*Speed limits in autonomous vehicular networks due to communication constraints*

# Speed Limits in Autonomous Vehicular Networks due to Communication Constraints

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***Abstract*—Increasing applications and decreasing sizes of autonomous vehicles is likely to result in a dense network of heterogeneous autonomous vehicles, each moving around to perform a separate task. Autonomous vehicles need to be aware of other vehicles in it's vicinity in order to successfully perform their tasks. Such network awareness is ensured by exchanging location and control information over wireless radio channels. However, wireless interference constraints limit the number of messages that can be exchanged between the vehicles. In this paper, we study the impact of such communication constraints on maximum speed in dense autonomous vehicular networks. We define *hazard rate* to be the fraction of times a vehicle enters a region, call it 'uncertainty region', where there is a positive chance of other vehicles being present. We show that such a performance measure follows a threshold behavior with respect to maximum speed $v$ as the network density $n$ increases to infinity. We show that, for a planar network, the hazard rate tends to $1$, if the maximum speed $v$ decreases slower than $n^{-3/2}$, and tends to $0$, if $v$ decreases faster than $n^{-3/2}$. For the *network hazard rate*, which is fraction of times any vehicle enters it's uncertainty region, the threshold is $n^{-2}$. For the spatial network, however, these thresholds turn out to be larger. This implies that it is better to plan autonomous vehicular networks, such as UAV networks, over a three dimensional space rather than a two dimensional one.**

## I. INTRODUCTION

In recent years, network of unmanned aerial vehicles (UAVs) have become prevalent, with applications ranging from surveillance, environment monitoring, product delivery, disaster monitoring and many more. Moreover, it is now possible to deploy very dense networks of 'micro' vehicles, with sizes as small as 10-50 cm [1], [2]. In such networks, the use of centralized control, and sophisticated sensing technology, to plan and control motion is not possible [3]; giving rise to the need for distributed control using wireless transmission [3], [4]. Wireless communications can be used to exchange position and control information. However, delays in exchanging such information can result in uncertainty, and potentially lead to collisions between vehicles that are not fully aware of each others location.

The amount of time that has passed since vehicles last exchanged location information is a crude measure of uncertainty regarding vehicles' position, as vehicles may have moved in the time that elapsed. The 'uncertainty region' is the region where a vehicle may have traveled to since the last position update, as illustrated in Figure 1. While ideally the uncertainty region should be kept very small, in dense networks this may not be possible due to limits in communications. In particular, wireless interference constraints limit the number of simultaneous transmissions that can take place [5]. As can be seen from Figure 1, the uncertainty region is a function of the vehicles' speed, and the time that has elapsed since the most recent update. Thus, if it is not possible to transmit position updates more frequently, vehicles may need to reduce their speed in order to avoid hazardous conditions. This situation is exacerbated in dense networks, where vehicles are in closer proximity to avoid each others uncertainty regions.

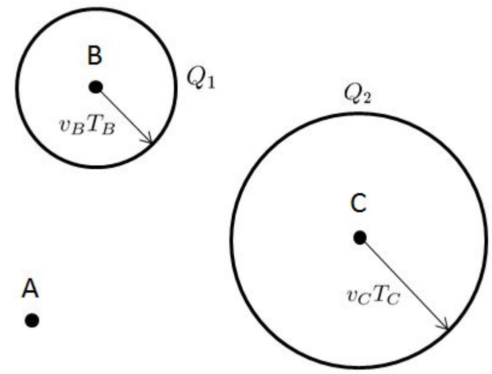


Fig. 1. Time snapshot of an autonomous vehicle network with vehicles A, B, and C. Vehicle B and C travel at maximum speed $v_B$ and $v_C$, respectively, and the amount of time that elapsed since vehicles B and C communicated their location to vehicle A is $T_B$ and $T_C$, respectively. From the point of view of vehicle A, vehicle B can be anywhere inside the circle $Q_1$ of radius $v_BT_B$; similarly vehicle C can be anywhere inside circle $Q_2$. The uncertainty region of vehicle $A$ is the total region covered by circles $Q_1$ and $Q_2$.

We consider a network of $n$ autonomous vehicles in a planar and spatial bounded region. The nodes move according to an independent, stationary, and ergodic random process, with maximum speed $v$. We define the *hazard rate* of a vehicle to be the fraction of times the vehicle is in it's uncertainty region and the *network hazard rate* to be the fraction of times any vehicle is in it's uncertainty region, and study the hazard rates as $n \to \infty$.

Our main result is that the hazard rates follow a threshold curve with respect to $v$ as $n \to \infty$. For the planar network, we show that, under any communication scheme, if $v$ decreases

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slower than $n^{-3/2}$ the hazard rate of any vehicle will go to 1 as $n \to \infty$. This means that every vehicle almost surely will be in it's uncertainty region. However, if $v$ decreases faster than $n^{-3/2}$, then a simple communication scheme ensures that the hazard rate of any vehicle will be 0 as $n \to \infty$, i.e., vehicle will not be in it's uncertainty region with probability one. Also, for the planar network, the speed threshold for the network hazard rate is $n^{-2}$. For the spatial network, the speed thresholds for both hazard rates are smaller than the planar case. We also show that in both cases a simple round-robin scheme, in which vehicles transmit in a preassigned order, attains the minimum hazard rates.

### A. Related Work

Dense network of communicating mobile nodes have been studied for communication delay and capacity [6]–[8]. In [7], it was first shown that mobility improves capacity. Subsequently, communication delay has been studied under various node mobility models such as Markov [9], [10], random way point [11], and Brownian [12]. General observation has been that increasing node speed improves communication delay. Such a relation between delay and node/vehicle speed is also known for load-carry-and-deliver or data ferrying protocols [13], [14]. However, a constraint on vehicle speed due to communication constraints has never been considered.

A critical speed limit for a collision-free trajectory through a dense forest was proved in [15]. The obstacles were modeled as static objects derived from a stationary marked point process. In our model, the obstacles, being other vehicles, are also in motion.

### B. Outline

The paper is organized as follows. We describe our system model in Section II for the planar network model. In Section III, we state and prove the main results with respect to an individual vehicle's hazard rate, and in Section IV, study the network-wide hazard rate. We discuss the spatial model and it's threshold results in Section V. We conclude in Section VI.

## II. Problem Definition

We consider a system with $n$ autonomous vehicles that move inside a square torus $S = [0,1]^2$. For the torus, the distance between two points $\mathbf{x} = (x_1, x_2) \in S$ and $\mathbf{y} = (y_1, y_2) \in S$ is given by

$$d(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{e} \in \{-1,0,1\}^2} \|\mathbf{x} + \mathbf{e} - \mathbf{y}\|_2, \tag{1}$$

where $\|\cdot\|_2$ is the Euclidean norm. Figure 2 illustrates the distance function $d$ on unit torus $\mathcal{S}$. We denote $N = \{1, 2, \ldots n\}$ to be the set of autonomous vehicles.

We also use the following notations. We use $\mathbf{P}[\cdot]$ and $\mathbf{E}[\cdot]$ to denote probability and expectation, respectively. For functions $f$ and $g$ we say $f(n) = O(g(n))$ if there exists a $C > 0$ such that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \le C$. We write $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

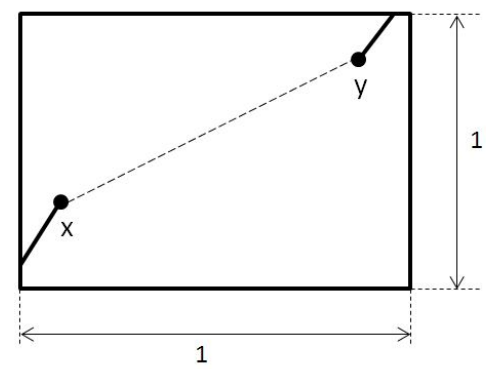


Fig. 2. Undotted black line traces the shortest distance path between points $\mathbf{x}$ and $\mathbf{y}$ on unit torus $\mathcal{S}$.

### A. Mobility Model

Each vehicle moves according to an independent, stationary, ergodic random process with uniform stationary distribution. We also assume that this motion is such that, if at time $t$ the vehicle is at location $\mathbf{x}$, then it's location at time $t + \tau$ can be anywhere inside the region $B(\mathbf{x}, v\tau)$, for all $t$ and $\tau > 0$. Thus, the variable $v$ is the maximum speed that vehicles in the network can possibly achieve. Random waypoint and Markov mobility are two examples of such motion models [9], [11], [16], [17].

### B. Communication Model

The autonomous vehicles exchange location information with each other over wireless radio channel. Each vehicle maintains two lists. The first list tracks the last received location of every vehicle and the second list tracks the time validity of this information. More precisely, a vehicle $i$ at time $t$ maintains lists:

$$\chi^i(t) = \left(x_1^i(t), x_2^i(t), \ldots x_n^i(t)\right), \tag{2}$$

where $x_k^i(t)$ denotes the last communicated location of vehicle $k$ to vehicle $i$ by time $t$; here $x_i^i(t)$ is the exact location of vehicle $i$, and

$$\Theta^i(t) = \left(\Delta_1^i(t), \Delta_2^i(t), \ldots \Delta_n^i(t)\right), \tag{3}$$

where $\Delta_k^i(t)$ is the time elapsed since vehicle $k$ was at location $x_k^i(t)$. This means that at time $t$, for vehicle $i$, the location of vehicle $k$ can be anywhere inside the circle of radius $v\Delta_k^i(t)$ centered at $x_k^i(t)$. In the absence of a new information packet from vehicle $k$ to vehicle $i$, $\Delta_k^i(t)$ increases linearly in $t$ at rate 1. On the other hand, if it receives a packet from vehicle $k$, $\Delta_k^i(t)$ is reset to zero. Since vehicle $i$ always knows it's location we set $\Delta_i^i(t) = 0$.

For simplicity, in this work, we assume a single cell broadcast channel model. When a single vehicle transmits a packet, all other vehicles can receive it correctly. A packet transmitted by a vehicle contains the vehicles current location. We consider a time slotted system [18]. Duration of

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each slot, denoted by $\delta$, equals the time required for a single packet transmission. A single packet can be transmitted in one slot. However, two or more packet transmissions during a slot leads to failure in packet reception due to wireless interference [5], [18].

A communication scheme is an agreed set of rules that determines when each vehicle transmits. We call a communication scheme *recurrent* if, in it, each vehicle transmits infinitely often. We say it is $O(n)$-*recurrent* if it is recurrent and each vehicle transmit every $O(n)$ time slots, *i.e.*, $\limsup_{k\to+\infty} \max_{i\in N} \tau^i(k) = O(n)$ for all $i \in N$, where $\tau^i(k)$ is the $k$th inter-transmission time between two transmissions of $i$. A *round robbin* scheme is one where, in slot $m$, vehicle $i_m = 1 + m \bmod (n-1) \in N$ transmits.

We consider communication schemes that are location independent. That is, vehicles do not use their location information to schedule transmissions. Thus, a real-world autonomous vehicular system can perform at least as good as the performance characterized here.

### C. Performance Measure

At time $t$, for vehicle $i$, vehicle $k$ can be anywhere inside the circle or ball with radius $v\Delta_k^i(t)$ centered at $x_k^i(t)$; denoted as $B\left(x_k^i(t), v\Delta_k^i(t)\right)$, where $B(x,r) = \{y \in S | d(x,y) < r\}$. This circle is called *the region of uncertainty* of vehicle $k$ with respect to vehicle $i$. Then, the *net uncertainty region* of vehicle $i$ with respect to all other vehicles is defined as

$$R^i(t) = \bigcup_{k=1}^{n} B\left(x_k^i(t), v\Delta_k^i(t)\right). \tag{4}$$

Informally speaking, the net uncertainty region for vehicle $i$ is the set of all locations that may include another vehicle which vehicle $i$ is unaware of. To guarantee location-awareness for vehicle $i$, we would like to make sure that vehicle $i$ does not lie inside it's own uncertainty region $R^i(t)$. In this way, vehicle $i$ can be aware of any vehicle that approaches its location.

We define $A^i(t)$ to be the event that vehicle $i$ lies in $R^i(t)$,

$$A^i(t) = \{x_i^i(t) \in R^i(t)\}, \tag{5}$$

and $\gamma_n^i$ to be the fraction of times vehicle $i$ lies in $R^i(t)$, *i.e.*,

$$\gamma_n^i = \mathbf{E}\left[\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbb{I}_{A^i(t)} dt\right], \tag{6}$$

where $\mathbb{I}_A$ is the indicator function for event $A$. Using dominated convergence theorem [19], we have

$$\gamma_n^i = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{P}\left[A^i(t)\right] dt. \tag{7}$$

We call $\gamma_n^i$ as the *hazard rate for vehicle* $i$. The hazard rate denotes the rate at which vehicle $i$ goes into its uncertainty region, hence the rate at which vehicle $i$ may miss another vehicle passing by its location without vehicle $i$ knowing.

In Section III, we minimize the hazard rate for a vehicle as $n \to \infty$ for different values of $v$. We also show that the simple round robbin scheme attains the minimum.

This individual location awareness does not entail location awareness for the entire network. For the latter, we also consider the event that any vehicle may lie in it's uncertainty region at time $t$:

$$A(t) = \bigcup_{i=1}^{n} A^i(t), \tag{8}$$

and define *network hazard rate* to be

$$\gamma_n = \mathbf{E}\left[\lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbb{I}_{A(t)} dt\right], \tag{9}$$

$$= \lim_{T\to+\infty} \frac{1}{T}\int_0^T \mathbf{P}\left[A(t)\right] dt. \tag{10}$$

In Section IV we minimize the network hazard rate.

Notice that the individual hazard rate and network hazard rate depend on the communication scheme used to exchange information. Thus, if $\mathcal{C}$ denotes a communication scheme, we shall also use $\gamma_n^i(\mathcal{C})$ and $\gamma_n(\mathcal{C})$ to denote the individual and network hazard rate, respectively, using scheme $\mathcal{C}$.

## III. Analysis for Individual Location Awareness

In this section we minimize the individual hazard rate as $n \to \infty$. We show that, in the limit, the hazard rate $\gamma_n^i$ exhibits a phase transition with respect to maximum speed $v$.

**Theorem** *1:* If $v$ scales in $n$ such that,

1) $vn^{3/2} \to \infty$ then for any communication scheme,

$$\lim_{n\to\infty} \gamma_n^i = 1, \tag{11}$$

for all $i \in N$.

2) $vn^{3/2} \to 0$ then for the round robbin scheme

$$\lim_{n\to\infty} \gamma_n^i = 0, \tag{12}$$

for all $i \in N$.

This result implies that the hazard rate for vehicle $i$, $\gamma_n^i$, follows a threshold behaviour with respect to maximum speed $v$, in the asymptotic as $n \to \infty$. Further, if any vehicle $i$ indents to avoid it's uncertainty region then the maximum speed should scale down faster than $n^{-3/2}$.

The key reason for $n^{-3/2}$ threshold is that the delays, $\Delta_k^i(t)$ for $i, k \in N$, grow at best linearly in $n$ for any communication scheme. When this is the case the area of each ball $B\left(x_k^i(t), v\Delta_k^i(t)\right)$ is $\pi v^2 n^2$. Since there are $n-1$ of them in the net uncertainty region $R^i(t)$, the area of $R^i(t)$ is roughly $\pi v^2 n^3$. Thus, when $vn^{3/2} \to 0$, the area of $R^i(t)$ goes to zero.

Theorem 1 also states that the round robbing scheme achieves the best performance, in the asymptotic as $n \to \infty$. This is because, the round robbin scheme ensures that

$\Delta_k^i(t) \le n\delta$ for all $i, k \in N$ as each vehicle transmits once every $n$ slots.

We make two key observations: First, in a single cell broadcast channel, when one vehicle transmits a packet every other vehicle receives that packet. As a result, the last location communicated by any vehicle $k$ be same for all other vehicles, i.e.,

$$x_k^i(t) = x_k^j(t), \tag{13}$$

for all $i, j \in N\backslash\{k\}$. This also implies that the time since location of vehicle $k$ was communicated will be same for all vehicles, i.e.,

$$\Delta_k^i(t) = \Delta_k^j(t), \tag{14}$$

for all $i, j \in N\backslash\{k\}$. Let us, therefore, denote

$$\Delta_k^*(t) = \Delta_k^i(t), \tag{15}$$

for any $i$ and all $k \in N\backslash\{i\}$. We then know that such a collection $\{\Delta_k^*(t)\}_{k\in N}$ is well defined.

The second observation is an invariant property that is satisfied by all communication schemes. Let $f_t$ denotes the fraction of vehicles $k$ with $\Delta_k^*(t)$ greater than $\delta\left\lceil\frac{n}{2}\right\rceil$. This is given by

$$f_t = \frac{1}{n}\sum_{k\in N}\mathbb{I}_{\left\{\Delta_k^*(t)>\delta\left\lceil\frac{n}{2}\right\rceil\right\}}. \tag{16}$$

The following lemma guarantees a lower bound on $f_t$ for any communication scheme.

**Lemma** *1:* For any communication scheme, if $t > n\delta$ then

$$f_t \ge \frac{1}{2} - \frac{1}{n}, \quad \text{a.s.} \tag{17}$$

*Proof:* Proof is given in Appendix A. ■

This shows that at least nearly half the vehicles have the delay, $\Delta_k^*(t)$, greater than $\left\lceil\frac{n}{2}\right\rceil\delta$. We now prove Theorem 1.

*Proof of Theorem 1:* We first prove the second part of the claim. Since the round robbin scheme does not depend on vehicle location, $\mathbf{P}\left[A^i(t)\right]$ can be written as

$$\mathbf{P}\left[A^i(t)\right] = \mathbf{P}\left[U_i \in \bigcup_{k\in N\backslash\{i\}} B\left(U_k, v\Delta_k^i(t)\right)\right], \tag{18}$$

where $(U_1, \ldots, U_n)$ are independent and identically distributed random variables, uniformly distributed over $\mathcal{S}$. Since, $\Delta_i^i(t) = 0$ we can also write $\mathbf{P}\left[A^i(t)\right]$ to be

$$\mathbf{P}\left[A^i(t)\right] = \mathbf{P}\left[V \in \bigcup_{j\in N} B\left(U_j, v\Delta_j^i(t)\right)\right], \tag{19}$$

where $V$ is another uniformly distributed random variable over $\mathcal{S}$ that is independent of all $U_j$. For round robbin scheme, the delays $\Delta_k^i(t)$ are bounded above by $n\delta$ as in any time duration of $n\delta$ there is at least once that every vehicle transmits. This upper bound on $\Delta_k^i(t)$ implies

$$\begin{aligned}
\mathbf{P}\left[A^i(t)\right] &\le \mathbf{P}\left[V \in \bigcup_{j\in N} B\left(U_j, nv\delta\right)\right], &(20)\\
&= \mathbf{P}\left[\bigcup_{j\in N}\left\{V \in B\left(U_j, nv\delta\right)\right\}\right], &(21)\\
&= 1 - \mathbf{P}\left[\bigcap_{j\in N}\left\{V \notin B\left(U_j, nv\delta\right)\right\}\right]. &(22)
\end{aligned}$$

The events $\{V \notin B\left(U_j, nv\delta\right)\}$ are independent because the $U_j$s and $V$ are independent. This implies,

$$\begin{aligned}
\mathbf{P}\left[A^i(t)\right] &\le 1 - \prod_{j\in N}\mathbf{P}\left[V \notin B\left(U_j, nv\delta\right)\right], &(23)\\
&= 1 - \prod_{j\in N}\left(1 - \pi(nv\delta)^2\right), &(24)\\
&= 1 - \left(1 - \pi(nv\delta)^2\right)^n, &(25)\\
&= \Theta\left(1 - e^{-c\left(n^{\frac{3}{2}}v\delta\right)^2}\right). &(26)
\end{aligned}$$

Thus, if $vn^{3/2} \to 0$ then $\mathbf{P}\left[A^i(t)\right] \to 0$ as $n \to +\infty$. Since $\gamma_n^i$ is a Cesaro mean of $\mathbf{P}\left[A^i(t)\right]$, we have $\gamma_n^i \to 0$ as $n \to 0$. This proves the second part of the result.

We now prove the first claim. Since our communication schemes are location independent, we still have (19) to be true. Take $t > n\delta$ and define $\tilde{\Delta}_k^i$ as follows:

$$\tilde{\Delta}_k^i(t) = \begin{cases} 0 & \text{if } \Delta_k^i(t) \le \delta\lceil\frac{n}{2}\rceil \\ \frac{n\delta}{2} & \text{otherwise}\end{cases} \tag{27}$$

We know from Lemma 1 that the number of $k \in [n]$ with $\tilde{\Delta}_k^i = \frac{n\delta}{2}$ is at least $\frac{n}{2} - 1$. Using this along with (19) we have

$$\begin{aligned}
\mathbf{P}\left[A^i(t)\right] &= \mathbf{P}\left[V \in \bigcup_{j\in N} B\left(U_j, v\Delta_j^i(t)\right)\right] &(28)\\
&\ge \mathbf{P}\left[V \in \bigcup_{j=1}^{\frac{n}{2}-1} B\left(U_j', \frac{nv\delta}{2}\right)\right], &(29)
\end{aligned}$$

where $U_j'$ are the locations corresponding to vehicles that have $\tilde{\Delta}_k^i(t) = \frac{n\delta}{2}$. Since the communication scheme is location independent, $U_j'$s would be independent and uniformly

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distributed over $\mathcal{S}$. This implies

$$\mathbf{P}\left[A^i(t)\right] \geq \mathbf{P}\left[V \in \bigcup_{j=1}^{\frac{n}{2}-1} B\left(U_j^{'}, \frac{nv\delta}{2}\right)\right], \tag{30}$$

$$= \mathbf{P}\left[\bigcup_{j=1}^{\frac{n}{2}-1} \left\{V \in B\left(U_j^{'}, \frac{nv\delta}{2}\right)\right\}\right], \tag{31}$$

$$= 1 - \mathbf{P}\left[\bigcap_{j=1}^{\frac{n}{2}-1} \left\{V \notin B\left(U_j^{'}, \frac{nv\delta}{2}\right)\right\}\right], \tag{32}$$

$$= 1 - \left(1 - \pi\left(\frac{nv\delta}{2}\right)^2\right)^{\frac{n}{2}-1}, \tag{33}$$

$$= \Theta\left(1 - e^{-c\left(n^{\frac{3}{2}}v\delta\right)^2}\right). \tag{34}$$

Thus, if $vn^{3/2} \to \infty$, then $\mathbf{P}\left[A^i(t)\right] \to 1$ as $n \to +\infty$. Since $\gamma_n^i$ is Cesaro mean of this sequence, $\gamma_n^i \to 1$ as $n \to \infty$. ■

## IV. Analysis for Network Location Awareness

In this section, we minimize the network hazard rate, $\gamma_n$, as $n \to \infty$. We show that, in the limit, $\gamma_n$ has a threshold behaviour.

**Theorem** *2:* If $v$ scales in $n$ such that

1) $vn^2 \to \infty$ then for any communication scheme

$$\lim_{n \to +\infty} \gamma_n = 1. \tag{35}$$

2) $vn^2 \to 0$ then for the round robbin scheme

$$\lim_{n \to +\infty} \gamma_n = 0. \tag{36}$$

This result implies that the network hazard rate also follows a threshold behaviour with respect to maximum speed $v$, as $n \to \infty$. The speed threshold for the network hazard rate is $n^{-2}$, which is smaller than the threshold for the hazard rate. Thus, the vehicles need to move more slowly if they want to ensure network wide location awareness.

The key reason is again that the delays, $\Delta_k^i(t)$ for all $i, k \in N$, grow at best linearly in $n$ for any communication scheme. A random geometric graph $\mathcal{G}(n, r)$ is a graph with $n$ nodes independent and uniformly distributed on $\mathcal{S}$ with an edge between two nodes located at $\mathbf{x}$ and $\mathbf{y}$ if $d(\mathbf{x}, \mathbf{y}) < r$. We can then roughly think of the event $A(t)$ to be the event that there exists an edge in a random geometric graph $\mathcal{G}(n, r)$ with $r \approx vn$; because $r$ approximates $\Delta_k^i(t)$. From [20], we know that the probability that there is an edge in $\mathcal{G}(n, r)$ goes to zero (or one) if $rn \to 0$ (or if $rn \to \infty$). This implies the $n^{-2}$ threshold for maximum speed $v$.

Theorem 2 also shows that the round robbin scheme attains the best performance. This is because, in the round robbin scheme, the delays grow linearly in $n$. We now prove Theorem 2.

*Proof of Theorem 2:* For any time $t \geq 0$, the probability $\mathbf{P}[A(t)]$ is given by

$$\mathbf{P}[A(t)] = \mathbf{P}\left[\bigcup_{i \in [n]} \left\{x_i^i(t) \in \bigcup_{k \in [n]\setminus\{i\}} B\left(x_k^i(t), v\Delta_k^i(t)\right)\right\}\right].$$

Since the communication schemes we consider are vehicle location independent we can write $\mathbf{P}[A(t)]$ as

$$\mathbf{P}[A(t)] = \mathbf{P}\left[\bigcup_{i \in [n]} \left\{U_i \in \bigcup_{k \in [n]\setminus\{i\}} B\left(U_k, v\Delta_k^i(t)\right)\right\}\right], \tag{37}$$

where $U_i$s are independent, uniformly distributed random variables over $\mathcal{S}$. For the round robbin scheme we also have $\Delta_k^i(t) \leq n\delta$. This implies,

$$\mathbf{P}[A(t)] \leq \mathbf{P}\left[\bigcup_{i \in [n]} \left\{U_i \in \bigcup_{k \in [n]\setminus\{i\}} B\left(U_k, nv\delta\right)\right\}\right]. \tag{38}$$

Now, note that if $\mathcal{G}(n, nv\delta)$ is a random geometric graph on the torus $\mathcal{S}$ then the event

$$\bigcup_{i \in [n]} \left\{U_i \in \bigcup_{k \in [n]\setminus\{i\}} B\left(U_k, nv\delta\right)\right\}, \tag{39}$$

is same as the event that there is at least one node in the graph $\mathcal{G}(n, nv\delta)$. Thus,

$$\mathbf{P}[A(t)] \leq \mathbf{P}[M \geq 1], \tag{40}$$

where $M$ is the number of edges in the graph $\mathcal{G}(n, nv\delta)$. For a random geometric graph $\mathcal{G}(n, r)$, $\mathbf{P}[M \geq 1] \to 0$ if $rn \to 0$ as $n \to +\infty$. Hence, if $vn^2 \to 0$ we have $\mathbf{P}[A(t)] \to 0$ as $n \to +\infty$. Since $\gamma_n$ is a Cesaro mean sequence of $\mathbf{P}[A(t)]$, we have $\gamma_n \to 0$. This proves the first part.

For the second part, define

$$\tilde{\Delta}_k^*(t) = \begin{cases} 0 & \text{if } \Delta_k^*(t) \leq \delta\lceil \frac{n}{2} \rceil \\ \frac{n\delta}{2} & \text{otherwise} \end{cases} \tag{41}$$

By Lemma 1 there are at least $\frac{n}{2} - 1$ vehicles at any time slot $t$ such that $\Delta_k^*(t) > \delta\lceil \frac{n}{2} \rceil$. Using (37) we get

$$\mathbf{P}[A(t)] = \mathbf{P}\left[\bigcup_{i \in N} \left\{U_i \in \bigcup_{k \in N\setminus\{i\}} B\left(U_k, v\Delta_k^i(t)\right)\right\}\right], \tag{42}$$

$$\geq \mathbf{P}\left[\bigcup_{i \in N} \left\{U_i \in \bigcup_{k \in N\setminus\{i\}} B\left(U_k, v\tilde{\Delta}_k^i\right)\right\}\right], \tag{43}$$

$$\geq \mathbf{P}\left[\bigcup_{i \in N} \left\{U_i^{'} \in \bigcup_{k \in \left[\frac{n}{2}-1\right]\setminus\{i\}} B\left(U_k^{'}, \frac{nv\delta}{2}\right)\right\}\right],$$

where $U_i^{'}$ denote location of vehicles with $\tilde{\Delta}_k^*(t) = \frac{n\delta}{2}$. Since the communication schemes under consideration are location independent, $U_i^{'}$s will be independent and uniformly distributed over $\mathcal{S}$. Since there are $\lceil \frac{n}{2} - 1 \rceil$ of them we have

$$\mathbf{P}\left[A(t)\right] \geq \mathbf{P}\left[\bigcup_{i \in N'} \left\{ V_i \in \bigcup_{k \in N' \setminus \{i\}} B\left(V_k, \frac{nv\delta}{2}\right)\right\}\right], \quad (44)$$

where $V_i$s, for $i \in N' = \{1, 2, \ldots \lceil \frac{n}{2} - 1 \rceil\}$, are independent and uniformly distributed over $\mathcal{S}$. We, therefore have,

$$\bigcup_{i \in N'} \left\{ V_i \in \bigcup_{k \in N' \setminus \{i\}} B\left(V_k, \frac{nv\delta}{2}\right)\right\} = \{M \geq 1\}, \quad (45)$$

where $M$ is the number of edges in the random geometric graph $\mathcal{G}\left(\lceil \frac{n}{2} - 1 \rceil, \frac{nv\delta}{2}\right)$. For $\mathcal{G}\left(n, r\right)$, $\mathbf{P}\left[M \geq 1\right] \to 1$ if $rn \to \infty$. Hence, if $vn^2 \to \infty$ we have $\mathbf{P}\left[A(t)\right] \to 1$ as $n \to \infty$. Since $\gamma_n$ is a Cesaro mean of $\mathbf{P}\left[A(t)\right]$, we have $\gamma_n \to 1$ as $n \to \infty$. ■

### A. Generalization to All $O(n)$-recurrent Schemes

It turns out that the results of Theorem 1 and 2 for the round robbin scheme hold for any $O(n)$-recurrent communication scheme.

**Corollary** *1:* For any $O(n)$-recurrent communication scheme, if $v$ scales in $n$ such that

1) $vn^{3/2} \to 0$ then
$$\lim_{n \to \infty} \gamma_n^i = 0, \quad (46)$$
for all $i \in N$.
2) $vn^2 \to 0$ then
$$\lim_{n \to \infty} \gamma_n = 0. \quad (47)$$

*Proof:* The proof is same as the proofs of Theorem 1 and 2 for the round robbin scheme. In those proofs, we only used the fact that $\Delta_k^i(t)$ is bounded above by $n\delta$ for all $i, k$. For an $O(n)$-recurrent scheme, $\Delta_k^i(t) \leq cn$ for all $i, k \in N$, all large $t$ and some positive constant $c$. Thus, the same proofs follow. ■

## V. Spatial Network Model

We now extend the threshold results proved in Theorem 1 and 2 to a similar network in three dimensional space. Consider $n$ autonomous vehicles inside the space $S_3 = [0, 1]^3$. The distance between two points $\mathbf{x} = (x_1, x_2, x_3) \in S_3$ and $\mathbf{y} = (y_1, y_2, y_3) \in S_3$ is given by

$$d_3\left(\mathbf{x}, \mathbf{y}\right) = \min_{\mathbf{e} \in \{-1, 0, 1\}^3} \left\|\mathbf{x} + \mathbf{e} - \mathbf{y}\right\|_2. \quad (48)$$

The rest of the system model is same as stated in Section II. In this case, we can again show that, both the individual and network hazard rate has a threshold behaviour with respect to maximum speed $v$. The thresholds, however, are different.

**Theorem** *3:* Hazard rates $\gamma_n^i$ and $\gamma_n$ have threshold behaviour with respect to $v$, as $n \to \infty$. This threshold for $\gamma_n^i$ is $n^{-4/3}$ and for $\gamma_n$ it is $n^{-5/3}$. And, the round robbin scheme attains the smallest hazard rates as $n \to \infty$.

TABLE I
Comparison of maximum speed thresholds for individual and network hazard rate in planar and spatial networks.

| Max. speed thresholds for | Planar Network | Spatial Network |
|---|---|---|
| individual hazard rate | $n^{-3/2}$ | $n^{-4/3}$ |
| network hazard rate | $n^{-2}$ | $n^{-5/3}$ |

This result shows that the speed thresholds for the three dimensional spatial network are larger than those for the planar network of Section II. We compare them in Table I. This implies that it is better to plan an autonomous vehicular network over a three dimensional space than a two dimensional one, as the former provides for higher mobility. We now state the proof of Theorem 3.

*Proof of Theorem 3:* The proof of the threshold for $\gamma_n^i$ is same as given in the proof of Theorem 1, except for minor modifications. In (24), instead of $\pi(nv\delta)^2$ we will have $\frac{4}{3}\pi(nv\delta)^3$. Similarly, in (33), instead of $\pi\left(\frac{nv\delta}{2}\right)^2$ we will have $\frac{4}{3}\pi\left(\frac{nv\delta}{2}\right)^3$. This will give the threshold of $n^{-4/3}$.

The proof of the threshold for $\gamma_n$ is same as given in the proof of Theorem 1 till equations (40) and (45). However, the random geometric graphs $\mathcal{G}\left(n, nv\delta\right)$ and $\mathcal{G}\left(\lceil \frac{n}{2} - 1 \rceil, \frac{nv\delta}{2}\right)$ are on space $S_3$ and not $S$. For a random geometric graph $\mathcal{G}(n, r)$ on $S_3$, the probability that there is at least an edge, $\mathbf{P}\left[M \geq 1\right] \to 0$ if $n^{2/3}r \to 0$ and $\mathbf{P}\left[M \geq 1\right] \to 1$ if $n^{2/3}r \to \infty$; see [20]. Using this in (40) and (45), we obtain the threshold $n^{-5/3}$. ■

## VI. Conclusions

We analyzed the impact of wireless interference constrains on maximum attainable speed in an autonomous vehicular network. We defined hazard rate, a measure of network safety, and showed that it follows a threshold behaviour with respect to maximum speed $v$ as $n \to \infty$. We saw that below the threshold, a simple round robbin scheme attained the minimum hazard rate as $n \to \infty$. The speed thresholds for the spatial network were proved to be higher than those for the planar network. This shows that planning autonomous vehicular networks over a three dimensional space can ensure greater safety or network awareness.

## Appendix

### A. Proof of Lemma 1

Let $n_t$ denote the number of nodes that transmitted in the previous $\lceil \frac{n}{2} \rceil$ slots before the current slot. Then $n_t \leq \lceil \frac{n}{2} \rceil \leq \frac{n}{2} + 1$. Also, none of these $n_t$ nodes can have $\Delta_k^* > \delta \lceil \frac{n}{2} \rceil$, while, all other $n - n_t$ nodes will have $\Delta_k^* > \delta \lceil \frac{n}{2} \rceil$. Hence,

$$f_t = \frac{n - n_t}{n} \geq \frac{n - (n/2) - 1}{n} = \frac{1}{2} - \frac{1}{n} \quad \text{a. s.}\,. \quad (49)$$

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