HIGH RESOLUTION WAVEFORMS SUITABLE FOR A MULTIPLE TARGET ENVIRONMENT

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ABSTRACT

The early use by radars of a uniform pulse train is reviewed and the fundamental nature of the ambiguities in time and frequency associated with this waveform are discussed. Consideration is given to the effect of variations in the parameters of a uniform pulse train on the location and size of the ambiguities in time and frequency.

Maximum reduction of these ambiguities is shown to be achievable through the use of a non-uniformly spaced pulse train restricted to having unequal time intervals between any two pair of pulses in the train. This results in equal amplitude sidelobes in time (reminiscent of the Barker phase codes) which are not increased by doppler frequency shifts. The ratio of the peak sidelobe level to the central response is 1:n volts where n is the number of pulses in the train.

It is shown that a completely uniform distribution of the interference volume of the ambiguity surface with this approach is, in general, impossible. Consequently, a constraint is imposed on the minimum distance to the first time sidelobes to permit solutions, albeit not optimum, to the waveform synthesis problem. Two specific techniques for generating these non-uniformly spaced trains are examined, and the minimum and maximum duration-bandwidth product ($TW/2\pi$) required for any n by these approaches is evaluated.

After establishing the desirability of a random variation in the interpulse spacings, a pseudo-random approach to specifying the interpulse spacings using the complete set of least residues modulo (n - 1) is examined.

The increase in resolution and reduction in interference achievable when a non-uniform spacing is combined with a variation in other pulse train parameters is evaluated. Target limitations to the maximum $TW/2\pi$ are discussed and the limits imposed by three different approaches to the design of the matched filter are considered.

With these limitations on $TW/2\pi$ and the present capability for synthesis it is presently practical to achieve a 100-pulse train, having a -40 db maximum sidelobe level, with $TW/2\pi \approx 25,000$.

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INTRODUCTION

Background

Throughout its thirty-year history, radar has had the problem of being unable to achieve simultaneous measurement of range and velocity free from ambiguity. Early attempts to overcome this problem were basically unsuccessful and the fundamental nature of this problem was soon well recognized. With the introduction of automatic data processing equipment, and the resultant elimination of the human eye and brain from the filtering and decision-making process, the consequence of this problem increased. No approach for reducing the severity of this difficulty has been generally adopted for existing radar systems.

This thesis examines the use of a non-uniformly spaced pulse train as a means for achieving high resolution and accuracy simultaneously in range and velocity, with freedom from ambiguity.

Contents of the Chapters

The contents of the chapters of this thesis are outlined below:

Chapter I considers the historical use of uniform pulse trains and the fundamental nature of the ambiguities associated with this waveform. Approaches to overcoming these ambiguities, and the limitations of these approaches, are examined.

Chapter II considers the effect of altering each of the parameters of the uniform pulse train. To examine the usefulness of the resultant waveforms, two criteria for performance are introduced: maximum freedom from interference and maximum freedom from ambiguity. It is concluded that a uniform pulse train, unambiguous in range and having the smallest probability of interference, and a non-uniformly spaced pulse train having the most uniform distribution of interference, respectively, gives maximum performance for these criteria.

Chapter III examines the general properties of non-uniformly spaced pulse trains. The ultimate limit to performance--a completely uniform distribution of the interference volume--is evaluated and found to be impossible to achieve for pulse trains longer than four pulses. A constraint is imposed on the distance to the first sidelobe in time and the maximum performance (minimum $TW/2\pi$ for a given number of pulses) for trains with relatively few pulses is examined.

Chapter IV examines a formal approach to selecting the interpulse spacings for minimum $TW/2\pi$. Two specific techniques for generating non-uniformly spaced trains are examined and the minimum and maximum performance of each technique is determined. The uniformity of distribution of the interference volume and the resulting spectra are also examined.

In Chapter V, the results of the previous chapter are used to determine the worst and best approach to generating the interpulse spacings. A random sequence of interpulse spacings is conceded to yield the best performance. Consequently, an approach which provides the opportunity for a random-like sequence of interpulse spacings is discussed and evaluated.

Chapter VI examines the performance achievable when a non-uniform spacing is combined with (a) an amplitude and density taper, (b) selective pulse dropping, (c) a pulse-to-pulse frequency shift, and (d) a modulation on each of the pulses. The increase in resolution, when a Barker code of order 13 is modulated on each of the pulses, is calculated.

Chapter VII considers some possible target limitations to the maximum $TW/2\pi$. Also evaluated are three approaches to the design of the matched filter receiver. These are the use of (a) a bank of n delay lines and an n x n phasing matrix, (b) a series of range gates and a bank of n narrow-band filters, and (c) a series of n re-circulating delay lines. The critical state-of-the-art hardware limitations of each approach are considered and the resultant limitation on signal $TW/2\pi$ is evaluated. An approach to compensating for timing errors and pulse distortion is examined.

Chapter VIII summarizes the target limitation, the state-of-the-art hardware limitations, and the waveform design limitations. It is concluded that a pulse train with 100 pulses having all time sidelobes below -40 db and requiring a $TW/2\pi \approx 25{,}000$ can be reasonably achieved.

Chapter IX briefly discussed several unsolved problems in the design of non-uniformly spaced pulse trains.

CHAPTER I

HISTORY OF THE USE AND STUDY OF PULSE TRAINS AND THEIR LIMITATIONS

1.1 Early Problems and Early Solutions

The uniform train of pulses as a radar signal is as old as radar itself and dates back to before World War II[1]. The outstanding reasons for its widespread use were a) simplicity of generation and reception, b) the inevitable energy per pulse and peak power limitations of existing transmitters, resulting in the necessity for integration of many pulses, and c) the capability it offered for discriminating between wanted and unwanted signals, based upon differences in range and velocity.

During World War II and the years immediately following there was little actual use made of waveforms other than the uniform pulse train, and indeed there was very little reason to investigate any. The availability of high power tubes at the centimeter wavelengths, the greater range performance achievable at these wavelengths and the higher doppler velocity of targets did produce problems of "blind speeds," and "second-time-around returns," but these problems were seldom severe enough to cause sufficient discomfort to motivate serious study of the problem. The two approaches available for alleviating the confusion created by second-time-around returns were a slight jittering of the prf and the alternative use of two widely separated operating frequencies[2]. The first approach did not actually eliminate the second-time around target. However because its location on the PPI was constantly jittering it could be more easily identified. With the second approach, the interpulse period on each operating frequency could be made twice as large and consequently the unambiguous range doubled. By proper filtering the second-time-around returns could be eliminated. However neither approach eliminated the blind speeds or blind ranges.

1.2 Ambiguity Function for the Uniform Pulse Train

The significance of the terms "blind speeds," blind ranges" and "second-time-around returns" and the general properties of pulse trains can be conveniently examined in terms of the ambiguity function, first introduced by Ville, Woodward and Davies and extensively studied by Siebert[3,4,5].

Given a signal $s(t) = u(t)e^{j\omega_o t}$, the ambiguity function

$$\theta(\tau,\omega) = \int_{-\infty}^{\infty} u(t - \tau/2)\, u^*(t + \tau/2)e^{j\omega t}dt \tag{1}$$

is the complex envelope of the response of the matched filter to the orignal signal, when that signal is shifted in time and frequency. The autocorrelation function, $\theta(\tau,0)$, and the spectrum, $\theta(0,\omega)$, are the response along the time and frequency axes of the ambiguity function.

Among the important properties of $\theta(\tau,\omega)$ are

$$\theta(0,0) = \text{Signal Energy} = \int_{-\infty}^{\infty} |\, u(t)\,|^2\, dt \tag{2}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\, \theta(\tau,\omega)\,|^2\, dt\; d\omega/2\pi = \text{Signal Energy} = \int_{-\infty}^{\infty} |\, u(t)\,|^2\, dt. \tag{3}$$

For this reason it is convenient to normalize the signal energy so that

$$\int_{-\infty}^{\infty} | u(t) |^2 dt = 1.$$

It is also convenient to define

$$\psi(\tau,\omega) = | \theta(\tau,\omega) |^2 . \qquad (4)$$

Figure 1 shows several representations of the ambiguity function for a uniform pulse train. The large responses, or ambiguities at multiples of Δ in time away from the central response correspond to the blind ranges* as well as to the second-, third-, fourth-, etc. time-around targets. The large responses at multiples of $\frac{1}{\Delta}$ in frequency away from the central response correspond to the doppler ambiguities or blind speeds*. The height

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*Blind Range $\equiv k \frac{\Delta c}{2} = k \frac{c}{2\text{prf}}$ Blind Speed $\equiv k \text{ prf} \frac{\lambda}{2} = k \frac{\lambda}{2\Delta}$

where λ = radar wavelength

c = speed of light

Δ = interpulse spacing = $\frac{1}{\text{prf}}$

k = any integer

Since the receiver was usually shut off when the transmitter was operating the signals returning to the radar at that time were not detected. Consequently the radar was blind to the ranges corresponding to these times.

In the pulse doppler radars all zero doppler returns were usually rejected in order to reduce the ground clutter. Any doppler frequency that was a multiple of the prf, being indistinguishable from zero doppler, was also rejected and hence the radar was blind to targets at these speeds.

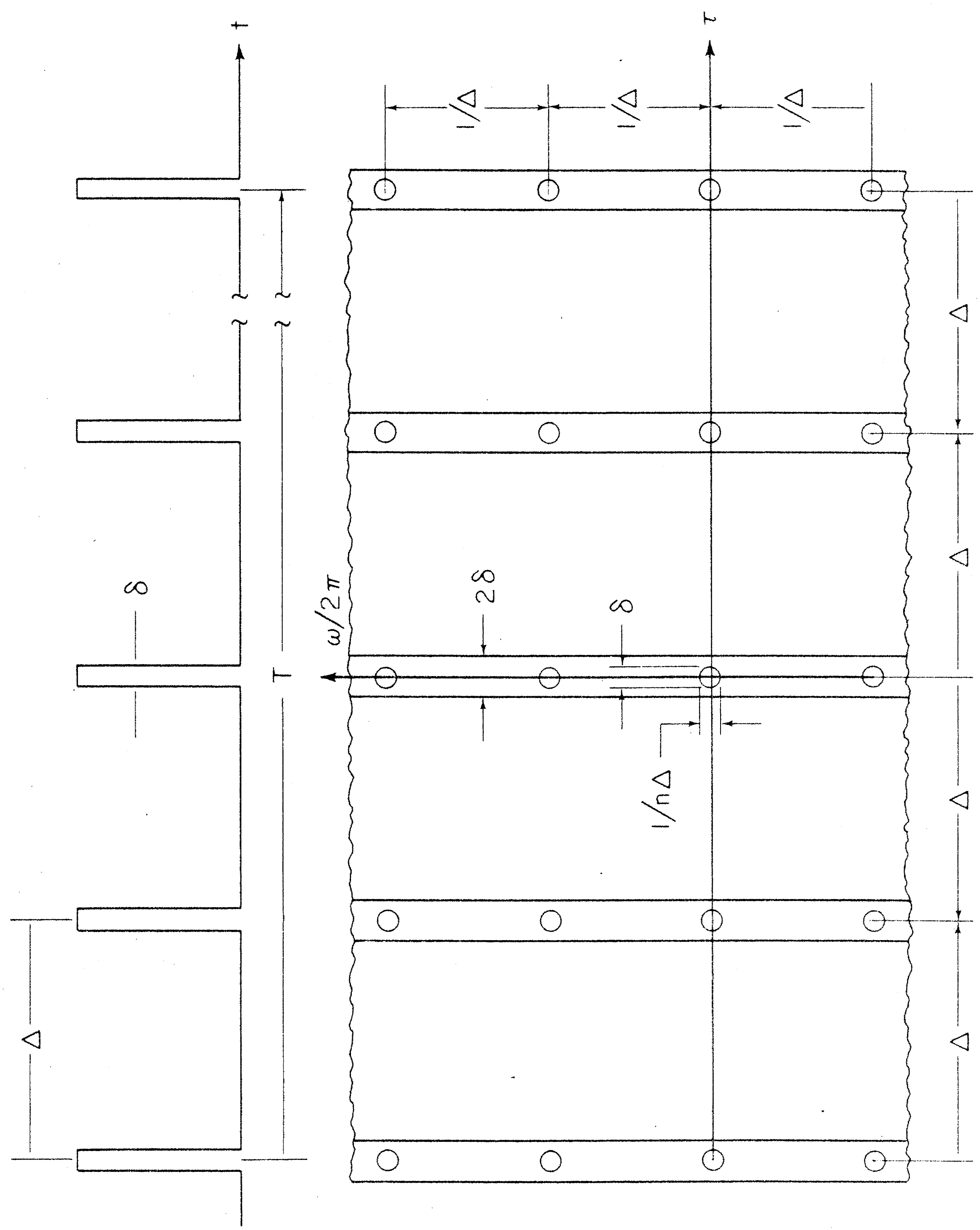

Fig. 1a: AMBIGUITY SURFACE FOR UNIFORM PULSE TRAIN

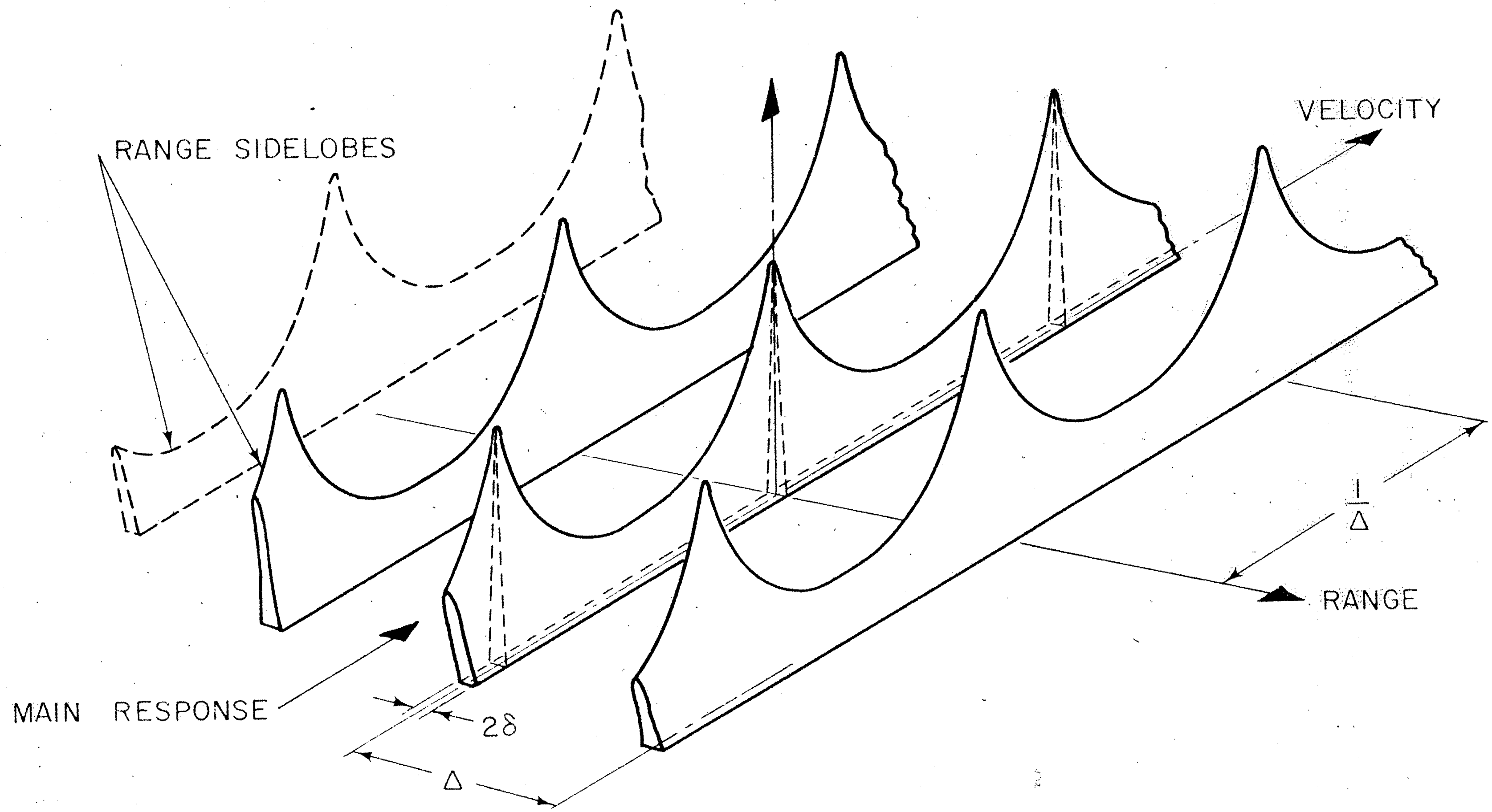

Fig.1b. Envelope of ambiguity surface of uniform burst.

of these frequency ambiguities decreases as $\frac{\sin \delta\omega/2}{\delta\omega/2}$ due to the finite width, δ, of the pulse.

Of fundamental importance is the fact that the "clear area" of the ambiguity surface has a duration in time of Δ and a width in frequency of $\frac{1}{\Delta}$, with a resultant clear area of 1, independent of the prf. Whenever the product of the time extent and the doppler spread of the target ensemble exceeds 1 then the entire target ensemble cannot be contained within the ambiguity-free central region and ambiguity in time and/or frequency must occur. Consequently, we can define a time extent-doppler bandwidth for the target as

$$T_R W_D / 2\pi \equiv \frac{2(r_1 - r_2)}{c} \cdot \frac{2(\nu_1 - \nu_2)}{\lambda} \tag{5}$$

where

r_1 = maximum target range

r_2 = minimum target range

ν_1 = maximum target velocity

ν_2 = minimum target velocity

λ = radar wavelength

For example if

r_1 = 120 miles

r_2 = 0 miles

ν_1 = +300 mph = 440 ft/sec

ν_2 = -300 mph = -440 ft/sec

λ = 1/3′ (S-band)

then

$$T_R W_D/2\pi = \frac{2 \times 120}{186{,}000} \times \frac{2 \times 880}{1/3} \approx 7$$

By selection of the prf, the signal can be designed to be ambiguous in one dimension only or be ambiguous in both but not (by adjusting the prf alone) ambiguity-free. Since the ambiguities in time and frequency are regularly spaced and are of approximately equal height then the total volume under $\psi(\tau,\omega)$, over the target space, or the interfering volume, is independent of the prf and is

$$\iint_{\text{Target space}} \psi(\tau,\omega)\,d\tau\,\frac{d\omega}{2\pi} \approx \frac{TW_{\text{Target}/2\pi}}{TW_{\text{Waveform}/2\pi}} \qquad (6)$$

assuming that the time extent and doppler bandwidth of the target environment are smaller than the duration and bandwidth of the signal. When the target extent in time and spread in frequency is greater than the signal duration and bandwidth, then all the ambiguities, and the whole volume under $\psi(\tau,\omega)$ is within the target space.

1.3 Historical Study of Pulse Trains

In the early and mid-Forties very little examination was made of variations of the uniform pulse train, and the variations actually considered, amplitude modulation, pulse width modulation, frequency and/or phase modulation, and modulation of the prf, were only applied in extremely specialized situations[2].

Amplitude modulation usually referred to cross-section variation due to changes in the target aspect or to a variation that was characteristic of the particular target itself. Propeller rotation on an airplane, for example, gives rise to a periodic variation in cross-section, which can be used to derive additional information about the target. The intentional use of amplitude modulation of the transmitted signal was limited to specialized communication systems, where the amplitude varied in accordance with the modulating function.

Pulsewidth modulation was a technique for amplitude (actually energy) tapering the transmitted pulse train by using a pulse to pulse variation of the duration of transmission. This technique had applications in specialized communication systems and was occasionally used to provide a more distinctive target return (vis-a-vis clutter) on the PPI.

In the literature of the late Forties and early Fifties* many authors[6,7,8,9] investigated the spectrum and autocorrelation function of a uniform pulse train when it is modulated by a) an amplitude variation

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*And even in current Russian Literature[10,11].

across the train, b) a non-uniform spacing, and c) an unequal pulse duration. These variations were invariably considered either to analyze the effects of random errors in these parameters on the uniform train performance or to examine the usefulness of these variations as modulation techniques (AM, PPM, or PDM) for telemetry and/or communication.

In the mid Fifties, when electronic data processing came into use the problems created by the ambiguities and blind regions inherent in the uniform pulse train became increasingly acute. A procedure was suggested for resolving these time and frequency ambiguities that required the sequential use of more than one prf[12,13]. Several prf's could be selected so that their first few blind ranges and blind speeds were slightly different. Consequently, the range and velocity where all prf's were ambiguous would be large enough so that all ranges and velocities could be determined unambiguously.

The limitation to this approach is the "de-ghosting" problem. In order to determine the real range and velocity of a target, the ambiguous range and velocity measured using the several prf's must be compared. In the presence of k targets, there could be as many as k returns from each prf, resulting in k^n correlations for n prf's. These correlations become unmanageable when more than a few targets are simultaneously observed. The solution often adopted in a multiple target environment was to accept an ambiguity in either range or in velocity, but not both. This usually resulted in an unambiguous range and several blind speeds. As a result while there was an inability to determine the doppler velocity unambiguously, the facility to reject stationary objects was still retained.

In the late Fifties, the problems created by the space age dominated most of radar thinking and only sporadic consideration was given to the use of pulse trains of any kind. In 1956, Siebert[14] discussed the advantages inherent in the use of a uniform pulse train "for those applications in which resolution capability is the pre-eminent design specification." More recently, Fowle, Kelly and Sheehan[15] discussed the high resolution achievable with a "properly designed" uniform pulse train. The phrase "properly designed" implied that the radar frequency was low enough so that the time extent-doppler bandwidth product of the target ensemble was less than unity, and the prf selected so that the "clear area" encompassed the entire target environment. They concluded that any change from the completely uniform pulse train would adversely affect the resolution and target handling capability of the waveform*. But often the target environment and the operating frequency were fixed by other constraints and consequently, the target $T_R W_D/2\pi$ could not be reduced below one. No solution is prescribed for this situation.

1.4 Conclusions

Siebert points out that, "When slight compromise can be tolerated in resolution performance, important advantages in other respects can be achieved by several variations in the coherent periodic pulse waveform..."[14]. It is these variations that must be taken advantage of if pulse trains are

*They also concluded, "It does not appear desirable to use complex modulation on the individual pulses in the train." The example in 6.4 shows this conclusion to be erroneous.

to find extensive use in large $T_R W_D/2\pi$ environments. Considering the renewed interest in, and need for, high resolution waveforms capable of operating in this type of target environment it is important to examine variations of the uniform pulse train and to determine the influence these variations have in altering the location and amplitude of the ambiguities in time and frequency. The target environment (i.e. number of targets, variation in scattering cross-section, distribution in range and velocity, etc.) within which the resultant waveforms will and will not prove useful can then be determined.

CHAPTER II

EFFECT OF VARIATION OF THE PARAMETERS OF THE UNIFORM PULSE TRAIN

A train of pulses is uniquely determined by the specification of the following parameters for each pulse in the train:

a) amplitude

b) phase

c) pulsewidth

d) polarization of transmission

e) bandwidth

f) center frequency

g) location

2.1 Amplitude Modulation

The use of a deliberate variation in the amplitude of pulses in the train has been suggested as a means for achieving lower near-in sidelobes along the frequency axis. This also results in a slight broadening of the doppler ambiguities. An amplitude taper alone results in no alteration in the location of the ambiguities in time and frequency and only slight reduction in their magnitude.

A form of amplitude taper is the selective dropping, from a uniform pulse train, of individual pulses in the train. In a target environment limited in number this approach can be used to "tag" parts of the train to resolve the range ambiguity. This approach was used in the radar observations of Venus made by the Millstone Hill radar of the Lincoln Laboratory.

The selective dropping of pulses can also be used to achieve a variation in the energy density (i.e. number of pulses per unit time) across the train. By dropping more pulses from the beginning and end of the train than from the middle, a symmetrical energy density taper can be created which will reduce the near in doppler sidelobes. The singular advantage of this approach over a multistepped variation in amplitude is the more efficient operation inherent in the use of only one transmitting level, and the elimination of the necessity for mismatching the receiver.

2.2 Phase Modulation

Extensive study has been made of coded pulse waveforms[16,17,18,19]. These waveforms are usually constructed by introducing 0° or 180° phase shifts, each increment in time δ, in a pulsed sinusoid of duration T^*. The phase code is presented as a sequence of one's and zero's, standing for a sequence of 0° or 180° phase shifts.

Although these phase codes were originally designed as a modulation for a continuous signal, they are also applicable for a train of pulses and result in

$$\psi_{continuous}(\tau = k\delta, 0) = \psi_{train}(\tau = k\Delta, 0) \qquad (7a)$$

and

$$\psi_{continuous}(\tau = k\delta, \omega) \approx \psi_{train}(\tau = k\Delta, \omega \cdot \frac{\delta}{\Delta}) \text{ for } \frac{\pi}{\Delta} \geq \omega \geq -\frac{\pi}{\Delta} \qquad (7b)$$

where k = 0, 1, 2, ..., n-1

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*Codes with three phase shifts, 0°, 120° and 240°, and codes with four phase shifts, 0°, 90°, 180° and 270° have also been examined.

In the search for phase codes having a uniformly low interference level throughout the ambiguity surface the most successful results were achieved through the use of null sequence codes[16], in which $TW/2\pi = 2^M - 1$. These codes result in

$$\psi(\tau,\omega) \approx \frac{\psi(0,0)}{\sqrt{2TW/2\pi}} \approx \frac{\psi(0,0)}{2^{\left(\frac{M+1}{2}\right)}} \qquad (8)$$

For the same code used with a pulse train, the same results would apply, except that the doppler ambiguities along the frequency axis are unaffected by the phase code. Note that the location of the non-zero portions of the ambiguity surface still lie in strips, 2δ wide in time, centered at $\tau = \pm k\Delta$.

Other phase codes, when used in a target environment where the maximum doppler spread, $f_{D\ Max} \ll \frac{1}{T}$, can provide superior performance. The most well known of these are the Barker Codes[17] which satisfy

$$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,0) \qquad (9)$$

The autocorrelation function for the largest known Barker Code (for which n=13) is shown in Figure 2. This phase code depends upon the coherent cancellation of pairs of sub-pulses having equal amplitude, frequency and duration, but 180° out of phase. But the frequency shift due to the target velocity results in a relative phase shift across the pulse, upsetting this coherent cancellation. Consequently the usefulness of these codes is limited to those target situations where the doppler spread is small and for

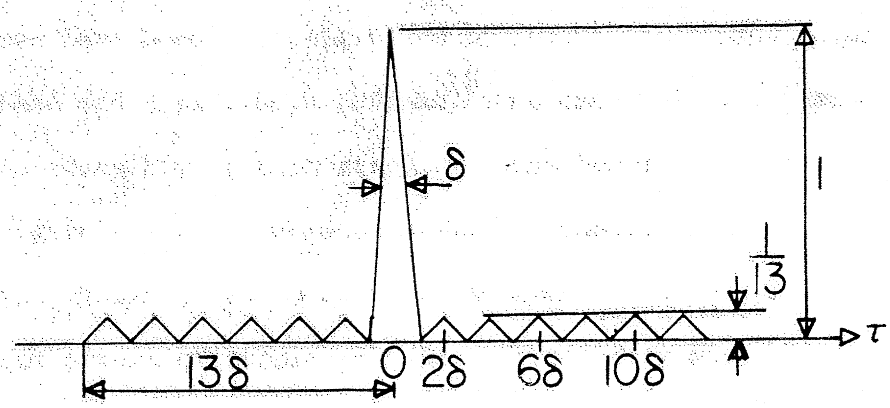

Fig 2 AUTOCORRELATION FUNCTION OF BARKER CODE OF ORDER 13

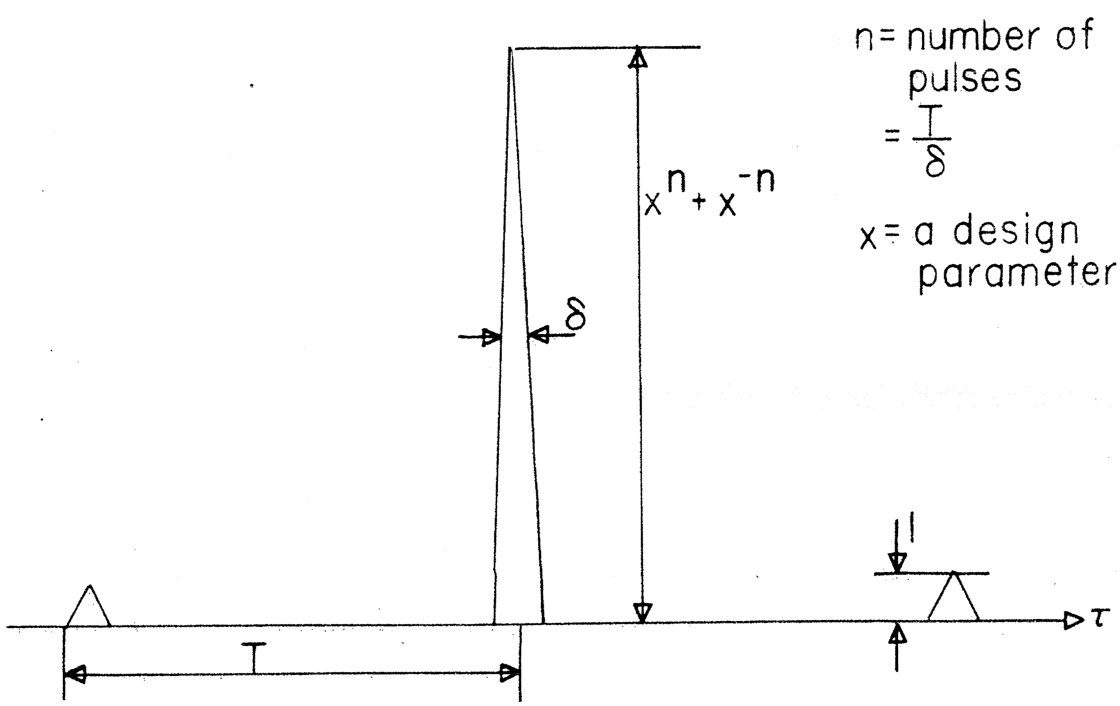

Fig 3 AUTOCORRELATION FUNCTION OF "IMPULSE-EQUIVALENT" PULSE TRAIN

which resolution in frequency is either not necessary or not possible.

There have been investigations of codes that employ a variation of pulse phase <u>and</u> amplitude. Huffman[18] has examined the class of amplitude and phase codes having zero response along the major part of the $\psi(\tau,0)$ axis. Figure 3 shows the autocorrelation function for a continuous signal of duration T using one such code. No examination has been made of the effects of errors or doppler shifts on the filter response.

Klauder[20] develops a waveform using the self-transform property of the Hermite polynomial, that has a circularly symmetrical ambiguity surface, without prominent peaks. The resultant envelope of the transmitted waveform is a bipolar signal very similar to several cycles of a sinusoid. The use of a sampled version of this signal is possible. Unfortunately, the performance of this waveform is rather poor, with first sidelobe levels of -8 db and a fall-off proportional to only $t^{-1/2}$. Nonetheless, when better ambiguity surfaces are achieved with amplitude and phase "tapers" the waveforms can be adopted to use with pulse trains.

Consequently when a phase code with a uniformly low ambiguity surface is used with a pulse train the peak of the ambiguities at $\tau = \pm k\Delta$ will be reduced although the response at other doppler frequencies, at the same time, will be increased. Using the best available codes results in

$$\psi(\tau = k\delta,\omega) \approx \frac{\psi(0,0)}{\sqrt{2TW/2\pi}} \quad . \tag{8}$$

The ambiguities along the zero doppler axis are unaffected by these phase codes.

2.3 Pulsewidth Modulation

When the transmitter is peak power limited, a variation in pulse width can be used to put an equivalent amplitude taper across the pulse train. If the spectrum of these pulses is not too different over the doppler frequencies of interest, then the effect on the ambiguity surface, over the target space, is virtually the same as that achieved with a direct amplitude taper, except for an increase in the time duration of the non-zero strips of the ambiguity surface around $\tau = k\Delta$.

It is possible to envision situations where the variations in pulsewidth became large enough so that the signal spectrum is considerably altered. However, unless a fairly large number of pulses have widths equal to the interpulse spacing and consequently, nulls at $\omega = 2\pi k/\Delta$, the ambiguity at $\psi(0,2\pi/\Delta)$ will not be severely affected. Further, the use of pulses this long will sharply reduce the range resolution. The major disadvantage of a pulse width variation across the train is that the range resolution of the signal will inevitably be determined by the longest pulse used.

2.4 Polarization Diversity

The transmission of pulses alternatively on two orthogonal polarizations could conceivably double the "clear area." If the target scattering matrix is diagonal and consequently the target does not depolarize the signal, then all the energy returned at any time will be received on one polarization only and hence, the two polarizations will be "blind" to each other. One use made of this property is the reduction of the clutter return from rain clouds[1,21]. Here transmission is on one sense of circular polarization and reception on the same sense. Since the spherical rain drops will reverse the sense of polarization and real targets can be expected to depolarize the transmitted signal (i.e. return energy on all polarizations) the target returns on the same sense of polarization will be enhanced vis-a-vis the rain.

To achieve a doubling of the clear area using this approach requires, in addition to a diagonal scattering matrix, that transmission and reception on orthogonal polarizations be available, and the Faraday rotation be very small or known. Rarely, in actual situations, is even one of these requirements met.

2.5 Bandwidth Variation

The use of pulses centered at the same frequency but varying in bandwidth across the train can provide a "decoupling" quite similar to that provided by a pulse-to-pulse frequency jump. Since the filter response to a signal at the same frequency as the filter is approximately proportional to the ratio of the two bandwidths, an increase in bandwidth of two to one on successive pulses reduces the first time sidelobes by 6 db, the second time sidelobes by 12 db, the third time sidelobes by 18 db, etc. However, the total bandwidth required is $\frac{2^{n-1}}{\delta}$, a prohibitively large bandwidth for large n.

The alternative transmission of two pulses, both at the same center frequency but one having 32 times the bandwidth of the other, results in approximately a 30 db reduction in all odd time sidelobes. To reduce the first and second, fourth and fifth, seventh and eighth, etc. sidelobes to the same 30 db level would require another factor of 32 in bandwidth on the third pulse or a thousand to one variation in bandwidth across three pulses! The same type of result can be achieved using an equivalent variation in pulse width. However, a variation in bandwidth does not increase the non-zero areas of the ambiguity surface. Neither the variation in bandwidth or the variation in pulsewidth alters the position or amplitude of the ambiguities along the frequency axis.

This approach is limited by the extravagant use of bandwidth. Further the entire system must be able to handle the large bandwidth, but the system resolution is not increased proportionately.

2.6 Frequency Shifting

In general, frequency jumping, pulse-to-pulse;

a) will not affect the location and magnitude of the ambiguities on the frequency axis,

b) will result in a reduction in the time duration of the central response from δ to approximately $2\pi/W$,

c) will not alter the volume under the central region of the ambiguity surface $[\ \delta \geq \tau \geq -\delta,\ \frac{\pi}{\Delta} \geq \omega \geq -\frac{\pi}{\Delta}\ ]$ or the resolution between two targets close in range and doppler, and

d) will result in a reduction in the ambiguities, at multiples of Δ in time.

Since the ambiguities along the frequency axis are unaffected, for this technique to be useful the spacing between pulses must be small enough so that all doppler components are unambiguous (i.e., $f_{D\ Max} < \frac{1}{\Delta}$). The resulting ambiguities at multiples of Δ are reduced due to the "orthogonality" of the different pulses.

Unfortunately, the effect of the frequency shift on the central response [$\psi(\delta \geq \tau \geq -\delta,0)$] and the effect on responses at $\psi[k\Delta,\omega]$ are intimately related. If the frequency steps are sufficiently large so that the filter response over much of the ambiguity surface is small, then the resultant signal spectrum will have large holes in it. As a result, there will be ambiguities in the central time response, i.e., several peaks along $\psi(\delta \geq \tau \geq -\delta,0)$. The presence of these ambiguities need not be a deterrent to the use of this approach, but it implies that the unambiguous range accuracy is determined by $\frac{2\pi}{\delta}$, the pulse spectral width, and not W, the signal spectral width. Conversely, if the resultant signal spectrum is completely filled and consequently there is only a single peak along $\psi(\delta \geq \tau \geq -\delta,0)$, then the pulse spectra will tend to overlap resulting in higher sidelobe levels.

One way of reducing these sidelobes is to use a pulse with a spectrum more narrowly confined than $\frac{\sin x}{x}$. Frequency shifting with pulses having a rectangular spectrum offers the possibility of very low sidelobes.

A technique for achieving a completely filled signal spectrum is a pulse-to-pulse step in frequency of $1/\delta$. This produces a signal extremely similar to linear FM. This can be seen by examining the spectrum, which is the sum of many $\frac{\sin \omega\, \delta/2}{\omega\, \delta/2}$ spectra, separated by $1/\delta$ in frequency. Despite the fact that all signals used are orthogonal, the maximum response of each filter to these orthogonal pulses is not zero but $\frac{1}{k\pi}$, where k is the number of spectral widths separating the signal and filter.

Unfortunately with a linear change in frequency with time, there will be a correlation between range and velocity[22]. In a pulse train, this coupling is $\frac{1}{\delta}$ cps/Δ seconds or $\delta\Delta$ second/cps. If the prf is selected slightly higher than the maximum doppler frequency, then the range variation, from minimum velocity to maximum velocity, is approximately δ.

To eliminate this objectionable time-frequency coupling, yet still retain a completely filled signal spectrum, the use of a random sequence of these same frequencies has been suggested. However, Kelly[23] has shown that the volume under the ambiguity surface within the central region is

$$\int_{0}^{\frac{2\pi}{\Delta}} \int_{-\delta}^{\delta} \psi(\tau,\omega)d\tau \, \frac{d\omega}{2\pi} = \frac{\delta}{T} \tag{10}$$

and is independent of the frequencies of the transmitted signal as long as the narrow band assumption is valid. Consequently, the use of a random sequence can only redistribute within the central region the same volume inherent in the use of <u>any</u> pulse train of duration T and pulsewidth δ, regardless of the frequencies used or their sequence. Even assuming that it is possible, by sufficiently randomizing the sequence of frequencies, to achieve a central region uncoupled in time and frequency, and of the shape shown in Figure 4 this volumetric constraint limits the lowest possible average sidelobe within this region. Since the total volume is δ/T and the volume in the central spike is $\approx 2\pi/WT$, the remaining volume must be dis-

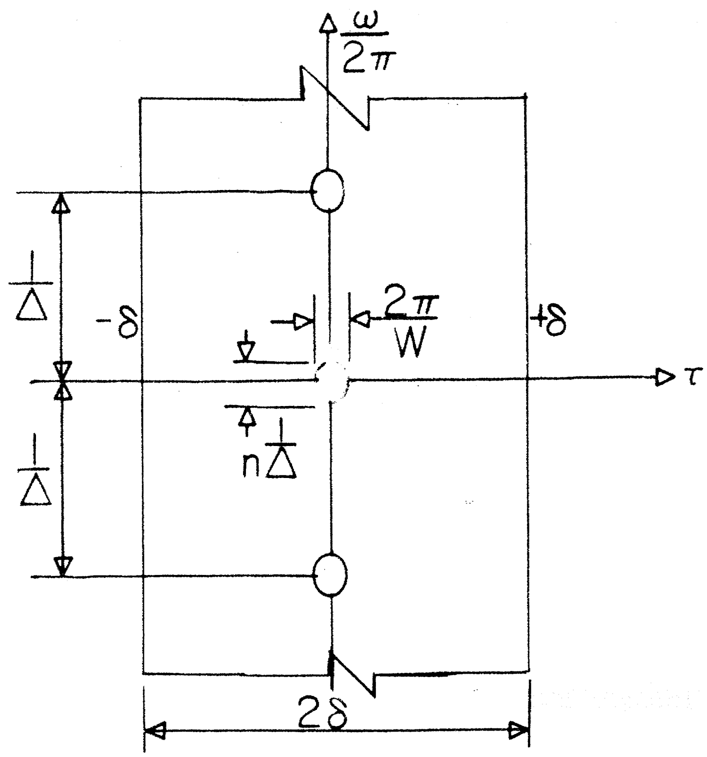

Fig 4 CENTRAL REGION OF AMBIGUITY FUNCTION FOR PULSE TRAIN OPTIMUMLY USING A PULSE-TO-PULSE FREQUENCY STEP

tributed throughout a region which is 2δ by $1/\Delta$ or an area of $2\delta/\Delta$. The lowest possible average sidelobe occurs when the volume is uniformly spread and results in

$$\overline{\psi(\tau,\omega)} = \frac{\delta/T - 2\pi/WT}{2\delta/\Delta}$$

or

$$\overline{\psi(\tau,\omega)} = \frac{1 - 2\pi/W\delta}{2n} \qquad (11)$$

Further, for n frequencies separated by $\frac{1}{\delta}$, $W/2\pi = \frac{n}{\delta}$ and therefore the average sidelobe level is

$$\overline{\psi(\tau,\omega)} = \frac{1 - \frac{1}{n}}{2n} \quad . \qquad (12)$$

A train of at least 50 pulses using 50 different frequencies will be required to make it possible to achieve a 20 db average sidelobe within this central region.

Consequently, frequency jumping will reduce the interference between two targets separated by more than δ. It does not enhance the ability of the waveform to resolve between two targets closely separated in time and/or frequency.

2.7 Position Modulation

A variation in the pulse period, or the location of the pulses, by altering the uniformly sampled character of the waveform alters the size and location of the ambiguities along the time and frequency axes. Further, with the interpulse period varied, non-zero portions of the ambiguity surface are no longer restricted to the regions where $\tau = k\Delta \pm \delta$.

It may be possible, by spacing the pulses randomly, to achieve a spectrum which is the result of a random addition of all the pulses at $\omega \neq 0$. This is quite similar to the effect of a random phase reversal on the time axis and should yield

$$\frac{\psi(0,0)}{n} \approx \psi(0,\omega) \ . \tag{13}$$

If it were possible, in addition to select the minimum interpulse spacing larger than the time extent of the target ensemble, then $\psi(\tau,\omega) = 0$ for $|\tau| \geq \delta$. The limitation to this approach is the necessarily low prf, the poor velocity resolution and the necessity for a long time on target. In many respects this approach is worse than a uniform pulse train having the same minimum spacing.

Another approach is to ensure that the pulse spacing is varied by at least a pulse width, for example, as shown in Figure 5, so that the distance between any two pulses in the train differs by at least δ from the distance between any other pair of pulses. When the pulse train is being autocorrelated, the correlation (for $|\tau| > \delta$) of each pulse with every other pulse will consequently occur for slightly different time shifts. This results in the autocorrelation function shown in Figure 6, with many non-overlapping time sidelobes (triangles for a constant frequency rectangular pulse), each with amplitude for $1/n$. The autocorrelation function for a uniform pulse train is also shown for comparison.

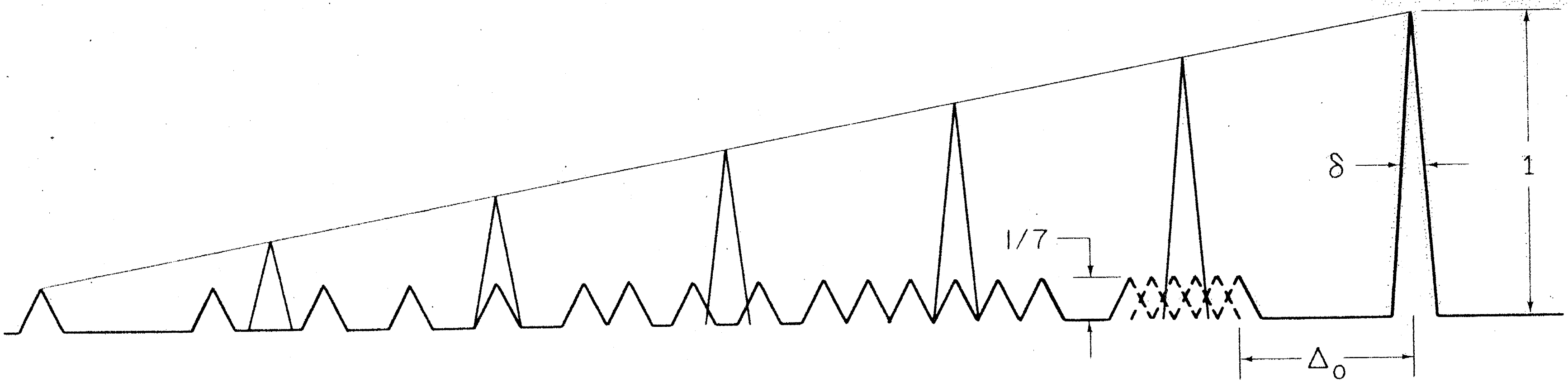

Fig. 6: AUTOCORRELATION FUNCTION FOR UNIFORMLY AND NON-UNIFORMLY SPACED TRAINS HAVING SEVEN PULSES AND EQUAL $TW/2\pi$

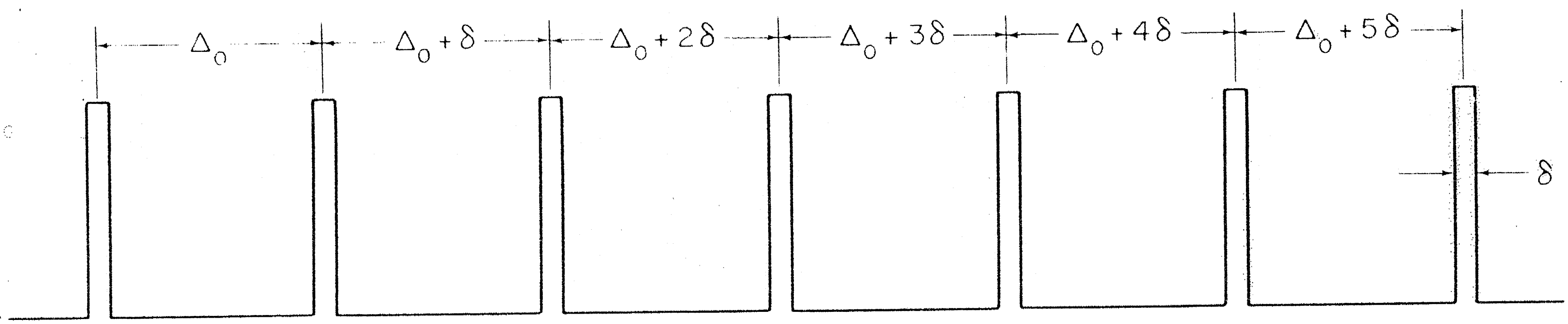

Fig. 5: NON-UNIFORMLY SPACED PULSE TRAIN WITH 7 PULSES.

Consequently, as long as the sidelobes are no closer together than δ

$$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,0) \text{ for } |\tau| > \delta$$

It is simple to show that this result holds for all ω and that this analysis accounts for the total volume under $\psi(\tau,\omega)$.

$$\Theta(\tau,\omega) = \int_{-\infty}^{\infty} u(t - \tfrac{\tau}{2})\, u^*(t + \tfrac{\tau}{2})\, e^{j\omega t}\, dt \qquad (1)$$

where

$$u(t) = \frac{1}{\sqrt{n}} \quad t_i - \frac{\delta}{2} \geq t \geq t_i + \frac{\delta}{2} \qquad \text{for } t_i = t_1, t_2, \ldots\, t_{n-1}\,.$$

$$= \quad \text{otherwise}$$

and

$$\int_{-\infty}^{\infty}\int |\Theta(\tau,\omega)|^2\, d\tau\, \frac{d\omega}{2\pi} = 1 \qquad (3)$$

By definition of the generating technique, $u(t - \frac{\tau}{2})\, u^*(t + \frac{\tau}{2})$ consists of, at most, the product of one pulse with another pulse. Hence, $\Theta(\tau,\omega)$, for each of the time sidelobes, is just the $\Theta(\tau,\omega)$ for a single, constant frequency, rectangular pulse sinusoid of duration δ (shown in Figure 7).

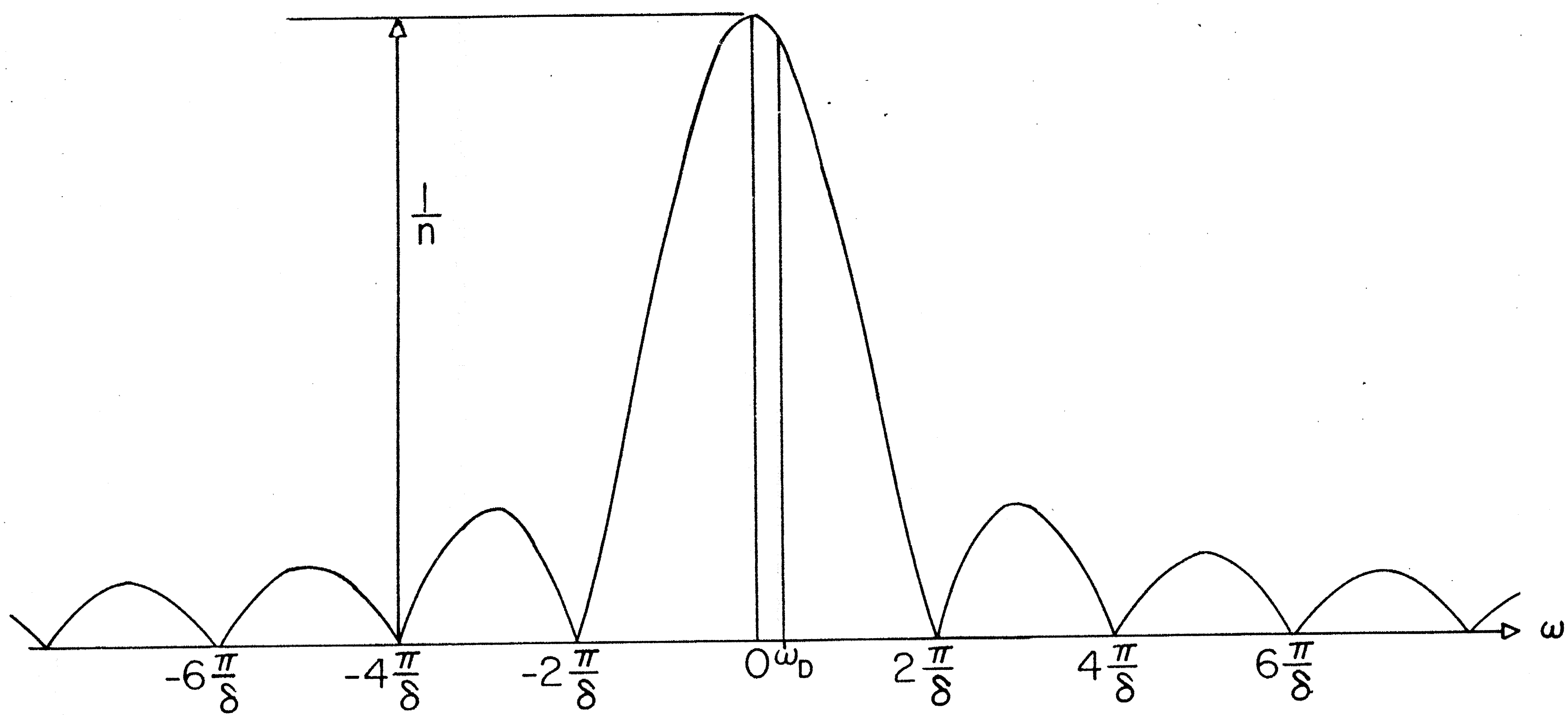

Fig. 7. THE RESPONSE WITH FREQUENCY OF THE PEAK OF ANY TIME SIDELOBE OF A NON UNIFORMLY SPACED PULSE TRAIN.

Therefore

$$\Theta(\tau = t_i - t_j, \omega) = \frac{1}{n} \frac{\sin \delta\omega/2}{\delta\omega/2} \tag{14}$$

which has its maximum value when $\omega = 0$. Consequently equation (10) holds for all values of ω, and

$$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,\omega) \qquad |\tau| \geq \delta \tag{15}$$

Further

$$\int_{-\infty}^{\infty} \int_{k\Delta-\delta}^{k\Delta+\delta} |\Theta(\tau,\omega)|^2 \, dt \, \frac{d\omega}{2\pi} = \frac{1}{n^2} \tag{16}$$

Since a sidelobe in time is produced by the correlation of each of the n pulses with n-1 other pulses, the total number of these sidelobes along the time axis is n(n-1). Therefore, the total volume contained in these sidelobes is

$$\int_{-\infty}^{\infty} \int_{|\tau| \geq \delta} |\Theta(\tau,\omega)|^2 \, d\tau \, \frac{d\omega}{2\pi} = \frac{1}{n^2} [n(n-1)] = 1 - \frac{1}{n} \tag{17}$$

The rest of the volume can be found in the $\Theta(\delta \geq \tau \geq -\delta,\omega)$ region where there are the equivalent of about Δ/δ ambiguities each of volume $\frac{1}{TW/2\pi}$ and consequently, a total volume in this region is

$$\int_{-\infty}^{\infty}\int_{-\delta}^{\delta} |\, \Theta(\tau,\omega)\, |^2 \, d\tau \, \frac{d\omega}{2\pi} = \frac{\Delta}{\delta}\left[\frac{1}{TW/2\pi}\right] = \frac{\Delta}{\delta}\left[\frac{1}{[n\Delta]\,[1/\delta]}\right] = \frac{1}{n} \qquad (18)$$

The total volume in the region $\delta \geq \tau \geq -\delta$, $\frac{\pi}{\Delta} \geq \omega \geq -\frac{\pi}{\Delta}$ is still $\frac{\delta}{n\Delta}$ or $\frac{1}{TW/2\pi}$. Contours of the envelope of the central regions of the ambiguity function for this type of waveform are shown in Figure 8. The restriction on pulse spacing; that no pair of pulses can be the same distance apart as any other pair of pulses and that the increment in spacing be $\geq \delta$, need not be severely limiting to the signal designer. Further, to the extent that this train "looks" like a uniformly spaced train, the near-in spectrum will have the $\frac{\sin nx}{\sin x}$ characteristic that would be expected of a uniform train.

With a waveform having a few dozen pulses in the train, a reasonable design approach is to use an "almost uniform" pulse spacing, since the $\frac{\sin nx}{\sin x}$ response has lower average sidelobes than would the spectrum for a random pulse spacing. Using this approach, the average prf should be high enough so that the first ambiguity at $\omega = 2\pi/\Delta$ is at a higher frequency than the highest doppler frequency of interest. With a very large number of pulses a random spacing could be used to yield not only the $\frac{1}{n^2}$ reduction in range but also a $\frac{1}{n}$ reduction in the doppler ambiguities.

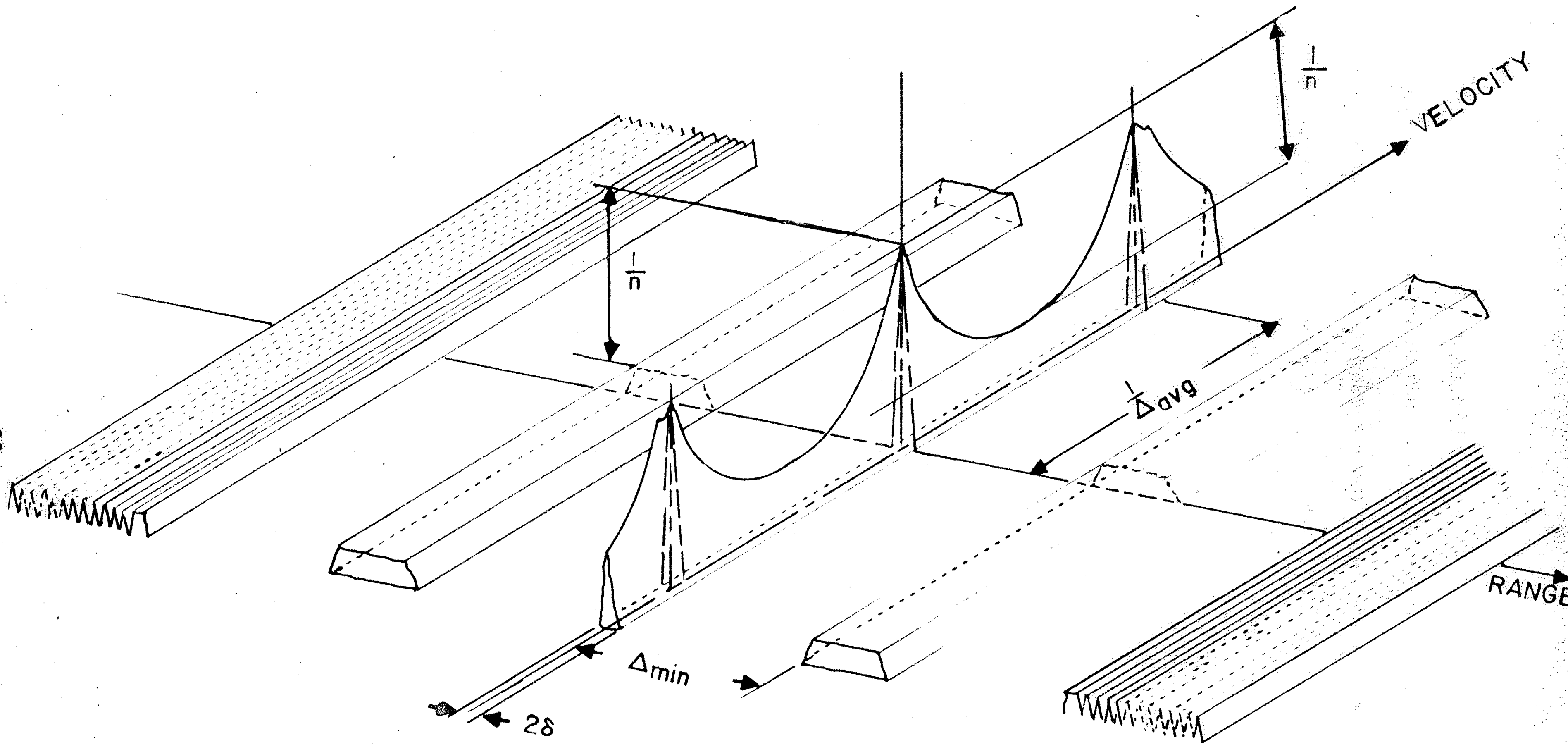

Fig. 8. Envelope of the ambiguity function for a non-uniformly spaced pulse train.

2.8 Criteria for Judging Performance

The use of one or more variations from a uniform pulse train appear to offer considerable freedom in the design of better (or at least different) ambiguity surfaces. However all suffer from the volumetric constraint imposed by the duration-bandwidth product of the waveform and the target. Given this nettlesome constraint only two criteria seem reasonable for judging the superiority of one ambiguity surface over another and consequently one design approach over another:

a. Maximum Freedom from Interference

For signals of equal duration-bandwidth product; if over a region where $\tau_a \geq \tau \geq \tau_b$ and $\omega_a \geq \omega \geq \omega_b$ the <u>area</u> for which $\psi_1(\tau,\omega)$ is not identically zero, is greater than the <u>area</u> for which $\psi_2(\tau,\omega)$ is not identically zero, then $\psi_2(\tau,\omega)$ is the better ambiguity surface over that region.

b. Maximum Freedom from Ambiguities

For signals of equal duration-bandwidth product; if over a region where $\tau_a \geq \tau \geq \tau_b$ and $\omega_a \geq \omega \geq \omega_b$

$$\psi_1(\tau,\omega)_{max} > \psi_2(\tau,\omega)_{max}$$

then $\psi_2(\tau,\omega)$ is the better ambiguity surface over that region.

The first criterion is applicable in an environment where the variation in cross section is extremely large or the target of interest is very small compared to the background interference. In this case, any object located where $\psi(\tau,\omega) \neq 0$ is assumed to obscure the target of interest.

The second criterion is applicable where the variation of target cross section is not excessively large. It is this criterion of "uniform volume spread" which is usually implied when the "ideal" or "optimum" ambiguity surface is discussed[14,24].

2.9 Conclusions

Given the target environment implied by the first criterion, it is important to minimize the non-zero areas of $\psi(\tau,\omega)$ within the target space. This requires that the pulse train have a uniform pulse spacing and uniform pulsewidth. If possible the interpulse spacing should exceed the time extent of the target ensemble. As noted before this limits interference to targets not differing in range by more than δ, but results in a multiple doppler ambiguity. Having done this, it is desirable to sharply amplitude taper the train to reduce the frequency sidelobes to a minimum and to make use of modulation on each of the pulses to increase the signal resolution.

Given the target environment implied by the second criterion, it is desirable to reduce the ambiguities to a uniform level. This requires spreading the volume out over large regions of the time-frequency space. The most effective way to do this is to vary the interpulse period so that the maximum number of time sidelobes are generated. This results in

$$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,\omega) \qquad \text{for } |\tau| \geq \delta \tag{15}$$

The addition of a pulse-to-pulse frequency jump can be used to reduce the time sidelobes while the use of modulation internal to each pulse will increase the resolution of the waveform.

Perhaps the most important conclusions that can be drawn are that, for the two extremes of target ensemble used to compare ambiguity surfaces the most effective solutions are, a) the use of the very old (and time honored) approach of a uniform train, unambiguous in range, ambiguous in velocity; and b) the use of a new approach, a non uniformly spaced pulse train unambiguous in either range or velocity. In view of the fact that very little new can be said about uniform pulse trains and very little has been said about non uniformly spaced pulse trains the rest of this study will be devoted to the latter subject.

CHAPTER III

GENERAL PROPERTIES OF NON UNIFORMLY SPACED PULSE TRAINS

3.1 Properties of Optimum Pulse Trains

Since the side-lobe (or interference) level over virtually the entire surface is directly related to the number of pulses in the train, it is reasonable to examine first the limits to this number. The maximum number of pulses, for any $TW/2\pi$ can be determined by examining the total number of sidelobes generated and constraining their maximum time extent to 2T.

Each pulse in the train correlates with all the n pulses in the train for a total of n^2 correlations. Since n of these correlations occur at $\tau = 0$, the total number of sidelobes, N, is

$$N = n^2 - n = n(n-1)$$

Since the time duration of each triangular sidelobe measured at the one-half voltage points is δ, then the minimum total time duration of all sidelobes if they were adjacent to one another is

$$\tau = n(n-1)\delta$$

Since the total duration of the autocorrelation function cannot exceed 2T,

$$2T \geq \tau = n(n-1)\delta$$

$$\frac{2T}{\delta} \geq n(n-1) \tag{19}$$

or

$$2TW/2\pi \geq n(n-1) \tag{20}$$

and for $n \gg 1$

$$\sqrt{2TW/2\pi} \geq n_{max} \tag{21}$$

If equation (20) is an equality then the autocorrelation function is as shown in Figure 9 and an optimum ambiguity surface (i.e. having a completely uniform volume spread with time) will have been achieved. Equation (19) can be rewritten

$$\frac{2}{\delta} \times \frac{T}{(n-1)} \geq n$$

and since $\frac{T}{n-1} = \Delta_{average}$

$$\frac{2\Delta_{average}}{\delta} \geq n. \tag{22}$$

Finally, since the average prf should exceed the doppler spread

$$\frac{1}{\Delta_{average}} > f_{DMax}$$

$$\therefore \frac{2}{\delta\, f_{DMax}} \geq n \tag{23}$$

If we were to limit the train duration bandwidth product to 5000 by limit-

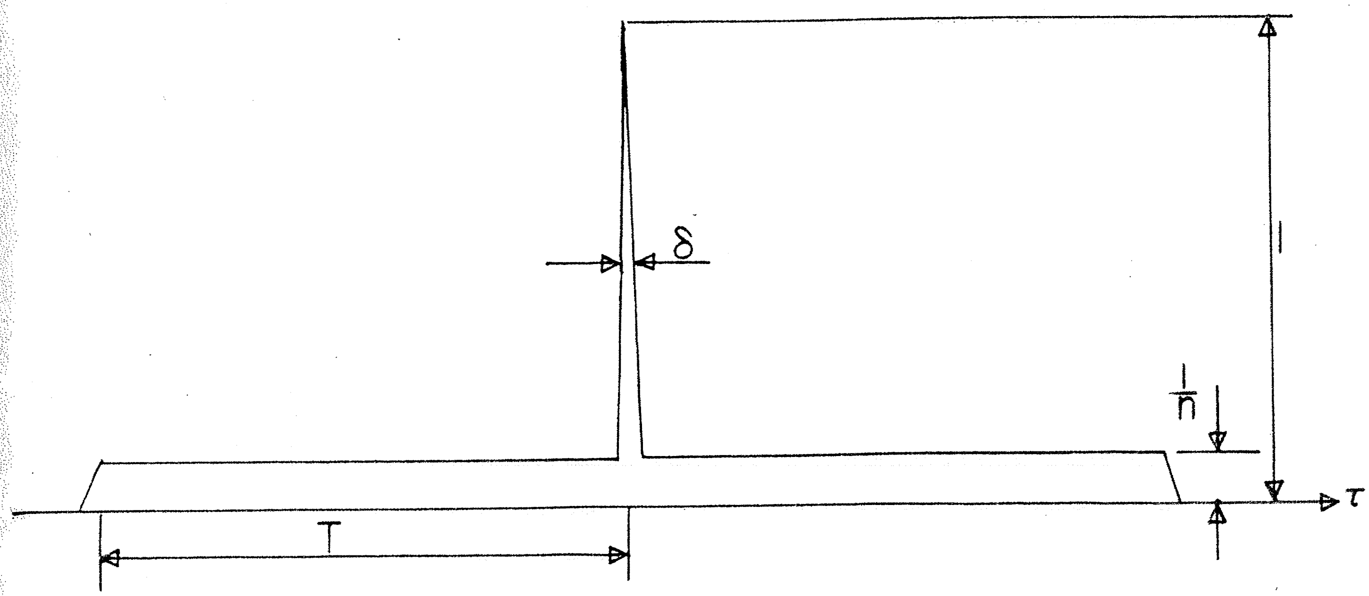

Fig. 9 AUTOCORRELATION FUNCTION OF OPTIMUM NON UNIFORMLY SPACED TRAIN OF n PULSES

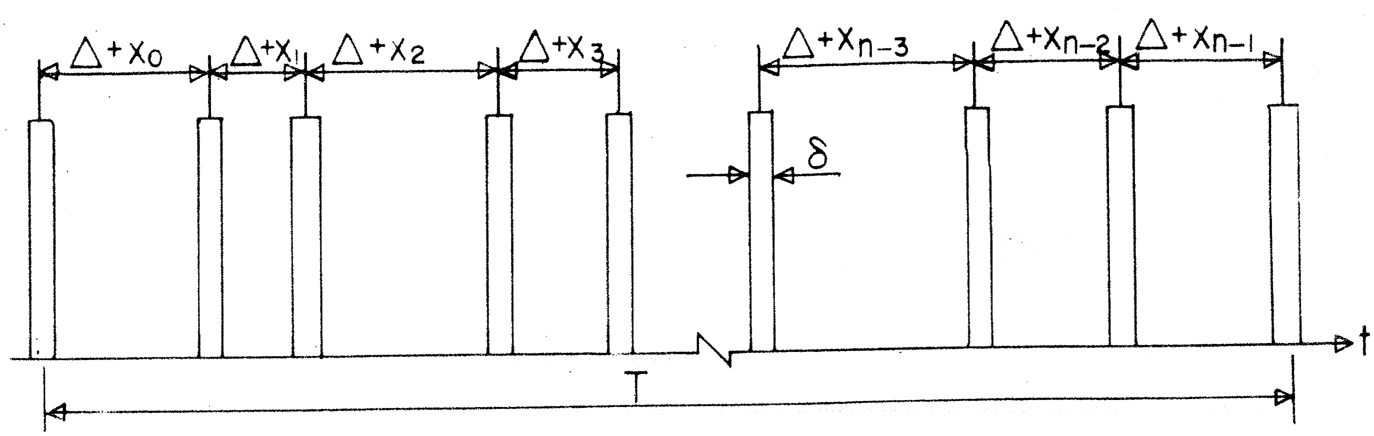

Fig 10 NON UNIFORMLY SPACED TRAIN WITH n PULSES

ing the train duration, T, to 5 milliseconds and the pulse width, δ, to 1.0 μsec then

$$n_{max} \leq \sqrt{\frac{(2)5,000x10^{-6}}{1x10^{-6}}} = 100 \text{ pulses}$$

and the best sidelobe performance achievable is $\frac{1}{(100)^2}$ or -40 db. In this case

$$\Delta_{average} \geq \frac{100\ (1x10^{-6})}{2} = 50 \ \mu\text{second}$$

and

$$\frac{1}{\Delta_{average}} = 20 \text{ kcps} \geq f_{DMax}$$

However, this performance, as will now be shown, cannot be actually achieved. First the spacings necessary to achieve a uniform volume spread will be calculated. Second, it will be shown that optimum pulse trains with more than four pulses are impossible.

Consider the train of n pulses shown in Figure 10. The total time duration of this train (in time units normalized to δ, the pulse width)* is

$$T = \sum_{i=1}^{i=n-1} X_i \quad \text{where the } X_i\text{'s are different integer multiples of } \delta .$$

*This normalization will be generally used throughout this thesis.

But from equation (19) to achieve a uniform volume spread

$$T = \frac{n(n-1)}{2}$$

$$\therefore \sum_{i=1}^{i=n-1} X_i = \frac{n(n-1)}{2} \tag{24}$$

However

$$\sum_{j=1}^{j=n-1} j = \frac{n(n-1)}{2}$$

Therefore the X_i's must be the first n-1 integers, since any other series of n-1 differing positive integers must have a higher sum than $\frac{n(n-1)}{2}$.

Therefore to achieve a uniform volume distribution with time the spacings between pulses must be some arrangement of the first n-1 integers such that no two pulses are the same distance apart as any other pair of pulses. A simple consideration of how these spacings can be arranged will show that for $n > 4$, this is impossible.

Consider somewhere in the sequence of interpulse spacings the spacing equal to 1. If the adjacent spacing is any integer except n-1 then the sum of these two adjacent spacings will be equal to another spacing and two sidelobes will occur at the same time. Therefore not only must the unit spacing be adjacent to the n-1 spacing but it must also be the first spacing since no spacing other than n-1 can be adjacent to it without causing an equality.

Since the spacings from 1 through n-1 must be used and the spacing between the first and third pulse is n, then all additional sums must be greater than n. The number 2, when summed with any number from 1 through n-1, will be greater than n only when that number is n-1. Therefore the two unit spacing must be also adjacent to the n-1 unit spacing. Since all the integers from 1 to n-1 must be used, the sequence, if it is to include 2, must terminate with 2. Consequently the maximum optimum sequence is a train of four pulses with the spacings and autocorrelation function shown in Figure 11.

3.2 Approaches to Non Optimum Pulse Trains

Since optimum ambiguity surfaces with a completely uniform volume spread cannot, in general, be generated using these non-uniformly spaced pulse trains three alternative approaches to further study suggest themselves:

a) relax the unit sidelobe level constraint and investigate achievable uniformity of volume spread using the spacings $1 \rightarrow n-1$;

b) retain the unit sidelobe level constraint, make use of the spacings from $1 \rightarrow m > n-1$, and determine which spacings yield n pulses with the minimum $TW/2\pi$;

c) retain the unit sidelobe level constraint, make use of the spacings $\Delta \rightarrow \Delta + n-2$ and examine the limits on n vs Δ and $TW/2\pi$.

Alternative (a) presents several difficulties, in particular the problem of determining what makes a reasonable criterion for uniformity of volume spread. The significance of using the spacings $1 \rightarrow n-1$ can also be

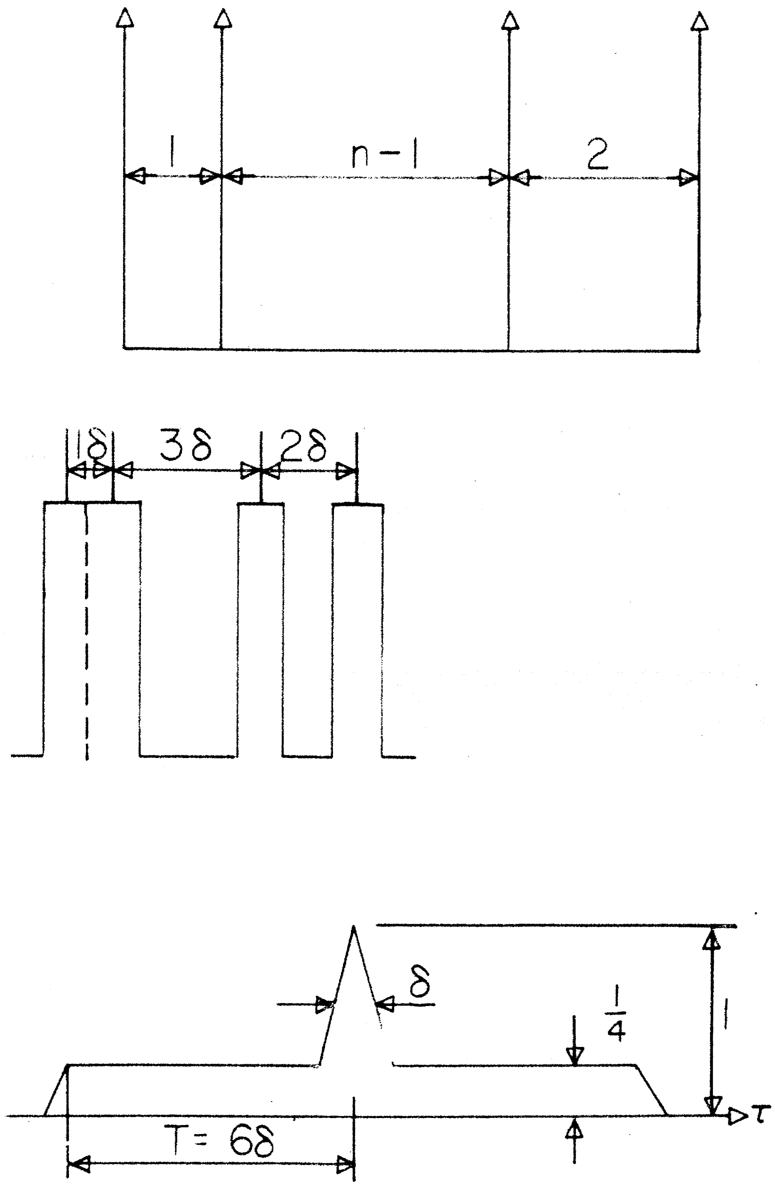

Fig 11 MAXIMUM LENGTH OPTIMUM PULSE TRAIN AND ITS AUTOCORRELATION FUNCTION

questioned since they relate only to the optimum pulse train.

Alternative (b) is a reasonable approach to finding waveforms with as high a performance as possible (n maximum) for as small a price as possible ($TW/2\pi$ minimized.) A limitation of this approach is that the large variation in interpulse period may result in a poorer spectrum than could be achieved with a more nearly uniform spacing.

Alternative (c) is equivalent to requiring, in addition to the unit sidelobe constraint, that the minimum $TW/2\pi$ be achieved given that the first sidelobe can be no closer to the central response than Δ. This is often important when closely spaced targets are to be examined and it is desired that these adjacent targets not interfere with one another. In radar astronomy the examination of a diffuse plasma might suggest such a requirement. With a reasonable selection of Δ, the clear area of the ambiguity surface (the area of infinite contrast) is virtually the same as for a uniform pulse train. Further, the addition of Δ to all spacings tends to make the train more uniform and improves the response along the doppler axis.

Considering the advantages inherent in the last of the three alternatives and the reasonableness of the additional constraint this analysis will be limited to that sub-class of possible spacings.

It should be clear that there will often be more than one waveform, or sequence of pulse spacings, which are optimum in the sense that, a) they have the smallest $TW/2\pi$ consistent with a clear distance Δ to the first sidelobe and, b) they have non-overlapping or interfering sidelobes in time.

Three of the more obvious approaches to generating a non uniformly spaced pulse train with this constraint are shown in Figure 12. The monotonic increases in interpulse period and the alternating increase in interpulse period shown in this Figure could just as easily have been decreases. This would merely have required redefining Δ -(n-2) as Δ', Δ -(n-3) as Δ' + 1, ..., Δ as Δ' + (n-2).

Which of several possible trains should be used would be determined by, a) the spectrum $\psi(0,\omega)$; $0 \leq \omega \leq 2\pi/\Delta$, b) the approach to the design of the matched filter, c) the sensitivity to errors and hardware limitations, and d) the possibility of increased freedom from interference by greater isolation of time sidelobes. These considerations will be discussed later.

3.3 Results for Trains with Few Pulses

Before discussing general approaches to obtaining the variations in interpulse period and the limits to the performance achievable with each of these approaches, it is interesting to examine the performance actually achievable for pulse trains short enough to permit an extensive examination of all combinations of spacings. Figure 13 shows a plot of the $TW/2\pi$ actually required for several values of n, the minimum (but generally unrealizable) $TW/2\pi$ and the functions $\frac{n^{5/2}}{4}$ and $\frac{n^2 \ln n}{4}$ vs the number of pulses, n. The values of $TW/2\pi$ actually required for several values of n and the sequence of spacings required are shown in Figure 14.

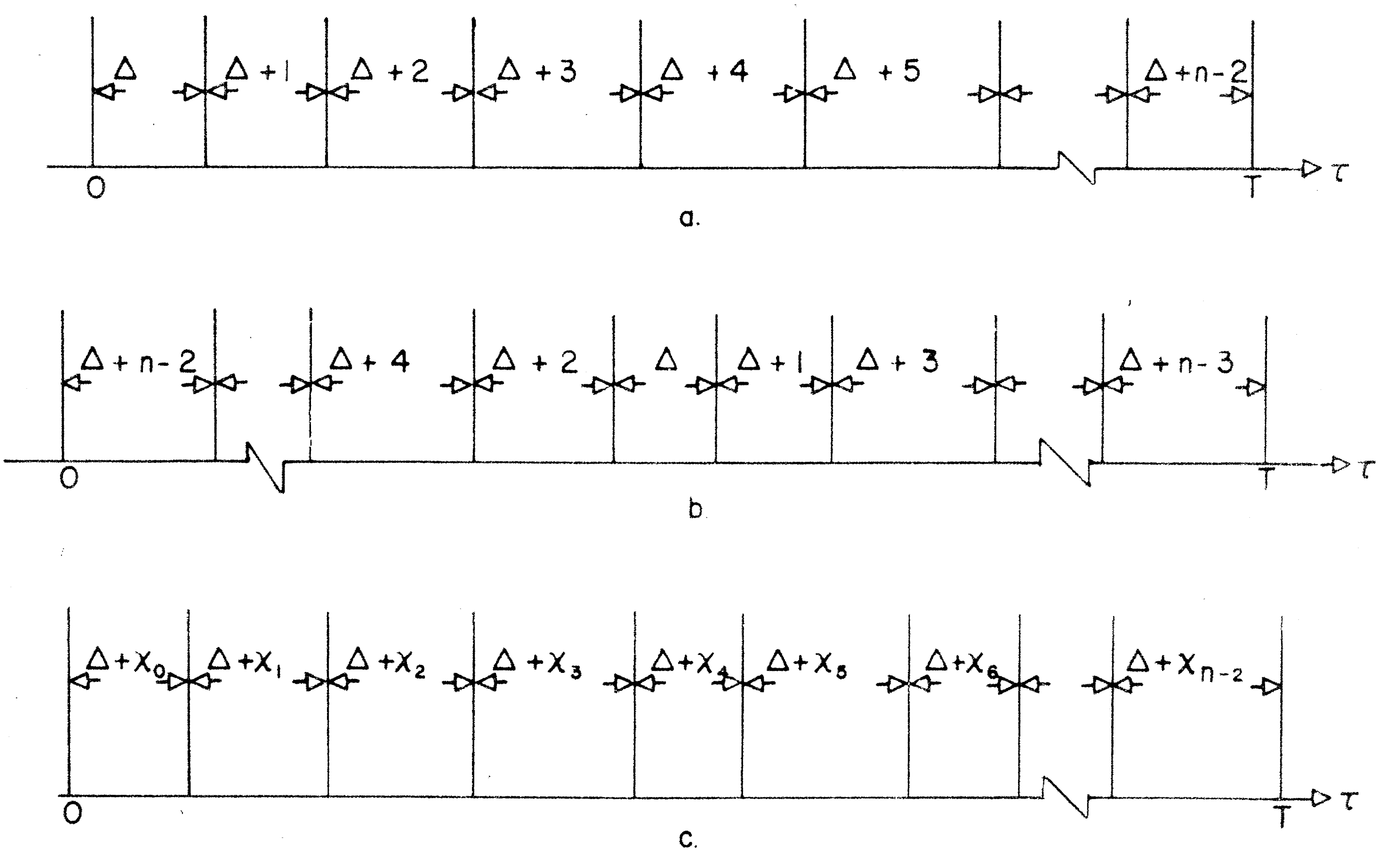

Fig 12 THREE APPROACHES TO GENERATING NON-UNIFORMLY SPACED PULSE TRAINS

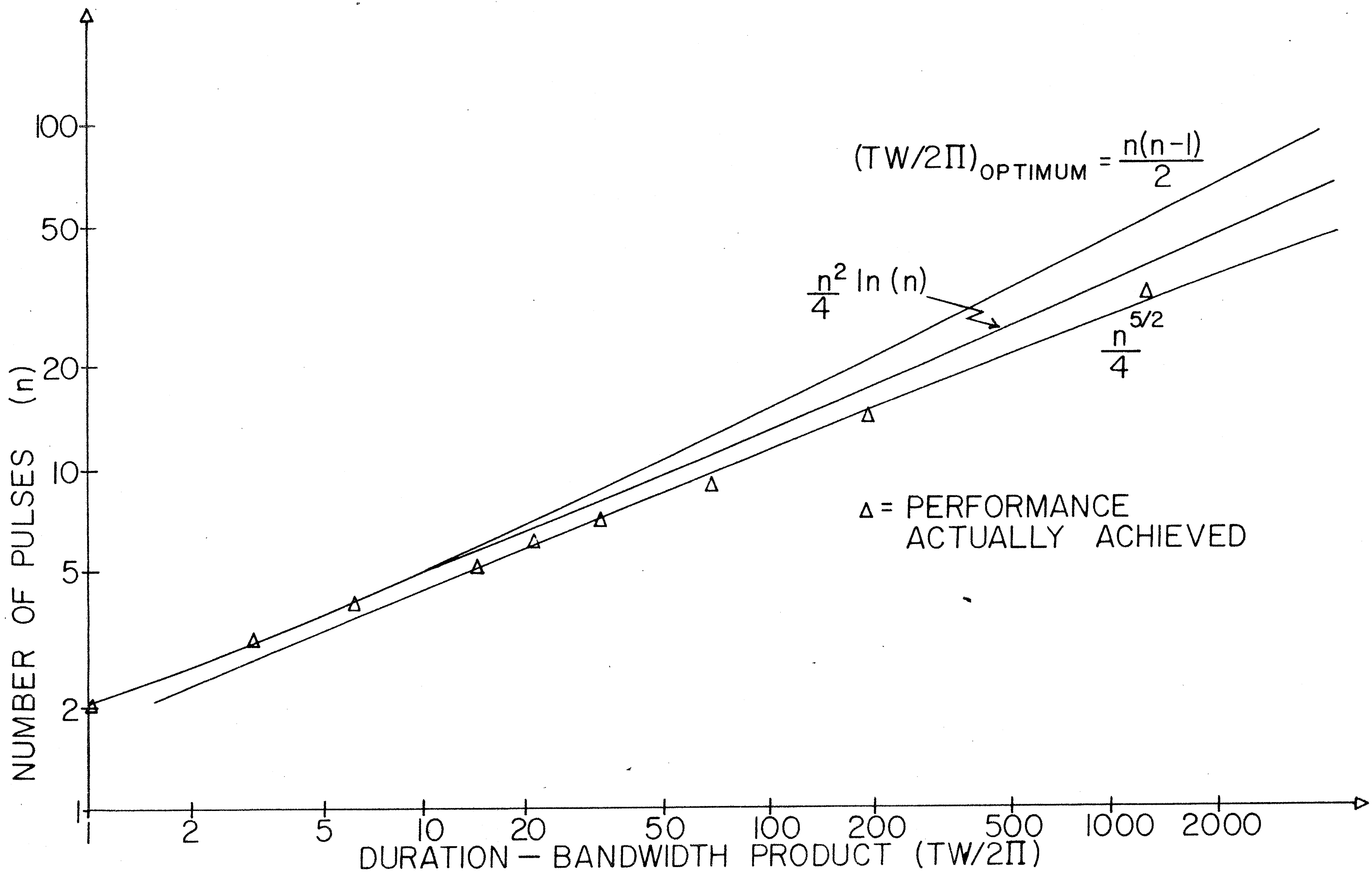

Fig. 13. NUMBER OF PULSES ACHIEVABLE AS A FUNCTION OF TW/2Π.

n	$(TW/2\pi)_{optimum}$	$(TW/2\pi)_{required}$	Sequence [Normalized to δ, the pulse width]
2	1	1	1
3	3	3	1, 2
4	6	6	1, 3, 2
5	10	14	2, 4, 3, 5
6	15	20	2, 6, 5, 4, 3
7	21	33	4, 7, 5, 8, 6, 3
8	28	49*	7, 4, 8, 5, 9, 6, 10
9	36	68*	6, 7, 8, 9, 10, 12, 11, 5
12	66	110*	5, 12, 8, 15, 11, 7, 14, 10, 6, 13, 9
14	91	195*	9, 14, 19, 11, 16, 21, 13, 18, 10, 15, 20, 12, 17

Figure 14: Tabulation of $TW/2\pi$ and sequence of spacings for small values of n

* These values are the best achievable after an extensive but not exhaustive search.

CHAPTER IV

ANALYSIS OF TWO SPECIFIC APPROACHES TO SELECTION OF INTERPULSE SPACINGS

4.1 The General Approach

To determine, in the general case, how the X_i's should be chosen to ensure non-overlap of time sidelobes it is necessary to compute the location of each of the time sidelobes. Figure 15 shows one approach to an orderly handling of this problem. The "matrix" of numbers is the time difference between all pulses in the train shown in Figure 12c. The times between adjacent pulses are located, sequentially, in column 1, the time between alternate pulses in column 2, the time between every third pulse in column 3, etc. With n pulses there will be n-1 first differences, and consequently n-1 numbers in column 1. There will be n-2 second differences and so n-2 numbers in column 2, etc., and in column n-1 there will be only one number, equal to the total time between the first and last pulses.

With the time difference matrix written in this form it is evident that within the first column no two numbers will be the same if all the X's are different; within the second column no two will be the same if the sums of consecutive X's, taken two at a time, are different; within the third column if the sums of consecutive X's taken three at a time are different; within the fourth column if taken four at a time; etc. Further, if the numbers within each of the columns are different then it is clear that some Δ is large enough to ensure that the maximum number in every column is smaller than the minimum number in the next higher column. Consequently, if the numbers within a column are unequal and the columns are ordered so

k	Column 1	Column 2	Column 3	Column 4	Column 5	etc.
n	$\Delta + \chi_{n-2}$	$2\Delta + \chi_{n-2} + \chi_{n-3}$	$3\Delta + \chi_{n-2} + \chi_{n-3} + \chi_{n-4}$	$4\Delta + \sum_{n-5}^{n-2} \chi_i$	$5\Delta + \sum_{n-6}^{n-2} \chi_i$	
.	.	.	.	.	.	
.	.	.	.	.	.	
.	.	.	.	.	.	
j	$\Delta + \chi_{j-2}$	$2\Delta + \chi_{j-2} + \chi_{j-3}$	$3\Delta + \chi_{j-2} + \chi_{j-3} + \chi_{j-4}$	$4\Delta + \sum_{i=j-5}^{j-2} \chi_i$	$5\Delta + \sum_{i=j-6}^{j-2}$	
.	.	.	.	.	.	
.	.	.	.	.	.	
.	.	.	.	.	.	
7	$\Delta + \chi_5$	$2\Delta + \chi_6 + \chi_5$	$3\Delta + \chi_3 + \chi_4 + \chi_5$	$4\Delta + \chi_2 + \chi_3 + \chi_4 + \chi_5$	$5\Delta + \sum_{i=1}^{5} \chi_i$	
6	$\Delta + \chi_4$	$2\Delta + \chi_3 + \chi_4$	$3\Delta + \chi_2 + \chi_3 + \chi_4$	$4\Delta + \chi_1 + \chi_2 + \chi_3 + \chi_4$	$5\Delta + \sum_{i=0}^{4} \chi_i$	
5	$\Delta + \chi_3$	$2\Delta + \chi_2 + \chi_3$	$3\Delta + \chi_1 + \chi_2 + \chi_3$	$4\Delta + \chi_0 + \chi_1 + \chi_2 + \chi_3$		
4	$\Delta + \chi_2$	$2\Delta + \chi_1 + \chi_2$	$3\Delta + \chi_0 + \chi_1 + \chi_2$			
3	$\Delta + \chi_1$	$2\Delta + \chi_0 + \chi_1$				
2	$\Delta + \chi_0$					

Figure 15: Time difference matrix for pulse train in Figure 12c

that all numbers in column 1 are less than all numbers in column 2, which in turn are less than all numbers in column 3, etc. then no two numbers will be alike and non-overlap of sidelobes will be assured.

The difference matrix also points up the fact that the numbers in successive columns are not independent of one another. If all the numbers in column 1 are different then all adjacent numbers within every other column are also different. If all the numbers in column 2 are different then all alternate numbers within each column are different. In general if all the numbers in column j are different, then all the elements j numbers apart within all the other columns are different. Consequently if the numbers within columns 1 through $\frac{n}{2}$ are different then the numbers within the rest of the columns are also different. The number of independent comparisons actually necessary to assure no equality within any of the columns can be rapidly determined.

Assume that Δ has been chosen so that the maximum number in each column is less than the minimum number in the next higher column or that

Column (a) Maximum < Column (a+1) Minimum for $[n-1 > a > 0]$

There are (n-1) terms in the first column. For all terms to be compared requires a total number of comparisons equal to

$$C_1 = (n-2) + (n-3) + (n-4) + \ldots\ldots + (1)$$

$$= (n-2) \cdot \left(\frac{n-2+1}{2}\right) = \frac{(n-1)(n-2)}{2}$$

There are n-2 terms in the second column. Inequality of all first column terms guarantee inequality of all adjacent second column terms. Therefore, for all unproven terms to be compared requires

$$C_2 = (n-4) + (n-5) + (n-6) + \ldots\ldots + (1)$$

$$= (n-4)\ (\frac{n-4+1}{2}) = \frac{(n-3)\ (n-4)}{2}$$

Similarly with the n-3 terms in the third column, all adjacent terms are unequal due to the first column comparison and all alternate terms unequal. due to the second column comparison. Therefore, the number of comparisons required is

$$C_3 = (n-6) + (n-7) + (n-8) + \ldots\ldots + 1$$

$$= (n-6)\ (\frac{n-6+1}{2}) = \frac{(n-5)\ (n-6)}{2}$$

Therefore, in Column (a)

$$C_a = \frac{(n-2a+1)\ (n-2a)}{2}$$

and when $a = \frac{n}{2}$ or $\frac{n+1}{2}$ $\quad C_a = 0$

As was indicated before the numbers in the columns from $\frac{n}{2}$ to n-1 are different providing the first $\frac{n}{2}$ are different.

Finally

$$C_T = \sum_{d=1}^{d=a_{Max}} C_d = \sum_{d=1}^{d=n/2} \frac{(n-2a+1)\ (n-2a)}{2}$$

$$C_T = \frac{n^3}{12} - \frac{n^2}{8} - \frac{n}{12} = \frac{n(n+1/2)\ (n-2)}{12} \qquad (25)$$

With this number of computations necessary to guarantee that no two numbers within any of the columns are equal, a cut-and-try approach, for example, choosing the X's at random and then making the necessary comparisons, is extremely impractical. Consequently some constraint must be brought to bear on the sequence of X's to permit an orderly examination of the limits on n as a function of $TW/2\pi$ and Δ.

4.2 Train with an Arithmetic Increase in Interpulse Spacing

4.21 Limits on $TW/2\pi$ and Δ vs n

Figure 16 shows the time difference matrix for the pulse train having an arithmetic increase in interpulse spacing, shown in Figure 12a. Since the numbers within each column are monotonically increasing, they cannot be equal. Consequently no two numbers in the difference matrix have the same value (and no two sidelobes occur at the same time) if Δ is selected so that the maximum value in column (a) is less than the minimum value in column (a+1) for $n-1 > a > 0$.

k	Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7
j	1[Δ+j-2]	2[Δ+j-2 1/2]	3[Δ+j-3]	4[Δ+j-3 1/2]	5[Δ+j-4]	6[Δ+j-4 1/2]	7[Δ+j-5]
10	$\Delta + 8$	$2\Delta + 15$	$3\Delta + 21$	$4\Delta + 26$	$5\Delta + 30$	$6\Delta + 33$	$7\Delta + 35$
9	$\Delta + 7$	$2\Delta + 13$	$3\Delta + 18$	$4\Delta + 22$	$5\Delta + 25$	$6\Delta + 27$	$7\Delta + 28$
8	$\Delta + 6$	$2\Delta + 11$	$3\Delta + 15$	$4\Delta + 18$	$5\Delta + 20$	$6\Delta + 21$	$7\Delta + 21$
7	$\Delta + 5$	$2\Delta + 9$	$3\Delta + 12$	$4\Delta + 14$	$5\Delta + 15$	$6\Delta + 15$	
6	$\Delta + 4$	$2\Delta + 7$	$3\Delta + 9$	$4\Delta + 10$	$5\Delta + 10$		
5	$\Delta + 3$	$2\Delta + 5$	$3\Delta + 6$	$4\Delta + 6$			
4	$\Delta + 2$	$2\Delta + 3$	$3\Delta + 3$				
3	$\Delta + 1$	$2\Delta + 1$					
2	Δ						

General Term $= a[\Delta + k - \frac{3 + a}{2}]$

where $a = 1, 2, 3, \ldots, n-1$ = column number

$k = a+1, a+2, a+3, \ldots, n$ = element number

Figure 16: Time difference matrix for pulse train in Figure 12a

Therefore if

$$\text{Column (a) Maximum} < \text{Column (a+1) Minimum} \tag{26}$$

then

$$a[\Delta + k - \frac{3 + a}{2}]_{Max} < (a+1)\ [\Delta + k - \frac{3 + a + 1}{2}]_{Min}$$

$$a[\Delta + n - \frac{3 + a}{2}] < (a+1)\ [\Delta + (a+1) - \frac{3 + a + 1}{2}]$$

and

$$n < \frac{\Delta}{a} + a + 2$$

for $n-1 > a > 0$

For any Δ, the tightest bound on n occurs when

$$\frac{d}{da}\ [\ \frac{\Delta}{a} + a + 2\ ] = 0$$

or

$$a = \sqrt{\Delta} \tag{27}$$

and for this value of (a) the lowest value of n, n_L, is

$$n_L < \frac{\Delta}{\sqrt{\Delta}} + \sqrt{\Delta} + 2 = 2\sqrt{\Delta} + 2 \leq n_L + 1$$

or

$$\left(\frac{n_L - 2}{2}\right)^2 < \Delta \leq \left(\frac{n_L - 1}{2}\right)^2 . \tag{28}$$

Since

$$TW/2\pi = \frac{T}{\delta} = \sum_{i=1}^{n-1} [\ \Delta + (i-1)\ ] = (n-1)\ \Delta + \frac{(n-1)(n-2)}{2} \tag{29}$$

$$\therefore (n_L-1)\left(\frac{n_L-2}{2}\right)^2 + \frac{(n_L-1)(n_L-2)}{2} < TW/2\pi \leq (n_L-1)\left(\frac{n_L-1}{2}\right)^2 + \frac{(n_L-1)(n_L-2)}{2}$$

$$\frac{n_L(n_L-1)(n_L-2)}{4} < TW/2\pi \leq \frac{(n_L-1)(n_L^2-3)}{4} \tag{30}$$

and for $n_L \gg 1$

$$\frac{n_L^{\ 3}}{4} \simeq TW/2\pi \tag{31}$$

Equation (30) bounds the minimum number of pulses obtainable for any $TW/2\pi$, using this approach to synthesizing non uniformly spaced pulse trains. To bound the maximum number of pulses achievable with this approach it is necessary to insure not only that the maximum in one column exceeds the minimum in the next but that a number in one column equals a number in another column. The monotonic increase by (a) of successive numbers in column (a) limits the maximum increase in the value of n_L to no more than (a) before an equality is assured.

Consequently

$$n_U = n_L + a$$

and

$$n_U < 2\sqrt{\Delta} + 2 + a \leq n_U + 1$$

or

$$n_U < 3\sqrt{\Delta} + 2 \leq n_U + 1$$

and consequently

$$\left(\frac{n_U-2}{3}\right)^2 < \Delta \leq \left(\frac{n_U-1}{3}\right)^2 \qquad (32)$$

Using equation (29) this results in

$$\frac{(n_U-2)(n_U-1)(n_U+2\ 1/2)}{9} < TW/2\pi \leq \frac{(n_U-1)(n_U^2 + 2\ 1/2\ n_U-8)}{9} \qquad (33)$$

and for $n_U \gg 1$

$$\frac{n_U^3}{9} \approx TW/2\pi \qquad (34)$$

Consequently, for large n, the bound on n^2, the maximum sidelobe level, is

$$\left(9TW/2\pi \right)^{2/3} \geq n^2 \geq \left(4TW/2\pi \right)^{2/3} \tag{35}$$

where $$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,\omega) \qquad \text{for } |\tau| \geq \delta \tag{15}$$

Plots of n vs Δ and $TW/2\pi$ for the upper and lower limits obtained here, as well as for the actual maximum values calculated for several values of n are shown in Figures 17 and 18. A detailed examination, after many such calculations, shows the lower bound on n to be a closer bound and probably a better index of actual performance than the upper bound.

The performance for a monotonically decreasing spacing between pulses is virtually the same as for the monotonic increase, with

$$\frac{n_L(n_L-1)(n_L-2)}{4} < TW/2\pi \leq \frac{(n_L-1)(n_L^2+1)}{4} \tag{36}$$

and for $n_{L,U} \gg 1$ also results in

$$\frac{n_U^3}{9} \geq TW/2\pi \geq \frac{n_L^2}{4} \tag{37}$$

The uniformity of volumn spread can be determined by comparing the

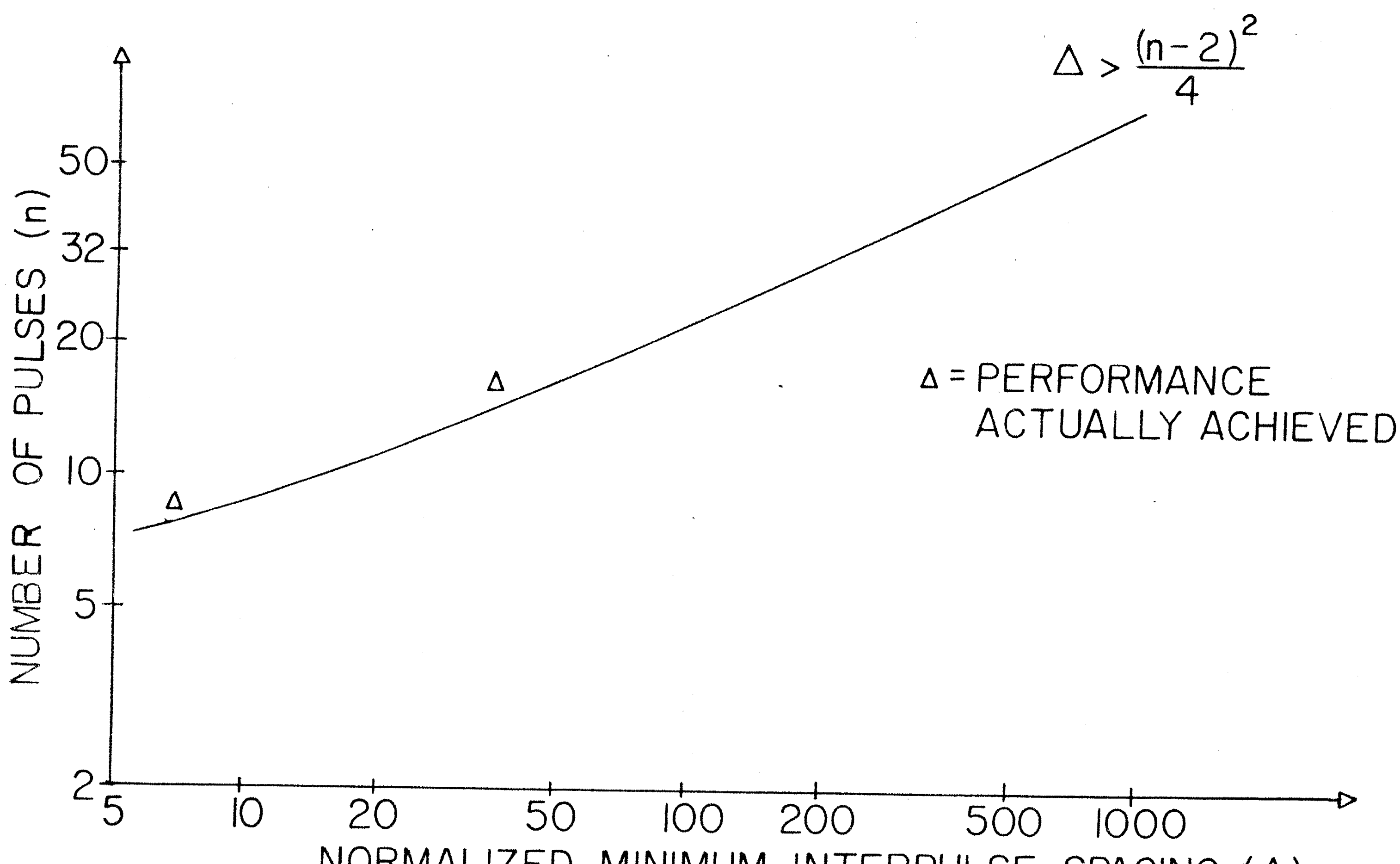

Fig. 17. NUMBER OF PULSES (n) AS A FUNCTION OF Δ FOR AN ARITHMETIC INCREASE IN INTERPULSE SPACING

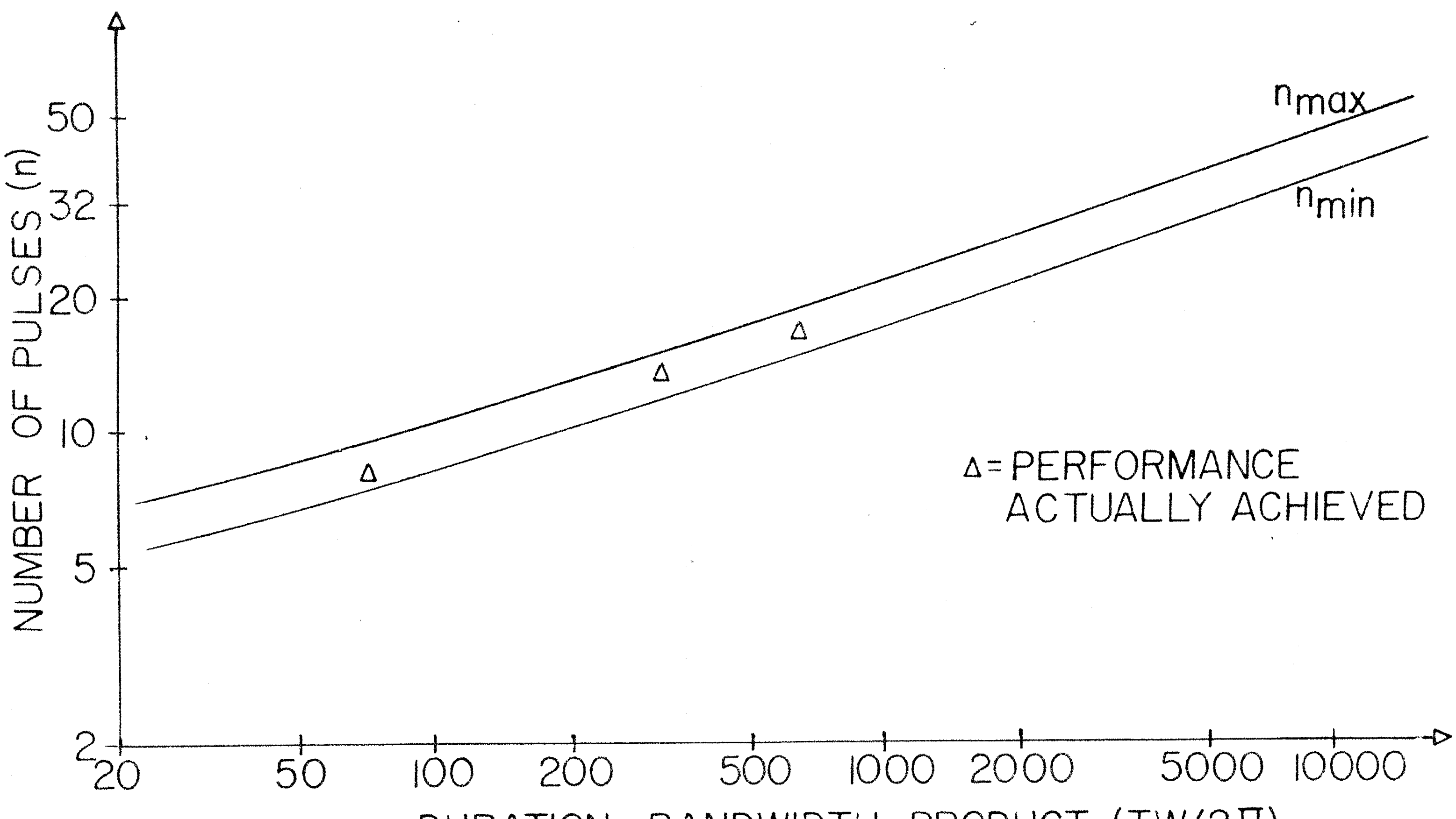

Fig. 18. NUMBER OF PULSES (n) AS A FUNCTION OF TW/2Π FOR AN ARITHMETIC INCREASE IN INTERPULSE SPACING

performance achieved with the optimum performance achievable.
For $\Delta \approx \frac{n^2}{4}$ and $TW/2\pi \approx \frac{n^3}{4}$, a completely uniform volume spread corresponds to

$$\psi(\tau,\omega) \leq \frac{\psi(0,0)}{2TW/2\pi} \simeq \frac{\psi(0,0)}{2\left(\frac{n^3}{4}\right)} = \frac{\psi(0,0)}{\frac{n^3}{2}}$$

But the actual performance is

$$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,\omega) \qquad \text{for } |\tau| \geq \delta \tag{15}$$

Consequently the efficiency of this technique towards achieving a uniform volume spread is

$$\eta = \frac{2}{n} \tag{38}$$

4.22 The Spectrum

The spectrum, $H(\omega)$, of a train of n impulses, with an arithmetic increase (or decrease) in interpulse spacing (shown in Figure 19) is

$$H(\omega) = \int_{-\infty}^{\infty} e^{j\omega t} \Big[u_o(t + \tfrac{\Delta}{2}) + u_o(t - \tfrac{\Delta}{2}) \qquad +$$

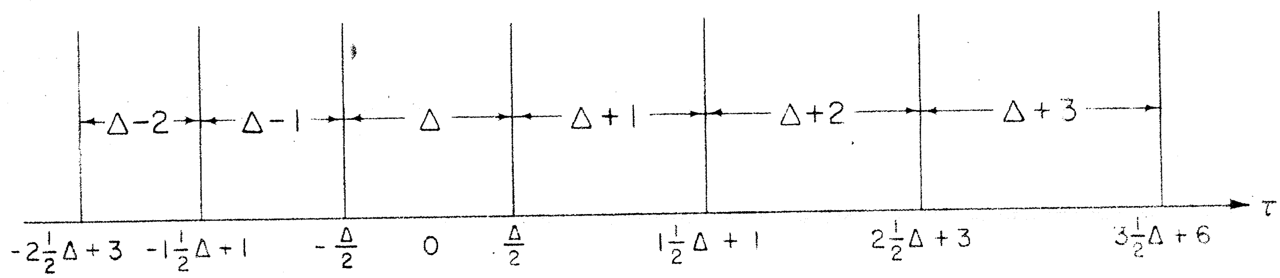

Fig. 19. Part of a pulse train with an arithmetic increase in interpulse spacing.

$$+ u_o(t + 1\tfrac{1}{2}\Delta + 1) + u_o(t - 1\tfrac{1}{2}\Delta + 1)$$

$$+ u_o(t + 2\tfrac{1}{2}\Delta + 3) + u_o\ (t - 2\tfrac{1}{2}\Delta + 3)$$

$$+ u_o(t + 3\tfrac{1}{2}\Delta + 6) + u_o(t - 3\tfrac{1}{2}\Delta + 6)$$

$$+ \;.....$$

$$+ u_o(t + \frac{n-1}{2}\ \Delta + \sum_{k=1}^{n} k) + u_o(t - \frac{n-1}{2}\ \Delta + \sum_{k=1}^{n} k)\ \Big]dt$$

$$\therefore H(\omega) = e^{j\omega\frac{\Delta}{2}} + e^{-j\omega\frac{\Delta}{2}}$$

$$+ e^{j\omega[1\frac{1}{2}\Delta + 1]} + e^{-j\omega[1\frac{1}{2} - 1]}$$

$$+ e^{j\omega[2\frac{1}{2}\Delta + 3]} + e^{-j\omega[2\frac{1}{2}\Delta - 3)}$$

$$+ e^{j\omega[3\frac{1}{2}\Delta + 6]} + e^{-j\omega[3\frac{1}{2}\Delta - 6]}$$

$$+ \;.....$$

$$+ e^{j\omega[\frac{n-1}{2}\Delta + \sum_{k=0}^{\frac{n-2}{2}} k]} + e^{-j\omega[\frac{n-1}{2}\Delta - \sum_{k=0}^{\frac{n-2}{2}} k\ ]}$$

Finally

$$H_m(\omega) = e^{j\omega\frac{\Delta}{2}} + e^{-j\omega\frac{\Delta}{2}}$$

$$+ \left[e^{j\omega\frac{3\Delta}{2}} + e^{-j\omega\frac{3\Delta}{2}} \right] e^{j\omega}$$

$$+ \left[e^{j\omega\frac{3\Delta}{2}} + e^{-j\omega\frac{5\Delta}{2}} \right] e^{j\omega 3}$$

$$+ \left[e^{j\omega\frac{7\Delta}{2}} + e^{-j\omega\frac{7\Delta}{2}} \right] e^{j\omega 6}$$

$$+ \cdots$$

$$+ \left[e^{j\omega\left(\frac{n-1}{2}\right)\Delta} + e^{-j\omega\left(\frac{n-1}{2}\right)\Delta} \right] e^{j\omega\sum_{k=0}^{\frac{n-2}{2}} k}$$

The only difference between the spectrum of this train and the spectrum of a pulse train with a constant interpulse period is the introduction of an equal phase shift, of the form $e^{j\omega\Sigma k}$, to symmetrically spaced pulses. This phase shift vs n is shown in Figure 20. For $\omega = 0$ all the phase terms are $1\underline{/0^\circ}$ and the spectrum is identical to that for a constant interpulse period train. As long as ω is small enough so that the maximum variation in phase is less than $\pm\frac{\pi}{4}$ from the average phase across the pulse train we may expect the two spectrums to be virtually indistinguishable. This requires that

$$\frac{\omega_D}{2} \sum_{k=0}^{\frac{n-2}{2}} k\delta \leq \frac{\pi}{4}$$

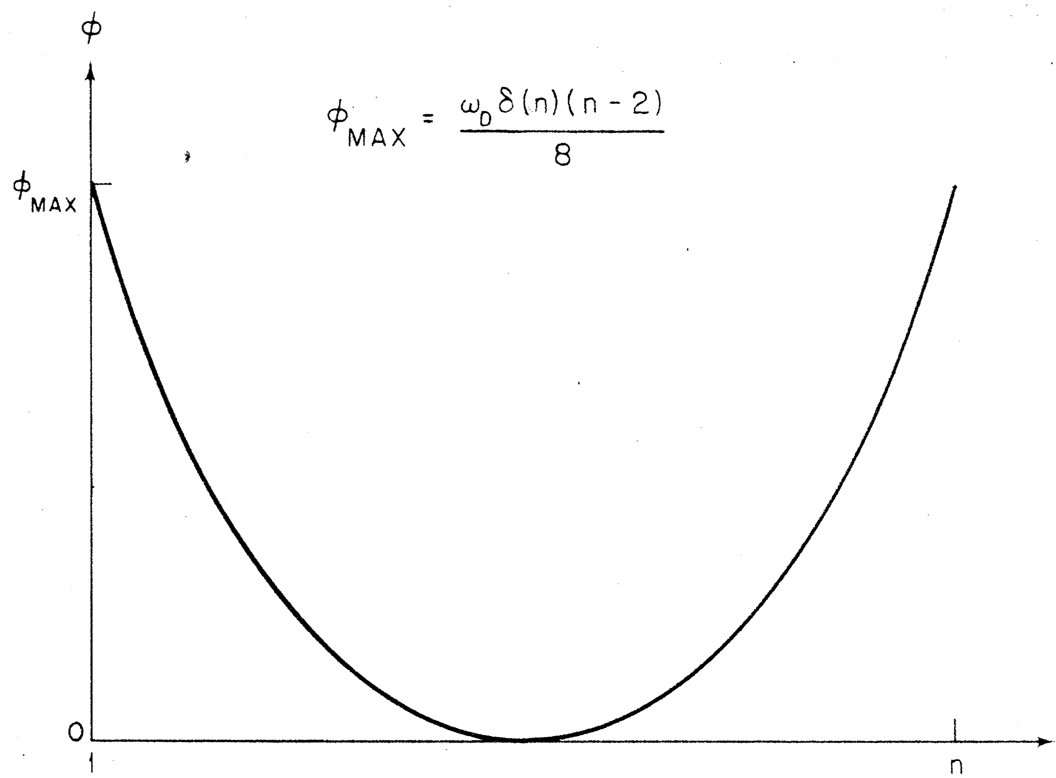

Fig. 20. Differential phase shift due to arithmetic increase in spacing as a function of pulse position for a train with n pulses.

or

$$f_D < \frac{1}{4\delta \sum_{k=0}^{\frac{n-2}{2}} k} = \frac{2}{\delta n(n-2)}$$

Since $n \approx \sqrt{4\Delta/\delta}$

then for large n

$$f_D < \frac{2}{\delta(4\Delta/\delta)} = \frac{1}{2\Delta}$$

Since, in the design of these waveforms it is reasonable to set $\Delta \simeq \frac{1}{f_{D\ Max}}$,

then

$$f_D < \frac{1}{2(1/f_{D\ Max})} = \frac{f_{D\ Max}}{2} \tag{40}$$

Hence we would expect a virtually exact correspondence in spectral response between a uniformly spaced train and a monotonically varying train as far as one-half of the distance to the first doppler ambiguity. This correspondence can be seen in Figure 21 which shows the actual spectrum of a train of 16 pulses with an arithmetic increase in interpulse spacing, and the spectrum of a uniformly spaced train of 16 pulses having the same duration.

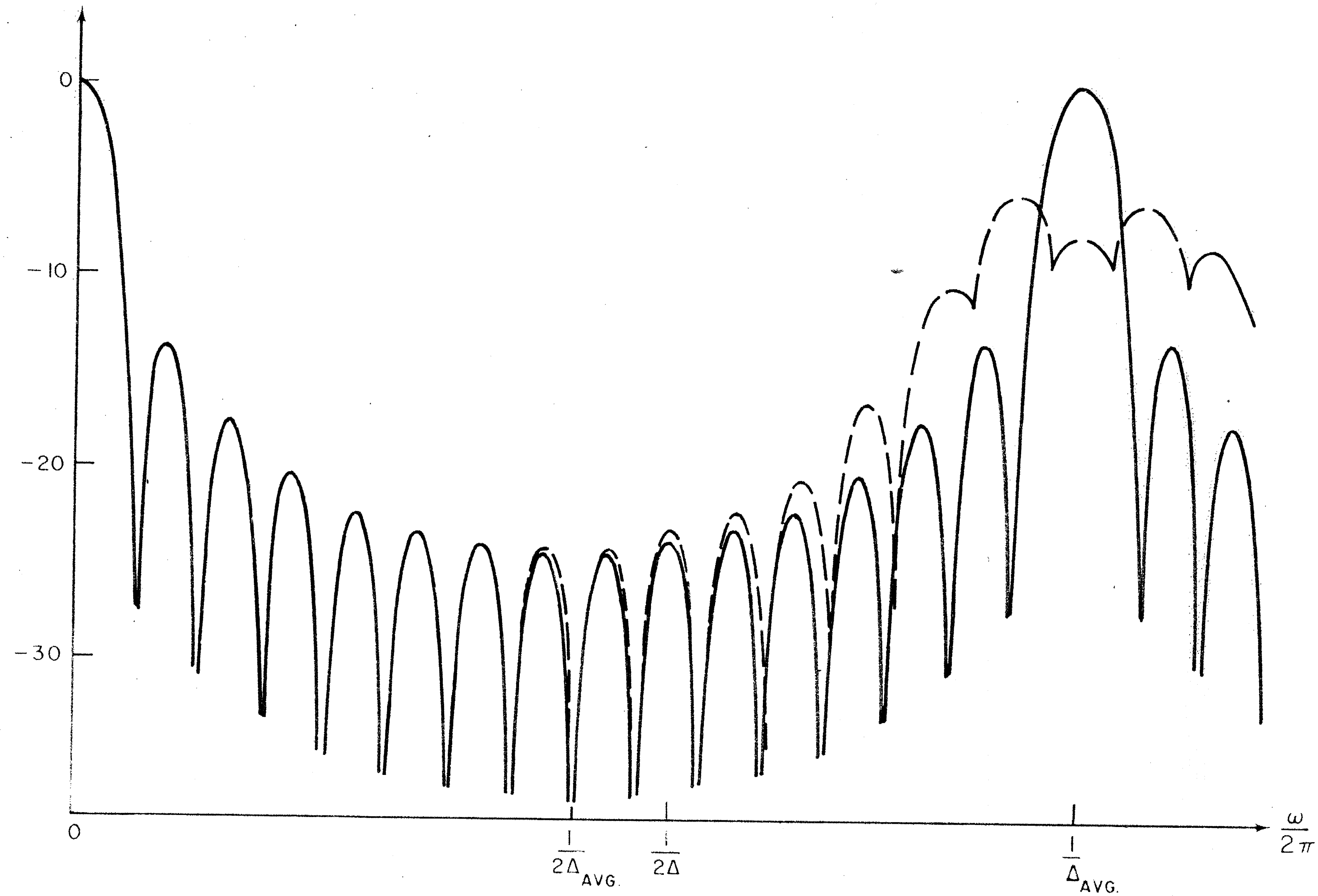

Fig. 21. Spectrum of uniformly spaced train and non-uniformly spaced train (Fig. 12a) with 16 pulses and equal $TW/2\pi$.

An analogy can be drawn between the effect of the non-uniform spacing on the spectrum and the effect of a linear sweep in frequency, coupled with an amplitude tapering of the low frequency (large interpulse spacing) components.

To determine the actual extent of the non uniformity consider the fractional variation in interpulse spacing.

$$\frac{\Delta_{Max} - \Delta_{Min}}{\Delta_{Min}} = \frac{[\Delta_{Min} + (n-2)] - \Delta_{Min}}{\Delta_{Min}} \approx \frac{n-2}{\frac{n^2}{4}} \approx \frac{4}{n}$$

Consequently the larger the value of n the smaller the fractional variation in interpulse period (and the smaller the equivalent frequency sweep and amplitude taper) and consequently the more nearly uniform the pulse train becomes. For $n > 16$ a spectrum even more like that of a uniform train will result.

4.3 <u>Train with Interpulse Spacing Arithmetically Increasing from Center</u>

4.31 Limits on $TW/2\pi$ and Δ vs n

The number of pulses achievable for any Δ or $TW/2\pi$, can be increased, using the pulse train spacing shown in Figure 12b rather than the arithmetic increase of interpulse period. Figure 22 shows the time difference matrix for this pulse train. The general term for the <u>minimum</u> in each column is

$$a\Delta + \sum_{0}^{a-1} c = a\Delta + \frac{a(a-1)}{2} \qquad \text{for } n > a > 0$$

k		Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
11		$\Delta + 9$	$2\Delta + 16$	$3\Delta + 21$	$4\Delta + 24$	$5\Delta + 25$	$6\Delta + 25$	$7\Delta + 27$	$8\Delta + 31$	$9\Delta + 37$
9		$\Delta + 7$	$2\Delta + 12$	$3\Delta + 15$	$4\Delta + 16$	$5\Delta + 16$	$6\Delta + 18$	$7\Delta + 22$	$8\Delta + 28$	
7		$\Delta + 5$	$2\Delta + 8$	$3\Delta + 9$	$4\Delta + 9$	$5\Delta + 11$	$6\Delta + 15$			
5		$\Delta + 3$	$2\Delta + 4$	$3\Delta + 4$	$4\Delta + 6$					
3		$\Delta + 1$	$2\Delta + 1$							
2		Δ								
4		$\Delta + 2$	$2\Delta + 2$	$3\Delta + 3$						
6		$\Delta + 4$	$2\Delta + 6$	$3\Delta + 6$	$4\Delta + 7$	$5\Delta + 10$				
8		$\Delta + 6$	$2\Delta + 10$	$3\Delta + 12$	$4\Delta + 12$	$5\Delta + 13$	$6\Delta + 16$	$7\Delta + 21$		
10		$\Delta + 8$	$2\Delta + 14$	$3\Delta + 18$	$4\Delta + 20$	$5\Delta + 20$	$6\Delta + 21$	$7\Delta + 24$	$8\Delta + 28$	$9\Delta + 36$
Min	=	Δ	$2\Delta + 1$	$3\Delta + 3$	$4\Delta + 6$	$5\Delta + 10$	$6\Delta + 15$	$7\Delta + 21$	$8\Delta + 28$	$9\Delta + 36$
or		$\Delta + \sum_{o}^{o} c$	$2\Delta + \sum_{o}^{1} c$	$3\Delta + \sum_{o}^{2} c$	$4\Delta + \sum_{o}^{3} c$	$5\Delta + \sum_{o}^{4} c$	$6\Delta + \sum_{o}^{5} c$			
Max	=	$1(\Delta-2+j)$	$2(\Delta-3+j)$	$3(\Delta-4+j)$	$4(\Delta-5+j)$	$5(\Delta-6+j)$	$6(\Delta-7+j)$	etc for $a \leq \frac{n}{2}$		

Figure 22: Time difference matrix for the pulse train shown in Figure 12b
[an increase in pulse spacing from the center of the train]

The general term for the maximum in each column is

$$[\,\Delta - (a + 1) + k\,] \qquad \text{for} \qquad \frac{n}{2} \geq a > 0.$$

The elements within any column are clearly unequal. Then no two elements in the difference matrix have the same value if

$$\text{Column (a) Maximum} < \text{Column (a+1) Minimum} \tag{26}$$

$$\therefore\ a[\,\Delta - (a + 1) + n\,] < (a+1)\,\Delta + \frac{(a+1-1)\,(a+1)}{2} \qquad [a \leq \tfrac{n}{2}]$$

$$n < \frac{\Delta}{a} + \frac{3}{2}\,(a + 1)$$

When

$$\frac{d}{da}\left[\,\frac{\Delta}{a} + \frac{3}{2}\,(a+1)\,\right] = 0$$

or

$$a = \left(\frac{2\Delta}{3}\right)^{1/2} \tag{41}$$

then n is most tightly bound.

$$\therefore n_L < \frac{\Delta}{\sqrt{\frac{2}{3}\Delta}} + \frac{3}{2}\sqrt{\frac{2\Delta}{3}} + \frac{3}{2} < n_L + 1$$

and

$$n_L < \sqrt{6\Delta} + \frac{3}{2} \leq n_L + 1$$

or

$$\frac{\left(n_L - \frac{3}{2}\right)^2}{6} < \Delta \leq \frac{\left(n_L - \frac{1}{2}\right)^2}{6} \tag{42}$$

and since

$$TW/2\pi = (n-1)\,\Delta + \frac{(n-2)(n-1)}{2} \tag{29}$$

then

$$\frac{\left(n_L - 1\right)\left(n_L^2 - 3\tfrac{3}{4}\right)}{6} < TW/2\pi \leq \frac{\left(n_L - 1\right)\left(n_L^2 + 2n_L - 5\tfrac{3}{4}\right)}{6} \tag{43}$$

For $n_L \gg 1$

$$\frac{n_L^3}{6} \approx TW/2\pi \tag{44}$$

The upper bound on the performance of this approach can be evaluated if, as before,

$$n_L + a = n_U$$

or since

$$n_u < \sqrt{6\Delta + \frac{3}{2}} + \sqrt{\frac{2}{3}\Delta} = \sqrt{\Delta}\left[\sqrt{6} + \sqrt{\frac{3}{2}}\right] + \frac{3}{2} \leq n_u + 1$$

or

$$\left(\frac{n_U - \frac{3}{2}}{\sqrt{6} + \frac{2}{3}}\right)^2 < \Delta \leq \left(\frac{n_U - \frac{1}{2}}{\sqrt{6} + \sqrt{\frac{2}{3}}}\right)^2 \qquad (45)$$

$$\therefore \frac{(n-1)(3n^2 + 7n - 25\text{-}1/4)}{32} < TW/2\pi \leq \frac{(n-1)(3n^2 + 13n - 31\text{-}1/4)}{32} \qquad (46)$$

and for $n_U \gg 1$

$$\frac{n_U^{\;3}}{10\frac{2}{3}} \approx \frac{TW}{2\pi} \qquad (47)$$

Consequently for large n the bound on n^2, the maximum sidelobe level is

$$\left(10\tfrac{2}{3}\, TW/2\pi\right)^{2/3} \geq n^2 \geq \left(6TW/2\pi\right)^{2/3} \qquad (48)$$

For $a \geq \frac{n}{2}$ the previously derived limits on Δ are not valid. For these larger values of (a) the general term for the maximum is:

for k even:

$$\text{General Term for Maximum} = a\Delta + \sum_{c=1 \text{ Even}}^{c=k-1} C_1 + \sum_{c_2=0 \text{ Odd}}^{c_2=2a-k-1} C_2$$

$$= a\Delta + \frac{k(k-2)}{4} + \frac{(2a-k)^2}{4}$$

$$= a\Delta + \frac{k^2-k(2a+1) + 2a^2}{2}$$

for k odd:

$$\text{General Term for Maximum} = a\Delta + \sum_{C_1=1 \text{ Odd}}^{C_1=k-2} C_1 + \sum_{C_2=0 \text{ Even}}^{C_2=2a-k-1} C_2$$

$$= a\Delta + \frac{(k-1)^2}{4} + \frac{[(2a-k)^2 - 1]}{4}$$

$$= a\Delta + \frac{k^2 - k\ (2a + 1) + 2a^2}{2}$$

Since both are the same only one inequality need be examined namely

Column (a) Maximum < Column (a+1) Minimum

$$a\Delta + \frac{n^2 - n(2a+1) + 2a^2}{2} < (a+1)\ \Delta + \frac{a(a+1)}{2}$$

or

$$\frac{(n-a)^2 - (n+a)}{2} < \Delta$$

The left hand side of the inequality is largest when a is smallest. But the minimum value of a for which this limit is valid is $\frac{n}{2}$ and so

$$\Delta > \frac{(n - \frac{n}{2})^2 - (n + \frac{n}{2})}{2} = \frac{n^2 - 12n}{8}$$

But the limit on Δ when $a \leq \frac{n}{2}$ is

$$\sqrt{6\Delta} + \frac{3}{2} > n$$

or

$$\Delta > \frac{n^2 - 3n + \frac{9}{4}}{6}$$

and since

$$\frac{n^2 - 3n + \frac{9}{4}}{6} > \frac{n^2 - 12n}{8}$$

for all n, then the tightest limit on Δ occurs when $a \leq \frac{n}{2}$, and equations (41) through (43) are valid for all (a). Figures 23 and 24 show plots of n vs Δ and $TW/2\pi$, respectively for the upper and lower limits as well as for the actual maximum values calculated for several values of n.

Detailed examination of these two figures shows that, as before, the lower bound is a slightly better index of actual performance. Again, as

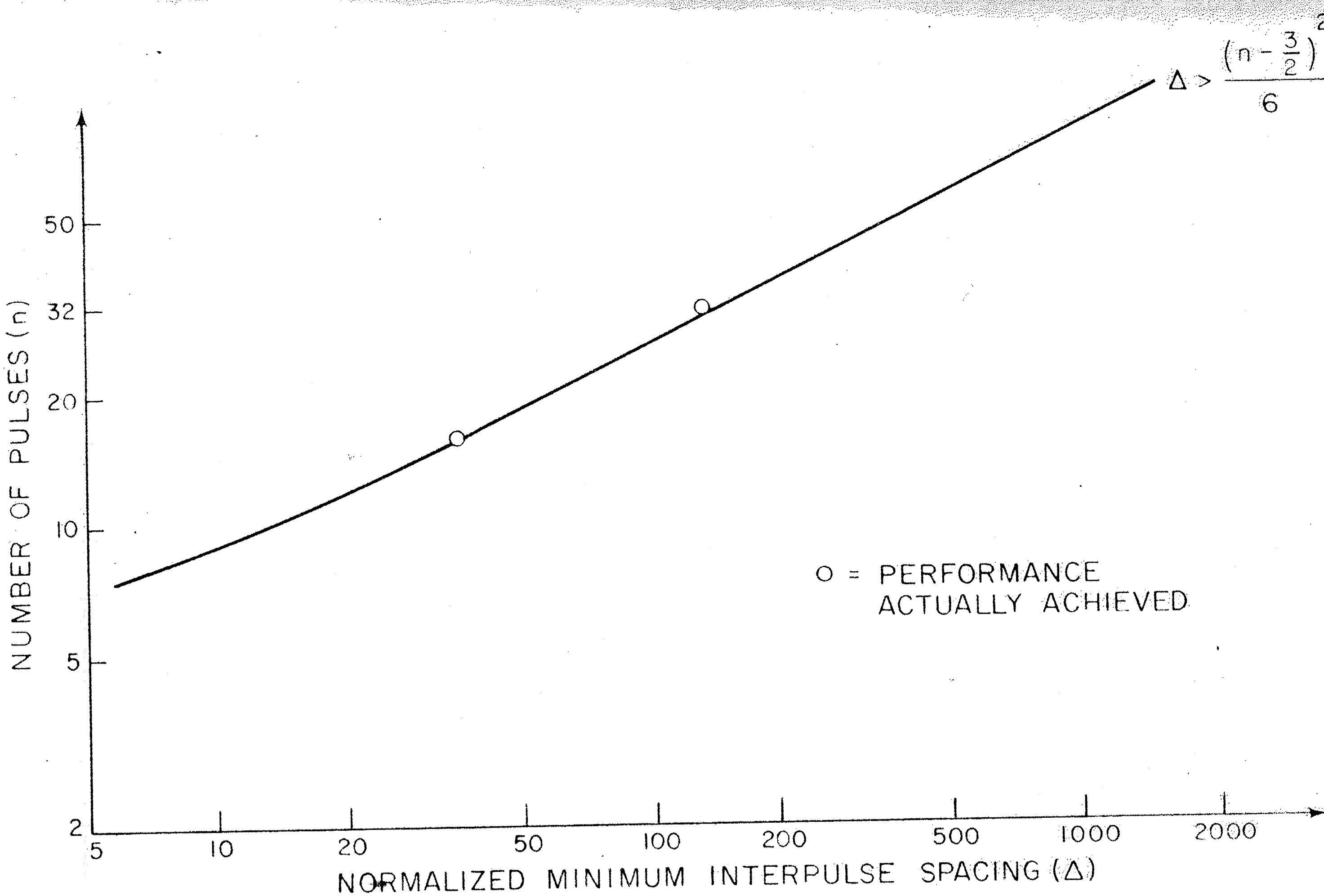

Fig. 23. Number of pulses (n) as a function of Δ for pulse train shown in Fig. 12 b.

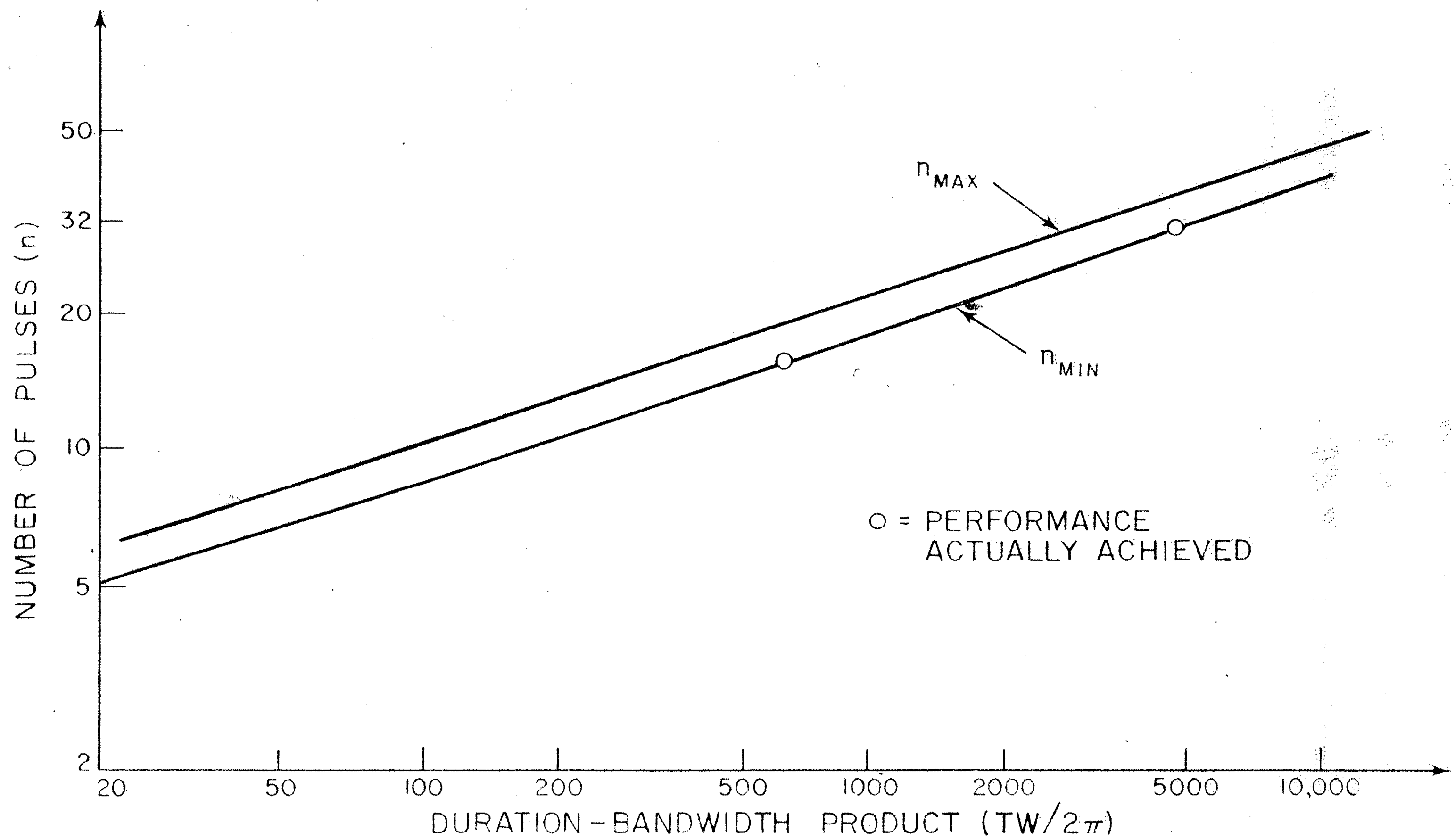

Fig. 24. Number of pulses (n) vs $TW/2\pi$ as a function of Δ for pulse train shown in Fig. 12b.

was true before, the performance for an alternate decrease in spacing is virtually the same as that derived for an alternate increase and goes to the same limits for $n \gg 1$. The efficiency of this approach toward achieving a uniform volume distribution, is

$$\eta = \frac{\frac{n^2}{2}}{\frac{n^3}{6}} = \frac{3}{n}$$

or an improvement of 3:2 over the previously examined train.

4.32 The Spectrum

The spectrum, $H(\omega)$, of a train of n impulses, with an alternating increase in interpulse spacing is

$$H(\omega) = \int_{-\infty}^{\infty} e^{j\omega t} \left[u_o(t + \frac{\Delta}{2}) + u_o(t - \frac{\Delta}{2}) \right.$$

$$+ u_o(t + 1\frac{1}{2}\Delta + 1) + u_o(t - 1\frac{1}{2}\Delta - 2)$$

$$+ u_o(t + 2\frac{1}{2}\Delta + 4) + u_o(t - 2\frac{1}{2}\Delta - 6)$$

$$+ u_o(t + 3\frac{1}{2}\Delta + 9) + u_o(t - 3\frac{1}{2}\Delta - 12)$$

$$+ \ldots\ldots +$$

$$\left. + u_o\left(t + \frac{n-1}{2}\Delta + \sum_{j=odd}^{j=n} j\right) + u_o\left(t - \frac{n-1}{2}\Delta - \sum_{j=even}^{j=n} j\right)\right] dt$$

$$\therefore H(\omega) = e^{j\omega \frac{\Delta}{2}} + e^{-j\omega \frac{\Delta}{2}}$$

$$+ e^{j\omega[1 \frac{1}{2} \Delta + 1]} + e^{-j\omega[1 \frac{1}{2} + 2]}$$

$$+ e^{j\omega[2 \frac{1}{2} \Delta + 4]} + e^{-j\omega[2 \frac{1}{2} \Delta + 6]}$$

$$+ e^{j\omega[3 \frac{1}{2} \Delta + 9]} + e^{j\omega[3 \frac{1}{2} \Delta + 12]}$$

$$+ e^{j\omega \left[\frac{n-1}{2} \Delta + \sum_{j=odd}^{j=n-2} j \right]} + e^{-j\omega \left[\frac{n-1}{2} \Delta + \sum_{j=even}^{j=n-2} j \right]}$$

Finally,

$$H(\omega) = e^{j\omega \frac{\Delta}{2}} + e^{-j\omega \frac{\Delta}{2}} + e^{j\omega(1\frac{1}{2} \Delta + 1)} + e^{j\omega\delta} e^{j\omega(1\frac{1}{2} \Delta + 1)}$$

$$+ e^{j\omega(2\frac{1}{2} \Delta + 4)} + e^{j\omega 2\delta} e^{j\omega(2\frac{1}{2} \Delta + 4)}$$

$$+ e^{j\omega(3\frac{1}{2} \Delta + 9)} + e^{\omega 3\delta} e^{j\omega(3\frac{1}{2} \Delta + 9)} + \ldots$$

$$+ e^{j\omega \left[(\frac{n-1}{2}) \Delta + \sum_{j=odd}^{n-3} j \right]}$$

$$+ e^{j\omega(\frac{n-2}{2})\delta} e^{j\omega \left[(\frac{n-1}{2}) \Delta + \sum_{j=odd}^{n-3} j \right]} \qquad (49)$$

If

$$\frac{n-1}{2}\,\Delta \gg \sum_{j=odd}^{n-3} j = \frac{(n-2)^2}{4}$$

$$\frac{(n-1)}{2}\,\Delta \approx \frac{(n-1)}{2} \cdot \frac{n^2}{6} \gg \frac{(n-2)^2}{4}$$

or

$$n \gg 1$$

then the only perturbation is the phase term effecting one of the pair of almost symmetrically located pulses. But the maximum value of this phase term, ϕ_{Max}, is

$$\phi_{Max} = \frac{\omega_D\,(n-2)}{2}$$

and since

$$\omega_D = 2\pi f_{DMax} \approx \frac{2\pi}{\Delta} \approx \frac{12\pi}{n^2}$$

then

$$\phi_{Max} = \frac{12\pi}{n^2} \cdot \left(\frac{n-2}{2}\right) = \frac{6\pi}{n} .$$

For $n = 16$

$$\phi_{Max} \approx 70^{\circ}$$

Figure 25 compares the spectrum of a 16 pulse non-uniformly spaced train and a 16 pulse uniformly spaced train of equal duration. The correspondence between these spectra is very close at least half way to the doppler ambiguity. Since all these phase perturbations decrease with n it can be expected that for larger values of n the spectrum will be even more like the $\frac{\text{Sin } nx}{\text{Sin } x}$ of a uniformly spaced train and the correspondence will remain close for even larger doppler shifts.

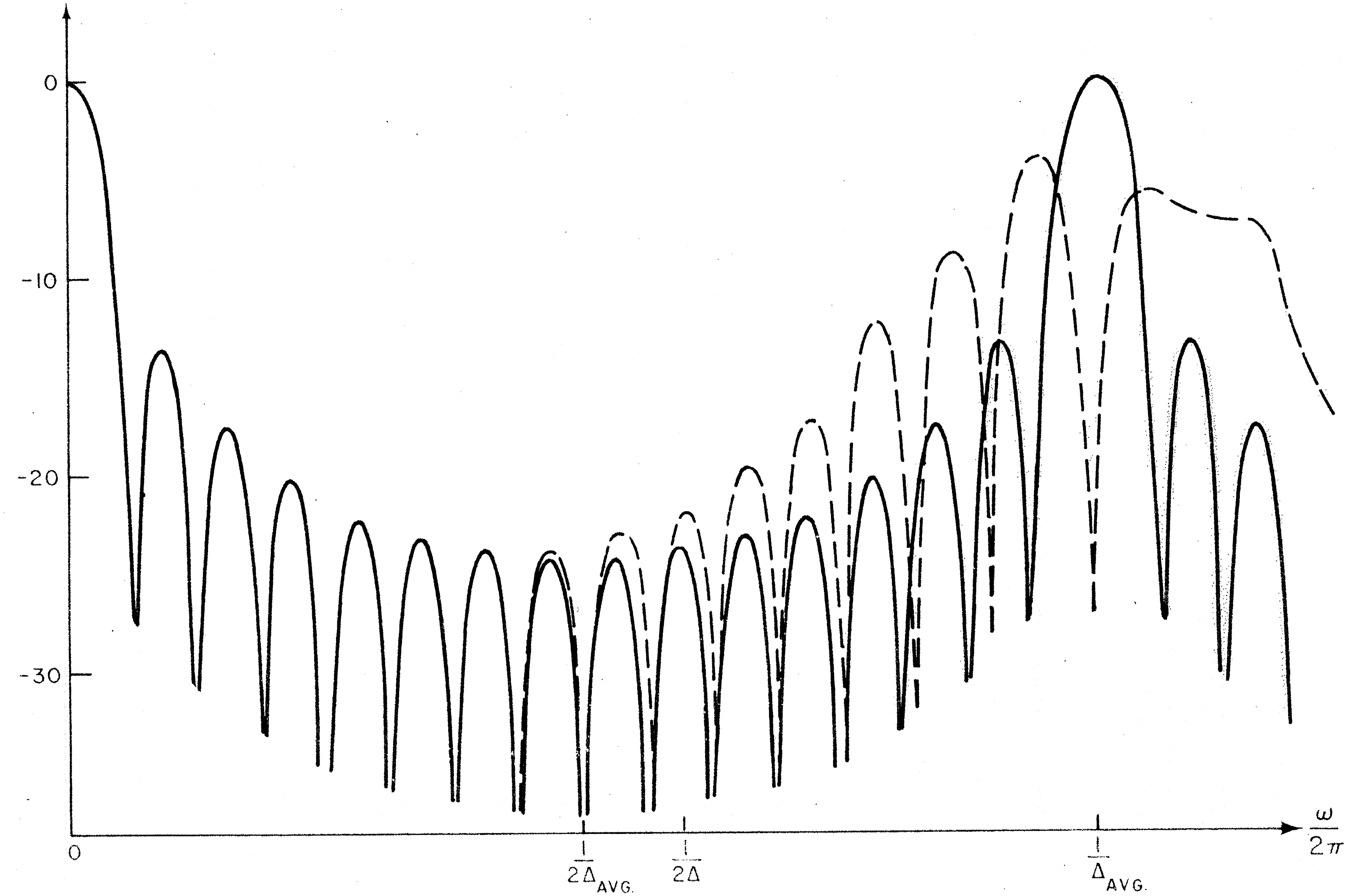

Fig. 25. Spectrum of uniformly spaced train and non-uniformly spaced train (Fig. 12b) with 16 pulses and equal $TW/2\pi$.

CHAPTER V

A PSEUDO-RANDOM SELECTION OF THE INTERPULSE SPACING

5.1 Evaluation of Conditions for Worst Performance and Best

In one sense it should not be surprising to find that an alternating increase in spacing yields better performance than a monotonic increase. It can be shown that of all the arrangements of the (n-1) numbers from Δ through $\Delta + (n-2)$ which result in a non-overlapping of sidelobes, the arrangement yielding the worst performance is a monotonic increase or decrease. Consider the criterion which determined Δ.

1. Column 1 Maximum < Column 2 Minimum

$$\Delta + X_i < 2\Delta + X_j + X_k$$

$$\text{or } X_i - (X_j + X_k) < \Delta$$

where X_i is the largest number in column 1 and $(X_j + X_k)$ is the smallest number in column 2.

2. Column 2 Maximum < Column 3 Minimum

$$2\Delta + X_\ell + X_m < 3\Delta + X_n + X_o + X_p$$

$$\text{or } (X_\ell + X_m) - (X_n + X_o + X_p) < \Delta$$

where $(X_\ell + X_m)$ is the largest number in column 2 and $(X_n + X_o + X_p)$ is the smallest number in column 3.

a. Column (a) Maximum < Column (a+1) Minimum

$$a\Delta + \sum_{i=k}^{k+a-1} X_i < (a+1)\,\Delta + \sum_{i=j}^{i=j+a} X_i$$

where $\sum_{i=k}^{k+a-1} X_i$ is the largest number in column a, and

$$\sum_{i=k}^{k+a-1} X_i - \sum_{i=j}^{i=j+a} X_i < \Delta$$

$\sum_{i=j}^{j+a} X_i$ is the smallest number in column (a+1).

But the largest sum of (a) consecutive X's is maximized and the smallest sum of (a+1) consecutive X's is minimized when the X's are chosen to be the arithmetic progression 0,1,2,3 ..... n-2. When the X's are chosen this way then Δ is maximized for any value of n and the result is the worst performance possible. In this case

$$\text{Max}\left[\sum_{i=k}^{k+a-1} X_i\right] = \sum_{i=n-a-1}^{i=n-2} X_i = \frac{a}{2}\left[2n - 3 - a\right]$$

and

$$\text{Min}\left[\sum_{i=j}^{i=j+a} X_i\right] = \sum_{i=o}^{i=a} X = \frac{a(a+1)}{2}$$

$$\frac{a}{2}\left[2n-3-a\right] - \frac{a(a+1)}{2} < \Delta$$

$$a\left[n-a-2\right] < \Delta$$

and $\frac{(n-2)^2}{4} < \Delta$ which occurs for $a = \frac{n-2}{2}$

But if an arithmetic increase in interpulse period results in the largest Δ and consequently the largest duration bandwidth product, what approach results in the smallest duration-bandwidth product? The answer appears to lie in the fact that Δ (and consequently $TW/2\pi$) is controlled by the difference between greatest and smallest numbers in adjacent columns. Consequently a reasonable approach is to try to minimize the highest number and maximize the lowest number within each column*. This can be done by distributing the high and low values of X more uniformly through the pulse train. However, if the distribution is too uniform then the sums of several adjacent spacings in different parts of the pulse train will tend to be the same. Consequently a not-too-uniform sequence of the integers from 0 through n-2 is required. The difficulty with a random selection, as, for example, from a table of random numbers, is that there is no guarantee that two elements within a particular column will not be equal. No juggling of Δ can eliminate such an equality.

Perhaps the best approach is to use a reasonable technique for generating the integers 0 through n-2 and examine what limitations are imposed on this technique by the requirement for non-equality within any column of the time difference matrix.

*Δ is actually only bounded by the difference between maxima and minima in adjacent columns. Interleaving of the numbers in adjacent columns is certainly permissible. However, when the density of sidelobes is fairly high, the probability of equality occuring as the columns are overlapped increases rapidly. The additional reduction in Δ under these circumstances will be quite small.

5.2 Use of the Complete Set of Least Residues Modulo (n-1)

5.21 Conditions on (n-1)

Let us consider, as the X_i's for the pulse train shown in Figure 12c, the sequence of (n-1) integers

$$X_i = [X_o + iq] \text{ modulo } (n-1) \tag{50}$$

$$\text{for } i = 0, 1, 2 \ldots n-2$$

If n-1 and q are relatively prime then the first n-1 numbers in the sequence will be some ordering of the integers from 0 to n-2. If we denote any number in column 1 as X_i then the sequence of successive numbers will be (from left to right)

				$X_i + 4q$
			$X_i + 3q$	or
		$X_i + 2q$	or	$X_i + 4q-(n-1)$
	$X_i + q$	or	$X_i + 3q-(n-1)$	or
X_i	or	$X_i + 2q-(n-1)$	or	$X_i + 4q-2(n-1)$ etc
	$X_i + q-(n-1)$	or	$X_i + 3q-2(n-1)$	or
		$X_i + 2q-2(n-1)$	or	$X_i + 4q-3(n-1)$
			$X_i + 3q-3(n-1)$	or
				$X_i + 4q-4(n-1)$

We know that all numbers in the first column are different due to the generating technique and the fact that n-1 and q are chosen relatively prime. Every number in the second column of the time difference matrix, being the sum of two adjacent first column terms, can be written as

$$2X_i + q$$

$$\text{or} \quad 2X_i + q-(n-1)$$

Clearly if $2X_i + q \neq 2X_j + q$ i

$$\neq 2X_j + q-(n-1) \qquad \text{ii}$$

and

$$2X_i + m-(n-1) \neq 2X_j + q \qquad \text{iii}$$

$$\neq 2X_j + q-(n-1) \qquad \text{iiii}$$

for all i and j then no two second column numbers can be equal.

Therefore if $X_i \neq X_j$ i, iiii

and $|X_i - X_j| \neq \frac{n-1}{2}$ ii, iii

no second column terms are alike. X_i and X_j are both first column terms and cannot be equal; if n-1 is odd, then $\frac{n-1}{2}$ is not an integer and

$$|X_i - X_j| \neq \frac{n-1}{2} .$$

Therefore for all second column numbers to be different

$$(n-1) \text{ modulo } 2 \neq 0 \qquad (51)$$

Every number in the third column of the time difference matrix is the distance between every third pulse, or the sum of three adjacent first column terms, and can be written as

$$3X_i + 3q$$

$$3X_i + 3q-(n-1)$$

$$3X_i + 3q-2(n-1)$$

or

$$3X_i + 3q-3(n-1)$$

If

$$\left.\begin{matrix} 3X_i + 3m \\ 3X_i + 3m-(n-1) \\ 3X_i + 3m-2(n-1) \\ 3X_i + 3m-3(n-1) \end{matrix}\right) \neq \left(\begin{matrix} 3X_j + 3m \\ 3X_j + 3m-(n-1) \\ 3X_j + 3m-2(n-1) \\ 3X_j + 3m-3(n-1) \end{matrix}\right.$$

then inequality of all third column numbers is assured.

Therefore if

$$|X_i - X_j| \neq \begin{cases} 0x \frac{n-1}{3} \\ 1x \frac{n-1}{3} \\ 2x \frac{n-1}{3} \\ 3x \frac{n-1}{3} \end{cases}$$

or $\left[\text{since } X_i - X_j < (n-1)\right]$ if $(n-1)$ modulo $3 \neq 0$ (52)

then all third column numbers are unequal.

In a similar way it can be shown that the condition for inequality of all column (a) numbers is

$$(n-1) \text{ modulo } a \neq 0 \tag{53}$$

Since this must be true for all (a) up to $\frac{n}{2}$, then

$$(n-1) \text{ modulo } 2,3,4,5 \;\ldots\ldots\; \frac{n}{2} \neq 0 \tag{54}$$

or

$$(n-1) \text{ must be a prime number.} \tag{55}$$

Therefore if the X's are selected as

$$X_i = [\, X_o + iq] \text{ modulo } (n-1) \tag{50}$$

where (n-1) is a prime number then inequality of the numbers within all columns is assured. Since there are a total of n-1 numbers with which to begin the sequence, and a total of n-2 different values of q, then there are $\frac{(n-2)(n-1)}{2}$ different sequences*, all of which have no two numbers alike within the same column of the time difference matrix. [These $\frac{(n-2)(n-1)}{2}$ sequences do not include all possible arrangements having this desired characteristic. Among the n! total arrangements possible, many other satisfactory ones may exist.]

5.22 Selection of X_o

It will be shown that the minimum values of Δ and $TW/2\pi$, for any q, will vary only slightly with X_o. Further, while the explicit dependence of Δ and $TW/2\pi$ on X_o is, at present, unknown, there is a straight forward approach for obtaining the X_o resulting in the smallest Δ and $TW/2\pi$ for any particular q.

Figure 26 shows the time difference matrix for (n-1) = 11, q = 4 and $X_o = 1$. Note that this is also the time difference matrix for (n-1) = 11, q = 7, and $X_o = 8$**. [This can be most easily seen by reversing the sequence of numbers within each of the columns.] Given this matrix a bound on Δ can be obtained by finding the largest value in each column and the smallest value in the next higher column. An upper bound to Δ is the lar-

*Each sequence appears twice, once as $X_o, X_1, X_2, \ldots, X_{n-2}$ and once as $X_{n-2}, X_{n-3}, \ldots, X_2, X_1, X_o$.

**In general the matrix for $[n-1;\ q';\ X_o']$ is also the matrix for $[n-1;\ n-1-q';\ (n-1-q' + X_o')$ modulo $(n-1)]$.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9	Column 10	Column 11
8	12	12	19	22	32	38	40	49	54	55
4	4	11	14	24	30	32	41	46	47	
0	7	10	20	26	28	37	42	43		
7	10	20	26	28	37	42	43			
3	13	19	21	30	35	36				
10	16	18	27	32	33					
6	8	17	22	23						
2	11	16	17							
9	14	15								
5	6									
1										

Figure 26: Time difference matrix for $n-1=11$; $q=4$; $\chi_o=1$ [and $n-1=11$; $q=7$; $\chi_o=8$]

gest difference plus one. Interweaving of the columns can be accomplished by determining if a value less than the upper bound will still not result in an equality of two numbers within the matrix.

To determine the minimum Δ for any value of X_o, given q, it is necessary to compute the complete time difference matrix, shown in Figure 27 for $n-1 = 7$; $q = 2,5$. Although the number of computations is doubled, the minimum Δ for two values of q and all values of X_o can be determined. The lowest matrix, within the solid lines, is for $q = 2$; $X_o = 0$, or $q = 5$; $X_o = 5$. The next matrix, although different from the first by only one line, is for $q = 2$; $X_o = 2$, or $q = 5$; $X_o = 0$. The next, different from the first by two lines, different from the second by only one line, is for $q = 2$; $X_o = 4$ or $q = 5$; $X_o = 2$, etc. Consequently all the matrices for $q = 2$ and 5 are available and can be evaluated.

With the high correlation between these matrices for successive values of X_o, and for q' and $(n-1-q')$, there will undoubtedly be a high correlation for the values of Δ and $TW/2\pi$ computed for each of these cases. Consequently it is reasonable to expect only a slight variation in Δ and $TW/2\pi$ with a variation in X_o.

5.23 Selection of q

Using the procedure just outlined, it would be necessary to calculate and evaluate $\frac{n-2}{2}$ of these complete time difference matrices to completely exhaust all possible sequences and to determine the minimum Δ and $TW/2\pi$ and the values of q and X_o which achieve this result. Since it is not possible, at present, to bound Δ, the calculation and evaluation of at least one time

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7
3	4	10	14	16	16	21
1	7	11	13	13	18	21
6	10	12	12	17	20	21
4	6	6	11	14	15	21
2	2	7	10	11	17	21
0	5	8	9	15	19	21
5	8	9	15	19	21	21
3	4	10	14	16	16	
1	7	11	13	13		
6	10	12	12			
4	6	6				
2	2					
0						

Figure 27: Complete time difference matrix for (n-1)=7;q=2, 5

difference matrix is necessary to assure a proper value of Δ. To require doubling this effort is conceivable. To require multiplying this effort by $\frac{n-2}{2}$ is not.

However this additional effort is not required to achieve a low $TW/2\pi$, but only to achieve the <u>minimum</u>. An examination of the effect of q on the minimum $TW/2\pi$ for $(n-1) = 11$ and 13, and on the randomness of the sequence of X's, indicates a reasonable approach for selecting a value of q resulting in a low (if not minimum) $TW/2\pi$.

Figure 28 shows a plot of the minimum $TW/2\pi$ vs q for $(n-1) = 11, 13$ and for $X_o = 0$. The general shape of these curves is reasonably representative. The symmetry expected about $q = \frac{n-1}{2}$ due to the high correlation between the numbers in the matrices for q' and in the matrices for n-1-q' is evident.

It should not be surprising, at this point, that the sequence for q = 1 results in the highest $TW/2\pi$. This sequence of interpulse spacings, Δ, Δ+1, Δ+2, Δ+3, etc. is identical to the sequence analyzed in 4.2 and shown in 5.1 to yield the worst performance. However it may be surprising that the best performance is not achieved for $q = \frac{n}{2}$, or its complement $\frac{n-2}{2}$, since these values might be expected to result in the most random ordering of the values of X. However these particular values of q result in a decided non-random distribution of the large and small values of X, as is shown for the sequence of X_i's for n-1 = 11; $q = \frac{n}{2}$; $X_o = 0$, where X_i's = 0, 6, 1, 7, 2, 8, 3, 9, 4, 10, 5.

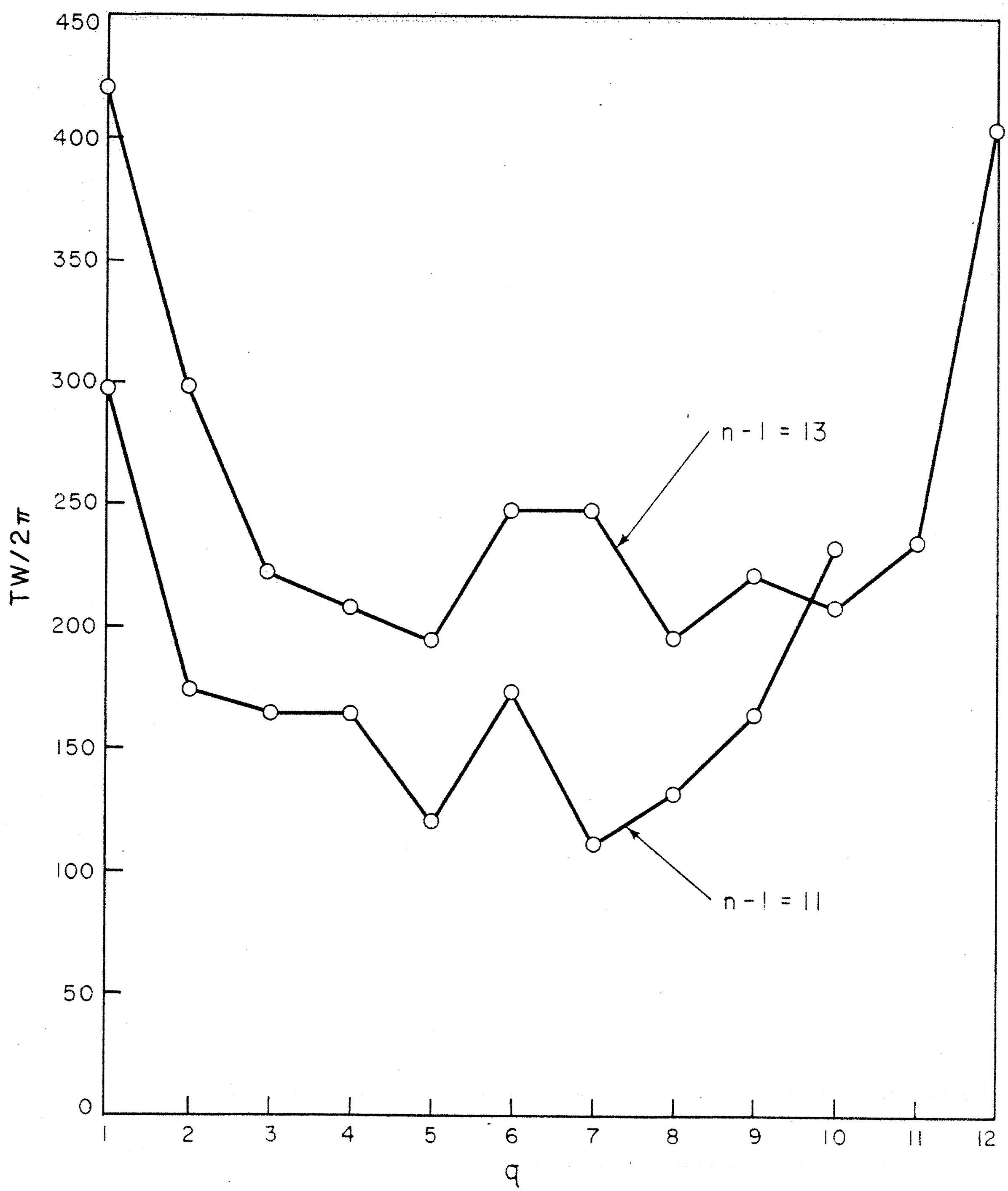

Fig. 28. Minimum $TW/2\pi$ as a function of q for n-1 = 11 and 13.

There are in fact several values of q, in particular $q = 1, \frac{n}{2}, \frac{n}{4}, \frac{n}{8}, \frac{n}{16}$, etc. and their complements, which similarly yield a sufficiently ordered sequence of numbers that makes it possible to determine the relationship between Δ, $TW/2\pi$, q, and n for them. In these specific cases it turns out that

$$\Delta \propto n^{3/2} \tag{56a}$$

$$\frac{TW}{2\pi} \propto n^{5/2} \tag{56b}$$

for $$q \approx n^{1/2} \tag{56c}$$

What is misleading about these results (which are offered without proof) is that they do not represent the conditions for achieving the minimum $TW/2\pi$. It is the same orderliness and non-randomness of the sequence of X's which enables the limits to be determined, that also limits the achievable performance. However the good results achievable even with these particular sequences, should be a lower limit to the performance attainable in general.

It appears, from Figure 28, that the minimum $TW/2\pi$ does not vary significantly for the values of $q \approx \frac{n-1}{2}$. Consequently it is reasonable to conclude that the selection of

$$q \approx \frac{n-1}{2} \tag{57}$$

will result in a $TW/2\pi$ reasonably close to the minimum*.

The performance achievable under these circumstances should be approximately

$$\Delta \approx \frac{n^{3/2}}{4} \qquad (58a)$$

and

$$TW/2\pi \approx \frac{n^{5/2}}{4} \qquad (58b)$$

but it appears unreasonable to expect to achieve

$$\Delta \approx \frac{n}{4} \log_e n \qquad (59a)$$

and

$$TW/2\pi \approx \frac{n^2}{4} \log_e n \qquad (59b)$$

Figure 13 shows a plot of equations (58b) and (59b) vs n, with the actually required $TW/2\pi$ for several values of n also shown.

5.3 Conclusion

The use of the complete set of least residues modulo (n-1) where

$$(n-1) \text{ is a prime number} \qquad (55)$$

will yield a sequence of X's resulting in non-equality of the numbers with-

*For large n, neither $\frac{n}{2}$ nor $\frac{n-2}{2}$ are suitable values for q.

in the columns of the time difference matrix. Although the best performance achievable with this approach is not yet adequately bounded it can be expected that when

$$q \approx \frac{n-1}{2} \tag{57}$$

then

$$\Delta \approx \frac{n^{3/2}}{4} \tag{58a}$$

and

$$TW/2\pi \approx \frac{n^{5/2}}{4} \tag{58b}$$

can be realized. The initial value of the sequence of X's does not appear to have much affect on the value of $TW/2\pi$ achieved.

CHAPTER VI

COMBINATION OF NON-UNIFORM SPACING WITH VARIATION IN OTHER PULSE TRAIN PARAMETERS

6.1 Amplitude and Density Tapering

The use of an amplitude taper with a non-uniformly spaced pulse train to reduce the frequency sidelobes has an undesirable effect on the time sidelobes. Despite the fact that the signal-to-noise ratio is reduced only 1 db by a cosine taper the time sidelobe level increases by 4 db. The use of a more severe amplitude taper, such as $(\text{cosine})^2$, results in only a $1\frac{1}{2}$ db drop in signal-to-noise but a 6 db increase in the time sidelobe level. This is a high price to pay.

If the reduced frequency sidelobe level is of extreme importance it can be achieved by use of an energy density taper. With an energy density taper the pulse spacings are not varied from Δ through $\Delta + n-2$, but are varied proportional to the desired amplitude taper, with small spacings corresponding to large amplitudes and vice versa. Consequently, the spacings in an energy density tapered train will be smaller in the middle of the train and larger at the ends, resulting in a higher concentration of energy in the middle of the train. The resultant spectrum is similar to that achievable with a direct amplitude taper.

Due to the large variation in spacing necessary to achieve a substantial taper the distance to the first sidelobe in time will have to be much less than could otherwise be achieved. Consequently, the area of infinite contrast is reduced. Even more limiting is that there are no presently known general approaches to achieving satisfactory energy density tapers.

The use of the symmetrical increase in interpulse spacing, evaluated in 4.3, does not actually result in much of a taper. The height of the pedestal, h, for this approach is

$$h = \frac{\Delta_{Min}}{\Delta_{Max}} \approx \frac{\frac{(n-2)^2}{6}}{\left[\frac{(n-2)^2}{6}\right] + n-2} \approx \frac{1}{1 + \frac{6}{n-2}}$$

To achieve a pedestal as modest as 0.5, there can be only 8 pulses in the train. Any reasonable density taper can be tried but only trial-and-error can guarantee the uniformly low time sidelobes.

An orderly, but inefficient, approach to density tapering can be achieved by pulse dropping. A signal with low TW/2π can be designed by the approach outlined in Chapter V. Then some fraction of these pulses, perhaps one-third, can be symmetrically dropped in accordance with the desired energy density taper. However, reducing the number of pulses by one-third results in a 2 db reduction in energy per pulse (and consequently in signal-to-noise ratio) and a 4 db increase in time sidelobe level.

Consequently the use of this approach to reducing the frequency sidelobes results in approximately the same loss in overall performance as would be expected from a direct amplitude taper of the original waveform. However the greater efficiency inherent in density tapering makes it the more attractive approach.

6.2 Selective Pulse Dropping

A more important use of pulse dropping is the selective dropping of pulses on transmit or receive to completely eliminate sidelobe interference. This technique requires altering the ambiguity surface so that the response at the time corresponding to an interferring target is absolutely zero.

Figure 29 shows the three possible locations of any two sidelobes in close proximity. When the two sidelobes overlap at the -6 db points, as shown in Figure 29a the two pulses which created these two sidelobes must be dropped to create a zero over this region. For the relationship shown in Figure 29b only one pulse ever need be dropped, unless errors in system timing result in some overlap in which case two pulses might have to be dropped. For the situation shown in Figure 29c, only one pulse would ever have to be dropped to provide zero response. While for large pulse trains the sidelobe overlap shown in Figure 29a will be the least likely of the three, to guarantee that it never occurs would require designing the waveform for a 2δ pulse width, but using a δ pulse width. This may require doubling the signal $TW/2\pi$.

Clearly, only a few pulses can be dropped before the pulse train spectrum begins to be adversely affected. The number of interfering targets which can be rejected will depend on the sidelobe structure of the ambiguity surface and the acceptable level of degradation in spectrum.

The most significant feature of this technique is that no knowledge of the amplitude, phase, or velocity and only approximate knowledge of the

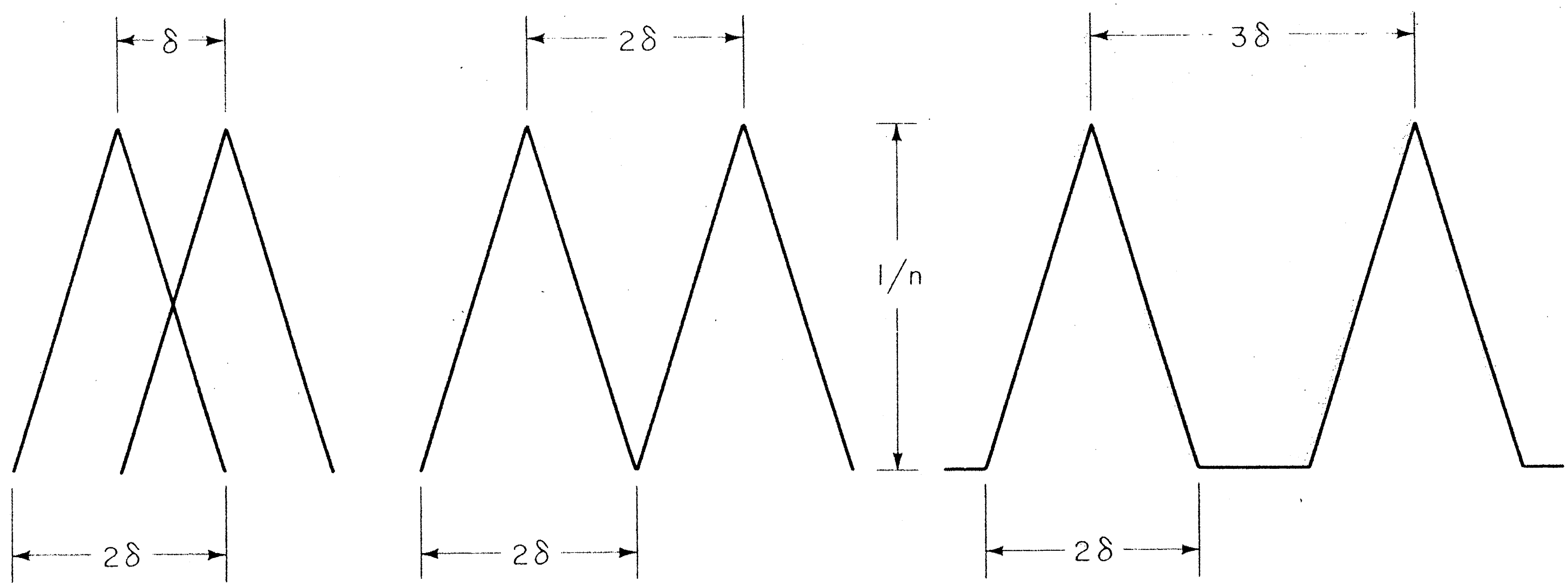

Fig 29: POSITION OF ADJACENT SIDELOBES FOR A NON-UNIFORMLY SPACED PULSE TRAIN

range of the interfering target is necessary. Further, this technique can even be applied in non-real time to the recorded (but unfiltered) signal return since the pulse dropping can be performed in the receiver as well as the transmitter.

6.3 Pulse-to-Pulse Frequency Shifts

A pulse-to-pulse frequency shift can be combined with the use of non-uniform spacing. The combination has many advantages but it also has all the disadvantages inherent in the use of a pulse-to-pulse frequency shift.

The most obvious sequence of frequencies, a linear shift in frequency by $\frac{1}{\delta}$, results in

$$\frac{\psi(0,0)}{(k\pi m)^2} \geq \psi(\tau,\omega) \tag{60}$$

where k is the number of spectral widths between signal and filter. This yields a minimum reduction in sidelobe level of 10 db. However, because of the inevitable range-doppler coupling the facility for resolving in velocity, or even measuring velocity, is lost. While the use of a sufficiently random sequence of these same frequencies may be able to eliminate this coupling, as was pointed out before in 2.6, frequency jumping does not alter the volume under the central region of the ambiguity surface and consequently does not increase the resolution between two targets close in range and velocity. Nonetheless, if $\frac{T}{\delta}$ is sufficiently large to assure satisfactory resolution between two neighboring targets, a random sequence of frequencies can be used to reduce the overall interference.

6.4 Modulation of the Individual Pulses

The use of a repetitive sequence of modulated pulses does provide a convenient way to reduce the interference volume within the central region and to increase the resolution of the waveform. Further, the use of modulation on each of the pulses is not restricted by the doppler bandwidth, since the pulse width is normally much less than Δ. Consequently the pulses themselves are operating in a narrow doppler environment, and any modulation techniques which were previously rejected for their inability to perform adequately in the presence of a large doppler shift can be useful here. The use, for example, of the Barker code of order 13 provides an interesting opportunity to achieve doppler resolution, fine resolution in range, and also the transmission of a pulse longer than the range resolution. With the use of this code on the individual pulses of a non uniformly spaced train

$$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,\omega) \qquad \text{for } |\tau| \geq \delta \tag{15}$$

and

$$\frac{\psi(0,0)}{13^2} \geq \psi(\tau,\omega) \qquad \text{for } |\tau| \geq \delta' \tag{61}$$

$$\text{where } \delta' = \frac{\delta}{13}$$

Most important is that the frequency response of the 12 sidelobe peaks are miniature replicas of the central peak. Consequently, when the spectrum is 23 db down from the peak response these sidelobes are also 23 db down from the sidelobe peak or more than 45 db down from the peak response.

The volume within the central region of the ambiguity surface can be shown to be significantly less as the result of the use of this phase code.

$$\int_{0}^{2\frac{\pi}{\Delta}} \int_{-\delta}^{\delta} \psi(\tau,\omega)\, dt\, d\omega/2\pi = \frac{\delta'}{T} \left[(1)^2 + 12\left(\frac{1}{13} \right)^2 \right] \approx 1.07 \frac{\delta'}{T}$$

$$\approx \frac{1}{12} \frac{\delta}{T} \quad . \tag{62}$$

Consequently there has been a 12:1 reduction in the interference volume.

With the use of this phase code the sidelobe structure of the ambiguity surface has been altered. There are still the same number of peaks with an amplitude of $\frac{1}{n^2}$, but their null-width has been reduced from 2δ to $2\delta/13$. Further, in the space between sidelobe peaks there is a plateau whose amplitude is 1/13th of the sidelobe peak, or about 22 db below the sidelobe peaks. Consequently there will be four different contrast ratios if the non uniform spacing and this phase code are combined; a) an infinite contrast ratio for $\Delta \geq |\tau| \geq \delta$ and at other points where there are no sidelobes, b) a contrast ratio of $\left(\frac{1}{13n} \right)^2$ occuring at $6n(n-1)$ points, c) a contrast ratio of $\left(\frac{1}{n} \right)^2$ occuring at $n(n-1)$ points, and d) a contrast ratio of $\left(\frac{1}{13} \right)^2$ for $\delta > |\tau| > \frac{\delta}{13}$.

6.5 Conclusions

The use of an amplitude taper with non uniformly spaced pulse trains increases the time sidelobe level and should be avoided. An energy density taper should be considered if reduction of frequency sidelobes is important. Elimination of interference through the use of pulse dropping should be used if the radar system is capable of adapting to each new target environment.

When a modulation is used on the pulses in the train the entire bandwidth of the signal must be handled, both on transmit and receive, instantaneously. When the frequency is jumped pulse to pulse the bandwidth can be handled piecemeal. However, it appears that it is the instantaneously used bandwidth which determines the interference volume within the central region of the ambiguity surface and consequently controls the resolution achievable. For enhanced resolution between two targets close in range and velocity, modulation of the pulses should be used rather than frequency jumping. If sufficient resolution exists between close targets then frequency jumping should be used to provide a general reduction in interference.

CHAPTER VII

TARGET-AND EQUIPMENT-IMPOSED LIMITATIONS TO THE SIGNAL $TW/2\pi$

7.1 Target Limitations

It is important to examine the limitations imposed on the waveform design by the target. A fundamental limit to the train duration is the time interval over which the target parameters are "stationary." This is equivalent to requiring that the quality or quantity being measured remain constant, or be varying in a known manner, during the measurement interval. Consequently in certain target environments the ability to generate, process, and record signals with large $TW/2\pi$, but of low bandwidth and long duration, may not really represent useful capability. For this reason large bandwidths are often necessary to achieve useful large duration-bandwidth products.

When a signal $e^{j\omega t}$ is sent out and strikes a moving target the returned signal $e^{j(1 + \frac{2\nu}{c})\omega t}$ is usually considered to be doppler shifted by the amount $\frac{2\nu}{c}\,\omega$. But ω is rarely a single frequency. If Δf is the bandwidth of the transmitted signal then the frequencies at the edges of the band, $\overline{\omega} \pm \pi\Delta f$, will suffer the maximum differential doppler frequency shift of $\pm \frac{2\nu}{c}\,\pi\Delta f$. The seriousness of this doppler spread is determined by the frequency resolution of the signal. Certainly when this spread equals the resolution in frequency the narrow band approximation is no longer valid. Therefore

$$\pm \frac{2\nu}{c}\,\pi\Delta f \leq \pm \frac{\pi}{T} \tag{63}$$

or

$$\frac{2\nu}{c} \Delta f \leq \frac{1}{T} \qquad (64)$$

for the narrow band assumption to hold.

$$\therefore \; T\Delta f = TW/2\pi \leq \left(\frac{2\nu}{c} \right)^{-1} \qquad (65)$$

This can be given a different interpretation by rewriting equation (64) as

$$\frac{2T\nu}{c} \leq \frac{1}{\Delta f} \; . \qquad (66)$$

Since $\frac{1}{\Delta f}$ is the time resolution of the signal, and $\frac{2T\nu}{c}$ is the difference in the actual duration of the returned train as a result of the velocity ν, for equation (66) to be satisfied the differential compression of the returned train due to target motion must be less than a resolution element in range (δ for a pulse train.)

For typical satellite velocities ($\approx$ 3.6 n. miles/sec)

$$TW/2\pi \leq 23{,}000. \qquad (67)$$

While analytically it is only a little harder to handle wide-band signals the actual matched filter becomes more complex.

The equipment-imposed limits to the signal duration-bandwidth product can be determined by examining three different hardware approaches using analog circuitry to the design of the matched filter for these non uniformly spaced pulse trains.

7.2 Use of Delay Lines and Phasing Matrix

Figure 30 shows the block diagram of a commonly used approach to the matched filter using a set of delay lines and a phasing matrix. With this approach each return from the target is delayed in time so that the n returns from the n pulses come out from the delay line and receiver simultaneously. At this point they are treated as samples of a continuous sinewave, where the effect of the doppler frequency is simply a constant phase shift pulse-to-pulse. A bank of filters is required to insure that all possible or expected doppler frequencies are covered. Using this approach no apriori knowledge of range or doppler frequency is required since all ranges and frequencies are examined.

The most critical elements to this approach are the delay lines themselves. Among the delay media possible are lumped constant, distributed constant, magnetostrictive, mercury, quartz and glass delay lines, coaxial cable, waveguide and a tape loop. Lumped constant and distributed constant delay lines have limited delay-to-rise time ratios and magnetostrictive delay lines have limited bandwidth capability. With a tape loop there is the problem of head spacing. To achieve a 1/4" differential spacing for the heads, at the unachieved head-to-tape speed of 5000 inches/sec, the minimum delay required must exceed 50 μsec. Waveguide and coaxial cable are not used for large duration signals because of the large physical size and length required to achieve even short delays; approximately 800' of waveguide or cable is required for each microsecond of delay. Of the three remaining techniques [mercury, quartz and glass delay lines] mercury delay

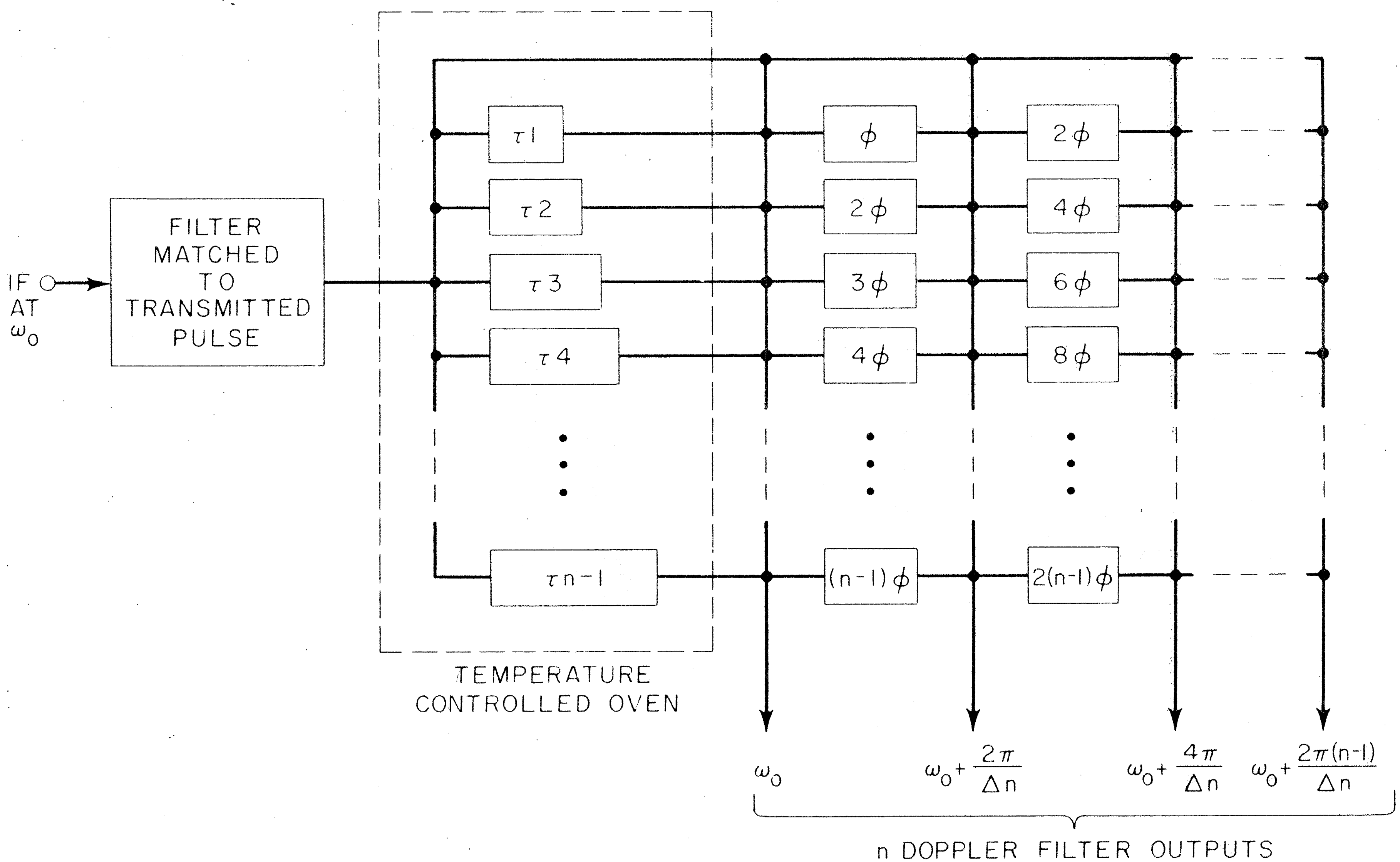

Fig. 30: BLOCK DIAGRAM OF DELAY LINE APPROACH TO MATCHED FILTER

lines have the highest temperature coefficient, the lowest bandwidth and the shortest time delay achievable of the three. Quartz which has been used extensively in this application has the capability for large bandwidth as well as large time delay but is unfortunately characterized by a fairly high temperature coefficient. Glass delay lines have been developed with temperature coefficients an order of magnitude smaller than quartz but the spurious responses and the attenuation are significantly higher and the delays presently achievable significantly smaller. Since quartz is widely used, and has proven to be a practical delay medium, it is reasonable to consider it as representative of the best delay medium available for large bandwidth, large $TW/2\pi$ signals, and to calculate the maximum $TW/2\pi$ for a system using these lines.

It is generally accepted that the maximum bandwidth of a system consisting of quartz delay lines, transducers, drivers and amplifiers is $W/2\pi_{max} \approx 0.4f_o$, where f_o is the center frequency of the line. The temperature coefficient of quartz is approximately 75 ppm/$^{\circ}$C and the present state-of-the-art limitation in short term temperature control is approximately 0.01°C variation within the temperature controlled oven. $\therefore \Delta T = T(.01 \times 75 \times 10^{-6}) = 0.75 \times 10^{-6} T$. To achieve coherent addition out of the lines the maximum differential change in phase, due to the differential temperature should be small, hopefully less than $\frac{\pi}{9}$ or 20°. Consequently

$$2\pi(\Delta T f_o) \leq \pi/9$$

or

$$f_o \leq \frac{1}{18\Delta T}$$

Since

$$\frac{W}{2\pi} \leq 0.4f_o$$

$$\therefore TW/2\pi \leq T \frac{0.4}{18\Delta T} \leq \frac{0.4T}{18 \times .75 \times 10^{-6}T} \approx 30{,}000 \qquad (68)$$

If glass delay lines can achieve lower attenuation, greater length and reduction of the spurious responses without sacrifice of temperature coefficient then their use will raise this limitation by an order of magnitude.

7.3 Use of Range Gates and Narrow-band Filters

The second approach, shown in Figure 31, involves the gating of the received IF signal into a bank of narrow band filters. The gate has a width of δ and is opened n times. If a target return from each of the pulses in the train is in each of the gates then the narrow band filters coherently integrate these returns. The use of a bank of filters insures that one of the filters is close to a match to the returned signal regardless of the doppler shift. The noise free input and output of this filter with time is also shown in Figure 31. A useful feature of this approach is that the narrow band filters store the signal output for a long time after the signal is gone. Consequently the resolution inherent in a bandwidth of $1/\delta$ is achieved but the data can be read out of the filter at a much lower bandwidth. The price paid for this is the requirement that the target appear

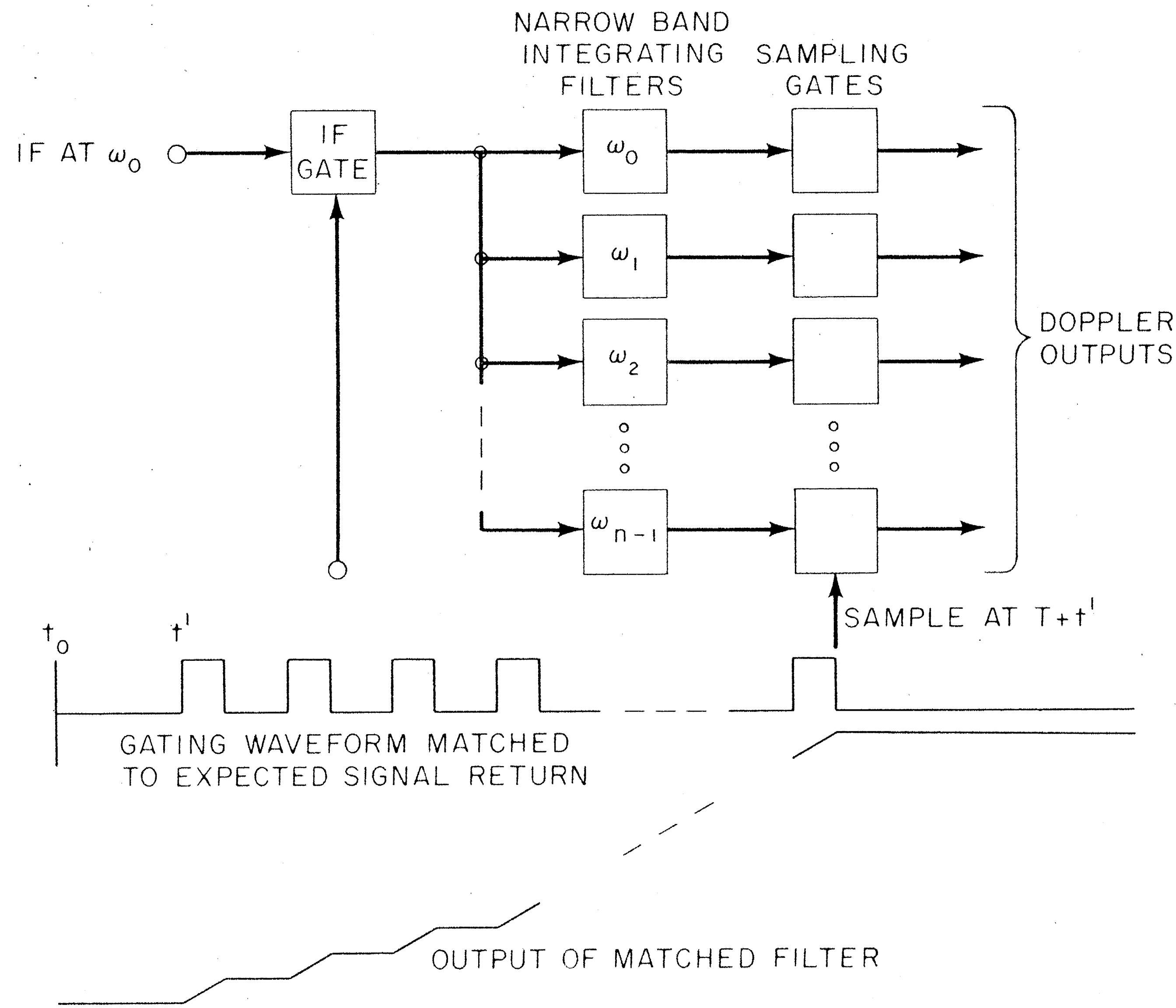

Fig. 31. BLOCK DIAGRAM OF RANGE-GATE APPROACH TO MATCHED FILTER

in the range gate. Consequently a series of range gates, spaced in time, must be provided to guarantee that targets over a given range of interest can be observed. The number of gates required is

$$N \approx \frac{\text{time extent of target}}{\delta} .$$

It can be shown that the Q of the integrating filter limits the maximum duration-bandwidth product that can be reasonably used with this approach.

For each integrating filter at frequency f_o with bandwidth Δf if $T\Delta f \leq 0.4$ then the loss in signal-to-noise due to non-zero bandwidth of the integrating filter is less than 1 db. Further it is essential that $\delta f_o \geq 2$ to provide at least two cycles of the carrier frequency within the pulse envelope. Gating, mixing and filtering become extremely difficult with fewer cycles.

Since
$$TW/2\pi = \frac{T}{\delta} \leq \frac{Tf_o}{2}$$

and
$$f_o = \Delta f Q$$

then
$$TW/2\pi \leq \frac{T(\Delta f \times Q)}{2} = \frac{Q(T\Delta f)}{2} \leq 0.2Q \qquad (69)$$

For frequencies up to a few hundred kcps the highest Q is obtained with magnetostrictive filters while at higher frequencies crystal filters achieve the highest Q. Both of these have a maximum Q of approximately 2×10^4 and result in a limit of

$$TW/2\pi \leq 4000 \tag{70}$$

7.4 <u>Use of Recirculating Delay Lines</u>

The use of a recirculating delay line for a uniform pulse train with a constant interpulse period Δ requires a single line of length Δ. With the non-uniformly spaced trains there must be the capability of having sequential delays, pulse-to-pulse, covering the interpulse intervals ranging from Δ to Δ_{max}. The approach shown in Figure 32 is one way to achieve this result. The length of the total delay is sequentially altered by the use of (n-1) short lines or a single line with (n-1) taps, and an electronic commutator which is stepped in accordance with the transmitted signal. Since the recirculating delay line shown in Figure 32 will coherently integrate returns at one frequency, a bank of n such lines is required to cover all expected frequencies. Because the smallest interpulse period is Δ the range extent over which targets can be observed is also Δ. Consequently if observation is required over a greater range extent several banks of these filters would be required.

The limitations to the successful use of this approach are the problems of temperature control of the quartz line and the difficulty of achieving a flat frequency response in the quartz lines and extra delay lines over the signal bandwidth. Since the first pulse recirculates for

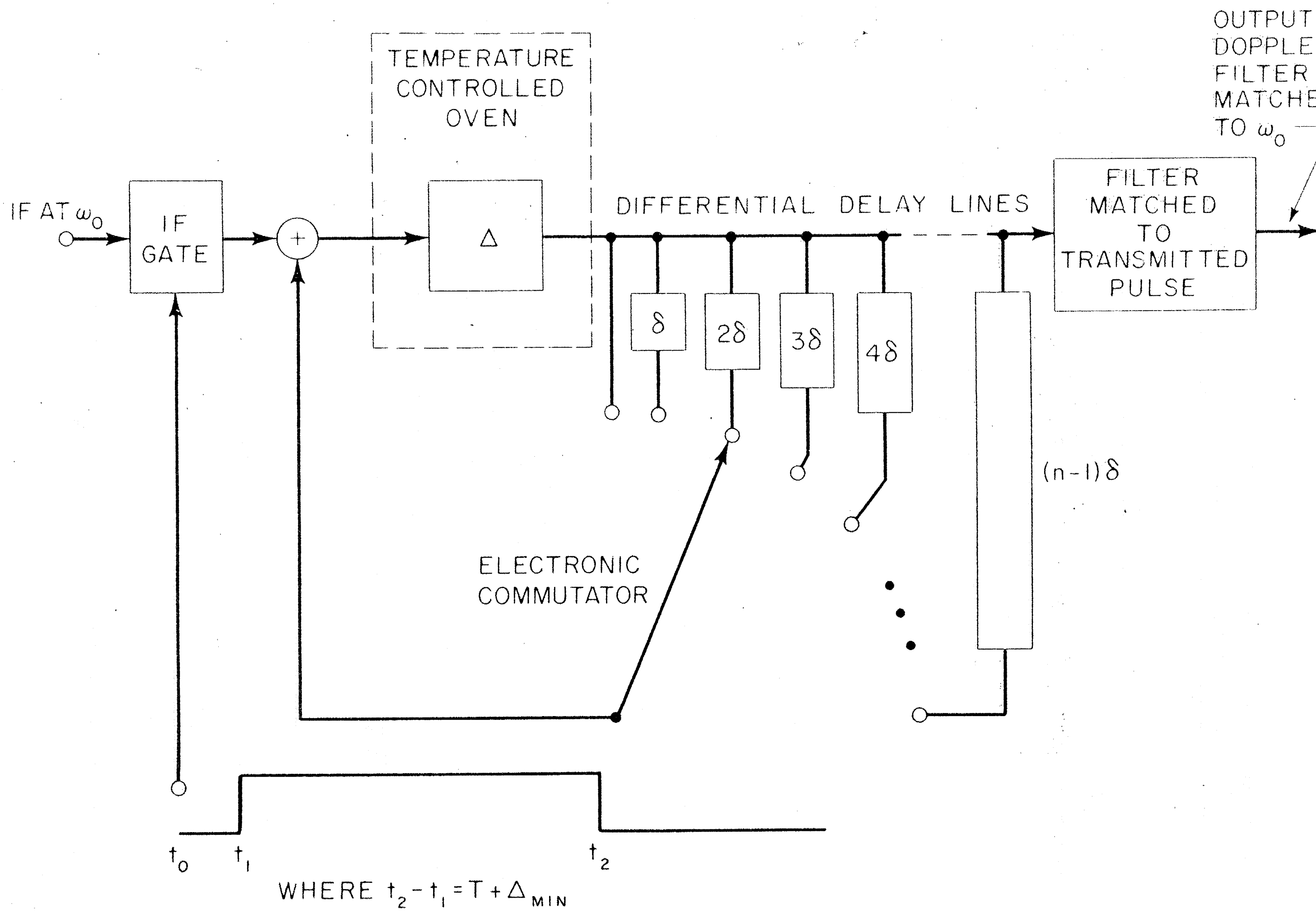

Fig. 32: BLOCK DIAGRAM OF RECIRCULATING DELAY LINE APPROACH TO MATCHED FILTER

n-1 times, the 1 db points of the delay line response become the (n-1) db points for this pulse. The difficulty of achieving feedback ratios near unity, without causing oscillation will result in a further degredation in signal-to-noise.

7.5 Digital Approach

It is entirely possible to digitize the IF signals first and then perform the filtering inside a computer. As mentioned earlier the limitation to the usefulness of this approach will be the time the target is "stationary," the maximum sampling rate of analog to digital converters ($\approx 30 \times 10^6$ bits/sec at present), and the maximum input data rate to a computer. The IBM 7090 can handle a maximum information rate (bandwidth) of about 200 Kc with about 6-bits of quantization. This can represent all the data from a single 200 Kc bandwidth channel or, with a suitable buffer storage, 10% of the data from a 2 megacycle bandwidth channel. In a target environment limited in terms of number of targets of interest this approach can prove very useful.

7.6 Compensation for Timing Errors and Pulse Distortion

In the design of these non-uniformly spaced pulse trains it has been assumed that the signal was a sequence of rectangular pulses of width δ. It was further assumed that the error in the time of transmission was zero. For these assumptions a variation of δ in the interpulse spacings is sufficient to ensure that no sidelobes are ever closer together than δ (as shown in Figure 29a). However, it would be wise to compensate for expected timing errors in the transmitter and for widening of the pulse due to limited transmitter and receiver bandwidth. In a well designed system equivalent broadening of the pulse due to these factors will be less than $\pm \frac{\delta}{8}$. Consequently a 25% increase in all the interpulse spacings should be adequate. If the complete isolation of all time sidelobes is required (as shown in Figure 29b) the train should be designed with a $2\frac{1}{4}\ \delta$ variation in interpulse spacing.

CHAPTER VIII

SUMMARY AND CONCLUSIONS

Non-uniformly spaced pulse trains with n pulses having a uniformly low sidelobe level, or

$$\frac{\psi(0,0)}{n^2} \geq \psi(\tau,\omega) \qquad |\tau| \geq \delta \tag{15}$$

will be generated when the sequence of interpulse spacings are

$$\Delta_i = \Delta + \delta\ [X_o + iq]\ \text{modulo}\ (n-1) \tag{50}$$

where

$$\begin{aligned} i &= 0,1,2,\ldots,\ n-2 \\ n &= \text{the number of pulses in the train} \\ n-1 &> q > 0 \\ n-1 &> X_o \geq 0 \\ n-1 &= \text{a prime number} \\ \delta &= \text{the pulse width} \end{aligned} \tag{55}$$

The duration of the train will be

$$T = (n-1)\left[\Delta + \delta\left(\frac{n-2}{2}\right)\right]. \tag{29}$$

The minimum interpulse spacing Δ, which is also the distance to the first time sidelobe, will be

$$\Delta \approx \frac{n^{3/2}}{4} \tag{58a}$$

and the duration-bandwidth product of the signal generated can be expected to be

$$TW/2\pi \approx \frac{n^{5/2}}{4} \tag{58b}$$

Optimum waveforms, which would require

$$TW/2\pi \approx \frac{n^2}{2} \tag{21}$$

are impossible for $n > 4$.

For a pulse train with 100 pulses the maximum sidelobe level will be -40 db and the required $TW/2\pi \approx 25{,}000$. For a pulse train with 140 pulses the maximum sidelobe level will be -43 db and $TW/2\pi \approx 60{,}000$.

The characteristics of the target ensemble will undoubtedly limit the maximum duration of the waveform. Further, for the narrow-band approximation to be valid

$$TW/2\pi \leq \left(\frac{2\nu}{c} \right)^{-1} \tag{65}$$

or for targets with satellite velocities

$$TW/2\pi \leq 23{,}000 \tag{67}$$

When $TW/2\pi$ increases beyond the point where the narrow band approximation is valid an increase in the difficulty of constructing the matched filter must be expected.

Analog approaches to realizing the matched filter for these pulse trains are presently limited by

a) the temperature coefficient of quartz delay lines and the present state-of-the-art capability of providing a uniform temperature environment;

b) the maximum Q available from crystal or magnetostrictive filters.

For these approaches (and the present state-of-the-art) the limits on $TW/2\pi$ are, respectively,

a) $TW/2\pi \leq 30{,}000$ (68)

b) $TW/2\pi \leq 0.2Q \leq 4{,}000$ (69, 70)

Consequently, given these limits to the maximum duration-bandwidth product it is reasonable to design non-uniformly spaced trains with 100 pulses having a -40 db sidelobe level and requiring a duration-bandwidth of the order of 25,000. Achievement of another 3 db reduction in sidelobe level will come dearly.

CHAPTER IX

RECOMMENDED AREAS FOR FURTHER STUDY

The use of non-uniform interpulse spacing having a greater density of pulses in the center of the train than at the ends will result in a significant reduction in the close-in sidelobes in doppler with no increase in the time sidelobes. However, no general approach now exists for selecting the interpulse spacings in this manner which simultaneously assures a non-overlapping of time sidelobes. It remains a cut-and-try process. A technique for realizing a cosine or $(\text{cosine})^2$ density taper, which assures uniformly low sidelobe levels in time without requiring an exhorbitant $TW/2\pi$, would be extremely useful.

The limits to the performance achievable with the use of the complete set of least residues modulo (n-1) are, at present, unknown. A detailed study from a Number Theory point of view should provide these limits and indicate the value of the parameter, q, which yields the best performance. Also, this approach can be used only if there is a prime number close to the desired number of pulses. It would be useful to have another approach which would yield equivalent or superior performance and was not similarly bound.

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