Mechanics of Swellable Elastomeric Seals

by

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Abstract
This thesis investigates the mechanics of swellable elastomeric seals for the purpose of hydraulic fracturing in oil and gas applications. The first component of the thesis is the development of a laboratory-scale apparatus for the visualization of swellable seals in situ up to the point of leakage. Experiments using this apparatus show that leakage is a result of large, nonuniform deformation that stretches the seal material tangential to the sealing surfaces and leads to a corresponding loss of traction normal to the sealing surfaces due to Poisson contraction. This phenomenon was investigated in two analogous seal systems—an O-ring and a rectangular swellable elastomer used to seal a rectangular channel. Both analog systems exhibit leakage due to the same mechanism. Corresponding finite element simulations predict a fluid leakage path that agrees qualitatively with experiments.

The second part of this thesis consists of an experimental investigation of the effect of geometry and metal support rings on the performance of swellable seal systems. Although this work is highly applied, it reveals two interesting results. The first is that mechanical supports, in the form of rigid metal support rings, provide most of the support for the applied differential pressure. Secondly, in some seals, changing the length of the rubber part of the seal does not significantly affect the maximum differential pressure that the seal can support.

Motivated by the experiments showing no dependence of critical leakage pressure on seal length, we conduct an analytic investigation of the combined effects of compressibility and aspect ratio on the performance of the seal system. We find an approximate, linear elastic Saint-Venant type solution that agrees well with nonlinear (finite deformation neo-Hookean) finite element simulations, indicating nonlinear effects are unimportant in the bulk of the seal, and only important at the high-pressure and low-pressure ends. Using finite element simulations, we characterize the energy release rates for the growth of cracks in the regions of high stress concentration at the ends of the seal. We show that, despite the linear Saint-Venant solution not being valid at the ends, it correlates the energy release rates obtained in the nonlinear finite element solutions.

Although the Saint-Venant solution enables understanding of the location where fracture will first occur, experimental observations indicate that fracture often happens on both ends of the seal. In order to understand this, we implement a user subroutine within the finite element software Abaqus to predict fracture initiation and propagation. Results indicate that, despite fracture initially occurring on either end, the growth of cracks leads to fracture on both ends of the seal, consistent with experimental observations.

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Chapter 1

Introduction

1.1 Motivation

The period from 2009 to 2017 has seen a dramatic revolution in the production of crude oil and natural gas in the United States, which has in turn affected global energy prices, as shown in Fig. 1-1. This sharp increase in production is due to the extraction of oil and gas from low-permeability hydrocarbon-bearing geologic formations conventionally called “shale.” Examples of these are the Bakken formation in North Dakota and Montana, the Marcellus formation in Ohio, Pennsylvania, and New York, the Eagle Ford formation in Texas and the Fayetteville formation in Arkansas. Although the existence of hydrocarbons in these formations has been known since the 1950s [72], they have only recently become economically viable to produce. This is both a function of economic market issues as well as technological advancements that have made extraction of hydrocarbons less costly.

Figure 1-1: United States production of crude oil [1] and price per barrel of Brent crude oil [2], as reported by United States Energy Information Administration. Price adjusted for inflation to January 2017 dollars using seasonally adjusted urban consumer price index (code CUSR0000SA0) [73].

The confluence of two key technologies has lowered the cost of extracting hydrocarbons from
these formations and made their extraction economically viable. The first is the ability to precisely drill long horizontal wells. Precision is necessary because these hydrocarbon-bearing layers are thin, typically on the order of 100 m in thickness and sometimes less than 50 m [72], and long horizontal wells are necessary to maximize contact area with the formation and decrease the distance hydrocarbons must flow through rock before reaching high hydraulic conductivity tubulars. In short, horizontal drilling means one can reach the same amount of hydrocarbon-bearing formation with a single horizontal well as would require tens or hundreds of vertical wells.

The second technology that has enabled economically viable extraction of hydrocarbons from these low-permeability formations is hydraulic fracturing. Hydraulic fracturing was first done in the 1940s [17], but has seen a surge in utilization in the 2000s for the stimulation of wells in low-permeability formations. Hydraulic fracturing involves the pumping of high-pressure fluid from the Earth’s surface down into the well. The tensile stress created by this high-pressure fluid causes a crack to initiate and propagate within the rock. The high-pressure fluid flows into this newly created crack, and the crack grows as more high-pressure fluid is pumped into the crack.

There are a variety of approaches used to prescribe the locations of the cracks and initiate their growth. Regardless of the approach used, the goal is always to increase the rate of extraction of hydrocarbons by minimizing the flow resistance experienced by each parcel of hydrocarbon as it flows from its native position to the surface. The cracks in the rock, once propped open with particles called proppant, form the path of least resistance (high permeability), and overall flow resistance is minimized by maximizing the number and length of the cracks (up to a certain point). In an analogy with heat transfer, one can think of the high hydraulic conductivity cracks as analogous to fins in a heat exchanger. The resistance to thermal energy transport is small in the fins, just as the resistance to hydrocarbon flow is small in the cracks. Resistance to thermal energy transport in the ambient fluid surrounding the fins is high. Similarly, resistance to hydrocarbon flow through the low-permeability rock surrounding the cracks is high. Finally, increasing the number and length of fins increases the rate of heat flux, just as increasing the number or length of the cracks increases the flux of hydrocarbons from the rock into the horizontal wellbore and eventually to the surface.

Indeed, if capital cost and operating cost were not a factor, it would be desirable to have hundreds or thousands of cracks evenly spaced along the length of a horizontal well [97]. However, because each additional crack involves an incremental cost in both time and materials, current wells are economically limited to a maximum of approximately 50 fractures per horizontal well.

Because it is desirable to have multiple cracks, each penetrating far into the rock, a mechanism for controlling the initiation and growth of each crack is required. In order to accomplish this, the fracturing is typically done in stages where a single crack is initiated and grown. Then, upon completing the growth of the crack in that stage, a new fracture stage is initiated and a new crack is grown. In this way, the cracks are generated sequentially, starting from the end of the horizontal well (toe) and working back toward the junction between the vertical well and the horizontal well (heel), until the specified number of cracks are created. In this process of sequential fracturing of stages, it is necessary to isolate each fracture stage from its adjacent stages. As an example, when trying to grow the second crack in a sequence of three adjacent cracks, it is undesirable to pump fluid into the previously generated first crack, or to inadvertently initiated a crack at the location of the third crack before finishing the second crack.

There are at least two common approaches for isolation of adjacent fracture stages from each other. Among the most robust of methods is depicted in Fig. 1-2.b and uses metal casing (tube) with outer diameter slightly smaller than the diameter of the hole drilled in the rock (wellbore). This casing is cemented in place in the well by squeezing cement into the annular gap between the casing and the wellbore. If the cement does not crack and is well-bonded to both the rock and the casing, it provides a robust barrier to fluid migration (both hydrocarbons from the formation and fluids pumped down from the Earth’s surface). In order to hydraulically fracture the rock in such a cased hole, it is necessary to perforate the casing and cement at specified locations using shaped charge explosives conveyed into the wellbore on a perforating gun. Following the plugging of the wellbore below the target crack location and perforating the casing and cement, high-pressure fluid is pumped from the surface to fracture the rock at the location of the perforation. Once the crack grows to sufficient length, the process is repeated with plugging of the well and perforating above the most
recently created fracture. In this way, the horizontal well is fractured from toe (bottom of the well) to heel (location where horizontal well protrudes from vertical well) in a repeated plug-perforate-fracture sequence. Following the fracture of all the stages, the plugs are milled away, leaving an unobstructed flow path inside the casing.

![Diagram](image)

Figure 1-2: Schematic of hydraulic fracturing. (a) Overview of typical horizontal well in low-permeability formation showing one of several possible horizontal wells extending from the vertical well. (b) Hydraulic fracturing using perforation through a cased hole. (c) Hydraulic fracturing of an open-hole completion using annular seals between the production tubing and rock formation. (Sketches not to scale because aspect ratios are large.)

Although the use of casing and cement is reliable for preventing unwanted leakage and fractures can be positioned with great precision, it has the disadvantage that it is often more costly than other methods of isolating adjacent fracture stages. Because the oil and gas industry is a commodity industry, driven primarily by economics, it is necessary to have lower-cost alternatives to the more expensive method of perforating of cased holes. One such low-cost alternative is termed an “open hole completion,” and it uses discrete seals to separate adjacent fracture stages [44], as depicted in Fig. 1-2c. In this method, a steel tube, called “production tubing” or “basepipe” is inserted into the wellbore. The annular gap between the wellbore and the production tubing is larger than the annular gap for a cased hole. Unlike the cased hole, this gap is not filled with cement, and is therefore filled with ambient fluid, which may be a mixture of hydrocarbons and aqueous solutions coming from the rock as well as hydrocarbons and/or aqueous solutions pumped from the surface. In order to isolate adjacent fracture stages from each other, rubber annular seals called “packers” are distributed along the length and mounted to the periphery of the production tubing [26].

Packers are similar to O-rings in both form and function. Like O-rings, they are toroidal annular elastomeric seals, but unlike O-rings with circular or square cross-sections, the revolved surface of the packer toroid is a rectangle with large aspect ratio, such that the length of the packer parallel to the axis of rotation is much larger than the thickness of the packer in the radial direction, as depicted in Fig. 1-3. Packers are deployed into the well by rigidly mounting them on the basepipe between metal support rings rigidly affixed to the basepipe. The packers are usually, although not always, chemically bonded on their inner surface to the basepipe. Upon mounting all the required packers to the basepipe, the basepipe is inserted into the wellbore.
Figure 1-3: Schematic of packer for sealing of annular gap between fracture stages. (a) Packer as inserted into hole on inner tubular/basepipe. There is a clearance gap between the metal support rings and the rock in order to allow for insertion into the hole. The metal support rings are rigidly attached to the inner metal tube (often using set screws). The entire assembly is conveyed into the hole by pushing the inner metal tube into the hole. (b) Packer once actuated to provide seal. Arrows indicate compressive tractions exerted by surroundings on the elastomeric seal. There are radial tractions due to contact with the rock and the inner tubular. There are axial tractions due to contact with the metal support rings at the ends of the seal. There may also be axial tractions due to frictional effects at the interfaces at $R_1$ and $R_3$ (not shown). Finally, there are hoop stresses shown in the cross-sectional view due to the free body cut through the seal. (c) Packer under applied differential fluid pressure. Arrows indicate tractions exerted by the surroundings on the elastomeric seal. In the pressurized state, there is friction between the seal and the inner metal tube. Applied fluid force on the left end is balanced by frictional forces along the length as well as axial force provided by the metal support ring on the low-pressure end.

Similar to O-ring seals, packers require an initial pre-compression in the radial direction prior to application of fluid differential pressure in the axial direction. Unlike O-ring seals, which are typically pre-compressed using an interference fit, packers cannot have an interference fit with the rock because large frictional forces would make it impossible to insert the basepipe into the wellbore. (Recall there may be on the order of 50 packers mounted on the basepipe, and friction between those packers and the wellbore would lead to damage of the surface of the packers and could lead to buckling of the basepipe due to axial compressive forces [63].) Instead, the pre-compression must be generated after the basepipe, with all of its packers, is inserted into the wellbore.

The actuation of packers to generate the required radial pre-compression can be accomplished in a number of different ways. A mechanical packer is actuated via axial compression of the ends of the seal, which is often done using pre-loaded springs or hydraulic actuation from the surface [18]. This
axial compression generates radial expansion and an interference fit between the seal and the rock. Inflatable packers are actuated by inflating bladders within the packer with fluid pumped from the surface. The inflation of the bladders, similar to inflating a balloon in a confined hole, generates the interference fit required for the seal [5, 25]. Although both these packer types are commonly used in industry, they both suffer from the drawback that they require external communication from the surface in order to execute the actuation.

A passively actuated packer requiring no intervention or stimuli from the surface offers the advantages of simplicity and lower costs over mechanical and inflatable packers. One such packer is the swell packer, which is actuated passively by absorbing ambient fluid from the surrounding wellbore. Such absorption occurs spontaneously, due to chemical affinity between the ambient fluid and the packer material. In its most simple form, the polymer chains in the elastomer matrix have a chemical affinity for the ambient fluid. (They are either non-polar polymer chains, such as those found in ethylene propylene diene monomer (EPDM) rubber, which readily absorb non-polar solvents such as octane, or they are polymers composed of polar monomers, such as those found in polyacrylamide hydrogels, and they readily absorb polar solvents such as water.) The absorption of solvent into the crosslinked elastomeric matrix creates a polymeric gel, which was first investigated by Huggins [42] and Flory [27]. In reality, the formulations are much more complicated than polymeric network composed of a single monomer, and much work is done on polymer blends and fillers, such as salts for modification of osmotic pressure, to tailor mechanical properties and swelling kinetics, but one can conceptually think of them as a polymeric gel. As a passive, spontaneously actuated device, swell packers provide a lower cost alternative to packers that require communication from the surface to actuate.

Once the packer is actuated, such that there is compressive traction exerted by the surrounding wellbore on the outer radius of the seal, the packer can support the applied differential pressure necessary to fracture the rock. As in an O-ring seal, a differential pressure applied to the packer leads to a compressive axial stress within the packer. Because the packer is confined by a metal support ring on the low-pressure end, this axial compression leads to a tendency for expansion in the radial direction due to the fact that the Poisson ratio is positive. However, because the seal is confined in the radial direction by the wellbore on the outside and the basepipe on the inside, it is unable to expand radially, and instead compressive radial tractions exerted by the wellbore and basepipe on the seal increase as fluid pressure is increased.

If the seal were perfectly incompressible and perfectly well-constrained by the low-pressure support ring, each incremental increase in fluid pressure would lead to a corresponding incremental increase in radial compressive traction exerted by the wellbore on the outer surface of the seal [66]. In such an idealized system, there would be no deformation corresponding to an increase in applied fluid pressure. The increase in tendency to leak due to higher fluid pressure would be offset by the increase in resistance to leakage due to the increase in compressive traction exerted by the wellbore on the seal. In such an ideal system, leakage would never occur.

In summary, horizontal drilling and hydraulic fracturing have revolutionized global supply of crude oil and natural gas, thereby affecting energy prices and the quality of life for billions of people. The process for extracting hydrocarbons from these low-permeability formations is complex, and there are many highly engineering systems involved. One such subsystem is the swell packer, whose failure, while not hazardous to health or the environment, adversely affects the rate of production of hydrocarbons and the economic viability of the well.

### 1.2 Technical Problem: Leakage of Swell Packer Seals

Although an ideal swell packer with infinite bulk modulus and no clearance gap will not leak in theory, real packers are far from ideal. In practice, swell packers must be tested both to determine how fast the seal swells and to determine how much differential pressure it can support before leakage.

In determining the swelling kinetics of a given elastomer in a given environment (ambient temperature, ambient solvent, pH, salinity, hydrogen sulfide concentration, etc.), the typical procedure
is to make experimental measurements using small scale samples in the environment of interest. Then the results of the small-scale experiments can be used to calibrate a constitutive model, such as that of Duda, et al. [23] or Chester and Anand [15], and that model can be used to predict the swelling kinetics and mechanical properties of the full-scale seal as a function of time. In this way, relatively inexpensive and quick experiments combined with the constitutive model can be used to predict the swelling kinetics and mechanical properties for a full-scale device.

However, an accurate constitutive model alone coupled with small-scale tests is not sufficient for the prediction of the maximum differential pressure that a full-scale packer can support prior to leakage. This question of leakage is a device-level problem, where there may be multiple mechanisms of leakage, and it is not known how the material properties contribute to these various mechanisms of leakage. The problem is analogous to predicting the maximum load that can be supported by a bridge knowing only the geometry and material properties of the steel and concrete of which it is composed. Without knowing the mechanism of bridge failure – do the steel reinforcing members in the concrete yield or does the cement delaminate from the steel reinforcements or does something else occur? – one cannot quantitatively predict the maximum load that the bridge can support. Similarly, it is not known what the mechanism of seal leakage is, and therefore it is impossible to quantitatively predict the maximum differential pressure a given seal design can support.

Instead, what is done in practice is that, once a candidate material formulation and seal geometry has been identified, one or more full-scale prototype seals are manufactured, swollen and tested to the point of leakage. (This is analogous to building a full-scale bridge and destructively testing it in order to determine the maximum load that it can support, albeit with far less cost for the seal than the bridge.) These full scale tests are particularly time-consuming and expensive because they involve swelling the seal in a pressure vessel at an elevated temperature for a duration of approximately one month. Dedicated testing facilities are required because of the risk of catastrophic rupture of the pressure vessel containing the seal. One or more technicians are required to perform the initial assembly and conduct the pressure test, in addition to the engineer or researcher who commissioned the test. Although bridge designers would scoff at such an endeavor, packer engineers accept it because there exists no alternative.

The development of new swell packer seals is largely trial and error. Like many engineering devices, there is a body of knowledge on the design and use of swell packers, but this body of knowledge predominantly takes the form of experience of the design engineers and internal corporate reports. The use of seals in general is more art and proprietary engineering than documented and disseminated science. Exceptions certainly exist. There has been a great deal of published work on the chemistry and material properties of elastomeric seal material for use in O-rings and gaskets [100, 75, 30, 24, 93, 94, 32, 8, 65]. Moving in the direction of applied mechanics, there has also been some work on the sealing forces generated by O-rings under confinement [45, 20, 29, 77, 12, 71, 45]. Although leakage in the absence of adhesion is generally thought to occur when the fluid pressure exceeds the normal contact traction between the seal and rigid substrate [66], as implemented in Abaqus’s pressure penetration boundary condition [90], there Persson and co-workers have recently examined the effect of surface roughness and percolation of fluid flow through the asperities in the substrate [78, 56, 54]. However, this work has not provided a conclusive understanding of the failure mechanism swell packer seals nor enabled prediction of the critical differential pressure at which the seal begins to leak.

Undoubtedly, part of the cause for such a slow evolution in understanding of swell packer systems is that these devices, at full scale, are large; experiments are expensive and time-consuming; minimal field data is available because the devices are intended to be permanent; and the focus of work on swell packers is commercial rather than fundamental science. Despite these obstacles, there has been a small pocket of intense research on swell packer devices conducted over the last decade at Schlumberger-Doll Research. Much of this work, led primarily by Robison [11, 58, 51, 86, 36, 37, 81, 69], focused on the material science of developing novel compounds for improved device performance. This intense work at Schlumberger also spawned research collaborations with academic institutions, among the most notable of which is the research group of Professor Zhigang Suo at Harvard, who was initially involved with constitutive model development for the kinetics of swelling of these swell packers, which, in the simplest form are crosslinked polymeric gels as studied by Huggins and Flory.
In addition to fundamental material science research performed at Schlumberger-Doll Research, an approach to testing of scale models of swell packers, colloquially termed ‘mini packers’, was developed by Maheshwari [59], and successively refined by Amarante, Han, Qu, Tu, Belaud, Godfrey and Druecke [80]. These scale models were approximately one quarter of full-scale size, with nearly all tests being performed inside pressure vessels having inner diameter of 4.83 cm (1.90 inch). All dimensions were scaled linearly based on the full-scale nominal diameter of the borehole and the small-scale inner diameter of the pressure vessel. These small-scale experiments were extremely appealing because they could be done on a bench in a laboratory, where capital cost of equipment was much smaller than that for full-scale testing. Furthermore, the small scale implies that swelling occurs much more quickly. Although solvent diffusion is non-Fickian, a coarse ‘back of the envelope’ Fickian approximation states that quarter scale models will swell in one sixteenth the time of full scale packers. Accounting for non-Fickian processes, whether this factor is 5 or 16 or 100, it is still substantially faster than full-scale testing, and swelling of multiple packers can be done concurrently, limited only by the number of pressure vessels available.

Despite the advantages of this small-scale packer test, it has the drawback that nobody knows how to use results from small-scale tests to quantitatively predict performance of full-scale seals. The small scale seal test’s primary use is in comparison, where seals of two materials are tested, and it is determined which material performs better in small-scale tests. Presumably this material will also perform better in full-scale seal systems, but, despite having a quantitative number for the critical differential pressure at small scale, it is not known how much pressure it will support at full scale. Similarly, this small scale seal test can be used to test mechanical designs of packers, and such work is contained herein. However, as with material testing, it can only be said that one mechanical design performs better at small scale and will likely perform better at large scale, but quantitative prediction of critical differential pressure is not possible.

Part of the underlying reason for an inability to predict critical differential pressure is that the leakage mechanism had not been known prior to this work. The full-scale and bench top testing of seal systems is done in thick-walled steel pressure vessels capable of supporting internal pressures of at least 52 MPa (7500 psi) and often much higher. Therefore, the information that these tests produces include input parameters, such as swell time, temperature, and applied pressure versus time, as well as output parameters, including fluid flow rate into the seal system necessary to maintain prescribed differential pressure. However, the nature of these tests is that the seal system itself is only observed at the beginning of the test before swelling and at the end of the test, after failure. (Although tests could be aborted prior to leakage, the fact is that this almost never occurs.) Any single experiment yields a critical differential pressure at which leakage begins to occur as well as observations of the seal after failure. In most post mortem observations of the failed seal, there is a visibly damaged region near the low-pressure end of the seal, and, at times, a visibly damaged region near the high-pressure end of the seal. However, there is nearly always a long region in the interior of the seal that appears undamaged and pristine. Such observations were made by Maheshwari and begged the question of how fluid leaks past this undamaged portion of the seal.

In addition to the observation of a nearly pristine midsection of the swellable seal after leakage, it was observed that the ends of the seal often were swollen to a larger diameter than the midsection. This was a phenomenon that was not accounted for in the one-dimensional model of Suo, which was the working model at the time this work was begun. The fact that the ends were swelling more, and the problem was not purely one-dimensional radial diffusion as predicted by the working model, called into question the predictions of swelling time required for the seal to make contact with the wellbore and sealing time required for the seal to swell to equilibrium.

An additional question that was raised with both full-scale and model testing is that, with the damage occurring exclusively at the ends, what is the effect of length of the seal. In the industry, the ability of a swell packer system to seal is often reported as a maximum differential pressure per unit length, in a presumption that the critical differential pressure scales linearly with length of the seal. Indeed, this is explicitly documented in some of the intellectual property surrounding these seals. However, somewhat contrary to this assertion is the observation that engineering devices on the low-pressure end in the form of anti-extrusion devices can markedly improve the critical differential pressure, which is somewhat paradoxical to the presumption that critical differential pressure scales
linearly with seal length. This raises the question of what is the effect of seal length on critical differential pressure, and specifically, can the maximum differential pressure supported by the seal be increased simply by increasing the seal length.

In summary, swell packers leak at finite differential pressures. There is currently no model to predict how much differential pressure a swell packer can support as a function of geometry, material properties and environmental factors. For each new swell packer design, full-scale tests must be conducted to measure the maximum differential pressure that the seal can support. Additionally, because the mechanism of failure is not known, one cannot intuit which material properties are the most important properties to optimize. Is stiffness more important than toughness?

The goal of this thesis is primarily to understand the mechanism of seal failure and incorporate that into a model that can be used to understand at what differential pressure the seal will leak. Then using that knowledge, we aim to provide guidance on improving seal designs to either enable them to support a larger differential pressure or to support the same differential pressure while costing less.

1.3 Scope of Thesis

The body of this thesis begins with an examination of the failure mechanism of long swellable seals. First an experimental apparatus for visualization of seal behavior under applied differential pressure is presented. The mechanism of seal failure is caused by non-axisymmetric deformation of the seal, which results in a decrease in compressive traction and eventual loss of contact between the seal and sealing surface. In addition to the small-scale annular swellable seal, two analogous systems – one of a swellable seal in plane strain and one of an O-ring seal in axisymmetric geometry – exhibit the same behavior. An analytic model for finite deformation of the plane-strain seal is presented along with finite element simulations showing the loss of sealing stress due to nonuniform deformation.

Following this work, we present an experimental investigation of the effect of seal length as well as mechanical support on the performance of swellable seal systems. The results show that, for the relatively smooth-walled pressure vessels used in the experiments, the critical differential pressure does not depend on length of the seal. Instead it depends primarily on the mechanical support provided by nearly-rigid metal rings on the low-pressure end of the seal. We show a configuration that nearly doubles the critical differential pressure supported for a given length of rubber. We further present a seal design with embedded mechanical supports along the length of the seal. This particular design provides a 70 percent improvement in critical differential pressure over an analogous seal without embedded mechanical support.

Following these experimental observations of the effect of length and mechanical support on the performance of a swell packer seal system, we present an analytic model for the effect of length on the performance of the seal. The analytic solution provides an asymptotic solution for stresses, strains and displacements over the bulk of the seal system, except very near the two ends of the seal. In this boundary layer region, the analytic solution breaks down and a boundary layer correction is yet to be determined. Nevertheless, we show that the bulk prediction of average axial stress versus position along the length of the seal can be used to predict average stresses at the two regions of high stress concentration. Using finite element simulations, it was found that these stresses are well-correlated with the energy release rates for the advancement of small cracks on the high-pressure and low-pressure ends in the regions of high stress concentration. Thus once a single finite element simulation is used to correlate stress with energy release rate, the analytic solution can be used to understand how changing the length of the seal affects energy release rates at the two ends. The model shows that, as the bulk modulus of the seal increases or the seal becomes shorter, the propensity for failure at the low-pressure end increases while that for failure at the high-pressure end decreases. The prediction of the trade-off between failure at the two ends predicted by the model is in agreement with that obtained from finite element calculations.

Finally, we present some preliminary results from a finite element model incorporating fracture propagation, as opposed to simply fracture initiation, which was predicted by the analytic model. The fracture propagation model allows for growth of interfacial cracks between the seal and the rigid
inner surface to which it is bonded, as well as propagation of cracks on the low-pressure end due to high stress concentrations at the extrusion gap. We foresee this model being used to better predict seal failure due to asymmetric deformation, as well as other industrially relevant phenomena, such as eccentricity of the seal in the wellbore.

The primary contributions of this thesis are threefold. First we demonstrate that nonuniform deformations lead to a loss of contact traction, which can allow fluid to locally and non-uniformly leak past the seal. Second, we demonstrate that mechanical supports of of primary importance in seal design, and incorporation of metal supports is far more important than increasing the length of the rubber in a seal. Third, we demonstrate that a linear Saint-Venant solution can be used to understand fundamentally nonlinear energy release rates in certain cases. Finally, although the implementation of computational fracture initiation and propagation based on strain energy density presented in Chap. 5 has not been found in the literature, it is hard to imagine that it does not already exist exactly in the form presented here. If it is indeed novel, it could provide a useful means for fracture propagation computations.
Chapter 2

Mechanism of Leakage

Observations of swell packers after leakage have raised an intriguing question. Typical swell packers observed after leakage, such as the model swell packer shown in Fig. 2-1, have some material fracture on the low-pressure end and sometimes have slight damage on the high-pressure end (due to swelling-induced extrusion); however, more than half the length of the packer is intact and undamaged. These observations raise the question: how does fluid leak across this undamaged portion of the seal?

A significant amount of prior work has been carried out to evaluate incremental material failure in seals due to mechanisms including chemical aging [93, 94], thermal degradation [100], compression set [30], swelling [20, 32] and extrusion [24]. However, the onset of any one of these material failure modes is generally not sufficient to make the seal leak, and the seal may be subjected to higher differential pressures or longer environmental exposures before there is bulk fluid leakage past the seal. Therefore, prediction of the onset of any one of these material failure modes is a conservative estimation of seal failure and leakage.

Recently, there has been a substantial effort by Persson [78, 56] and collaborators examining fluid leakage between a seal and sealing surface due to surface roughness. Persson, as well as Bottiglione, et al. [10], use the framework of percolation theory and contact mechanics to predict how fluid flow rate increases as surface roughness and driving differential pressure increase and decreases with increasing sealing stress and fluid viscosity. Additionally, the experiments of Lorenz and Persson [56, 54] show remarkably good agreement with their theory.

While this theory has been supported by their experiments for slow flow rates over rough surfaces, it does not fully capture the physics of interest in many seal leakage problems. As an example, the plots of pressure and flow rate for the characteristic swell packer experiment shown in Fig. 2-1 reveal that flow rate is not linearly proportional to applied differential pressure, as suggested by lubrication layer fluid dynamics scaling in the theories of Persson and Bottiglione, et al. The experiments of Liu, et al [52] also show that flow rate is not linearly dependent on differential pressure. Although Lorenz and Persson refined their theory to incorporate the effect of fluid pressure on the reduction of contact pressure between the seal and the rigid substrate [55], thereby resulting in a nonlinear dependence of flow rate on differential pressure, the percolation theory suggests that the flow rate can be made arbitrarily small by making the surfaces sufficiently smooth and the sealing pressure sufficiently high. While this may be true in certain idealized situations, typical applications generally have another failure regime resulting in leakage at flow rates much larger than those predicted by percolation.

The complementary piece of the story on seal leakage involves the change in contact pressure between the seal and sealing surface throughout the lifetime of the seal. Incremental material failure modes, such as compression set, mentioned previously lead to a decrease in sealing stress between the seal and sealing surface. Some of these material failure modes occur gradually with increasing time, such as material degradation, and others, such as extrusion, can occur abruptly due to over pressurization. One of these mechanisms leading to seal leakage is elastic deformation.

We will present in situ experimental observations for three different seal systems, shown in Fig. 2-2, as differential pressure is applied up to the point of leakage. Each seal undergoes large deformations
Figure 2-1: Characteristic differential pressure testing of model swell packer made of Dow-Corning Sylgard 184 PDMS swollen in LVT-200 mineral oil inside a pressure vessel with 4.83 cm inner diameter. The hydraulic pressure was provided by a Teledyne Isco Model 260D syringe pump, and pressure, flow rate and displaced volume were measured at the pump, upstream of the high-pressure end of the seal. (a) Prescribed differential pressure as a function of time. (b) Volumetric flow rate, \( Q \), provided by the pump to the high-pressure end of the packer and displaced volume, \( V = \int Q \, dt \), of the pump. The first spike in volumetric flow rate at \( t = 10 \text{ min.} \) is due to excess headspace in the pressure vessel and corresponds to rigid-body motion when applied pressure overcomes friction and gravity, and the entire packer assembly is displaced upward and seated against the low-pressure end of the pressure vessel. The second spike in volumetric flow rate at \( t = 110 \text{ min.} \) corresponds to rigid-body displacement of the packer assembly when a protrusion on the low-pressure end of the pressure vessel permanently deformed the low-pressure end of the aluminum cylinder on which the packer was mounted, and the packer assembly shifted further within the pressure vessel. (See deformation in photo.) Failure occurred at \( t = 210 \text{ min.} \), at which point solvent was discharged from the low-pressure end of the pressure vessel, indicating a leak across the seal. The small nonzero flow rate and displaced volume occurring between 110 min. and failure may be due to a combination of compliance of the hydraulic system, elastic deformation of the seal and small fluid leakage past the seal. The photo shows damage characteristic of that observed for typical swell packers following differential pressure testing. High pressure was applied on the left side. There is slight axisymmetric damage on the high-pressure end and substantial nonaxisymmetric damage on the low-pressure end.

while still effectively preventing fluid flow. Eventually, as pressure is increased, the seal begins to leak and bulk fluid flow across the seal between the seal and the glass sealing surface is observed. The difference between the three cases is the mode by which nonuniform elastic deformation is enabled. Following a description of observations for each of these cases, a finite element model corresponding to the third case is presented. The leak path predicted by the model shows good qualitative agreement with the location of the leak path observed in the experiment. Finally, a discussion of the proposed leakage mechanism is presented.
2.1 Experimental Observations

Three different seal geometries, sketched in Fig. 2-2, have been studied experimentally. All three seal systems exhibit large elastic deformation prior to fluid flow across the seal, with the essential difference between the three cases being the mode by which this deformation is enabled. The first system is a scale model of an annular swell packer. The seal undergoes large deformations because the applied differential pressure causes material on the low-pressure end to fracture and extrude in

Figure 2-2: Three seal geometries showing seal schematic, sealed state prior to significant differential pressure, and large, nonuniform deformation at onset of leakage. (a) Schematic of annular swellable seal. (b) Annular swellable seal prior to application of differential pressure. (c) Annular swellable seal at point of leakage. (d) Schematic of O-ring seal. (e) O-ring seal prior to leakage. (f) O-ring seal shortly after instant of leakage. (g) Schematic of planar swellable seal, analogous to annular swellable seal but in a flat channel. (h) Swellable planar seal prior to application of differential pressure. (i) Swellable planar seal at point of leakage.
a nonaxisymmetric manner. The second system is an O-ring in a groove. Large deformation occurs elastically and nonaxisymmetrically as the O-ring is elastically extruded out of the groove and into the gap between the inner cylinder and outer glass tube. The third system is a rectangular solid swellable seal used to seal a channel with rectangular cross-section. This geometry was chosen for ease of visualization and analysis.

2.1.1 Annular Swellable Seal

To visually observe the failure of the swell packer seal system in situ, a scale model system (approximately one sixteenth scale of a 1.59 mm ID by 206 mm OD by 457 mm length packer in a 216 mm diameter well) was constructed using a model swellable polymer. The model swell packer was molded from Elite Double 8, a crosslinked silicone-based rubber from Zhermack, with a 9.3 mm inside diameter, 12.0 mm outside diameter, and 25.4 mm length. The packer was molded directly onto a 9.3 mm outer diameter cylinder with 1.6 mm diameter holes drilled radially, which provided fingers of seal material extending into the inner cylinder following the industrial design of Lenné and Vaidya [48]. (In the absence of chemical bonding between the seal material and inner cylinder, these fingers where intended to provide mechanical bonding between the seal and the inner cylinder, but were sheared off upon application of differential pressure, failing to achieve their purpose of preventing slip between the seal and the inner cylinder.) The inner cylinder with attached annular swellable seal was inserted into the 12.5 mm inside diameter gauge glass tube and allowed to swell at room temperature for 48 hours in n-octane from Sigma Aldrich (Product 296988). Upon swelling to equilibrium, the seal system was removed from the octane and connected to a Teledyne Isco Model 500D syringe pump containing deionized water used to apply a differential pressure across the seal. The applied differential pressure was increased stepwise in steps of 69 kPa each minute until the seal could no longer support the fluid pressure. Video and still images of the seal throughout pressurization, leakage and disassembly were recorded as well as pressure, displaced volume and volumetric flow rate provided by the syringe pump.

Figure 2-2(b) shows the seal in the swollen and sealed state prior to application of significant differential pressure. Prior to large deformation, the seal is axisymmetrically swollen, filling the annulus and exerting sealing pressure on both the inner metallic cylinder as well as the outer glass tube.

Figure 2-2(c) shows the model swell packer at the instant of leakage. There is substantial, nonaxisymmetric extrusion of the elastomer between the glass and the metallic support ring on the low-pressure end. The material that is extruded into the thin gap is fractured off from the remainder of the packer, as shown in Fig. 2-3. Although impossible to see from this perspective, there is no substantial extrusion on the back side 180° opposite the field of view.

On the high-pressure end of the seal, there is large, nonaxisymmetric deformation. The high-pressure end is deformed into a cusp with the same azimuthal position as the maximum extrusion on the low-pressure end. Although this cusp is very prominent, it is thought to arise from compressive swelling-induced hoop stresses, and is in fact observed in slightly different geometries purely due to swelling in the absence of applied differential pressure. Cusps for this case and other geometries are discussed further in Section 2.2.3. The important point is that the cusp is a nonaxisymmetric deformation that results in an elongation of the contact line between the seal, sealing surface and fluid.

Figure 2-3: Elastic recovery of the high-pressure end of a swellable annular seal

Figure 2-3 shows the seal upon removal from the glass tube. The high-pressure end of the
seal fully recovers elastically. Although difficult to observe in the image, the outer diameter of the seal is larger than the inner diameter of the glass tube against which it sealed. The material removed from the seal due to fracture remains in the glass tube. We see that the fracture was highly non-axisymmetric. The fact that the high-pressure end of the seal fully recovers is consistent with observations in Fig. 2-1, in which there is a length of the seal that is completely intact over the entire circumference, and there is no evident leak path across this length of the seal. Despite the fact that there is no evident leak path over this intact portion of the seal, the in situ observations reveal that there was large asymmetric deformation that allowed fluid to leak across this length of the seal.

2.1.2 O-Ring

In light of the observations of leakage and full elastic recovery of the high pressure end observed for the swell packer above, an O-ring seal system was investigated. The purpose of examining this system was to decouple the material fracture observed in the preceding case from large asymmetric deformation and show that large asymmetric deformation, even in the absence of material fracture, ultimately leads to seal leakage. Best practices for O-ring system design, such as those found in the Parker O-ring handbook [73] were intentionally not followed, and the O-ring seal system was designed to have a very large gap between the inner cylinder and outer glass tube in order to promote seal leakage at pressures lower than the pressure rating of the glass tube.

A size 012 O-ring with Shore A hardness of 70 was mounted in a groove of diameter 9.8 mm and width 2.4 mm cut into a 10.1 mm diameter solid cylinder. The assembly was inserted into a glass tube with nominal inner diameter of 12.5 mm using centralizers on either end of the O-ring to maintain concentricity. A differential pressure was applied to the O-ring with a syringe pump using as a working fluid a solution of water and fluorescein dye to enhance contrast and visibility.

The deformation of the O-ring during pressurization is shown in Fig. 2-2(e) and (f). Initially there is negligible displacement and deformation of the O-ring system. When sufficient pressure was applied, in this case approximately 380 kPa gauge (55 psig), the O-ring slipped out of the groove asymmetrically and the differential pressure across the O-ring subsequently decreased to 210 kPa gauge (30 psig). As more fluid was pumped in at constant flow rate, the O-ring deformed further at nearly constant differential pressure. Eventually, the O-ring began to leak at a differential pressure of approximately 215 kPa gauge (31 psig) as shown in Fig. 2-2(f). Upon disassembly of the seal system, the O-ring elastically returned to its original state and there was no visible permanent damage.

Detailed photos at the instant of first fluid leakage across the O-ring seal are shown in Fig. 2-4. Fig. 2-4(a) is taken at the instant before leakage and the next three are taken following leakage. The photos show that the leakage does not occur at the tip of the deformed O-ring where deformation is maximum, but it occurs somewhere between the point of maximum deformation and the section of O-ring that remains in the groove. This is analogous to the leak path observed in the subsequent section.

Figure 2-4: Progression of leakage in deformed O-ring. The images show a reflective patch indicative of early leakage above and to the right of the tip where maximum O-ring displacement occurs. The fluid leaks not at the tip where O-ring displacement is maximum but at some intermediate location. This is analogous to the location of the leak path observed in the rectangular seal shown on the right side of Fig. 2-2 and discussed in §2.1.3.

The experiments show that the O-ring can support a differential fluid pressure without leakage while in a state of non-axisymmetric deformation. However, further increasing differential pressure
and deformation of the O-ring results in leakage of fluid past the seal. Upon disassembly of the seal system, the O-ring completely recovers elastically with no permanent damage.

2.1.3 Flat Swellable Seal

The preceding experiments of the annular swell packer and the O-ring show that asymmetric elastic deformation can lead to leakage of seals. In the case of the swell packer, the asymmetric deformation was enabled by asymmetric material fracture, while in the case of the O-ring, the source of asymmetry was slight geometric imperfections in the assembly, such as imperfect concentricity or lack of circularity in the glass tube. To better control these imperfections, to enable visualization of the entire seal system simultaneously and to facilitate analysis, a third seal system with rectangular Cartesian geometry was examined.

In this system, a channel of depth 3.4 mm and width 38.1 mm was sealed by a swellable seal made of two-part Elite Double 32 silicone-based elastomer mixed in a 1:1 mass ratio. The seal was cast in the channel, filling the width, and had an initial thickness of 2.3 mm and an initial length in the streamwise direction of 12.7 mm. The seal was mechanically bonded to the walls by fingers of the elastomer, which also served to prevent leakage around the main body of the seal. A glass cover was positioned on the top surface of the seal to constrain swelling, form the second sealing surface and enable in situ visualization. The seal was swollen in n-octane for two days at room temperature.

Figure 2-2(h) shows the seal geometry in the sealed state before pressurization, and Figure 2-2(i) shows the seal at the instant of leakage. In the sealed state before pressurization, a cusp can be seen near the center of the seal on the top side. This cusp occurs due to the compressive stresses generated by swelling in a geometry that is fully confined in the spanwise direction and constrained in the thickness direction. The maximum compressive stresses are generated in the spanwise direction, due to the complete confinement, which explains the cusp orientation [38, 40, 98]. The resulting deformation is not perfectly symmetric due to frictional effects between the polymer and the sealing surfaces.

Following swelling, a differential pressure was applied across the seal using a solution of fluorescein and water. There were undissolved fluorescein particles in suspension to provide a trace of the leak path as shown in Fig. 2-5. As the pressure increased, the crease on the high-pressure side grew elastically. Eventually the seal leaked and the leak path trace is shown in Fig. 2-5. The path of the fluid across the seal was not that with the minimum path length along the centerline of the seal through the cusp. Rather, the leak path originated at a point near the seal-wall interface on the high-pressure side and extended to the centerline of the seal on the low-pressure end.

The experimental results show that the seal supports a nontrivial differential pressure despite being in a highly deformed state. At low applied fluid pressures, the seal deforms slightly but still does not leak. As the pressure is increased, the deformation of the seal increases while still preventing leakage of fluid across the seal. Eventually a critical differential pressure is reached at which point the seal is highly deformed and fluid leaks across the seal. Upon removal of the differential pressure from the seal, the seal recovers elastically.

2.2 Analysis and Discussion

The preceding experiments all exhibit large deformations of the seal material prior to any fluid leakage across the seal. The deformations are nonuniform and involve stretching as well as shear. More specifically, they all exhibit an elongation of the three-phase contact line at the interface of the fluid, seal and sealing surface. This elongation of the contact line means that the seal material is stretched in a direction tangential to the contact line. In the first two cases of annular seals, the deformations depend strongly on azimuthal position. In the third case, the deformation of the rectangular seal is nonuniform due to the bonding of the lateral boundaries to the sides of the channel while allowing the middle of the seal to remain unsupported. To support these observations, we conducted a finite element analysis of a rectangular elastomeric seal to explain both the mechanism of leakage and the location of the leak path observed in the experiments.
2.2.1 Model of Rectangular Seal

A finite element model of a rectangular solid seal was analyzed using Abaqus. The purpose of this model was to reproduce the salient features of the flat swellable seal experiment and illustrate the mechanism of leakage proposed by this work while maintaining a model of minimum complexity. A nearly incompressible isothermal neo-Hookean constitutive model with strain energy of the form 
\[ U = C_{10} (\bar{T}_1 - 3) + \frac{1}{2} (\bar{J} - 1)^2 \]
with \( C_{10} = 0.5 \) and \( D_1 = 0.01 \) was chosen as the nonlinear material model, giving a small-strain shear modulus of \( \mu_0 = 1 \). (Abaqus is inherently unitless. We can therefore think of this as a shear modulus of \( \mu_0 = 1 \) MPa, which is the order of the shear modulus of the swollen seal investigated in the previous section.) Since this material model does not incorporate the solvent diffusion and swelling of the experimental system, the initial sealing stress prior to application of differential pressure was generated by initially compressing the model seal. The model seal having sides of \( L_x = 1 \), \( L_y = 2 \) and \( L_z = 0.2 \), was compressed to a spanwise stretch ratio of \( \lambda_y = 0.7 \) and an out-of-plane stretch ratio of \( \lambda_z = 0.95 \). (Again, since Abaqus does not use specific units, we can choose any scale for these lengths. A scale of \( L_x = 1 \times (2.72 \text{ cm}) \), \( L_y = 2 \times (2.72 \text{ cm}) \), \( L_z = 0.2 \times (2.72 \text{ cm}) \) approximately corresponds to the scale of the experiments.) These dimensions were chosen to approximate the aspect ratio of the experimental seal in its swollen configuration. Following this initial compression, a differential pressure of \( \Delta P = 0.4 \) (\( \Delta P = 0.4 \) MPa using the scaling above) was prescribed in the direction of the axis of the channel (\( x \)-direction). The problem is symmetric about the centerline of the seal and calculations were performed on half of the domain.

![Figure 2-5: Contours of out-of-plane stress predicted by finite element model of rectangular neo-Hookean seal under differential pressure loading. Yellow indicates least compressive out-of-plane stress. Dark green contours indicate largest compressive contact stress and, therefore, path of greatest resistance. The fluid path of least resistance is normal to the contours of out-of-plane contact stress and corresponds to the experimentally observed path. A trace of the computed path is overlaid on the contour plot.](image)

The plot in Fig. 2-5 shows contours of out-of-plane stress. Along the line of symmetry (left edge of plot) on the high-pressure end and along the boundary on the low-pressure end the stress is most compressive, which provides the largest barrier to fluid leakage between the seal and sealing surface. The stress is most tensile on the high-pressure end near the lateral boundary and on the low-pressure end near the centerline. There is a saddle point in the contact stress.

As shown in the percolation experiments of Lorenz and Persson [56], leak rate increases as contact stress decreases (becomes more tensile). Thus we expect the path of fluid flow to follow the path of minimum compressive contact stress and the fluid flow to be orthogonal to lines of constant contact stress. The path of least resistance for fluid flow was calculated using gradient methods and is overlaid on the plot of contours of contact stress. Qualitatively the fluid path in the experiments and model are quite similar. The finite element model supports experimental observations that leakage occurs due to elastic deformation and the leak path occurs along the path of least compressive sealing stress.

Although the leak path is qualitatively similar between the experimentally observed leak path and
that predicted by the finite element model, the deformed shapes of the seals do not agree particularly well. The most notable difference is the cusp that forms at the centerline on the high-pressure end of the experiments which is not present in the finite element results. We expect that this is due in part to the oversimplified neo-Hookean constitutive model and insufficient spanwise compressive strain, and that better agreement would be obtained using a gel swelling model, such as that of Chester and Anand [13].

2.2.2 Leakage Mechanism

In each of the experimental cases, large deformation preceded leakage. The seals were highly deformed while still not leaking under nontrivial applied differential pressure. Eventually, as the pressure was increased and the deformation increased correspondingly, the seals leaked. The deformation of the seals was nonuniform and involved the stretching of the three-phase contact line, making the contact line longer and thereby also stretching the material points along the contact line in a direction tangential to the contact line.

This can be clearly seen in the latter two experiments. For the O-ring in Fig. 2-4, the original length of the three-phase contact line between fluid, seal and outer sealing surface was $l_{\text{contact}} = \pi D$, where $D$ is the inner diameter of the glass tube against which the O-ring sealed. However, as the O-ring was deformed and extruded out of its gap, the circumference of the O-ring increased as did the length of the contact line. Likewise, in the case of the rectangular solid seal, the length of the initial contact line was simply the width of the channel. However, as the seal deformed into the vee shown in Fig. 2-5, the length of the contact line between the green seal, glass sealing surface and fluid clearly increased. In so doing, the material elements along this contact line were also stretched in a direction tangential to the contact line.

A positive value of Poisson ratio tells us that a material stretched in one direction has a tendency to contract in orthogonal directions. Thus, as the seal material is stretched in the direction tangential to the contact line, it has a tendency to contract in the direction perpendicular to the sealing surfaces. This tendency for contraction results in a decrease in compressive sealing stress between the seal and sealing surface. This mechanism is illustrated schematically in Fig. 2-6. However, because the stretching is nonuniform, the resulting changes in compressive sealing stress are nonuniform, and the final compressive sealing stresses are complicated, as illustrated by the results of the finite element model shown in Fig. 2-5.

To provide a more definitive example of this, we consider the deformation of a characteristic material element in a flat seal. A sketch of the characteristic element in the deformed state is shown in the lower left-hand corner of Fig. 2-6 and in the deformed configuration in the lower right-hand corner of the same figure. In the undeformed, sealed configuration, we denote the sealing stress between the seal and sealing surface as $\sigma_0 = -\sigma_{zz}$. We then consider the deformation of the seal, presumably due to differential pressure loading, and parameterize the resulting motion by a stretch, $\lambda$, a rotation, $\theta$, and a shear, $\gamma$, each sketched in the lower right-hand corner of Fig. 26.

For a neo-Hookean material with strain energy density function $\Psi = \frac{E}{2} (\lambda^2_1 + \lambda^2_2 + \lambda^2_3 - 3)$ and corresponding principal stresses of the form

$$\sigma_i = -p + \mu\lambda_i^2$$

it is shown in the appendix that the contact stress between the seal and sealing surface is independent of the shear, $\gamma$, and the rotation, $\theta$. Instead, the change in contact stress depends solely on the stretch, $\lambda$, with the compressive sealing stress decreasing (becoming more tensile) as stretch increases.

A plot of the dependence of normal stress, $\sigma_{zz}$, versus the stretch is shown in Fig. 2-7 for several values of initial contact stress. As the stretch, $\lambda$, increases, the stress becomes more tensile leading to increased tendency for leakage.

In summary, nonuniform deformation of the seal results in an elongation of the contact line and stretching of material elements in a direction tangential to the contact line. This stretching of material elements tangential to the contact line corresponds to a decrease of compressive contact stress (sealing stress) in a direction perpendicular to the sealing surface. As the squeezing pressure
Figure 2-6: Mechanism by which nonuniform deformation decreases compressive sealing stress. Fluid pressure creates nonuniform deformation and stretching, leading to a decrease in compressive sealing stress.

Figure 2-7: Normal stress as a function of stretch for parameterized deformation of a neo-Hookean solid. The sequence of curves represent different starting compressive sealing stresses. All of the curves are plotted for $\sigma_1 = 0$. Nonzero values of $\sigma_1$ shift the curves up or down.

between the seal and sealing surface decreases, the tendency for leakage increases. Eventually there is a point when pressure is sufficiently large and compressive contact stress sufficiently small such that leakage occurs.
2.2.3 Note on Swelling Instabilities and Cusps

In experiments on swellable seals in both an annular and rectangular geometry, cusps were observed. For the annular case, the cusp was observed only after application of differential pressure and extrusion on the low-pressure end, whereas it was observed prior to application of differential pressure in the flat rectangular case. In a separate experiment depicted in Fig. 2-8, a cusp was observed on an annular swellable seal mounted on a hollow inner cylinder such that pressure was the same on either side of the seal.

These observations suggest that the cusps observed in swellable seals are caused by compressive stresses generated by swelling and are analogous to cusps observed by Hohlfeld [38] and others [40, 98] where the compressive stresses are applied via external compression. A review of cusps and other instabilities in gels is provided by Dervaux and Ben Amar [21].

While the formation of cusps is initially remarkable, the experiment of the flat swellable seal shows that leakage does not occur at the tip of the cusp. Therefore, from the perspective of seal leakage, this cusp formation is only important because it provides a means of nonuniform deformation.

Figure 2-8: Cusp formation due to swelling of an annular seal between a glass tube and a hollow inner cylinder. No differential pressure has been applied, and the cusp arises purely as a result of swelling.

2.3 Conclusions

A mechanism of soft seal leakage has been proposed. In this mechanism, nonuniform elastic deformation of the seal leads to stretching of the seal material in a direction tangential to the contact line between the seal, sealing surface and fluid. This stretching of the seal leads to a decrease in compressive sealing stress between the seal and sealing surface. Intuition and prior work tell us that a decrease in sealing stress increases the tendency for bulk fluid leakage past the seal. As the fluid pressure increases and the sealing stress between the seal and sealing surface decreases, the fluid pressure eventually overwhelms the compressive sealing stress and bulk fluid leakage occurs. Post-failure diagnosis of seal leakage is complicated by the fact that the seal recovers elastically upon disassembly, thereby obscuring this mode of seal failure.

This mechanism has been experimentally observed in situ for three different seal systems. A well-confined annular swell packer underwent nonuniform deformation only after nonuniform extrusion and fracture of the seal material. An O-ring underwent nonuniform deformation in the absence of material failure due to nonaxisymmetric elastic extrusion into the large gap between the inner and outer cylinders. Finally, a rectangular seal underwent nonuniform elastic deformation due to support only at the lateral boundaries. Regardless of the cause of nonuniform elastic deformation, the deformation resulted in a stretching of the seal material in a direction tangential to the contact line. As applied differential pressure was increased, the deformation increased and eventually bulk fluid leakage past the seal was observed. Upon disassembly, each of the seals recovered elastically such that no permanent leak path was evident.

A finite element analysis of the rectangular seal showed that applied differential pressure leads to nonuniform deformation as well as nonuniform sealing stress between the seal and sealing sur-
face. The contour of minimum sealing stress predicted by the model is qualitatively similar to the experimentally observed leak path for the analogous experimental case.

This work highlights the need for proper design of new seal systems to prevent nonuniform elastic deformation. Well-confined O-ring seal systems already accomplish this. However, this leakage mechanism should be kept in mind when designing future seal systems. This mechanism may be especially important for very stiff seals, where a small amount of elastic deformation can lead to a large decrease in compressive sealing stress between the seal and sealing surface. This work provides an answer to the question of how fluid can leak past a seal when there is no apparent leak path upon disassembly of the seal system. Finally, this work underscores the importance of contact stress between the seal and sealing surface, the loss of which is important in other failure modes (i.e. compression set) even in the absence of large elastic deformation.
Chapter 3
Seal Stacking

In this chapter we present a purely experimental investigation of the effect of the geometry and configuration of swell packer systems. Specifically, we investigate two seemingly conflicting conjectures thought to be true in the oil and gas industry. The first conjecture is that the differential pressure supported by a swell packer seal system scales linearly with the length of the seal. This is explicitly documented in intellectual property [48], and it is also implicitly perpetuated by the industry when reporting maximum differential pressure in terms of pressure per unit length of the seal.

The second conjecture is that anti-extrusion devices at the low-pressure end of the seal increase the differential pressure that the seal system can support. Although this is not well-documented in literature, there are intellectual property claims regarding this concept [9].

Indeed, in industry there are swell packer seals of different varieties. One type is bonded to an inner metallic cylinder, and is conventionally referred to as “bonded-to-pipe”. Another variety is a highly modular design, where the elastomeric element is simply slid onto the inner pipe and fixed in place with support rings on either end. This variety is typically called “slip on”. In addition to these designs, there are other hybrid designs, such as one where the elastomer is bonded onto a rigid metal sleeve, and then that sleeve is slid onto an inner tube, which provides both the modularity of a slip-on packer as well as the mechanical integrity of a bonded-to-pipe packer [48, 3].

It is possible that each conjecture is true for a different configuration. For a slip on packer, in the limit when friction between the elastomer and sealing surfaces is small, a simple free-body diagram shows that the force due to applied differential pressure must be supported by the low-pressure metal support. On the other hand, in the bonded-to-pipe configuration, or when friction between the elastomer and sealing surfaces is large, the applied load at the high-pressure end is distributed via friction and shear tractions to the sealing surfaces and less is transmitted to the low-pressure end.

In this investigation, we do not attempt to discern which conjecture is correct for a given interfacial boundary condition (bonded versus slip-on, coefficient of friction, ...). Instead, we hold the interfacial boundary conditions fixed and attempt to determine the effect of length of the seal and configuration of mechanical supports on the maximum differential pressure that the seal can hold. Despite the highly applied nature of these experiments, some unpredicted results were obtained, and these results provide motivation for the investigation of length and compressibility in the subsequent chapter.

3.1 Experimental Procedure

In this sequence of experiments, five different configurations of seal system were investigated. The seals were made of a proprietary, EPDM-based, oilfield grade, carbon-black filled rubber. They were bonded to an inner steel shell at the time of manufacture by an industrial vendor, and these modular elements were slid onto a solid 17-4 PH stainless steel cylinder and fixed in place with metal support rings. The seals were then swollen for either two or three weeks in LVT 200 mineral oil at 82°C
and tested. The testing procedure is based on that developed by Maheshwari at Schlumberger-Doll Research for the testing of “mini packers” [59].

Figure 3-1: Schematic of experimental system used to test small-scale seal systems. Originally developed by Maheshwari [59] and later refined for the present work.

### 3.1.1 Configurations

Five different configurations of seal elements were investigated. Each of these was based on an elastomeric seal rigidly bonded to a steel shell having holes through it, as sketched in Fig. 3-2. A thin layer of rubber lined the inside of the steel shell and was intended to swell to create a seal between the thin shell and the steel basepipe. For each of the configurations, the steel shell had an inner diameter of 3.31 cm (1.305 inch) and outer diameter of 3.49 cm (1.375 inch). The outer diameter of rubber bonded to the outer surface of the steel shell was 4.61 cm (1.816 inch) and the inner diameter of the rubber on the inside of the steel shell was 3.07 cm (1.207 inch). The seal elements had a length of 10.18 cm (4.01 inch) in all but one configuration, where elements of half the standard length were used. In each of the configurations, the elements were swollen inside a pressure vessel with an inner diameter of 4.83 cm (1.90 inch). All mechanical support rings used had outer diameters of 4.61 cm (1.816 inch), corresponding to the outer diameter of the unswollen elastomer, and the rings were axially fixed in place either using set screws or shaft collars, which sat in grooves on the inner mandrel, to prevent rigid-body axial motion of the seals. A sketch of the typical element along with relevant geometries is shown in Fig. 3-2. A summary of the relevant dimensions is given in Table 3.1.

The swellable elastomer sealing elements described above were assembled on the solid steel mandrel in one of five configurations, to examine the effects of configuration (proximity, mechanical supports, length) on the maximum differential pressure. A summary of the configurations is provided in Table 3.2.

In the control configuration, a single element was fixed to the mandrel, and stainless steel support rings of outer diameter 4.61 cm (1.816 inch) were affixed immediately adjacent to the elastomeric sealing element.

In the second configuration, two of these 10 cm long sealing elements were mounted on a solid mandrel with a space of 10 cm between the low-pressure end of one and the high-pressure end of the other. Each elastomeric element had stainless steel support rings on either end.

In the third configuration, the two elements, each of length 10 cm, were mounted on the steel mandrel immediately adjacent to each other with no support between them. In some sense, this
Figure 3-2: Sketch of swellable element cross-section. Labeled dimensions are in inches.

Table 3.1: Dimensions of laboratory-scale sealing experiment investigating effect of seal configuration on performance of seal system

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mandrel OD</td>
<td>3.06 cm (1.207 in)</td>
</tr>
<tr>
<td>Steel Shell ID</td>
<td>3.31 cm (1.305 in)</td>
</tr>
<tr>
<td>Steel Shell OD</td>
<td>3.49 cm (1.375 in)</td>
</tr>
<tr>
<td>Unswollen Rubber OD</td>
<td>4.61 cm (1.816 in)</td>
</tr>
<tr>
<td>Pressure Vessel ID</td>
<td>4.83 cm (1.90 in)</td>
</tr>
<tr>
<td>Control Element Length</td>
<td>10.2 cm (4.01 in)</td>
</tr>
<tr>
<td>Half Element Length</td>
<td>5.10 cm (2.00 in)</td>
</tr>
</tbody>
</table>

Figure 3-3: Packer stacking control configuration. Single element of length 10.2 cm.
approximated the behavior of a single element that was 20 cm long. However, the fact that the
interface between the two seal elements was a contact interface, as opposed to a rigidly bonded
interface, meant that the interactions were due to normal contact traction and frictional shear
traction, as opposed to equivalent displacements if they had been perfectly bonded.

In the fourth configuration, two elements each with length of 10 cm, were affixed to the mandrel
with a rigid metal support rings between them. The rigid metal support rings were held in place
with a smaller diameter ring mounted to the mandrel with set screws.

In the fifth and final configuration, four packer elements, each of length 5 cm, were mounted
on the mandrel. Between each element, a single metal support ring with outer diameter 4.61 cm
(1.816 inch) was placed. At the two ends of the seal stack, metal support rings of the same diameter
were affixed to the mandrel with set screws on the high-pressure end and with a shaft collar on the
low-pressure end. Because the elastomeric seal elements were bonded onto inner steel cores, the
axial position of all components in the seal stack were fixed.

3.1.2 Test Procedure

Upon assembly of the sealing elements onto the solid mandrel, the seal assembly was inserted into
a High Pressure Equipment Co. confined gasket pressure vessel (Model 802554) having an inner
diameter of 4.83 cm (1.90 inches) and test length of 43.1 cm (17.01 inches) and sealed. The swelling
solvent, consisting of LVT 200 low-viscosity mineral oil, was pumped into the vessel using an Isco
Teledyne Model 260D syringe pump, which both provides fluid at specified pressure as well as
Table 3.2: Configurations for seal stacking experiment.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Control</td>
<td>1 Element</td>
</tr>
<tr>
<td></td>
<td>10 cm length</td>
</tr>
<tr>
<td>2. Stacked - Independent</td>
<td>2 Elements</td>
</tr>
<tr>
<td></td>
<td>Each 10 cm length</td>
</tr>
<tr>
<td></td>
<td>Spaced 10 cm apart</td>
</tr>
<tr>
<td>3. Stacked - Adjacent - No Intermediate Support</td>
<td>2 Elements</td>
</tr>
<tr>
<td></td>
<td>Each 10 cm length</td>
</tr>
<tr>
<td></td>
<td>Stacked adjacent</td>
</tr>
<tr>
<td>4. Stacked - Intermediate Support</td>
<td>2 Elements</td>
</tr>
<tr>
<td></td>
<td>Each 10 cm length</td>
</tr>
<tr>
<td></td>
<td>Stacked intermediate support</td>
</tr>
<tr>
<td>5. Half-Elements - Stacked</td>
<td>4 Elements</td>
</tr>
<tr>
<td></td>
<td>Each 5 cm length</td>
</tr>
<tr>
<td></td>
<td>Intermediate support</td>
</tr>
</tbody>
</table>

measures pressure, volumetric flow rate, and displaced volume. The pressure vessel was heated using an electric resistance heater (BriskHeat model BSAT201008) with temperature controlled using a Micromatic E54 Brewer's Edge Controller II thermostat with a thermocouple inserted in the thermocouple port at the low-pressure end of the pressure vessel. Heating and swelling was maintained either for two weeks or three weeks.

Upon swelling for either two weeks or three weeks, the seal system was subjected to applied differential pressure using the syringe pump. In the test procedure, an applied pressure of 50 psi was initially applied. Pressure was then ramped at a constant rate of 10 psi/min until there was a surge in fluid flow rate and a corresponding drop in differential pressure, at which point it was determined that the seal had failed. The maximum pressure during this test procedure is used as the critical differential pressure. The entire test lasts for a few hours and depends on the maximum differential pressure that the seal system supports. Characteristic curves for pressure, volumetric flow rate, and total pumped volume (integral of volumetric flow rate) are shown in Fig. 2-1.

Following failure to support differential pressure, the pump was stopped and the heat was turned off. Upon cooling, the pressure vessel was disassembled and the seal assembly was pushed out of the pressure vessel using a screw press, pushing on the low-pressure end of the mandrel such that the direction of extrusion was the same as that experienced during application of hydraulic differential pressure. Qualitative observations of the damaged seal were made upon removal from the pressure vessel.

3.2 Results

As is the nature of experimentation, the time cost of these experiments was high and fewer replications than ideal were conducted for each configuration. It is readily acknowledged that the statistical significance of these results may be questioned. Nevertheless, the results show some interesting trends. Four replications of the control configuration were tested, and two replications for each of the other configurations were tested.

A summary of the differential pressure supported for configuration is shown in Fig. 3-8. Table 3.3 contains a tabular summary of the results.
Table 3.3: Results for investigation of seal stacking configurations.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Total Length of Rubber</th>
<th>Differential Pressure</th>
<th>Differential Pressure per Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Control</td>
<td>10.2 cm (4.01 in)</td>
<td>9.6 ± 0.4 MPa (1400 ± 60 psi)</td>
<td>95 ± 4 MPa/m (350 ± 15 psi/m)</td>
</tr>
<tr>
<td>2. Stacked - Independent</td>
<td>20.4 cm (8.02 in)</td>
<td>14.1 ± 1.4 MPa (2040 ± 200 psi)</td>
<td>69 ± 7 MPa/m (255 ± 25 psi/m)</td>
</tr>
<tr>
<td>3. Stacked - Adjacent</td>
<td>20.4 cm (8.02 in)</td>
<td>9.5 ± 0.7 MPa (1380 ± 95 psi)</td>
<td>47 ± 3 MPa/m (170 ± 12 psi/m)</td>
</tr>
<tr>
<td>4. Stacked - Intermediate Support</td>
<td>20.4 cm (8.02 in)</td>
<td>19.3 ± 0.8 MPa (2800 ± 110 psi)</td>
<td>95 ± 4 MPa/m (350 ± 14 psi/m)</td>
</tr>
<tr>
<td>5. Stacked - Half Length Elements</td>
<td>20.4 cm (8.02 in)</td>
<td>32.3 ± 0.7 MPa (4680 ± 100 psi)</td>
<td>160 ± 3 MPa/m (580 ± 13 psi/m)</td>
</tr>
</tbody>
</table>

3.3 Discussion

The results show that, for these experimental conditions with a relatively smooth outer sealing surface, maximum differential pressure supported by the seal does not scale linearly with length. Instead, mechanical support plays a much more significant role than the total length of the rubber sealing elements.

A comparison between the control configuration (one element 10 cm in length) and the third configuration, where two elements were stacked adjacent to each other reveals that both configurations support essentially the same differential pressure, despite the fact that the third configuration has double the length of elastomeric seal elements. If one were to draw conclusions purely from this experiment, one might conclude that length of the seal does not at all affect the differential pressure that the seal system can support.

A comparison between the control configuration and the fourth configuration, in which two elements are stacked with an intermediate support ring between the two elements, shows that the
fourth configuration can support twice the differential pressure as the control configuration. Normalized per unit length, the differential pressures supported are the same in the two cases. This is consistent with the comparison of first and third configurations, and the observation that length of the sealing element does not affect critical differential pressure. Rather, it is the mechanical supports that improve performance, and doubling the number of mechanical supports doubles the amount of differential pressure that the seal system can support.

One might then ask why there is a difference between the second configuration, in which two seal elements, each with support rings on either end, are stacked with some space between, and the fourth configuration, where the space between the two elements is minimized. Although we do not have a conclusive answer, one possible cause is the fact that the fluid between the two sealing elements is not perfectly incompressible. Rather, the bulk modulus of typical mineral oils is on the order of $K \sim 2 \text{ GPa}$. The increased length of mineral oil filling the volume between the two seals in Configuration 2, as compared with Configuration 4, means that the low-pressure seal provides less mechanical support to the high-pressure seal, and therefore, there is less synergy between the two. In other words, the compressible oil between the two sealing elements effectively decouples the two seals.

### 3.3.1 Lumped Parameter Analysis of Seal Stacking

In order to understand the effect of seal element spacing, let us idealize the system as a lumped parameter system. We consider the system sketched in Fig. 3-9. First a free body diagram of each of the seal elements is drawn. On the left element, and applied fluid pressure, $P_0$, is imposed. This is the differential pressure that the system of seals is required to support. The applied load is $F_0 = P_0 A_c = P_0 \pi (b^2 - a^2)$, where $A_c$ is the cross-sectional area of the seal, $b$ the inner diameter of the steel pressure vessel and $a$ the outer diameter of the inner solid steel cylinder (mandrel). The left seal element supports this applied load by a combination of shear traction and axial force supplied by both the metal support ring and the fluid on the low-pressure end. The force exerted by the fluid on the low-pressure end of the first seal element is $F_{\text{fluid}} = P_{\text{fluid}} A_c$, where we denote the pressure of the fluid between the two sealing elements as $P_{\text{fluid}}$.

![Figure 3-9](image_url)

Figure 3-9: Schematic showing lumped parameter model for stacking of swellable elastomeric seals. (a) Schematic of stacking configuration. (b) Free body diagram of sealing elements in stacked configuration. (c) Lumped parameter spring model.
This pressure in the intermediate fluid between the two sealing elements is what couples the two elements together. In the quasistatic limit, the force $F_{\text{fluid}}$ exerted by the intermediate fluid on the low pressure end of the left seal element is equal and opposite to the force exerted by the same intermediate fluid on the high-pressure end of the right seal element. This applied fluid force must in turn be supported by shear traction on the inner and outer surfaces of the seal element (due to chemical bonding and/or friction) and by the axial force supported by the metal support ring.

For both the high-pressure element and the low-pressure element, we model the shear traction and the axial force due to the metal support rings as linear springs. In general, this is a poor assumption, but it provides a leading-order analytically tractable model to explain some of the observed experimental effects. The linear spring used to model shear traction has a stiffness $k_{\text{shear}}$, such that the shear force is given by

$$F_{\text{shear},1} = k_{\text{shear}} \pi_A$$  \hspace{1cm} (3.1a)

$$F_{\text{shear},2} = k_{\text{shear}} \pi_C$$  \hspace{1cm} (3.1b)

where $\pi_A = A_c^{-1} \int u_A dA_c$ is the characteristic displacement given by the area-averaged displacement on the high-pressure end of the left sealing element and $\pi_C$ is the characteristic displacement of the high-pressure end of the right sealing element. This is a rather crude approximation because the shear traction is distributed along the length of the sealing element, and it is proportional to the shear strain along the length. Here we are approximating the total shear force as proportional to the displacement at the high-pressure end. We will remedy this crudeness in the subsequent chapter.

In each sealing element, the axial force due to the metal support ring is again approximated as a linear spring, such that the forces due to the metal support rings are

$$F_{\text{support},1} = k_{\text{support}} \pi_B$$  \hspace{1cm} (3.2a)

$$F_{\text{support},2} = k_{\text{support}} \pi_D$$  \hspace{1cm} (3.2b)

Finally, both the elastomer and the ambient solvent between the sealing elements are compressible. For an elastic body under infinitesimal uniaxial strain, the effective spring stiffness due to bulk compression is given as

$$k_{\text{bulk}} = \frac{F}{u} = (2\mu + \lambda) A_c \frac{KA_c}{L} \sim \frac{KA_c}{L}$$  \hspace{1cm} (3.3)

where $F$ is the total applied force, $u$ the displacement, $\mu$ the shear modulus, $\lambda$ a Lamé modulus, $A_c$ the cross-sectional area, $L$ the length parallel to the axis of the applied strain, $K$ the bulk modulus, and the approximation holds when the shear modulus, $\mu$, is much less than the bulk modulus, as in the case of liquids and typical elastomers. The compressive forces in the two elastomer elements, due to compression, are

$$F_{\text{bulk},1} = k_{\text{bulk}} (\pi_A - \pi_B)$$  \hspace{1cm} (3.4a)

$$F_{\text{bulk},2} = k_{\text{bulk}} (\pi_C - \pi_D)$$  \hspace{1cm} (3.4b)

and the compressive force that the intermediate fluid exerts on the low-pressure end of the left sealing element and the high-pressure end of the right sealing element is

$$F_{\text{fluid}} = k_{\text{fluid}} (\pi_B - \pi_C)$$  \hspace{1cm} (3.5)

Using this lumped parameter system, we can write the governing equations for quasi-static equilibrium by summing forces acting on each node to give

$$\begin{bmatrix}
  k_{\text{shear}} + k_{\text{bulk}} & -k_{\text{bulk}} & 0 & 0 \\
  -k_{\text{bulk}} & k_{\text{bulk}} + k_{\text{support}} + k_{\text{fluid}} & -k_{\text{fluid}} & 0 \\
  0 & -k_{\text{fluid}} & k_{\text{fluid}} + k_{\text{shear}} + k_{\text{bulk}} & -k_{\text{bulk}} \\
  0 & 0 & -k_{\text{bulk}} & k_{\text{bulk}} + k_{\text{support}}
\end{bmatrix}
\begin{bmatrix}
  \pi_A \\
  \pi_B \\
  \pi_C \\
  \pi_D
\end{bmatrix} =
\begin{bmatrix}
  F_0 \\
  0 \\
  0 \\
  0
\end{bmatrix}$$  \hspace{1cm} (3.6)
Fig. 3-10 shows characteristic plots for the solution of the lumped parameter model formulated in Eqn. (3.6). The top plot shows the distribution of forces as a function of the effective stiffness of the intermediate fluid, $k_{\text{fluid}}$. When the fluid is highly compressible or very long, having small stiffness, the high-pressure seal, denoted ‘1’, is effectively decoupled from the low-pressure seal. The entirety of the applied load is supported by the high-pressure seal. However, as the stiffness of the intermediate fluid is increased, presumably by decreasing the distance between the two sealing elements, the coupling between the high-pressure element and the low-pressure element increases. More load is supported by both shear and extrusion in the low-pressure element.

![Graph showing distribution of forces as a function of effective stiffness of the intermediate fluid.]

Figure 3-10: Representative solution of the lumped parameter model showing that, as the fluid becomes less compressible and effectively stiffer, there is a stronger coupling between the high-pressure element and the low-pressure element. Parameter values for the results are $k_{\text{shear}}/k_{\text{bulk}} = 0.1$, $k_{\text{support}}/k_{\text{bulk}} = 0.2$. Applied load is normalized such that the maximum value of $\pi_A$ is unity.

Although Fig. 3-10 does not show it because the values of stiffnesses of the elastomer and intermediate fluid are modest, if both the seal elements and the intermediate fluid had extremely large values of effective stiffness compared with the effective stiffnesses for shear and extrusion, the forces supported by the low-pressure rings in both the high-pressure and low-pressure elements would be equivalent, and the forces supported by shear in both the high-pressure and low-pressure seal elements would be equal. Furthermore, the ratio between shear and extrusion forces would equal...
the ratio between the shear stiffness, $k_{\text{shear}}$ and the metal ring support stiffness, $k_{\text{support}}$.

$$
\lim_{k_{\text{bulk}}, k_{\text{fluid}} \to \infty} F_{\text{support, } 1} = \lim_{k_{\text{bulk}}, k_{\text{fluid}} \to \infty} F_{\text{support, } 2} = k_{\text{support}} \lim_{k_{\text{bulk}}, k_{\text{fluid}} \to \infty} F_{\text{shear, } 1} = k_{\text{support}} \lim_{k_{\text{bulk}}, k_{\text{fluid}} \to \infty} F_{\text{shear, } 2}
$$

(3.7)

Stated another way, if the bulk moduli of the fluid and elastomer are infinitely large, then all of the displacement in the system either comes from shearing or from extrusion past the supports. All displacement on the low-pressure end of the left seal is transmitted to the high-pressure end of the right seal, because the intermediate fluid is incompressible. The magnitudes of the shearing and support displacements are inversely proportional to their respective stiffnesses. Both seals behave the same as each other, and the applied load is evenly distributed over the two seals.

### 3.3.2 Stacking Many Short Sealing Elements Together

The final interesting result from the sequence of stacking experiments comes from the results of the fifth configuration, for which four seal elements, each 5 cm long, were stacked on a mandrel, separated by metal support rings. This stacked configuration supported an average differential pressure of 32.3 MPa (4680 psi), which is a factor of 3.3 times larger than the control configuration and 67 percent better than the fourth configuration (two stacked elements with intermediate support ring). This result solidifies our conclusion that, in these relatively smooth-walled pressure vessels, the differential pressure depends more strongly on the low-pressure mechanical supports and less strongly on the length of the sealing element(s).

Indeed, the seal performed so well that, in both of the replications for this configuration, it was not the elastomeric part of the system that failed. Rather the failure was due to the buckling of the steel shell inside the elastomer, similar to the mechanism of the crushing of an aluminum can. From a design perspective, if it had been known a priori that the steel shell would buckle, the pattern of radial holes throughout the shell, used for diffusion of solvent when swelling, could be designed more intelligently to reduce this tendency for buckling. Fig. 3-11 shows a picture of the system after failure, revealing the buckling of this low-pressure element, which was responsible for supporting the entirety of the applied load.

![Photograph of post-failure seal system revealing that steel shell of the low-pressure sealing element (right-most element) buckled.](image)

We can again analyze this stacking configuration using the lumped model of above. Here we denote the effective stiffnesses of each of the low-pressure support rings as $k_{\text{support}}$ and the effective stiffnesses of the shear along each of the sealing elements as $k_{\text{shear}}$. The effective stiffness of each elastomeric element due to volumetric compression of the elastomer is denoted as $k_{\text{bulk}}$ and the effective stiffness of the fluid between each elastomeric element is denoted $k_{\text{fluid}}$. Then the linear system for $N$ elements corresponding to that given in Eqn. (3.6) is
where the first two rows and the last two rows are given in Eqn. 3.6, and the interior rows are provided in Eqn. (3.8).

If we consider the limiting case when the sealing elements are short, such that their bulk stiffness $k_{\text{bulk}} = (2\mu + \lambda)A_e/L$ is large, and also assume that there is very little fluid between adjacent sealing elements such that $k_{\text{fluid}}$ is large, then the applied fluid force, $F_0$, will be evenly distributed over the $N$ sealing elements.

$$F_{\text{shear}} = \frac{F_0}{N} \frac{k_{\text{shear}}}{k_{\text{shear}} + k_{\text{support}}} \quad (3.9a)$$

$$F_{\text{support}} = \frac{F_0}{N} \frac{k_{\text{support}}}{k_{\text{shear}} + k_{\text{support}}} \quad (3.9b)$$

Therefore, doubling the number of seal elements should decrease the force, and in turn the stress, at each element by a factor of two. Continuing the assumption of linearity, if we prescribe a critical stress at failure or critical force at failure ($F_{\text{shear,crit}}$ or $F_{\text{support,crit}}$) in an element, the total force, $F_{0,\text{max}}$, that the system can support will scale with the number of seal elements.

$$F_{0,\text{max}} \propto NF_{\text{support,crit}} \propto NF_{\text{shear,crit}} \quad (3.10)$$

Although the experimental results do not quantitatively support this coarse, back-of-the-envelope approximation, they support it qualitatively. Furthermore, the experiments in this configuration were flawed in that the mode of failure was unlike anything seen in the other configurations. Here the steel core buckled before the elastomeric part of the seal failed. It remains to be determined whether the performance of the seal does scale linearly with the number of mechanical supports.

### 3.4 Extension – Swellable Packer Element with Embedded Mechanical Support

In light of observation that increasing the number of metal support rings in a seal stack improved performance of the seal system, a new design for a swellable elastomeric seal was designed, built and tested. The new design incorporated a series of metal support rings embedded within the elastomer and rigidly attached to the inner metal cylinder. Aside from the attempt at improved performance, there were a number of logistical reasons for embedding support rings within the elastomer. One of these was that the oil and gas industry is averse to risk, and any new product that has a visual appearance different than existing products immediately receives increased scrutiny. Embedding rings within the elastomer means that the visual appearance of the seal system can be identical to existing seals, but with improved performance. Another reason for embedding rings within the elastomer, as opposed to simply combining modular elements with intermediate rings is that the rings could be machined directly onto the inner pipe, thereby preventing the buckling experienced in the experiments done using the fifth configuration above. Finally, embedding the rings within the rubber meant that, when transfer molding the elastomeric seals during manufacturing, the fact that the internal support rings had a smaller diameter than the elastomer would aid in the flow of the elastomer around the outer edges of the metal support rings. However, regardless of all these minor issues, the purpose of the design was to improve performance by increasing the maximum amount of differential pressure supported by the seal system.
3.4.1 Design

Photographs of the control design along with the proposed improved design are shown in Fig. 3-12. The sealing elements were molded using polydimethylsiloxane (Dow-Corning Sylgard 184) which was prepared according to the manufacturer’s recommended recipe of 10:1 base polymer to curative on a mass basis. The reason for choosing this, as opposed to an oilfield grade polymer was that this could be molded in-house. In both configurations, the PDMS was bonded to an aluminum mandrel using a commercially available primer (Dow-Corning PR-1200 RTV Red). In the control configuration, the outer diameter of the mandrel was 3.06 cm (1.205 in) and the outer diameter of the as-molded seal was 4.61 cm (1.816 in). The total length of the seal was 10.2 cm (4.00 in). In the configuration with the embedded rings, the inner diameter, outer diameter and length were the same. The only difference was that three rings were embedded within the body of the elastomer. Each support ring had an outer diameter of 4.45 cm (1.75 in) and a thickness of 0.318 cm (0.125 in). Each support ring had four holes of diameter 0.318 cm (0.125 in) in order to facilitate filling the entire mold and removing all the bubbles during molding. A engineering drawing of the mandrel for the embedded support design is included in Fig. 3-12.

![Design of sealing element with embedded support rings within the elastomeric element.](image)

**Figure 3-12:** Design of sealing element with embedded support rings within the elastomeric element. (Top) Photographs of design with embedded rings and control design. (Bottom) Engineering drawing of mandrel with embedded rings machined onto it.

3.4.2 Results

Due to time constraints, only two replications of the control configuration and one replication of the embedded ring configuration were tested. The results are summarized in Table 3.4. Despite the
limited number of replications, the results indicate that embedding mechanical supports within the swell packer improves performance, and may be an effective method for deploying modular sealing elements with enhanced performance.

<table>
<thead>
<tr>
<th>Description</th>
<th>Critical Differential Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Test 1</td>
<td>14.9 MPa (2165 psi)</td>
</tr>
<tr>
<td>Control Test 2</td>
<td>17.3 MPa (2513 psi)</td>
</tr>
<tr>
<td>Embedded Rings</td>
<td>26.3 MPa (3817 psi)</td>
</tr>
</tbody>
</table>

The critical differential pressure at leakage was significantly higher than that for both replications of the control configuration. There is a rather large discrepancy between the critical differential pressure for the two control configurations. One possible cause is that, in Control Test 1, the seal was swollen for 5.2 days, but it was swollen for 6.7 days in Control Test 2. The importance of swell duration on critical differential pressure has been recognized, but it is still not yet conclusively understood. Part of the reason that the configuration with embedded rings does not support four times as much pressure as the control configuration is that the outer diameter of the embedded rings is only 4.45 cm (1.75 inch), while the outer diameter of the metal support ring on the low-pressure end in both configurations is 4.62 cm (1.82 in). It is widely known that, a smaller diameter support ring and hence a larger extrusion gap, supports less differential pressure than a configuration with a larger diameter support ring and smaller extrusion gap. Because the extrusion gap for the three embedded rings is large, it is reasonable to expect that the load will not be evenly distributed.

### 3.5 Conclusions

The experiments on the commercial carbon black filled EPDM elastomer show that, when tested in relatively smooth-walled pressure vessels, the length of the elastomer has relatively little effect on the differential pressure supported by the seal system. This is true even though the elastomers were bonded to the inner mandrel, effectively preventing axial displacement along this inner surface. The result is that shear along this inner surface accounts for very little of the load transferred from the applied pressure to the basepipe. Instead, it is the metal annular support rings that bear the majority of the load. Increasing the number of these support rings increases the maximum differential pressure that the seal system can support. For Configuration 3, the critical differential pressure was doubled compared to that of the control configuration. For the fifth configuration, where there were four seal elements, the critical differential pressure was increased by a factor of 3.3 over the control configuration. In that case, failure occurred due to the buckling of the steel shell and not initially due to failure of the elastomer. If the steel shell did not buckle, the critical differential pressure would be greater than 3.3 times larger than that for the control configuration. One might speculate that it would be four times larger. Without more experiments, we do not know. In short, in smooth-walled pressure vessels for configurations tested, length of the elastomeric material is less important than number and spacing of metal support rings.
Chapter 4

The Combined Effect of Seal Length and Bulk Modulus on the Tendency for Fracture at the Ends of the Seal

The results from the previous chapter indicate that mechanical supports are of primary importance in determining the maximum differential pressure that a seal system can support. Furthermore, in comparing the control configuration to that in which two seals were stacked adjacent to each other, with no intermediate metal support, the maximum differential pressure was the same for both configurations, despite one configuration having twice as much rubber as the other. Although stacking two elements adjacent to each other is not identical to a single element that is twice as long, because the boundary conditions at the interface are traction boundary conditions (friction and contact) rather than displacement boundary conditions, it is expected that both systems are comparable and that we can understand the stacking effect in one system by analyzing the effect of length for a single element. Furthermore, the question of length of a single sealing element is important in the design of these seals, and understanding the stress, strain and failure mechanisms as a function of length is one consideration for design engineers.¹

In order for seal length to have an effect on the behavior of the seal, we must either incorporate the effect of finite compressibility or the effect of friction, or both. In the absence of compressibility, any incremental volume displacement on the high-pressure end of the seal is directly transmitted to the low-pressure end of the seal. In the absence of friction, any load applied to the high-pressure end of the seal is directly transmitted to the low-pressure end. In both these cases, the effect of length is mitigated.

It is well-known in the literature and practice of seal engineering that finite compressibility is fundamentally important for the performance of short seals, such as O-rings [43]. The highly confined nature of O-ring seals means that, despite the bulk modulus being orders of magnitude larger than the shear modulus, volumetric deformations are of comparable importance to volume-preserving isochoric deformations. Despite the realization that compressibility is important for short seals, it has been largely overlooked in recent models of long seals for oil and gas applications [11, 16, 22, 51, 53, 57, 58]. Part of the reason is that the focus has been on the use of gels as seals. These gel-based seal investigations have followed the work on use of gels as seals in microfluidic [6] and the use of gels in biologic applications. In both biologic and microfluidic applications, the hydrostatic pressure experienced by the gel is small, and the gel can be treated as perfectly incompressible. However, in oil and gas sealing applications, both the ambient hydrostatic pressure as well as the applied differential pressure are quite large, often exceeding the shear modulus of the

¹Another motivator for long seals is the irregularity of the wellbore and unusually large wellbore diameters called washouts. In order to seal over irregularly shaped wellbores, long seals are often used to span these washouts and form a seal on either side.
material. In these situations, the effect of finite bulk modulus becomes significant.\(^2\)

In this chapter we present analytic models for long, slightly compressible seals in both plane strain and axisymmetric configurations. The models are linear elastic Saint-Venant type models. Agreement between the models and finite deformation, hyperelastic finite element simulations indicate the effect of nonlinearity is not important in the bulk of the seal. Both the inability of the Saint-Venant solution to satisfy the boundary conditions and the neglect of nonlinearity mean that there is poor agreement between the analytic model and the finite element solutions near the high-pressure and low-pressure ends of the seal. Despite this poor agreement near the ends, we show that stress measures predicted by the analytic solution correlate with energy release rates for short cracks in the high stress concentration regions near the ends of the seal. The correlations are reasonable over a wide range of aspect ratios and ratios of bulk modulus to shear modulus, and they can be used to understand the effect of seal length on the tendency for failure at the two ends.

### 4.1 Approximate Plane Strain Solution

Here we present the analysis of the effect of length and compressibility for a seal under plane strain deformation. Plane strain deformation is an approximation of axisymmetric geometry, presented below, in the limit when the thickness of the seal is much less than the radius of curvature. Although this solution neglects the effect of curvature, it presents a more clear picture of the relevant scales in the problem. Specifically, we show that there is an important parameter that is a function of both the ratio of elastic moduli, \(\mu/\lambda\), and the aspect ratio, \(L/H\).

Upon deriving the solution for the plane strain case and fully exploring its implications, we derive the approximate solution in the axisymmetric case in §4.5. The axisymmetric solution will have the same intrinsic dependence on material properties and aspect ratio as the plane strain case, and there is simply one additional parameter to account for the effect of curvature. We later show that when curvature becomes unimportant, the axisymmetric solution simplifies to the plane strain solution. Therefore, understanding of the plane strain solution facilitates interpretation of the axisymmetric solution.

#### 4.1.1 Boundary Value Problem

We consider the deformation of an elastic body on the domain

\[
\Omega = \{(X^*, Y^*) | 0 \leq \chi^* \leq L, 0 \leq Y^* \leq H\}
\]

as depicted in Fig. 4-1. The displacement of a material point located at position \(X^*\) in the undeformed configuration is given by

\[
u^* = x^* - X^* = u^* (X^*, Y^*) \vec{e}_1 + v^* (x, y) \vec{e}_2
\]

where \(X^* = \chi^* (X^*)\) is the position of the material point \(X^*\) in the deformed configuration, as is convention in solid mechanics texts [39, 95, 34]. The formality and rigor of this notation is omitted here, and the reader is referred to these texts for the complete details.

The infinitesimal strain tensor, \(\epsilon^*\), is

\[
\epsilon^* = \frac{1}{2} \left( \text{Grad}^* \cdot u^* + (\text{Grad}^* \cdot u^*)^T \right), \quad \epsilon_{ij}^* = \frac{1}{2} \left( \frac{\partial u_i^*}{\partial X_j^*} + \frac{\partial u_j^*}{\partial X_i^*} \right)
\]

where \(\text{Grad}^*\) denotes the gradient with respect to the material coordinates, \(X^*\).\(^3\) We treat the

\(^2\)Contrary to the common assumption that elastomers are incompressible, the bulk modulus of rubbers is often on the order of \(K_{\text{rubber}} \approx 2\) GPa, which is well below the bulk modulus of steel, which is on the order of \(K_{\text{steel}} \approx 160\) GPa. Therefore, on an absolute scale, rubber is far more compressible than steel.

\(^3\)For infinitesimal displacements, the gradient, divergence or curl with respect to the material point \(X^*\) is indistinguishable, to leading order, from that with respect to the deformed coordinate, \(x^*\).
material as an isotropic linear elastic solid with constitutive relation

\[
\sigma^* = 2\mu \varepsilon^* + \lambda (\text{tr} \varepsilon^*) I, \quad \sigma^*_{ij} = 2\mu \varepsilon^*_{ij} + \lambda \epsilon_{kk} \delta_{ij}
\]  
(4.4)

where \( \mu \) and \( \lambda \) are the Lamé moduli and \( I \) is the identity tensor. The material parameter \( \mu \) is the shear modulus, sometimes written as \( G \). The other modulus, \( \lambda \), can be written in terms of more familiar material properties as

\[
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = K - \frac{2}{3}\mu
\]  
(4.5)

where \( E \), \( \nu \) and \( K \) are the Young’s modulus, Poisson ratio and bulk modulus, respectively. For elastomers, for which the bulk modulus is much larger than the shear modulus, \( K \gg \mu \), we see that \( \lambda \) is nearly equal to \( K \).

\[
\lambda \approx K \text{ when } K \gg \mu
\]  
(4.6)

This will be relevant in the work that follows.

Finally, conservation of linear momentum, in the limit of equilibrium and the absence of body forces, requires the divergence of the Cauchy stress to vanish.

\[
\text{Div}^* \sigma^* = 0, \quad \frac{\partial \sigma^*_{ij}}{\partial X_j} = 0
\]  
(4.7)

Combining the above equations for strain, stress as a function of strain (constitutive) and equilibrium, and assuming the elastic moduli are spatially uniform, gives the Navier displacement equations

\[
(\lambda + \mu) \text{Grad}^* (\text{Div}^* \boldsymbol{u}^*) + \mu \text{Div}^* (\text{Grad}^* \boldsymbol{u}^*) = 0
\]  
(4.8)

or in component form

\[
(\lambda + \mu) \frac{\partial}{\partial X^*} \left( \frac{\partial u^*_i}{\partial X^*} + \frac{\partial v^*_i}{\partial Y^*} \right) + \lambda \left( \frac{\partial^2 u^*_i}{\partial X^* \partial X^*} + \frac{\partial^2 u^*_i}{\partial Y^* \partial X^*} \right) = 0
\]  
(4.9a)

\[
(\lambda + \mu) \frac{\partial}{\partial Y^*} \left( \frac{\partial u^*_i}{\partial X^*} + \frac{\partial v^*_i}{\partial Y^*} \right) + \lambda \left( \frac{\partial^2 v^*_i}{\partial X^* \partial X^*} + \frac{\partial^2 v^*_i}{\partial Y^* \partial X^*} \right) = 0
\]  
(4.9b)

The boundary conditions we consider here are those of no displacement on the inner surface of the seal at \( Y^* = 0 \) in accordance with seals that are bonded on their inner surface to a rigid substrate.

\[
u^* = 0 \text{ on } Y^* = 0
\]  
(4.10)

On the outer surface at \( Y^* = H \) there is typically frictional contact between the seal and the surface against which it seals. In open-hole wellbores, we expect the frictional contact to be large such that

![Shear-free Contact: \( v^* = \sigma_{xy}^* = 0 \)]
zero displacement is a reasonable approximation. In laboratory testing in relatively smooth-walled pressure vessels, qualitative observations indicate that friction is often small. Here we implement a boundary condition consistent with the negligible friction observations in laboratory testing. Requiring the seal to remain in contact with the sealing surface gives boundary conditions

\[
\begin{align*}
\sigma_{12}^* &= 0 \\
v^* &= 0
\end{align*}
\] on \( Y^* = H \)

\( \iff \)

\( u^* = v^* = 0 \) on \( Y^* = 2H \) \hspace{1cm} (4.11)

Remaining B.C.s

Symmetric

about \( Y^* = H \)

In this plane strain configuration, the solution for this boundary condition is equivalent, via symmetry, to a seal of thickness \( 2H \) with zero displacement on the upper surface and a shear-free symmetry plane at \( Y^* = H \), and in this way, the solution obtained below can be used for either shear-free or no slip boundaries on the upper surface.

On the high-pressure end of the seal, hydrostatic fluid pressure, \( p_0 \), is applied, such that the boundary conditions on the high-pressure end are approximated as

\[
\begin{align*}
\sigma_{11}^* &= -p_0 \\
\sigma_{12}^* &= 0
\end{align*}
\] on \( X^* = 0 \) \hspace{1cm} (4.12)

Finally, the low-pressure boundary condition in realistic seals is complicated. In real seals, there is typically a nearly rigid metallic support ring that fills the majority of the annular gap between sealing surfaces, as depicted in Fig. 1-3. However, because of the need to assemble the seal, typically kilometers below the surface of the earth, a gap between the metallic support ring and the outer hole is required. It is this gap that allows the material to extrude and the seal to fail when pressurized. Furthermore, it is the corner of the metallic support ring at the transition between well-supported to no support that provides a large stress concentration that contributes to the fracture of the seal. Neglecting the curvature at the corner of the metallic support ring, we might suppose the boundary conditions to be

\[
\begin{align*}
u^* &= 0 \text{ on } X^* = L, Y^* \leq h_{\text{support}} \hspace{1cm} (4.13a) \\
\sigma_{11}^* &= 0 \text{ on } X^* = L, Y^* \geq h_{\text{support}} \hspace{1cm} (4.13b) \\
\sigma_{12}^* &= 0 \text{ on } X^* = L, 0 \leq Y^* \leq H \hspace{1cm} (4.13c)
\end{align*}
\]

where \( h_{\text{support}} \) is the height of the metallic support ring. Although these boundary conditions are reasonable when friction is unimportant, they are not convenient to solve.\(^4\)

Instead of solving the problem for this discontinuous boundary condition, we can think of the low-pressure metallic support as a deformable beam, spanning the entire height of the seal, but having variable stiffness, \( EI \left( Y^* \right) \).

\[
\begin{align*}
\sigma_{11}^* + \frac{\partial^2}{\partial Y^*^2} \left( EI \frac{\partial^2 u^*}{\partial Y^*^2} \right) &= 0 \text{ on } X^* = L \\
\sigma_{12}^* &= 0
\end{align*}
\] \hspace{1cm} (4.14)

where we might parameterize the bending stiffness as

\[
EI \left( Y^* \right) = \frac{1}{2} \left( EI \right)_{\text{max}} \text{erfc} \left( \frac{Y^* - h_{\text{support}}}{s} \right) \hspace{1cm} (4.15)
\]

where \( s \) is a parameterized width of transition from completely supported to completely unsupported and \( EI_{\text{max}} \) is the maximum bending stiffness, equal to infinity in the limit of theoretically rigid

\(^4\)It is likely that the problem can be solved analytically using conformal mapping or methods of Muskhelishvili \cite{Muskhelishvili}. 

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supports.

Alternately, an even simpler approximation that we might make (and which turns out to be
analytically feasible in light of the simplifications to be made below) is one where we treat the
low-pressure boundary condition as an elastic foundation of uniform stiffness $k^*$ such that the boundary
conditions take the form

$$
\begin{align*}
\sigma_{11}^* + k^* u^* &= 0 \\
\sigma_{12}^* &= 0 
\end{align*}
$$
on $X^* = L$

We now employ conventional fluid dynamics scaling arguments to reduce Eqns. (4.9a) and (4.9b) to
analytically tractable equations.

### 4.1.2 Scaling, Approximations and Simplification of Field Equations

Together the field equations, Eqn. (4.9a) and (4.9b), subject to the boundary conditions in Eqn.
(4.10), (4.11), (4.12) and (4.16), constitute the boundary value problem of interest. We now employ
conventional scaling arguments to reduce the boundary value problem to a system that can be readily
solved. We make the following assumptions:

1. The aspect ratio is large: $L/H \gg 1$

2. The material is nearly incompressible: $\mu/K \approx \mu/\lambda \ll 1$

The first of these is reasonable given the design of typical swellable seals. In the oil and gas
industry, the thickness, $H$, of these seals is often on the order of $H \sim 2.5$ cm (1 in), and the length
take on values ranging from $L \sim 46$ cm (18 in) up to $L \sim 610$ cm (20 ft), giving the range of aspect
ratios $18 \leq L/H \leq 240$.

The second is reasonable because the gel material of these seals is composed of cross-linked rubber,
which has a bulk modulus several orders of magnitude larger than its shear modulus, and a diffuse
solvent phase which, as a Newtonian fluid, has zero shear modulus, and a bulk modulus infinitely
larger than its shear modulus. Thus the assumption of near-incompressibility is reasonable for these
gels.

If we were to assume the material to be perfectly incompressible, we would impose a kinematic
constraint requiring the divergence of the displacement field to vanish,

$$\text{Div}^* \mathbf{u}^* = 0, \quad \frac{\partial u^*}{\partial x^*} = 0, \quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

If we scale the independent, spatial variables with the size of the domain, as in

$$x = \frac{x^*}{L}, \quad y = \frac{y^*}{H}$$

and we scale the dependent displacement variables with unknown scales, $U$ and $V$, as

$$u = \frac{u^*}{U}, \quad v = \frac{v^*}{V}$$

then perfect incompressibility requires

$$\frac{U}{L} \frac{\partial u}{\partial x} + \frac{V}{H} \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{U}{L} \sim \frac{V}{H}.$$

We now relax this constraint of perfect incompressibility and allow for the possibility of slight
compressibility. The relationship between the trace of stress and the trace of strain is

$$p^* \equiv -\frac{1}{3} \text{tr} \sigma^* = -K \text{tr} \epsilon^* = -K \text{Div}^* \mathbf{u}, \quad p^* = -\frac{1}{3} \sigma_{kk} = -K \epsilon_{kk} = -K \frac{\partial u_k^*}{\partial x_k^*}$$

where $p^* = p^*(x^*, y^*)$ is the mechanical pressure and $K = \lambda + 2\mu/3$ is the bulk modulus. Non-
dimensionalizing the mechanical pressure, $p^*$ by the applied fluid pressure, $p_0$, and evaluating the
above equation in nondimensional form gives

\[ p_0 p = -K \left( \frac{U}{L} \frac{\partial u}{\partial x} + \frac{V}{H} \frac{\partial v}{\partial y} \right) \]  \hspace{1cm} (4.21b)

which gives the scaling

\[ p_0 \sim K \left( \frac{U}{L} + \frac{V}{H} \right) \]  \hspace{1cm} (4.21c)

Retaining the requirement of Eqn. (4.20) requiring that each term on the right-hand side of (4.21c) balance each other, which holds in the limit as \( K \) becomes extremely large, then each term on the right-hand side of Eqn. (4.21c) must individually scale with the left-hand side. Thus we obtain the scalings for \( U \) and \( V \) of

\[ U = \frac{p_0 L}{K}, \quad V = \frac{p_0 H}{K} \]  \hspace{1cm} (4.22)

which makes physical sense in that they are linearly proportional to applied pressure, \( p_0 \), which is driving the displacement and they are inversely proportional to the bulk modulus, \( K \), which is resisting displacement due to volumetric contraction. Then, in nondimensional form, the relationship between mechanical pressure and volumetric expansion is

\[ p = - \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]  \hspace{1cm} (4.23)

Substituting the scalings of Eqn. (4.18) and (4.22) into the field equations given by Eqn. (4.9a) and (4.9b), dividing the first field equation through by \( \lambda L \) and the second through by \( \lambda H \) gives

\[ \frac{p_0}{K} \left[ \frac{\mu}{\lambda} \left( \frac{\partial^2 u}{\partial x^2} + \frac{L^2}{H^2} \frac{\partial^2 u}{\partial y^2} \right) + \left( 1 + \frac{\mu}{\lambda} \right) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] = 0 \]  \hspace{1cm} (4.24a)

\[ \frac{p_0}{K} \left[ \frac{\mu}{\lambda} \left( \frac{H^2}{L^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \left( 1 + \frac{\mu}{\lambda} \right) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] = 0. \]  \hspace{1cm} (4.24b)

Imposing the assumptions of long aspect ratio, \( H^2/L^2 \ll 1 \), and slight compressibility, \( \mu/\lambda \ll 1 \), we eliminate several terms and the equations simplify to

\[ \frac{\mu}{\lambda} \frac{L^2}{H^2} \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \]  \hspace{1cm} (4.25a)

\[ \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \]  \hspace{1cm} (4.25b)

where we have divided through by the leading coefficient \( p_0/K \), and we have retained the product of the small term, \( \mu/\lambda \ll 1 \), with a large term, \( L^2/H^2 \gg 1 \) in the first equation. In typical systems, this term could have a range of values, and we choose not to make further simplifying assumptions by neglecting this term. Retaining it as currently formulated without making further assumptions regarding its magnitude allows for the possibility that it is large or small, and yields a solution valid for arbitrary values of \( \left( \mu/\lambda \right) \left( L^2/H^2 \right) \).
4.1.3 Nondimensionalization of Boundary Conditions

The boundary conditions in Eqn. (4.10), (4.11), (4.12) and (4.16) are non-dimensionalized according to the preceding scaling as

\[
\begin{align*}
\sigma_{11} &= -1 \\
\frac{L}{H} \frac{\partial u}{\partial y} + \frac{H}{L} \frac{\partial v}{\partial x} &= 0 & \text{on } x = 0 \\
\frac{H}{L} \frac{\partial u}{\partial y} + \frac{L}{H} \frac{\partial v}{\partial x} &= 0 & \text{on } y = 0 \\
\sigma_{11} + \frac{k^* L}{K} u &= 0 & \text{on } x = 1 \\
\frac{L}{H} \frac{\partial u}{\partial y} + \frac{H}{L} \frac{\partial v}{\partial x} &= 0 & \text{on } y = 1.
\end{align*}
\]

(4.26a)

(4.26b)

(4.26c)

(4.26d)

where \( \sigma = \sigma^*/p_0 \) with nonzero components

\[
\begin{align*}
\sigma_{11} &= \frac{1}{K} \left( (2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y} \right) \\
\sigma_{22} &= \frac{1}{K} \left( (2\mu + \lambda) \frac{\partial v}{\partial x} + \lambda \frac{\partial u}{\partial y} \right) \\
\sigma_{12} &= \mu \left( \frac{L}{H} \frac{\partial u}{\partial y} + \frac{H}{L} \frac{\partial v}{\partial x} \right) \\
\sigma_{33} &= \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right).
\end{align*}
\]

(4.27a)

(4.27b)

(4.27c)

(4.27d)

We note that the shear-free boundary condition on the surface \( y = 1 \) can be simplified because \( v = 0 \forall x \) implies that \( dv/dx = 0 \), therefore leaving

\[
\frac{du}{dy} = 0 \text{ on } y = 1.
\]

(4.28)

4.1.4 Approximate Solution

Eqn. (4.25b) can be rewritten using the relationship between the traces of stress and strain, as given in Eqn. (4.23), as

\[
\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\frac{\partial}{\partial y} (p) = 0
\]

(4.29)

which states that, as in the analogous Poiseuille fluid problem, the pressure does not vary in the transverse direction, and only varies in the longitudinal direction.

\[
p = p(x)
\]

(4.30)

Substituting this solution \( p(x) \) into the longitudinal equilibrium equation given by Eqn. (4.25a) and integrating twice with respect to \( y \) gives

\[
\frac{\mu}{\lambda} \frac{L^2}{H^2} u(x, y) = \frac{1}{2} \frac{dp}{dx} y^2 + f_1(x) y + f_2(x)
\]

(4.31)
where \( f_1(x) \) and \( f_2(x) \) are constants of integration. Imposing the zero longitudinal displacement boundary condition on \( y^* = y = 0 \) from Eqn. (4.26b₁) gives

\[
f_2(x) = 0.
\]  
(4.32)

We now return to the \( y \)--equilibrium equation of Eqn. (4.25b) and utilize the fact that the divergence of the displacement is negative pressure.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -p(x)
\]  
(4.33)

Substituting the expression for \( u \) found in Eqn. (4.31)

\[
\frac{\lambda}{\mu} \frac{H^2}{L^2} \left( \frac{1}{2} \frac{d^2 p}{dx^2} y_2 + \frac{df_1}{dx} y \right) + \frac{dv}{dy} = -p(x)
\]  
(4.34)

and integrating with respect to \( y \) gives

\[
\frac{\lambda}{\mu} \frac{H^2}{L^2} \left( \frac{1}{6} \frac{d^2 p}{dx^2} y_3 + \frac{1}{2} \frac{df_1}{dx} y^2 \right) + v = -p(x) y + f_3(x).
\]  
(4.35)

The zero displacement boundary condition along \( y = 0 \) requires the normal displacement to vanish, giving

\[
f_3(x) = 0.
\]  
(4.36)

At this point we have the general solution

\[
u = \frac{\lambda}{\mu} \frac{H^2}{L^2} \left( \frac{1}{2} \frac{d p}{dx} y_2 + f_1 y \right)
\]  
(4.37a)

\[
v = - \frac{\lambda}{\mu} \frac{H^2}{L^2} \left( \frac{1}{6} \frac{d^2 p}{dx^2} y_3 + \frac{1}{2} \frac{df_1}{dx} y^2 \right) - py
\]  
(4.37b)

We still have two unknown functions of horizontal position: \( p(x) \) and \( f_1(x) \), and we still have two boundary conditions along the long surface at \( y^* = H \) that we have not yet used.²

Eqn. (4.26d₁) requires the normal displacement on the upper surface at \( y^* = H \) to vanish. Recall the corresponding value of the nondimensional coordinate on this surface is \( y = y^*/H = 1 \), giving \( v(y = 1) = 0 \). Imposing this boundary condition gives

\[
\frac{\lambda}{\mu} \frac{H^2}{L^2} \left( \frac{1}{6} \frac{d^2 p}{dx^2} + \frac{1}{2} \frac{df_1}{dx} \right) = -p
\]  
(4.38)

We will later find it useful to have the \( x \)--derivative of this also, which, since it is a function only of \( x \) and independent of \( y \) must hold for all values of \( y \).

\[
\frac{\lambda}{\mu} \frac{H^2}{L^2} \left( \frac{1}{6} \frac{d^3 p}{dx^3} + \frac{1}{2} \frac{d^2 f_1}{dx^2} \right) = \frac{d p}{dx} \quad \Rightarrow \quad \frac{d^2 f_1}{dx^2} = 2 \left( -\frac{\mu}{\lambda} \frac{L^2}{H^2} \frac{d p}{dx} - \frac{1}{6} \frac{d^3 p}{dx^3} \right)
\]  
(4.39)

The remaining boundary condition is the shear-free boundary condition along the upper surface of the domain at \( y^* = H \) \( (y = 1) \) given by Eqn. (4.26d₂). In nondimensional form, it is given as

\[
\frac{\mu \rho_0}{K} \left( \frac{L}{H} \frac{\partial u}{\partial y} + \frac{H}{L} \frac{\partial v}{\partial x} \right) = 0 \text{ on } y = 1.
\]  
(4.40)

²We also have two boundary conditions at each of the surfaces \( x^* = 0 \) and \( x^* = L \), but we will return to those later. They are not immediately relevant in obtaining the general form of the Saint-Venant solution, which must exactly satisfy boundary conditions along the long sides of the domain given by \( y^* = 0 \) and \( y^* = H \).
Omitting the prefactor and substituting the general solutions for the displacement components gives

$$\frac{L}{H} \frac{\lambda H^2}{\mu L^2} \left( \frac{dp}{dx} y + f_1 \right) + \frac{H}{L} \left( \frac{\lambda H^2}{\mu L^2} \left( \frac{1}{6} \frac{d^3 p}{dx^3} y^3 + \frac{1}{2} \frac{d^2 f_1}{dx^2} y^2 \right) - \frac{dp}{dx} y \right) = 0 \text{ on } y = 1. \quad (4.41)$$

Substituting the \( x \)-derivative of the normal displacement boundary condition, as given in Eqn. (4.39), gives

$$\frac{\lambda H}{\mu L} \left( \frac{dp}{dx} + f_1 \right) = 0 \quad \Rightarrow \quad f_1(x) = -\frac{dp}{dx} \quad (4.42)$$

Finally, returning to the boundary condition requiring zero normal displacement along \( y = 1 \) gives

$$v(y = 1) = \frac{1}{3} \frac{\lambda H^2}{\mu L^2} \frac{d^2 p}{dx^2} - p = 0 \quad (4.43)$$

which is a differential equation governing \( p(x) \) with general solution

$$p(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x) \quad (4.44)$$

where the dimensionless parameter governing the decay of pressure along the length of the body is

$$\alpha = \sqrt{3} \frac{\mu L^2}{\lambda H^2} \quad (4.45)$$

At this point we have the solution

$$u(x, y) = \frac{\lambda H^2}{\mu L^2} \frac{dp}{dx} \left( \frac{1}{2} y^2 - y \right) \quad (4.46a)$$

$$v(x, y) = p(x) \left( -\frac{1}{2} y^3 + \frac{3}{2} y^2 - y \right) \quad (4.46b)$$

$$p(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x) \quad (4.46c)$$

All that remains is to determine the constants \( C_1 \) and \( C_2 \) using the boundary conditions on \( x = 0 \) and \( x = 1 \). However, we note that we have two free parameters (the values of \( C_1 \) and \( C_2 \)) and four spatially distributed boundary conditions: normal and shear-free tractions on \( x = 0 \) and \( x = 1 \) over \( 0 \leq y \leq 1 \). The problem, as formulated cannot be solved, and the reason is that we neglected terms of order \( \mu/\lambda \) and \( H^2/L^2 \). These additional terms are required to satisfy the boundary conditions in a pointwise sense on the boundaries at \( x = 0 \) and \( x = 1 \).

This is a classic boundary layer problem. It can be understood by first understanding the analogous linear ordinary differential equation boundary layer problem ("toy problem") given by Bender and Orszag [7].

$$\epsilon \frac{d^2 f}{dx^2} + (1 + \epsilon) \frac{df}{dx} + f(x) = 0 \quad (4.47)$$

$$f(0) = 0$$

$$f(1) = 1$$

In this math problem, \( \epsilon \) is a small parameter. Neglecting it, (as we have done with terms involving \( \mu/\lambda \) and \( H^2/L^2 \) in the problem of interest), gives the differential equation

$$(1 + \epsilon) \frac{df_o}{dx} + f_o = 0 \quad (4.48)$$

where \( f_o \) is the outer solution, which can be solved (as we were able to solve the boundary value problem of interest above) to obtain

$$f_o(x) = C \exp \left( \frac{-1}{1 + \epsilon} x \right) \quad (4.49)$$

where \( C = \exp(1/(1 + \epsilon)) \) is a single constant of integration determined by the boundary condition at \( x = 1 \). This solution satisfies the approximate differential equation in the limit as \( \epsilon \to 0 \). However, discarding this highest derivative means that we are only able to satisfy one of the two boundary conditions. What we have derived here, as in the problem of interest above, is an outer solution that does an excellent job matching the exact solution far from the boundaries, but does not have
enough free parameters to satisfy all the boundary conditions.

\[
f_i(x_i) = \frac{D_1}{1 + \epsilon} + D_2 \exp\left(-(1 + \epsilon)x_i\right)
\]

Therefore, it is necessary to find an inner solution, near \( x = 0 \), where the first term, \( \epsilon f'' \), is leading order, and then match with the outer solution. The procedure for finding this inner solution and then matching it with the outer solution to obtain a single, uniformly valid solution is known as the procedure of Matched Asymptotic Expansions. To do so involves perturbing the coordinate, \( x \), such that term with the highest derivative again becomes leading order. An inner coordinate, \( x_i = x/\epsilon \) is created. Then an inner solution of the form

\[
f_i(x_i) = \frac{D_1}{1 + \epsilon} + D_2 \exp\left(-(1 + \epsilon)x_i\right)
\]

is found. Matching the inner and outer solutions as

\[
\lim_{x_i \to \infty} f_i = \lim_{x \to 0} f_o
\]

yields an overlap solution \( f_{\text{overlap}} = C = D_1/(1 + \epsilon) \). Subtracting the overlap gives the uniformly valid solution

\[
f_{\text{uniform}} = f_i + f_o - f_{\text{overlap}} = C \left(\exp\left(-\frac{1}{1 + \epsilon} x\right) - \exp\left(-\frac{1 + \epsilon}{\epsilon} x\right)\right)
\]

Complete details are found in Bender and Orszag’s Chapter 9. The important observation is that the approximate solution obtained for the two-dimensional seal is analogous to the outer solution obtained in this toy problem.

Returning to the problem at hand governing the deformation of an elastic body under plane strain, we forgo the asymptotic analysis and the finding of inner solutions at the two ends of the domain. Instead, we take the approach of Saint-Venant and assert that, far from the boundaries, the details of the boundary conditions are unimportant provided the resultant forces and moments are the same between the approximate and exact problems. In this way, we impose boundary conditions that agree, in the integral sense, with those formulated in Eqs. (4.26a) and (4.26c). Noting that we have only two free parameters, we choose to solve the axial stress boundary conditions on each end, and neglect the shear-free boundary conditions. Asymptotic inner solutions would be needed to correct for the shear-free boundaries.

The axial boundary conditions integrated over the cross-sectional area \(^6\) are

\[
\int_0^1 \sigma_{11} dy = -1 \text{ on } x = 0
\]

\[
\int_0^1 \left(\sigma_{11} + ku\right) dy = 0 \text{ on } x = 1
\]

\(^6\)The integrand changes slightly in the axisymmetric case due to the fact that \( dA^* = 2\pi R^* dR^* \). Refer to §4.5 for the full details for axisymmetric geometry.
In terms of the general solution of Eqn. (4.46), the \( \sigma_{11} \) component of the stress tensor is

\[
\sigma_{11} = \frac{1}{K} \left( 2\mu \left( \frac{3}{2} y^2 - 3y \right) - \lambda \right) p(x)
\]  

(4.55)

The averaged value over the cross-section of this component of stress and the axial component of displacement are

\[
\bar{\sigma}_{11}(x) = \int_0^1 \sigma_{11} dy = -\frac{2\mu + \lambda}{K} p(x) = -\frac{\lambda + 2\mu}{K} (C_1 \cosh \alpha x + C_2 \sinh \alpha x)
\]  

(4.56)

\[
\bar{\pi}(x) = \int_0^1 ud y = -\frac{1}{3\mu} \left( \frac{H^2}{L^2} \right) d p = -\frac{1}{\alpha} (C_1 \sinh \alpha x + C_2 \cosh \alpha x)
\]  

(4.57)

The boundary conditions require

\[
\bar{\sigma}_{11}(0) = -\frac{\lambda + 2\mu}{K} C_1 = -1
\]  

(4.58)

\[
\bar{\sigma}_{11}(1) + \bar{\pi}(1) = -\frac{\lambda + 2\mu}{K} (C_1 \cosh \alpha + C_2 \sinh \alpha) - \frac{k}{\alpha} (C_1 \sinh \alpha + C_2 \cosh \alpha) = 0
\]  

(4.59)

which have solutions

\[
C_1 = \frac{K}{\lambda + 2\mu}, \quad C_2 = -\frac{K}{\lambda + 2\mu} \left( \frac{\lambda + 2\mu}{K} \cosh \alpha + \frac{k}{\alpha} \sinh \alpha \right)
\]  

(4.60)

### 4.1.5 Finite Element Determination of \( k^* \)

In the analytic solution presented in the above, the low-pressure support at \( x^* = L \) (\( z^* = L \) in axisymmetric geometry) is parameterized as a linear spring with stiffness coefficient \( k^* \) such that the applied stress is related to the deformation at the low-pressure surface through

\[
\bar{\sigma}_{11} + k^* \bar{u} = 0 \quad \text{on} \quad x^* = L
\]  

(4.61a)

for the plane-strain case and correspondingly

\[
\bar{\sigma}_{zz} + k^* \bar{u}_z = 0 \quad \text{on} \quad z^* = L
\]  

(4.61b)

for the axisymmetric configuration, where the overline denotes the average value over the cross-section. The entire analytic solution has only this single adjustable parameter, which is not solved for explicitly as part of the model. However, a single finite element simulation for a given geometry \((h, H, r_c)\) and shear modulus \( \mu \) can be used to determine \( k^* \). Then this value can be used as the length and bulk modulus of the seal are varied.

In order to determine this effective stiffness of the low-pressure support, we conduct a single finite element analysis of a short elastomeric seal, similar to that sketched in Fig. 4-3, having a length, \( L \), that is on the order of four times the thickness, \( H \), of the seal. (The factor of four is empirically found to be sufficiently far from the low-pressure support such that the deformation is nearly uniform on the high-pressure surface.)

The objective of this calibration is to quantify the relation between stress and deformation at the low-pressure support. In order to separate the effect of shear on the lateral surfaces from the effect of mechanical support on the low-pressure surface, we prescribe shear-free boundary conditions on both the lower and upper surfaces at \( y^* = 0 \) and \( y^* = H \). (In the corresponding axisymmetric configuration, shear-free boundaries at the inner radius, \( r^* = R_1 \), and outer radius, \( r^* = R_3 \), are prescribed.) The input to the finite element simulation is the applied load, \( p_0 \), and the output given is the resulting nodal displacements on the high-pressure end at \( x^* = 0 \) (\( z^* = 0 \)) in axisymmetric

---

7 Note that we have observed no dependence of \( k^* \) on bulk modulus for these nearly incompressible materials. However, the precise dependence of \( k^* \) on material properties is unknown.
geometry). To calculate an effective stiffness in plane strain, we evaluate the average displacement of the high-pressure end at $x^* = 0$ by numerically integrating the nodal displacements.

$$
\bar{u}_0^* = \frac{1}{H} \int_0^H u^* (x = 0, y) \, dy \quad (4.62)
$$

The total displacement on the high-pressure end is the result of both the decrease in volume of the body due to compression as well as isochoric extrusion into the low-pressure gap. The stiffness, $k^*$, gives a relation between the part of displacement due to isochoric extrusion. We therefore partition the total displacement on the high-pressure end into that which is caused by bulk compression, and that which is due to isochoric extrusion.

$$
\bar{u}_0 = \bar{u}_{0,\text{bulk}} + \bar{u}_{0,\text{extrusion}} \quad (4.63)
$$

The bulk deformation is the deformation that would result if the low-pressure end were infinitely stiff, and there were only uniform uniaxial strain due to volume reduction. Since the bulk modulus is large and the volume change small, we expect an infinitesimal deformation solution for uniaxial compression to give reasonable estimate for $\bar{u}_{0,\text{bulk}}$. The infinitesimal displacement solution for uniaxial strain gives

$$
\sigma_{11}^* = (2\mu + \lambda) \epsilon_{11}^* = \frac{E (1 - \nu)}{(1 + \nu) (1 - 2\nu)} \epsilon_{11}^* \quad (4.64)
$$

For uniform applied stress $\sigma_{11}^* = -p_0$, integrating the infinitesimal strain $\epsilon_{11}^* = \partial u^*/\partial x^*$ over the length of the body gives

$$
\bar{u}_{0,\text{bulk}}^* = u^* (x^* = 0, y) = \frac{p_0 L}{2\mu + \lambda} = \frac{p_0 L (1 + \nu) (1 - 2\nu)}{E (1 - \nu)} \quad (4.65)
$$

where $\mu$ and $\lambda$ are the small-strain Lamé moduli (tangent moduli at zero strain), and we have prescribed zero displacement, $u^* = 0$, at the low-pressure end of the seal, $x^* = L$. Finally, the stiffness is evaluated as

$$
k^* = \frac{p_0}{\bar{u}_{0,\text{extrusion}}^*} = \frac{p_0}{\bar{u}_0^* - \bar{u}_{0,\text{bulk}}^*} \quad (4.66)
$$

Fig. 4-4 shows results for calculation of the low-pressure stiffness for a seal of thickness $H = 2.5$ cm, neo-Hookean material with small-strain shear modulus $\mu = 613$ kPa, low-pressure support filling 90 percent of the seal height and a corner radius of $r_c = 0.188$ mm. The results are shown for three significantly different small-strain bulk moduli in order to demonstrate that the results of support stiffness, $k^*$, are insensitive to bulk modulus. Despite the vastly different total average displacement, $\bar{u}_0$, for the three simulations, subtracting the bulk displacement, $\bar{u}_{0,\text{bulk}}$, yields an isochoric deformation, $\bar{u}_{0,\text{extrusion}}$, that is insensitive to bulk modulus.

Although the linear fit of extrusional displacement versus applied pressure in Fig. 4-4b appears to be fairly good, there is undoubtedly some underlying curvature of the data points that is not captured by the linear fit. The reason lies in the nonlinearity of the low-pressure boundary due to
both contact geometry and material nonlinearities. Recall the applied load is on the same order as the shear modulus of the material. In order to investigate this effect, we approximate the derivative, $d\bar{\pi}_{\text{extrusion}} / dp_0$ using finite differences and calculate the effective stiffness of the low-pressure support as a function of applied pressure.

$$k^*(p_0) = \frac{d\bar{\pi}_{\text{extrusion}}}{dp_0} \approx \frac{\bar{\pi}_{\text{extrusion}}(p_0 + \Delta p) - \bar{\pi}_{\text{extrusion}}(p_0)}{\Delta p}$$  (4.67)

The results of this calculation are plotted in Fig. 4-5. We see that there is indeed some dependence of

Figure 4-5: Dependence of actual support stiffness on applied pressure for finite deformation neo-Hookean material. Note that the infinitesimal pressure (linear) prediction in the plot of $5.4 \times 10^9$ Pa/m agrees reasonably well with the prediction of $k_{\text{Linear}}^* = 5.3 \times 10^9$ Pa/m based on the linear formulation in Eqn. (4.68) below.

the effective stiffness of the low-pressure support on the applied pressure. This variation with applied pressure is not captured in the purely linear model above. However, it could be captured within the framework of the above model by simply allowing $k^*$ to depend on the applied force/traction at the low-pressure end. Doing so would give nonlinear equations for $C_1$ and $C_2$, which would need to be solved iteratively. This has not been done in this thesis in the interest of capturing more important
phenomena, such as fracture propagation in the next chapter.

**Low-Pressure Stiffness Calculated from Linear Elastic Model**

In the preceding finite element calculations, we considered finite deformation of a neo-Hookean material, which is complicated because the contact condition on the low-pressure end changes as the material is elastically extruded into the gap. For the case of finite deformation, we have made no attempt to map the stiffness of the support as a function of the geometry and material parameters. However, we have attempted to do so, via finite element analysis, for the linear limit (which can likely alternate be solved analytically, perhaps using complex analysis techniques of Muskhelishvili.) For infinitesimal linear elastic deformations, we empirically find via a parametric finite element investigation the low-pressure support stiffness for plane strain to be of the form

\[ k^* = 2.5 \frac{H \mu}{(H - (h - r_c))^2} = 2.5 \frac{H \mu}{g_{\text{eff}}} \]  (4.68)

where the prefactor 2.5 is determined from curve fitting and we define the effective gap, \( g_{\text{eff}} \), to be the height between the outer sealing surface and the lower corner of the fillet on the low-pressure support. Plots of finite element simulations versus this empirical result are shown in Fig. 4-6. Note that we have considered only a very narrow range of Poisson ratio near the incompressible limit of \( \nu \to 1/2 \), which is of interest for elastomers. We do not know how Poisson ratio will affect the effective stiffness if allowed to deviate far from \( \nu \approx 1/2 \).

Note that the behavior of the effective stiffness for finite deformations is complicated by the fact that the seal deforms around the corner radius. Therefore, the effective gap, \( g_{\text{eff}} \), may be better approximated by \( H - h \) rather than \( H - (h - r_c) \) in the infinitesimal deformation case. For finite deformation cases, the effective stiffness for a material undergoing infinitesimal deformation can be used as a starting point, and a single finite element analysis can determine the more precise value for a given geometry and shear modulus.

### 4.1.6 Alternate Approximation of Coefficients – A Ritz Approach

As an alternative to imposing the axial boundary conditions on \( x^* = 0 \) and \( x^* = L \) to arrive at the two coefficients, \( C_1 \) and \( C_2 \) in Eqn. (4.60), we ask the question if we could find better coefficients if we implemented boundary conditions using an energetic formulation rather than the Saint-Venant formulation. In this way, we use the general solution of Eqn. (4.46) as two distinct basis functions for our approximate solution and use the Ritz method to find the best amplitudes, \( C_1 \) and \( C_2 \), such that the energy is minimized.

The low-pressure boundary condition is somewhat non-standard because it is a mixed, Robin type boundary condition where traction and displacement are coupled. This is not treated in prototype problems in graduate mechanics courses, where boundaries of domains are specified either as Dirichlet boundaries on which displacements are known, or Neumann boundaries on which tractions are known. Because of this, we formulate a “toy” problem in Appendix C, where we consider a one-dimensional elastic body coupled to a spring. In summary, the mixed boundary condition is effectively an energy storage device (which is how we formulated it in the first place), and the energy due to deformation of this spring boundary condition must be included in the free energy of the system.

The total elastic energy in the body can be written

\[ U_{\text{body}} = \int_0^L \int_0^H \frac{1}{2} \epsilon^* \sigma^* dy^* dx^* = \int_0^L \int_0^H \frac{1}{2} (\epsilon^*_{11} \sigma^*_{11} + \epsilon^*_{22} \sigma^*_{22} + 2 \epsilon^*_{12} \sigma^*_{12}) dy^* dx^* \]  (4.69)

The internal energy stored in the linear spring is

\[ U_{\text{spring}} = \int_0^H \frac{1}{2} k^* (u^*(x^* = L, y^*))^2 dy^* = \frac{3Hk^*L^2 r_0^2 (C_1 \sinh(\alpha) + C_2 \cosh(\alpha))^2}{5\alpha^2 K^2} \]  (4.70)
The work that the high-pressure force does on the body is

\[ W_{\text{nonconservative}} = \int_0^H p_0 u^* (x^* = 0, y^*) dy^* = -\frac{C_2 H L p_0^2}{\alpha K} \quad (4.71) \]

The high-pressure boundary is the only surface on which nonconservative work is done on the body. On the lower, no-slip boundary, there is no displacement, so the shear does no work. On the upper boundary, there is no friction, so no work is done.

For this purely elastic body, the work done by the applied pressure must equal the strain energy stored in the body (including the low-pressure boundary). The energy functional

\[ \Phi(C_1, C_2) = U_{\text{body}} + U_{\text{spring}} - W_{\text{nonconservative}} \quad (4.72) \]

represents the mismatch between work done and energy stored, and we seek to minimize this mismatch by choosing values the best values of \( C_1 \) and \( C_2 \). Taking the derivative of the functional with respect to these amplitude coefficients and setting them equal to zero gives the equations necessary
to solve for $C_1$ and $C_2$.

$$\frac{\partial \Phi}{\partial C_1} = 0$$
$$\frac{\partial \Phi}{\partial C_2} = 0$$

(4.73)

The calculations are tedious. We therefore provide Mathematica code that can be used to generate the appropriate intermediate results and solve for the coefficients $C_1$ and $C_2$.

$$u_1 := 1/\alpha \sinh[\alpha x + 1/2(3/2(y/\gamma - 2) - 3 + \gamma)]$$
$$u_2 := 1/\alpha \cosh[\alpha x + 1/2(3/2(y/\gamma - 2) - 3 + \gamma)]$$
$$v_1 := \cosh[\alpha x + 1/2(-1/2(y/\gamma - 3 + 3/2(y/\gamma - 2) - 3\gamma))]$$
$$v_2 := \sinh[\alpha x + 1/2(-1/2(y/\gamma - 3 + 3/2(y/\gamma - 2) - 3\gamma))]$$

$$u := p_0 x x x (C_1 + u_1 + C_2 + u_2)$$
$$v := p_0 \sinh[1 + (C_1 + v_1 + C_2 + v_2)]$$

$$\text{exx} := 0[u, x]$$
$$\text{eyy} := 0[v, y]$$
$$\text{esx} := 0[0, x] + 0[u, y]$$

$$\text{szx} := (2-mu + \text{lambda}) \text{exx} + \text{lambda} \text{eyy}$$
$$\text{szy} := (2-mu + \text{lambda}) \text{eyy} + \text{lambda} \text{exx}$$
$$\text{sgy} := 2-mu$$

$$\text{Psi} := 1/2 \epsilon (\text{szx} + \text{eyy} + \text{szx} + \text{sgy} + 2 \text{esx} + \text{szx})$$

$$\text{Yo} := \text{Integrate}[\text{Psi}, \{x, 0, L\}, \{y, 0, H\} \}$$

$$\text{ybar} := \text{Integrate}[\text{yo}, \{x, 0, L\} \}$$

$$\text{Etot} := \text{Yo} + \text{Ypr} - \text{VFr}$$

$$\text{Eqn} := 0\{\text{Etot}, 0\}$$

$$\text{Soln} := \text{Simplify}[\text{Solve}\{\text{Eqn} = 0, \text{Eqn2} = 0, \{C_1, C_2\}\}] / \{\text{alpha} \rightarrow \text{Sqrt}[3 \mu / \text{lambda} \}] \}$$

The coefficients $C_1$ and $C_2$ depend on three dimensionless parameters:

$$\alpha \equiv \sqrt{\frac{3 \mu L}{H}} \frac{K}{\lambda + 2\mu} = \frac{3 \lambda + 2\mu}{3 \lambda + 6\mu} k = \frac{k^* L}{K}$$

(4.74)

which can be seen from the Saint-Venant solution in Eqn. (4.60). We make no effort to exhaustively compare the Saint-Venant and Ritz solutions over the entire parameter space. Instead, we compare results for two select cases. First, the case most relevant for later finite element calculations on energy release rate is one for which we fix $\mu = 1.0 \times 10^6 [\text{Pa}]$, $\lambda = 2.0 \times 10^9 [\text{Pa}]$, and $L = 1[\text{m}]$ and vary $\alpha$ by varying the thickness, $H$. The results for $C_1$ and $C_2$ are compared for the Ritz and Saint-Venant solutions in Fig. 4-7. The results show that the two solution procedures give slightly different values for $C_2$ while the values for $C_1$ are nearly identical (note the scale on the plot). These coefficients are consistent with our understanding that, as the seal becomes longer and $\alpha$ increases, the low-pressure boundary condition has less importance and the seal behaves as if it were infinitely long. This means that the superposition of the hyperbolic functions should result in a pure exponential decay, which corresponds to $C_2 = -C_1$.

As indicated in the Saint-Venant solution for $C_1$ and $C_2$ in Eqn. (4.60), the values also depend on the low-pressure stiffness, $k$. Here we consider a smaller value of $k^*$ than presented in Fig. Fig: Ritz vs Saint-Venant - Realistic kstar. Fig. 4-8 shows the comparison between the coefficients $C_1$ and $C_2$ as computed using Saint-Venant and Ritz methods for $k^* = 1.0 \times 10^9 \text{ Pa/m}$, shear modulus $\mu = 1.0 \times 10^9 [\text{Pa}]$, Lamé modulus $\lambda = 2.0 \times 10^9 [\text{Pa}]$ and length $L = 1[\text{m}]$. We observe similar behavior for large values of $\alpha$ to that observed in Fig. Fig: Ritz vs Saint-Venant - Realistic kstar, but there is a slight overshoot in $C_2$ near $\alpha = 1$ at which point $C_2$ dips below the value $C_2 = -1$. It is important to note that part of the reason for the larger discrepancy at low values of $\alpha$ is that, for these fixed ratios of $\mu/\lambda$, lower values of $\alpha$ correspond to lower aspect ratios, $L/H$. The solution was derived assuming large aspect ratio, $L/H \gg 1$, and as this ratio decreases, the solution becomes increasingly poor because the boundary layers no longer comprise a small portion of the length of the seal, and instead are a significant fraction of the overall length.

Understanding the difference between the Saint-Venant and Ritz solutions for $C_1$ and $C_2$ in all this
is the importance of shear in the model. We derived the approximate solution assuming the shear modulus, and therefore shear stress, was small. In deriving the coefficients using the Saint-Venant approach, we continue to assume the shear is small. However, in deriving the coefficients using the Ritz approach, we simply started with the basis functions derived using the small shear, long aspect ratio approximation. Then, when plugging them into the strain energy formula, there was no assumption of small shear modulus or large aspect ratio. Thus the importance of shear was better incorporated into the Ritz solution of the coefficients. This is the reason that we have different values of $C_1$ and $C_2$ for small values of $\alpha$ (relatively large values of $H$) but they converge with the Saint-Venant values as $\alpha$ becomes large. In short, Ritz is telling us that Saint-Venant neglected the importance of shear when $\alpha$ is small, and Ritz is trying to account for it by adjusting the coefficients $C_1$ and $C_2$.

In summary, the Ritz method yields slightly different and presumably better (from an energy perspective) values than the Saint-Venant solution, but there is not a large difference between the coefficients in the two cases.

Figure 4-7: Comparison of Saint-Venant and Ritz solutions for the coefficients $C_1$ and $C_2$ in the general solution of Eqn. (4.46) for $\mu = 1.0 \times 10^9$ [Pa], $\lambda = 2.0 \times 10^9$ [Pa], $k^* = 5.6 \times 10^9$ [Pa/m] and $L = 1$[m].

Figure 4-8: Comparison of Saint-Venant and Ritz solutions for the coefficients $C_1$ and $C_2$ in the general solution of Eqn. (4.46) for $\lambda/\mu = 2000$ and $k^* = 1.0 \times 10^9$ [Pa/m].
4.1.7 Validation – Comparison with Large-Deformation Finite Element Simulations

The asymptotic solution does not satisfy the shear-free boundary conditions at \( x^* = 0 \) and \( x^* = L \), and is therefore valid only in the bulk of the domain, \( H \lesssim x^* \lesssim L - H \), sufficiently far from the boundary layers at the ends, which are of thickness \( O(H) \). The utility of the analytic solution can be seen by comparing key dependent variables with those predicted by more realistic finite element solutions.

Fig. 4-9 compares results of the approximate analytic solution to detailed finite element calculations (finite deformation, neo-Hookean hyperelastic material) for a case when the aspect ratio is \( L/H = 20 \), the ratio of small strain elastic moduli is \( \mu/\lambda = 2.5 \times 10^{-3} \), the applied fluid pressure is five times the shear moduli, \( p_0/\mu = 5 \), and the low-pressure support ring fills three quarters of the gap between the inner and outer surfaces of the seal. Note that when referring to material properties in finite element simulations, we use \( \mu \) and \( \lambda \) to denote the small-strain limits of the elastic moduli for finite element simulations of finite deformation hyperelasticity.

Three quantities of interest to seal engineers are plotted. The first of these is the sealing stress between the outer surface of the seal and the rigid surface against which it seals. Lorenz and Persson [56] have shown that leak rate past the seal increases with a decrease in this contact stress, and the normal contact traction is of primary importance in modeling the flow of fluid between deformable surfaces, as done using the pressure penetration boundary condition in Abaqus/Standard [90]. Sealing stress is well-predicted by the analytic solution except in the boundary layer regions near the ends given by \( x \lesssim H/L \) and \( x \gtrsim 1 - H/L \). Note that the present solution does not include the effect of pre-compression of the seal, which is required for sealing, but, for a linear system, pre-compression can be accounted for by superposition.

![Figure 4-9](image_url)

Figure 4-9: Comparison between finite element results and approximate analytic solution of several variables relevant for seal performance. (a) Sealing stress between upper surface of seal and top rigid surface at \( y^* = H \). (b) Axial force, \( F_x = (p_0 H)^{-1} F^*_x \), along the length of the seal. (c) Shear stress at interface where seal is bonded to inner rigid surface at \( y^* = 0 \).
Similarly, as one might intuit and as we show below, the transfer of axial force,
\[ F_x^* (x^*) \equiv -\int_0^H \sigma_{11}^* (x^*, y^*) \, dy^*, \]  
along the length of the seal from the high-pressure end to the low-pressure end is important for predicting the onset of failure on both ends. Note that the force is defined to be positive when acting in the \( +\hat{e}_1 \) direction on a cut through the body with outward unit normal \( \hat{n} = -\hat{e}_1 \). (This is simply for convenience, because the axial stress, \( \sigma_{11}^* \), is compressive and therefore negative.) As shown in Fig. 4-9.b, the analytic solution is able to predict axial force along the length of the seal.

Finally, one might be interested in shear stress along the inner surface where the seal is bonded to a rigid substrate. By equilibrium in the longitudinal direction and the shear-free boundary condition on the upper surface, the shear traction is equal to the slope of the axial force curve.

\[ \int_0^H \left( \frac{\partial \sigma_{11}^*}{\partial x^*} + \frac{\partial \sigma_{12}^*}{\partial y^*} \right) \, dy^* = 0 \Rightarrow \frac{dF_x^*}{dx^*} = \sigma_{12}^* \bigg|_{y^*=0}. \]  

Fig. 4-9.c shows that the agreement is reasonable, except in the boundary layer regions at the ends.

### 4.2 Fracture Mechanics – Energy Release Rates on High- and Low-Pressure Ends (Plane Strain)

In experimental observations of typical oilfield seals, large cracks are often observed on the low-pressure end of the seal. Additionally, delamination between the seal and the rigid substrate is sometimes observed on the high-pressure end [22], as depicted schematically in Fig. 4-10. In what follows, we presume that there are small, preexisting cracks in the material, both on the high-pressure end at the interface between the seal and the substrate and at the low-pressure end near the extrusion gap. The precise geometry of these cracks is generally unknown. We investigate, through the use of finite element simulations, the combined effects of seal length and bulk modulus on the tendency for growth of these cracks.

#### 4.2.1 Finite Element Simulations of Crack Initiation

The parameter of interest for crack mechanics is the energy release rate, \( G \), which gives the incremental decrease in elastic strain energy per incremental increase in crack surface area as the crack advances. Classic energy arguments pioneered by Griffith [33] state that the crack will advance when the energy release rate exceeds a critical value, \( G_{\text{crit}} \), which is a material property, such that

\[ G \geq G_{\text{crit}}. \]
To investigate these tendencies for crack propagation, we conduct a sequence of finite element simulations for the finite deformation of a neo-Hookean material under plane strain. The bulk modulus and length of the seal are varied. We use the finite element program Abaqus/Standard with plane strain elements (CPE4H) to calculate the energy release rate \cite{76} for the growth of an interfacial crack on the high-pressure end. We use the same simulations to calculate the energy release rate for the growth of each crack in an array of cracks on the low-pressure end. Fig. 4-10 illustrates the system of interest, where there is a single interfacial crack on the high-pressure end and a sequence of \( N_{\text{cracks}} = 51 \) cracks introduced into the mesh on the low-pressure end.

This array of a large number of cracks on the low-pressure end was used because the finite element simulations account for finite deformation, meaning that the cracks are allowed to advect with the material through space, and it is not known \textit{a priori} which crack will produce the largest energy release rate at any particular value of applied pressure. Physically, the details of the cracks on the low-pressure end are unknown in real seal systems. Cracks are not intentionally added in design and manufacture, but imperfections exist in reality.

On the high-pressure end, a single interfacial crack of length \( L_{c,\text{HP}} = 10 \mu m \) is introduced at the interface between the elastic body and the rigid substrate. We impose boundary conditions on this crack of zero shear traction and zero normal displacement. The choice of these is subjective. The choice of zero normal displacement was made because it was presumed that, in realistic applications, the seal will be initially pre-compressed in the direction normal to the inner and outer sealing surfaces. This pre-compression will be caused by swelling for the case of swell packer seals or mechanical compression upon assembly or due to application of axial load, as in the case of mechanical packers. Regardless, it is presumed that, in order for a seal to be maintained, it is necessary that the compressive traction normal to the sealing surface be larger than the applied hydrostatic load from fluid pressure. Therefore, we presume this crack does not open in Mode I fracture.

The choice of no shear in the tangential direction is arbitrary. There will certainly be friction in realistic situations. However, friction has been neglected throughout the analytic model at the other surfaces, and we therefore neglect it here. In more applied finite element modeling, this assumption can easily be relaxed.

On the low-pressure end, we introduce a sequence of \( N_{\text{cracks}} = 51 \) small cracks, each of length \( L_{c,\text{LP}} = 10 \mu m \), perpendicular to the undeformed low-pressure surface as shown in Fig. 4-10. These cracks are uniformly spaced in the direction parallel to the face of the low-pressure support with spacing \( s = 20 \mu m \). The reason for the introduction of such a large number of cracks is that in physical systems, we expect there to be small cracks throughout the material, and we generally do not know the geometry of these pre-existing defects. The reason we care about these is that, at some critical pressure, one of the cracks will begin to grow. If we knew exactly which crack would begin to grow, we could omit many of the other cracks because the stress concentrations around their crack tips are locally confined. However, for the finite deformation finite element analysis, we do not know \textit{a priori} which crack will experience the maximum energy release rate. For small loads, and small deformations, the crack experiencing the largest energy release rate is one that is located near the corner of the low-pressure support in the undeformed configuration. However, as pressure is increased, this particular crack elastically extrudes into the extrusion gap, and a different crack experiences a larger energy release rate than the first. This transfer of peak energy release rate from one crack to another continues as pressure and elastic deformation increase. Because of this finite deformation, it is not a single material point (single crack) which always experiences the maximum energy release rate. Therefore, in order to calculate the tendency, at any given pressure, for crack growth on the low-pressure end, the maximum value over all 51 cracks is taken for each value of applied pressure.

\[
G_{\text{LP}} (p_0) = \max_i (G_{\text{LP},i} (p_0)) \quad (i = 1, 2, \cdots, N_{\text{cracks}})
\] (4.78)

Although this close spacing of cracks leads to shielding, giving a lower energy release rate for a particular crack than if there were no neighboring cracks, the crack geometry and shielding effect are held fixed in all simulations.
Twenty-five Percent Extrusion Gap

In the first parametric investigation, we computationally investigate the effect of aspect ratio and ratio of elastic moduli for seals with a low-pressure support ring that fills seventy-five percent of the annulus between the inner and outer scaling surfaces, leaving an extrusion gap that is twenty-five percent of the channel height. The geometry for this set of seals is given in Table 4.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.01875</td>
</tr>
<tr>
<td>$H$</td>
<td>0.025</td>
</tr>
<tr>
<td>$r_c$</td>
<td>0.000188</td>
</tr>
<tr>
<td>$L_{c,HP}$</td>
<td>$1.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>$L_{c,LP}$</td>
<td>$1.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>$s$</td>
<td>$2.0 \times 10^{-9}$</td>
</tr>
<tr>
<td>Mesh Size (Locally Refined)</td>
<td>$1.0 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

The results for the computed energy release rate for the growth of an interfacial crack on the high-pressure end are given in Fig. 4-11. The inset of the plot shows energy release rates versus applied pressure for four different aspect ratios and three different ratios of elastic moduli. The results of energy release rate versus applied pressure, $P_0$, are approximately quadratic. This is consistent with linear elastic fracture mechanics, which states that energy scales with the square of the applied load.

Close inspection of the inset plot reveals that increasing the aspect ratio or decreasing the Lamé modulus, $\lambda$, while holding shear modulus $\mu$ fixed increases the energy release rate for the growth of an interfacial crack at the high-pressure end of the seal.

Plotting the energy release rate versus the horizontal gradient of axial force, $dF^*/dx^*$, on the high-pressure end collapses these curves onto a single master curve. More will be said about the theory behind this scaling in §4.3.

On the low-pressure end, the maximum energy release rate is plotted as a function of applied differential pressure in the inset of Fig. 4-12. The maximum energy release rate is calculated as the maximum value over all 51 cracks for that applied pressure as indicated in Eqn. 4.78. The aspect ratios and ratios of elastic moduli are the same as those used in Fig. 4-11.

The results show the converse of Fig. 4-11. Specifically, increasing the aspect ratio of the seal or decreasing the bulk modulus lowers the energy release rate on the low-pressure end. The reason is that, as the seal is made longer or more compliant (lower bulk modulus), more of the load applied at the high-pressure end is transmitted via shear to the rigid substrate, and less is transmitted to the low-pressure end. In the limit of an infinitely long seal (or one with zero bulk modulus), all the load would be transmitted via shear to the basepipe and there would be no load at the low-pressure support; therefore, there would be zero energy release rate for the cracks on the low-pressure end.

The primary axes of Fig. 4-12 show the low-pressure energy release rate versus axial force transmitted to the low-pressure end, $F_x^*$ ($x^* = L$). The scaling shows that, despite the fact that the curve is irregularly shaped, the energy release rate scales well with the axial force transmitted to the low-pressure end. The theory underlying this scaling is discussed in §4.3.

Ten Percent Extrusion Gap

We conduct a second sequence of computations investigating the effects of aspect ratio and ratio of elastic moduli. The only difference between this parametric investigation and the previous one is that the extrusion gap in this investigation is 10 percent of the seal thickness whereas it was previously 25 percent of the seal thickness. Thus only the low-pressure boundary condition is altered. The
location of each crack is altered and the mesh is locally refined accordingly in order to capture the stress concentrations near the corner on the low-pressure end. The geometry for these computations is given in Table 4.2.

### Table 4.2: Plane strain geometry – 10 percent gap

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
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</tr>
<tr>
<td>$H$</td>
<td>0.025</td>
</tr>
<tr>
<td>$r_c$</td>
<td>0.000188</td>
</tr>
<tr>
<td>$L_{c,HP}$</td>
<td>$1.0 \times 10^{-9}$</td>
</tr>
<tr>
<td>$L_{c,LP}$</td>
<td>$1.0 \times 10^{-9}$</td>
</tr>
<tr>
<td>$s$</td>
<td>$2.0 \times 10^{-9}$</td>
</tr>
<tr>
<td>Mesh Size (Locally Refined)</td>
<td>$1.0 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

Fig. 4-13 shows the computed energy release rates for the growth of an interfacial crack at the high-pressure end for various aspect ratios and ratios of elastic moduli. In the inset, the energy release rate is plotted versus applied pressure. In the primary axes, the energy release rate is plotted versus the gradient of axial force at the high-pressure end. As in Fig. 4-11, we see that the gradient of axial force correlates the energy release rates over a range of aspect ratios and ratios of elastic moduli. The right plot in Fig. 4-13 shows that, for small applied loads, the energy release rate is quadratic in applied pressure, as we would expect from linear elastic fracture mechanics.

Fig. 4-14 shows an excerpt of the data in Fig. 4-13. The left plot shows the computed energy release rate for the growth of an interfacial crack on the high-pressure end for a fixed aspect ratio over a range of bulk moduli. The results show that as the bulk modulus increases, the the energy release rate for growth of a high-pressure crack decreases. The reason is that a higher bulk modulus

![Figure 4-11: Energy release rate for the growth of an interfacial crack at the high-pressure end for an extrusion gap that is 25 percent of the seal thickness. Inset: Energy release rate versus applied pressure. The axial force gradient, $dF_a/dx^*$, given by Eqn. (4.76) correlates energy release rates over a range of aspect ratios and elastic moduli.](image-url)

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Figure 4-12: Energy release rate for the growth of cracks on the low-pressure end. Inset gives energy release rate versus applied pressure on the high-pressure end. Primary graph shows that energy release rate is dependent only upon axial load transmitted to low-pressure end, regardless of length or material compressibility. The odd shape of the curves results from finite deformation.

Figure 4-13: Energy release rate versus shear stress at the leading edge. The shear stress is calculated entirely from the analytic model, with the lone fitting parameter being the low-pressure support stiffness, $k^*$, which was inferred independently. The point corresponding to the energy release rate of $G = 0.5 \text{ J/m}^2$ is denoted on the plot because it is used as the failure criterion in Fig. 7 of the paper. The right plot shows that the slope for modest pressures is 2 as predicted by linear elastic fracture mechanics.

implies that the seal is stiffer, and therefore more of the applied load is supported by the low-pressure support.

The left plot of Fig. 4-13 gives the computed energy release rates for the growth of an interfacial crack on the high-pressure end for a fixed ratio of elastic moduli for several different aspect ratios. For a fixed ratio of elastic moduli, making the seal longer increases the energy release rate for the growth of the high-pressure interfacial crack. However, comparison of the curves for aspect ratios of 50 and 100 indicates that there is not much change in high-pressure energy release rate when
Figure 4-14: High pressure energy release rate as a function of elastic moduli for a fixed aspect ratio (left) and as a function of length for a fixed bulk modulus (right).

doubling the seal from an aspect ratio of 50 to 100. This suggests that once the seal is sufficiently long, further increases in length may not significantly alter the energy release rate on the high-pressure end. In effect, the seal behaves as if it were infinitely long and the effect of the low-pressure boundary is not felt on the high-pressure end.

Fig. 4-15 shows the energy release rate for the growth of cracks on the low-pressure end both as a function of applied pressure, \( p_0 \), in the inset and as a function of transmitted axial force, \( F_x^*(L) \), as computed using the Saint-Venant solution.

Figure 4-15: Energy release rate versus axial force at the low-pressure end. The axial force is calculated entirely from the analytic model, with the lone fitting parameter being the low-pressure support stiffness, \( k^* \), which was calculated independently. The point corresponding to the energy release rate of \( G = 0.5 \text{ J/m}^2 \) is denoted on the plot because it is used as the failure criterion in §4.4.

Fig. 4-16 is a subset of Fig. 4-15 showing that the energy release rate for the growth of cracks on the low-pressure end increases as the bulk modulus increases, while holding aspect ratio fixed. As expected, energy release rates decrease as the seal becomes longer, while holding elastic moduli fixed.

Fig. 4-17 shows plots of the behavior of the low-pressure end of the seal, when there is no shear
Figure 4-16: Low pressure energy release rate as a function of elastic moduli for a fixed aspect ratio (left) and as a function of length for a fixed bulk modulus (right).

on the bottom surface at $y = 0$. This is the configuration used to evaluate the low-pressure stiffness, $k^*$, in Section 4.1.5 above. Fig. 4-17.a shows that the energy release rate for growth of cracks at the low-pressure end, when neglecting the effects over the bulk of the seal, are insensitive to the bulk modulus.

Figure 4-17: Energy release rate for growth of cracks on low-pressure end, for shear-free inner and outer sealing surfaces for a ten percent extrusion gap. (a) Effect of bulk modulus on energy release rates showing $G_{LP}$ is relatively insensitive to $\lambda/\mu$. (b) Energy release rate for several individual cracks, $G_{LP,i}$, on low-pressure end as a function of applied pressure for $\lambda/\mu = 1.5 \times 10^5$. $G_{LP}$ is the locus of maximum values over all the cracks for each value of applied pressure.

Fig. 4-17.b shows the energy release rates for several individual cracks, $G_{LP,i}$ for a ratio of small-strain elastic moduli of $\lambda/\mu = 1.5 \times 10^5$. The results show that, for very small applied loads and small deformations, crack numbers 37, 38, 39 and 40 yield the largest energy release rates. As pressure is increased past approximately $p_0 = 400$ kPa, crack 51 yields the largest energy release rates. The reason for this is shielding from other cracks. Crack 51 is the top-most crack in the array of cracks, and therefore has only neighboring cracks below it. Therefore, it receives only half the shielding of the other cracks, and its energy release rate is larger, over a range of pressures, than cracks that are more shielded. Finally, at the highest pressures, cracks 24 and 25 yield the largest energy release rates. This transfer among cracks {37-40}, 51 and {24-25} is due to finite deformation. Together, the finite deformation and contact make this local energy release problem nonlinear and contribute to the peculiar shape of the resulting maximum energy release rate curve.
4.3 Scaling of Energy Release Rates – Interpretation in Light of Saint-Venant Solution

High-Pressure End

In physical seal systems, there is a radial compressive stress that is needed to prevent the flow of fluid between the seal and its mating surfaces. This stress acts perpendicular to the plane of the small interfacial crack on the high-pressure end and serves to close the crack. Therefore, the driving force for crack propagation is the shearing between the seal and the rigid substrate, which gives rise to a Mode II shear failure at the interface. Although the behavior of interfacial cracks between dissimilar materials is nontrivial [83], we expect both the energy release rate and the Mode II stress intensity factor to be proportional to the shear stress at the high-pressure corner

\[ G_{HP} \sim \frac{K_{II}^2}{E'} \sim \frac{\sigma_{12}^2 a}{E'} \]  \hspace{1cm} (4.79)

where \( a \) is the crack length, \( G_{HP} \) is the energy release rate for the growth of a crack at the high-pressure end, \( E' = E/(1 - \nu^2) = 4\mu(\lambda + \mu)/(\lambda + 2\mu) \approx 4\mu \) is an effective Young’s modulus for plane strain, and \( \sigma_{12} \) is some characteristic (far-field) shear stress evaluated near the high-pressure corner at \((x^*, y^*) = (0, 0)\). Despite the singular nature of the shear stress at the crack tip in linear elasticity theory, we take a coarse-grained approach and treat the shear stress ahead of the crack as the gradient of axial force given in Eqn. (4.76). Then we expect a quadratic dependence of energy release rate on gradient of force, as

\[ G_{HP} \sim \left( \frac{dF^*}{dx^*} \bigg|_{x^*=0} \right)^2. \]  \hspace{1cm} (4.80)

Fig. 4-11 shows the energy release rate computed by the finite element simulations versus the analytically predicted gradient of axial force at the leading edge, \( dF^*/dx^* \bigg|_{x^*=0} \), for several aspect ratios and two drastically different bulk moduli. The results show that this scaling, and the approximate analytic solution for the axial force gradient correlate the computed energy release rates over a wide range of aspect ratios and ratios of elastic moduli.

Aside from the collapse of all the energy release rates versus shear stress, it is worthwhile to note that, in the inset of Fig. 4-11, all the energy release rate curves for highly compressible seals with the ratio of elastic moduli \( \lambda/\mu = 150 \) fall on top of each other, despite the widely varying aspect ratios. This indicates that the energy release rate for the growth of a crack on the high-pressure end of the seal is insensitive to seal length when the bulk modulus is low and the seal is highly compressible. The reason is that, for highly compressible seals, the seal does not feel the effect of the low-pressure boundary condition, and the entirety of the applied differential pressure is transferred to the rigid substrate via shear. The seal is effectively infinitely long from the high-pressure crack’s perspective. Thus, for a fixed bulk modulus, increasing seal length beyond a certain value does not affect the tendency for crack growth on the high-pressure end and does not improve seal performance.

Low-Pressure End

On the low-pressure end, we expect the energy release rate for the growth of a small crack to depend on the load transmitted from the high-pressure end along the length of the seal to the extrusion gap at the low-pressure end. The approximate analytic solution above provides a prediction of the distribution of axial load, \( F^*(x^*) \), and we therefore attempt to scale energy release rate with transmitted axial force as

\[ G_{LP} \sim F^*(L) \]  \hspace{1cm} (4.81)

Fig. 4-12 shows that, despite the fact that the energy release rate versus applied pressure is not quadratic, as one would expect from linear theory and that we do not observe here due to finite deformations, the energy release rate on the low-pressure end is well-predicted by the transmitted axial force, which is calculated from the approximate analytic solution.
The shapes of both energy release rate versus applied pressure, \( p_0 \), and energy release rate versus transmitted axial force, \( \dot{F}_x |_{r^* = \sqrt{L}} \), are a manifestation of finite deformation in the simulations. As the pressure increases, the seal material is pushed into the extrusion gap. Therefore, a small preexisting crack originally located at the corner of the support ring where the stress concentration is highest is extruded into a region where the stress concentration is lower, and a new crack moves into the region of highest stress concentration. The curve of energy release rate is the locus of the instantaneously maximum (evaluated over all the cracks) energy release rate

\[
\mathcal{G}_{LP} (p_0) = \max_i \mathcal{G}_{LP,i} (p_0)
\]  

(4.82)

where \( \mathcal{G}_{LP,i} (p_0) \) is the energy release rate of the \( i \)th crack at applied pressure \( p_0 \). In typical linear elastic fracture mechanics, the energy release rate scales with the square of the applied load. Here, the scaling is initially weaker than that because finite deformation extrusion provides an additional degree of freedom, thereby relieving some of the increase in strain energy at the crack tips.

Most importantly, we do not claim that the shape of this curve is universal. Undoubtedly it depends on the geometry of the low-pressure support and the geometry of the cracks in the low-pressure region. However, the collapse of all of these curves over a range of aspect ratios and elastic moduli ratios, when scaled with transmitted axial load, is the key finding.

**Qualitative Explanation of Observed Correlation**

Here we provide a qualitative explanation for the reason the analytic solution provides good correlations of energy release rates despite having poor agreement with detailed finite element simulations at both ends of the seal. The reasoning is analogous to the theory of matched asymptotic expansions and conventional fracture mechanics arguments regarding ratios of length scales. We have already shown through finite element simulations that the deformations in the bulk of the seal are small, due to the highly confined nature of the seal, and have utilized Saint-Venant’s principle to obtain a solution in the bulk without accurately resolving the details in the boundary layer regions at the ends of the seal. The bulk solution is insensitive to the detailed solution at the high-pressure end, and therefore is insensitive to the details of the stress and displacement fields surrounding the crack tip for a short interfacial crack. Similarly, on the low-pressure end, the boundary condition has been parameterized as a linear spring, and the bulk solution is therefore only sensitive to the overall stiffness of the low-pressure support and insensitive to the details of the crack tip stress and displacement fields.

On the other hand, the inner (boundary layer) solutions (which we have only computed numerically) near the crack on the high-pressure end and near the cracks on the low-pressure end are highly dependent on the details of the crack geometry as well as the far-field loading of the crack. Although in this analysis we do not claim to know the detailed geometry of the cracks (crack length, spacing, etc.), it is clear that once this geometry is fixed, the only variables affecting the stress intensity factor and energy release rate are the far-field loading and material properties. The energy release rate for these cracks is dependent only on shear modulus or Young’s modulus, and is not strongly affected by the bulk modulus, as illustrated by Eqn. (4.79), where the Poisson ratio is always very near to its incompressible limit: \( \nu \approx 1/2 \).

Having fixed the crack geometry and the shear modulus, the only remaining parameter affecting energy release rate is the far-field loading on the cracks, which is precisely what is provided by the approximate analytic solution in the bulk. Therefore, holding the shear modulus and local geometry around the cracks fixed, the outer analytic solution provides far-field loading on the crack and can be used to correlate energy release rates for the growth of cracks when the length and bulk modulus of the seal are varied.

Fig. 4-18 schematically illustrates this separation of length scales and matching. On the high-pressure end, the far-field loading condition is dependent both on the applied pressure, \( p_0 \), and on the traction exerted by the bulk of the seal on the high-pressure end. For Mode II fracture, the far-field stress of importance is this gradient of axial force given by the difference between the applied pressure \( p_0 \) and the traction on the hypothetical cut dividing the high-pressure end from the bulk.
of the seal

\[ \sigma_{\infty,(HP)} \sim \lim_{r \to \infty} \frac{p_0 + \int_0^1 t_{HP}(x, y) \cdot \hat{e}_x \, dy}{x} \]

where \( r \) is the radial coordinate from the tip of the interface crack on the high-pressure end. However, from the theory of matched asymptotic expansions, we know that the limit of the outer, bulk solution as \( x^+ \to 0 \) must equal the limit of the inner, local crack solution as \( r \to \infty \).

\[ \lim_{z \to \infty} \sigma_{HP} = \lim_{x^+ \to 0} \sigma_{\text{bulk}} \]

\[ \Rightarrow \sigma_{\infty,HP} \sim \lim_{x^+ \to 0} \frac{-\sigma_{xx}(x^+ = 0) + \int_0^1 \sigma_{xx(\text{bulk})}(x^+, y) \, dy}{x^+} \quad (4.83) \]

This is the gradient of average axial force, or, using Eqn. (4.76), this is equivalent to the shear stress in the bulk predicted by the approximate analytic solution.

![Diagram](image)

Figure 4.18: Schematic illustrating the boundary value problem near the crack tips and the separation of the problem into boundary layers, on which the solution is evaluated numerically, and an outer region over which the approximate analytic solution is reasonable.

Similarly, on the low-pressure end, the far-field stress of importance is the traction in the horizontal direction applied far from the crack as \( \rho/b \to \infty \) where \( \rho \) is the radial coordinate from the tip of the low-pressure crack and \( b \) is the length of the low-pressure crack. As in the analysis for the high-pressure end, the theory of matched asymptotic expansions states that the outer limit of stresses in the low-pressure solution must equal the inner limit of stresses in the bulk solution

\[ \lim_{x^+ \to 1} \sigma_{(\text{Bulk})} = \lim_{z \to \infty} \sigma_{(LP)} \]

which indicates that the bulk solution evaluated at the low-pressure end gives the proper scaling of the far-field loading for a small crack at the low-pressure end.

### 4.4 Discussion and Implications of Fracture Initiation on High- and Low-Pressure Ends

In preceding sections we derived an approximate analytic solution for the distribution of stresses and displacements in a long, slightly compressible seal. We have also shown that the energy release rate for the growth of a small interfacial crack on the high-pressure end of the seal is dependent on the coarse-grained shear stress at the high-pressure end, which was predicted from the approximate Saint-Venant solution. Finally, we have shown that the energy release rate for the growth of a small crack on the low-pressure end depends on the axial force transmitted from the high-pressure end to the low-pressure end by the seal. Here, we put together these pieces of information and use them to analyze the effects of bulk modulus and seal length on seal performance. Specifically, we show that decreasing the bulk modulus decreases the tendency for crack growth on the low-pressure end, which implies that the seal is more likely to begin to fail on the high-pressure end. Similarly, increasing the seal length makes the seal effectively more compressible and decreases the energy release rate for growth of a crack on the low-pressure end. We examine this behavior and compare analytic
predictions with a small parametric finite element investigation illustrating these points.

The analytic solution above gave approximations for the stress distribution in the body, which was used to calculate the total axial force, \( F' \), its gradient with respect to \( x \), which were used to correlate energy release rate in Figs. 4-11 and 4-12. In dimensional form, the analytically predicted shear stress on the high-pressure end and the transmitted axial force on the low-pressure end were, respectively,

\[
\frac{dF'}{dx'} \bigg|_{x'=0} = -p_0 \frac{H \lambda}{L K} C_2 \\
F' \bigg|_{x'=L} = -p_0 H \frac{3 \lambda + 6 \mu}{3 \lambda + 2 \mu} (C_1 \cosh \alpha + C_2 \sinh \alpha). \tag{4.84a}
\]

As stated above, given the material properties and geometry of the seal, this solution has a single free parameter, \( k^* \), which characterizes the stiffness of the low-pressure support. For a neo-Hookean seal with a small strain shear modulus of \( \mu = 6.13 \times 10^6 \) Pa and geometry given in Table 4.2, the low-pressure support stiffness is found, via finite element analysis, to be \( k^* = 5.59 \times 10^6 \pm 4.3 \times 10^7 \) Pa/m.

Using this value of \( k^* \), the stresses in Eqn. (4.84) are known. Fig. 4-19 shows a plot of the normalized stress on the high-pressure end, \( p_0^{-1} \frac{dF}{dx} \) and the normalized stress on the low-pressure end, \( p_0^{-1} H^{-1} F' \bigg|_{x'=L} \), both as a function of bulk modulus and aspect ratio. The results show that, on the high-pressure end, the shear stress decreases with increasing bulk modulus because, as the bulk modulus increases, more of the applied load is supported by the low-pressure support and less by shear along the bonded interface. Similarly, as the length of the seal decreases, more load is transferred to the low-pressure support and less is supported by shear. The opposite trends in axial stress are observed at the low-pressure end of the seal.

In addition to the analytic solution, the preceding section demonstrated that the energy release rates on the high- and low-pressure ends depend on these stress measures. For the sake of clarity of discussion, let us denote the functional dependence of the energy release rates on these stress measures as

\[
G_{HP} = \phi_{HP} \left( \frac{dF'}{dx'} \bigg|_{x'=0} \right); \text{Crack Length, } \mu, \cdots \tag{4.85a}
\]
\[
G_{LP} = \phi_{LP} \left( F' \bigg|_{x'=L} \right); \text{Crack Length, Gap, } \mu, \cdots \tag{4.85b}
\]

where the dependence on variables such as crack length could be further investigated using finite element analysis but are simply held fixed in the current study. The functional dependence of energy release rate on stress, as denoted by \( \phi_{HP} \) and \( \phi_{LP} \), can be determined from a single finite element analysis, as illustrated in Figs. 4-11 and 4-12, and then used for varying seal lengths and bulk moduli.

Although this current investigation makes no attempt to experimentally characterize the critical energy release rate for growth of cracks on either end of a seal, using hypothetical values allows us to explore the effects of aspect ratio and bulk modulus. If we computationally determine the dependence of energy release rates on the stress measures of Eqn. (4.84), then we can graphically invert these functions to obtain stress as a function of energy release rate. Specifically, we are interested in stresses corresponding to critical energy release rates, \( G_{HP, \text{crit}} \) and \( G_{LP, \text{crit}} \) at which cracks begin to grow. Inverting Eqn. (4.85) while holding constant crack length, gap, etc., gives

\[
\frac{dF'}{dx} \bigg|_{x'=0, \text{crit}} = \phi_{HP}^{-1} \left( G_{HP, \text{crit}} \right) \tag{4.86a}
\]
\[
F' \bigg|_{x'=L, \text{crit}} = \phi_{LP}^{-1} \left( G_{LP, \text{crit}} \right). \tag{4.86b}
\]

Furthermore, substituting into Eqn. (4.84) allows us to determine the pressure required to reach the
Figure 4.19: Stress as a function of aspect ratio and ratio of elastic moduli. ×: Finite element simulations where failure first occurs on high-pressure (HP) end. ○: Finite element simulations where failure first occurs on the low-pressure (LP) end. (a) Shear stress, $p_0^{-1}dF^*/dx^*(x = 0)$, at the high-pressure end given by the approximate analytic solution. (b) Average axial stress, $p_0^{-1}H^{-1}F_x^*(a^* = L)$, at the low-pressure end. (Upper left region has no values because the parameter region gives unrealistically large values of $\alpha$ here, such that asymptotic solution is not valid.) Solid lines indicate contours along which the tendency for failure at high-pressure end relative to low-pressure end is constant for fixed value of $k$ and fixed critical energy release rates.

critical energy release rate at each of the two ends of the seal.

$$p_{\text{HP crit}} = \frac{\phi_{\text{HP}}^{-1}(\mathcal{G}_{\text{HP crit}})}{-\alpha \frac{H}{L} \frac{\lambda}{K} C_2}$$  \hspace{1cm} (4.87a)

$$p_{\text{LP crit}} = \frac{\phi_{\text{LP}}^{-1}(\mathcal{G}_{\text{LP crit}})}{\frac{H(2\mu + \lambda)}{K}} \left( C_1 \cosh \alpha + C_2 \sinh \alpha \right).$$ \hspace{1cm} (4.87b)

The minimum of these two is the pressure at which the seal will begin to fracture. If the critical pressure on the high-pressure end is smaller, a crack on the high-pressure end will begin to grow before a crack on the low-pressure end will advance. Conversely, if the critical pressure on the low-pressure end is smaller, a crack there will begin to grow first. The point at which cracks on both
the high-pressure end and low-pressure end are equally likely to advance is given by the equality of these two critical pressures, which can be solved for via a numeric root-finding algorithm, such as Newton-Raphson [79] or MATLAB’s fminsearch [47].

In the absence of knowledge of values of critical energy release rates, $G_{HP,\text{crit}}$ and $G_{LP,\text{crit}}$, we choose hypothetical values and examine the resulting behavior. Because of convergence issues and the inability to push the finite element simulations to high energy release rates for cracks with length of 10 $\mu$m, as given in Table 4.2, critical energy release rates of $G_{HP,\text{crit}} = G_{LP,\text{crit}} = 0.5$ J/m$^2$ were chosen for the purpose of illustrating the combined effects of length and compressibility. Despite the fact that these are low for realistic systems, the qualitative behavior obtained from them is representative. Using plots analogous to those of Figs. 4-11 and 4-12 for the geometry of Table 4.2, the resulting critical shear stress values are $dF^*/dx^*_{x^*=0,\text{crit}} = \phi_{HP}^{-1} (0.5$ J/m$^2$) $\approx 2.6 \times 10^4$ N/m and $F^*|_{x^*=L,\text{crit}} = \phi_{LP}^{-1} (0.5$ J/m$^2$) $\approx 1.4 \times 10^4$ N. Then using these values and equating the critical pressure on the low-pressure end with that on the high-pressure end gives the solid black curve shown in the plots of Fig. 4-19. This is the predicted curve on which failure at the low-pressure end is equally favorable as failure at the high-pressure end. To the left of this curve, stress on the high-pressure end increases, while stress on the low-pressure end decreases, making failure on the high-pressure end more favorable. Conversely, to the right of this curve, failure on the low-pressure end is more probable.

Superposed on this plot are symbols denoting results of finite element simulations. In these finite element simulations, energy release rates were calculated for the growth of small cracks on both the high-pressure and low-pressure ends of the seal. The same failure criteria of $G_{HP,\text{crit}} = G_{LP,\text{crit}} = 0.5$ J/m$^2$ were used for the finite element simulations, and once the computed energy release rate exceeded the critical energy release rate, the seal was deemed to have failed, with the location of failure given by the end that first reached the critical energy release rate. In the plot, the symbols $\times$ denote simulations in which the failure first occurred on the high-pressure end and the symbols $\circ$ denote those simulations in which the failure first occurred on the low-pressure end. The predictions by the set of finite element simulations predicting whether the seal will fail at the high-pressure end or the low-pressure end agree well with the predictions made from the analytic solution.

These results are consistent with expectations regarding the effect of aspect ratio and compressibility. For a given aspect ratio, increasing the bulk modulus shifts the location of failure from the high-pressure end to the low-pressure end because a high bulk modulus means more axial load is transferred to the low-pressure end and less is transferred via shear to the rigid substrate. Similarly, for a fixed bulk modulus, increasing the length (aspect ratio) increases the tendency for failure on the high-pressure end. The reason is that, as the seal becomes longer, less of the applied load is supported by the low-pressure support, and more of it is transferred by shear to the rigid substrate. This shear stress is the source of failure on the high-pressure end.

### 4.5 Derivation of Approximate Axisymmetric Solution

The annular geometry is defined in Fig. 4-20, where the domain of the boundary value problem is $r^* = (r^*, z^*)$ such that $R_1 \leq r^* \leq R_3$ and $0 \leq z^* \leq L$. The displacement vector in this domain is
\[ u^*(\mathbf{r}^*) = u^*_r(\mathbf{r}^*, z^*) \hat{e}_r + u^*_z(\mathbf{r}^*, z^*) \hat{e}_z \] (4.88)

The field equations and boundary conditions are analogous to those presented above for the plane strain case. We nondimensionalize the independent variables as

\[ r^* = R_1 + Hr \] (4.89)

where \( H \equiv R_3 - R_1 \), giving \( 0 \leq r \leq 1 \). We define a nondimensional ratio of lengths

\[ \beta \equiv \frac{H}{R_1} \] (4.90)

which gives the importance of curvature. As \( \beta \) becomes small, the effect of curvature decreases and, in the limit as \( \beta \to 0 \), the plane-strain case is recovered. Then the independent variables are nondimensionalized as

\[ r^* = R_1 (1 + \beta r) \] (4.91a)

\[ z^* = Lz \] (4.91b)

The dependent displacement variables are nondimensionalized as

\[ u^*_r = U u_r, \quad u^*_z = W u_z \] (4.92)

with

\[ U = \frac{p_0 H}{K}, \quad W = \frac{p_0 L}{K} \] (4.93)

Then the nondimensionalized equilibrium equations are

\[ \frac{\partial}{\partial r} \left( 2 \mu \frac{\partial u_r}{\partial r} + \nu \frac{\partial u_r}{\partial z} \right) + \frac{\mu}{\lambda} \frac{\partial^2 u_r}{\partial z^2} + \frac{\mu}{\lambda} \frac{\partial^2 u_z}{\partial r^2} + 2 \beta \frac{\mu}{\lambda} \frac{\partial^2 u_r}{\partial r^2} \frac{\partial u_z}{\partial r} \left( \frac{\partial u_r}{\partial r} + \frac{\beta u_r}{1 + \beta r} \right) = 0 \] (4.94a)

\[ \frac{\mu}{\lambda} \frac{\partial}{\partial z} \left( \frac{\partial u_r}{\partial z} + \frac{L^2}{H^2} \frac{\partial u_z}{\partial r} \right) + \frac{\partial}{\partial z} \left( \frac{\mu}{\lambda} \frac{\partial u_z}{\partial z} + \nu \frac{\partial u_z}{\partial r} \right) + \beta \frac{\mu}{\lambda} \frac{\partial^2 u_z}{\partial r^2} \frac{\partial u_z}{\partial z} \left( \frac{1 + \beta r}{1 + \beta r} \right) = 0 \] (4.94b)

We again ignore small terms of order \( H/L \ll 1, \mu/\lambda \ll 1 \) while retaining terms of leading order \( (\mu/\lambda)(L^2/H^2) \sim 1 \) giving the leading-order governing equations

\[ \frac{\partial (\nu \frac{\partial u_r}{\partial r})}{\partial r} = -\frac{\partial p}{\partial r} = 0 \] (4.95a)

\[ \frac{\mu}{\lambda} \frac{L^2}{H^2} \frac{\partial^2 u_z}{\partial r^2} + \frac{\partial}{\partial z} (\nu \frac{\partial u_z}{\partial r}) + \beta \frac{2 \mu L^2}{H^2} \frac{\partial u_z}{\partial z} \frac{\partial u_z}{\partial r} \left( \frac{1 + \beta r}{1 + \beta r} \right) = 0 \] (4.95b)

Upon integration and application of boundary conditions on the surfaces \( r^* = R_1, R_3 \), we arrive at the solution

\[ u_r(r, z) = p(z) \{ -(2 + \beta) \beta^2 r(r - 1) \left[ 4 + \beta (2 + 4r) + \beta^2 r(r + 1) \right] + 4(1 + \beta)^2 \left[ 2 + \beta (2 + r) \ln(1 + \beta r) \right. \]

\[ -r (1 + \beta)^2 (2 + \beta) \ln(1 + \beta r) \} / \]

\[ \{ 2(1 + \beta r) \left[ 4(1 + \beta)^2 \ln(1 + \beta) - \beta (4 + 14\beta + 12\beta^2 + 3\beta^3) \right] \} \] (4.96a)

\[ u_z(r, z) = \frac{H^2}{L^2} \left( \frac{\beta r (2 + \beta r) - 2 (1 + \beta)^2 \ln(1 + \beta r)}{4\beta^2} \right) \frac{dp}{dz} \] (4.96b)
with
\[ p(z) = D_1 \cosh(\gamma z) + D_2 \sinh(\gamma z) \] (4.97)

where
\[ \gamma = f(\beta) \sqrt{\frac{L^2 \mu}{H^2 \lambda}} \] (4.98a)
\[ f(\beta) = \frac{2\sqrt{2}\beta^{3/2}\sqrt{\beta + 2}}{\sqrt{4(\beta + 1)^3 \ln(\beta + 1) - \beta (3\beta^3 + 12\beta^2 + 14\beta + 4)}} \] (4.98b)

where the Taylor series expansion of \( f(\beta) \) for values of \( \beta \) near zero in the plane-strain limit is
\[ f(\beta) = \sqrt{3} \left( 1 - \frac{1}{4} \beta + \frac{23}{160} \beta^2 + O(\beta^3) \right) \] (4.99)

which can be clearly seen to have the limiting value of \( f(\beta) \to \sqrt{3} \) and \( \gamma \to \alpha \) in the plane-strain limit of \( \beta \to 0 \).

As in the plane strain case, there are two constants of integration and four axial boundary conditions. We choose to satisfy the boundary conditions normal to the surfaces at \( z^* = 0, L \) in the integral sense.

\[ \int_{R_1}^{R_3} \sigma_{zz}^* 2\pi r^* dr^* = -\pi \left( R_3^2 - R_1^2 \right) p_0 \quad \text{on} \quad z^* = 0 \]
\[ \Rightarrow 2R_1 H \int_0^1 \sigma_{zz} (r, z = 0) (1 + \beta r) dr = - \left( R_3^2 - R_1^2 \right) \] (4.100a)

\[ \int_{R_1}^{R_3} 2\pi r^* \left( \sigma_{zz}^* + k^* u_z^* \right) dr^* = 0 \quad \text{on} \quad z^* = L \]
\[ \Rightarrow \int_0^1 \left[ \sigma_{zz} (r, z = 1) + k u_z (r, z = 1) \right] (1 + \beta r) dr = 0 \] (4.100b)

where \( k = \frac{k' L}{K} \) is the nondimensional low-pressure stiffness. Together, these boundary conditions specify the values of the integration constants as

\[ D_1 = \frac{K}{\lambda + 2\mu} \] (4.101a)
\[ D_2 = -\frac{\cosh \gamma + \frac{kK}{\gamma(\lambda + 2\mu)} \sinh \gamma}{\frac{\lambda + 2\mu}{K} \sinh \gamma + \frac{k}{2} \cosh \gamma} \] (4.101b)

Fig. 4-21 provides a comparison between finite element simulations and approximate analytic solution results for profiles of key stress measures along the length of the seal. The input parameters for the finite element simulations are given in Table 4.3, and are for a value of \( \beta = 1 \) indicating the effect of curvature is leading order. The results are analogous to those in Fig. 4-9, which were for the plane strain case. As in the plane-strain case, the approximate analytic results agree exceptionally well with the finite element results except in the boundary layer regions near the ends of the seal for \( z^* < H \) and \( L - H < z^* < L \).

### 4.6 Summary

In this chapter we investigated the combined effects of aspect ratio, \( L/H \), and ratio elastic moduli, \( \mu/\lambda \), on the tendency for fracture initiation at the two regions of high stress concentration – one
Figure 4-21: Comparison between finite element prediction and approximate analytic solution for axisymmetric geometry. Analog to plane strain comparison in Fig. 4-9. The horizontal axis is stretched to show the detailed differences between the analytic and computational solutions at the ends. Finite element results for a neo-Hookean material undergoing finite deformations. The relevant parameters for the simulation are given in Table 4.3. (a) Sealing stress between upper surface of seal and top rigid surface at $r^* = R_3$. (b) Axial force, $F_z$, along the length of the seal. (c) Shear stress at interface where seal is bonded to inner rigid surface at $r^* = R_1$.

Table 4.3: Parameter values for validation of axisymmetric analytic solution with finite element simulation.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer Radius: $R_3/R_1$</td>
<td>2</td>
</tr>
<tr>
<td>Thickness: $H/R_1$</td>
<td>1</td>
</tr>
<tr>
<td>Length: $L/R_1$</td>
<td>20</td>
</tr>
<tr>
<td>LP Support Height: $h/H$</td>
<td>1</td>
</tr>
<tr>
<td>LP Support Stiffness: $k^*$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Curvature: $\beta$</td>
<td>1</td>
</tr>
<tr>
<td>Bulk Modulus: $K/\mu$</td>
<td>100</td>
</tr>
<tr>
<td>Applied Load: $p_0/\mu$</td>
<td>0.5</td>
</tr>
<tr>
<td>Solution Parameter: $f(\beta)$</td>
<td>1.433</td>
</tr>
<tr>
<td>Length Scaling (Plane Strain): $\alpha$</td>
<td>3.476</td>
</tr>
<tr>
<td>Length Scaling (Axisymmetric): $\gamma$</td>
<td>2.917</td>
</tr>
</tbody>
</table>

on the high-pressure end at the interface between the seal and the substrate and one on the low-pressure end at the corner of the support ring near the extrusion gap. We first found an approximate, linear, analytic Saint-Venant type solution to the boundary value problem of interest. We showed that, with a single finite element calculation to calibrate the stiffness of the low-pressure support, $k^*$, we could use the approximate solution to predict deformation and stresses in a long, slightly compressible hyperelastic seal over a range of aspect ratios and bulk moduli. The agreement between the approximate linear solution and the nonlinear finite element simulations in the bulk away from the seal ends indicates that nonlinearity is unimportant away from the ends of the seal due to the
highly confined nature of the seal and resulting small deformations.

However, the approximate analytic solution was unable to capture the behavior of the seal near the high-pressure and low-pressure ends because of boundary layer effects. Recall we implemented boundary conditions in a weak, Saint-Venant, sense and did not impose shear-free boundaries on the two ends. Despite the inability of the approximate solution to faithfully capture behavior near the ends in precisely the regions of high stress concentration, we show that coarse-grained measures predicted by the analytic solution correlate the energy release rates for growth of cracks in these two regions. The reasoning can be argued from a mathematical perspective using the idea of matched asymptotic expansions, where the far-field stress on the boundary layer region is equal to the stress in the bulk solution in the limit as the boundary layer is approached. A more physical explanation comes from the idea of fracture mechanics where the behavior of stress fields and energy release rates near the crack tip depends only on some far-field loading of the crack, and the Saint-Venant solution provides precisely this far-field loading for cracks at both ends of the structure.

Finally, we provided the solution for the analogous problem in axisymmetric geometry. We showed that again the linear Saint-Venant solution agrees well with finite element simulations of hyperelastic seals under finite deformation. One possible next step of this work would be to introduce cracks into the axisymmetric solution and calculate energy release rates for these cracks. Following the analysis for the plane-strain case, one should attempt to correlate energy release rates with the far-field stress measures of axial force and axial gradient of axial force. Although this is a logical next step and ought to be done, there is a conceptual problem with doing so in that axisymmetric finite element simulations with cracks implies that the cracks are axisymmetric. While this may in fact be true for the initial interfacial crack on the high-pressure end, it is doubtful that any pre-existing cracks on the low-pressure end are initially axisymmetric. Furthermore, the growth of these cracks is non-axisymmetric, as shown in experiments in Chapter 2, and calculating energy release rates for axisymmetric crack growth is inconsistent with the physics of the problem. Nevertheless, it should be done as the next incremental step.
Chapter 5

Computational Modeling of Fracture Propagation in Long Hyperelastic Seals

In the preceding chapter we examined the effect of aspect ratio and ratio of bulk modulus to shear modulus on the effect of tendency for fracture to first occur on the high-pressure end or the low-pressure end. Although this tells us important information about the location of first failure and the critical pressure at which we can expect the beginning of the demise of the seal, the fact is that in experiments, failure is often observed on both the high-pressure and low-pressure ends of the seal. Delamination is observed on the high-pressure end, and fracture is observed on the low-pressure end where material is extruded into the gap between the support ring and the hole. Furthermore, as shown in Chapter 2, the onset of material fracture occurs much earlier than the leakage of the seal, and knowing at what applied pressure a crack first begins to grow does not indicate when the seal first begins to leak. Therefore, the results in the previous chapter regarding fracture initiation are necessary but insufficient in understanding the device-level failure and loss of sealing.

Here we present efforts in understanding not only fracture initiation, but fracture propagation through the seal. This problem is not analytically tractable for a number of reasons including the fact that the crack paths on the low-pressure end are not known a priori, deformations are large, contact is important, and geometries are not simple. It is therefore necessary to resort to numerical simulations of the problem. A finite element model for fracture propagation meeting the following criterion is sought:

1. valid for finite deformations of hyperelastic materials
2. does not require a priori knowledge of the crack paths
3. valid for mixed mode fracture
4. able to accommodate friction
5. insensitive to mesh

The first criterion is that it can be used for nonlinear problems with both geometric and material nonlinearity. At present, we ignore the presence of solvent in these seals because the time-scale for diffusion over the length of the seal is much longer than the timescale of loading of these in practice. It is acknowledged that the timescale for diffusion in very small regions, such as the region of high stress concentration at the crack tip, is the same as that for loading, and we expect the diffusion of solvent at the crack tip to affect the physics of crack propagation. Solvent diffusion is a dissipative mechanism, and such mechanisms have been shown to increase fracture toughness of gels [101].

This requirement of nonlinearity excludes potential computational methods such as the extended finite element method (XFEM), and various dynamic fracture propagation methods, which rely upon
knowing crack tip stress fields (modulo a stress intensity factor) and utilizing superposition to enrich elements at the crack tip.

Related to this requirement of applicability for nonlinearity is the practical concern that, if the method in any way erodes the stiffness of elements due to a damage model, the decreased stiffness must not result in excessive element distortion leading the failure to converge. From a mathematical perspective, setting a given element stiffness equal to zero will give a rank-deficient stiffness matrix and yield non-unique solutions for nodal positions on these elements, and thus failure of the finite element iteration to converge. Thus care needs to be used if a constitutive model with damage-eroded elastic moduli is used.

The second requirement is that the method can be used when the crack path is not known \textit{a priori}. Although the fracture that is observed on the high-pressure end of the seal is often a delamination (interfacial crack) between the rigid substrate and the rubber seal, the cracks that form on the low-pressure end propagate through the interior of the rubber, and the precise path of propagation is not known beforehand. Indeed, the path of propagation will undoubtedly depend on the geometry of the low-pressure support ring, and may also depend on the material properties, aspect ratio, friction coefficient and other parameters. In short, fracture propagation methods, such as the use of traction-separation boundary conditions \cite{49} are adequate for the interfacial layer and will be used in the system-level model below, but, unless used between every element, they are inadequate for the bulk material.

The third requirement is that the fracture propagation model must be valid for multi-mode fracture. The fracture at the low-pressure end most closely approximate Mode II fracture, in which adjacent layers are sheared past each other. In the seal, the layer of material with no axial support is sheared past the adjacent layer with axial support. However, once failure initiates, it has been observed that it does not occur axisymmetrically, and there may be a Mode III type fracture due to azimuthal variations. Additionally, the fact that these seals are in both radial and axial compression poses an additional modeling challenge related to contact of newly formed surfaces. It is essential that, as the fracture propagates, the newly formed faces of the crack do not interpenetrate.

In the analysis of the previous chapter, friction was ignored. In real systems it is certainly present and its effect may be leading order. Therefore, we must be able to account for its effect in the computational model.

Finally, it is desirable for the results of the computations to be insensitive to the mesh. In numerical analysis, we expect that the solution will converge to the real solution as the mesh is refined. However, finite element solutions are expected to give approximations that are optimum (in an energy norm) for nonzero mesh size. Therefore, we require that the results be insensitive to mesh size. This issue will arise when choosing the proper failure criterion.

There are a number of different methods that satisfy the above constraints and this is an active area of research. One class of methods involves domain decomposition and/or adaptive mesh refinement near the crack tip, which must be sequentially refined as the crack propagates \cite{50}. Another class of solutions for crack propagation involve the use of phase-field models in which the damage is smeared out over multiple elements. Phase field approaches to fracture is a highly active area of research \cite{62, 61, 82, 92}.

Here we utilize an alternate approach, which readily lends itself to implementation within the commercial finite element package ABAQUS through the use of user subroutines, and thereby takes advantage of ABAQUS's framework for modeling contact, friction and finite deformation. Specifically, we write a user subroutine (VUMAT) for use in ABAQUS/Explicit, which contains a damage initiation criterion, a damage evolution criterion, and an element deletion criterion. The reason for using such a model is given by the constraints enumerated above. With a VUMAT, elements can be deleted before excessive distortion occurs. Contact between crack faces created by deleting elements can be implemented in three dimensional simulations (but not in two-dimensional simulations). Arbitrary constitutive relations for finite deformation can be implemented. Here we implement a neo-Hookean constitutive model and an Arruda-Boyce eight-chain model, implemented as the five-term series expansion of the inverse Langevin function, as done in ABAQUS.

The outline of this chapter is as follows. First we present work on developing the constitutive model for failure based on a maximum principal stretch criterion. This work shows that the VUMATs
for neo-Hookean and Arruda-Boyce materials agree with those implemented in Abaqus. We present the behavior of a single element simulations for this implementation based on principal stretch. Then we present the investigation of the effect of element size on fracture initiation for a single edge notch tension test under plane strain. The results show that a fracture initiation criterion based on maximum principal stretch does not converge, as expected, because the maximum principal stretch in an element near the crack tip depends on how well-resolved the stress concentration is, and therefore, the failure initiation criterion depends on element size. These results are used to motivate a local energy-based fracture initiation criterion. A comparison between fracture initiation predicted by the local fracture criterion and that predicted by a global energy release rate calculation is made for both single edge notch tension testing and Mode II shearing. We then present simulations of the extrusion process characteristic of the low-pressure end of elastomeric seals. In these simulations we implement the contact conditions for non-interpenetrating fracture faces under radial compression, as experienced in realistic applications. Finally, we present a realistic model of an annular seal under applied differential pressure with a traction-separation failure criterion at the interface between the rigid substrate and the inner surface, and the element deletion fracture propagation method on the low-pressure end of the seal.

### 5.1 Thermodynamics of Damage

In this section we follow classical continuum mechanics arguments pioneered by Truesdell and Noll [95], which are outlined in Gurtin, Fried and Anand [34]. The first law of thermodynamics states that energy must be conserved. Considering a material part, we can write the first law in variational form as

$$\delta E + \delta K = \delta W + \delta Q$$

(5.1)

where $\delta E$ is the variation in internal energy, $\delta K$ the variation in kinetic energy, $\delta W$ the variation in work done on the part, and $\delta Q$ the variation in heat into the part. The internal energy can be written as the integral over the part, $P$, in a deformed configuration at time $t$ as

$$\delta E = \delta \left( \int_{P_t} \rho \varepsilon \, dv \right)$$

(5.2)

where $\varepsilon$ is the specific internal energy (per unit mass) in the deformed configuration and $\rho$ is the mass density in the deformed configuration. Using the fact that $dv = Jdv_R$ and $\rho = J^{-1}\rho_R$, where $dv_R$ is an incremental unit of volume in the reference configuration and $J = \det \mathbf{F}$ is the volume ratio and $\rho_R$ is the density in the reference configuration, we have

$$\delta E = \delta \left( \int_{P_t} \rho \varepsilon \, dv \right) = \delta \left( \int_{P_R} \rho_R \varepsilon_R \, dv_R \right) = \int_{P_R} \rho_R \delta (\varepsilon_R) \, dv_R = \int_{P_t} \rho \delta \varepsilon \, dv$$

(5.3)

as the variation in the internal energy of the part, where we recognize that the internal energy, when normalized by mass in both the reference and deformed configuration are the same because we are considering a Lagrangian description of the material: $\varepsilon = \varepsilon_R$. Similarly, the variation in kinetic energy of the part can be written

$$\delta K = \int_{P_t} \rho \dot{u} \cdot \delta \mathbf{u} \, dv$$

(5.4)

where $\dot{u} = \partial \mathbf{u} / \partial t \big|_X$ is the Lagrangian time derivative of the displacement $\mathbf{u}(X, t) = \mathbf{x}(X, t) - X$. We can write the work done on the part as a combination of that due to surface tractions and that due to internal body forces as

$$\delta W = \int_{\partial P_t} (\mathbf{T} \cdot \mathbf{n}) \, da + \int_{P_t} (\mathbf{b}_0 \cdot \delta \mathbf{u}) \, dv$$

(5.5)

where $\mathbf{T}$ is the Cauchy stress, $\mathbf{n}$ the outward unit normal, $\delta \mathbf{u}$ the variation in displacement, and $\mathbf{b}_0$ the body force per unit deformed volume. We have used Cauchy’s hypothesis that the traction on
the surface, \( t \), is equal to the linear mapping of the Cauchy stress tensor acting on the outward unit normal: \( t = T n \), where the Cauchy stress tensor is symmetric in order to satisfy conservation of angular momentum: \( T = T^T \). Using the divergence theorem, we rewrite the variation in work done on the material part as

\[
\delta W = \int_{P_t} (\text{div } T \cdot \delta \mathbf{u} + T : \text{grad } \delta \mathbf{u} + b_0 \cdot \delta \mathbf{u}) \, dv \\
= \int_{P_t} ((\text{div } T + b_0) \cdot \delta \mathbf{u} + T : \text{sym} (\text{grad } \delta \mathbf{u})) \, dv 
\]

where we have used the fact that the Cauchy stress tensor is symmetric, and the inner product of a symmetric tensor with a skew tensor is zero. (Note that this is analogous to the conventional expression of \( T : D \), where \( D = \text{sym } L \) is the symmetric part of the velocity gradient, \( L = \text{grad } \dot{\mathbf{u}} \))

\[
\int_{\partial P_t} Tn \cdot \delta n da = \int_{\partial P_t} T_{ij} n_j \delta u_i da \\
= \int_{\partial P_t} T^T \delta u \cdot nda \\
= \int_{P_t} \text{div } (T^T \delta u) \, dv \\
= \int_{P_t} \left( \frac{\partial T^T_{ij}}{\partial x_i} \delta u_j + T_{ij} \frac{\partial \delta v_j}{\partial x_i} \right) \, dv \\
= \int_{P_t} (\text{div } T \cdot \delta \mathbf{u} + T : \text{grad } \delta \mathbf{u}) \, dv \\
= \int_{P_t} (\text{div } T \cdot \delta \mathbf{u} + T : (\text{sym} (\text{grad } \delta \mathbf{u}) + \text{skw}(\text{grad } \delta \mathbf{u}))) \, dv \\
= \int_{P_t} (\text{div } T \cdot \delta \mathbf{u} + T : \text{sym} (\text{grad } \delta \mathbf{u})) \, dv 
\]

Finally, the variation in heat into the part can be written

\[
\delta Q = - \int_{\partial P_t} \delta q \cdot nda + \int_{P_t} \delta q dv 
\]

where \( \delta q \) is the variation of the heat flux vector with dimensions of energy per area, and \( \delta q \) is the variation in volumetric heat into the system, due to a variety of effects possibly including Joule heating or radiation absorption (from nuclear decay, UV curing of elastomers, ...), and having dimensions of energy per unit volume. Again using the divergence theorem, we write this as variation in heat as

\[
\delta Q = \int_{P_t} (- \text{div } \delta q + \delta q) \, dv 
\]

Putting together the variations in each of the four terms, we have

\[
\delta E \equiv \int_{P_t} \left( \rho \delta \varepsilon + \rho \ddot{\mathbf{u}} \cdot \delta \mathbf{u} - ((\text{div } T + b_0) \cdot \delta \mathbf{u} + T : \text{sym} (\text{grad } \delta \mathbf{u}) + \text{div } \delta q - \delta q) \right) \, dv = 0. \] (5.10)

We use the principles of variational calculus to write a variational indicator as the integral of this energy over an arbitrary time interval with the restriction that the variations in the individual generalized coordinates (\( \delta \mathbf{u}, \delta q, \delta q, \delta \varepsilon \)) vanish at the endpoints of the time domain [31, 35]. Then integration by parts (with respect to time) on the kinetic energy term, imposing the constraint of no variations at the time end points, yields the equation

\[
\int_{P_t} \left( \rho \delta \varepsilon - ((\text{div } T + b_0 - \rho \ddot{\mathbf{u}}) \cdot \delta \mathbf{u} - T : \text{sym} (\text{grad } \delta \mathbf{u}) + \text{div } \delta q - \delta q) \right) \, dv = 0, \] (5.11)
where the canceled term is identically zero by conservation of linear momentum. Since this equation is valid for an arbitrary part, \( P_t \), the integrand must vanish, yielding
\[
\rho \delta \varepsilon - \mathbf{T} : \operatorname{sym} (\operatorname{grad} \delta \mathbf{u}) + \operatorname{div} \delta \mathbf{q} = 0, \tag{5.12}
\]
which is the variational, as opposed to the time derivative, form of Gurtin, Fried and Anand’s Eqn. (26.8) [34].

The second law of thermodynamics states that entropy production must be positive. A balance of entropy for a part requires
\[
\delta S_{\text{part}} = \delta S_{\text{net transfer}} + \delta S_{\text{generated}} \tag{5.13}
\]
where \( \delta S_{\text{part}} \) is the variational change in entropy in an arbitrary part, \( P \) at time \( t \), \( \delta S_{\text{net transfer}} \) is the net transfer of entropy into the part, and \( \delta S_{\text{generated}} \) is the variational entropy generated in the part. The second law requires \( \delta S_{\text{generated}} \geq 0 \). The entropy in the part at any given time is equal to
\[
S_{\text{part}} = \int_{P_t} \rho \eta \, dv \tag{5.14}
\]
where \( \eta \) is the specific entropy per unit mass. Using the same argument as for Eqn. (5.3), the variation in entropy of the part can be written
\[
\delta S_{\text{part}} = \delta \left( \int_{P_t} \rho \eta \, dv \right) = \delta \left( \int_{P_t} \rho_R \eta_R \, dv_R \right) = \int_{P_t} \rho_R \delta \eta_R \, dv_R = \int_{P_t} \rho \delta \eta \, dv. \tag{5.15}
\]
Following Clausius, entropy entering the system does so along with heat modulated by the local temperature, \( \vartheta \), at which the heat crosses the boundary. Thus the entropy transfer is
\[
\delta S_{\text{net transfer}} = - \int_{\partial P_t} \frac{\delta \mathbf{q}}{\vartheta} \cdot \mathbf{n} \, da + \int_{P_t} \frac{\delta q}{\vartheta} \, dv = \int_{P_t} \left( - \operatorname{div} \left( \frac{\delta \mathbf{q}}{\vartheta} \right) + \frac{\delta q}{\vartheta} \right) \, dv \tag{5.16}
\]
where we have used the divergence theorem to rewrite the surface integral as a volume integral. Finally, we may write the total entropy generated in the part as the integral of the local generation at each point in the part as
\[
\delta S_{\text{generated}} = \int_{P_t} \rho \delta \eta_{\text{generated}} \, dv \tag{5.17}
\]
where \( \delta \eta_{\text{generated}} \) is the variational entropy generation per unit mass, which is required to be non-negative at every point in the body.
\[
\delta \eta_{\text{generated}} \geq 0 \tag{5.18}
\]
Combining each of the terms in the entropy balance given by Eqn. (5.13) gives
\[
\int_{P_t} \left( \rho (\delta \eta - \delta \eta_{\text{generated}}) + \operatorname{div} \left( \frac{\delta \mathbf{q}}{\vartheta} \right) - \frac{\delta q}{\vartheta} \right) \, dv = 0 \tag{5.19}
\]
which is valid for arbitrary material part \( P_t \) and therefore must hold locally at every point in the body, giving the strong form of the entropy balance as
\[
\rho \delta \eta = - \operatorname{div} \left( \frac{\delta \mathbf{q}}{\vartheta} \right) + \frac{\delta q}{\vartheta} + \rho \delta \eta_{\text{generated}} \tag{5.20}
\]
with the second law requiring \( \delta \eta_{\text{generated}} \geq 0 \). Combining the entropy balance and the second law and assuming non-negative density, \( \rho \geq 0 \), gives the local entropy imbalance
\[
\rho \delta \eta + \operatorname{div} \left( \frac{\delta \mathbf{q}}{\vartheta} \right) - \frac{\delta q}{\vartheta} = \rho \delta \eta + \frac{\delta q}{\vartheta} - \frac{\delta \mathbf{q} \cdot \operatorname{grad} \vartheta}{\vartheta^2} - \frac{\delta q}{\vartheta} = \rho \delta \eta_{\text{generated}} \geq 0, \tag{5.21}
\]
which is the variational form of the entropy imbalance given in Gurtin, Fried and Anand’s Eqn. (27.13) [34]. Combining this with the first law, given in Eqn. (5.12), gives

\[ \rho (\delta \varepsilon - \partial \delta \eta) - \mathbf{T} : \text{sym} \left( \text{grad} \, \delta \mathbf{u} \right) + \frac{\delta \mathbf{q} \cdot \text{grad} \, \vartheta}{\vartheta} = -\rho \partial \delta \eta_{\text{generated}} \leq 0, \]  

(5.22)

which is a free energy imbalance.

At this point we depart from the structure of Gurtin, Fried and Anand. We hypothesize that the internal energy, \( \varepsilon \), is composed of a internal energy due to thermal and strain effects, \( u \), as well as surface energy due to the formation of micro-surfaces. On a continuum level, we consider the concentration of these microsurfaces as the surface area of the microcracks per unit mass

\[ d_R \equiv \frac{\text{Surface Area of Cracks}}{\text{Mass}} \]  

(5.23)

where we define the surface area of cracks with respect to the reference configuration. In this way, the surface energy in the part is

\[ \varepsilon_{\text{surface}} = \int_{\mathcal{P}_1} \rho R \Gamma_R dR \, dv_R = \int_{\mathcal{P}} \rho \Gamma \, dv \implies \Gamma_R dR = \Gamma d \]  

(5.24)

where \( \Gamma_R \) is the energy per unit surface area in the reference configuration and is a material property and \( d \) is the area density of cracks in the deformed configuration per unit mass. The specific internal energy is written

\[ \varepsilon = u + \Gamma d \]  

(5.25)

where \( u \) is the internal energy (due to strain and temperature) per unit mass. Then the Helmholtz free energy per unit mass, \( \psi \), is

\[ \psi \equiv u - \partial \eta \]  

(5.26)

and the free energy imbalance is written

\[ \rho (\delta \psi + \Gamma \delta d + \eta \delta \vartheta) - \mathbf{T} : \text{sym} \left( \text{grad} \, \delta \mathbf{u} \right) + \frac{\delta \mathbf{q} \cdot \text{grad} \, \vartheta}{\vartheta} = -\rho \partial \delta \eta_{\text{generated}} \leq 0, \]  

(5.27)

Following Gurtin, Fried and Anand [34], we define a free energy per unit reference volume, \( \psi_R \), as well as internal energy and entropy per unit reference volume as

\[ \psi_R \equiv \rho_R \psi = J \rho \psi, \quad \varepsilon_R \equiv \rho_R \varepsilon, \quad \eta_R \equiv \rho_R \eta \]  

(5.28)

such that \( \psi_R = \varepsilon_R - \partial \eta_R \). Following the procedures of Gurtin, Fried and Anand’s §31, we can write the free energy imbalance as

\[ \delta \psi_R + \Gamma \delta d_R + \eta_R \delta \vartheta - \mathbf{T}_R : \delta \mathbf{F} + \frac{\delta \mathbf{q}_R \cdot \text{Grad} \, \vartheta}{\vartheta} = -\partial \delta \eta_{R_{\text{generated}}} \leq 0, \]  

(5.29)

We write the referential free energy as a function of the generalized coordinates \( C \), \( d_R \) and \( \vartheta \) as

\[ \psi_R = \hat{\psi}_R (C, d_R, \vartheta) \]  

(5.30)

where \( C = F^T F = U^2 \) is the right Cauchy-Green deformation tensor. Then the variation in the referential Helmholtz potential energy is

\[ \delta \hat{\psi}_R = \frac{\partial \hat{\psi}_R}{\partial C} : \delta C + \frac{\partial \hat{\psi}_R}{\partial \vartheta} \delta \vartheta + \frac{\partial \hat{\psi}_R}{\partial d_R} \delta d_R. \]  

(5.31)

We recognize that the inner product of the first Piola-Kirchhoff stress with the variation in deformation gradient is equal to half the inner product of the second Piola-Kirchhoff stress with the variation
in the right Cauchy-Green deformation tensor

\[ T_R : \delta F = \frac{1}{2} T_{RR} : \delta C \]  

(5.32)

where the second Piola-Kirchhoff stress is related to the first as

\[ T_{RR} = F^{-1} T_R. \]  

(5.33)

Then the free energy imbalance can be written as

\[
\left( \frac{\partial \hat{\psi}_R}{\partial C} - \frac{1}{2} T_{RR} \right) : \delta C + \left( \frac{\partial \hat{\psi}_R}{\partial \vartheta} + \eta_R \right) \delta \vartheta + \left( \frac{\partial \hat{\psi}_R}{\partial d_R} + \Gamma \right) \delta d_R + \frac{\delta q_R \cdot \text{Grad} \vartheta}{\vartheta} = -\vartheta \delta \eta_{R,\text{generated}} \leq 0.
\]  

(5.34)

The generalized coordinates on which the free energy depends, \( \vartheta, d_R \) and \( C \), as well as the heat quantities \( q \) and \( q_R \), can be specified independently of each other. Therefore, each term must satisfy the inequality independently of all the others. This states that the second Piola stress is twice the gradient of free energy with respect to the right Cauchy-Green deformation tensor.

\[ T_{RR} = 2 \frac{\partial \hat{\psi}_R}{\partial C} \]  

(5.35)

It also gives the result that the entropy is the gradient of Helmholtz energy with respect to temperature, holding the deformation and crack density constant,

\[ \frac{\partial \hat{\psi}_R}{\partial \vartheta} \bigg|_{C,d_R} = -\eta, \]  

(5.36)

as required by fundamental rules of thermodynamics. Thirdly, the free energy imbalance requires the variational increment in heat flux, \( \delta q_R \), dotted with the temperature gradient be negative,

\[ \delta q_R \cdot \text{Grad} \vartheta \leq 0 \]  

(5.37)

which gives the well-known result that heat cannot flow up a temperature gradient. Finally, for systems of interest in this thesis, we restrict ourselves to cases when damage is a monotonically increasing function and the material cannot self-heal. (If self-healing were considered, we would need to consider additional physics, such as chemical reactions, such that the entropy production in a healing system would still be non-negative due to the change in free energy of the reacting components.) In the present case, since \( \delta dd_R \geq 0 \), it is necessary that the gradient of free energy with respect to the surface area of cracks be less than or equal to the surface energy of the material.

\[ \frac{\partial \hat{\psi}_R}{\partial d_R} + \Gamma \leq 0 \]  

(5.38)

If this inequality is satisfied, then it is possible for \( \delta d_R \) to be positive. However, if this inequality is not satisfied, then the surface area of cracks cannot increase and \( \delta d_R = 0 \). This criterion is necessary but not sufficient, and provides only a thermodynamic constraint. In reality, there may be an energy barrier to fracture, as is common in transition state theories of chemical reactions (of which fracture is indeed a subset). In such theories, there is a difference in potential energy between initial state and final state, but there is an energy barrier at the transition state, which must be overcome, and only by providing a sufficiently large differential free energy, or sufficiently large thermal fluctuations, can the barrier be traversed. This is sketched qualitatively in Fig. 5-1.

Furthermore, it is widely known in the field of non-equilibrium thermodynamics [46] that the rate at which a process occurs is a function of the driving potential. In linearized theories, the flux
Figure 5-1: Schematic illustration of non-equilibrium overpotential required to drive crack generation at nonzero rates through the transition state.

is linearly proportional to the gradient of potential, as in Fourier’s

\[ q = -k \text{grad} \vartheta \]  \hspace{1cm} (5.39a)

and Fick’s

\[ j = -D \text{grad} \mu \]  \hspace{1cm} (5.39b)

laws, and the rate of irreversibility, or entropy production, is the product of the driving potential and the flux, which we see from free energy imbalance of Eqn. (5.34), where, in the absence of variational deformation, crack growth and temperature change, the entropy production is given by

\[ \frac{\delta q_R \cdot \text{Grad} \vartheta}{\vartheta} = -\partial \delta \eta_{R,\text{generated}} \]

Similarly, in the linearized limit such as that argued by Onsager much to the disdain of Truesdell [67], we might expect

\[ \dot{d}_R \propto -\frac{\partial \psi_R}{\partial d_R} - \Gamma. \]  \hspace{1cm} (5.40)

**Isothermal System**

This is simply a conjecture and outside the scope of this thesis. Instead, the matter at hand is to specify the free energy function and the rate variables as functions of the generalized coordinates. First, we restrict our attention to isothermal conditions, such that temperature is no longer a variable, and we need not worry about heat. Then the isothermal free energy imbalance leads is given by

\[ \left( \frac{\partial \psi_R}{\partial C} - \frac{1}{2} T_{RR} \right) : \delta C + \left( \frac{\partial \psi_R}{\partial d_R} + \Gamma \right) \delta d_R = -\partial \delta \eta_{R,\text{generated}} \leq 0. \]  \hspace{1cm} (5.41)

We have already made the simplifying assumption that damage can be characterized by a scalar damage variable, \( d_R \). Here we choose a multiplicative decomposition of the free energy function,
such that when there is no damage, the free energy is equal to that for a generic hyperelastic material (in this case we consider neo-Hookean and Arruda-Boyce models). In the limit when the material is completely damaged, the material is able to support no elastic free energy inside, and there is a monotonic transition between the two limits. For simplicity, we choose a linear dependence on $d_R$, but this could easily be modified if one had experimental justification for doing so. We write the free energy in the reference configuration as

$$
\psi_R = \hat{\psi}_R (C, d_R) = (1 - L_c d_R) \hat{\psi}_{R,\text{hyper}} (C)
$$

(5.42)

where $\hat{\psi}_{\text{hyper}} (C)$ is a generic hyperelastic free energy function and $L_c$ is some constant characteristic length necessary to yield the correct dimensions because $d_R$ has dimensions of area divided by volume, or inverse length. In this way, the stresses are equal to those for a hyperelastic material modulated by the damage.

$$
T = (1 - L_c d_R) T_{\text{hyper}} = 2 J^{-1} (1 - L_c d_R) F \frac{\partial \hat{\psi}_{\text{hyper}}}{\partial C} F^T
$$

$$
T_R = (1 - L_c d_R) T_{R,\text{hyper}} = 2 (1 - L_c d_R) \frac{\partial \hat{\psi}_{\text{hyper}}}{\partial C}
$$

$$
T_{RR} = (1 - L_c d_R) T_{RR,\text{hyper}} = 2 (1 - L_c d_R) \frac{\partial \hat{\psi}_{\text{hyper}}}{\partial C}
$$

(5.43)

All that remains is to specify a functional relationship for $d_R$. The evolution of $\delta d_R$ must satisfy the free energy inequality

$$
(\Gamma - L_c \hat{\psi}_{R,\text{hyper}} (C)) \delta d_R = -\partial \delta \eta_{R,\text{generated}} \leq 0
$$

(5.44)

giving the requirement that $L_c \hat{\psi}_{\text{hyper}} \geq \Gamma$ if the crack is to grow.

We present two attempts for specifying this functional relationship in the sections that follow. The first is a naive approach, based on maximum principal stretch, where failure initiation is expected to occur when the maximum principal stretch reaches a critical value. Although seemingly consistent with the idea that elastomers are composed of polymeric chains, which will rupture once they are stretched beyond a critical value based on bond length, number of bonds in a chain, etc. [92, 60], this approach does not lend itself to computational methods. Specifically, the results obtained for failure initiation are strongly dependent on mesh size, as expected. We then consider a failure criterion based on strain energy density, and show that the results for failure initiation are independent of mesh size, yielding a viable computational approach. Both attempts were conducted prior to the formal free energy analysis above, and it turns out that the second approach below is an obvious choice given the above derivations.

### 5.2 Hyperelastic Models without Damage

The objective of this section is to develop Abaqus user subroutines for the behavior of hyperelastic materials in the absence of damage. Although hyperelastic constitutive models are standard, there are slight variations in the treatment of the compressibility term for materials which are not perfectly incompressible. It is fairly standard to decompose the deformation into and isochoric deformation, denoted as $F = J^{-1/3} F$. However, the formulation of the compressible part of the free energy, and whether it is formulated as $K/2 (J - 1)^2$ or $K/2 (\ln J)^2$ or some other formulation yields minor differences. Indeed there are a variety of options, and Holzapfel [39] provides an approachable introduction to some possibilities. We do not presume to know the proper formulation that best agrees with experiments, and for nearly incompressible materials, it makes only little difference. Instead, we seek to match the results of constitutive models built into Abaqus. Upon matching Abaqus’s material models in the absence of damage, we proceed to implement a damage model in subsequent sections.
5.2.1 Kinematics

The position of a material point in the reference configuration is given by the vector \( \mathbf{X} \). In the deformed configuration, the position of this point is \( \mathbf{x} = \chi(\mathbf{X}, t) \) where \( \chi \) is the mapping from the reference configuration to the deformed configuration at time \( t \) and is termed the motion. The displacement, \( \mathbf{u} \), is the difference between the position in the deformed configuration and the reference configuration given by \( \mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \). The deformation gradient, \( \mathbf{F} \), is given by the gradient, with respect to the reference configuration, of the motion, such that its components are

\[
F_{ij} = \frac{\partial \chi_i}{\partial X_j}
\]

The right and left Cauchy-Green tensors are

\[
\mathbf{C} = \mathbf{F}^T \mathbf{F}, \quad \mathbf{B} = \mathbf{F} \mathbf{F}^T
\]

respectively. Then the principal stretches, \( \lambda_a, a = 1, 2, 3 \) are the square roots of the eigenvalues of these tensors, and the principal directions are the eigenvectors. Note that the eigenvalues of both \( \mathbf{B} \) and \( \mathbf{C} \) are the same, while the principal directions differ by a rotation.

\[
\mathbf{C} \mathbf{N}_a = \lambda_a^2 \mathbf{N}_a, \quad \mathbf{B} \mathbf{n}_a = \lambda_a^2 \mathbf{n}_a
\]

Because the right and left Cauchy-Green tensors are symmetric, their spectral decompositions are

\[
\mathbf{C} = \sum_{a=1}^{3} \lambda_a^2 \left( \mathbf{N}_a \otimes \mathbf{N}_a \right), \quad \mathbf{B} = \sum_{a=1}^{3} \lambda_a^2 \left( \mathbf{n}_a \otimes \mathbf{n}_a \right)
\]

Similarly, we can define the right and left stretch tensors as

\[
\mathbf{U} = (\mathbf{C})^{\frac{1}{2}} = \sum_{a=1}^{3} \lambda_a \left( \mathbf{N}_a \otimes \mathbf{N}_a \right), \quad \mathbf{V} = (\mathbf{B})^{\frac{1}{2}} = \sum_{a=1}^{3} \lambda_a \left( \mathbf{n}_a \otimes \mathbf{n}_a \right)
\]

respectively. The rotation tensor, \( \mathbf{R} \), is a proper orthogonal tensor that either first rotates the material before stretch or rotates the material following stretch as

\[
\mathbf{F} = \mathbf{VR} = \mathbf{RU}
\]

It also provides the mapping between the principal directions in the spatial and referential configurations.

\[
\mathbf{n}_a = \mathbf{R} \mathbf{N}_a
\]

Finally, the volume ratio, \( J \), gives the ratio of volume in the deformed configuration to volume in the initial configuration.

\[
J \equiv \det \mathbf{F}
\]

For isotropic materials, the free energy function depends only on the invariants of the left and right Cauchy-Green tensors. The invariants are

\[
I_1 \equiv \tilde{I}_1(\mathbf{B}) = \text{tr} \mathbf{B} = \sum_{a=1}^{3} \lambda_a^2
\]

\[
I_2 \equiv \tilde{I}_2(\mathbf{B}) = \frac{1}{2} \left( (\text{tr} \mathbf{B})^2 - \text{tr} \left( \mathbf{B}^2 \right) \right) = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2
\]

\[
I_3 \equiv \tilde{I}_3(\mathbf{B}) = \det \mathbf{B} = \lambda_1^2 \lambda_2^2 \lambda_3^2
\]

For nearly incompressible materials, which is often the case for elastomers, the deformation gradient can be decomposed into isochoric and isotropic volumetric components via a multiplicative
decomposition [28].

\[ \mathbf{F} = J^{\frac{3}{2}} \mathbf{F} \]

where the isochoric deformation, \( \mathbf{F} \), preserves volume and the volumetric component, \( J^{\frac{3}{2}} \mathbf{I} \), is isotropic. Then all the tensors derived from the deformation gradient can be decomposed into their volumetric and isochoric components. The left Cauchy-Green tensor is decomposed as

\[ \mathbf{B} = J^{\frac{3}{2}} \mathbf{B} \]

where \( \mathbf{B} \) is the isochoric left Cauchy-Green tensor. The eigenvalues of the isochoric left Cauchy-Green tensor are

\[ \lambda_a^2 = J^{-\frac{3}{2}} \lambda_a^2 \]

The principal invariants of the isochoric left Cauchy-Green tensor are

\[ I_1 \equiv \tilde{I}_1 (\mathbf{B}) = \text{tr} \mathbf{B} = \sum_{a=1}^{3} \lambda_a^2 = J^{-\frac{3}{2}} I_1 \]

\[ I_2 \equiv \tilde{I}_2 (\mathbf{B}) = \frac{1}{2} \left( (\text{tr} \mathbf{B})^2 - \text{tr} (\mathbf{B}^2) \right) = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 = J^{-\frac{3}{2}} I_2 \]

\[ I_3 \equiv \tilde{I}_3 (\mathbf{B}) = \text{det} \mathbf{B} = \lambda_1^2 \lambda_2^2 \lambda_3^2 = J^{-2} I_3 \]

### 5.2.2 Stress in Isotropic Hyperelastic Materials

For isotropic hyperelastic constitutive relations, the free energy function per unit volume, \( \Psi \), is a function of the invariants of the Cauchy-Green tensors. Using this free energy formulation, the stress is the derivative of the free energy with respect to the deformation. Specifically, the second Piola-Kirchhoff stress tensor \( \mathbf{S} \) is given by

\[ \mathbf{S} = 2 \frac{\partial \Psi}{\partial \mathbf{C}} \]

and the Cauchy stress, \( \sigma \), is related to the second Piola-Kirchhoff stress by

\[ \mathbf{S} = J \mathbf{F}^{-1} \sigma \mathbf{F}^T, \quad \sigma = J^{-1} \mathbf{F} \mathbf{S} \mathbf{F}^T \]

Then the Cauchy stress is

\[ \sigma = 2 J^{-1} \left( I_1 \frac{\partial \Psi}{\partial I_1} \mathbf{I} + \frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) \mathbf{B} - \frac{\partial \Psi}{\partial I_2} \mathbf{B}^2 \]  

(5.45)

### 5.2.3 neo-Hookean Constitutive Relation

One form of a neoHookean free energy function for a slightly compressible material is

\[ \Psi_{\text{neoHookean}} = C_1 (I_1 - 3) + D_1 (J - 1)^2 \]

\[ = C_1 \left( I_3^{-\frac{4}{3}} I_1 - 3 \right) + D_1 \left( I_3^\frac{4}{3} - 1 \right)^2 \]

where the first term, with coefficient \( C_1 \), accounts for the volume-preserving deformation and the second term, with coefficient \( D_1 \), accounts for the isotropic dilational/volumetric deformation. Then, putting this into the expression for stress in Eq. (5.45) gives

\[ \sigma_{\text{neoHookean}} = 2 J^{-1} C_1 \left( \frac{ \mathbf{B} - \frac{1}{3} I_1 }{3} \right) + 2 D_1 (J - 1) \mathbf{I} \]

where the small-strain shear modulus is \( \mu_0 = 2 C_1 \) and the small-strain bulk modulus is \( K_0 = 2 D_1 \).
5.2.4 Arruda-Boyce Eight-Chain Constitutive Relation

A better hyperelastic free energy formulation comes from the classic eight chain model put forth by Arruda and Boyce in their seminal work[4]. In taking the Taylor series expansion of this model, it becomes a specialized case of the generic reduced polynomial free energy function of the form

\[ \Psi_{\text{reduced poly}} = \sum_{i=1}^{N} C_i \left( T_i - 3^i \right) + D \left( \frac{J^2 - 1}{2} - \ln J \right) \]

\[ = \sum_{i=1}^{N} C_i \left( \left( I_3 - \frac{1}{2} I_1 \right)^i - 3^i \right) + D \left( \frac{I_3 - 1 - \ln(I_3)}{2} \right) \]

For the conventional five-term series expansion of the inverse Langevin function used in the Arruda-Boyce model, the terms in the generic reduced polynomial free energy function take the values

\[ N = 5, \quad C_1 = \frac{1}{2} \mu, \quad C_2 = \frac{\mu}{20 \lambda_m^2}, \quad C_3 = \frac{11 \mu}{1050 \lambda_m^4}, \quad C_4 = \frac{19 \mu}{7000 \lambda_m^6}, \quad C_5 = \frac{519 \mu}{673750 \lambda_m^8}, \quad D = \frac{K_0}{2} \]

where \( \lambda_m \) is the locking stretch of the polymer chains, and \( \mu \) is approximately equal to the small-strain shear modulus (although not exactly equivalent).

Substituting the reduced polynomial free energy function into Eq. (5.45) gives Cauchy stress of

\[ \sigma_{\text{reduced poly}} = 2J^{-1} \left( \sum_{i=1}^{N} iC_i (\bar{T}_i)^{i-1} \right) \left( B - \frac{1}{3} I_3 \right) + J^{-1} D (I_3 - 1) I \]

5.2.5 Validation – Comparison with Standard Hyperelastic Models in Absence of Failure

In order to verify that both the neo-Hookean and Arruda-Boyce constitutive models above are implemented correctly in Abaqus user subroutines (VUMATs), we compare finite element simulations using the user subroutines with both theoretical predictions and finite element simulations using built-in Abaqus models. These subroutines are adapted from the neo-Hookean subroutine (VUMAT) downloaded from Chester’s research webpage [14].

Uniaxial Stress Case

In the first test case, we consider uniaxial stress of a single element, as depicted schematically in Fig. 5-2. For the set of basis vectors drawn, the displacement vector becomes

\[ \mathbf{u} = u_1 (x_1) \mathbf{e}_1 + u_2 (x_2) \mathbf{e}_2 + u_3 (x_3) \mathbf{e}_3 \]

with deformation gradient

\[ \begin{bmatrix} \mathbf{F} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \]

where

\[ \lambda_1 = 1 + \frac{\partial u_1}{\partial X_1}, \quad \lambda_2 = 1 + \frac{\partial u_2}{\partial X_2}, \quad \lambda_3 = 1 + \frac{\partial u_3}{\partial X_3} \]

The volume ratio is given by

\[ J = \lambda_1 \lambda_2 \lambda_3 \]
Figure 5-2: Schematic illustrating kinematics of uniaxial stress.

For this motion in this set of basis vectors, there is no rotation, and the components of the left Cauchy-Green tensor are equal to those of the right:

$$[\mathbf{B}] = [\mathbf{C}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad [\overline{\mathbf{B}}] = [\overline{\mathbf{C}}] = J^{-\frac{2}{3}} \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

with invariants

$$I_1(\overline{\mathbf{B}}) = J^{-\frac{2}{3}} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{(\lambda_1^2 \lambda_2^2 \lambda_3^2)^{\frac{2}{3}}}$$

$$I_3(\overline{\mathbf{B}}) = \frac{\lambda_1^2 \lambda_2^2 \lambda_3^2}{J^{\frac{2}{3}}} = 1$$

For a prescribed axial stretch, $\lambda_3$, it is then simply necessary to calculate the remaining two stretch components, $\lambda_1$ and $\lambda_2$, by imposing the zero stress condition.

For the neo-Hookean constitutive model, with Cauchy stress given by

$$\mathbf{\sigma} = 2J^{-1}C_1 \left( \overline{\mathbf{B}} - \frac{1}{3}I_1 \mathbf{I} \right) + 2D_1 (J - 1) \mathbf{I}$$

the three normal components of the stress tensor are

$$\sigma_{11} = 2C_1 \left( \frac{\lambda_1^2 - \frac{1}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}{(\lambda_1 \lambda_2 \lambda_3)^{\frac{2}{3}}} \right) + 2D_1 (\lambda_1 \lambda_2 \lambda_3 - 1)$$

$$\sigma_{22} = 2C_1 \left( \frac{\lambda_2^2 - \frac{1}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}{(\lambda_1 \lambda_2 \lambda_3)^{\frac{2}{3}}} \right) + 2D_1 (\lambda_1 \lambda_2 \lambda_3 - 1)$$

$$\sigma_{33} = 2C_1 \left( \frac{\lambda_3^2 - \frac{1}{3} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}{(\lambda_1 \lambda_2 \lambda_3)^{\frac{2}{3}}} \right) + 2D_1 (\lambda_1 \lambda_2 \lambda_3 - 1)$$
Setting $\sigma_{11} = \sigma_{22} = 0$ gives the requirement that $\lambda_1 = \lambda_2$ and the resulting equation

$$0 = 2C_1 \left( \frac{\lambda_1^2 - \frac{1}{3} \left( 2\lambda_1^2 + \lambda_3^2 \right)}{(\lambda_1^2 \lambda_3)^{\frac{5}{2}}} \right) + 2D_1 \left( \lambda_1^2 \lambda_3 - 1 \right)$$

is a relationship between $\lambda_1$ and $\lambda_3$ necessary for stress-free lateral surfaces. The equation is nonlinear, and, for specified axial stretch, $\lambda_3$, can be solved using an iterative numeric scheme, such as Newton-Raphson, or MATLAB's root command.

Similarly, for a reduced polynomial free energy function, the Cauchy stress is

$$\sigma = 2J^{-1} \left\{ \frac{D}{2} (I_3 - 1) \mathbf{I} + \left( \sum_{i=1}^{N} iC_i (T_1)^{i-1} \right) \left( \mathbf{B} - \frac{1}{3} T_1 \mathbf{I} \right) \right\}$$

In component form, for uniaxial stress in the chosen basis, this becomes

$$\sigma_{11} = 0 = 2J^{-1} \left\{ \frac{D}{2} (I_3 - 1) + \left( \sum_{i=1}^{N} iC_i (T_1)^{i-1} \right) \left( \frac{\lambda_1}{J^2} - \frac{1}{3} T_1 \right) \right\}$$

$$\sigma_{22} = 0 = 2J^{-1} \left\{ \frac{D}{2} (I_3 - 1) + \left( \sum_{i=1}^{N} iC_i (T_1)^{i-1} \right) \left( \frac{\lambda_2}{J^2} - \frac{1}{3} T_1 \right) \right\}$$

$$\sigma_{33} = 2J^{-1} \left\{ \frac{D}{2} (I_3 - 1) + \left( \sum_{i=1}^{N} iC_i (T_1)^{i-1} \right) \left( \frac{\lambda_3}{J^2} - \frac{1}{3} T_1 \right) \right\}$$

where we have specified that there is no stress in the lateral directions. Then subtracting the first from the second gives

$$\lambda_1 = \lambda_2$$

The resulting equation

$$0 = 2J^{-1} \left\{ \frac{D}{2} (I_3 - 1) + \left( \sum_{i=1}^{N} iC_i (T_1)^{i-1} \right) \left( \frac{\lambda_1}{J^2} - \frac{1}{3} T_1 \right) \right\}$$

can be numerically solved for $\lambda_1$ for specified values of $\lambda_3$. We note that numeric solution is necessary because the material is slightly compressible, and the closed-form analytic solution for incompressible uniaxial stress is not valid. However, this incompressible solution is used as the initial guess when iteratively solving the nonlinear equation.

In order to verify that the constitutive models are implemented correctly in the user subroutines, we compare the results for a single element finite element simulation using Abaqus for both the user subroutine and the constitutive model built into Abaqus. The kinematics of the single element are prescribed, such that there is uniform deformation in the direction aligned with the $\mathbf{e}_3$ direction, and no stress in the two orthogonal directions. The user subroutines are implemented for the Abaqus/Explicit finite element package, which is an explicit solver incorporating transient dynamics and the effect of inertia. In order to compare with the quasi-static theory outlined above, we prescribe a small deformation rate such that the resulting behavior is quasi-static. Fully integrated linear eight node hexahedral elements (C3D8) are used. Linear elements, rather than quadratic elements, are chosen because element deletion is of principal importance, and the ability to better resolve material failure using smaller elements is judged more important than high-order interpolation on coarser grids that may be offered by quadratic (or higher order) elements.

Figure 5-3 shows a comparison for the stress predicted by the newly implemented VUMAT with that predicted by the constitutive models incorporated within Abaqus for both neo-Hookean and Arruda-Boyce models. The results for the finite element simulations with the user subroutines agree with theory and finite element simulations using standard Abaqus models, which merely indicates the constitutive model was properly implemented in the user-defined subroutines (VUMATs).
Figure 5-3: Stress versus stretch for uniaxial stress for neo-Hookean and Arruda-Boyce constitutive models. The results show that the stress versus stretch for the VUMAT agree with models built into Abaqus and corresponding theoretical solutions. For both neo-Hookean and Arruda-Boyce, the shear modulus parameter was chosen to be $\mu = 1.0 \times 10^3 \text{ [Pa]}$. For the Arruda-Boyce model, the locking stretch was specified to be $\lambda_m = 5$. Two different bulk moduli, $K = 1.0 \times 10^5 \text{ [Pa]}$ and $K = 1.0 \times 10^7 \text{ [Pa]}$ were used, as indicated in the legend.

Simple Shear

We perform a second test to verify that the neo-Hookean and Arruda-Boyce constitutive models are properly implemented in the user subroutines. In this test, we consider the case of simple shear, as sketched in Fig. 5-4.

![Simple Shear Schematic](image)

In the basis chosen in the sketch, we consider displacement defined as

$$ u = u(X_3) \hat{e}_1 = \gamma X_3 \hat{e}_1 $$
such that the deformed position is
\[ x = X + u = X + \gamma X_3 \hat{e}_1 \]
and the deformation gradient, in the chosen basis, becomes
\[
[F] = \begin{bmatrix}
1 & 0 & \gamma \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
with volume ratio
\[ J = \det F = 1 \]
The left and right Cauchy-Green tensors are,
\[
B = FF^T = \begin{bmatrix}
1 + \gamma^2 & 0 & \gamma \\
0 & 1 & 0 \\
\gamma & 0 & 1
\end{bmatrix}, \quad C = F^TF = \begin{bmatrix}
1 & 0 & \gamma \\
0 & 1 & 0 \\
\gamma & 0 & 1 + \gamma^2
\end{bmatrix}
\]
The principal stretches are
\[
\lambda_1 = \sqrt{\frac{2 + \gamma^2 - \gamma \sqrt{4 + \gamma^2}}{2}}, \quad \lambda_2 = 1, \quad \lambda_3 = \sqrt{\frac{2 + \gamma^2 + \gamma \sqrt{4 + \gamma^2}}{2}}
\]
with principal directions in the deformed configuration
\[
\alpha_1 = \frac{1}{\sqrt{4 + \left(\gamma - \sqrt{4 + \gamma^2}\right)^2}} \begin{bmatrix}
\gamma - \sqrt{4 + \gamma^2} \\
0 \\
2
\end{bmatrix}, \quad \alpha_2 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad \alpha_3 = \frac{1}{\sqrt{4 + \left(\gamma + \sqrt{4 + \gamma^2}\right)^2}} \begin{bmatrix}
\gamma + \sqrt{4 + \gamma^2} \\
0 \\
2
\end{bmatrix}
\]

Fig. 5-5 shows a comparison of the results for a finite element simulation consisting of a single element for the user subroutine as well as the constitutive models incorporated into Abaqus. The results are compared with those for the theory. The results simply indicate that the constitutive models were properly implemented within the user subroutines.

5.3 Maximum Principal Stretch Failure Criterion

The preceding section confirmed that, in the absence of damage, the user subroutines for neo-Hookean and Arruda-Boyce materials yield results that agree with constitutive models built into Abaqus. We now implement a damage model into these hyperelastic constitutive models.

5.3.1 Damage Model Based on Maximum Principal Stretch

The purpose for developing these subroutines is to predict fracture propagation through structures. The previous section indicates that, in the absence of failure, the user subroutines yield results that are consistent with those of models already incorporated within Abaqus. In this section, we present a damage initiation and damage evolution criterion based on the maximum principal stretch.
Figure 5-5: Shear stress for simple shear showing the results of the user subroutines compared with those for Abaqus standard models and theoretical solutions. The Arruda-Boyce locking stretch parameter for these simulations is $\lambda_m = 5$.

Specifically, we choose the coefficient multiplying the free energy and stress terms of the form

$$1 - L_c d_R = \begin{cases} 1, & \lambda_{\text{max}} \leq \lambda_{c,1} \\ \exp \left( - \frac{\lambda_{\text{max}} - \lambda_{c,1}}{\lambda_d} \right), & \lambda_{c,1} \leq \lambda_{\text{max}} \leq \lambda_{c,2} \\ 0, & \lambda_{\text{max}} \geq \lambda_{c,2} \end{cases}$$

where $\lambda_{\text{max}}$ is the maximum principal stretch, $\lambda_{c,1}$ the critical stretch at which damage initiates, $\lambda_{c,2}$ the critical stretch at which complete failure occurs and $\lambda_d$ the decay rate.

We choose this damage model based on the physical intuition that polymer chains have a finite length between crosslinks, and therefore have a finite extensibility before they are fully elongated and undergo locking. Upon further stretching, the bonds are able to deform to a certain extent, and then, through a stochastic process, one of the bonds will rupture, thereby decreasing the force supported by the polymer network. This physical reasoning is the basis for this choice of maximum principal stretch as a failure criterion. Indeed, the idea of maximum principal stretch, or principal strain, or principal stress is far from novel, and maximum principal stress is often used as a failure criterion for brittle fracture.

In the computational implementation of this failure, the maximum principal stretch is evaluated at each material point in each element within the mesh, and the modulation of the computed stress is imposed at these points. Once a material point reaches the failure stretch, $\lambda_{\text{max}} = \lambda_{c,2}$, the element deletion flag is activated for that material point. Upon all material points being flagged for deletion within a given element, the element is removed from the mesh, thereby preventing errors pertaining to loss of stiffness and excessive mesh distortion in these damaged elements [91].

### 5.3.2 Uniaxial Stress Case

Fig. 5-6 contains plots of axial stress versus axial stretch for uniaxial stress of neo-Hookean and Arruda-Boyce hyperelastic materials with material damage given by Eqs. (5.42) and (5.47). The left plot shows results for fully integrated hexahedral elements with eight integration points (C3D8 elements in Abaqus) and the right plot shows the same results for reduced integration elements (C3D8R) having only a single integration point. In both plots, the Arruda-Boyce and neo-Hookean materials with damage follow the theoretical values until axial stretch, $\lambda_3$, reaches the prescribed value of failure stretch, $\lambda_{c,1}$. When the stretch reaches this initial failure stretch, the stress decays
despite the stretch still increasing, due to the exponential decay of the modulus, as given by Eqn. (5.43). Two different values of the decay rate, \( \lambda_d \), were considered. We see that for \( \lambda_d = 1.0 \), the decay of stress is more gradual than for \( \lambda_d = 0.1 \).

![Graph](image)

Figure 5-6: Axial stress versus axial stretch for uniaxial stress illustrating the effect of eroding the modulus to simulate failure. (a) Single element with full integration using eight integration points for eight node hexahedral element. (Abaqus element C3D8) (b) Single element with reduced integration using a single integration point for an eight node hexahedral element. (Abaqus element C3D8R)

In all cases considered, the maximum stretch at total failure, \( \lambda_{c,2} \), was prescribed to have a value of nine. We see that for the full integration element in Fig. 5-6.a, the axial stress goes to zero at stretches less than this value. However, for the reduced integration element in Fig. 5-6.b, the axial stress does not vanish until \( \lambda_3 = \lambda_{c,2} = 9 \) is reached. A note on the reason for this is contained in §5.3.3.

For both types of elements, the elements behave as purely hyperelastic elements up until \( \lambda_3 = \lambda_{c,1} \), at which point the stress begins to decay exponentially as expected. The difference between the two element types manifests itself in the maximum stretch before complete failure of the element.

### 5.3.3 A Note on Instability in Single Element Uniaxial Stress

It was observed in the single element simulation of uniaxial stress for the fully integrated elements (C3D8), the entire element failed at a smaller maximum principle stretch, \( \lambda_3 \) than the prescribed complete failure stretch, \( \lambda_{c,2} \). However, in the case of the reduced integration element with a single integration point (C3D8R), the element only failed upon the maximum principle stretch, \( \lambda_3 \), reaching the prescribed total failure stretch, \( \lambda_{c,2} \). Although this is not of particular concern in the present application, where the elastomers fracture unstably with rapid, uncontrolled crack growth, it does merit analysis and discussion from a fundamental perspective.

To understand the instability physically, we consider a two-dimensional analog model of the fully integrated eight-node brick element. In this analog model, we envision the integration points as springs, as sketched in Fig. 5-7. In this two-dimensional model, we envision the entire system to have two degrees of freedom such that there is a displacement, \( u_0 \), and a rotation \( \theta \) in the plane. Then the elongation of the left spring is \( u_1 = u_0 - L\theta \) and the elongation of the right spring is \( u_2 = u_0 + L\theta \) for small displacements. We consider a strain-softening spring consistent with our constitutive model for damage evolution. Letting the springs have stiffness

\[
k = k_0 \exp \left( -\frac{u}{d} \right),
\]

(5.48)
then the total potential energy stored in the system is
\[ V = d k_0 \left( 2d - e^{-\frac{\theta^2}{2d}} (d + u_1) - e^{-\frac{\theta^2}{2d}} (d + u_2) \right). \] (5.49)

Then we see that, for this conservative system (the real system is dissipative because of crack growth, but this system simply has strain-softening springs, which are still elastic, albeit nonlinearly so), the second derivative of potential energy with respect to the angle \( \theta \) about the value \( \theta = 0 \) is
\[ \frac{\partial^2 V}{\partial \theta^2} \bigg|_{\theta=0} = 2k_0L^2e^{-\frac{u_0}{d}} \frac{d - u_0}{d} \] (5.50)
which means that, provided \( d > u_0 \), the system is stable to perturbations in the variable \( \theta \). However, once \( u_0 \) is greater than \( d \), the system is no longer stable to these perturbations, which is a mechanistic explanation for the observed instability in the fully integrated eight-node brick elements.

5.3.4 Simple Shear

Further validation of the implementation of the failure model was conducted under simple shear. The results for shear stress, \( \sigma_{13} \), and normal stress, \( \sigma_{11} \), are shown in Fig. 5-8. The results simply indicate that the formulation described in Eqs. (5.45) and (5.47) was properly implemented in the user subroutine. The start of the decay depends on the parameter, \( \lambda_{c,1} \). The decay rate depends on the parameter \( \lambda_{c,2} \). Although the parameter \( \lambda_{c,2} \) was not varied in these simulations, the results show that complete failure of the element corresponds to a maximum principal stretch of \( \lambda_{max} = \lambda_{c,2} \).

It was noted in the case of uniaxial stress that in the fully integrated eight node hexahedral element with eight integration points, there was an instability that caused the element to fail when the maximum principal stretch was less than the prescribed critical stretch \( \lambda_{c,2} \). However, for reduced integration elements having only a single integration point per element, this instability was not observed. Here in the case of simple shear, we do not observe this instability for the fully integrated eight note hexahedral element. The results shown in Fig. 5-8 are indeed for fully integrated elements with eight integration points, and the stress decays up to the point of prescribed critical stretch, \( \lambda_{c,2} \), at which point the element is deleted from the mesh and all stresses are set equal to zero. A comparison between the fully integration and reduced integration elements in Fig. 5-9 shows no difference between reduced integration elements (C3D8R) and fully integrated elements (C3DS).

5.3.5 Non-Monotonic Loading – Progressive Damage

The similarities between this implementation of failure, and the progressive degradation of modulus in Mullins effect damage [74] raises the question of how this material model compares with a model
Figure 5-8: Stress-strain behavior for simple shear with the user subroutine defined to have exponential decay of shear and bulk modulus as a function of maximum principal stretch. The results examine the effect of the parameters $\lambda_{c,1}$ and $\lambda_d$ in the model. (a.) Shear stress, $\sigma_{13}$. (b.) Normal stress, $\sigma_{11}$.

Figure 5-9: Comparison of stress-strain behavior under simple shear between reduced integration and fully integrated elements.

of Mullins effect damage. Although it is not of particular interest for the current application of elastomer seals for hydraulic fracturing, which are typically not subjected to cyclic loading, it is of fundamental interest. We therefore briefly investigate the behavior of this constitutive model under
cyclic loading with progressively larger strains.

Unlike the Mullins effect damage, the current constitutive model erodes both the shear modulus and the bulk modulus simultaneously, whereas the damage in the model of Ogden and Roxburgh [74] only erodes the shear modulus. Whether to erode only the shear modulus or both the shear and bulk modulus was one of the questions we had contemplated while implementing this model. Ultimately it was chosen to erode both moduli in this work. The reason is that, upon rupture, the failed material can support neither isochoric deformation nor dilational deformation, thus implying both shear and bulk moduli should be zero, and the intent of this model is to capture the state prior to the onset of failure, and the state after element deletion, with little regard for the transition between the two. In experiments, the transition between zero damage and completely failed is often abrupt with no observable plasticity, and should therefore be modeled with a very rapidly decaying function, which in our implementation implies that $\lambda_d$ should be very small.

In order to investigate Mullins type behavior, we consider cyclic uniaxial loading of a single element of an Arruda-Boyce type elastomer with the maximum principle stretch damage model of Eqn. (5.47). We consider a loading profile composed of four cycles with maximum axial stretches of $\lambda_3 = 4, 6, 8$ and 9. The results are shown in Fig. 5-10. Because of the erosion of modulus, energy is dissipated in a given cycle, and the elastic modulus of each subsequent cycle is smaller than that for the previous because the stretch of each subsequent cycle is larger than for the previous. We have not exhaustively investigated Mullins effect behavior and the goal of this model is not to model Mullins effect. Despite this, the erosion of modulus implemented in this damage model results in a decrease in elastic modulus upon loading, as observed in the Mullins effect.

![Graph showing cyclic loading](image)

Figure 5-10: Effect of cyclic loading on progressive damage of single element for Arruda-Boyce constitutive model with $\mu = 1.0 \times 10^3$, $K = 1.0 \times 10^8$ and $\lambda_m = 5$. The parameters for the damage model are $\lambda_{c,1} = 5.0$, $\lambda_{c,2} = 9.0$ and $\lambda_d = 1.0$.

### 5.3.6 Effect of Mesh Size on Failure Prediction

It is readily acknowledged that deleting entire elements introduces the characteristic element size as an artificial length scale into the problem. On one hand, from a purely computational perspective, a numerical method is deemed to be convergent if it is both stable and the error goes to zero as
the discretization is successively refined toward an element size of zero. On the other hand, finite element methods, which are weak solutions, are expected to predict physics independent of the mesh size – the predicted physics are expected to be accurate for finite mesh size even though the error scales with mesh size. Here we examine the effect of mesh size on failure results for the proposed maximum principal stretch failure criterion.

Uniaxial Tension of a Single Edge Notch Tensile Specimen

Inspired by the geometry of Mao, Talamini and Anand [60], we consider the fracture of a thin specimen in uniaxial tension giving Mode I loading on the crack tip. Although Mao, Talamini and Anand consider the case of plane stress, here we consider the case of plane strain because, even for thin specimens, the behavior at the crack tip is under a state of asymptotic plane strain [70].

![Single edge notch tension specimen](image)

Figure 5-11: Single edge notch tension specimen.

We investigate the displacement required for crack propagation a specimen of width $w = 2$ cm, height $h = 10$ cm and crack length $a = 4$ mm (20 percent of the width). We model the specimen as a neo-Hookean material with shear modulus $\mu = 1$ MPa, bulk modulus $K = 500$ MPa, maximum principle stretch at onset of failure, $\lambda_{c,1} = 1.5$, maximum principle stretch at complete failure (element deletion), $\lambda_{c,2} = 2.0$, exponential decay constant $\lambda_d = 1.0$ and density $\rho = 1000$ kg/m$^3$. Abaqus/Explicit with fixed mass scaling of 10, distortion control and hourglass control using elements of type CPE4R was used to conduct the simulations. We conduct four different simulations with varying mesh sizes where the smallest elements in the mesh were of size $L_{\text{element}} = 1$ mm, 500 $\mu$m, 200 $\mu$m and 100 $\mu$m. In each of the cases the smallest elements were located near the midplane, at the tip of the crack, and the height of the elements (but not the width) was increased by a factor of 10 in going from the midplane to the top and bottom surfaces of the specimen. In this way, the aspect ratio of elements at the top and bottom of the domain was 10 (height/width). For reference, a picture of the domain and mesh for the characteristic element size at the midplane of $L_{\text{element}} = 500$ $\mu$m in the undeformed configuration is shown in Fig. 5-12.

Fig. 5-13 shows the engineering stress versus engineering strain for the stretching and tearing of the Mode I tensions specimen depicted in Figs. 5-11 and 5-12. Engineering stress was calculated by summing the nodal reaction forces on the top nodes and dividing by the width, $w$, of the specimen. The displacements of the top surface of the specimen were imposed in the finite element simulation, and engineering strain was calculated by the quotient of this imposed axial displacement with the original specimen height, $h$. 

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Figure 5-12: Mesh for Mode I tension specimen with crack 20 percent of the width, \( w \), of the body for the coarsest mesh with characteristic element size near the crack of \( L_{\text{element}} = 1 \) mm. Nodes along the crack are highlighted with grey circles. (Rotated 90° counter-clockwise.)

The results in the left image show that, while the specimen is intact, the stress-strain curves are independent of mesh size. However, the onset of failure, as indicated by the sharp drop in stress, is strongly dependent on the mesh size. The right figure shows that mesh size dependence of failure stress depends approximately on the square root of the mesh size. This square root dependence is consistent with approximations from linear elastic fracture mechanics coupled with this maximum principle stretch failure criterion.

For a prescribed displacement of the top boundary, \( u_0 \), we expect the stress near the crack tip to be proportional to the far-field stress, \( \sigma_\infty \), and inversely proportional to the square root of the radial distance from the crack tip:

\[
\sigma = \frac{\sigma_\infty}{\sqrt{r}} \tag{5.2.1}
\]

We expect the maximum principle stretch, \( \lambda_1 \), to be proportional to the stress, such that near the crack tip where failure is expected, we have

\[
\lambda_1 \propto \frac{\sigma}{E} \Rightarrow \frac{\lambda_1}{\sqrt{r}} = \frac{\sigma_\infty}{E \sqrt{r}} \tag{5.2.2}
\]

When boundary value problem is discretized and solved using finite elements, the finite elements the ability of the finite element solution to resolve this singularity is based on the mesh size. Considering the finite element solution is a weak solution, it does not blow up at the singularity, but integrates it in a weak form. Thus the maximum computed stress in the discretized solution should scale inversely with the square root of the mesh size.

\[
\max (\sigma_{\text{discretized}}) \propto \frac{\sigma_\infty}{L_{\text{element}}} \tag{5.2.3}
\]

At the point when the simulation reaches the failure criterion, we have

\[
\lambda_{c,1} = \lambda_{1,\text{crit}} \Rightarrow \frac{\sigma_{\text{crit}}}{E} = \frac{\sigma_{\infty,\text{crit}}}{E L_{\text{element}}} \Rightarrow \sigma_{\infty,\text{crit}} = E \lambda_{c,1} L_{\text{element}} \tag{5.2.4}
\]

Figure 5-13: Effect of mesh size on stress-strain curves for fracture of plane strain specimen with crack 20 percent of the width of the specimen. (Left) Traces of engineering stress versus engineering strain for varying mesh size. (Right) Failure stress versus crack size.

The right plot in Fig. 5-13 shows that the far-field applied stress, \( \sigma_\infty = F_2/w \) indeed scales very
nearly with the square root of the grid size, \( \Delta x \), as

\[
\frac{\sigma_\infty}{E} = \frac{F_2}{w} = 8.02 (\Delta x)^{0.48}
\]

(5.55)

where \( \Delta x \) is expressed in meters, where this equation was found using linear regression for four simulations with different grid size. It is evident from dimensional analysis that this is not the complete picture because we have a dimensional quantity, grid size, raised to an exponential power. It is likely that this grid size should be scaled with either crack length, \( a \), or specimen width, \( w \), in order to obtain a non-dimensional relationship. However, no effort is made to do this because the results show that failure stress, and therefore failure strain, are strongly dependent on the grid size, contrary to original hopes. Thus we need an alternate failure criterion if we hope to achieve mesh independence.

5.4 Strain Energy Density Failure Criterion

In light of the fact that a failure initiation criterion for \( d_R \) based on maximum principal stretch yielded predictions of crack initiation that were strongly dependent on mesh size, we seek an alternate formulation of \( d_R \).

The inspiration for this model, based on strain energy density, lies in the fundamental energy arguments for fracture put forth by Griffith [33], and, despite the fact that we first formulated this model phenomenologically, it agrees with the derivation done after the fact, which was presented in Section 5.1.

It should be noted that this model is quite naive, and it is highly unlikely that it is novel. However, we have not found any work in the literature in which this approach was used. Indeed, countless others have used strain energy density in some form, but the regularization with element size is the component that we have not been able to find in literature. George Sih [87, 88] is among the earliest to propose strain energy density as a method for predicting whether a crack will advance and in what direction it will do so for mixed mode fracture. However, his derivation is rather concoluted, and first uses linear elastic fracture mechanics analysis at the crack tip to later derive strain energy density at the tip. Such a method is unsuitable for the current application because the assumptions of linear elasticity are violated both due to the hyperelastic material response as well as the large deformations involved. Herein we take the continuum approach. There exists a finite strain energy density everywhere in the material.\(^1\) We approximate this strain energy density over a finite region using finite elements, which yield a best approximation in the \( L_2 \) norm. We use this finite element approximation of strain energy density near the crack and, if such a measure is independent of mesh size near the crack, then it will yield a potentially viable numerical method.

5.4.1 Theory

We consider the finite element discretization near a stress concentration, as depicted schematically in Fig. 5-14. The strain energy in any given element is the integral of the strain energy density over the volume of the element.

\[
U_{\text{element}} = \int_{P_t} \psi dv = \int_P \psi_R dv_R
\]

(5.56)

where we choose the part, \( P_t \), to correspond to the element. For rectangular prism elements, such as those shown in the sketch, having undeformed dimensions \( a_0, b_0 \) and \( c_0 \) and dimensions in the deformed configuration of \( a, b \) and \( c \), we can write the strain energy in the element as

\[
U_{\text{element}} = \bar{\psi}abc = \bar{\psi}_R a_0 b_0 c_0.
\]

(5.57)

\(^1\)Strain energy density is finite because, in reality, there is no \( r^{-1/2} \) singularity in stress. Instead, these singularities are ameliorated by plasticity, in the case of ductile materials, or nonlinearity in both the material properties and geometry (finite radius of curvature upon deformation) in brittle elastomers.
where $\bar{\psi}$ is the average strain energy density of the element in the deformed configuration, and $\bar{\psi}_R$ the corresponding value in the reference configuration.

When the element at the leading edge of the crack is removed, new surface is generated. For the planar configuration shown here, surface area on the top and bottom of the removed element totaling $\Delta A_{\text{surface}} = 2ab$ in the deformed configuration or $\Delta A_{R,\text{surface}} = 2a_0b_0$ in the reference configuration is generated. If the surface energy in the reference configuration is $\Gamma$, the energy argument of Griffith requires

$$\psi_R a_0 b_0 c_0 \geq 2\Gamma a_0 b_0 \implies 2\Gamma - \psi_R c_0 \leq 0$$

(5.58)

in order for the fracture to be energetically favorable. Then, for this discretized case, it makes sense to choose the characteristic length $L_c = c_0/2$ as half the characteristic length of the element in the direction perpendicular to the plane of the crack. This scaling analysis accounts only for surfaces parallel to the surface of the crack, and neglects any affects due to generation of new surfaces due to lengthening of the crack front. Therefore, this analysis is reasonable for quasi-planar cracks but it is unclear of its validity for fully three-dimensional cracks.

Although the energy argument given in Eqn. (5.58) is justified for fracture initiation, it is unclear what the proper evolution rule for fracture evolution ought to be. In vulcanized natural rubber, such as that used by Rivlin and Thomas [85], there is a noticeable progressive slow tearing up until the point of catastrophic rupture. In our observations using silicone-based elastomers, there is no slow tearing, and onset of failure coincides with catastrophic ruputer of the entire specimen. Although the explanation of this phenomenon is outside the scope of this thesis, it is perhaps correlated with the fact that locking stretches are typically much smaller in PDMS materials ($\lambda_{\text{lock,PDMS}} \sim 2$) [41] than in vulcanized natural rubber ($\lambda_{\text{lock,natural}} \sim 5$) [4], and corresponds to shorter distances between crosslinks in PDMS.

Because we lack sufficient experimental data, we impose an exponential decay in the prefactor $(1 - L_c d_R)$ as done previously, with the exception that the rate of decay is now based on the hyperelastic strain energy density. The chosen form is

$$1 - L_c d_R = \begin{cases} 
1, & L_c \psi_{R,\text{hyper}} \leq \Gamma_1 \\
\exp \left( -\frac{L_c \psi_{R,\text{hyper}} - \Gamma_1}{\Gamma_d} \right), & \Gamma_1 \leq L_c \psi_{R,\text{hyper}} \leq \Gamma_2 \\
0, & L_c \psi_{R,\text{hyper}} \geq \Gamma_{c,2} 
\end{cases}$$

(5.59)
Note that the strain energy density used in the damage initiation and evolution of Eqn. (5.59) is that of the purely hyperelastic material, without damage. In this way, even though the real free energy is modulated by the damage as in Eqn. (5.42), the evolution of the damage parameter is dependent only upon the right Cauchy-Green deformation tensor, $C$, and independent of $d_R$.

### 5.4.2 Implementation

In complicated cases of loading, it is not known a priori the characteristic element dimension perpendicular to the plane of the advancing crack. In fact, at the local (element) level, there is no knowledge of a crack-like entity, which is a global, macroscopic feature, and the failure criterion is implemented locally. In order to do so, we assume that the elements are approximately equaxed, such that the characteristic length dimension in any direction is approximately the same. Then we use the default characteristic length scale provided by Abaqus through the `charLength()` variable, $L_{\text{element}}^{(\text{Abaqus})}$, to scale the strain energy density at impose the failure initiation and evolution criterion. The default characteristic element length provided by Abaqus is the cube root of the element volume in the deformed configuration. (This distinction between deformed and reference configuration is negligible when the bulk modulus is large, but becomes increasingly important as the ratio of bulk modulus to shear modulus is lowered.)

$$L_{\text{element}}^{(\text{Abaqus})} = V_{\text{element}}^{\frac{1}{3}}$$

The referential strain energy density is the standard hyperelastic strain energy density, as discussed in §5.2. The resulting form of the implementation is

$$1 - L_c d_R = \begin{cases} 
1, & \exp \left( -L_{\text{element}}^{(\text{Abaqus})} \psi_{R,\text{hyper}} - \Gamma_1 \right) \leq \Gamma_1 \\
0, & \Gamma_1 \leq L_{\text{element}}^{(\text{Abaqus})} \psi_{R,\text{hyper}} \leq \Gamma_2 \\
L_{\text{element}}^{(\text{Abaqus})} \psi_{R,\text{hyper}} \geq \Gamma_{c,2} & \Gamma_2 \leq L_{\text{element}}^{(\text{Abaqus})} \psi_{R,\text{hyper}} \leq \Gamma_{c,2} 
\end{cases}$$

Note that, as implemented here, the characteristic element size is the characteristic size of the element in the deformed configuration. For consistency, it ought to be the characteristic element size in the reference configuration, and should be modified as such in future implementations. However, as noted above, for nearly incompressible materials, the discrepancy is slight.

### 5.4.3 Convergence

We first investigate whether the prediction for failure converges as the mesh is refined for a single edge notch tension specimen, depicted in Fig. 5-11, with crack length that is twenty percent of the specimen width, such that $a/w = 0.2$. We consider a neo-Hookean material with shear modulus $\mu = 1$ MPa and bulk modulus $K = 500$ MPa, with failure parameters $\Gamma_1 = 1000$ J/m$^2$, $\Gamma_2 = 2000$ J/m$^2$ and $\Gamma_d = 1000$ J/m$^2$. Fig. 5-15 shows the engineering stress-strain diagram for the plane strain specimen under uniaxial stress, as well as the effect of mesh size on the convergence of the failure stress. We see that, unlike the results for the maximum principal stretch criterion depicted in Fig. 5-13, which were strongly dependent on mesh size, the predicted failure stress depends only weakly on mesh size and appears to converge to a non-zero value as the mesh size is refined.

In light of the apparent convergence for a single crack of length twenty percent of the specimen width, we investigate the convergence behavior for cracks five, ten and forty percent of the specimen width for the same neo-Hookean material with identical failure parameters as used for Fig. 5-15. Fig. 5-16 shows the critical stress and critical strain at failure for for the four different crack lengths as a function of mesh size. In each of the four cases, the critical values of stress and strain appear to converge to nonzero values (as expected) as the mesh size is refined, indicating a reasonable numerical method.
Figure 5-15: Effect of mesh size on stress-strain curves for fracture of plane strain specimen with crack 20 percent of the width of the specimen and modified failure criterion of Eqn. (5.61). (Left) Traces of engineering stress versus engineering strain for varying mesh size. (Right) Failure stress versus crack size.

Figure 5-16: Critical stress and strain at failure for four different crack lengths in single edge notch tension testing.

5.4.4 Correlation with Energy Release Rates

In order for this energy-based failure model to be useful for prediction of failure initiation, it must correlate with energy release rates predicted by standard calculations, such as the J-integral [13, 84] and the virtual crack extension method [76]. We compare the strain energy density multiplied by characteristic element length, computed in the finite element simulations for various crack lengths presented above, with static finite element calculations of energy release rate using the virtual crack extension method.

Fig. 5-17 provides a comparison of the energy release rates calculated using both the virtual crack extension method as well as the elemental strain energy calculation described above. The left figure shows the energy release rates versus applied strain for the single-edge notch tension test specimen with geometry specified in §5.3.6 composed of neo-Hookean material with a shear modulus of $\mu = 1$ MPa and a bulk modulus of $K = 500$ MPa. Note that the energy release rate computed from the static simulations with the virtual crack closure method has been divided by a factor of 4 in order to match with energy release rates calculated using the elemental strain energy density approach. The reason for the factor of 4 is not evident. A factor of 2 may be expected from the derivation Eqn. (5.58), but the source of the additional factor of 2 has not been found. Nevertheless,
if one knows that a rubber fails when the energy release rate reaches a critical value of \( J_{\text{crit}} \), this correlation shows that it will also fail when the scaled strain energy density, \( J_{\text{crit}} \) \( \psi_{B} \), reaches a value of \( J_{\text{crit}}/4 \).

![Graph showing energy release rate vs applied strain for varying crack lengths.](image)

Figure 5-17: A comparison of energy release rate predicted with virtual crack extension with that due to elemental strain energy density. (Left) Calculated energy release rates versus applied strain for varying crack lengths. (Right) Predicted failure strain versus crack length for varying failure criterion. (Data from simulations with mesh size of \( \Delta x/w = 0.01 \) above.)

This type of criterion is implemented in the right side of Fig. 5-17, where we impose three distinct critical energy release rates, \( J_{\text{crit}} \), equal to 4000, 2000 and 1000 J/m\(^2\), and compare the predicted strains at failure for the two methods.

### 5.5 System Level Model

Now that we have implemented fracture propagation models within Abaqus/Explicit, we use them to further investigate the ideas of seal length and compressibility investigated in the preceding chapter. Here we investigate the effect of bulk modulus on the location of fracture initiation and the corresponding propagation of these cracks.

#### 5.5.1 Quasi-Two Dimensional Model for Fracture Initiation and Propagation – Boundary Value Problem

We consider a quasi-two dimensional model of a seal with an aspect ratio of \( L/H = 10 \) and investigate the effect of bulk modulus on the location of fracture initiation and also the evolution of crack growth. Because the crack path on the high-pressure end is known \textit{a priori} – it is an interfacial crack – we use a traction separation model along the interface to model this crack opening, and we use the element deletion method described above to model crack growth on the low-pressure end.

The quasi-two dimensional model having seal length \( L = 1 \) m, thickness \( H = L/10 = 10 \) cm, support ring of height \( h = 9 \) cm support ring corner radius of \( r_{c} = 2 \) mm and depth \( w = 1 \) cm. The reason for constructing a three-dimensional model, instead of a two-dimensional model is twofold. First, from a practical standpoint, the the contact method implemented in Abaqus/Explicit 6.14 for newly created surfaces, due to element deletion, is only implemented in three dimensions. Thus, in a two-dimensional model, we cannot prevent interpenetration of newly created crack faces generated by element deletion.

The second reason for creating a three-dimensional model is that we are marching toward an annular, three-dimensional geometry of interest in real applications. Such an annular three-dimensional geometry will permit investigations of breaking of axial symmetry, known to be important from the work in Chapter 2 of this thesis, as well as effects like eccentricity of the seal inside the hole.
swellable seals are typically deployed in horizontal branches of oil and gas wells, the seals must support weight of the pipe interconnecting the seals, giving rise to an asymmetric gravitational force acting on the seals. It is not known what effect this has on the performance of the seal. Finally, in laboratory tests on cement-filled rubber seals, post-failure observations indicate that the axis of the seal and mandrel is not aligned with the axis of the pressure vessel. Thus there may be asymmetry in both the radial position as well as the angular orientation of the seal within the well. Although an exhaustive investigation of these effects is outside the scope of the current investigation, it is the hope that the tools developed here may be used to answer some of these questions.

![Figure 5-18: Schematic for quasi-two-dimensional system investigating fracture initiation and propagation. (a) Characteristic dimensions of seal material. (b) Geometry of confining supports.](image)

In the model, the seal body is composed of three distinct material models. There is a thin single layer of cohesive elements at the inner surface \( z = 0 \) whose apparent thickness in the mesh is 1 mm but whose constitutive behavior is based on a thickness of \( h_{\text{cohesive}} = 10 \) mm. The density of these elements is prescribed to be 1000 times that of water. (The reason for the larger thickness and density is the desire to increase the timestep for explicit calculations, which is limited by the CFL condition [19] and based on the time for the fastest elastic wave \( (P\text{-wave}) \) to traverse an element: \( t_{\text{wave}} \sim L_{\text{element}} \sqrt{\rho / K} \).) These cohesive elements are governed by a traction separation constitutive model, such that they deform and fail when either the normal or tangential tractions exceed threshold values. Their constitutive response due to shear is depicted in Fig. 5-19.

![Figure 5-19: Constitutive response of traction separation elements.](image)

Outside this layer of cohesive elements single layer of cohesive elements, the elements on the low-pressure half of the domain \( (L/2 \leq x \leq L, \quad h_{\text{cohesive}} \leq z \leq H) \) are hyperelastic neo-Hookean with shear modulus \( \mu = 1 \) MPa and bulk modulus of either \( K = 100 \) MPa or \( K = 1 \) GPa. They obey a maximum principal stretch failure model, presented in §5.3, with failure initiation occurring when \( \lambda_{\text{max}} = \lambda_{c,1} = 1.5 \), total failure occurring at \( \lambda_{c,2} = 2.0 \) and exponential decay constant \( \lambda_d = 500 \). Finally, the elements on the high-pressure end are purely hyperelastic, governed by ABAQUS’s neo-Hookean constitutive relation with \( C_{10} = 1/2G = 0.5 \) MPa and \( D_1 = 2/K = 2 \times 10^{-8} \) Pa\(^{-1} \) or
Table 5.1: Simulation Parameters for Quasi-2D Model of Seal

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cohesive Elements</strong></td>
<td></td>
</tr>
<tr>
<td>Normal Stiffness $E_{nn}$ [MPa]</td>
<td>5.98</td>
</tr>
<tr>
<td>Shear Stiffness $E_{ss} = E_{tt}$ [MPa]</td>
<td>2.0</td>
</tr>
<tr>
<td>Isotropic Damage Initiation $\sigma_{\text{initiation}}$ [MPa]</td>
<td>1.0</td>
</tr>
<tr>
<td>Damage Evolution (Energy Density) $U_{\text{failure}}$ [J/m$^3$]</td>
<td>$1.0 \times 10^3$</td>
</tr>
<tr>
<td>Density $\rho$ [kg/m$^3$]</td>
<td>$1 \times 10^6$</td>
</tr>
<tr>
<td>Thickness $h_{\text{Coh}}$ [m]</td>
<td>0.01</td>
</tr>
<tr>
<td><strong>Low-Pressure Elements</strong></td>
<td></td>
</tr>
<tr>
<td>Shear Modulus $\mu$ [MPa]</td>
<td>1</td>
</tr>
<tr>
<td>Bulk Modulus $K$ [MPa]</td>
<td>100 or 1000</td>
</tr>
<tr>
<td>Density $\rho$ [kg/m$^3$]</td>
<td>1000</td>
</tr>
<tr>
<td><strong>High-Pressure Elements</strong></td>
<td></td>
</tr>
<tr>
<td>neo-Hookean Shear Coefficient $C_{10}$ [MPa]</td>
<td>0.5</td>
</tr>
<tr>
<td>neo-Hookean Bulk Coefficient $D_1$ [Pa$^{-1}$]</td>
<td>$2 \times 10^{-8}$ or $2 \times 10^{-9}$</td>
</tr>
<tr>
<td>Density $\rho$ [kg/m$^3$]</td>
<td>1000</td>
</tr>
</tbody>
</table>

$D_1 = 2 \times 10^{-9}$ Pa$^{-1}$. The density of both bulk materials are set to $\rho = 1000$ kg/m$^3$. These properties are summarized in Table 5.1.

5.5.2 Quasi-Two Dimensional Model for Fracture Initiation and Propagation – Results

In this section we compare the simulation results for the case when the bulk modulus is 100 times larger than the shear modulus with that for the case when it is 1000 times larger. The results show that for the low bulk modulus case, fracture initiates on the high-pressure end at an applied pressure of $p_0 = 3.2$ MPa. For the higher bulk modulus case, fracture first occurs on the low-pressure end at an applied pressure of $p_0 = 3.3$ MPa. This is consistent with the conclusions from the preceding chapter that increasing the bulk modulus implies that more of the load is supported by the low-pressure end and fracture is more likely to occur there first. Fig. 5-20

Fig. 5-20 shows contour plots of maximum principal logarithmic strain at an applied pressure of $p_0 = 3.3$ MPa at the instant when the low-bulk modulus seal begins to fail due to Mode II fracture on the high-pressure end.

For the small bulk modulus case when $K = 100$ MPa, failure on the low-pressure end begins when the pressure reaches $p_0 = 4.1$ MPa. However, for the corresponding applied pressure in the case with the larger bulk modulus, the crack on the low-pressure end is already growing, but there is not yet any damage on the high-pressure end of the seal. The deformation and damage for the two cases under an applied load of $p_0 = 4.1$ MPa is depicted in Fig. 5-21.

At an applied pressure of $p_0 = 4.2$ MPa, the damage on the low-pressure end of the seal with bulk modulus $K = 100$ MPa begins to grow. Both seals at this applied pressure are shown in Fig. 5-22.

As the pressure is increased to $p_0 = 5.0$ MPa, the cracks on both ends of the seal with bulk modulus $K = 100$ MPa grow. For the seal with higher bulk modulus, the crack on the low-pressure end continues to grow with no damage occurring on the high-pressure end. The contours of maximum principal logarithmic strain at this point are shown in Fig. 5-23.

We note that the failure occurring on the high-pressure end of the low bulk modulus seal depicted in the top plot of Fig. 5-23 is a dynamic process as currently modeled. One can see the shock wave angled to the left and upward extending from the moving crack front. It is also notable that the deformed high-pressure surface is not parabolic, as one might expect from either a short crack, as in Fig. 5-20, and also is not a flat surface as one might expect from a fairly long static crack. The reason is the dynamic effects of the crack propagation. It is unclear whether, in this particular
configuration, the crack will propagate unstably as in dynamic fracture, or whether the dynamic effects exhibited here are a result of the explicit finite element procedure combined with mass scaling implemented to enable larger timesteps. The detailed investigation of whether this process is static or dynamic, and what effect friction might have on the stability of crack growth is outside the scope of this investigation.

Once the applied pressure reaches $p_0 = 6.3$ MPa, the seal with bulk modulus $K = 1000$ MPa begins to fracture on the high-pressure end. This is illustrated in the contour plot of maximum principal strain in Fig. 5-24. Here we see that there is extensive damage on the low-pressure end at the instant when an interfacial crack on the high-pressure end begins to grow.

A summary of the applied pressures required for damage initiation (beginning of element failure) as well as the pressure at which cracks begin to grow is found in Table 5.2.

Figure 5-20: Failure initiation for the two different cases for an applied pressure of $p_0 = 3.3$ MPa. (Top) Bulk modulus is $K = 100$ MPa and failure initiates at the high-pressure end. (Bottom) Bulk modulus is $K = 1000$ MPa and failure initiates at the low-pressure end. Note there is more deformation on the low-pressure end and less on the high-pressure end as compared with the top contour plot.

Figure 5-21: Deformation and failure of two seals under applied load of $p_0 = 4.1$ MPa. (Top) Bulk modulus is $K = 100$ MPa and failure initiates at the low-pressure end. (Bottom) Bulk modulus is $K = 1000$ MPa and crack growth occurs on the low-pressure end. On the high-pressure end, there is not yet any failure.
Figure 5-22: Deformation and failure of two seals under applied load of $p_0 = 4.2$ MPa. (Top) Bulk modulus is $K = 100$ MPa and failure initiates at the low-pressure end. (Bottom) Bulk modulus is $K = 1000$ MPa and crack growth occurs on the low-pressure end. On the high-pressure end, there is not yet any failure.

Figure 5-23: Deformation and failure of two seals under applied load of $p_0 = 5.0$ MPa. (Top) Bulk modulus is $K = 100$ MPa and failure initiates at the low-pressure end. (Bottom) Bulk modulus is $K = 1000$ MPa and crack growth occurs on the low-pressure end. On the high-pressure end, there is not yet any failure.

Figure 5-24: Deformation and failure of two seals under applied load of $p_0 = 6.3$ MPa.

5.5.3 Fracture Propagation in Seals – Discussion

Although the geometry and material properties chosen for the preceding simulations were not representative of any single particular seal design, they exhibit behavior consistent with realistic seals.
Table 5.2: Applied pressure required for failure

<table>
<thead>
<tr>
<th></th>
<th>$K = 10^8$ Pa</th>
<th>$K = 10^9$ Pa</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Damage on High-Pressure End</td>
<td>3.2 MPa</td>
<td>5.8 MPa</td>
</tr>
<tr>
<td>First Damage on Low-Pressure End</td>
<td>4.1 MPa</td>
<td>3.3 MPa</td>
</tr>
<tr>
<td>First Element Deletion on High-Pressure End</td>
<td>3.7 MPa</td>
<td>6.3 MPa</td>
</tr>
<tr>
<td>First Element Deletion on Low-Pressure End</td>
<td>4.2 MPa</td>
<td>3.4 MPa</td>
</tr>
</tbody>
</table>

Similar to the results of the preceding chapter, we see that seals with higher bulk modulus are more likely to fail at the low-pressure end because the applied force is transmitted along the length of the seal to the low-pressure support, and is not distributed via shear to the rigid substrate.

Secondly, the failure propagation shows that, although fracture may originate at one end, the growth of cracks on that end weaken the seal. Fracture at the low-pressure end makes the low-pressure support effectively more compliant (lower $k^*$), thereby resulting in an increase in load transmitted via shear to the rigid substrate. This increased shear contributes to an increase in tendency for Mode II interfacial fracture at the high-pressure end.

Conversely, growth of Mode II interface fractures on the high-pressure end makes the seal effectively shorter. Therefore, less load is distributed to the rigid substrate via shear, and more is transmitted to the low-pressure support. Therefore, growth of an interfacial crack on the high-pressure end increases the tendency for cracks to grow on the low-pressure end.

In this way, regardless of whether cracks first occur on one end or the other, it is likely, upon further increase in applied load, they will eventually occur on the other end. This is consistent with experimental observations, such as those in Chapter 3.
Chapter 6

Summary, Conclusions and Future Work

6.1 Summary

In this thesis we investigated several aspects of failure of seals composed of crosslinked polymeric gels. In Chapter 2 we investigated the mechanism of seal leakage and specifically how a seal could be undamaged over the majority of its length upon removal from an opaque test fixture, yet still have leaked during applied differential pressure. In situ observations through a transparent test fixture show that non-axisymmetric deformations lead to stretching in the azimuthal direction and consequently thinning in the radial direction, thereby reducing the compressive sealing traction and allowing fluid to seep between the seal and sealing surface. Furthermore, we showed the same mechanism in two analogous systems — an O-ring and a planar seal. In these analogous systems, the same nonuniform deformation leading to a thinning in the direction normal to the sealing surface was observed, despite the fact that the seal material sustained no permanent damage. Thus, non-axisymmetric behavior, often due to fracture in well-designed and well-manufactured seals, leads to loss of sealing stress and fluid leakage past the seal.

In Chapter 3 we undertook a purely experimental investigation of the effect of configuration of sealing elements on the performance of the seal. The primary findings of this chapter are twofold. First, mechanical supports, in the form of rigid metal rings, are essential in supporting differential pressure. Adding more metal rings improves performance, presumably up until a certain point. However, that point was not encountered in the configurations tested. Using this finding regarding improvement of performance, a novel packer with embedded metal support rings was designed, fabricated and tested. Although only a single test was performed, the new design held 50 percent more differential pressure than a seal of the same overall length (and same material) without embedded mechanical supports.

The second finding of this section is that compressibility of the seal, and any intermediate fluid between adjacent seal elements, is important. Specifically, longer seals do not necessarily perform better and the compressibility of fluid between adjacent seal elements decreases the effectiveness of distribution of load over the seal elements. This gives reason to consider the effect of length of the seal on seal performance, which was done in Chapter 4.

In Chapter 4 we considered the combined effect of bulk modulus and aspect ratio on the performance of the seal. We presented a linear, Saint-Venant solution for stresses and deformations in the seal. The linear model compared well with finite element simulations for finite deformation, neo-Hookean material, which indicates that nonlinearity is unimportant in the interior of the seal. The reason nonlinearity is unimportant is that the seal is highly confined, and the confinement means that the importance of isochoric deformations is reduced, such that dilational deformations are as important as isochoric deformations, which the Saint-Venant solution is able to capture.

Although the Saint-Venant solution does not accurately approximate the real solution near the
high-pressure and low-pressure ends of the seal, in the mathematical boundary layers, the coarse-grained stress measures predicted by the Saint-Venant solution can be used to correlate the computationally predicted energy release rates for the growth of short cracks on the high-pressure and low-pressure ends of the seal. The results show that very long seals, or seals with low bulk modulus (relative to shear modulus) tend to first fail on the high-pressure end. Therefore, making them longer does not result in an increase in the maximum differential pressure they can support prior to leakage.

Although the combination of the Saint-Venant solution with numerically evaluated energy release rates enables the prediction of the location at which a crack will begin to grow, it does not predict what happens after crack growth starts. Because material damage is often observed on both ends of the seal, we created a model that incorporates fracture propagation as well as fracture initiation. To do so, we wrote Abaqus/Explicit user subroutines (VUMATs) to erode the elastic moduli for neo-Hookean and Arruda-Boyce hyperelastic materials as a function of a failure criterion. We implemented a failure criterion based on maximum principal stretch, and showed that, although giving reasonable results for behavior of a single element, the prediction of structural failure of a single edge notch tension specimen was strongly dependent on mesh size. In an effort to eliminate mesh dependence, we implemented a strain energy density criterion, regularized by the characteristic scale of the element, which was based on surface energy arguments of Griffith. The results of the energy-based failure criterion are nearly insensitive to mesh size, and unlike the maximum principal stretch criterion, converge to finite failure strains as the mesh is refined. Upon implementation of these two failure models, we use the former to model a quasi-two dimensional seal system and show that changing the bulk modulus affects whether cracks begin on the high-pressure or low-pressure end of the seal. Despite differing initiation locations, ultimately cracks appear on both ends of the seal, consistent with experimental observations. This fracture propagation model is particularly important for understanding the behavior of the seal after fracture initiates because, despite the fact that material failure occurs, the device does not immediately leak. Therefore, understanding the behavior of the device between the point of first fracture and the point of first leakage is critical, and must be done with a computational model such as the fracture propagation model presented here.

6.2 Conclusions

In conclusion, the failure and leakage of swellable elastomeric seals is highly complicated. Unlike initial hypotheses related to lubrication layer flows or flow through porous gels, the failure behavior of these seals is dominated by fracture and large deformation. In this work, we have almost entirely neglected the gel-like behavior of these seals, treating them as purely hyperelastic materials. The reason for this is that the timescale for application of differential pressure is much faster than the timescale for bulk solvent redistribution throughout the seal, which typically takes place over several weeks or even months. However, in the vicinity of the crack tip, solvent redistributes itself on these short length scales over the same duration as the loading and fracture propagation. Regrettably we have neglected this phenomena. (However, Anand [99] is embarking on this path, and will hopefully provide great insight in the near future.)

Despite investigating only hyperelastic seals, this thesis makes the following contributions to the state of knowledge. First, it demonstrates that non-axisymmetric behavior can lead to stretching and loss of sealing stress between the seal and sealing surfaces. This deformation is often caused by either elastic or fracture-based extrusion of the seal material into the gap between the low-pressure support and the sealing surface.

Second, this thesis demonstrates that an approximate solution not valid in the vicinity of a crack can still be used to predict energy release rates provided the approximate solution yields meaningful stress measures far from the crack. With current usage of finite element analysis, this has questionable value. However, a physical understanding and analytic solution are valuable for reducing the design space for which finite element simulations must be run, and also for interpreting results of finite element simulations. Although the use of Saint-Venant solutions to correlate energy release rates may not be as valuable now as it would have been prior to widespread use of finite
element simulations, it still provides significant value in understanding the effect of key material and geometric parameters on fracture initiation.

Finally, this thesis provides a numerical implementation for a failure criterion that converges with mesh refinement for a single-edge notch test, and whose failure prediction is correlated with energy release rates calculated using the virtual crack extension method. The advantage of this criterion over the strain energy density criterion of Shi [87] is that it does not rely upon linear elastic solutions, and instead can be implemented for a variety of hyperelastic materials. One of its primary drawbacks is that it is implemented in an explicit code, for which timestep size must scale directly with element size and inversely with bulk modulus due to the CFL stability criterion. However, it is possible to run these finite element simulations first as an implicit static step without failure, followed by an explicit dynamic step incorporating failure. In this way, the solution immediately before failure can be found by Abaqus/Standard, and only the fracture propagation analysis must be done with Abaqus/Explicit. This has been implemented using the *Import command in the Abaqus/Explicit job to import results from the Abaqus/Standard analysis.

A second criticism, fundamental to the approach of element deletion, is that deleting elements from the mesh does not conserve mass, momentum or energy. Upon deletion of these elements, these element-born properties are removed from the system. This is indeed true. However, other methods, such as adaptive mesh refinement or even phase field models require very fine meshes in order to resolve the crack. If such refined meshes are used with this method, only a very small percentage of the system mass, momentum and energy will be lost with the deletion of these elements.

Despite its drawbacks, this approach has the clear advantage that it is fairly easy to implement within the commercial software package Abaqus, and thus can be used as an industrial tool for the design and failure prediction of a variety of rubber-based devices.

6.3 Future Work

The future work on this project can be divided into the applied oilfield work and the more fundamental academic work. The original goal of this project, driven by the oilfield side of the stakeholders, was to develop a method for prediction of the maximum differential pressure that the seal can support as a function of geometry, material properties, environmental conditions and other relevant parameters. Although we have begun this work, and made some important observations regarding this, we have not developed a single modeling tool that can be used to predict performance. It seems that, between this work, the work of Suo, the work of Chester, and the ongoing work of Anand on fracture of gels, all the pieces are in place; however, the unified model is just beyond the scope of this current thesis.

The scope of academic work to be done is far more daunting. The computational prediction of fracture propagation seems to be the most valuable, despite being well-studied since the dawn of computers. The extended finite element method of Belytschko [64] was a pioneering method for mesh-independent prediction of fracture propagation. More recently, phase field models, such as those of Miche [62], Linder [82], Anand [92] and others have become popular for predicting fracture in nonlinear materials. There is also work on modeling of fracture mechanics in the peridynamics community initiated by the revolutionary work of Silling [89]. Regardless of the approach chosen, robust computational tools will be essential for modeling and prediction of fracture propagation in a variety of systems.
Bibliography


Appendix A

Large Deformation of Seal Element

We consider the parameterization of deformation of a characteristic element of a seal having unstressed referential dimensions $L_{x(\text{ref})}$ in the streamwise direction, $L_{y(\text{ref})}$ in the spanwise direction and $L_{z(\text{ref})}$ in the sealing direction. We examine the plane-strain deformation of this unstressed seal element into an element that seals the channel depicted in the lower right corner of Fig. 2-6. In this configuration, the seal element has a dimension $L_0$ in the spanwise direction and the stretch ratio in the spanwise direction in the sealed configuration is given by

$$\lambda_0 = \frac{L_0}{L_{y(\text{ref})}} < 1. \quad (A.1)$$

In order to conserve volume for the plane-strain compression, the stretch ratio in the streamwise direction is expressed as

$$\frac{L_{X(0)}}{L_{X(\text{ref})}} = \frac{1}{\lambda_0} > 1. \quad (A.2)$$

From the sealed but unsheared configuration denoted with a subscript 0 and depicted in the lower left of Fig. 2-6, we parameterize the motion of the element resulting from application of differential pressure as the superposition of a stretch, $\lambda$, a shear, $\gamma$, and a rotation, $\theta$, all of which are depicted in the lower right image of Fig. 2-6. Then the motion that maps a point in the referential element at $X$ to a point in the deformed element $x$ is given by

$$x = FX, \quad (A.3)$$

where the deformation gradient, $F$, is the superposition of the deformations due to initial compression $F_{\lambda_0}$, stretch $F_\lambda$, shear $F_\gamma$, and rotation, $F_\theta$, and is expressed as

$$F = F_\theta F_\gamma F_\lambda F_{\lambda_0} \quad (A.4)$$

with

$$[F_{\lambda_0}] = \begin{bmatrix} \frac{1}{\lambda_0} & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [F_\lambda] = \begin{bmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (A.5)$$

$$[F_\gamma] = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [F_\theta] = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The principal stretches, $\lambda_i$, are then the square roots of the eigenvalues of the right Cauchy-Green tensor $C \equiv F^T F$ and the principal directions, $N_i$, are the eigenvectors. Solving the eigenvalue problem

$$CN_i = \lambda_i^2 N_i \quad (A.6)$$
yields the squares of the principal stretches

\[
\lambda_1^2 = 1
\]
\[
\lambda_2^2 = \frac{1 + \gamma^2 + \lambda^4 \lambda_0^4 - \sqrt{(1 + \gamma^2 + \lambda^4 \lambda_0^4)^2 - 4 \lambda^4 \lambda_0^4}}{2 \lambda^2 \lambda_0^2}
\]
(A.7)
\[
\lambda_3^2 = \frac{1 + \gamma^2 + \lambda^4 \lambda_0^4 + \sqrt{(1 + \gamma^2 + \lambda^4 \lambda_0^4)^2 - 4 \lambda^4 \lambda_0^4}}{2 \lambda^2 \lambda_0^2}
\]

with corresponding principal directions

\[
[N_1] = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[
[N_2] = \begin{bmatrix}
\frac{-1 - \gamma^2 + \lambda^4 \lambda_0^4 + \sqrt{\gamma^4 + 2 \gamma^2 (1 + \lambda^4 \lambda_0^4) + (1 + \lambda^4 \lambda_0^4)^2}}{2 \gamma \lambda^2 \lambda_0^2} & 1 \\
1 & 0
\end{bmatrix}
\]

(A.8)

\[
[N_3] = \begin{bmatrix}
\frac{-1 - \gamma^2 + \lambda^4 \lambda_0^4 - \sqrt{\gamma^4 + 2 \gamma^2 (1 + \lambda^4 \lambda_0^4) + (1 + \lambda^4 \lambda_0^4)^2}}{2 \gamma \lambda^2 \lambda_0^2} & 1 \\
1 & 0
\end{bmatrix}
\]

where these principal direction have not been normalized to have unit lengths. The normalized principal directions are

\[
\tilde{N}_i = \frac{N_i}{|N_i|}, \quad i = 1, 2, 3
\]
(A.9)

We define the tensor with principal directions as its columns as

\[
[N] = \begin{bmatrix}
\tilde{N}_1 \\
\tilde{N}_2 \\
\tilde{N}_3
\end{bmatrix}
\]

(A.10)

and the tensor with principal stretches along the diagonals

\[
[\lambda] = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]
(A.11)

such that the original eigenvalue problem could be expressed as \(CN = N\lambda\lambda\). Then the right stretch tensor, \(U\), defined such that \(U^2 = C\) can be obtained by:

\[
U = \text{NN}^{-1} = \text{NN}^T = \sum_{i=1}^{3} \lambda_i \left( \tilde{N}_i \otimes \tilde{N}_i \right)
\]
(A.12)

and the rotation tensor \(R\) can be obtained by

\[
R = FU^{-1}
\]
(A.13)
This rotation tensor can be used to transform the principal directions, \( \hat{N}_i \), in the referential configuration to principal directions in the deformed configuration via

\[
\hat{n}_i = R \hat{N}_i, \quad n = RN
\]  

(A.14)

Finally, we note that an outward unit normal, \( \hat{M} \), in the reference configuration is transformed into an outward unit normal in the deformed configuration by

\[
\hat{m} = \frac{F^{-1} \hat{M}}{|F^{-1} \hat{M}|}
\]  

(A.15)

Thus the kinematics for the parameterized deformation of an element of a seal have been fully described.

We now examine the stresses resulting from this parameterized deformation. For simplicity, we choose an incompressible neo-Hookean constitutive model, which illustrates all the salient features of nonlinear elasticity while still being analytically tractable. More complicated constitutive models can be substituted and should be used when modeling real swellable seals. For an incompressible neo-Hookean material, the strain energy density is given by

\[
\Psi = \mu \left( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \right),
\]  

(A.16)

where \( \mu \) is the elastic modulus. Then the principal stresses, in terms of the principal stretches, are

\[
\sigma_i = -p + \frac{\partial \Psi}{\partial \lambda_i} = -p + \mu \lambda_i^2, \quad i = 1, 2, 3,
\]  

(A.17)

where \( p \) is a Lagrange multiplier used to enforce incompressibility and is determined from a stress boundary condition. For an isotropic material, the principal directions for stress are aligned with the principal directions for strain, and we may write the Cauchy stress tensor, \( \sigma \) as

\[
\sigma = \sum_{i=1}^{3} \sigma_i (\hat{n}_i \otimes \hat{n}_i) = n \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} n^T
\]  

(A.18)

Then the traction vector on an outward unit normal, \( \hat{m} \), is

\[
t = \sigma \hat{m}
\]  

(A.19)

Imposing the stress boundary condition that the normal compressive stress on the boundary originally in the \( X \)-direction is \( \sigma_1 \) as depicted in Fig. 2-6, the Lagrange multiplier \( p \) can be solved for by solving

\[
t \cdot \hat{m}_X = \hat{m}_X^T \sigma \hat{m}_X = -\sigma_1
\]  

(A.20)

where \( \hat{m}_X \) is the outward unit normal to the deformed surface whose undeformed outward unit normal, \( \hat{M}_X \), was in the \( X \)-direction. Then the outward unit normal stress in the sealing direction plotted in Fig. 2-7 can be expressed as

\[
\sigma_{zz} = \hat{m}_Z^T \sigma \hat{m}_Z
\]  

(A.21)

where \( \hat{m}_Z = \hat{M}_Z \) are the outward unit normal in the sealing direction and are equal because of the plane-strain assumption.
Appendix B

Source Code for System-Level Model

B.1 Abaqus Input File

*Heading
** Job name: job_contact Model name: Model-1
** Generated by: Abaqus/CAE 6.14-1
*Preprint, echo=YES, model=YES, history=NO, contact=NO
**---------------------------------------------------------
** PARAMETERS
**---------------------------------------------------------
*Parameter
p0 = 2.0e+7
G = 1.0e6
K = 1.0e8
rho = 1000.0
C10 = 0.5*G
D1 = 2./K
DamInit = 1.0e6
DamEvol = 1.0e+3
fric = 0.0000
MaxStretch1 = 1.50  # Element Erosion
MaxStretch2 = 2.00  # Element Deletion
tmax = 500.
nOutput = 200
L1 = 0.50  # Length of High-Pressure End
L2 = 0.50  # Length of Low-Pressure End
H = 0.10  # Height of Elastic Elements
W = 0.01  # Depth of Body
h = 0.001  # Thickness of cohesive elements
Hh = H + h

nx1 = 101  # Number of nodes in x on HP end
nx2 = 101  # Number of nodes in x on LP end
ny = 9    # Number of nodes in y
nx = 161  # Number of nodes in z
nx = nx1+nx2-1

hsupport = 0.09  # Height of gauge ring
rsupport = 0.002  # Corner radius - gauge rings
# Calculated Parameters
# Geometry
L1L2 = L1 + L2
L1L2r = L1L2 + r_support
L1L2H = L1L2 + H
h_r = h_support - r_support

# Node/Element Numbering Parameters
nxny = (nx1+nx2-1)*ny
nxny1 = nx*ny + 1
nxnynz = nxny*nz
nxny_nz_1 = nxny*(nz-1)

nx1_1 = nx1 - 1  # Number of elements in a row - Left Half
nx2_1 = nx2 - 1  # Number of elements in a row - Right Half
nx1nx2_1 = nx1+nx2-1  # Number of nodes in a row
nx1nx2_2 = nx1*nx2-2  # Number of elements in a row
ny_1 = ny - 1  # Number of elements in y
br1 = 1.02  # Bias Ratio - Left Half
br2 = 2.0 - br1  # Bias Ratio - Right Half
nx_1 = nz-1  # Number of elements in z
# nx_1_ny_1 = (nx1+nx2-2)*(ny-1)  # Number of elements in a layer

NEL_Layer = (nx1+nx2-1-1)*(ny-1)
NEL_Layer1 = NEL_Layer + 1

NEL_CornerLP = NEL_Layer + nx1

NEL_LeftBackTop = NEL_Layer*nz - (nx-2)
NEL_RightFrontBottom = NEL_Layer + nx - 1
NEL_RightBackTop = (nx-1)*(ny-1)*nz  # Bottom layer is cohesive. (nz-1) elements in elastic
nx_1 = nx - 1

# Node Numbers
N1 = 1
N2 = 2
N3 = nx1
N4 = nx1+nx2-1
N5 = N4 + 1
N6 = N5 + 1
N7 = (nx1+nx2-1)*(ny-1)+1
N8 = N7+nx1-1
N9 = nxny

N11 = nxny+N1
N12 = nxny+N2
N13 = nxny+N3
N14 = nxny+N4
N15 = nxny+N5
N16 = nxny+N6
N17 = nxny+N7
N18 = nxny+N8
N19 = nxny+N9
N21 = nxny+N11
N22 = nxny+N12
N23 = nxny+N13
N24 = nxny+N14
N25 = nxny+N15
N26 = nxny+N16
N27 = nxny+N17
N28 = nxny+N18
N29 = nxny+N19

N31 = nxny_nz_1+N11
N32 = nxny_nz_1+N12
N33 = nxny_nz_1+N13
N34 = nxny_nz_1+N14
N35 = nxny_nz_1+N15
N36 = nxny_nz_1+N16
N37 = nxny_nz_1+N17
N38 = nxny_nz_1+N18
N39 = nxny_nz_1+N19

N101 = nxny + nx1
N102 = N101 + 1
N103 = nxny + nx1 + nx2 + nx1 - 1
N104 = N103 + 1
N201 = N101+nxny
N202 = N102+nxny
N203 = N103+nxny
N204 = N104+nxny

***-------------------------------------------***
*** PART - SEAL
***-------------------------------------------***
*Part, name=Seal

***-------------------------------------------***
*** Bottom Layer - Cohesive
***-------------------------------------------***
*Node, nset=N1
  <N1>, 0., 0., 0.
*Node, nset=N3
  <N3>, <L1>, 0., 0.
*Node, nset=N4
  <N4>, <L1L2>, 0., 0.
*Node, nset=N7
  <N7>, 0., <W>, 0.
*Node, nset=N8
  <N8>, <L1>, <W>, 0.
*Node, nset=N9
  <N9>, <L1L2>, <W>, 0.
*Nfill, bias=<br2>, nset=Edge13
  N1, N3, <nx1_1>, 1
*Nfill, bias=<br1>, nset=Edge34
  N3, N4, <nx2_1>, 1
*Nfill, bias=<br2>, nset=Edge78
  N7, N8, <nx1_1>, 1
*NFill, bias=<br1>, nset=Edge89
N8, N9, <nx2_1>, 1
*NFill, nset=LeftBottom
Edge13, Edge78, <ny_1>, <nx1nx2_1>
*NFill, nset=RightBottom
Edge34, Edge89, <ny_1>, <nx1nx2_1>
*Nset, nset=bottom, generate
  1, <nxny>, 1

** INTERFACIAL LAYER
**

*Node, nset=N11
 <N11>, 0., 0., <h>
*Node, nset=N13
 <N13>, <L1>, 0., <h>
*Node, nset=N14
 <N14>, <LIL2>, 0., <h>
*Node, nset=N17
 <N17>, 0., <W>, <h>
*Node, nset=N18
 <N18>, <L1>, <W>, <h>
*Node, nset=N19
 <N19>, <LIL2>, <W>, <h>
*NFill, bias=<br2>, nset=Edge1113
N11, N13, <nx1_1>, 1
*NFill, bias=<br1>, nset=Edge1314
N13, N14, <nx2_1>, 1
*NFill, bias=<br2>, nset=Edge1718
N17, N18, <nx1_1>, 1
*NFill, bias=<br1>, nset=Edge1819
N18, N19, <nx2_1>, 1
*NFill, nset=LeftInterface
Edge1113, Edge1718, <ny_1>, <nx1nx2_1>
*NFill, nset=RightInterface
Edge1314, Edge1819, <ny_1>, <nx1nx2_1>

**

** Elastic - 1 Layer Above Interface
**

**

** Elastic - Top Surface
**

*Node, nset=N31
 <N31>, 0., 0., <Hh>
*Node, nset=N33
 <N33>, <L1>, 0., <Hh>
*Node, nset=N34
 <N34>, <LIL2>, 0., <Hh>
*Node, nset=N37
 <N37>, 0., <W>, <Hh>
*Node, nset=N38
 <N38>, <L1>, <W>, <Hh>
*Node, nset=N39
<N39>, <L1L2>, <W>, <Hh>
*Nfill, bias=<br2>, nset=Edge3133
N31, N33, <nx1_1>, 1
*Nfill, bias=<br1>, nset=Edge3334
N33, N34, <nx2_1>, 1
*Nfill, bias=<br2>, nset=Edge3738
N37, N38, <nx1_1>, 1
*Nfill, bias=<br1>, nset=Edge3839
N38, N39, <nx2_1>, 1
*Nfill, nset=LeftTop
Edge3133, Edge3738, <ny_1>, <nx1nx2_1>
*Nfill, nset=RightTop
Edge3334, Edge3839, <ny_1>, <nx1nx2_1>
*Nset, nset=Top, generate
  <N31>, <N39>, 1
*Nset, nset=HP, generate
  <N11>, <N37>, <nx>
*Nset, nset=LP, generate
  <N14>, <N39>, <nx>

** Nodes - Elastic - Interior - Fill with Z-direction sweep

*NFill, nset=Left
LeftInterface, LeftTop, <nx_1>, <nxny>
*NFill, nset=Right
RightInterface, RightTop, <nx_1>, <nxny>

** Nodes - Define Front and Back Faces

*NFill, bias=<br2>, nset=LeftFrontFace
LeftFrontEdge, MiddleFrontEdge, <nx1_1>
*NFill, bias=<br1>, nset=RightFrontFace
MiddleFrontEdge, RightFrontEdge, <nx1_1>
*NFill, bias=<br2>, nset=LeftBackFace
LeftBackEdge, MiddleBackEdge, <nx1_1>
*NFill, bias=<br1>, nset=RightBackFace
MiddleBackEdge, RightBackEdge, <nx1_1>

**

** ELEMENTS - COHESIVE

**
*ELEMENT, type=COH3D8
1, <N1>, <N2>, <N6>, <N11>, <N12>, <N16>, <N15>
*Elgen, elset = COH
1, <nx1nx2_2>, 1, 1, <ny_1>, <nx1nx2_1>, <nx1nx2_2>, 1

** ELEMENTS - ELASTIC

*ELEMENT, type=C3D8
<N11>, <N12>, <N16>, <N15>, <N21>, <N22>, <N26>, <N25>
*Elgen, elset = ElasticLeft
<N11>, <nx1_1>, 1, 1, <ny_1>, <nx1nx2_1>, <nx1nx2_2>, <nz_1>, <nxny>, <NEL_Layer>
*Element, type=C3D8, elset=ElasticRight
<N11>, <N12>, <N16>, <N15>, <N21>, <N22>, <N26>, <N25>
*Elgen, elset = ElasticRight
<N11>, <nx2_1>, 1, 1, <ny_1>, <nx1nx2_1>, <nx1nx2_2>, <nz_1>, <nxny>, <NEL_Layer>

** ELEMENT SETS - ELASTIC

*Elset, elset=HP, generate
<N11>, <NEL_LeftBackTop>, <nx_1>
*Elset, elset=LP, generate
<N11>, <NEL_RightBackTop>, <nx_1>

** SURFACES

*Surface, name=InteriorRight, Type=Element
ElasticRight, Interior
*Surface, name=HP, Type=Element
HP, S6
*Surface, name=LP, Type=Element
LP, S4
*Surface, name=SurfAll
ElasticLeft
ElasticLeft, interior
ElasticRight
ElasticRight, interior

** SECTION - COHESIVE

*Cohesive Section, elset=COH, material=Coh, response=TRACTION SEPARATION, thickness=SPECIFIED
0.01,

** SECTION - Elastic

*Solid Section, elset=ElasticLeft, material=Hyper1
*Solid Section, elset=ElasticRight, material=Hyper2

** END PART - SEAL

*End Part

** ASSEMBLY

*Part, name=Basepipe
*End Part
**
*Part, name=Casing
*End Part
*Assembly, name=Assem1
*Instance, name=Seal, part=Seal
*End Instance
**
*Instance, name=Basepipe, part=Basepipe
  0, 0, 0
  0, 0, 0, 1, 0, 0, 90
*Node
1, 0., 0., 0.
*NSet, nset=RP
1
*Surface, Name=Surf, type=Cylinder
Start, -<H>, <h_support>
  Line, -<r_support>, <h_support>
    CIRCL, 0.0, <h_r>, -<r_support>, <h_r>
  Line, 0.0, 0.0
  Line, <L1L2>, 0.0
  Line, <L1L2>, <h_r>
  CIRCL, <L1L2>, <h_support>, <L1L2>, <h_r>
  Line, <L1L2H>, <h_support>
  *Rigid Body, ref node=RP, analytical surface=Surf
*End Instance
**
*Instance, name=Casing, part=Casing
  0, 0, 0
  0, 0, 0, 1, 0, 0, 90
*Node
1, 0., 0., <Hh>
*NSet, nset=RP
1
*Surface, Name=Surf, type=Cylinder
Start, <L1L2H>, <Hh>
  Line, -<H>, <Hh>
  *Rigid Body, ref node=RP, analytical surface=Surf
*End Instance
*End assembly
**
** MATERIAL - COHESIVE
**
*Material, name=Coh
*Damage Initiation, criterion=MATS <DamInit>, <DamInit>, <DamInit>
*Damage Evolution, type=ENERGY <DamEvol>,
*Density
1e+06,
*Elastic, type=TRACTION
5.9801e+06, 2e+06, 2e+06
**
** MATERIAL - ELASTIC
*Material, name=Hyper1

*Density
1000.,

*Hyperelastic, neo hooke
<C10>, <D1>

** MATERIAL - USER

*Material, name=Hyper2

*Density

<rho>,

*User material,Constants=4

<G>, <K>, <MaxStretch1>, <MaxStretch2>

*Depvar, delete=3

3

1,Stretch

2,FailTime

3,Active

** INTERACTION PROPERTIES

** Surface Interaction, name=IntProp-Frictionless

*Friction

<frc>

*Surface Behavior, pressure-overclosure=HARD

** AMPLITUDES

*Amplitude, name=Amp1

0., 0., <tmax>, 1.

** STEP 1

** STEP: Step-1-Slack

**

*Step, name=Step-1-Slack, nlgeom=YES

*Dynamic, Explicit

0 , <tmax>

*Bulk Viscosity

0.06, 1.2

*Fixed Mass Scaling, dt=0.01, type=below min

** BOUNDARY CONDITIONS

*Boundary

Basepipe.RP, 1, 6, 0.0

Casing.RP, 1, 6, 0.0

Seal.LeftBottom, 1, 3, 0.0

Seal.RightBottom, 1, 3, 0.0

Seal.LeftFrontFace, 2, 2, 0.0

Seal.RightFrontFace, 2, 2, 0.0
Seal.LeftBackFace, 2, 2, 0.0
Seal.RightBackFace, 2, 2, 0.0

** APPLIED LOADS

*Dsload, amplitude=Amp
Seal.HP, P, <p0>

** CONTACT

*Contact, op=NEW
*Contact Inclusions, ALL EXTERIOR
*Contact Inclusions
  Seal.SurfAll, Seal.SurfAll
*Contact Inclusions
  Seal.SurfAll, Casing.Surf
  Seal.SurfAll, Basepipe.Surf
*Contact Property Assignment
  , , IntProp-Frictionless

** OUTPUT

*Restart, write, number interval=1, time marks=NO
*Output, field, variable=PRESELECT, number interval=<nOutput>, time marks=YES
*Output, field, number interval=<nOutput>, time marks=YES
*Element Output
  SDV, STATUS
*Element Output, elset=SEAL.COH
  SDEG, STATUS
*Output, history, variable=PRESELECT
*End Step

**

B.2 Fortran User Subroutine – Maximum Principal Stretch

******************************************************************************
| A single large deformation runs for an isotropic nearly
| incompressible Neo-Hookean material |
| Shawn Chester, September 2008, Implemented in Abaqus 6.7-1 |
| Modifications: Modified to implement failure of elements based on |
| maximum principal stretch. Shear and bulk moduli are varied as |
| maximum principal stretch is increased past a first threshold, |
| MaxStretch1. Upon reaching a second threshold, MaxStretch2, |
| elastic moduli are set to zero and the material point is deleted |
| from the mesh. Upon deletion of all material points in an |
| element, the element is deleted from the mesh. If implemented with |
| 2-D elements and appropriate contact conditions, the deletion of |
| the element can generate new surfaces, which are constrained not |
| to intersect with other surfaces. |
| Modified by: R. Druke |

******************************************************************************

State Variables

| maxShear (n,1) | shear | Maximum principal stretch (at current step) |
| maxStreth (n,1) | radius | Time at which failure begins |
| maxShear (n,2) | Elactive | Element is active 1: Element is active 0: Element is deleted |

Material Property Vector

| Gshear = prop(1) | Gshear | Shear modulus |
| Khulb = prop(2) | Khulb | Bulk modulus |
| MaxStretch1 = prop(3) | MaxStretch1 | Maximum Principal Stretch: Start of erosion |
| MaxStretch2 = prop(4) | MaxStretch2 | Delete element |

*
SUBROUTINE VUSB
  + mblock, ndir, mahr, mmaser, mfiws, nprops, ktaus, 
  + smagTime, totalTime, d1, ennme, coordM, charlength, 
  + props, density, avrginc, ralSpinc, 
  + smag3D, stresch3D, desgrad3D, fiel3Dold, 
  + smrasold, mmasrod, ener3rangen, ener3nm, 
  + mmasnew, mmasnew, ener3rn3new, ener3nmnew

include 'vaha_parm.inc'

dimension props(nprops), density(chblock), coordM(chblock,*)
  + charlength(chblock), avrginc(chblock,ndir+mahr), 
  + ralSpinc(chblock,ndir+mahr), smag3D(chblock), 
  + stresch3D(chblock,ndir+mahr), 
  + desgrad3D(chblock,ndir+mahr), 
  + fiel3Dold(chblock,mfiws), smrasold(chblock,ndir+mahr), 
  + mmasrod(chblock,mmaser), ener3rangen(chblock), 
  + ener3nm(chblock), mmasnew(chblock), 
  + ener3rn3new(chblock,ndir+mahr), 
  + mmasnewnew(chblock,mmaser), 
  + ener3rn3new(chblock), ener3nmnew(chblock)

character*80 commons

integer j, j, j, j, j, j, j, k, km1

real*8 tden(3,3), f_p(3,3), f_au(3,3), u_a(3,3), u_au(3,3), f_r(3,3)
real*8 f_pau(3,3), n_au(3,3), n_au(3,3), n_pau(3,3), n_pau(3,3), n_au(3,3)

real*8 g0, a, g0, a, t_au(3,3), t_au(3,3), t_au(3,3), t_au(3,3)

real*8 g, d1, d2, t_au(3,3), t_au(3,3), t_au(3,3)

real*8 g0, d1, d2, t_au(3,3), t_au(3,3), t_au(3,3)

real*8 t_r, t_au(3,3), t_au(3,3), t_au(3,3)

real*8 mni, mi, mi, effs(3,3), effs(3,3), effs(3,3)

real*8 wta(3,3), wta(3,3), wta(3,3)

real*8 stresch, wta(3,3), wta(3,3)

character*40 words

real*8 Gmasr, khblock, MaxStrech1, MaxStrech2

! Parameters
!
real*8 zero, one, two, three, half, mfour, P1, tmem
parameter (zero=0.0, one=1.0, two=2.0, three=3.0, half=0.5, mfour=4.0, P1=1.0, tmem=0.0)

! Identity matrix for later use.
!
call com(1, tden)
!
! Lead material properties needed here
!
Gmasr = props(1)
khblock = props(2)
MaxStrech1 = props(3)
MaxStrech2 = props(4)
!
! START LOOP OVER MATERIAL POINTS:
!
do km=1, mblock

  ! Copy old and new deformation gradients
  !
  f_p(1,1) = desgrad3D(km, 1)
  !
  if(mmahr, km, 2) then
    ! 2D case
    f_p(1,1) = desgrad3D(km, 5)
  else
    ! 3D case
    f_p(5,5) = desgrad3D(km, 5)
  end if

! Compute the relative volume change
!
call svd(F_mau, desf)
!
! Compute the directional left Cauchy-Green tensor
! and its deviator
!
Sdev = (det(F_mau/three))#maxsig(F_mau, transpose(F_mau))

Sdev = Sdev(1,1) + Sdev(2,2) + Sdev(3,3)
!
! Compute the effective stretch
!
efts = daqr(tmem#Sdev)
!
! Compute maximum principal stretches
!
! Compute the Cauchy stress
!
end do

C 11

EROSION OF MODULUS

t0 = 1.00E9 ! time after which keep track of failure
ModFactor = 1.0D ! Factor by which to multiply modulus
if (stateOld(kx,2) .le. u0) then  ! Check whether max stretch
  ...  
end if

C ELEMENT DELETION
if (stretch .gt. MaxStretch2) then 
  ...  
end if

! ABAQUS/Explicit uses stress measure (transpose(X) T B)
! call m3inv(U_eig,UInv)
! ...  

ends  ! end loop over material points

end subroutine runex

! SUBROUTINES ...
Appendix C

Example Problem: Simple Ritz Analysis

For the purpose of illustrating the Ritz method when there are mixed boundary conditions (traction and displacement are coupled), we consider the uniaxial strain of a one-dimensional uniform tie of length $L$ and cross-sectional area $A_c$ connected on the left end at $X = 0$ to a linear elastic spring of stiffness $k$ and free length $L_0$ and under an applied force $F$ on the right end at position $X = L$, as illustrated in Fig. C-1. (We recognize that, in the static limit considered here, this system is identical to two idealized springs in series.)

Figure C-1: Schematic of deformable body connected to a spring under applied loading, $F$, used to illustrate the variational method of Ritz. The deformable body undergoes uniaxial strain.

C.1 Direct Solution

The position of material points in the reference configuration when $F = 0$ is given by the independent variable $X \in [0, L]$. The position of the material points in the deformed configuration is given by $x = \chi (X)$ and the displacement is $u = x - X$. The infinitesimal displacement strain is

$$\epsilon_{11} = \frac{du}{dX}$$

For an isotropic linear elastic material with constitutive relation $\sigma = 2\mu \epsilon + \lambda (\text{tr}\epsilon) \mathbf{I}$ under uniaxial strain, the nonzero stress tensor components are

$$\sigma_{11} = (2\mu + \lambda) \epsilon_{11}, \quad \sigma_{22} = \sigma_{33} = \lambda \epsilon_{11}$$

where $\mu$ and $\lambda$ are the Lamé moduli, and $\mathbf{I}$ is the second-order identity tensor. The equilibrium equation, $\nabla \cdot \sigma = 0$, in one dimension requires

$$\frac{d\sigma_{11}}{dX} = \frac{d}{dx} ((2\mu + \lambda) \epsilon_{11}) = 0$$

(C.3)
For spatially uniform material properties the resulting displacement equation is

\[
\frac{d^2 u}{dX^2} = 0
\]  

(C.4)

subject to the boundary conditions

\[
\begin{align*}
\sigma_{xx} A_c + ku &= 0 \text{ on } X = 0 \\
\sigma_{xx} A_c &= F \text{ on } X = L
\end{align*}
\]  

(C.5)

where \( A_c \) is the cross-sectional area and the boundary conditions are applied in the average sense (as opposed to point-wise, and consistent with our one-dimensional assumption) over the cross-sectional area. Integrating the field equation

\[
u = AX + B
\]  

(C.6)

and application of the boundary conditions

\[
A_c \sigma_{xx} + k (u - L_0) = A_c (2\mu + \lambda) \frac{du}{dX} + k (u - L_0) = 0 \text{ on } X = 0
\]

\[
A_c (2\mu + \lambda) \frac{du}{dX} = F \text{ on } X = L
\]

(C.7)

gives

\[
A_c (2\mu + \lambda) A + k (B - L_0) = 0
\]

\[
A_c (2\mu + \lambda) A = F
\]  

(C.8)

giving the solution

\[
u = L_0 + \frac{F}{k} + \frac{F}{A_c (2\mu + \lambda)} X
\]  

(C.9)

### C.2 Variational Approach

Suppose we define \( w \) to be the Ritz approximation to the displacement, and expand this approximation as a power series in the dependent variable \( X \) as

\[
w = q_0 + q_1 X + q_2 X^2 + \cdots + q_j X^j + \cdots
\]  

(C.10)

The approximate strain tensor, \( \gamma \), has only a single nonzero component

\[
\gamma_{11} = \frac{du}{dX} = q_1 + 2q_2 X + \cdots + jq_j X^{j-1} + \cdots
\]  

(C.11)

The approximate stress tensor, \( \Sigma \), has nonzero components

\[
\Sigma_{11} = (2\mu + \lambda) \epsilon_{11}, \quad \Sigma_{22} = \Sigma_{33} = \lambda \epsilon_{11}
\]  

(C.12)

The internal energy stored in the deformable tie is

\[
U_{tie} = A_c \int_0^L \frac{1}{2} \gamma_{11} \Sigma_{11} dX
\]

\[
= A_c (2\mu + \lambda) \int_0^L \frac{1}{2} \gamma_{11}^2 dX
\]  

(C.13)

\[
= A_c (2\mu + \lambda) \int_0^L \frac{1}{2} \left( q_1 + 2q_2 X + \cdots + jq_j X^{j-1} + \cdots \right)^2 dX
\]
The internal energy stored in the spring is

\[ U_{\text{spring}} = \frac{1}{2} k (w(0) - L_0)^2 = \frac{1}{2} F (w(0) - L_0) = \frac{1}{2} k (q_0 - L_0)^2 \] (C.14)

The work done by the applied force is

\[ W_{\text{nonconserv}} = F w(L) = F (q_0 + q_1 L + q_2 L^2 + \cdots) \] (C.15)

The energy functional is

\[ \Phi(q_i) = U_{\text{tie}} + U_{\text{spring}} - W_{\text{nonconserv}} \]

\[ = A_c(2\mu + \lambda) \int_0^L \left( q_1 + 2q_2 X + \cdots + jq_j X^{j-1} + \cdots \right) dX - FL \]

\[ = A_c(2\mu + \lambda) \left( q_1 L + q_2 L^2 + \cdots + q_j L^j + \cdots \right) - FL \]

\[ = L \left[ A_c(2\mu + \lambda) \left( q_1 + q_2 L + \cdots + q_j L^j + \cdots \right) - F \right] \]

\[ = A_c(2\mu + \lambda) \int_0^L \left( q_1 + 2q_2 X + \cdots + jq_j X^{j-1} + \cdots \right) 2X dX - FL^2 \]

\[ = A_c(2\mu + \lambda) \left( q_1 L^2 + \frac{4}{3} q_2 L^3 + \cdots + \frac{2j}{j+1} q_j L^{j+1} + \cdots \right) - FL^2 \]

\[ = L^2 \left[ A_c(2\mu + \lambda) \left( q_1 + \frac{4}{3} q_2 L + \cdots + \frac{2j}{j+1} q_j L^{j-1} + \cdots \right) - F \right] \] (C.16)

which we seek to minimize. To do so, we evaluate derivatives of the energy functional with respect to the generalized displacements, \( q_i \), and set each equal to zero to find the local minimum or maximum.

\[ \frac{\partial \Phi}{\partial q_i} = 0, \; i = 1, 2, \cdots \] (C.17)

\[ \frac{\partial \Phi}{\partial q_0} = k (q_0 - L_0) - F = 0 \]

\[ \frac{\partial \Phi}{\partial q_1} = A_c(2\mu + \lambda) \int_0^L (q_1 + 2q_2 X + \cdots + jq_j X^{j-1} + \cdots) dX - FL \]

\[ = A_c(2\mu + \lambda) \left( q_1 L + q_2 L^2 + \cdots + q_j L^j + \cdots \right) - FL \]

\[ = L \left[ A_c(2\mu + \lambda) \left( q_1 + q_2 L + \cdots + q_j L^j + \cdots \right) - F \right] \]

\[ = A_c(2\mu + \lambda) \left( q_1 + 2q_2 X + \cdots + jq_j X^{j-1} + \cdots \right) 2X dX - FL^2 \]

\[ = A_c(2\mu + \lambda) \left( q_1 L^2 + \frac{4}{3} q_2 L^3 + \cdots + \frac{2j}{j+1} q_j L^{j+1} + \cdots \right) - FL^2 \]

\[ = L^2 \left[ A_c(2\mu + \lambda) \left( q_1 + \frac{4}{3} q_2 L + \cdots + \frac{2j}{j+1} q_j L^{j-1} + \cdots \right) - F \right] \] (C.18)

The first of these gives the rigid-body displacement due to the extension of the spring.

\[ q_0 = L_0 + \frac{F}{k} \] (C.19)

The remaining \( N \) equations for \( q_i, \; i = 1, 2, \cdots, N \) are given by

\[ q_1 + \frac{2j}{j+1} q_2 L + \cdots + \frac{ij}{i + j - 1} q_j L^{j-1} + \cdots = \frac{F}{A_c(2\mu + \lambda)}, \; i = 1, 2, \cdots, N, \; j = 1, 2, \cdots, N \] (C.20)
By inspection it can be seen that setting

\[ q_1 = \frac{F}{A_c (2\mu + \lambda)} \quad (C.21) \]

yields a homogeneous algebraic system for \( q_i, i = 2, 3, \ldots, N \), which can generally be solved by the trivial solution \( q_2 = q_3 = \cdots = q_N = 0 \). Thus we arrive at an approximate solution, \( w \), identical to the solution found using the direct approach in the preceding section.

\[ w = L_0 + \left( \frac{F}{k} \right)_{q_0} + \left( \frac{F}{A_c (2\mu + \lambda)} \right)_{q_1} X \quad (C.22) \]
Appendix D

Deformation and Leakage of Gasket Composed of Square O-Ring

The investigation of the leakage mechanism discussed in Chapter 2 illustrated that non-uniform stretching parallel to the sealing surfaces leads to more tensile/less compressive normal tractions between between the seal and the sealing surfaces. However, this mechanistic explanation of seal leakage is incomplete without a quantitative criterion for seal leakage.

In the literature there is already a leakage theory based on contact mechanics and percolation, which has been developed largely by Persson and his coworkers [78, 56, 54]. This theory states that the flow rate past a seal is directly proportional to the differential pressure across the seal and the surface roughness of asperities between the seal and the substrate against which it seals, and the flow rate is inversely proportional to the magnitude of the compressive traction normal to the seal and sealing surface at their interface. Although their theory agrees remarkably well with their experiments for rough surfaces, relatively low compressive contact tractions, and relatively small differential pressures, it exhibits a characteristic mechanism that is not observed in seals investigated in this work. Specifically, it states that there is a steady flow of fluid regardless of how small the applied differential pressure or how large the compressive sealing traction, and the magnitude of such a continuous flow is modulated by the normal contact traction and differential pressure. This is contrary to the observations in this work where there is negligible flow past the seal up to a critical differential pressure, above which there is a precipitous increase in flow rate past the seal. Admittedly, the observations in this work are confounded with material failure in the form of fracture at the low-pressure support of the seal. Nevertheless, one might question whether the theory of Persson and coworkers is valid only for low compressive contact tractions and low differential pressures, and whether a different leakage criterion is valid for large compressive tractions, relatively smooth surfaces and high differential pressures.

During discussions at the time of the writing of Druecke et al. [22], it was proposed that a rational mechanics argument for leakage in the absence of adhesion for surfaces of negligible roughness should require that fluid pressure exceeds the compressive contact traction normal to the sealing interface, formulated as:

\[ p > -\sigma \hat{n} \cdot n \text{ at contact line} \iff \text{Contact Line Advances} \]  

(D.1)

where \( p \) is the fluid pressure, \( \sigma \) the Cauchy stress and \( \hat{n} \) the outward unit normal pointing from the elastomeric seal toward the sealing surface. The hypothesis was that, for quasi-static leakage, in the absence of surface tension effects, the contact line between the leaking fluid, the rigid substrate, and the deformable elastomeric seal would advance if this criterion were satisfied. If it were not satisfied, the contact line would tend to recede, such that the fluid front moved up the pressure gradient away from the low-pressure end. If the fluid pressure were equal to the normal component of the compressive traction, the fluid and seal would be in equilibrium, with neither advancement nor recession of the contact line.

Unbeknownst at the time, such a leakage criterion, which follows directly from the above equilib-
Figure D-1: Schematic illustrating mechanics near a contact line in the absence of adhesion. (a) Fluid pressure applied upstream of the contact line. Contact with a rigid substrate downstream of the contact line. (b) Free-body diagram of the seal in the vicinity of the contact line illustrating quasistatic fluid normal to seal surface, contact traction (normal and tangential) and bulk forces from the surrounding seal acting on the section near the contact line.

In order to investigate this, we considered the axial deformation of an O-ring with square cross-section and subsequent internal pressurization of the O-ring. A schematic for this system is shown in Fig. D-2. In this set-up, the O-ring functions as a gasket.

Figure D-2: Schematic of experimental set-up allowing control of axial force and internal pressure of an O-ring with square cross-section functioning as a gasket.

Although only a handful of experiments were conducted – insufficient to precisely conclude the proper leakage criterion – we were able to visualize the seal as leakage occurred. Frames of a video of this leakage process are shown in Fig. D-3. The observations of leakage indicate that, despite the fact that the O-ring and loading were initially axisymmetric (within manufacturing tolerances), the leakage was markedly non-axisymmetric. This is inconsistent with Persson's contact mechanics theory which states that, provided surface roughness and normal contact stress are uniform, leakage past the seal should also be uniform. Furthermore, the non-axisymmetric nature of the observations suggests an instability and the breaking of axial symmetry. Although perhaps unimportant from a practical perspective for design and leakage of O-rings and gaskets, it is an interesting coupled fluid-structure stability problem.
Figure D-3: Leakage of O-ring seal under axial compression and internal pressurization. (Top Left) Initial leakage. A small droplet of fluorescein-dyed water can be seen on the outer periphery of the seal. (Top Right) Jetting of fluid. (Bottom Left) Jet instability. (Bottom Right) Jet breakup.
Appendix E

An Examination of the Relative Magnitudes of the Two Small Parameters in the Asymptotic Analysis of Chapter 4: Proper Limits

A question has been raised regarding the asymptotic analysis. Specifically, concern regarding the magnitude of the term containing the product of a small parameter, $\mu/\lambda$, with a large parameter, $L^2/H^2$, is expressed. Here we conduct a more rigorous asymptotic analysis to appease the concerns of the reviewer.

We start with the governing nondimensional field equations as given in Eqn. (4.24)

$$
\epsilon_{\text{mat}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon_{\text{geom}}} \frac{\partial^2 u}{\partial y^2} \right) + (1 + \epsilon_{\text{mat}}) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (\text{E.1a})
$$

$$
\epsilon_{\text{mat}} \left( \epsilon_{\text{geom}} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + (1 + \epsilon_{\text{mat}}) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0. \quad (\text{E.1b})
$$

subject to the non-dimensional boundary conditions given by Eqn. (4.26)

$$
\frac{\lambda}{K} \left( (1 + 2\epsilon_{\text{mat}}) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -1
\quad \text{on } x = 0 \quad (\text{E.2a})
$$

$$
\frac{\partial u}{\partial y} + \epsilon_{\text{geom}} \frac{\partial v}{\partial x} = 0
\quad \text{on } x = 0
$$

$$
\begin{cases}
  u = 0 \\
  v = 0
\end{cases}
\quad \text{on } y = 0 \quad (\text{E.2b})
$$

$$
\frac{\lambda}{K} \left( (1 + 2\epsilon_{\text{mat}}) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + k^* \frac{L}{K} \frac{\partial v}{\partial y} = 0
\quad \text{on } x = 1 \quad (\text{E.2c})
$$

$$
\frac{\partial u}{\partial y} + \epsilon_{\text{geom}} \frac{\partial v}{\partial x} = 0
\quad \text{on } x = 1
$$

$$
\begin{cases}
  v = 0 \\
  \frac{\partial v}{\partial y} = 0
\end{cases}
\quad \text{on } y = 1 \quad (\text{E.2d})
$$

where we have defined the small parameters $\epsilon_{\text{mat}} \equiv \mu/\lambda$ as the small parameter related to the elastic moduli of the material and $\epsilon_{\text{geom}} \equiv H^2/L^2$ as the small parameter related to the long aspect ratio. Additionally, we have used Eqn. (4.27a) to write the stress tensor component $\sigma_{11}$ in terms of displacements. Finally, we have recognized that, because $v = 0$ on $y = 1$, then $dv/dx$ must also
equal zero on this surface, and the shear-free boundary condition reduces to a condition on \( du/dy \).

### E.1 Boundary Layers

At this point we must decide how to deal with the fact that there are two small parameters in the problem. These two small parameters are quite distinct from each other, both in a physical sense – one can change the length of the seal without changing its material, or one can change the ratio of elastic moduli without changing geometry – and in a mathematical sense. From a mathematical perspective, the small parameter \( \epsilon \text{geom} \) gives rise to a boundary layer problem, as discussed in §4.1.4. However, within the context of the boundary layer problem, we can conduct a regular asymptotic expansion with regard to the small parameter \( \epsilon \text{mat} \). We do so here using the method of matched asymptotic expansions.

There are boundary layers at either end of the domain. Therefore, it is necessary to perturb the geometry to obtain local coordinates at each of these ends. Using the knowledge that the width of the boundary layer in dimensional coordinates scales as the thickness, \( H \), we define\(^1\) the coordinate in the high-pressure boundary layer near \( x = 0 \) (at the left end) to be

\[
x_L = (\epsilon \text{geom})^{-\frac{1}{2}} x
\]

and the coordinate in the low-pressure boundary layer near \( x = 1 \) (at the right end) to be

\[
x_R = (\epsilon \text{geom})^{-\frac{1}{2}} (1 - x)
\]

Derivatives with respect to \( x \) become

\[
\frac{\partial}{\partial x} \to (\epsilon \text{geom})^{-\frac{1}{2}} \frac{\partial}{\partial x_L}, \quad \frac{\partial}{\partial x} \to - (\epsilon \text{geom})^{-\frac{1}{2}} \frac{\partial}{\partial x_R}
\]

in the left and right boundary layer, respectively.

In the left boundary layer, the governing field equations become

\[
\epsilon \text{mat} \left( \frac{1}{\epsilon \text{geom}} \frac{\partial^2 u_L}{\partial x_L^2} + \frac{1}{\epsilon \text{geom}} \frac{\partial^2 u_L}{\partial y^2} \right) + (1 + \epsilon \text{mat}) \left( \frac{1}{\epsilon \text{geom}} \frac{\partial^2 u_L}{\partial x_L^2} + \frac{1}{(\epsilon \text{geom})^\frac{3}{2}} \frac{\partial^2 v_L}{\partial x_L \partial y} \right) = 0 \tag{E.6a}
\]

\[
\epsilon \text{mat} \left( \frac{\partial^2 v_L}{\partial x_L^2} + \frac{\partial^2 v_L}{\partial y^2} \right) + (1 + \epsilon \text{mat}) \left( \frac{1}{(\epsilon \text{geom})^\frac{3}{2}} \frac{\partial^2 u_L}{\partial x_L \partial y_L} + \frac{\partial^2 v_L}{\partial y^2} \right) = 0 \tag{E.6b}
\]

\(^1\)In order to see that this is the correct scaling, we can say that the outer edge of the left boundary layer occurs at \( x^* = O(H) \). This corresponds to \( x = x^*/L = O(H/L) \). Even though \( x \) is small at this point, we want the value of the inner, stretched coordinate, \( x_L \), to be leading order at the edge of the boundary layer. Therefore, when \( x = O(H/L) \), we want \( x_L = O(1) \) giving \( x_L = O(L/H) \) \( x = O(\epsilon \text{geom})^{-\frac{1}{2}} x \). A more physical argument is that, in the boundary layer, \( x_L \) must scale like \( H \) so terms like \( d^2 u/dx_L^2 \) must be the same order as \( d^2 u/dy^2 \). Similar arguments can be made for the boundary layer at the other end with proper accounting of the sign.
subject to boundary conditions

\[
\left. \begin{array}{l}
\frac{\lambda}{K} \left( 1 + 2\epsilon_{\text{mat}} \frac{\partial u_L}{\partial y} + \frac{\partial v_L}{\partial y} \right) = -1 \\
\frac{\partial u_L}{\partial y} + (\epsilon_{\text{geom}})^{\frac{1}{2}} \frac{\partial x_L}{\partial x_L} = 0 
\end{array} \right\} \text{on } x_L = 0 \quad (E.7a)
\]

\[
\left. \begin{array}{l}
u_L = 0 \\
v_L = 0 
\end{array} \right\} \text{on } y = 0 \quad (E.7b)
\]

\[
\left. \begin{array}{l}
u_L = 0 \\
\frac{\partial u_L}{\partial y} = 0 
\end{array} \right\} \text{on } y = 1 \quad (E.7c)
\]

Note that we will also need to impose matching conditions to the bulk solution following the method of matched asymptotic expansions \([7, 96]\).

The equations governing the inner solution in the right boundary layer are

\[
\epsilon_{\text{mat}} \left( \frac{1}{\epsilon_{\text{geom}}} \frac{\partial^2 u_R}{\partial x_R^2} + \frac{1}{\epsilon_{\text{geom}}} \frac{\partial^2 u_R}{\partial y^2} \right) + (1 + \epsilon_{\text{mat}}) \left( \frac{1}{\epsilon_{\text{geom}}} \frac{\partial^2 u_R}{\partial x_R^2} - \frac{1}{(\epsilon_{\text{geom}})^{\frac{1}{2}}} \frac{\partial^2 v_R}{\partial x_R \partial y} \right) = 0 \quad (E.8a)
\]

\[
\epsilon_{\text{mat}} \left( \frac{\partial^2 v_R}{\partial x_R^2} + \frac{\partial^2 v_R}{\partial y^2} \right) + (1 + \epsilon_{\text{mat}}) \left( -\frac{1}{(\epsilon_{\text{geom}})^{\frac{1}{2}}} \frac{\partial^2 u_R}{\partial x_R \partial y} + \frac{\partial^2 v_R}{\partial y^2} \right) = 0 \quad (E.8b)
\]

subject to boundary conditions

\[
\left. \begin{array}{l}
u_R = 0 \\
u_R = 0 
\end{array} \right\} \text{on } y = 0 \quad (E.9a)
\]

\[
\left. \begin{array}{l}rac{\lambda}{K} \left( -\frac{1 + 2\epsilon_{\text{mat}} \frac{\partial u_R}{\partial y} + \frac{\partial v_R}{\partial y}}{(\epsilon_{\text{geom}})^{\frac{1}{2}}} \frac{\partial x_R}{\partial x_R} \right) + \frac{k^* L}{K} u_R = 0 \\
\frac{\partial u_R}{\partial y} - (\epsilon_{\text{geom}})^{\frac{1}{2}} \frac{\partial v_R}{\partial x_R} = 0 
\end{array} \right\} \text{on } x = 1 \quad (E.9b)
\]

\[
\left. \begin{array}{l}
u_R = 0 \\
\frac{\partial u_R}{\partial y} = 0 
\end{array} \right\} \text{on } y = 1 \quad (E.9c)
\]

where will still require matching conditions with the bulk solution in order to form a well-posed problem.

**Remark:** In each of the boundary layer problems, the presence of the square root of the geometry parameter indicates that, if there were no small material parameter \(\epsilon_{\text{mat}}\), we might expect an asymptotic series of the form

\[
u_L = u_{L,0} + (\epsilon_{\text{geom}})^{\frac{1}{2}} u_{L,1} + \epsilon_{\text{geom}} u_{L,2} + \cdots \quad (E.10)
\]

However, because there is also a small material parameter, \(\epsilon_{\text{mat}}\), which is orthogonal to the geometric parameter, we should expect each term in the above series to be expanded also in terms of the material parameter, \(\epsilon_{\text{mat}}\), such that

\[
u_{L,0} = u_{L,0,0} + \epsilon_{\text{mat}} u_{L,0,1} + \epsilon_{\text{mat}}^2 u_{L,0,2} + \cdots
\]

\[
u_{L,1} = u_{L,1,0} + \epsilon_{\text{mat}} u_{L,1,1} + \epsilon_{\text{mat}}^2 u_{L,1,2} + \cdots
\]

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E.2 Outer Asymptotic Solution Valid in Bulk of Domain

We now turn our attention to the outer asymptotic solution governing the bulk of the domain far from the boundary layers. Here, to be precise, we first make a choice of the relative magnitudes of the small parameters, $\epsilon_{\text{mat}}$ and $\epsilon_{\text{geom}}$ (although they are really completely independent), and then conduct the analysis accordingly. There are three distinct cases of the limiting values of the ratio of these parameters: zero, a constant and infinity. We consider each case separately.

E.2.1 $\frac{\epsilon_{\text{mat}}}{\epsilon_{\text{geom}}} = \mathcal{O}(1)$

This is the case considered in the §4.1.4. The leading-order problem reduces to

\begin{equation}
\frac{\epsilon_{\text{mat}}}{\epsilon_{\text{geom}}} \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0
\end{equation}

subject to the boundary conditions along the long surfaces of

\begin{equation}
\begin{aligned}
&u = 0 \quad \text{on } y = 0 \\
v = 0 \quad &\text{on } y = 1
\end{aligned}
\end{equation}

Note we have omitted the boundary conditions on $x = 0$ and $x = 1$ because the boundary conditions in the $x-$direction on the bulk solution must be obtained by matching with the boundary layer solutions (or via the Saint-Venant or Ritz approximations in §4.1.4). The general solution, upon imposing boundary conditions on $y = 0$ and $y = 1$, is

\begin{equation}
\begin{aligned}
&u(x, y) = \frac{\epsilon_{\text{geom}}}{\epsilon_{\text{mat}}} \frac{dp}{dx} \left( \frac{1}{2} y^2 - y \right) \\
v(x, y) = &p(x) \left( -\frac{1}{2} y^3 + \frac{3}{2} y^2 - y \right) \\
p(x) = &C_1 \cosh \left( \sqrt{\frac{3 \epsilon_{\text{mat}}}{\epsilon_{\text{geom}}}} x \right) + C_2 \sinh \left( \sqrt{\frac{3 \epsilon_{\text{mat}}}{\epsilon_{\text{geom}}}} x \right)
\end{aligned}
\end{equation}

where $C_1$ and $C_2$ are determined from the boundary conditions at $x = 0$ and $x = 1$ using either the Saint-Venant approach or the Ritz method. Because both $\epsilon_{\text{mat}}$ and $\epsilon_{\text{geom}}$ are small but non-zero, this formulation is valid for any small, positive, real values of the parameters $\epsilon_{\text{geom}}$ and $\epsilon_{\text{mat}}$.

For potential use later when we consider the limit when $\frac{\epsilon_{\text{mat}}}{\epsilon_{\text{geom}}} = \mathcal{O}(0)$, we define the parameter $\xi$ as the ratio of the two small parameters, such that

\begin{equation}
\xi = \frac{\epsilon_{\text{mat}}}{\epsilon_{\text{geom}}}
\end{equation}

and expand this solution as a Taylor series in powers of $\xi$ about $\xi = 0$, giving

\begin{equation}
\begin{aligned}
&u(x, y) = \left( C_2 3 \xi \xi^0 0! + C_1 3 \xi \xi^0 1! + C_2 3 \xi \xi^1 2! + C_1 3 \xi \xi^1 3! + \cdots \right) \left( \frac{1}{2} y^2 - y \right) \\
v(x, y) = &\left( C_1 + C_2 3 \xi \xi^1 2! + C_1 3 \xi \xi^1 3! + \cdots \right) \left( -\frac{1}{2} y^3 + \frac{3}{2} y^2 - y \right)
\end{aligned}
\end{equation}
E.2.2 \( \frac{\epsilon_{\text{mat}}}{\epsilon_{\text{geom}}} = \mathcal{O} (0) \)

The fact that the small parameter \( \epsilon_{\text{mat}} \) is never zero (because the bulk modulus is always finite) combined with the fact that the two parameters can be adjusted independently obviates the need for this analysis. In other words, the ratio \( \epsilon_{\text{mat}}/\epsilon_{\text{geom}} \) is always bounded to be strictly greater than zero and strictly less than infinity, and making \( \epsilon_{\text{geom}} \) smaller does not cause \( \epsilon_{\text{mat}} \) to become smaller. Nevertheless, we conduct this analysis for completeness.

The governing field equations are

\[
\epsilon_{\text{mat}} \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon_{\text{geom}}} \frac{\partial^2 u}{\partial y^2} \right) + (1 + \epsilon_{\text{mat}}) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (E.17a)
\]

\[
\epsilon_{\text{mat}} \left( \epsilon_{\text{geom}} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + (1 + \epsilon_{\text{mat}}) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (E.17b)
\]

subject to boundary conditions

\[
\begin{align*}
u &= 0 \\
v &= 0 \\
\frac{\partial u}{\partial y} &= 0
\end{align*} \quad \text{on } y = 0 \quad (E.18a)
\]

\[
\begin{align*}
v &= 0 \\
\frac{\partial u}{\partial y} &= 0
\end{align*} \quad \text{on } y = 1 \quad (E.18b)
\]

Let us sacrifice generality for the sake of tractability and assume here that \( \epsilon_{\text{mat}} \) is proportional to \( (\epsilon_{\text{geom}})^2 \) as they both approach zero.

\[
\epsilon_{\text{mat}} = \mathcal{O} \left( \epsilon_{\text{geom}}^2 \right) \quad \text{as } \epsilon_{\text{geom}} \rightarrow 0 \quad (E.19)
\]

More specifically, let us assume that, in this limit, there is a constant, \( K \), that relates them such that

\[
\lim_{\epsilon_{\text{geom}} \rightarrow 0} \left( \frac{\epsilon_{\text{mat}}}{(\epsilon_{\text{geom}})^2} \right) = K \quad \Rightarrow \quad \epsilon_{\text{mat}} = K (\epsilon_{\text{geom}})^2 \quad \text{as } \epsilon_{\text{geom}} \rightarrow 0 \quad (E.20)
\]

Making this assumption and substitution yields the governing field equations

\[
K (\epsilon_{\text{geom}})^2 \frac{\partial^2 u}{\partial x^2} + K \epsilon_{\text{geom}} \frac{\partial^2 u}{\partial y^2} + \left( 1 + K (\epsilon_{\text{geom}})^2 \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) = 0 \quad (E.21a)
\]

\[
K (\epsilon_{\text{geom}})^3 \frac{\partial^2 v}{\partial x^2} + K (\epsilon_{\text{geom}})^2 \frac{\partial^2 v}{\partial y^2} + \left( 1 + K (\epsilon_{\text{geom}})^2 \right) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (E.21b)
\]

Then we seek an asymptotic solution of the form

\[
\begin{align*}
u &= u_0 + \epsilon_{\text{geom}} u_1 + (\epsilon_{\text{geom}})^2 u_2 + (\epsilon_{\text{geom}})^3 u_3 + \cdots \\
v &= v_0 + \epsilon_{\text{geom}} v_1 + (\epsilon_{\text{geom}})^2 v_2 + (\epsilon_{\text{geom}})^3 v_3 + \cdots
\end{align*} \quad (E.22)
\]

**Leading Order**

The leading order field equations are

\[
\mathcal{O} (1) : \begin{cases}
\frac{\partial}{\partial x} \left( \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) = 0 \\
\frac{\partial}{\partial y} \left( \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) = 0
\end{cases} \quad (E.23)
\]
subject to boundary conditions

\[ \begin{align*}
  u_0 &= 0 \\
  v_0 &= 0 \quad \text{on } y = 0 \quad \text{(E.24a)} \\
  v_0 &= 0 \\
  \frac{\partial u_0}{\partial y} &= 0 \quad \text{on } y = 1. \quad \text{(E.24b)}
\end{align*} \]

Together, the field equations imply that the divergence of the leading-order displacement field must be a constant.

\[ \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = C \quad \text{(E. 25)} \]

We observe that leading-order terms in the Taylor series expansion of the solution from §4.1.4, as given in Eqn. (E.16), satisfies these equations. Thus we have the general solution

\[ \begin{align*}
  u_0 &= (a + 3bx) \left( \frac{1}{2} y^2 - y \right) \quad \text{(E.26a)} \\
  v_0 &= b \left( -\frac{1}{2} y^3 + \frac{3}{2} y^2 - y \right) \quad \text{(E.26b)}
\end{align*} \]

**First Correction**

The governing equations for the first correction are

\[ \mathcal{O} (\epsilon_{\text{geom}}) : \begin{aligned}
  \frac{\partial}{\partial x} \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) &= -K \frac{\partial u_0^2}{\partial y^2} \\
  \frac{\partial}{\partial y} \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) &= 0
\end{aligned} \quad \text{(E. 27)} \]

subject to the same form of the homogeneous boundary conditions as in the leading-order problem.

\[ \begin{align*}
  u_1 &= 0 \\
  v_1 &= 0 \quad \text{on } y = 0 \quad \text{(E.28a)} \\
  v_1 &= 0 \\
  \frac{\partial u_1}{\partial y} &= 0 \quad \text{on } y = 1. \quad \text{(E.28b)}
\end{align*} \]

Defining \( p_1 = -\text{div } u_1 \), integrating the second field equation gives

\[ p_1 = p_1(x) \quad \text{(E. 29)} \]

Upon substituting \( p_1(x) \) into the first field equation and integration, we obtain

\[ p(x) = Kax + \frac{3}{2} Kbx^2 + d \quad \text{(E. 30)} \]
where $d$ is a constant of integration. Then, attempting solutions of the form\(^2\)

\[ u_1 = \phi_1(x) \left( \frac{1}{2} y^2 - y \right) \]
\[ v_1 = \psi_1(x) \left( -\frac{1}{2} y^3 + \frac{3}{2} y^2 - y \right) \tag{E.31} \]

we obtain the solution

\[ u_1 = 3 \left( c + dx + \frac{1}{2} Kax^2 + \frac{1}{2} bKx^3 \right) \left( \frac{1}{2} y^2 - y \right) \]
\[ v_1 = \left( d + Kax + \frac{3}{2} bKx^2 \right) \left( \frac{1}{2} y^3 + \frac{3}{2} y^2 - y \right) \tag{E.32} \]

The governing equations for the first correction are

\[
O (\epsilon_{\text{geom}}) : \begin{cases}
\frac{\partial}{\partial x} \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = -K \frac{\partial u_0}{\partial y} \\
\frac{\partial}{\partial y} \left( \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = 0 
\end{cases}
\]

subject to the same form of the homogeneous boundary conditions as in the leading-order problem.

\[
\begin{aligned}
 & \{ u_1 = 0 \} \quad \text{on } y = 0 \\
 & \{ v_1 = 0 \} \quad \text{on } y = 0 \\
 & \{ \frac{\partial u_1}{\partial y} = 0 \} \quad \text{on } y = 1. 
\end{aligned} \tag{E.34a} \tag{E.34b}
\]

**Second Correction**

The governing equations for the second correction are

\[
O \left( (\epsilon_{\text{geom}})^2 \right) : \begin{cases}
\frac{\partial}{\partial x} \left( \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} \right) = -K \left( \frac{\partial u_1^2}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y} + 2 \frac{\partial^2 u_0}{\partial x^2} \right) \\
= -3K \left( c + dx + \frac{1}{2} aKx^2 + \frac{1}{2} bKx^3 \right) 
\end{cases} \tag{E.35} \\
\frac{\partial}{\partial y} \left( \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} \right) = -K \left( 2 \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 u_0}{\partial x \partial y} \right) \\
= 3bK (y - 1)
\]

\(^2\)chosen to satisfied the boundary conditions, and based on the Taylor series expansion above
subject to the same form of the homogeneous boundary conditions as in the leading-order problem.

\[
\begin{align*}
  u_2 &= 0 \\
  v_2 &= 0 \\
  v_2 &= 0 \\
  \frac{\partial u_2}{\partial y} &= 0
\end{align*}
\]  \text{on } y = 0 \quad \text{(E.36a)}
\]

\[
\begin{align*}
  \text{on } y = 1.
\end{align*}
\]  \text{on } y = 1 \quad \text{(E.36b)}
\]

Although we have not solved this second correction, it is speculated that the same functional form of the above solution in the $y$–direction should be tried. This will likely give two additional terms in the series expansion in $x$.

**Remark:** The terms obtained from the first correction to the leading order solution are consistent with the Taylor series expansion in Eqn. (E.16). Without performing the matching (or Saint-Venant or Ritz approximations), one cannot determine the coefficients, $a, b, c, d, \ldots$ in these successive approximations. Because there are more free parameters (four coefficients $a$ through $d$) here than in the solution of §4.1.4, a Ritz solution with these functions should perform as well or better than the Ritz solution using only the two coefficients $C_1$ and $C_2$ in the hyperbolic sine and cosine formulation.

### E.2.3 \[ \frac{\epsilon_{\text{mat}}}{\epsilon_{\text{geom}}} = O(\infty) \]

We now consider the final case for which the asymptotic limit of the ratio of $\epsilon_{\text{mat}}/\epsilon_{\text{geom}}$ is infinity as $\epsilon_{\text{geom}}$ approaches zero. The limiting case of $\epsilon_{\text{geom}} \to 0$ implies that the length, $L$, becomes infinitely larger than the thickness, $H$. This absence of the the longitudinal length scale implies that our original scaling of the dependent variables is no longer valid. Specifically, $u^*$ cannot scale with $L$ because $L$ is on the order of infinity.

We attempt to solve an analogous problem directly. Consider a semi-infinite planar strip on the domain $\Omega = \{(x, y) \in \mathbb{R}^2 | 0 \leq x^* < \infty, -1 \leq y^* \leq 1\}$ with no displacement on the the upper and lower surfaces and symmetric normal traction on the end at $x = 0$. Then the Navier equations governing the displacement field, $u = u(x, y)e_x + v(x, y)e_y$ in the linear elastic body with homogenous material properties in the absence of body forces are

\[
(\lambda + \mu) \text{grad} \left( \text{div } \mathbf{u} \right) + \mu \text{div} \text{grad } \mathbf{u} = 0
\]  \text{subject to boundary conditions \quad (E.37)}
\]

subject to boundary conditions

\[
\begin{align*}
  u &= v = 0 \text{ on } y = \pm 1 \\
  \sigma_{xx} &= \sigma_{xx}(y) \text{ on } x = 0 \\
  \sigma_{xy} &= 0 \text{ on } x = 0
\end{align*}
\]  \text{on } x = 0 \quad \text{(E.38a)}
\]

\text{on } x = 0 \quad \text{(E.38b)}
\]

where $\sigma_{xx}(y)$ is some prescribed symmetric function of $y$.

We expect the solution to exponentially decay with increasing $x$. We try an ansatz of the form

\[
\begin{align*}
  u(x, y) &= f(y) \exp(-sx) \\
  v(x, y) &= g(y) \exp(-sx)
\end{align*}
\]  \text{subject to boundary conditions \quad (E.39)}
\]

Substituting into the governing field equations yields

\[
\begin{align*}
  \epsilon f'' + (1 + 2\epsilon) s^2 f &= (1 + \epsilon) s g' \\
  (1 + 2\epsilon) g'' + \epsilon s^2 g &= (1 + \epsilon) s f''
\end{align*}
\]  \text{subject to boundary conditions \quad (E.40)}
\]

where $\epsilon \equiv \mu/\lambda$ is the ratio of elastic moduli. These equations have analytic solutions, despite the fact that they are rather messy. We solve these in Mathematica. Upon imposing the four boundary
conditions of zero displacement on \( y = \pm 1 \) given by

\[
f(-1) = f(1) = g(-1) = g(1) = 0
\]  \tag{E.41}

we obtain the equation for the eigenvalues of

\[
(1 + 3\epsilon) (\exp (4is) - 1) - 4is (1 + \epsilon) \exp (2is) = 0
\]  \tag{E.42}

indicating that the eigenvalues, \( s \), will be complex. Defining the complex eigenvalues \( s = s_r + is_i \) where \( s_r, s_i \in \mathbb{R} \), the following two equations must be solved

\[
(1 + \epsilon) s_r - \frac{1}{2} (1 + 3\epsilon) \cosh (2s_i) \sin (2s_r) = 0
\]  \tag{E.43a}

\[
(1 + \epsilon) s_i - \frac{1}{2} (1 + 3\epsilon) \sinh (2s_i) \cos (2s_r) = 0,
\]  \tag{E.43b}

which generally must be done numerically. Upon finding the eigenvalues, the modes must be superposed using the boundary conditions. This work is admittedly incomplete.