQUATION 8.44

Eq. 8.44

$$\int_0^{\pi/2} I_3 d\tau' \approx 0 \qquad \text{for } \zeta \ll 1 \qquad \text{(8.44 repeated)}$$

is made meaningful within the context of its usage by the statement

$$\frac{\int_0^{\pi/2} I_3 \, d\tau'}{\int_{\pi/2}^{T'} I_3 d\tau'} \lll 1 \qquad \text{for } \zeta \ll 1, \; T' \geqslant \frac{3\pi}{2}$$

(8.45 repeated)

where for the ensembles of stress histories considered in Section 8, I_3 is given by Eq. 8.17. From Eqs. 4.12 and 4.13

$$E \left[\Delta D_0 \, \Delta D_\ell \right] = (\Delta t)^2 \, I_3 \qquad \text{(F.1)}$$

where Δt is taken small enough to satisfy the assumptions of Section 2. Since the damage increments ΔD are always non-negative it follows from Eq. F.1 that I_3 is always non-negative and that its integral over any positive interval $a \leqslant \tau' \leqslant b$ is also non-negative. Hence satisfaction of the condition

$$\frac{\int_0^{\pi/2} I_3 \, d\tau'}{\int_{\pi/2}^{\frac{3}{2}\pi} I_3 d\tau'} \lll 1 \quad \text{for} \quad \zeta \ll 1 \qquad \text{(F.2)}$$

is sufficient to ensure the satisfaction of Eq. 8.45. In order to simplify the arguments to follow we change the dimensionless time variable $\tau' = \omega_n \tau$ in Eq. F.2

back into real time. Hence satisfaction of

$$\frac{\int_0^{\frac{\pi}{2\omega_n}} I_3 \, d\tau}{\int_{\frac{\pi}{2\omega_n}}^{\frac{3\pi}{2\omega_n}} I_3 \, d\tau} \lll 1 \quad \text{for} \quad \zeta \ll 1 \tag{F.3}$$

implies Eq. 8.44. In order to show that Eq. F.3 is satisfied for the ensembles of stress histories considered in Section 8 we shall use physical arguments utilizing the interpretation of I_3 given by Eq. F.1. Recalling that the integrals over τ^{ℓ} in Eq. 8.20 occurred as the result of limiting process in which $\Delta t \rightarrow 0$ (Section 4 , in particular Eq. 4.15), using Eqs. 4.5 and F.1 we can write the two integrals in F.3 as

$$\int_0^{\frac{\pi}{2\omega_n}} I_3(\tau) \, d\tau \approx \sum_{\ell=1}^{\frac{\pi}{2\omega_n \Delta t}} I_3(\ell \Delta t)\Delta t = \sum_{\ell=1}^{\frac{\pi}{2\omega_n \Delta t}} \frac{1}{\Delta t} \; E\left[\Delta D_0 \;\; \Delta D_\ell\right] \tag{F.4}$$

and

$$\int_{\frac{\pi}{2\omega_n}}^{\frac{3\pi}{2\omega_n}} I_3(\tau) d\tau \approx \sum_{\ell=1+\frac{\pi}{2\omega_n \Delta t}}^{\frac{3\pi}{2\omega_n \Delta t}} I_3(\ell\Delta t)\Delta t = \sum_{\ell=1+\frac{\pi}{2\omega_n \Delta t}}^{\frac{3\pi}{2\omega_n \Delta t}} \frac{1}{\Delta t} \; E\left[\Delta D_0 \;\; \Delta D_\ell\right] \tag{F.5}$$

The arguments to be used become more understandable if the expectations appearing in Eqs. F.4 and F.5 are conditioned on the event that a stress zero crossing occurs in the interval $0 < t < \Delta t$. We denote this event by Y_o. Then $E\left[\Delta D_o \; \Delta D_\ell \mid Y_o\right]$ is the mathematical expectation of the product of the damage increments ΔD_o and ΔD_ℓ, incurred in the time intervals $0 < t < \Delta t$ and $\ell \Delta t < t < (\ell+1)\Delta t$ respectively given that a stress zero crossing occurs in the interval $0 < t < \Delta t$. If we denote the probability of the event Y_o by $P_{Y_o} \Delta t$ (for small Δt the probability of this event is proportional to Δt) then

$$E\left[\Delta D_o \; \Delta D_\ell\right] = P_{Y_o} \Delta t \; E\left[\Delta D_o \, \Delta D_\ell \mid Y_o\right] \tag{F.6}$$

since $E\left[\Delta D_o \; \Delta D_\ell \mid Y_o^*\right] = 0$ where Y_o^* is the complement of the event Y_o. Substituting Eq. F.6 into F.4 and F.5 and the result into Eq. F.3 transfers the condition F.3 to

$$\frac{\displaystyle\sum_{\ell=1}^{\frac{\pi}{2\omega_n \Delta t}} E\left[\Delta D_o \; \Delta D_\ell \mid Y_o\right]}{\displaystyle\sum_{\ell=1+\frac{\pi}{2\omega_n \Delta t}}^{\frac{3\pi}{2\omega_n \Delta t}} E\left[\Delta D_o \Delta D_\ell \mid Y_o\right]} \lll 1 \quad \text{for } \zeta \ll 1 \tag{F.7}$$

and since the expectation and summation operations may be interchanged, Eq. F.7 is equivalent to

$$\frac{E\left[\left\{\displaystyle\sum_{\ell=1}^{\frac{\pi}{2\omega_n \Delta t}} \Delta D_o \; \Delta D_\ell\right\} \Big| \; Y_o\right]}{E\left[\left\{\displaystyle\sum_{\ell=1+\frac{\pi}{2\omega_n \Delta t}}^{\frac{3\pi}{2\omega_n \Delta t}} \Delta D_o \; \Delta D_\ell\right\} \Big| \; Y_o\right]} \lll 1 \quad \text{for } \zeta \ll 1 \tag{F.8}$$

which then implies Eq. 8.44.

We shall now give physical interpretations of the numerator and denominator appearing in Eq. F.8. It is recalled that for $\zeta \ll 1$ the stress sample functions of the ensembles considered in Section 8 have the usual narrow band characteristics (i.e. the appearance of sinusoids with slowly and randomly varying amplitude and phase). From Eq. 8.4 the average time between successive stress zero crossings is $1/\bar{N}_o = \pi/\omega_n$ seconds. Hence, for any sample function, given that a zero crossing occurs within the interval $0 < t < \Delta t$, i.e. event Y_o, we can be fairly sure that the deviation from π/ω_n of the time until the next zero crossing will be small in comparison to π/ω_n. (An exception to this might be found in cases where the slope at the event Y_o is very small in comparison with the average slope encountered at zero crossings). <u>In particular, for any stress sample function satisfying condition Y_o, we can be almost certain that the time between event Y_o and the next zero crossing will be between $\frac{\pi}{2\omega_n}$ and $\frac{3\pi}{2\omega_n}$ provided the slope at event Y_o is not exceptionally small.</u> On the basis of this statement Eq. F.8 follows. Consider the numerator appearing in Eq. F.8. In order for a sample function satisfying condition Y_o to contribute a non-zero increment to $\sum_{\ell=1}^{\frac{\pi}{2\omega_n \Delta t}} \Delta D_o \, \Delta D_\ell$ it is necessary that a zero crossing fall within at least one interval $\ell\Delta t < t < (\ell+1)\Delta t$ for $1 \leq \ell \leq \frac{\pi}{2\omega_n \Delta t}$ or given that Y_o occurs another zero crossing must fall within the interval $\Delta t < t < \left(\frac{\pi}{2\omega_n} + \Delta t\right)$. However the above underlined statement indicates that the probability of this happening is very very small unless the slope at event Y_o is exceptionally small. However, if this slope is small, the damage increment ΔD_o which is associated with event Y_o will also be exceptionally small, and from the narrow-band character of the sample functions so will be the damage increment associated with the next zero crossing. Hence most sample functions satisfying Y_o will give no contribution to $\sum_{\ell=1}^{\frac{\pi}{2\omega_n \Delta t}} \Delta D_o \quad \Delta D_\ell$ and for those that do the contributing products $\Delta D_o \; \Delta D_\ell$ will be exceptionally minute. Thus the mathematical expectation of this summation, the numerator appearing in Eq. F.8 is indeed very small. Now consider the denominator. Given that event Y_o occurs there is a contribution to $\sum_{\ell=1+\frac{\pi}{2\omega_n \Delta t}}^{\frac{3\pi}{2\omega_n \Delta t}} \Delta D_o \quad \Delta D_\ell$ if a zero crossing occurs within $(\frac{\pi}{2\omega_n} + \Delta t) < t < (\frac{3\pi}{2\omega_n} + \Delta t)$. From the above underlined statement we are almost certain that every sample function will contribute to this summation. Hence the mathematical expecation of this summation given in the denominator of Eq. F.8 is at least an order of magnitude larger than the numerator and the plausibility of the inequality is established. Eq. F.8 automatically implies Eq. 8.44

G. ON THE ASSUMPTIONS USED IN OBTAINING EQ. 8.50

In this appendix we show, using the small ξ approximations for A_1, A_2 and A_4, Eqs. 8.22, 8.23 and 8.24 that as $\xi \rightarrow 0$

$$\int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \tau' \left[\frac{A_1^{\alpha+3/2}(\xi,\tau')}{A_2^{\alpha+2}(\xi,\tau')} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; A_3(\xi,\tau')) - 1 \right] d\tau' \sim$$

$$\sim \int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \tau' \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; e^{-2\xi\tau'}) - 1 \right] d\tau' \tag{G.1}$$

for $m = 1, 2, 3 \ldots$.

Using the small approximations for A_1 A_2 and A_4 from Eq. 8.25 the integrand in the left hand side of Eq. G.1 is

$$\tau' \left[(1-e^{-2\xi\tau'})^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; e^{-2\xi\tau'} \cos^2\tau') - 1 \right] \tag{G.2}$$

As $\xi \rightarrow 0$ the change in $e^{-2\xi\tau'}$ over any interval $(m-\frac{1}{2})\pi \leqslant \tau' \leqslant (m+\frac{1}{2})\pi$ approaches zero. Hence let

$$e^{-2\xi\tau'} = B \tag{G.3}$$

where B is a constant less than unity. The integrand (G.2) then becomes

$$\tau' \left[(1-B)^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau') - 1 \right] \tag{G.4}$$

The quantity $\left[(1-B)^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau') - 1\right]$ is an even function about $\tau' = m\pi$.

Hence, we have

$$\int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \tau' \left[(1-B)^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau') - 1\right] d\tau'$$
$$= m\pi \int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \left[(1-B)^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau') - 1\right] d\tau' \tag{G.5}$$

However from Section 8 (see the work from Eqs. 8.27 to 8.36) it follows that

$$\int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \left[(1-B)^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau') - 1\right] d\tau' =$$
$$= \pi\left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; B) - 1\right] \tag{G.6}$$

and hence from Eq. G.5

$$\int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \tau' \left[(1-B)^{\alpha+1} F(1+\frac{\alpha}{2}, 1+\frac{\alpha}{2}; \frac{1}{2}; B\cos^2\tau') - 1\right] d\tau' =$$
$$= m\pi^2 \left[F(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; B) - 1\right] \tag{G.7}$$

On the other hand, treating $e^{-2\zeta\tau'}$ as a constant, B, in the right hand side of Eq. G.1 gives

$$\int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \tau' \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; B\right) - 1\right] d\tau' =$$

$$= \left[F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; B\right) - 1\right] \int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} \tau' \, d\tau'$$

$$= m\pi^2 \left[F\left(-\frac{\alpha}{2}, \frac{\alpha}{2}; 1; B\right) - 1\right] \qquad (G.8)$$

Comparison of Eqs. G.7 and G.8 shows that the left and right hand sides of Eq. G.1 are equal when using the small ξ approximations for A_1, A_2 and A_4 and letting $e^{-2\xi\tau'} = B$, a constant, which effectively happens as $\xi \rightarrow 0$.

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BIOGRAPHICAL NOTE

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