THE PICARD SCHEME OF A CURVE AND ITS COMPACTIFICATION

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ABSTRACT

In the first part of this work we show that the Picard scheme P of a curve X (reduced, but not necessarily irreducible) can be constructed from the Picard scheme of the normalization of X by a sequence of $\mathbb{G}_m$- and $\mathbb{G}_a$-extensions.

Next, we study the compactification $\overline{P}$ of P for an integral curve X defined as the moduli space of torsion-free, rank-1 sheaves on X. We show that if X lies on a smooth surface, the boundary points of P in $\overline{P}$ are singular points. If the δ-invariant of the normalization map of X is at most one at each point, we find the orbits of $\overline{P}$ under the action of P. Moreover, we describe the analytic structure of the singularities in this case, and we show how the singularities are distributed on the orbits. If X has ordinary double points as only singularities, we give an explicit construction of $\overline{P}$.

In the case that X does not lie on a smooth surface, we show that $\overline{P}$ is reducible. In the last chapter we extend this result to the moduli space $M(n,d)$ of semi-stable, torsion-free, rank-n sheaves of degree d on X. We show that if X does not lie on a smooth surface, then $M(n,\ell n)$, $\ell \in \mathbb{Z}$, is reducible.

Thesis Supervisor: Steven L. Kleiman
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CONTENTS

INTRODUCTION

Throughout this work k denotes an algebraically closed field. We will use the word curve to mean a reduced projective k-scheme of pure dimension 1, and by a point we mean a closed point. For other basic concepts of algebraic geometry, we use the terminology of [14].

In the first part of this work we study the component P of the Picard scheme of a curve X , which parameterizes invertible O_X-Modules of degree o . If X is smooth, P is a projective group variety. If X has singularities, P is quasi-projective. We show how P can be constructed from the Picard scheme of the normalization of X by a sequence of extensions by $\mathbb{G}_a$- and $\mathbb{G}_m$-bundles. We obtain this construction by showing that the normalization map of X can be written as a composition of maps where the δ-invariant changes by 1 [Theorem 1.2.4]. Then we prove that if $X' \to X$ is a surjective map of curves such that $\delta(X',X) = 1$, the Picard scheme of X is isomorphic to the Picard scheme of X' , or it is a $\mathbb{G}_m$- or $\mathbb{G}_a$-extension of the Picard scheme of X' [Theorem 3.2.1].

There exists a natural compactification $\overline{P}$ of P , where the points of $\overline{P}$ corresponds to torsion-free, rank-1 sheaves on X, if X is irreducible [2]. A main part of this work is devoted to an investigation of the properties of $\overline{P}$.

If X lies on a smooth surface, Altman, Iarrobino and Kleiman [1] proved that $\overline{P}$ is irreducible. We show the converse: $\overline{P}$ is reducible if X does not lie on a smooth surface [Theorem 5.2.4].

In the case that $\overline{P}$ is irreducible, we show that the boundary points of P in $\overline{P}$ are singular points [Theorem 6.1.3]. In the special case that the δ-invariant of the normalization map of X is at most 1 at each point, we find the orbits of $\overline{P}$ under the action of P. Moreover, we describe the analytic structure of the singularities of $\overline{P}$, and we show how the singularities are distributed on the orbits [Proposition 6.2.2].

If X has m ordinary double points as only singularities, we describe how $\overline{P}$ can be constructed from the Picard scheme of the normalization of X. More precisely, if $Y' \to Y$ is a desingularization of one of the nodes, we show that $\overline{P}_Y$ is obtained from a $\mathbb{P}^1$-bundle over $\overline{P}_{Y'}$ by identification of two sections via a translation by a point of $\mathrm{Pic}^o_{Y'}$ [Proposition 7.2.2].

Newstead [19] has verified that there exists a projective scheme $M(n,d)$, which parameterizes semi-stable, torsion-free, rank-n sheaves of degree d on an irreducible curve X. If X lies on a smooth surface, Rego [23] proved that $M(n,d)$ is irreducible. In the last chapter we show that $M(n,\ell n)$, $\ell \in \mathbb{Z}$, is reducible if X does not lie on a smooth surface [Theorem 8.3.2].

We now give a more detailed description of how the material is organized. In Chapter I we prove that the normalization map of a curve can be written as a composition of maps where the δ-invariant changes by 1. A main ingredient in the proof of this result is a modification of a method used by Serre to construct singular, irreducible curves from their normalization.

The presentation functor $\underline{\text{Pres}}_{X'/X}$, where $X' \to X$ is a surjective morphism of curves such that $\delta(X',X) = 1$, is introduced in Chapter II. We show that it is represented by a $\mathbb{P}^1$-bundle over $\text{Pic}^0_{X'}$ if X and X' have the same number of connected components.

In Section 2.3 we define a subfunctor $\underline{\text{StPres}}_{X'/X}$ of $\underline{\text{Pres}}_{X'/X}$, which is represented by a $\mathbb{G}_m$- or $\mathbb{G}_a$-bundle over $\text{Pic}^0_{X'}$ if X and X' have the same number of connected components and by $\mathrm{Pic}^{\mathrm{o}}_{X'}$ otherwise. In Chapter III we show that $\mathrm{StPres}_{X'/X}$ is isomorphic to $\mathrm{Pic}^{\mathrm{o}}_{X}$, and hence the Picard scheme of a curve has the structure of $\mathbb{G}_m$- and $\mathbb{G}_a$-extensions of the Picard scheme of the normalization of X .

In Chapter IV we recall basic facts about the functor $\underline{\mathrm{Pic}}^{=\mathrm{o}}_{X}$ of torsion-free, rank-1 sheaves on X and the Abel map

$$A^n : \mathrm{Quot}^n(\omega/X/k) \to \mathrm{Pic}^{=\mathrm{o}}_{X} .$$

We also give a short discussion of the problem of compactifying $\mathrm{Pic}^{\mathrm{o}}_{X}$ in the case that X is reducible. In Section 4.3 we give examples of cuspidal plane curves C such that there exists a point of $\mathrm{Pic}^{=\mathrm{o}}_{C}$ where the tangent cone is not a complete intersection. We explain how these examples show that the program we had for explicit constructions of compactifications of the Picard scheme fails.

In Chapter V we show that $\mathrm{Pic}^{=\mathrm{o}}_{X}$ is reducible if X does not lie on a smooth surface. This is done in two steps. We show that $\mathrm{Quot}^2(\omega/X/k)$ is reducible if X does not lie on a smooth surface. Then we prove that this implies reducibility of $\mathrm{Quot}^n(\omega/X/k)$, $n \geq 2$, and so the smoothness of the Abel map

$$A^n : \mathrm{Quot}^n(\omega/X/k) \to \mathrm{Pic}_X^{\overset{=}{o}}$$

shows that $\mathrm{Pic}_X^{\overset{=}{o}}$ is reducible.

In Chapter VI we study $\mathrm{Pic}_X^{\overset{=}{o}}$ in the case that X lies on a smooth surface. Using the description of the singular locus of $\mathrm{Hilb}^n(X/k)$ of [8], we prove that the boundary points of Pic_X^o in $\mathrm{Pic}_X^{\overset{=}{o}}$ are singular points. If the δ-invariant of the normalization map of X is at most 1 at each point, we show that $\mathrm{Pic}_X^{\overset{=}{o}}$ has $\binom{\delta}{k}$ orbits (under the action of Pic_X^o) of codimension k, $1 \leq k \leq \delta(\overline{X},X)$. We also give the analytic structure of the singularities of $\mathrm{Pic}_X^{\overset{=}{o}}$ and determine how the singularities are distributed on the orbits.

Chapter VII includes a generalization $\underline{\mathrm{G\,Pres}}_{Y'/Y}$ of the presentation functor introduced in Chapter II where $Y' \to Y$ is a surjective, birational morphism of irreducible curves. The source of a generalized presentation is taken to be a torsion-free, rank-1 sheaf on Y' . We show that $\underline{\mathrm{G\,Pres}}_{Y'/Y}$ is represented by a projective k-scheme. We use generalized presentations to describe explicitely the structure of $\mathrm{Pic}_X^{\overset{=}{o}}$ in the case that X has ordinary double points as only singularities as follows: If $Y' \to Y$ is a desingularization of one of the singularities of X , $\mathrm{G\,Pres}_{Y'/Y}$ is a $\mathbb{P}^1$-bundle over $\mathrm{Pic}_{Y'}^{\overset{=}{o}}$, and

$\mathrm{Pic}_Y^{=o}$ is obtained from this $\mathbb{P}^1$-bundle by identifying two sections via a translation in $\mathrm{Pic}_{Y'}^{=o}$.

Some of the techniques we use in Chapter VIII to prove reducibility of the moduli space $M(n, \ell n)$ of semi-stable, torsion-free sheaves of rank n and degree ℓn are similar to the one used in Chapter V. We show that $\mathrm{Quot}_{ss}^n(\omega^n/X/k)$ is reducible if X is not Gorenstein and $\mathrm{Quot}_{ss}^{2n}(\omega^n/X/k)$ is reducible if X is Gorenstein but X does not lie on a smooth surface (Quot_{ss} denotes the open subscheme of Quot consisting of quotients N such that $\ker(\omega^n \to N)$ is semi-stable). Since we have no smooth Abel map at hand, we devise other methods to derive reducibility of $M(n, \ell n)$.

I am grateful to my advisor Steven Kleiman for his help preparing this material.

CHAPTER I.

The normalization map for curves.

Let X be a curve (reduced, but not necessarily irreducible). In this chapter we prove that the normalization map

$$f : \overline{X} \to X$$

can be written as a composition

$$\overline{X} = X_r \xrightarrow{f_r} X_{r-1} \to \dots \xrightarrow{f_1} X_o = X$$

such that the δ-invariant of each f_i is one.

Both Artin [5] and Oort [21] have constructed a factorization of f ; Oort in the case that X is irreducible and Artin for X reducible. However, in their factorization the δ-invariant does not always change by one.

The main ingredient in our proof of the breaking up of f is a modification of the method used by Serre to construct singular, irreducible curves from their normalization [25, Prop. 2, page 69]. We generalize Serre's procedure so that we can construct quotients by a finite set-theoretic equivalence relation of a k-scheme, which is reduced, but which need neither be nonsingular nor irreducible.

The generalization of Serre's method to schemes of dimension greater than one allows the construction of a quotient by an equivalence relation defined by an involution on a closed subscheme. As an application we construct a quotient of a $\mathbb{P}^1$-bundle over $\overline{\overline{\mathrm{Pic}}}{}^{o}_{X'}$, which we in Chapter VII will prove is the compactification of Pic^{o}_{X} . Here X is an irreducible curve with ordinary double points as only singularities, and X' is the desingularization of one of the double points.

1.1.

Let X be a locally noetherian k-scheme, and let Z be a closed subscheme of X such that no component of X is contained in Z . Let

$$R \rightrightarrows Z$$

be a finite equivalence relation in the category of sets. It induces an equivalence relation

$$R \rightrightarrows X .$$

We denote by Y the quotient of X by R . The quotient topology gives Y the structure of a topological space. In this section we will deduce that Y can be given the structure of a reduced scheme in many ways.

First we introduce some notation. Let $\mathcal{R}(X)$ denote the sheaf of total quotient rings of X [11, Ch. I, Def. 8.3.1]. Since X is locally noetherian and reduced, the map

$$O_X \to \mathcal{R}(X)$$

is injective [11, Ch. I, Prop. 8.3.7].

For a closed point $Q \in Y$ we put

$$O_Q = \bigcap_{P \in f^{-1}(Q)} O_{X,P}$$

where the intersection takes place in $\mathcal{R}(X)$ and where $f : X \to Y$ denotes the projection.

Let d be a fixed positive integer. For each closed point $Q \in f(Z)$, fix a local ring O'_Q such that

$$(*) \quad k \oplus r_Q^d \subseteq O'_Q \subseteq k \oplus r_Q$$

where r_Q denotes the radical of O_Q , i.e. the intersection of the maximal ideals of O_Q .

For $Q \in Y$, $Q \notin f(Z)$ we set

$$(**) \quad O'_Q = O_Q \ .$$

<u>Proposition 1.1.1</u>. Let X , R , Y and O'_Q be as above. Suppose that X can be covered by open affine subsets,

which are R-stable. Then Y can be given the structure of a locally noetherian, reduced k-scheme such that

$$O_{Y,Q} \simeq O'_Q ,$$

and there is a natural projection morphism $p : X \to Y$.

Moreover, if X is proper over k , then Y is proper over k .

Proof. Serre's proof of [25, Prop. 2, page 69] carries over to the above situation with only minor modifications.

1.2.

Let $f : X' \to X$ be a surjective, birational morphism of curves. We recall that the δ-invariant of f at a point $Q \in X$, $\delta(X',X,Q)$, is defined by

$$\delta(X',X,Q) = \dim_k(O_Q/O_{X,Q})$$

where $O_Q = \bigcap_{P\in f^{-1}(Q)} O_{X',P}$. We set $\delta(X',X) = \sum_{Q\in X} \delta(X',X,Q)$.

Let $Q_1,\ldots,Q_r$ be the points of X such that $\delta(X',X,Q_i) \neq 0$ and let S be the points of $\bigcup_{i=1}^{r} f^{-1}(Q_i)$. We denote by R the equivalence relation on S in the category of sets, which intentifies the points in S mapping to the same point of X . Since S is a finite set of points, we can find an open covering $\{U_i\}$ of X' such that U_i are R-stable. Hence we can apply Proposition 1.1.1 to deduce:

Lemma 1.2.1. Let $f : X' \to X$ be a surjective, birational morphism of curves. Then there exists a curve Y and morphisms

$$g : X' \to Y \ , \ h : Y \to X$$

such that $f = h \cdot g$, h is a homeomorphism and $O_{Y,Q} = k \oplus r_Q$ for all $Q \in Y$ (r_Q is the radical of O_Q) .

The next two lemmas show that we can break up g and h in steps where δ changes by one.

Lemma 1.2.2. Let $g : X' \to X$ be as in Lemma 1.2.1. Then there exists a factorization

$$X' = X'_s \xrightarrow{g_s} X'_{s-1} \to \ldots \to X'_1 \xrightarrow{g_1} X'_o = Y$$

of g such that $\delta(X'_i, X'_{i-1}) = 1$.

Proof. Let P_1 and P_2 be two different points of X' , which map to the same point Q of X . Let X'_{s-1} be the quotient of X' in the category of sets by the equivalence relation, which indentifies P_1 and P_2 . By Proposition 1.1.1, X'_{s-1} can be given the structure of a curve with a morphism

$$g_s : X' \to X'_{s-1}$$

such that g_s is an isomorphism on $X'\backslash\{P_1,P_2\}$ and such that

$$O_{X'_{s-1},Q} \simeq k \oplus r_Q$$

where r_Q is the radical of $O_{X',P_1} \cap O_{X',P_2}$.

Set $A_1 = O_{X',P_1}$ and $A_2 = O_{X',P_2}$ and denote by m_1 and m_2 the maximal ideals of A_1 and A_2 . The natural surjection

$$A_1 \cap A_2 \to (A_1/m_1) \oplus (A_2/m_2)$$

has kernel $m_1 \cap m_2$ and so

$$\dim_k(A_1 \cap A_2/m_1 \cap m_2) = 2 .$$

Hence we get that

$$\dim_k(A_1 \cap A_2/k \oplus (m_1 \cap m_2)) = 1 ,$$

which shows that $\delta(X',X'_{s-1}) = 1$.

We repeat the procedure for the natural morphism $g' : X'_{s-1} \to Y$ to construct X'_{s-2} . After $s = \delta(X',Y)$ steps we reach the curve Y .

<u>Lemma 1.2.3.</u> Let $h : Y \to X$ be as in Lemma 1.2.1. Then there exists a factorization

$$Y = Y_t \xrightarrow{h_t} Y_{t-1} \to \cdots \to Y_1 \xrightarrow{h_1} Y_o = X$$

of h such that $\delta(Y_i, Y_{i-1}) = 1$.

<u>Proof.</u> Let P be a point of Y where h is not an isomorphism and set $Q = h(P)$. Let m denote the maximal ideal of $O_{X,Q}$ and let C denote the conductor of $O_{X,Q}$ in $O_{Y,P}$.

If $C \neq m$, we have that

$$(\square) \qquad m \neq mO_{Y,P}$$

since the conductor is the largest ideal in $O_{X,Q}$, which is also an ideal of $O_{Y,P}$. There exists a curve Y' , homeomorphic to Y and isomorphic to Y outside P , such that

$$O_{Y',P} \simeq k \oplus mO_{Y,P}$$

[Proposition 1.1.1]. From $(\square)$ it follows that $\delta(Y,Y') < \delta(Y,X)$, so we may assume, using induction on $\delta(Y,X)$, that the conductor C is equal to m .

Set $A = O_{X,Q}$ and $B = O_{Y,P}$ and denote by M the maximal ideal of B . Since h is birational, B/m is an artinian ring. Hence the exists a number ℓ such that

$$M^{\ell} \subseteq m \subseteq M .$$

Let u be an element of M such that $u \notin m$ and $u^2 \in m$ and set

$$A' = A[u] .$$

Since $mB = m$, every element in A' can be written as $a + cu$, $a \in A$ and $c \in k$, so $\dim_k(A'/A) = 1$.

There exists a curve Y_{t-1} and a morphism $h_t : Y \to Y_{t-1}$ such that h_t is a homeomorphism and $h_t|Y \setminus P$ is an isomorphism, and such that $O_{Y_{t-1},P} \simeq A'$ [Proposition 1.1.1]. Since $\delta(Y,Y_{t-1}) = 1$, the lemma is proved using induction on $\delta(Y,X)$.

Let $X_1,\ldots,X_r$ denote the irreducible components of X and let $\overline{X}_i$ denote the normalization of X_i. We define the normalization $\overline{X}$ of X to be

$$\overline{X} = \bigoplus_{i=1}^{r} \overline{X}_i .$$

The three previous lemmas give the following result:

<u>Theorem 1.2.4.</u> Let $f : \overline{X} \to X$ be the normalization map of the curve X. Then f has a decomposition

$$\overline{X} = X_t \to X_{t-1} \to \cdots \to X_1 \to X_o = X$$

such that $\delta(X_i,X_{i-1}) = 1$.

1.3.

Let $W = \mathrm{Spec}(B)$ be an affine scheme and let $\sigma : W \to W$ be an involution (i.e. $\sigma^2 = \mathrm{id}$) . Let $y \in W$ and let U be an open subset of W such that y , $\sigma(y) \in U$ ..

Lemma 1.3.1. There exists an element $b \in B$ such that the principal open subset $U' = \mathrm{Spec}(B_b)$ is σ-stable and $y \in U' \subseteq U$.

Proof. By shrinking U , if necessary, we may assume that $U = \mathrm{Spec}(B_s)$, $s \in B$. Put $b = s\sigma^*(s)$ and set $U' = \mathrm{Spec}(B_b)$ where σ^* denotes the comorphism $\sigma^* : O_W \to O_W$. Then $U' = U \cap \sigma(U)$ so U' is σ-stable and $y \in U'$.

Let Z be a locally noetherian and reduced projective k-scheme. Let $T \subseteq Z$ be a closed subscheme such that no component of Z is contained in T . Suppose we have an involution

$$\sigma : T \to T .$$

Lemma 1.3.2. For each point $y \in T$ there exists an affine open subset $U = \mathrm{Spec}(A)$ of Z such that $y \in U$ and $T \cap U$ is σ-stable.

Proof. Since σ is an involution on T, we can find an open affine subset

$$V = \operatorname{Spec}(B)$$

of T, which is σ-stable and such that $y \in V$. Indeed, let Ω be an affine open subset of T such that $\{y, \sigma(y)\} \subseteq \Omega$ and set $V = \Omega \cap \sigma(\Omega)$. Clearly V is σ-stable, and V is affine since T is separated [12, Ch. I, Prop. 5.5.6].

We choose an affine open subset

$$U_1 = \operatorname{Spec}(A_1)$$

of Z such that $U_1 \cap T \subseteq V$ and such that $\{y, \sigma(y)\} \subseteq U_1$. Then $U_1 \cap T$ is of the form

$$U_1 \cap T = \operatorname{Spec}(B_1)$$

where $B_1 = A_1/I_1$ for an ideal $I_1 \subseteq A_1$. There exists an element $b \in B_1$ such that

$$U' = \operatorname{Spec}(B_{1,b})$$

is σ-stable and such that $y \in U'$ [Lemma 1.3.1]. Let a be an element of A_1 such that the residue class of a modulo I_1 is equal to b. Set $A = A_{1,a}$ and put $U = \operatorname{Spec}(A)$. The assertion now follows since $U \cap T = U'$ and U' is σ-stable with $y \in U'$.

Denote by i the inclusion $T \subseteq Z$. The two morphisms i and $i \circ \sigma$ define a finite equivalence relation on Z in the category of sets. As in Section 1.1, let Y denote the quotient of Z with the quotient topology. For each closed point $Q \in Y$, let O'_Q be local rings, which satisfy the relations (*) and (**) of Section 1.1.

Proposition 1.3.3. Y can be given the structure of a reduced, proper k-scheme such that $O_{Y,Q} \simeq O'_Q$ for every closed point $Q \in Y$.

Proof. In order to apply Proposition 1.1.1, we must show that Z can be covered by affine open subsets, which are stable with respect to the equivalence relation defined by i and $i \circ \sigma$. That is an immediate consequence of the fact that there is an open, affine covering $\{U_i\}$ of Z such that $U_i \cap T$ is σ-stable [Lemma 1.3.2].

Let X' be an irreducible curve and denote by $\overline{P} = \overline{\overline{\text{Pic}}}{}^{0}_{X'}$ the scheme parameterizing torsion-free, rank-1 sheaves on X' of degree 0 [2, Theorem (8.5), (ii)]. Let L be a universal relatively torsion-free, rank-1 sheaf on $X' \times \overline{P}$.

Let Q_1 and Q_2 be different, nonsingular points of X' and denote by $L(Q_i)$ the pullback of L to $\overline{P}$ by the morphism $\overline{P} \simeq \overline{P} \times Q_i \to \overline{P} \times X'$. Let V be the $\mathbb{P}^1$-bundle

$$V = \mathrm{Proj}(L(Q_1) \oplus L(Q_2))$$

over $\overline{P}$, and set $V_i = \mathrm{Proj}(L(Q_i))$, $i = 1,2$.

We define morphisms

$$\psi_1 : V_2 \to V_1 , \psi_2 : V_1 \to V_2$$

by $\psi_1 = \varphi_1 \bullet g_2 \bullet \varphi_1^{-1}$ and $\psi_2 = \varphi_2 \bullet g_1 \bullet \varphi_2^{-1}$ where $\varphi_i : \overline{P} \to V_i$, $i = 1,2$, are the natural isomorphisms, g_1 the isomorphism on $\overline{P}$ defined by translation by $Q_1 - Q_2$ and g_2 the isomorphism defined by translation by $Q_2 - Q_1$.

The projections $L(Q_1) \oplus L(Q_2) \to L(Q_i)$ give rise to closed embeddings $V_i \to V$ [12, Ch. II, Rem. 4.3.6]. Let T denote the union of V_1 and V_2. Then

$$\sigma = \psi_1 \oplus \psi_2 : T \to T$$

defines an involution on T. Let $\widetilde{V}$ be the quotient (as toplogical space) of V by the equivalence relation given by σ. We get the following corollary of Proposition 1.3.3:

<u>Corollary 1.3.4.</u> $\widetilde{V}$ can be given the structure of a reduced k-scheme such that

$$O_{\widetilde{V},Q} \simeq k \oplus r_Q$$

for all closed points $Q \in \widetilde{V}$ where r_Q denotes the radical of $\bigcap_{Q' \to Q} O_{V,Q'}$.

<u>Remark 1.3.5.</u> Let X be an irreducible curve with ordinary nodes as only singularities and let X' be the desingularization of one of the double points. In Chapter VII we will show that the scheme $\widetilde{V}$ constructed in Corollary 1.3.4 is the compactification of Pic°_{X} .

CHAPTER II.

The presentation functor.

Let $f : X' \to X$ be a surjective, birational morphism of curves such that $\delta(X',X) = 1$. Let $Q \in X$ denote the point such that $\delta(X',X,Q) = 1$. We define the presentation functor $\underline{\mathrm{Pres}}_{X'/X}$ as follows: For each k-scheme S, let $\underline{\mathrm{Pres}}_{X'/X}(S)$ be the set of surjective O_{X_S}-Module homomorphisms

$$\varphi : (f_S)_* L \to N$$

where L is an invertible $O_{X'_S}$-Module of degree 0, N is an invertible O_S-Module and $\mathrm{Supp} N = Q \times S$.

A similar functor was first introduced by Oda and Seshadri [20, Section 12]. Our definition is more general since they only defined a functor suitable for their purpose, i.e. the case where Q is an ordinary node or a point where two components meet.

We show that $\underline{\mathrm{Pres}}_{X'/X}$ is represented by a $\mathbb{P}^1$-bundle over $\mathrm{Pic}^o_{X'}$ if X' and X have the same number of connected components. Oda and Seshadri claim that their presentation functor is always representable [20, Prop. 12.1]. However, they also need the hypothesis that X' and X have the same number of connected components.

In Section 2.3 we define a subfunctor $\underline{\text{StPres}}_{X'/X}$ of $\underline{\text{Pres}}_{X'/X}$, which we will show is isomorphic to $\underline{\text{Pic}}^{\text{o}}_{X}$ in Chapter III. We show that $\underline{\text{StPres}}_{X'/X}$ is represented by a $\mathbb{G}_a$- or $\mathbb{G}_m$-bundle over $\text{Pic}^{\text{o}}_{X'}$, if X and X' have the same number of connected components and by $\text{Pic}^{\text{o}}_{X'}$ otherwise.

2.1

Let $X = \cup X_i$ be a curve and denote by $\underline{\text{Pic}}^{\text{o}}_{X}$ the functor of invertible O_X-Modules of degree 0 , i.e. for each k-scheme S ,

$$\underline{\text{Pic}}^{\text{o}}_{X}(S)$$

is the set of equivalence classes of invertible O_{X_S}-Modules L such that $\chi(X_i, L(s)|_{X_i}) = \chi(X_i, O_{X_i})$ for each closed point $s \in S$ where χ denotes the Euler characteristic. Two invertible O_{X_S}-Modules L and L' are considered equivalent if there exists an invertible O_S-Module N and an isomorphism

$$L' \simeq L \otimes_{O_S} N \ .$$

Let $f : X' \to X$ be a surjective, birational morphism of curves such that $\delta(X',X) = 1$, and let $Q \in X$ denote the point such that $\delta(X',X,Q) = 1$.

Definition 2.1.1. Let S be a k-scheme. By a presentation over S we mean a surjective O_{X_S}-Module homomorphism

$$\varphi : (f_S)_* L \to N$$

where $L \in \underline{\mathrm{Pic}}^o_{X'}(S)$, $\mathrm{Supp} N = Q \times S$ and N is an invertible O_S-Module.

A presentation

$$\varphi' : (f_S)_* L' \to N'$$

is equivalent to φ if there exists an $O_{X'_S}$-isomorphism $\alpha : L \to L' \otimes_{O_S} T$, where T is an invertible O_S-Module, and an O_{X_S}-isomorphism $\beta : N \to N' \otimes_{O_S} T$ such that the diagram

$$\begin{array}{ccc} (f_S)_* L & \xrightarrow{\varphi} & N \\ {\scriptstyle (f_S)_*\alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ (f_S)_*(L' \otimes_{O_S} T) & \xrightarrow[\varphi' \otimes \mathrm{id}]{} & N' \otimes_{O_S} T \end{array}$$

commutes.

Let $S' \to S$ be a morphism of k-schemes. The pullback

$$\varphi_{S'} : [(f_S)_* L]_{S'} \to N_{S'}$$

of φ is a surjective $O_{X_{S'}}$-homomorphism. $N_{S'}$ is an invertible $O_{S'}$-Module, and since f_S is affine, there is a canonical isomorphism

$$[(f_S)_* L]_{S'} \simeq (f_{S'})_*(L_{S'})$$

[11, Ch. I, Prop. 9.3.2]. Hence the pullback $\varphi_{S'}$ of φ is a presentation over S', and the pullback of equivalent presentations are equivalent. Thus we can make the following definition:

Definition 2.1.2. Let $\underline{\text{Pres}}_{X'/X}$ be the functor defined as follows: For each k-scheme S, let

$$\underline{\text{Pres}}_{X'/X}(S)$$

be the set of equivalence classes of presentations over S. If $S' \to S$ is a morphism of k-schemes, the map $\underline{\text{Pres}}_{X'/X}(S) \to \underline{\text{Pres}}_{X'/X}(S')$ is given by pullback.

2.2

Let Y be a k-scheme and let E be a locally free sheaf on Y of rank $n + 1$. We define a contravariant functor $F(E/Y)$ from the category of k-schemes to the category of sets as follows: For each k-scheme T, let

$$F(E/Y)(T)$$

be the set of equivalence classes of pairs (N,φ) consisting of an invertible O_T-Module N and a surjective O_{Y_T}-Module homomorphism

$$\varphi : E_T \to N .$$

Two pairs (N,φ) and (N',φ') are equivalent if there exists an O_{Y_T}-isomorphism $\tau : N \to N'$ such that $\varphi' = \tau \circ \varphi$.

Let $S(E)$ denote the symmetric algebra of E and set $\mathbb{P}(E) = \mathrm{Proj}(S(E))$. Defined like this, $\mathbb{P}(E)$ comes with a projection $\pi : \mathbb{P}(E) \to Y$ and a tantological invertible sheaf $O(1)$ such that there is a natural surjective $O_{\mathbb{P}(E)}$-homomorphism $\pi^*E \to O(1)$ [12, Ch. II, Prop. 4.1.6].

The functor $F(E/Y)$ is represented by the $\mathbb{P}^n$-bundle $\mathbb{P}(E)$ over E , and the universal pair is $(O(1),\Phi)$ where $\Phi : \pi^*E \to O(1)$ is the canonical surjection [12, Ch. II, Prop. 4.2.3].

Proposition 2.2.1. Let $f : X' \to X$ be a surjective, birational morphism of curves such that $\delta(X',X) = 1$ and such that X' and X have the same number of connected components. Suppose that $\underline{\mathrm{Pic}}^{o}_{X'}$ is represented by a scheme P. Then $\underline{\mathrm{Pres}}_{X'/X}$ is represented by a $\mathbb{P}^1$-bundle over P.

<u>Proof.</u> Let $\mathcal{P}$ be a universal invertible sheaf on X'_P. Let Q be the point such that $\delta(X',X,Q) = 1$ and set

$$E = [(f_P)_*\mathcal{P}](Q)$$

where $[(f_P)_*\mathcal{P}](Q)$ denotes the pullback of $(f_P)_*\mathcal{P}$ to P by the morphism $P \simeq Q \times P \to X \times P$. Then E is a locally free O_P-Module of rank 2. We show that $\underline{\mathrm{Pres}}_{X'/X}$ is isomorphic to $F(E/P)$.

Let

$$\varphi : (f_S)_*L \to N$$

be a presentation over S. There exists a morphism $q : S \to P$, an invertible O_S-Module T and an isomorphism

$$\alpha : (q_{X'})^*\mathcal{P} \simeq L \otimes_{O_S} T .$$

The presentation

$$(f_S)_*(L \otimes_{O_S} T) \to N \otimes_{O_S} T$$

is equivalent to φ . Hence the presentation φ gives rise to a morphism $q : S \to P$ and a surjective O_{X_S}-homomorphism

$$\varphi_1 : (f_S)_*[(q_{X'})^*\mathcal{P}] \to M$$

where $\mathrm{Supp}M = Q \times S$ and M is an invertible O_S-Module. Since f_P is affine,

$$(f_S)_*[(q_{X'})^*\mathcal{P}] \simeq (q_X)^*[(f_P)_*\mathcal{P}]$$

[11, Ch. I, Prop. 9.3.2] so φ_1 corresponds to a homomorphism

$$\varphi_2 : (q_X)^*[(f_P)_*\mathcal{P}] \to M .$$

Let m denote the ideal of Q in O_X . Since $\mathrm{Supp}M = Q \times S$ and M is an invertible O_S-Module, $m \otimes O_S$ is the annihilator of M in O_{X_S} . Therefore φ_2 factors through the O_S-homomorphism

$$\varphi_3 : ((q_X)^*[(f_P)_*\mathcal{P}])(Q) \to M .$$

The commutative diagram

$$\begin{array}{ccc} X_S & \longleftarrow & Q \times S \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q} \\ X_P & \longleftarrow & Q \times P \end{array}$$

shows that

$$((q_X)^*[(f_P)_*\mathcal{P}])(Q) \simeq q^*([(f_P)_*\mathcal{P}](Q))$$

so φ_3 corresponds to an O_{X_S}-homomorphism

$$\varphi_4 : q^*E \to M ,$$

which is an element of $F(E/P)(S)$.

Let α' be another isomorphism

$$\alpha' : (q_{X'})^*\mathcal{P} \simeq L \otimes_{O_S} T .$$

It gives rise to a surjective O_{X_S}-homomorphism φ_1' and a commutative diagram

$$\begin{array}{ccc}
(f_S)_*[(q_{X'})^*\mathcal{P}] & \xrightarrow{\varphi_1'} & M \\
{\scriptstyle (f_S)_*(\alpha' \circ \alpha^{-1})} \downarrow & \nearrow {\scriptstyle \varphi_1} & \\
(f_S)_*[(q_{X'})^*\mathcal{P}] & &
\end{array}$$

Let Z denote the connected component of X containing Q and set $Z' = f^{-1}(Z)$. Since X' and X have the same number of connected components, Z' is connected and the isomorphism $\alpha' \circ \alpha^{-1}|_{Z'}$ is given by multiplication by an element $s \in O_S^*(S)$ [2, Lemma 5.4]. Hence we have a commutative diagram

$$\begin{array}{ccc} q^*E & \xrightarrow{\varphi_4} & M \\ & \underset{\varphi_4'}{\searrow} & \downarrow s \\ & & M \end{array}$$

so φ_4 and φ_4' define the same element of $F(E/P)(S)$.

The map

$$\rho : \underline{\mathrm{Pres}}_{X'/X} \to F(E/P)$$

defined above is a map of functors, and the map, which sends an element $q^*E \to M$ of $F(E/P)(S)$ to the presentation $(f_S)_*[(q_{X'})^*\mathcal{P}] \to M$, is an inverse of ρ .

2.3

We keep the same notation as in Section 2.1. Let S be a k-scheme. If L is an invertible O_{X_S}-Module, then $[(f_S)_*L](Q)$ is a locally free O_S-Module of rank 2, which splits as follows:

<u>Case 1.</u> There is only one point $Q' \in X'$ such that $f(Q') = Q$. Then $[(f_S)_*L] \simeq L(Q') \oplus L'$ where L' is an invertible O_S-Module. Indeed, let m and m' denote the ideals of Q and Q' . Since $\delta(X',X) = 1$, m is

the conductor of O_X in $O_{X'}$ [10, Ch. III, Rem. 1.3], and there is a canonical k-isomorphism

$$O_{X'/m} \simeq (O_{X'/m'}) \oplus (m'/m) .$$

Hence there is a canonical O_S-isomorphism

$$O_{X'_S/m_S} \simeq (O_{X'_S/m'_S}) \oplus (m'_S/m_S) .$$

The morphism f_S is affine, so there exists a canonical O_S-isomorphism

$$[(f_S)_*L](Q) \simeq L \otimes_{O_{X'_S}} (O_{X'_S/m_S})$$

[11, Ch. I, Prop. 9.3.2]. Hence we get a canonical splitting

$$[(f_S)_*L](Q) \simeq L(Q') \oplus L'$$

where $L' = L \otimes_{O_{X'_S}} (m'_S/m_S)$.

<u>Case 2.</u> There are two points Q_1 , $Q_2 \in X'$ such that $f(Q_1) = f(Q_2) = Q$. Then there is a canonical O_S-isomorphism

$$[(f_S)_*L](Q) \simeq L(Q_1) \oplus L(Q_2) .$$

The proof of this splitting is similar to that given in Case 1.

Suppose that $\underline{\mathrm{Pic}}^{\circ}_{X'}$ is represented by a scheme P and let $\mathcal{P}$ be a universal invertible sheaf on $X' \times P$. Using the splitting of $[(f_P)_*\mathcal{P}](Q)$ deduced above, Prop. 2.2.1 can be formulated as follows:

<u>Proposition 2.3.1.</u> $\underline{\mathrm{Pres}}_{X'/X}$ is represented by the $\mathbb{P}^1$-bundle

$$\mathbb{P}(\mathcal{P}(Q') \oplus \mathcal{P}')$$

in Case 1 and by the $\mathbb{P}^1$-bundle

$$\mathbb{P}(\mathcal{P}(Q_1) \oplus \mathcal{P}(Q_2))$$

in Case 2 if X' and X have the same number of connected components.

Let

$$\varphi : (f_S)_*L \to N$$

be a presentation over S. We say that φ is a <u>strict presentation</u> if $L' \to N$ is surjective (Case 1) or if $L(Q_1) \to N$ and $L(Q_2) \to N$ are both surjective (Case 2).

<u>Definition 2.3.2.</u> Let $\underline{\mathrm{StPres}}_{X'/X}$ be the subfunctor of $\underline{\mathrm{Pres}}_{X'/X}$ defined as follows: For each k-scheme

S , let

$$\underline{\mathrm{StPres}}_{X'/X}(S)$$

be the set of equivalence classes of strict presentations over S .

Proposition 2.3.3.

(a). $\underline{\mathrm{StPres}}_{X'/X}$ is represented by the $\mathbb{G}_a$-bundle

$$\mathbb{P}(\mathcal{O}(Q') \oplus \mathcal{O}') \setminus \mathbb{P}(\mathcal{O}(Q_1))$$

over P in Case 1.

(b). $\underline{\mathrm{StPres}}_{X'/X}$ is represented by the $\mathbb{G}_m$-bundle

$$\mathbb{P}(\mathcal{O}(Q_1) \oplus \mathcal{O}(Q_2)) \setminus (\mathbb{P}(\mathcal{O}(Q_1)) \cup \mathbb{P}(\mathcal{O}(Q_2)))$$

over P in Case 2 if X' and X have the same number of connected components.

(c). $\underline{\mathrm{StPres}}_{X'/X}$ is represented by P if X' and X do not have the same number of connected components.

Proof. (a). Let

$$\varphi : f_*L \to k$$

be a presentation over k and let $q : \mathrm{Spec}(k) \to P$ be a morphism such that $L \simeq (q_X)^*\mathcal{O}$. As in the proof of

Prop. 2.2.1, φ corresponds to a k-homomorphism

$$q^*(\mathcal{O}(Q')) \oplus q^*\mathcal{O}' \rightarrow k \ .$$

The presentation φ is not strict if and only if $q^*\mathcal{O}' \rightarrow k$ is zero, i.e. if and only if we have a commutative diagram

$$\begin{array}{ccc} q^*(\mathcal{O}(Q')) \oplus q^*\mathcal{O}' & \longrightarrow & k \\ \downarrow & \nearrow & \\ q^*(\mathcal{O}(Q')) & & \end{array}$$

where all the maps are surjective. Therefore φ is not strict if and only if the morphism $\mathrm{Spec}(k) \rightarrow \mathbb{P}(\mathcal{O}(Q') \oplus \mathcal{O}')$ corresponding to φ factors though the closed embedding $\mathbb{P}(\mathcal{O}(Q') \rightarrow \mathbb{P}(\mathcal{O}(Q') \oplus \mathcal{O}')$ determined by the surjective O_P-homomorphism $\mathcal{O}(Q') \oplus \mathcal{O}' \rightarrow \mathcal{O}(Q')$.

A presentation over a k-scheme S is strict if and only if the restriction to each closed point of S is a strict presentation. Hence a morphism $h : S \rightarrow \mathbb{P}(\mathcal{O}(Q') \oplus \mathcal{O}')$ corresponds to a strict presentation if and only if h factors through the open subset $\mathbb{P}(\mathcal{O}(Q') \oplus \mathcal{O}') \setminus \mathbb{P}(\mathcal{O}(Q'))$.

(b). The proof is similar to the one given for case (a). So the basic ingredient in the proof is the representability of $\underline{\text{Pres}}_{X'/X}$ by a $\mathbb{P}^1$-bundle, and therefore we need the hypothesis that X' and X have the same number of connected components.

(c). Set $E_1 = \mathcal{P}(Q_1)$ and $E_2 = \mathcal{P}(Q_2)$. We will show that $\underline{\text{StPres}}_{X'/X}$ is isomorphic to $F(E_1/P) \times_{\underline{\text{Pic}}^o_{X'}} F(E_2/P)$.

Let S be a k-scheme and let

$$\varphi : (f_S)_* L \to N$$

be a strict presentation over S. There exists a morphism $q : S \to P$, an invertible O_S-Module T and an isomorphism

$$\alpha : (q_{X'})^* \mathcal{P} \simeq L \otimes_{O_S} T .$$

As in the proof of Proposition 2.2.1, we get a surjective O_S-homomorphism

$$q^* E_1 \oplus q^* E_2 \to M = N \otimes_{O_S} T ,$$

and therefore surjective maps

$$\psi_1 : q^* E_1 \to M \quad \text{and} \quad \psi_2 : q^* E_2 \to M$$

because φ is strict.

Let α' be another isomorphism $(q_X')^*\mathcal{P} \simeq L \otimes_{O_S} T$. It gives rise to a surjective O_S-homomorphism $\varphi' : q^*E_1 \oplus q^*E_2 \to M$. Since Q_1 and Q_2 lie on different connected components of X', the isomorphism $(f_S)_*(\alpha' \circ \alpha^{-1})$ gives rise to an isomorphism

$$\psi : q^*E_1 \oplus q^*E_2 \to q^*E_1 \oplus q^*E_2$$

given by multiplication by $s_1 \in O_S^*(S)$ on q^*E_1 and mutiplication by $s_2 \in O_S^*(S)$ on q^*E_2 such that the diagram

$$\begin{array}{ccc} q^*E_1 \oplus q^*E_2 & \xrightarrow{\varphi'} & M \\ {\scriptstyle s_1 \oplus s_2}\downarrow & \nearrow{\scriptstyle \varphi} & \\ q^*E_1 \oplus q^*E_2 & & \end{array}$$

commutes. Hence we have commutative diagrams

$$\begin{array}{ccc} q^*E_1 & \xrightarrow{\psi_1} & M \\ & {\scriptstyle \psi_1'}\searrow & \downarrow{\scriptstyle s_1} \\ & & M \end{array}$$

and

$$\begin{array}{ccc} q^*E_2 & \xrightarrow{\psi_2} & M \\ & {\scriptstyle \psi_2'} \searrow & \downarrow {\scriptstyle s_2} \\ & & M \end{array}$$

and so φ and φ' give rise to the same element of

$$F(E_1/P)(S) \times_{\underline{\mathrm{Pic}}^{o}_{X'}(S)} F(E_2/P)(S) \ .$$

Hence we have defined a map of functors

$$\rho : \underline{\mathrm{StPres}}_{X'/X} \to F(E_1P) \times_{\underline{\mathrm{Pic}}^{o}_{X'}} F(E_2/P) \ .$$

Let $\psi_1 : q^*E_1 \to N$, $\psi_2 : q^*E_2 \to N$ and $\psi_1' : q^*E_1 \to N$, $\psi_2' : q^*E_2 \to N$ be surjective maps such that (ψ_1 , ψ_2) and (ψ_1' , ψ_2') define the same element of

$$F(E_1/P) \times_{\underline{\mathrm{Pic}}^{o}_{X'}(S)} F(E_2/P) \ .$$

We have commutative diagrams

$$\begin{array}{ccc} q^*E_1 & \xrightarrow{\psi_1} & N \\ & {\scriptstyle \psi_1'} \searrow & \downarrow {\scriptstyle s_1} \\ & & N \end{array}$$

and

$$\begin{array}{ccc} q^*E_2 & \xrightarrow{\psi_2} & N \\ & \searrow^{\psi_2'} & \downarrow s_2 \\ & & N \end{array}$$

where $s_1 , s_2 \in O_S^*(S)$. The pairs (ψ_1 , ψ_2) and (ψ_1' , ψ_2') give rise to strict presentations

$$\varphi , \varphi' : (f_S)_*[(q_{X'})^*\mathcal{P}] \to q^*E_1 \oplus q^*E_2 \to N .$$

Let α denote the $O_{X_S'}$-isomorphism of $(q_{X'})^*\mathcal{P}$ defined by s_1 on the connected component of X_S' containing Q_1 , by s_2 on the connected component containing Q_2 and by 1 on the other components. Then we have a commutative diagram

$$\begin{array}{ccc} (f_S)_*[(q_{X'})^*\mathcal{P}] & \xrightarrow{\varphi'} & N \\ (f_S)_*\alpha \downarrow & \nearrow_{\varphi} & \\ (f_S)_*[(q_{X'})^*\mathcal{P}] & & \end{array}$$

and φ and φ' define the same element of $\underline{\mathrm{StPres}}_{X'/X}(S)$. Hence we have defined a map

$$F(E_1/P) \times_{\underline{\mathrm{Pic}}_X^o} F(E_2/P) \to \underline{\mathrm{StPres}}_{X'/X} ,$$

which is an inverse of ρ .

The assertion of (c) follows since $F(E_1/P)$ and $F(E_2/P)$ are represented by schemes isomorphic to P .

CHAPTER III.

A construction of the Picard scheme of a curve.

In [13] Grothendieck showed the existence of the Picard scheme of a projective k-scheme [13, Exp. 232, Cor. 6.6]. Oort [21] proved that the Picard scheme of an irreducible curve X can be constructed from the Picard scheme of the normalization of X by a sequence of extensions by $(\mathbb{G}_m)^n$- and $(\mathbb{G}_a)^n$-bundles. In the special case that the curve has n singularities, which are all ordinary nodes, Oda and Seshadri used the presentation functor to construct Pic^o_X as a $(\mathbb{G}_m)^n$-extension of $\mathrm{Pic}^o_{\overline{X}}$ [20, Cor. 12.4].

In this chapter we prove that the Picard scheme of a curve X (not necessarily irreducible) can be constructed from the Picard scheme of the normalization of X by a sequence of $\mathbb{G}_m$- and $\mathbb{G}_a$-extensions. Our procedure differs notably from that of [21] since we, inspired by Oda and Seshadri, make the presentation functor play an essential role in our proof. We show that if $f : Y' \to Y$ is a birational, surjective morphism of curves such that $\delta(Y',Y) = 1$, then $\underline{\mathrm{Pic}}^o_Y$ is isomorphic to $\underline{\mathrm{StPres}}_{Y'/Y}$. If $\underline{\mathrm{Pic}}^o_{Y'}$ is represented by a scheme P , $\underline{\mathrm{StPres}}_{Y'/Y}$ is represented by a $\mathbb{G}_a$- or $\mathbb{G}_m$-bundle over P or by P [Proposition 2.3.3]. Since the normalization map of X can be written as a composition of maps where δ changes by one [Theorem 1.2.4], we obtain a stepwise construction of Pic°_{X} from $\mathrm{Pic}^{\circ}_{X'}$.

3.1.

Let X be a curve and denote by $\mathcal{R}(X)$ the sheaf of total quotient rings of O_X . Let F be an O_X-Module. We recall that the kernel $T(F)$ of the natural map

$$F \to F \otimes_{O_X} \mathcal{R}(X) ,$$

obtained by tensoring the map $O_X \to \mathcal{R}(X)$, is called the <u>sheaf of torsion</u> of F , and F is called <u>torsion-free</u> if $T(F) = 0$.

Let $f : X' \to X$ be a birational, surjective morphism of curves such that $\delta(X',X) = 1$. Let $\varphi : f_*L \to N$ be a presentation over k and put $I = \ker\varphi$. The commutative diagram

$$\begin{array}{ccc} f^*I & \longrightarrow & f^*I \otimes_{O_{X'}} \mathcal{R}(X') \\ \sigma_1 \downarrow & & \downarrow \\ L & \xrightarrow{\sigma_2} & L \otimes_{O_{X'}} \mathcal{R}(X') \end{array}$$

where σ_2 is injective, shows that σ_1 factors through a map

$$\sigma : \mathcal{L}(I) \to L$$

where $\mathcal{L}(I) = f^*I/T(f^*I)$. Moreover, $K = \ker\sigma$ is a torsion-free sheaf because it is a subsheaf of a torsion-free sheaf, and $K_g = 0$ for all generic points g of X' . Hence $K = 0$ and σ is injective.

Lemma 3.1.1. I is invertible if and only if $\mathcal{L}(I) \simeq L$.

Proof. If I is invertible, then $\mathcal{L}(I) \simeq f^*I$ and $\mathcal{L}(I) \simeq L$ because $\chi(X',f^*I) = \chi(X',L)$.

Conversely, suppose that $\mathcal{L}(I) \simeq L$. Let $U = \mathrm{Spec}(A)$ be an affine neighbourhood of the point $Q \in X$ where $\delta(X',X,Q) = 1$, and set $U' = \mathrm{Spec}(A')$ where $U' = f^{-1}(U)$. Let M be an A-module such that $\widetilde{M} \simeq I|_U$ and N an A'-module such that $\widetilde{N} = L|_{U'}$. Then $M \otimes A'/T(M \otimes A') \simeq N$, and by [10, Ch. I, 2.6], there exists an element $m \in M$ such that N is generated by $\overline{m \otimes 1}$ as A'-module. Let I' be the invertible O_X-Module defined by $I'|_{X\backslash Q} \simeq I|_{X\backslash Q}$ and $I'|_U = \widetilde{M}'$ where M' is the sub-module of M generated by m . Then $\mathcal{L}(I') \simeq f^*I' \simeq L$ and so $\chi(X,I) = \chi(X,I')$. Hence, since $I' \subseteq I$, $I' \simeq I$ and I is invertible.

Lemma 3.1.2. Let S be a k-scheme and let $\varphi \in \underline{\mathrm{Pres}}_{X'/X}(S)$. Then $\varphi \in \underline{\mathrm{StPres}}_{X'/X}(S)$ if and only if $\ker\varphi$ is an invertible O_{X_S}-Module.

Proof. Set $I = \ker\varphi$. Then I is invertible if and only if $I(s)$ is invertible for all closed points $s \in S$. Also, φ is a strict presentation if and only if $\varphi(s)$ is a strict presentation for all closed points $s \in S$ [Nakayama's Lemma]. Hence it is enough to prove the lemma in the case that $S = \mathrm{Spec}(k)$.

Let $\varphi : f_*L \to N$ be a presentation over k, and let

$$g : f_*\mathcal{L}(I) \to f_*L$$

be the natural homomorphism $\mathcal{L}(I) \to L$ considered as an O_X-homomorphism. We have a commutative diagram

$$\begin{array}{ccc} I & \longrightarrow & f_*L \\ & \searrow \quad \nearrow & \\ & f_*\mathcal{L}(I) & \end{array}$$

where all the maps are injective. Hence there is a homomorphism $\gamma : N \to \mathrm{coker}g$ and a commutative diagram

$$(*) \qquad \begin{array}{ccc} f_*L & \xrightarrow{\varphi} & N \\ & \searrow \quad \swarrow \gamma & \\ & \mathrm{coker}g & \end{array}$$

where all the maps are surjective.

Suppose that f is a morphism as in Case 1 [see Sect. 2.3]. Then g restricted to Q splits in a sum

$$g(Q') \oplus g' : \mathcal{L}(I)(Q') \oplus \mathcal{L}(I)' \to \mathcal{L}(Q') \oplus L' ,$$

and diagram $(*)$ restricted to Q gives a diagram

$$(**) \qquad \begin{array}{ccc} L(Q') \oplus L' & \longrightarrow & N \\ & \searrow \quad \swarrow & \\ & \operatorname{coker} g(Q') \oplus \operatorname{coker} g & \end{array}$$

where all the maps are surjective.

The presentation φ is strict if and only if $L' \to N$ is surjective. Diagram $(**)$ shows that $L' \to N$ is surjective if and only if the composition

$$L' \to L(Q') \oplus L' \to \operatorname{coker} g(Q') \oplus \operatorname{coker} g'$$

is surjective, i.e. if and only if $\operatorname{coker}(Q') = 0$. By Nakayama's Lemma, $\operatorname{coker} g(Q') = 0$ if and only if g is an isomorphism. Hence Lemma 3.1.2 shows that φ is strict if and only if $I = \ker\varphi$ is invertible.

The proof for a morphism f as in Case 2 is similar to the proof given above.

3.2.

Let S be a k-scheme and let

$$\varphi : (f_S)_* L \to N$$

be a presentation over S. It is easy to check, using [12, Ch. III, Prop. 6.5.8], that $\ker\varphi$ is S-flat and that the formation of the kernel of a presentation commutes with base change. If φ is a strict presentation, $\ker\varphi$ is invertible [Lemma 3.1.2], and it is an immediate consequences of the additivity of the Euler characteristic on short exact sequences that $\ker\varphi \in \underline{\mathrm{Pic}}^0_X(S)$. Hence the map

$$\underline{K} : \underline{\mathrm{StPres}}_{X'/X} \to \underline{\mathrm{Pic}}^0_X ,$$

which sends a presentation φ to $\ker\varphi$, is a map of functors.

Let I be an invertible O_{X_S}-Module of degree 0. Tensoring the natural surjection

$$(f_S)_* O_{X'_S} \to (f_S)_* O_{X'_S}/O_{X_S}$$

by I over O_{X_S} and using the projection formula [14, Ch. II, Ex. 5.1 (d)] gives a presentation

$$\varphi : (f_S)_*(f_S^* I) \to N .$$

By Lemma 3.1.2, φ is a strict presentation, and we have defined a map

$$\underline{\gamma} : \underline{\mathrm{Pic}}_X^0 \to \underline{\mathrm{StPres}}_{X'/X} ,$$

which is easily seen to be functional.

The kernel of the presentation $\psi = \underline{\gamma}(I)$ is isomorphic to I so $\underline{K} \circ \underline{\gamma} = \mathrm{id}$. Moreover, there is an isomorphism $\alpha : f_S^* I \to L$ of $O_{X'_S}$-Modules such that the diagram

$$\begin{array}{ccc} \ker\psi & \subset & (f_S)_*(f_S^* I) \\ \wr\downarrow & & \downarrow \alpha \\ I & \subset & L \end{array}$$

commutes. Hence φ and ψ are equivalent presentations and $\underline{\gamma} \circ \underline{K} = \mathrm{id}$. Thus the functors $\underline{\mathrm{Pic}}_X^0$ and $\underline{\mathrm{StPres}}_{X'/X}$ are isomorphic. From Proposition 2.3.3 we get the following theorem:

<u>Theorem 3.2.1.</u> Let $f : X' \to X$ be a surjective, birational morphism of curves such that $\delta(X',X) = 1$, and denote by Q the point of X such that $\delta(X',X,Q) = 1$. Suppose that $\underline{\mathrm{Pic}}_{X'}^0$ is represented by a scheme P and let $\mathcal{P}$ be a universal invertible sheaf on $X' \times P$.

(i). If X and X' do not have the same number of connected components, then $\underline{\mathrm{Pic}}_X^0$ is represented by P .

(ii). If there are two points Q_1 , $Q_2 \in X'$, which map to Q , and X' and X have the same number of connected components, then $\underline{\mathrm{Pic}}_X^0$ is represented by the $\mathbb{G}_m$-bundle

$$\mathbb{P}(\mathcal{O}(Q_1) \oplus \mathcal{O}(Q_2))\setminus(\mathbb{P}(\mathcal{O}(Q_1)) \cup \mathbb{P}(\mathcal{O}(Q_2)))$$

over P .

(iii). If there is only one point $Q' \in X'$, which map to Q , then $\underline{\mathrm{Pic}}_X^0$ is represented by the $\mathbb{G}_a$-bundle

$$\mathbb{P}(\mathcal{O}(Q') \oplus \mathcal{O}')\setminus\mathbb{P}(\mathcal{O}(Q'))$$

over P .

The theorem above together with the breaking up of the normalization map proved in Section 1.2 [Theorem 1.2.4] gives the corollary:

Corollary 3.2.2. The Picard scheme of a curve can be constructed from the Picard scheme of the normalization of the curve by a sequence of extensions by $\mathbb{G}_m$- and $\mathbb{G}_a$-bundles.

3.3.

Let $X = \bigcup_{i=1}^{\ell} X_i$ be a curve with ℓ irreducible components and r connected components. Using the additivity of the Euler characteristic, it is easy to see that the arithmetic genus $p(X) = 1 - \chi(X, O_X)$ is given by

$$p(X) = \sum_{i=1}^{\ell} p(\overline{X}_i) + \delta - \ell + 1$$

where $\overline{X}_i$ denotes the normalization of X_i and $\delta = \delta(\overline{X}, X)$.

From Theorem 3.2.1 it follows that

$$\dim \mathrm{Pic}^o_X = \dim \mathrm{Pic}^o_{\overline{X}} + \delta - (\ell - r) ,$$

and since $\dim \mathrm{Pic}^o_{\overline{X}} = \sum_{i=1}^{\ell} p(\overline{X}_i)$, we get the following formula for the dimension of Pic^o_X :

<u>Proposition 3.3.1.</u> $\dim \mathrm{Pic}^o_X = p(X) + r - 1$.

The formula of Prop. 3.3.1 can also be deduced from the fact that $\dim \mathrm{Pic}^o_X = \dim_k H^1(X, O_X)$, which is proved by Grothendieck [13, Exp. 236, Prop. 2.10 (iii)].

CHAPTER IV.

On the representability of $\underline{\text{Pic}}_X^{=0}$.

The Picard scheme Pic_X^0 of a smooth curve is a projective variety over k . If X has singularities, Pic_X^0 is not proper over k . Compactifications of the Picard scheme have been studied by many authors using different methods [see [2], [10] and [20] for a historical overview]. Altman and Kleiman [2] showed that if X is an irreducible curve, then the functor $\underline{\text{Pic}}_X^{=0}$ of torsion-free, rank-1 sheaves on X is represented by a projective k-scheme. We use their work as a basic reference in the upcoming chapters.

In this chapter we discuss the problem of compactifying Pic_X^0 for a reducible curve. Oda and Seshadri [10] constructed compactifications of Pic_X^0 for a class of reducible curves using geometric invariant theory. The breaking up of the normalization map in steps $X' \to X$ such that $\delta(X',X) = 1$ and the construction of Pic_X^0 as a $\mathbb{G}_m$- or $\mathbb{G}_a$-bundle over $\text{Pic}_{X'}^0$, suggests the possibility of a compactification of Pic_X^0 as a fibration over the compactification of $\text{Pic}_{X'}^0$. We give examples, which illustrates the difficulties met in carrying out such a construction.

Even for an irreducible curve we are interested in a new construction of $\mathrm{Pic}_X^{=0}$, which will give more information on the structure of the singularities of $\mathrm{Pic}_X^{=0}$. For instance, Kleiman has privately pointed out that all the properties of the Abel map

$$\mathrm{Hilb}^d(C/k) \to \mathrm{Pic}_C^0$$

proved in [16] for a smooth, irreducible curve C, can be proved for the Abel-Altman-Kleiman map

$$\mathrm{Quot}^d(\omega/X/k) \to \mathrm{Pic}_X^{=0}$$

for an arbitrary integral curve X if we know that the tangent cone of $\mathrm{Pic}_X^{=0}$ at each point is Cohen-Macaulay.

The stronger assertion, that the tangent cone is a complete intersection, does not hold. In Section 4.3 we give an example of a plane, irreducible curve and a point of $\mathrm{Pic}_X^{=0}$ where the tangent cone is not a complete intersection.

4.1

Let X be an irreducible curve. A coherent, torsion-free O_X-Module F is said to have rank n if $F_g \simeq O_{X,g}^n$ where g denotes the generic point of X. The degree of F , $\deg F$, is defined by

$$\deg F = \chi(X,F) - n\chi(X,O_X) .$$

Let $Y \to S$ be a morphism of k-schemes such that the fibers $Y(s)$ are integral curves for all closed points $s \in S$. An O_Y-Module I is called relatively torsion-free, rank-n over S if it is S-flat and if the pullback $I(s)$ of I to $Y(s)$ is a torsion-free, rank-n sheaf for all closed points $s \in S$.

We define a contravariant functor $\underline{\mathrm{Pic}}_X^{=}$ as follows: For each k-scheme S , let

$$\underline{\mathrm{Pic}}_X^{=}(S)$$

denote the set of equivalence classes of O_{X_S}-Modules, which are relatively torsion-free, rank-1 over S , where I and J are considered equivalent if there exists an invertible O_S-Module N and an isomorphism

$$I \otimes_{O_S} N \simeq J .$$

If $S' \to S$ is a morphism of k-schemes, the map $\underline{\mathrm{Pic}}_X^{=}(S) \to \underline{\mathrm{Pic}}_X^{=}(S')$ is given by pullback.

Let d be an integer. We define subfunctors $\underline{\mathrm{Pic}}_X^{=d}$ of $\underline{\mathrm{Pic}}_X^{=}$ as follows: For each k-scheme S , let

$$\underline{\mathrm{Pic}}_X^{=d}(S)$$

be the elements I of $\underline{\mathrm{Pic}}_X^{=}(S)$ such that $\deg I(s) = d$ for all closed points $s \in S$. It is proved in [2] that the functor $\underline{\mathrm{Pic}}_X^{=d}$ is represented by a projective k-scheme $\mathrm{Pic}_X^{=d}$ [2, Theorem (8.5) (ii)].

Let ω denote the dualizing sheaf on X. Let S be a k-scheme and fix a positive integer n. Let F be an element of $\mathrm{Quot}^n(\omega/X/k)$ and denote by $I(F)$ the kernel of the natural surjection

$$\omega_S \to F .$$

Let s be a closed point of S. The formation of $I(F)$ commutes with base change, so $I(F)(s) \subseteq \omega$. Since ω is a torsion-free, rank-1 sheaf on X [4, 2.8, page 8], it follows that $I(F)(s)$ is torsion-free, rank-1. By the additivity of the Euler characteristic on short exact sequences, we get that

$$\chi(I(F)(s)) = \chi(\omega) - n ,$$

so $I(F)$ is an element of $\underline{\mathrm{Pic}}_X^{=d}(S)$ where $d = \chi(\omega) - \chi(O_X) - n$. The map of functors

$$\underline{A}^n : \underline{\mathrm{Quot}}^n(\omega/X/k) \to \underline{\mathrm{Pic}}_X^{=d} ,$$

which sends a quotient F to $I(F)$, defines a morphism of schemes

$$\underline{A}^n : \mathrm{Quot}^n(\omega/X/k) \to \mathrm{Pic}_X^{=d} .$$

We call this map the <u>Abel map</u> associated to ω .

It is proved by Altman and Kleiman [2, Theorem (8.4) (v), Lemma (5.17) (ii) and Theorem (4.2)] that A^d is smooth and the fibers are projective spaces if and only if $d \geq 2p - 1$. Here p denotes the arithmetic genus of X . In fact Altman and Kleiman used the fact that the fibers of $\underline{A}^n$ are linear systems of quotients of ω , which are represented by projective spaces, to construct $\mathrm{Pic}_X^{=d}$ as a quotient of $\mathrm{Quot}^n(\omega/X/k)$ by a smooth and proper equivalence relation.

4.2.

The methods used by Altman and Kleiman to represent $\underline{\mathrm{Pic}}_X^{=d}$ for an irreducible curve X do not immediately extend to the case that X is reducible.

Let $X' \to X$ be a partial normalization of X such that $\delta(X',X) = 1$. Suppose we have constructed a compactification $\overline{P}_{X'}$ of $\mathrm{Pic}_{X'}^{o}$. We can try to construct a compactification $\overline{P}_X$ of Pic_X^{o} along the following lines: First we extend the $\mathbb{P}^1$-bundle $Pres_{X'/X}$ over $Pic^o_{X'}$ to a $\mathbb{P}^1$-bundle over $\overline{P}_{X'}$, and we construct $\overline{P}_X$ as a quotient of this $\mathbb{P}^1$-bundle by identifications of points in the fibers.

The first identifications to try are the following: If $X' \to X$ is an identification of two points of X' , we identify the point at infinity with the origin in the same fiber such that $\overline{P}_X$ is a fibration over $\overline{P}_{X'}$ by nodal cubic curves. If $X' \to X$ is an infinitesimal identification, we make an infinitesimal identification in each fiber such that $\overline{P}_X$ is a fibration over $\overline{P}_{X'}$ by cuspidal cubic curves.

However, examples show that the constructions indicated above cannot be carried out. First, suppose that X has one ordinary double point as only singularity and that the normalization X' has genus 1 . Then $\overline{P}_X$ is obtained from the $\mathbb{P}^1$-bundle $Pres_{X'/X}$ over $\overline{P}_{X'}$ by identifying two sections via a translation of $\overline{P}_{X'}$ by the point of $Pic^o_{X'}$ corresponding to $O_{X'}[Q_2 - Q_1]$ [20, Example (1), page 83]. Hence $\overline{P}_X$ is not a fibration over $\overline{P}_{X'}$.

The example of Oda and Seshadri mentioned above, suggests that $\overline{P}_X$ can be constructed as a quotient of a $\mathbb{P}^1$-bundle either by identifying two sections via a translation in $\overline{P}_{X'}$ or by making an infinitesimal identification in one section via an infinitesimal translation in $\overline{P}_{X'}$. If such a construction is possible, the tangent cone at a point of $\overline{P}_X$ will be a complete intersection since it depends only on the analytic structure of that point [4, Prop. 1.19].

However, in Section 4.3 we give an example of a plane, irreducible curve X and a point of $\overline{\overline{\mathrm{Pic}}}{}^{o}_{X}$ where the tangent cone is not a complete intersection.

4.3.

Let S be a smooth surface and let q be a closed subscheme of S of length n , which is supported at one point $Q \in S$. Set $\mathcal{H} = \mathrm{Hilb}^n(S/k)$ and let v denote the point of $\mathcal{H}$ corresponding to q . Then

$$A = O_{\mathcal{H},v}$$

is a regular, local ring of dimension $2n$ [1, Prop. (3)].

Let

$$W \subseteq S \times \mathcal{H}$$

denote the universal subscheme and set

$$R = O_{W,(Q,v)} \cdot$$

Then R is a free A-module of rank n since the projection $p : W \to \mathcal{V}$ is flat of degree n. Denote by m the maximal ideal of A. Since $p^{-1}(v) = q$, R/mR is a k-vector space of dimension n. We lift a basis $\overline{v}_1, \ldots, \overline{v}_n$ of R/mR to a basis $v_1, \ldots, v_n$ of R as an A-module.

Let C be a closed subscheme of S such that $q \subseteq C$. Let $\mathrm{Spec}(A_1)$ be an open affine subset of S containing q, and suppose that C is given by an equation $F \in A_1$ in this open subset. We denote by f the image of F in R by the natural homomorphism $A_1 \to R$. There exist elements $a_1, \ldots, a_n \in A$ such that

$$f = a_1 v_1 + \ldots + a_n v_n .$$

Lemma 4.3.1. Set $H = \mathrm{Hilb}^n(C/k)$ and denote by z the point of H corresponding to q. Then $O_{H,z} \simeq A/(a_1, \ldots, a_n)$.

Proof. Let K denote the kernel of the natural map $O_{S\times\mathcal{V}} \to O_{C\times\mathcal{V}}$, and let

$$u : K \to O_W$$

denote the composition of the inclusion $K \subseteq O_{S\times\mathcal{V}}$ and the surjection $O_{S\times\mathcal{V}} \to O_W$.

Let $T \to \mathcal{V}$ be a morphism of k-schemes. It corresponds to an element of $\underline{\mathrm{Hilb}}^n(S/k)(T)$, which is an element of $\underline{\mathrm{Hilb}}^n(C/k)(T)$ if and only if the map

$$u_T : K_T \to O_{W_T}$$

is zero. By [11, Ch. I, Prop. 9.7.9.1] there exists a closed subscheme $\mathcal{V}_o$ of $\mathcal{V}$ such that $T \to \mathcal{V}$ factors through $\mathcal{V}_o$ if and only if u_T is zero. Hence $\mathrm{Hilb}^n(C/k) = \mathcal{V}_o$, and $O_{H,z} = A/I$ for an ideal $I \subseteq A$.

The stalk of the map u at (Q,v) is the natural map

$$u_{(Q,v)} : F \otimes A \to R ,$$

and since $f = a_1v_1 + \ldots + a_nv_n$ is the image of F in R, $I = (a_1,\ldots,a_n)$ and $O_{H,z} = A/(a_1,\ldots,a_n)$.

Proposition 4.3.2. Fix an integer $n \geq 2$ and let $e \geq 3n + 1$ be an odd integer. Let X be the plane curve given by the equation

$$(T_1/T_o)^2 - (T_2/T_o)^e$$

in the open subset $\mathrm{Spec}k[T_1/T_o, T_2/T_o]$ of $\mathbb{P}^2 = \mathrm{Proj}k[T_o,T_1,T_2]$. Let z be the point of $\mathrm{Hilb}^n(X/k)$ corresponding to the closed subscheme of X given by the ideal (T_1,T_2^n). Then the tangent cone of $\mathrm{Hilb}^n(X/k)$ at z is not a complete intersection.

<u>Proof.</u> Set $t_1 = T_1/T_o$ and $t_2 = T_2/T_o$. Using the same notation as in the beginning of this section with $S = \mathbb{P}^2$ and q the closed subscheme of S given by the ideal (T_1, T_2^n) , we get that

$$1, t_2, \dots, t_2^{n-1}$$

is a basis of R over A . We write $t_2^n \in R$ as

$$t_2^n = c_o + c_1 t_2 + \dots + c_{n-1} t_2^{n-1}$$

where $c_i \in A$. Since q is a closed subscheme of the 1-dimensional subscheme of $\mathbb{P}^2$ defined by the equation $T_2^n = 0$, $A/(c_o, \dots, c_{n-1}) \neq 0$ [Lemma 4.3.1], so c_i are contained in the maximal ideal m of A .

An easy calculation shows that

$$\text{(i)} \quad t_2^{rn+1} = d_o + d_1 t_2 + \dots + d_{n-1} t_2^{n-1}$$

where $d_i \in m^r$.

Write

$$t_1 = V_o + V_1 t_2 + \dots + V_{n-1} t_2^{n-1}$$

where $V_i \in A$. Let C_1 be the line in $\mathbb{P}^2$ defined by the ideal (T_1) . Then $\mathrm{Hilb}^n(C_1/k)$ is a nonsingular scheme of dimension n [1, Lemma (1)], so $V_o,\ldots,V_{n-1}$ is a part of a regular system of parameters of m [Lemma 4.3.1].

An easy calculation shows that

$$\text{(ii)}\quad \begin{aligned} t_1^2 &= V_o^2 + h_o + (2V_oV_1 + h_1)t_2 + \ldots + \\ &+ (\sum_{i+j=\ell} V_iV_j + h_\ell)t_2^\ell + \ldots + (\sum_{i+j=n-1} V_iV_j + h_{n-1})t_2^{n-1} \end{aligned}$$

where $h_o,\ldots,h_{n-1} \in m^3$.

Using (i) and (ii) we write $t_1^2 - t_2^e$ as

$$\begin{aligned} t_1^2 - t_2^e = V_o^2 + g_o + \ldots + (\sum_{i+j=\ell} V_iV_j + g_\ell)t_2^\ell + \\ + \ldots + (\sum_{i+j=n-1} V_iV_j + g_{n-1})t_2^{n-1} \end{aligned}$$

where $g_i \in m^3$. Hence the local ring B of $\mathrm{Hilb}^n(X/k)$ at the point z is of the form

$$B = A/I$$

where $I = (V_o^2 + g_o,\ldots,\sum_{i+j=\ell} V_iV_j + g_\ell,\ldots,\sum_{i+j=n-1} V_iV_j + g_{n-1})$ [Lemma 4.3.1].

Let I^* be the ideal of A generated by the leading forms of the elements of I . Set

$$J = (V_o^2, V_oV_1,\ldots,\sum_{i+j=\ell} V_iV_j,\ldots,\sum_{i+j=n-1} V_iV_j) .$$

Since $g_i \in m^3$, we have an inclusion

$$J \subseteq I^* .$$

Let M denote the maximal ideal of B . There is an isomorphism

$$gr_M(B) \simeq A/I^*$$

[18, Ch. III, §3], and therefore

$$ht(I^*) = n$$

since $\dim gr_M(B) = \dim B = n$ [1, Cor. (7)].

The ideal J is contained in $(V_o, V_1, \ldots, V_{n-2})$, so $htJ \leq n - 1$. Hence I^* is of the form

$$I^* = (V_o^2, \ldots, \sum_{i+j=n-1} V_i V_j, H_1, \ldots H_s)$$

where $H_i \in m^3$.

It is easy to see that $V_o^2, \ldots, \sum_{i+j=\ell} V_i V_j, \ldots, \sum_{i+j=n-1} V_i V_j$ is a minimal set of generators of J , and therefore a minimal set of generators of I^* has more than n elements. Thus $gr_M(B) \simeq A/I^*$ is not a complete intersection.

The plane curve X defined by $t_1^2 - t_2^e$ has arithmetic genus $(e - 1)(e - 2)/2$. We plan to use the Abel map

$$\mathrm{Quot}^d(\omega/X/k) \to \mathrm{Pic}_X^{=0}$$

to show the existence of a point of $\mathrm{Pic}_X^{=0}$ where the tangent cone is not a complete intersection. Since this map is smooth if and only if $d \geq (e - 1)(e - 2) - 1$, we need the following lemma:

Lemma 4.3.3. Let C be a curve. Fix positive numbers n_1 and n_2. Set $H_1 = \mathrm{Hilb}^{n_1}(C/k)$, $H_2 = \mathrm{Hilb}^{n_2}(C/k)$ and $H = \mathrm{Hilb}^{n_1+n_2}(C/k)$. Let q_1 and q_2 be closed subschemes of C of length n_1 and n_2 such that $\mathrm{Supp} q_1 \cap \mathrm{Supp} q_2 = \emptyset$. Denote by v_1 and v_2 the points of H_1 and H_2 corresponding to q_1 and q_2 and by v the point of H corresponding to $q_1 \cup q_2$. Then

$$\hat{O}_{H,v} \simeq \hat{O}_{H_1,v_1} \otimes_k \hat{O}_{H_2,v_2} .$$

Proof. Let σ be a close subscheme of C of length n. We define a functor Def_σ from the category of local, artinian k-algebras with residue field k to the category of sets as follows:

$$\mathrm{Def}_\sigma(A)$$

is the set of subschemes $D \subseteq C \times \mathrm{Spec}(A)$ such that the projection $f : D \to \mathrm{Spec}(A)$ is flat and $f^{-1}(\mathrm{Spec}(k)) = \sigma$. This functor is prorepresentable [24, Def. on page 208] by $\hat{O}_{H',\Sigma}$ where Σ is the point of $H' = \mathrm{Hilb}^n(C/k)$ corresponding to σ.

Let A be a local, artinian k-algebra, and let D be an element of $\mathrm{Def}_q(A)$. Since A is henselian [12, Ch. IV, Prop. 18.5.11], D can be written as

$$D = D_1 \oplus D_2$$

where $D_i \in \mathrm{Def}_{q_i}(A)$ [12, Ch. IV, Thm. 18.5.11 (c)]. Hence the functor Def_q can be written at

$$\mathrm{Def}_q = \mathrm{Def}_{q_1} \oplus \mathrm{Def}_{q_2} ,$$

and therefore

$$\hat{O}_{H,v} \simeq \hat{O}_{H_1,v_1} \otimes_k \hat{O}_{H_2,v_2} .$$

Fix an integer $d \geq (e - 1)(e - 2) - 1$. There exists a point of $\mathrm{Hilb}^d(X/k)$ where the tangent cone is not a complete intersection [Prop. 4.3.2], so, by Lemma 4.3.3, there exists a point y of $\mathrm{Hilb}^d(X/k)$ where the tangent cone C_1 is not a complete intersection. Since the Abel map $A^d : \mathrm{Hilb}^d(X/k) \to \mathrm{Pic}_X^{=0}$ is smooth, we have that

$$C_1 \simeq C_2[U_1,\ldots,U_\ell]$$

where C_2 is the tangent cone of $\mathrm{Pic}_X^{=0}$ at $A^d(y)$ and U_i are independent variables over k [4, Thm. 3.2]. Hence the tangent cone of $\mathrm{Pic}_X^{=0}$ at the point $A^d(y)$ is not a complete intersection and we have proved:

Proposition 4.3.4. Let X be as in Proposition 4.3.2. Then there exists a point of $\mathrm{Pic}_X^{=0}$ where the tangent cone is not a complete intersection.

Remark 4.3.5. Set $n = 2$ in Prop. 4.3.2. In this case we can show that

$$I^* = (V_0^2, V_0V_1, V_1t_0 - V_0t_1)$$

where $t_i \in m^3$, and I^* is generated by the maximal minors of

$$\begin{pmatrix} V_0 & t_0 & t_1 \\ 0 & V_0 & V_1 \end{pmatrix}.$$

Hence A/I^* is Cohen-Macaulay [15, Cor. 4].

It is an open question if the tangent cone at each point of $\mathrm{Pic}_X^{=0}$ is Cohen-Macaulay if X lies on a smooth surface.

CHAPTER V.

Reducibility of the compactified Picard scheme.

Let X be an irreducible curve of arithmetitic genus p. Set $\overline{P} = \underline{\underline{\mathrm{Pic}}}^{o}_{X}$. Altman, Iarrobino and Kleiman proved an irreducibility theorem [1, Theorem (9)]: $\overline{P}$ is irreducible if X lies on a smooth surface, or equivalently, if the embedding dimension at each point of X is at most two [3, Corollary (9)]. They also constructed an example [1, Example (13)] of an X, which is a complete intersection in $\mathbb{P}^3$ and for which $\overline{P}$ is reducible. The example suggests that the converse of the theorem holds, and in this chapter we prove that if X does not lie on a smooth surface, then $\overline{P}$ is reducible.

Rego [22] asserted the reducibility theorem and offered a sketchy proof. First he showed that $\mathrm{Hilb}^2(X/k)$ is reducible if X does not lie on a smooth surface. Then, if X is also Gorenstein, he concluded that $\overline{P}$ is reducible from the fact that the Abel map

$$\mathrm{Hilb}^n(X/k) \to \overline{P}$$

is smooth for large n. This map is no longer smooth if X is not Gorenstein, and so Rego devised other methods to obtain reducibility in general.

However, Altman and Kleiman [2] developed a theory in which $\mathrm{Quot}^n(\omega/X/k)$, where ω is the dualizing sheaf of X , replaces $\mathrm{Hilb}^n(X/k)$ as the source of an Abel map

$$A^n : \mathrm{Quot}^n(\omega/X/k) \to \overline{P} .$$

Whether or not X is Gorenstein, A^n is smooth and its fibers are projective spaces for all $n \geq 2p - 1$. Hence $\overline{P}$ will be reducible if $\mathrm{Quot}^n(\omega/X/k)$ is reducible for large n .

This reducibility is proved below in two steps. First, we show that if $\mathrm{Quot}^m(\omega/X/k)$ is reducible, then $\mathrm{Quot}^n(\omega/X/k)$ is reducible for $n \geq m$ [Proposition 5.1.2]. Secondly, we show that if X does not lie on a smooth surface, then $\mathrm{Quot}^d(\omega/X/k)$ is reducible for small d , in fact for $d = 2$ if X is Gorenstein, and for $d = 1$ if X is not Gorenstein [Proposition 5.2.1].

5.1.

Fix a torsion-free, rank-1 sheaf G on X . Denote by U the open subscheme of X consisting of nonsingular points. There is an open subscheme Q^n_U of $\mathrm{Quot}^n(G/X/k)$, which parameterizes quotients of G with support contained in U [13, Exp. 221, 4a]. Since $G|_U$ is invertible, Q_U^n is isomorphic to $\mathrm{Hilb}^n(U/k)$, so Q_U^n is irreducible of dimension n [1, Lemma (1)]. Hence $\mathrm{Quot}^n(G/X/k)$ is irreducible if and only if Q_U^n is dense in $\mathrm{Quot}^n(G/X/k)$. Using the valuative criterion [12, Ch. II, Prop. 7.1.4 (i)], we therefore get Lemma 5.1.1 below:

Lemma 5.1.1. $\mathrm{Quot}^n(G/X/k)$ is irreducible if and only if, for all quotients F of G of length n , there exists a scheme $T = \mathrm{Spec}(A)$, where A is a complete, discrete valuation ring, and a T-flat quotient $\overline{F}$ of G_T such that

$$\overline{F}(t) \simeq F$$

and

$$\mathrm{Supp}\overline{F}(\eta) \subseteq U_T(\eta) .$$

Here t and η denote the closed and generic points of T .

Proposition 5.1.2. If $\mathrm{Quot}^n(G/X/k)$ is irreducible, then $\mathrm{Quot}^m(G/X/k)$ is irreducible for all $m < n$.

Proof. Let F be a quotient of G of length m . Let I denote the kernel of the natural map $G \to F$, and let $x_1,\ldots,x_{n-m}$ be different nonsingular points of X such that $x_i \notin \mathrm{Supp}F$ for $i = 1,\ldots,n-m$. Then

$$F' = G/M_1\cdots M_{n-m}I$$

where M_i denotes the ideal of x_i, is a quotient of G of length n. By Lemma 5.1.1 there exists a complete, discrete valuation ring A and a quotient $\overline{F}'$ of G_T, $T = \mathrm{Spec}(A)$, with all the properties listed in that lemma and such that

$$\overline{F}'(t) \simeq F' .$$

Let W be the closed subscheme of X_T defined by the annihilator of $\overline{F}'$, i.e. W is defined by the sheaf of ideals J where J is the kernel of the natural homomorphism

$$O_{X_T} \to \underline{\mathrm{Hom}}_{X_T}(\overline{F}',\overline{F}') .$$

The remaining part of the proof proceeds by steps.

Step 1. We have an inclusion

$$x_1 \cup \cdots \cup x_{n-m} \cup V \subseteq W(t)$$

where V is the closed subscheme of X defined by the annihilator of F.

<u>Proof.</u> Restricting the exact sequence

$$0 \to J \to O_{X_T} \to \mathrm{Hom}_{X_T}(\overline{F}',\overline{F}')$$

to $X_T(t) \simeq X$ gives a sequence

$$J(t) \to O_X \to \mathrm{Hom}_X(F',F') \ .$$

The image of $J(t)$ in O_X is the ideal defining $W(t)$ as a subscheme of X. Hence the subscheme of X defined by the annihilator of F' is contained in $W(t)$, and this proves Step 1.

<u>Step 2.</u> W can be written as

$$W = W_1 \oplus \cdots \oplus W_{n-m} \oplus W'$$

where $x_i \in W_i(t)$ and $V \subseteq W'(t)$.

<u>Proof.</u> A is a henselian ring [12, Ch. IV, Prop. 18.5.14], and hence the asserted decomposition follows from [12, Ch. IV, Thm. 18.5.11 (c)].

<u>Step 3.</u> Let i denote the inclusion $W' \subseteq X_T$. Define $\overline{F}$ by

$$\overline{F} = i_* i^* \overline{F}' \ .$$

Then $\overline{F}$ is a T-flat quotient of $\overline{F}'$.

Proof. Let x be a closed point of X_T. If $x \notin W'$, then $\overline{F}_x = (0)$. If $x \in W'$, then

$$\overline{F}_x \simeq \overline{F}'_x / J_x \overline{F}'_x ,$$

so $\overline{F}_x = \overline{F}'_x$ since J_x is the annihilator of $\overline{F}'_x$ in $O_{X_T, x}$. It follows that the natural map

$$\overline{F}' \to \overline{F}$$

is surjective and that $\overline{F}$ is T-flat.

Step 4. $\overline{F}(t) \simeq F$ and $\mathrm{Supp}\overline{F}(\eta) \subseteq U_T(\eta)$.

Proof. $\mathrm{Supp}\overline{F}'(\eta) \subseteq U_T(\eta)$ by the definition of $\overline{F}'$, so $\mathrm{Supp}\overline{F}(\eta) \subseteq U_T(\eta)$ since $\overline{F}$ is a quotient of $\overline{F}'$ [Step 3].

Since $i : W' \subseteq X_T$ is an affine morphism, the commutative diagram

$$\begin{array}{ccc} W' & \overset{i}{\subseteq} & X_T \\ \cup| & & \cup| \\ W'(t) & \underset{i(t)}{\subseteq} & X \end{array}$$

shows that

$$\overline{F}(t) \simeq i(t)_* i(t)^* \overline{F}'(t) .$$

Hence we get that

$$\overline{F}(t) \simeq G/M_1 \cdots M_{n-m} I + CG$$

where C is the ideal defining $W'(t)$ as a closed subscheme of X. By Step 2,

$$C \subseteq \mathrm{Ann}_{O_X}(F)$$

and therefore $CG \subseteq I$, so we have an inclusion

$$M_1 \cdots M_{n-m} I + CG \subseteq I .$$

Since $x_i \notin V$, the ideals $M_1 \cdots M_{n-m}$ and C are co-maximal, and hence we also have inclusions

$$I \subseteq M_1 \cdots M_{n-m} I + CI \subseteq M_1 \cdots M_{n-m} I + CG .$$

It follows that $\overline{F}(t) \simeq G/I = F$.

<u>Step 5.</u> $\mathrm{Quot}^m(G/X/k)$ is irreducible.

<u>Proof.</u> Let F be a quotient of G of length m. Let $T = \mathrm{Spec}(A)$, A a complete, discrete valuation ring, and let $\overline{F}$ be the quotient of G_T constructed in Step 3. By Step 4, $\overline{F}(t) \simeq F$ and $\mathrm{Supp}\overline{F}(\eta) \subseteq U_T(\eta)$. Hence the assertion follows from Lemma 5.1.1.

5.2.

Let ω denote the dualizing sheaf of X .

<u>Proposition 5.2.1.</u> Let x be a closed point of X and denote by M the ideal defining x .

(a). If $\dim_k(\omega/M\omega) \geq 2$, then $\mathrm{Quot}^1(\omega/X/k)$ is reducible.

(b). If $\dim_k(\omega/M\omega) = 1$ and $\dim_k(M/M^2) \geq 3$, then $\mathrm{Quot}^2(\omega/X/k)$ is reducible.

<u>Proof.</u> (a). Set $\omega_1 = \omega/M\omega$. Obviously, the functors $\underline{\mathrm{Quot}}^1(\omega_1/X/k)$ and $\underline{\mathrm{Grass}}_1(\omega_1/k)$ are isomorphic. Since $\dim_k(\omega_1) \geq 2$, $\mathrm{Grass}_1(\omega_1/k)$ has dimension at least 1 . Hence, since $\mathrm{Quot}^1(\omega_1/X/k)$ is a closed subscheme of $\mathrm{Quot}^1(\omega/X/k)$, we therefore get

$$\dim\mathrm{Quot}^1(\omega/X/k) \geq 1 .$$

If equality holds, $\mathrm{Quot}^1(\omega/X/k)$ is reducible since $\mathrm{Quot}^1(\omega_1/X/k)$ is a closed 1-dimensional subscheme. If equality fails, then the closure of Q_U^n is a component of $\mathrm{Quot}^1(\omega/X/k)$ of dimension 1 , and so $\mathrm{Quot}^1(\omega/X/k)$ is reducible.

(b). Since ω is torsion-free [4, 2.8, page 8], ω is invertible at x because $\dim_k(\omega/M\omega) = 1$. Since

$\dim_k(M/M^2) \geq 3$, we get that

$$\dim_k(M\omega/M^2\omega) \geq 3 .$$

Set $\omega_2 = \omega/M^2\omega$. A vector subspace of $M\omega/M^2\omega$ of codimension 1 corresponds to a quotient of ω_2 of length 2. It is not hard to see that this correspondence extends to families of quotients and vector subspaces, so that $\underline{\text{Grass}}_1([M\omega/M^2\omega]/k)$ can be considered as a subfunctor of $\underline{\text{Quot}}^2(\omega_2/X/k)$. Hence, since a proper monomorphism is a closed embedding [12, Ch. IV, Prop. 8.11.5], $\text{Quot}^2(\omega_2/X/k)$ contains $\text{Grass}_1([M\omega/M^2\omega]/k)$. Since the latter has dimension at least two, reasoning as in the proof of (a) we conclude that $\text{Quot}^2(\omega/X/k)$ is reducible.

We say that X has embedding dimension n at x if $\dim_k(M/M^2) = n$. Since an integral curve with embedding dimension at most 2 at each point can be embedded in a smooth surface [3, Cor. (9)], we have that X lies on a smooth surface if and only if the embedding dimension at each point is at most 2 .

As an immediate consequence of Proposition 5.1.2 and Proposition 5.2.1 we get:

<u>Proposition 5.2.2.</u> If X does not lie on a smooth surface, then $\text{Quot}^2(\omega/X/k)$ is reducible for $d \geq 2$.

<u>Lemma 5.2.3.</u> Suppose that $\overline{P}$ is irreducible. Then $\mathrm{Quot}^d(\omega/X/k)$ is irreducible for all $d \geq 1$.

<u>Proof.</u> The Abel map

$$A^d : \mathrm{Quot}^d(\omega/X/k) \rightarrow \overline{P}$$

is smooth with integral fibers if $d \geq 2p - 1$. Therefore $\mathrm{Quot}^d(\omega/X/k)$ is connected and hence irreducible for $d \geq 2p - 1$ [4, Theorem 1.8]. It follows from Proposition 5.1.2 that $\mathrm{Quot}^d(\omega/X/k)$ is irreducible for all $d \geq 1$.

<u>Theorem 5.2.4.</u> If X does not lie on a smooth surface, then the compactified Picard scheme $\overline{P}$ is reducible.

<u>Proof.</u> Proposition 5.2.2 gives that $\mathrm{Quot}^d(\omega/X/k)$ is reducible for $d \geq 2$. Hence, by Lemma 5.2.3, $\overline{P}$ is reducible.

CHAPTER VI

Results on the boundary points of $\mathrm{Pic}_X^{=0}$.

Let X be a curve lying on a smooth surface (or equivalently, $\mathrm{Pic}_X^{=0}$ is irreducible). Briacon, Granger and Speder [8] showed that the singular points of $\mathrm{Hilb}^n(X/k)$ are exactly the points corresponding to subschemes of X defined by ideals, which are not principal. Using the smoothness of the Abel map

$$A^n : \mathrm{Hilb}^n(X/k) \to \mathrm{Pic}_X^{=0}$$

for large n , we get that a point of $\mathrm{Pic}_X^{=0}$, which does not lie in Pic_X^0 , is a singular point of $\mathrm{Pic}_X^{=0}$.

In Section 6.2 we study the orbits of $\mathrm{Pic}_X^{=0}$ under the action of Pic_X^0 defined by tensor product. In the case that $\delta(\overline{X},X,Q)$ is at most one at each point $Q \in X$, we show that there are $\binom{\delta}{\ell}$ orbits of codimension ℓ in $\mathrm{Pic}_X^{=0}$ for each ℓ , $1 \leq \ell \leq \delta(\overline{X},X)$. Here $\overline{X}$ denotes the normalization of X .

D' Souza [10] studied the analytic structure of $\mathrm{Pic}_X^{=0}$ in the case that the singularities of X are ordinary double points. He showed that the completion of the local ring of $\mathrm{Pic}_X^{=0}$ at a singular point is of the form

$$k[[T_1,\ldots,T_r]]/(T_1T_2,\ldots,T_{2\ell-1}T_{2\ell})$$

where ℓ is an integer less or equal to the number of singular points of X .

We determine the analytic structure of the singularities of $\mathrm{Pic}_X^{=o}$ in the case that $\delta(\overline{X},X,Q)$ is at most one at each point $Q \in X$, and we show how the singularities are distributed on the $\sum_{\ell=1}^{\delta}\binom{\delta}{\ell}$ orbits of $\mathrm{Pic}_X^{=o}$. The completion of the local ring at a point in an orbit of codimension ℓ is of the form

$$k[[T_1,\ldots,T_r]]/(T_1T_2,\ldots,T_{2s-1}T_{2s},T_{2s+1}^2-T_{2s+2}^3,\ldots,T_{2\ell-1}^2-T_{2\ell}^3)$$

where s is a number less or equal to the number of nodes on X .

6.1.

Let X be a curve lying on a smooth surface S . In the characterization of the singularities of $\mathrm{Hilb}^n(X/k)$ in [8], Briaçon, Granger and Speder used a theory of "flattening" developed by Hironaka and Tessier. However, in a remark they pointed out that one can avoid the use of "flattening" by using the fact that an ideal of height 2 in a regular, 2-dimensional ring can be generated by the maximal minors of an $n \times (n+1)$ matrix. Following this approach, the proof of [8, Prop. II.2] becomes short and elegant.

Lemma 6.1.1. [8, Ch. II, Remarque]. Let A be a regular, local ring of dimension 2, and let $I \subseteq A$ be an ideal of height 2. Let

$$\varphi : I \to A/I$$

be an A-module homomorphism. Then, if I is not a complete intersection in A, $\varphi(I)$ is contained in M/I where M denotes the maximal ideal of A.

Proof. Set $p + 1 = \dim_k(I/MI)$, and lift a basis $\bar{i}_o,\ldots,\bar{i}_p$ of I/MI to a set of generators $i_o,\ldots,i_p$ of I. Let $\varphi \in \mathrm{Hom}_A(I,A/I)$ and suppose that $\varphi(i_o) \not\subseteq M/I$. Let a_t be an element of A such that the residue class of a_t modulo I is equal to $\varphi(i_t)/\varphi(i_o)$, $t = 1,\ldots,p$. Then $i'_o,\ldots,i'_p$, where $i'_o = i_o$ and $i'_t = i_t - a_t i_o$, is a minimal set of generators of I, and by [9, Thm. 5], $i'_o,\ldots,i'_p$ are the maximal minors of an $(p + 1) \times p$ matrix $R = (r_{ij})$, $r_{ij} \in A$. Since $i'_o,\ldots,i'_p$ form a minimal set of generators of I, therefore $r_{ij} \in M$.

If $p \geq 2$, $(i'_1,\ldots,i'_p) \subseteq M(r_{o1},\ldots,r_{op})$, so there exists an integer j such that $r_{oj} \not\subseteq I$ because none of $i'_1,\ldots,i'_p$ is in MI. On the other hand, $r_{oj}i'_o + \ldots + r_{pj}i'_p = 0$ implies that $r_{oj} \in I$, which is a contradiction, so if $\varphi(i'_o) \not\subseteq M/I$, I is a complete intersection.

<u>Proposition 6.1.2.</u> (Briaçon, Granger, Speder). Let q be a point of $H = \mathrm{Hilb}^n(X/k)$ such that the closed subscheme σ_q of X corresponding to q is not defined by an invertible ideal. Then q is a singular point of H .

<u>Proof.</u> Suppose that σ_q can be written as a disjoint union $\sigma_q = \sigma_1 \cup \ldots \cup \sigma_\ell$. Then $\hat{O}_{H,q} = \hat{O}_{H_1,q_1} \otimes \ldots \otimes \hat{O}_{H_1,q_n}$ where q_i is the point of $H_i = \mathrm{Hilb}^{n_i}(X/k)$ corresponding to σ_i [Lemma 4.3.3]. Hence we may assume that σ is supported at one point Q of X .

Set $A = O_{S,Q}$ and denote by M the maximal ideal of A . Then $O_{X,Q} = A/(f) = \overline{A}$ for an element $f \in A$. We denote by I the ideal in A corresponding to σ_q , and we set $\overline{I} = I/(f)$.

Let $\varphi \in \mathrm{Hom}_A(I,A/I)$. If I is a complete intersection generated by f_1,f_2 , then f is of the form $f = a_1f_1 + a_2f_2$, and $a_1,a_2 \in M$ because $\overline{I}$ is not a principal ideal. Hence $\varphi(f) \in M/I$. On the other hand, if I is not a complete intersection, then $\varphi(f) \in M/I$ by [Lemma 6.1.1].

The Zariski tangent spaces of $\mathrm{Hilb}^n(S/k)$ and $\mathrm{Hilb}^n(X/k)$ at q are isomorphic to $\mathrm{Hom}_A(I,A/I)$ and

$\mathrm{Hom}_{\overline{A}}(\overline{I},\overline{A}/\overline{I})$ [13, Exp. 221, Cor. 5.3]. The vector subspace $\mathrm{Hom}_{\overline{A}}(\overline{I},\overline{A}/\overline{I})$ of $\mathrm{Hom}_A(I,A/I)$ consists of elements $\varphi \in \mathrm{Hom}_A(I,A/I)$ such that $\varphi(f) = 0$. Since S is smooth

$$\dim_k \mathrm{Hom}_A(I,A/I) = 2n$$

[1, Prop. (3)].

Let $\beta = \{\varphi_1,\ldots,\varphi_{2n}\}$ be a basis of $\mathrm{Hom}_A(I,A/I)$. Since $\varphi_i(f) \in M/I$, $\varphi_i(f) = \sum_{j=1}^{n-1} b_{ij}t_j$ where $b_{ij} \in k$ and $t_1,\ldots,t_{n-1}$ is a basis of M/I . Set $B_\ell = (b_{1\ell},\ldots,b_{2n\ell})$, $\ell = 1,\ldots,n-1$. An element $\varphi \in \mathrm{Hom}_A(I,A/I)$ lies in $\mathrm{Hom}_{\overline{A}}(\overline{I},\overline{A}/\overline{I})$ if and only if the coordinates of φ relative to β is an element of the orthogonal space of $B_1,\ldots,B_{n-1}$. Hence

$$\dim_k \mathrm{Hom}_{\overline{A}}(\overline{I},\overline{A}/\overline{I}) \geq n + 1$$

and since $\dim H = n$ [1, Cor. (7)], q is a singular point of H .

<u>Theorem 6.1.3.</u> The boundary points of Pic_X^o in the compactification $\overline{\overline{\mathrm{Pic}}}{}_X^o$ are singular points.

<u>Proof.</u> Let p denote the arithmetic genus of X an fix an integer $n \geq 2p - 1$. Let q be a point of

$Hilb^n(X/k)$, which map to a boundary point of $Pic_X^{=o}$ by the Abel map

$$A^n : Hilb^n(X/k) \to Pic_X^{=o} .$$

The subscheme of X corresponding to q is defined by an ideal, which is not invertible. By Prop. 6.1.2, q is a singular point of $Hilb^n(X/k)$, and since A^n is smooth, $A^n(q)$ is a singular point of $Pic_X^{=o}$. Since A^n is surjective, all the boundary points of $Pic_X^{=o}$ are singular.

6.2.

Let X be an irreducible curve with m singularities $Q_1,\ldots,Q_m$ and suppose that $\delta(\overline{X},X,Q_i) = 1$. Let X' be the desingularization of ℓ of the points of X , say $Q_1,\ldots,Q_\ell$. Denote by $M_1,\ldots,M_\ell$ the ideals of $Q_1,\ldots,Q_\ell$. Set $M = M_1 \otimes \ldots \otimes M_\ell$ and put $I = M \otimes_{O_X} J$ where J is an invertible O_X-Module of degree ℓ . Denote by Q the point of $Pic_X^{=o}$ corresponding to I .

<u>Lemma 6.2.1.</u> The orbit $O(q)$ of q under the action of Pic_X^o has codimension ℓ in $Pic_X^{=o}$.

<u>Proof.</u> Since M is the conductor of O_X in $O_{X'}$ [10, Ch. III, Rem. 1.3], I is an $O_{X'}$-Module and the

tensor product defines a map

$$\psi : \mathrm{Pic}^{\circ}_{X'} \times q \to \mathrm{Pic}^{=\circ}_{X} .$$

Since every invertible $O_{X'}$-Module L is of the form $F \otimes_{O_X} O_{X'}$ where F is an invertible O_X-Module, the image of ψ is equal to $O(q)$.

Suppose that $I \otimes_{O_{X'}} L \simeq I \otimes_{O_{X'}} L'$ where L and L' are invertible $O_{X'}$-Modules of degree 0 . Since J is an invertible O_X-Module, tensoring by J^{-1} gives an isomorphism

$$M \otimes_{O_{X'}} L \simeq M \otimes_{O_{X'}} L' .$$

But M is an invertible $O_{X'}$-Module, so $L \simeq L'$, and therefore the morphism ψ has zero-dimensional fibers. Hence $\dim O(q) = \dim \mathrm{Pic}^{\circ}_{X'}$, and

$$\dim \mathrm{Pic}^{\circ}_{X'} = \dim \mathrm{Pic}^{=\circ}_{X} - \ell$$

because Pic°_{X} is dense in $\mathrm{Pic}^{=\circ}_{X}$ [1, Thm. (9)]. It follows that $O(q)$ has codimension ℓ in Pic°_{X} .

<u>Proposition 6.2.2.</u> $\mathrm{Pic}^{=\circ}_{X}$ has $\binom{m}{\ell}$ orbits of codimension ℓ , each given by the action of Pic°_{X} on a point q of $\mathrm{Pic}^{=\circ}_{X}$ corresponding to a torsion-free, rank-1 sheaf on X of the form $I = M_{t_1} \otimes \ldots \otimes M_{t_\ell} \otimes J$ where J is an invertible O_X-Module of degree ℓ .

The completion of the local ring of $\mathrm{Pic}_X^{=0}$ at a point of $O(q)$ is of the form

$$k[[T_1,\ldots,T_v]]/(T_1T_2,\ldots,T_{2s-1}T_{2s},T_{2s+1}^2 - T_{2s+2}^3,\ldots,T_{2\ell-1}^2 - T_{2\ell}^3)$$

where s is the number of nodes and $\ell - s$ is the number of cusps among the points $Q_{t_1},\ldots,Q_{t_\ell}$.

Proof. Let F be a torsion-free, rank-1 sheaf on X. There exists an invertible O_X-Module L such that $F \otimes_{O_X} L \subseteq O_X$ [2, Lemma 3.3]. Let $Q_{t_1},\ldots,Q_{t_\ell}$ be the points of X where F is not invertible. Then $F \otimes_{O_X} L$ is of the form $M_{t_1} \otimes \ldots \otimes M_{t_\ell} \otimes I'$ where I' is invertible [10, Ch. III, Lemma 1.4]. Hence every torsion-free, rank-1 sheaf I on X, which is not invertible at $Q_{t_1},\ldots,Q_{t_\ell}$ is of the form

$$I = M_{t_1} \otimes \ldots \otimes M_{t_\ell} \otimes J .$$

There are $\binom{m}{\ell}$ different subsets of $\{Q_1,\ldots,Q_m\}$ consisting of ℓ points, and hence there are $\binom{m}{\ell}$ orbits $O(q)$ of points q corresponding to torsion-free, rank-1 sheaves on X, which are not invertible at ℓ points. Each such orbit has codimension ℓ in $\mathrm{Pic}_X^{=0}$ [Lemma 6.2.1].

The point of $\mathrm{Pic}_X^{=0}$ corresponding to

$$I = M_{t_1} \otimes \ldots \otimes M_{t_\ell} \otimes J$$

is in the image of the Abel map A^n of a point q' of $\mathrm{Hilb}^n(X/k)$ corresponding to a subscheme $Q_1 \cup \dots \cup Q_\ell \cup V$ where $Q_i \not\subset V$. Using Lemma 4.3.3 and the fact that $\mathrm{Hilb}^1(X/k) \simeq X$ [2, Lemma 8.7], we get that the completion of the local ring of $\mathrm{Hilb}^n(X/k)$ at q is isomorphic to

$$k[[T_1,\dots,T_r]]/(T_1T_2,\dots,T_{2s-1}T_{2s},T_{2s+1}^2 - T_{2s+2}^3,\dots,T_{2\ell-1}^2 - T_{2\ell}^3) .$$

Hence, since A^n is smooth for large n, the completion of the local ring of $\mathrm{Pic}_X^{=0}$ at q is of the desired form.

CHAPTER VII.

The structure of compactifications.

Let X be an irreducible curve of arithmetic genus p. In some special cases the structure of $\mathrm{Pic}_X^{=0}$ is known. For example, if $p = 1$, $\mathrm{Pic}_X^{=0} \simeq X$ [2, Example 8.9 (iii)]. If $p = 2$ and X has one ordinary node as only singularity, then Oda and Seshadri [20, Ex. (1), page 83] showed that $\mathrm{Pic}_X^{=0}$ is obtained from the $\mathbb{P}^1$-bundle $\mathrm{Pres}_{X'/X}$ over $\mathrm{Pic}_{\overline{X}}^{0} \simeq \overline{X}$ as follows: Let Q_1 and Q_2 be the points of $\overline{X}$, which map to the singular point of X. Then $\mathrm{Pic}_X^{=0}$ is obtained from $\mathrm{Pres}_{\overline{X}/X} \simeq \mathbb{P}(\mathcal{O}_{\overline{X}} \oplus \mathcal{O}_{\overline{X}})$ by identifying the 0-section and the ∞-section via the translation in $\overline{X}$ by its point $Q_1 - Q_2$.

In this chapter we give an explicit construction of $\mathrm{Pic}_X^{=0}$ in the case that X has ordinary nodes as only singularities. The main tool in this construction is a generalized presentation functor $\underline{\mathrm{GPres}}_{Y'/Y}$ where $Y' \to Y$ is a surjective, birational morphism of curves. The source I of a generalized presentation

$$(f_S)_* I \to N$$

over S lies in $\underline{\mathrm{Pic}}^{=0}_{X'}(S)$. If $X' \to X$ is the desingularization of one of the points of X , we show that $\mathrm{GPres}_{X'/X}$ is a $\mathbb{P}^1$-bundle over $\mathrm{Pic}^{=0}_{X'}$ and that $\mathrm{Pic}^{=0}_{X}$ is obtained from this $\mathbb{P}^1$-bundle by identifying two sections via a translation in $\mathrm{Pic}^{=0}_{X'}$.

In the last section of this chapter we study $\mathrm{Pic}^{=0}_{X}$ for a curve X such that $\delta(\overline{X},X) = 2$. We give an explicit description of the underlying topological space of $\mathrm{Pic}^{=0}_{X}$ in the case that $p = 2$, $\overline{X} = \mathbb{P}^1$ and X has only one singularity, which is a tacnode.

7.1.

Let $f : X' \to X$ be a surjective, birational morphism of irreducible curves. Denote by C the conductor of O_X in $O_{X'}$ and set $\delta = \delta(X',X)$. Let S be a k-scheme and F an O_{X_S}-Module. We denote by CF the image of $C \otimes_{X_S} F \to F$. A <u>generalized presentation</u> over S is a surjective O_{X_S}-Module homomorphism

$$\varphi : (f_S)_* I \to N$$

where $I \in \underline{\mathrm{Pic}}^{=0}_{X'}(S)$, $CI \subseteq \ker\varphi$ and N is a locally free O_S-Module of rank δ . Equivalent presentations and the pullback $\varphi_{S'}$ by a k-morphism $S' \to S$ are defined as in Section 2.1.

Definition 7.1.1. We define a functor $\underline{GPres}_{X'/X}$ as follows: For each k-scheme S , let

$$\underline{GPres}_{X'/X}(S)$$

be the set of isomorphism classes of generalized presentations over S .

Set $P = \mathrm{Pic}^{=0}_{X'}$ and let $\mathcal{P}$ denote a universal torsion-free, rank-1 sheaf on $X' \times P$.

Proposition 7.1.2. The functor $\underline{GPres}_{X'/X}$ is represented by a projective scheme over P .

Proof. Let Z denote the closed subscheme of X defined by the conductor C and denote by $i : Z \to X$ the inclusion. Let S be a k-scheme and F an O_{X_S}-Module. We denote by F(C) the pullback i_S^*F .

We will show that $\underline{GPres}_{X'/X}$ is isomorphic to $\mathrm{Quot}^{\delta}([(f_P)_*\mathcal{P}](C)/Z \times P/P)$.

Let

$$\varphi : (f_S)_*L \to N$$

be an element of $\underline{GPres}_{X'/X}(S)$. There exists a morphism $q : S \to P$, an invertible O_S-Module T and an isomorphism

$$\alpha : (q_{X'})^*\mathcal{P} \to L \otimes_{O_S} T .$$

The presentation

$$(f_S)_*(L \otimes_{O_S} T) \to N \otimes_{O_S} T$$

is equivalent to φ . Hence the generalized presentation φ gives rise to a morphism $q : S \to P$ and a generalized presentation

$$\varphi_1 : (f_S)_*[(q_{X'})^*\mathcal{P}] \to M .$$

As in the proof of Proposition 2.2.1, φ_1 corresponds to a surjective O_{X_S}-homomorphism

$$\varphi_2 : (q_Z)^*[(f_P)_*\mathcal{P}](C) \to M ,$$

which is an element of $\underline{\text{Quot}}^{\delta}([(f_P)_*\mathcal{P}](C)/Z \times P/P)(S)$. Moreover, if

$$\alpha' : (q_{X'})^*\mathcal{P} \to L \otimes_{O_S} T$$

is another isomorphism, we get an element φ_2' in $\underline{\text{Quot}}^{\delta}([(f_P)_*\mathcal{P}](C)/Z \times P/P)(S)$, which is equivalent to φ_2 . Hence we have a map

$$\rho : \underline{\text{GPres}}_{X'/X} \to \underline{\text{Quot}}^{\delta}([(f_P)_*\mathcal{P}](C)/Z \times P/P)(S) .$$

It is easy to see that the map

$$\underline{\text{Quot}}^{\delta}([(f_P)_*\mathcal{P}](C)/Z \times P/P) \to \underline{\text{GPres}}_{X'/X} ,$$

which sends

$$(q_Z)^*([(f_P)_*\mathcal{P}](C)) \to M$$

to the generalized presentation

$$(f_S)_*[(q_{X'})^*\mathcal{P}] \to M$$

is an inverse of ρ .

The proof of the proposition is now completed since $\underline{\text{Quot}}^{\delta}([(f_P)_*\mathcal{P}](C)/Z \times P/P)$ is represented by a projective scheme over P [13, Exp. 221, Thm. 3.2].

<u>Corollary 7.1.3.</u> Let $f : X' \to X$ be the desingularization of one point Q of X , and suppose that C is equal to the maximal ideal M of Q . Then $\text{GPres}_{X'/X}$ is a $\mathbb{P}^{\delta}$ -bundle over P .

<u>Proof.</u> The functor $\underline{\text{Quot}}^{\delta}([(f_P)_*\mathcal{P}](M)/Q \times P/P)$ is isomorphic to $\underline{\text{Grass}}_{\delta}([(f_P)_*\mathcal{P}](M)/P)$. Since $[(f_P)_*\mathcal{P}](M)$ is a locally free O_P-Module of rank $\delta + 1$, $\underline{\text{Grass}}_{\delta}([(f_P)_*\mathcal{P}](M)/P)$ is represented by a $\mathbb{P}^{\delta}$ -bundle over P [17, Prop. 1.2 and Prop. 1.6].

Let

$$\underline{K} : \underline{GPres}_{X'/X} \to \underline{Pic}_X^{=o}$$

be the map, which sends a generalized presentation φ to $\ker\varphi$. The corresponding morphism

$$K : GPres_{X'/X} \to Pic_X^{=o}$$

is an isomorphism on $K^{-1}(Pic_X^o)$ [see Section 3.2 for the same property of the morphism $Pres_{X'/X} \to Pic_X^{=o}$].

Remark 7.1.4. The morphism $K : GPres_{X'/X} \to Pic_X^{=o}$ need not be surjective. For example, let X be a curve with one singularity Q such that $\delta(\overline{X},X,Q) = 2$ and such that there are three points $P_1,P_2,P_3 \in \overline{X}$, which map to Q. Then the conductor of O_X in $O_{\overline{X}}$ is the maximal ideal M of Q, and so $GPres_{\overline{X}/X}$ is a $\mathbb{P}^2$-bundle over $Pic_{\overline{X}}^o$ [Corollary 7.1.3]. Hence $GPres_{\overline{X}/X}$ is irreducible. On the other hand, since length of $(O_X/M) \neq$ length of $(O_{X'}/O_X)$, X is not Gorenstein [6, Cor. 6.5]. Therefore $Pic_X^{=o}$ is reducible by Theorem 5.2.4, and so K is not surjective.

In the next section we consider the case where X lies on a smooth surface. Then $Pic_X^{=o}$ is irreducible and K is surjective.

7.2.

Let X be an irreducible curve with ordinary double points as only singularities, and let $f : X' \to X$ be the desingularization of one of the double points $Q \in X$. We denote by Q_1 and Q_2 the points of X', which map to Q.

Suppose that $\overline{\overline{\mathrm{Pic}}}^{o}_{X'}$ is represented by a scheme P, and let $\mathcal{P}$ be a universal torsion-free, rank-1 sheaf on $X' \times P$.

Lemma 7.2.1. The underlying topological space of $\overline{\overline{\mathrm{Pic}}}^{o}_{X}$ is obtained by identifying the two sections $\mathbb{P}(\mathcal{P}(Q_1))$ and $\mathbb{P}(\mathcal{P}(Q_2))$ of the $\mathbb{P}^1$-bundle $\mathrm{GPres}_{X'/X} = \mathbb{P}(\mathcal{P}(Q_1) \oplus \mathcal{P}(Q_2))$ over P via a translation in P by the point of $\mathrm{Pic}^{o}_{X'}$ corresponding to $O_{X'}[Q_2 - Q_1]$.

Proof. Let

$$I \xrightarrow{\varphi} N$$

and

$$I' \xrightarrow{\varphi'} N'$$

be two generalized presentations over k. Set $J = \ker\varphi$ and $J' = \ker\varphi'$ and suppose that J is O_X-isomorphic to J' . If J and J' are invertible, then $\varphi = \varphi'$ because $K|_{K^{-1}(\mathrm{Pic}_X^o)}$ is an isomorphism onto Pic_X^o .

Suppose that J and J' are not invertible at Q . Then J and J' are $O_{X'}$-Modules of the form $J = I[-Q_i]$ and $J' = I[-Q_j]$ [10, Ch. III, Cor. 1.5]. Hence φ and φ' are of the form

$$I \xrightarrow{\varphi} I(Q_i) \ , \ I \xrightarrow{\varphi'} I(Q_j) \ .$$

If $i = j$, then $I \simeq I'$ because $I[-Q_i] \simeq I'[-Q_i]$, and hence $\varphi = \varphi'$.

Suppose that $i \neq j$, say $i = 1$ and $j = 2$. Since $I[-Q_1] \simeq I'[-Q_2]$, $I' \simeq I[Q_2 - Q_1]$. The point $q \in \mathbb{P}(\theta(Q_1))$ corresponding to φ is identified with the point $q' \in \mathbb{P}(\theta(Q_2))$ corresponding to φ' . Hence $\mathrm{Pic}_X^{=o}$ is obtained from the $\mathbb{P}^1$-bundle $\mathrm{GPres}_{X'/X}$ by identifying $\mathbb{P}(\theta(Q_1))$ and $\mathbb{P}(\theta(Q_2))$ via the translation in P by the point of $\mathrm{Pic}_{X'}^o$ corresponding to $O_{X'}[Q_2 - Q_1]$.

The quotient of $\mathrm{GPres}_{X'/X}$ in the category of topological spaces formed in Lemma 7.2.1, can be given the structure of a reduced k-scheme in many ways [Proposition 1.3.3]. However, in the case that X has ordinary double points as only singularities, we know the analytic structure of the singularities of $\underline{\underline{\mathrm{Pic}}}_X^o$ [Proposition 6.2.2], and this allows us to determine the scheme structure of $\underline{\underline{\mathrm{Pic}}}_X^o$ as follows:

Let O' be an orbit of $\underline{\underline{\mathrm{Pic}}}_{X'}^o$ of codimension ℓ. The completion of the local ring of $\underline{\underline{\mathrm{Pic}}}_{X'}^o$ at a point of O' is isomorphic to

$$k[[T_1,\ldots,T_v]]/(T_1T_2,\ldots,T_{2\ell-1}T_{2\ell})$$

[Proposition 6.2.2].

Set $V_i = \mathbb{P}(\mathcal{O}(Q_i))$ and $V = \pi^{-1}(O')$ where $\pi : \mathrm{GPres}_{X'/X} \to \underline{\underline{\mathrm{Pic}}}_{X'}^o$ denotes the natural projection. The identification of $V \cap V_1$ and $V \cap V_2$ is an orbit O of $\underline{\underline{\mathrm{Pic}}}_X^o$. Indeed, an O_X-Module corresponding to a point of $K(V \cap V_i)$ is an $O_{X'}$-Module [see the proof of Lemma 7.2.1], and every invertible $O_{X'}$-Module is of the form $L \otimes_{O_X} O_{X'}$ where L is an invertible O_X-Module. Moreover, O has codimension $\ell + 1$ in $\underline{\underline{\mathrm{Pic}}}_X^o$ since $\dim\underline{\underline{\mathrm{Pic}}}_X^o = \dim\underline{\underline{\mathrm{Pic}}}_{X'}^o + 1$.

The completion of the local ring of $\underline{\underline{\mathrm{Pic}}}_X^o$ at a point in O is isomorphic to

$$k[[T_1,\ldots,T_v]]/(T_1T_2,\ldots,T_{2\ell+1}T_{2\ell+2})$$

[Proposition 6.2.2]. Hence the δ-invariant of the morphism $K : GPres_{X'/X} \to Pic^{=o}_X$ is at most one at each point of $Pic^{=o}_X$. We have proved the following proposition:

<u>Proposition 7.2.2.</u> Let X be a curve with ordinary nodes as only singularities, and let

$$X_m = \overline{X} \to X_{m-1} \to \dots \to X_o = X$$

be a factorization of $\overline{X} \to X$ such that $\delta(X_i, X_{i-1}) = 1$. Then $Pic^{=o}_X$ can be constructed from $Pic^{o}_{\overline{X}}$ in m steps as follows: Suppose we have constructed $Pic^{=o}_{X_i}$. Then the underlying topological space of $Pic^{=o}_{X_{i-1}}$ is the quotient of $GPres_{X_i/X_{i-1}}$ constructed in Lemma 7.2.1, and if q_1 and q_2 are two points of $GPres_{X_i/X_{i-1}}$, which are identified to one point, the local ring of the resulting point of $Pic^{=o}_{X_{i-1}}$ is isomorphic to $k \oplus m_{q_1} \cap m_{q_2}$ where m_{q_i} denotes the ideal of q_i .

7.3.

Let X be an irreducible curve of arithmetic genus 2 such that the normalization $\overline{X}$ is equal to $\mathbb{P}^1$. Suppose that X has only one singular point, which is a tacnode. We can construct such a curve in the following way: Let X' be the plane, cubic nodal curve. Locally, X' is given by $\mathrm{Spec}(A)$ where

$$A = k[u_1,u_2] = k[U_1,U_2]/U_2^2 - U_1(U_1 + 1) .$$

Let ψ denote the composition

$$\psi : k[U_1,U_1U_2] \subseteq k[U_1,U_2] \to A .$$

The image of ψ is a subalgebra A' of A , and

$$\dim_k(A/A') = 1$$

because the elements of A not in A' are of the form cu_2 , $c \in k$.

Set $m = A' \cap (u_1,u_2)$. By Proposition 1.1.1, there exists a curve X , which is homeomorphic to X' , and which has one singular point Q where the local ring is isomorphic to A'_m .

The restriction of the morphism

$$K : \mathrm{Pres}_{X'/X} \to \mathrm{Pic}^{=0}_X$$

to $\mathrm{StPres}_{X'/X}$ is an isomorphism onto Pic^0_X [Lemma 3.1.2]. Let $\varphi \in \underline{\mathrm{Pres}}_{X'/X}(k)$, $\varphi \notin \underline{\mathrm{StPres}}_{X'/X}(k)$. Then φ is of the form

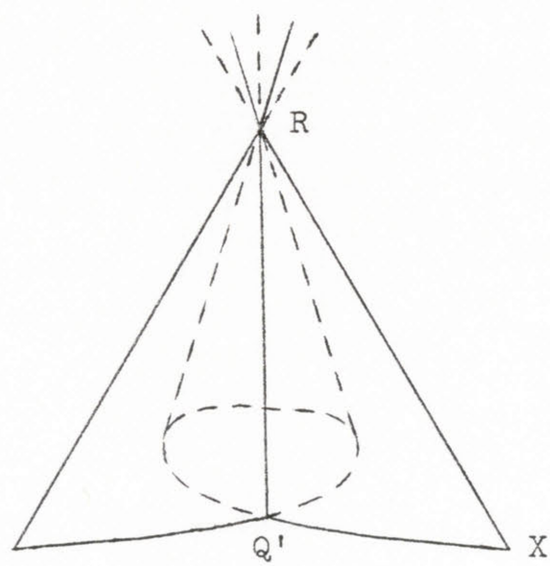
$$\varphi : f_*L \to L(Q')$$

where Q' is the singular point of X' and $L \in \underline{\mathrm{Pic}}^0_{X'}(k)$. Suppose that φ' is another presentation over k of the form

$$\varphi' : f_*L' \to L'(Q')$$

The $O_{X'}$-Modules $L'[-Q']$ and $L[-Q']$ are torsion-free, rank-1 of degree -1, which are not invertible. Since $\mathrm{Pic}^{=-1}_{X'} \simeq X'$ [2, Example 8.9 (iii)], $L'[-Q']$ and $L[-Q']$ correspond to the same point of $\mathrm{Pic}^{=-1}_{X'}$. Therefore $L'[-Q']$ is isomorphic to $L[-Q']$ as $O_{X'}$-Modules (and as O_X-Modules) and $K(q) = K(q')$ where q and q' are the points of $\mathrm{Pres}_{X'/X}$ corresponding to φ and φ'. Hence the image of K in $\mathrm{Pic}^{=0}_X$ is the cone over $\mathrm{Pic}^0_{X'} \simeq X' \setminus Q'$ obtained by identifying one section of the $\mathbb{P}^1$-bundle $\mathrm{Pres}_{X'/X}$ over $X' \setminus Q'$ to one point R.

The complement of Pic^0_X in $\mathrm{Pic}^{=0}_X$ is an irreducible scheme of codimension 1 [22, Theorem B], which passes through R. Therefore the underlying topological space of $\mathrm{Pic}^{=0}_X$ is a cone over $\mathrm{Pic}^{=0}_{X'} \simeq X'$.



CHAPTER VIII.

Reducibility of the moduli space of semi-stable, torsion-free sheaves on a singular curve.

Let X be a singular, integral curve. It has been verified by Newstead [19, Ch. 5, Thm. 5.8'] that there exists a projective scheme $M(n,d)$, which is a coarse moduli space for semi-stable, torsion-free O_X-Modules of rank n and degree d . The points of $M(n,d)$ corresponding to locally free O_X-Modules, form an open, irreducible subset [19, Rem. 5.9 (i)].

Rego [23] proved that if X lies on a smooth surface, then $M(n,d)$ is irreducible. Every torsion-free, rank-n sheaf on X is contained in O_X^n (by twisting if nescessary), and Rego obtained the irreducibility of $M(n,d)$ by showing that $\mathrm{Quot}^m(O_X^n/X/k)$ is irreducible for all $m \geq 1$ if X lies on a smooth surface.

In this chapter we prove that $M(n,\ell n)$, $\ell \in \mathbb{Z}$, is reducible if X does not lie on a smooth surface. Since every torsion-free, rank-1 sheaf is semi-stable, $M(1,0) = \overline{\mathrm{Pic}}_X^o$, and so we obtain another proof of Theorem 5.2.4.

The first step in the proof of reducibility of $M(n, \ell n)$ is to show that

$$\mathrm{Quot}^{tn}_{ss}(\omega^n/X/k)$$

is reducible for small t, in fact, for $t = 1$ if ω is not invertible and for $t = 2$ if ω is invertible. Here $\mathrm{Quot}_{ss}(\omega^n/X/k)$ denotes the open subscheme of $\mathrm{Quot}(\omega^n/X/k)$ parameterizing quotients N such that $\ker(\omega^n \to N)$ is semi-stable.

We show that the open subset $Q^{tn}_{F,ss}$ of $\mathrm{Quot}^{tn}_{ss}(\omega^n/X/k)$, parameterizing quotients N such that $\ker(\omega^n \to N)$ is locally free, is irreducible. Then, if q is a point of $\mathrm{Quot}^{tn}_{ss}(\omega^n/X/k)$, which does not lie on the component containing $Q^{tn}_{F,ss}$, the corresponding quotient N of ω^n has the property that $I = \ker(\omega^n \to N)$ is not deformable to a locally free sheaf. The degree of I is $n(2p - 2 - t)$ where p is the genus of X, and we get that $M(n, n(2p - 2 - t))$ is reducible.

Let $\ell \in \mathbb{Z}$. Tensoring by an invertible O_X-Module L with

$$\deg L = \ell + 2 + t - 2p ,$$

defines an isomorphism

$$M(n,n(2p-2-t)) \simeq M(n,\ell n) .$$

Hence $M(n,\ell n)$ is reducible for all $\ell \in \mathbb{Z}$.

8.1.

Let A be a complete, discrete valuation ring and set $S = \mathrm{Spec}(A)$. Denote by s any η the closed and generic points of S . Let I be an O_X-Module. An O_{X_S}-Module $\tilde{I}$ is called a <u>deformation</u> of I if it is S-flat and if

$$\tilde{I}(s) \simeq I .$$

We say that I can be deformed to a locally free sheaf if there exists a deformation $\tilde{I}$ of I such that $\tilde{I}(\eta)$ is locally free.

Let ω denote the dualizing sheaf on X , and denote by U the open subscheme of X consisting of nonsingular points. Let

$$Q_U^m$$

denote the open subscheme of $\mathrm{Quot}^m(\omega^n/X/k)$, which parameterizes quotients of ω^n with support contained in U . Rego [23, Prop. 1.2.0] showed that Q_U^m is irreducible of dimension nm . His proof runs as follows: Consider the map

$$\Lambda : \mathrm{Quot}^m(O_U^n/U/k) \to \mathrm{Hilb}^m(U/k)$$

defined by sending a quotient N of O_U^n to the subscheme of U defined by the ideal $\overset{n}{\Lambda}(\ker[O_U^n \to N])$. The fibers of Λ at points in the open subscheme H_{sm} of $\mathrm{Hilb}^m(U/k)$, corresponding to smooth subschemes of U, are isomorphic to $(\mathbb{P}^{n-1})^m$. Since $\mathrm{Hilb}^m(U/k)$ is irreducible of dimension m [1, Lemma (1)], the open subscheme $\Lambda^{-1}(H_{sm})$, which parameterizes quotients of O_U^n with support at m distinct points, is irreducible of dimension nm. Since every quotient of O_U^n of length m can be deformed to a quotient supported at m distinct points, $\Lambda^{-1}(H_{sm})$ is dense in $\mathrm{Quot}^m(O_U^n/U/k)$.

Clearly, $\mathrm{Quot}^m(\omega^n/X/k)$ is irreducible if and only if for each quotient F of ω^n of length m there exists a deformation $\widetilde{F}$ of F such that $\mathrm{Supp}\widetilde{F}(\eta) \subseteq U_S(\eta)$.

<u>Lemma 8.1.1.</u> Let x be a point of X and denote by M the ideal defining x.

(a) If $\dim_k(\omega/M\omega) \geq 2$, then $\mathrm{Quot}^n(\omega^n/X/k)$ is reducible.

(b) If $\dim_k(\omega/M\omega) = 1$ and if $\dim_k(M/M^2) \geq 3$, then $\mathrm{Quot}^{2n}(\omega^n/X/k)$ is reducible.

Proof. (a) Set $\omega_1 = \omega/M\omega$. Obviously the functors $\underline{\text{Quot}}^n(\omega_1^n/X/k)$ and $\underline{\text{Grass}}_n(\omega_1^n/k)$ are isomorphic. Since $\dim_k(\omega_1) \geq 2$, $\text{Grass}_n(\omega_1^n/k)$ has dimension at least n^2 . Hence, since $\text{Quot}^n(\omega_1^n/X/k)$ is a closed subscheme of $\text{Quot}^n(\omega^n/X/k)$, we therefore get

$$\dim\text{Quot}^n(\omega^n/X/k) \geq n^2 .$$

If equality holds, then $\text{Quot}^n(\omega^n/X/k)$ is reducible because $\text{Quot}^n(\omega_1/X/k)$ is a closed subscheme of dimension n^2 , which is obviously different from $\text{Quot}^n(\omega^n/X/k)$. If equality fails, the closure of Q_U^n in $\text{Quot}^n(\omega^n/X/k)$ is a component of dimension n^2 , and so $\text{Quot}^n(\omega^n/X/k)$ is reducible.

(b) Since ω is torsion-free, rank-1 [4, 2.8, page 8], ω is invertible at x because $\dim_k(\omega/M\omega) = 1$. Since $\dim_k(M/M^2) \geq 3$, we get that

$$\dim_k(M\omega/M^2\omega) \geq 3 .$$

Set $\omega_2 = \omega/M^2\omega$. A vector subspace of $(M\omega/M^2\omega)^n$ of codimension n corresponds to a quotient of ω_2^n of length $2n$. It is not hard to see that this correspondence extends to families of quotients and vector subspaces, so $\underline{\text{Grass}}_n([M\omega/M^2\omega]^n/k)$ can be considered as a subfunctor of $\text{Quot}^{2n}(\omega_2^n/X/k)$. Hence, since a proper monomorphism is a closed embedding [12, Ch. IV, Prop. 8.11.5], $\mathrm{Quot}^{2n}(\omega_2^n/X/k)$ contains $\mathrm{Grass}_n([M\omega/M^2\omega]^n/k)$. Since $\dim_k(M\omega/M^2\omega) \geq 3$, the latter has dimension at least $2n^2$, and reasoning as in the proof of (a), we conclude that $\mathrm{Quot}^{2n}(\omega^n/X/k)$ is reducible.

Let I be a torsion-free sheaf on X and set

$$\mu(I) = \deg I/\mathrm{rk} I .$$

We say that I is <u>semi-stable</u> if for all subsheaves $I' \subset I$, $\mu(I') \leq \mu(I)$.

<u>Lemma 8.1.2.</u> Let $I_1,\ldots,I_n$ be torsion-free, rank-1 sheaves on X such that $\deg I_1 = \ldots = \deg I_n = d$. Then

$$T = \bigoplus_{i=1}^{n} I_i$$

is a semi-stable, torsion-free, rank-n sheaf.

<u>Proof.</u> Let J be a subsheaf of T of rank r , and let $T_1,\ldots,T_t$ be the set of all subsheaves of T of the form $\bigoplus_{i=1}^{k} I_{n_i}$. We denote by

$$f_j : J \to T_j$$

the composition of the inclusion $J \subset T$ and the natural projection $T \to T_j$.

Let g denote the generic point of X. There exists an integer ℓ, $1 \leq \ell \leq t$, such that the map $f_{\ell,g}$ of $O_{X,g}$-vector spaces is an isomorphism. Hence

$$f_\ell : J \to T_\ell$$

is injective, and the cokernel of f_ℓ is supported at a finite set of points. The additivity of the Euler characteristic gives that $\deg J \leq \deg T_\ell = rd$, and therefore

$$\mu(J) \leq \mu(T) = d .$$

Set $Q = \mathrm{Quot}^m(\omega^n/X/k)$ and let $\mathcal{T}$ be a universal quotient on $X \times Q$. The points $q \in Q$ such that $[\ker(\omega_Q^n \to \mathcal{T})](q)$ is semi-stable, form an open subset Q_{ss} of Q [19, Ch. 5, §3, Rem., page 136]. Hence the subfunctor of $\underline{\mathrm{Quot}}^m(\omega^n/X/k)$ of quotients N such that $\ker(\omega^n \to N)$ is semi-stable, is represented by an open subscheme $\mathrm{Quot}^m_{ss}(\omega^n/X/k)$ of $\mathrm{Quot}^m(\omega^n/X/k)$.

<u>Proposition 8.1.3.</u> Let x be a point of X and denote by M the ideal defining x.

(a) If $\dim_k(\omega/M\omega) \geq 2$, then $\mathrm{Quot}^n_{ss}(\omega^n/X/k)$ is reducible.

(b) If $\dim_k(\omega/M\omega) = 1$ and if $\dim_k(M/M^2) \geq 3$, then $\mathrm{Quot}^{2n}_{ss}(\omega^n/X/k)$ is reducible.

Proof. (a). Set

$$\mathrm{Grass}_n^{ss}(\omega_1^n/k) = \mathrm{Grass}_n(\omega_1^n/k) \cap \mathrm{Quot}_{ss}^n(\omega^n/X/k)$$

where $\mathrm{Grass}_n(\omega_1/k)$ is the subscheme of $\mathrm{Quot}^n(\omega^n/X/k)$ defined in the proof of part (a) of Lemma 8.1.1. Let V be a vector subspace of ω_1 of colength 1. Then V^n corresponds to a point of $\mathrm{Grass}_n(\omega_1^n/k)$, which, by Lemma 8.1.2, lies in $\mathrm{Grass}_n^{ss}(\omega_1^n/k)$. Hence

$$\dim\mathrm{Grass}_n^{ss}(\omega_1^n/k) \geq n^2$$

and the arguments used to prove Lemma 8.1.1 (a) shows that $\mathrm{Quot}_{ss}^n(\omega^n/X/k)$ is reducible.
(b). A similar modification of the proof of part (b) of Lemma 8.1.1 gives that $\mathrm{Quot}_{ss}^{2n}(\omega^n/X/k)$ is reducible.

8.2.

The first lemma below was originally proved by Grothendieck [12, Ch.0_4, Prop. 19.1.10]. It is proved by Oda and Sehadri [20, Lemma in Appendix] in the following version:

Lemma 8.2.1. Let $A \to B$ be a local homomorphism of noetherian local rings. Let N and L be finite B-modules with L A-flat. Then a B-homomorphism

$$f : N \to L$$

is injective with A-flat cokernel if and only if

$$f \otimes_A K : N \otimes_A K \to L \otimes_A K$$

is injective where K denotes the residue field of A.

Let $A \to B$ be a flat homomorphism of local noetherian rings. If F is a B-module, we denote by $\overline{F}$ the A-module $F \otimes_A K$ where K is the residue field of A .

Let N be a finite B-module such that $\mathrm{Ext}^1_{\overline{B}}(\overline{N},\overline{B}) = 0$. Under this hypothesis Oda and Seshadri showed that

$$\mathrm{Hom}_B(N,B) \otimes_A K \simeq \mathrm{Hom}_{\overline{B}}(\overline{N},\overline{B})$$

[19, Corollary of Appendix]. However, their proof gives the more general result:

<u>Lemma 8.2.2.</u> Suppose that

$$\mathrm{Ext}^1_{\overline{B}}(\overline{N},\overline{L}) = 0 .$$

Then there is an isomorphism

$$\mathrm{Hom}_B(N,L) \otimes_A K \simeq \mathrm{Hom}_{\overline{B}}(\overline{N},\overline{L})$$

where N and L are finite B-modules with L A-flat.

As an immediate consequence of the two previous lemmas we get the proposition:

<u>Proposition 8.2.3.</u> Set $S = \mathrm{Spec}(A)$, A a local k-algebra, and let $Y \to S$ be a flat morphism of affine schemes. Let N and L be coherent O_Y-Modules with L flat over S . Suppose that

$$\mathrm{Ext}^1_{Y(s)}(N(s),L(s)) = 0$$

where s denotes the closed point of S . Then there is an isomorphism

$$\mathrm{Hom}_Y(N,L)(s) \simeq \mathrm{Hom}_{Y(s)}(N(s),L(s)) .$$

Moreover, if $\psi : N(s) \to L(s)$ is injective and $\varphi : N \to L$ is a homomorphism such that $\varphi(s) = \psi$, then φ is injective.

Next we give a criterion for vanishing of Ext^1-groupes, which we will use later.

<u>Lemma 8.2.4.</u> Let ω denote the dualizing sheaf of X , and let N be a torsion-free, rank-n sheaf. Then for all points $x \in X$ we have that

$$\mathrm{Ext}^1_{O_{X,x}}(N_x,\omega_x) = 0 .$$

<u>Proof.</u> Let I be an O_X-ideal, $I \neq O_X$, and set $G = \overset{n}{\oplus} I$. Let t_o be a number such that $\underline{Hom}_X(N,G)(t)$ is generated by global sections if $t \geq t_o$. Since there exists an isomorphism $N_g \simeq G_g$, where g denotes the generic point of X, there is an injective map

$$\alpha(t) : N(-t) \to G$$

for $t \geq t_o$.

If $H^o(X,N(-t)) \neq 0$, there is a non-zero map

$$\beta : O_X \to N(-t) .$$

Then $\alpha(t) \circ \beta$ gives a non-zero map $O_X \to G$, and hence a non-zero map $O_X \to I$. Since $\chi(I(n)) < \chi(O_X(n))$, $n \geq 0$, there is no non-zero map $O_X \to I$ [2, Prop. 3.4, (ii) (b)]. Hence we get that $H^o(X,N(-t)) = 0$.

By duality

$$\mathrm{Ext}^1_X(N(-t),\omega) \simeq H^o(X,N(-t)) ,$$

so

$$\mathrm{Ext}^1_X(N(-t),\omega^n) = 0$$

for $t \geq t_o$.

Let t_1 be an integer such that $\underline{\mathrm{Ext}}^1_X(N(-t),\omega^n)$ is generated by global sections for $t \geq t_1$. If $t \geq \max(t_o,t_1)$, then $\underline{\mathrm{Ext}}^1_X(N(-t),\omega^n)_x = 0$ for all points $x \in X$. Since

$$\underline{\mathrm{Ext}}^1_X(N(-t),\omega^n)_x \simeq \mathrm{Ext}^1_{O_{X,x}}(N_x,\omega^n_x)$$

[14, Prop. 6.8], the assertion follows.

8.3.

Let Q^m_F denote the open subscheme of $\mathrm{Quot}^m(\omega^n/X/k)$, which parameterizes quotients N of ω^n such that the kernel of $\omega^n \to N$ is locally-free.

Lemma 8.3.1. Q^m_F is irreducible.

Proof. Let q_1 and q_2 be two points of Q^m_F and denote by N_1 and N_2 the quotients of ω^n corresponding to q_1 and q_2. Set $I_i = \ker(\omega^n \to N_i)$. There exists a family F of locally free, rank-n sheaves over an irreducible scheme T such that $I_i = F(t_i)$ for closed points $t_1,t_2 \in T$ [19, Ch. 5, remark on page 136].

Let A be a discrete valuation ring and set $S = \mathrm{Spec}(A)$. Denote by s and η the closed and generic points of S. There exist maps $g_1,g_2 : S \to T$

such that $g_i(s) = t_i$ and $g_1(\eta) = g_2(\eta)$ [12, Ch. II, Prop. 7.1.4 (i)]. The pullbacks of F to S by g_1 and g_2 give families F_1 and F_2 over S such that $F_1(\eta) = F_2(\eta)$ and $F_i(s) \simeq I_i$.

Let V be an open subset of X such that $\mathrm{Supp}(N_1) \cup \mathrm{Supp}(N_2) \subseteq V$. By Proposition 8.2.3, there exist maps

$$h_1, h_2 : S \to Q_V^m(\omega^n/X/k)$$

such that $h_i(s) = q_i$ and $h_1(\eta) = h_2(\eta)$. Hence q_1 and q_2 lie on the same irreducible component of $Q_V^m(\omega^n/X/k)$ and therefore on the same component of $\mathrm{Quot}^m(\omega^n/X/k)$.

We are now ready to prove the main result of this chapter.

Theorem 8.3.2. If X does not lie on a smooth surface, then $M(n, \ell n)$, $\ell \in \mathbb{Z}$, is reducible.

Proof. $\mathrm{Quot}_{ss}^{tn}(\omega^n/X/k)$ is reducible for $t = 1$ if X is Gorenstein and for $t = 2$ if X is not Gorenstein [Proposition 8.1.3]. Since $Q_{F,ss}^{tn} = \mathrm{Quot}_{ss}^{tn}(\omega^n/X/k) \cap Q_F^{tn}$ is irreducible [Lemma 8.3.1], $\overline{Q}_{F,ss}^{tn} \neq \mathrm{Quot}_{ss}^{tn}(\omega^n/X/k)$.

Let $q \in \mathrm{Quot}^{tn}_{ss}(\omega^n/X/k)$, $q \notin \overline{Q}^{tn}_{F,ss}$. Let N denote the quotient of ω^n corresponding to q , and denote by I the kernel of the map $\omega^n \to N$. Suppose that I can be deformed to a locally free sheaf over $S = \mathrm{Spec}(A)$, A a complete, discrete valuation ring.

Let V be an affine open subset of X such that $\mathrm{Supp} N \subseteq V$ and denote by $Q^{tn}_{V,ss}$ the open subscheme of $\mathrm{Quot}^{tn}_{ss}(\omega^n/X/k)$, which parameterizes quotients of ω^n with support contained in V . Put $J = I|_V$. Since I can be deformed to a locally free sheaf over S , there exists a deformation $\widetilde{J}$ of J to a locally free sheaf over S . By Proposition 8.2.3 and Lemma 8.2.4, the inclusion

$$J \subseteq (\omega/V)^n$$

lifts to an injection

$$\alpha : J \to (\omega/V)^n_S \ .$$

The cokernel of α is S-flat [Lemma 8.2.1] so it corresponds to a morphism

$$S \to Q^{tn}_{V,ss}$$

such that the generic point of S maps to $Q^{tn}_{V,ss} \cap Q^{tn}_{F,ss}$. This implies that $q \in \overline{Q}_{F,ss}$, and we have a contradiction since q was chosen not to lie in $\overline{Q}^{tn}_{F,ss}$. Hence I is a torsion-free, rank-n sheaf of degree $n(2p - 2 - t)$, which can not be deformed to a locally free sheaf, and therefore $M(n,n(2p - 2 - t))$ is reducible.

If I is torsion-free of rank n and L is an invertible O_X-Module, then $\deg(I \otimes L) = \deg I + n\deg L$ [19, page 131]. Tensoring by an invertible O_X-Module L with

$$\deg L = \ell + 2 + t - 2p ,$$

$\ell \in \mathbb{Z}$, defines an isomorphism

$$M(n,n(2p - 2 - t)) \simeq M(n,\ell n) .$$

Hence $M(n,\ell n)$ is reducible for all $\ell \in \mathbb{Z}$.

Remark 8.3.3. Suppose that X does not lie on a smooth surface. Then there exists a torsion-free, rank-1 sheaf I_1 on X, which has no deformation to a locally free sheaf [Theorem 5.2.4].

Set

$$I = I_1 \oplus I_2 \oplus \ldots \oplus I_n$$

where I_i , $i = 2,\ldots,n$ are torsion-free, rank-1 and $\deg I_i = \deg I_1$. If every deformation $\widetilde{I}$ of I can be written as

$$\widetilde{I} = \widetilde{I}_1 \oplus \ldots \oplus \widetilde{I}_n$$

where $\widetilde{I}_i$ is a deformation of I_i , then I is a semi-stable, torsion-free, rank-n sheaf, which has no deformation to a locally free sheaf. Hence, if such decompositions of deformations hold, reducibility of $M(n,n\ell)$ will follow from reducibility of $M(1,d)$.

However, the next proposition shows that this is not the case.

Proposition 8.3.4. Let A be a local k-algebra, which is an integral domain of dimension 1, and suppose that A is not regular. Then there exists a torsion-free A-module I_1 of rank 1, a free A-module I_2 and a $k[[T]]$-flat $A[[T]]$-module $\widetilde{I}$ such that

$$\widetilde{I} \otimes_{k[[T]]} k \simeq I_1 \oplus I_2 ,$$

but $\widetilde{I}$ does not have a decomposition

$$\widetilde{I} = \widetilde{I}_1 \oplus \widetilde{I}_2$$

where $\tilde{I}_i$ is a deformation of I_i. Here T is an independent variable over k.

<u>Proof.</u> Let m denote the maximal ideal of A. Since A is not regular, there exist elements f_1, f_2 of m such that

$$\dim_k((f_1,f_2)/m(f_1,f_2)) = 2 .$$

Set $B = A[[T]]$ and let K be the submodule of B^3 generated by the element (f_1,f_2,T). Let K' denote the submodule of A^3 generated by $(f_1,f_2,0)$ and set

$$\tilde{I} = B^3/K \quad \text{and} \quad I = A^3/K' .$$

Then

$$\tilde{I} \otimes_{k[[T]]} k \simeq I$$

and $\tilde{I}$ is $k[[T]]$-flat [Lemma 8.2.1].

Let K'' be the submodule of A^2 generated by (f_1,f_2). Then $I_1 = A^2/K''$ is a torsion-free A-module of rank 1 and I can be written as

$$I = I_1 \oplus I_2$$

where I_2 is free of rank 1.

We will show that there is no decomposition of $\tilde{I}$ of the form

$$\tilde{I} = \tilde{I}_1 \oplus \tilde{I}_2 \ .$$

where $\tilde{I}_i$ are deformations of I_i . We proceed as in the proof of [8, Prop. 1.2]:

For a B-module M , let $\gamma(B)$ denote the least number of elements required to generate M . Suppose that $\tilde{I}$ can be written as

$$\tilde{I} = \tilde{I}_1 \oplus \tilde{I}_2 \ .$$

We have the following formulas:

$$\gamma(\tilde{I}_1) + \gamma(\tilde{I}_2) = \gamma(\tilde{I}) \leq 3 \quad [8, \text{Lemma } 1.3]$$

$$\text{rank } \tilde{I}_1 + \text{rank } \tilde{I}_2 = \text{rank } \tilde{I}$$

and

$$\text{rank } \tilde{I}_1 \leq \gamma(\tilde{I}_1) \ , \ \text{rank } \tilde{I}_2 \leq \gamma(\tilde{I}_2) \ .$$

From these conditions we conclude that either rank $\tilde{I} = \gamma(\tilde{I})$, rank $\tilde{I}_1 = \gamma(\tilde{I}_1)$ or rank $\tilde{I}_2 = \gamma(\tilde{I}_2)$, i.e. either $\tilde{I}_1, \tilde{I}_2$ or $\tilde{I}$ is free.

$\widetilde{I}$ is not free since $\widetilde{I} \otimes_{k[[T]]} k \simeq I$ and I is not a free A-module. Suppose, therefore, that $\widetilde{I}_1$, say, is free. Projecting $\widetilde{I} \to \widetilde{I}_1$ with kernel $\widetilde{I}_2$ induces a map $f : B^3 \to \widetilde{I}_1$, which thus splits. Since $\alpha \in \ker f$, α belongs to a proper summand of B^3. Hence to some new basis of B^3, α has at least one zero coordinate. But the ideal (f_1, f_2, T) in B is generated by the coordinates of α relative to any basis of B^3. Therefore, since $\gamma(f_1, f_2, T) = 3$, no coordinate of α vanish. Hence the assumption that $\widetilde{I}$ can be written as $\widetilde{I}_1 \oplus \widetilde{I}_2$ leads to a contradiction.

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