TRANSPORT METHODS AND UNIVERSALITY FOR β-ENSEMBLES

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Diplome de l'Ecole Normale Superieure (2013)

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Submitted to the Department of Mathematics
in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

Massachusetts Institute of Technology
February 2018

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Abstract

In this thesis, we investigate the local and global properties of the eigenvalues of β-ensembles. A lot of attention has been drawn recently on the universal properties of β-ensembles, and how their local statistics relate to those of Gaussian ensembles. We use transport methods to prove universality of the eigenvalue gaps in the bulk and at the edge, in the single cut and multicut regimes. In a different direction, we also prove Central Limit Theorems for the linear statistics of β-ensembles at the macroscopic and mesoscopic scales.

Acknowledgements

This thesis was written during my time as a PhD student at MIT. First and foremost, I would like to thank my advisor Alice Guionnet. Her mathematical creativity, expertise and enthusiasm have been both extremely helpful and motivational. All her insightful comments, ideas and guidance were greatly appreciated. I would also like to express my gratitude to the members of my defense committee Alexei Borodin and Vadim Gorin. I am very grateful to the people that have contributed to my mathematical education, my mentors and professors at Ecole Normale Superieure Thierry Bodineau, Jean Bertoin and Wendelin Werner who gave me my first exposure to Probability Theory, and Amir Dembo who mentored me during my visit at Stanford University in 2013. I have benefited from the insights and advice from many bright researchers and feel especially endebted to Sylvia Serfaty and Paul Bourgade whom I have visited at multiple occasions at NYU. I am deeply grateful to the people from the Math Academic Services who have helped me in many ways and made my stay as a student at the department as pleasant as possible.

I would like to thank my friends from the mathematical community Alex, Alisa, Andrew, Asad, Ewain, Francisco, Thomas, Nina, Raphaël, Sylvain and Yi with whom I had the pleasure discussing mathematics and other things. Finally, I would like to thank the people that are the closest to me and have continuously shown their love and support, my fiancée Claire, my best friends Ron, Thomas and Julian, my brother and sister Adrien and Céline and my parents Gérard and Mahshid.

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Chapter 1

Introduction

1.1 Background

The question of universality in Random Matrix Theory arose in the 50's with the pioneering work of E. Wigner [87] when he observed that the gaps of energy levels of large nuclei tend to follow the same law, irrespective of the material. According to quantum mechanics, energy levels of large nuclei should be described by the eigenvalues of a Hamiltonian, an Hermitian operator acting on a Hilbert space. Wigner had the idea to model this operator by a large Hermitian matrix with random entries, and look at its eigenvalues. With appropriate scaling, one would find that the spacings of such eigenvalues would match empirically those of the energy levels. This point of view was adopted subsequently by F. Dyson (see [38] and also [65]):

The statistical theory will not predict the detailed sequence of levels in anyone nucleus, but it will describe the general appearance and the degree of irregularity of the level structure that is expected to occur in any nucleus which is too complicated to be understood in detail.

From there emerged the universality conjecture in Random Matrix Theory, known as the Wigner - Mehta - Dyson conjecture, which states that the distribution of eigenvalue gaps are the same for random matrices within the same symmetry class. In Random Matrix Theory, there are several ensembles of matrices for which the question of universality has been extensively studied.

1.1.1 Random Matrix Ensembles

Wigner Ensembles

Wigner matrices correspond to random symmetric or Hermitian matrices with independent and identically distributed entries (up to the symmetry), with mean 0 and variance equal to $1/N$ where N denotes the size of the matrix. In this context, universality refers to universality with respect to the law of the entries. Wigner matrices with Gaussian entries were the first to be studied (see [38]).

Invariant Ensembles

These correspond to symmetric or Hermitian random matrices sampled from the probability distribution

$$d\mathbb{P}_V^N(M) = \frac{1}{Z_V^N} e^{-N\ trV(M)} dM \tag{1.1.1}$$

where $dM = \prod_{i\leq j} dM_{i,j}$ in the symmetric case, $dM = \prod_i dM_{i,i} \prod_{i<j} d\mathcal{R}M_{i,j} d\mathcal{I}M_{i,j}$ in the Hermitian case and V is a continuous potential with some growth assumptions. The normalization constant Z_V^N is called the partition function, and $\log Z_V^N$ the free energy. These ensembles are called invariant since they are invariant by resp. orthgonal and unitary conjugaison. The probability density of the eigenvalues from these ensembles can be explicitely computed and has the form

$$\mathbb{P}_{V,\beta}^N(d\lambda_1, \dots, d\lambda_N) := \frac{1}{Z_{V,\beta}^N} \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)}\, d\lambda_1 \dots d\lambda_N, \tag{1.1.2}$$

where

$$Z_{V,\beta}^N := \int \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)}\, d\lambda_1 \dots d\lambda_N \tag{1.1.3}$$

with $\beta = 1, 2$ respectively.

β-Ensembles

The previous density can be seen as the density of a gas with logarithmic repulsion, a confining potential V and at inverse temperature $\beta > 0$. This system is often referred as the β-ensemble, or Coulomb gas, and it will be the central object under study in this Thesis. In this context, universality refers to universality with respect the potential, the factor β corresponding to the symmetry class. β-ensembles generalize invariant ensembles and note that for generic values of β, there is not necessarily an underlying matrix model.

The Gaussian Ensembles (resp. Gaussian Orthogonal Ensemble, Gaussian Unitary Ensemble), represent the canonical exemple for these ensembles and correspond to random symmetric or Hermitian matrices with independant Gaussian entries (up to the symmetry). They also correspond to invariant ensembles with potential $V(\lambda) = \beta\lambda^2/4$ with $\beta = 1, 2$ respectively. The reader can refer to [65] for an extensive study of the Gaussian Ensembles. There are several questions that are of great importance in Random Matrix Theory, and we discuss some of them.

1.1.2 Some questions of interest

Law of Large Numbers

In the case of the Gaussian Ensembles, Wigner showed from a direct computation of the moments that the empirical distribution of the rescaled eigenvalues

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i} \tag{1.1.4}$$

converge in expectation towards a probability measure called the Wigner semicircle law and with explicit density

$$\mu_{sc}(x) = \frac{1}{2\pi}\sqrt{4-x^2}\ . \tag{1.1.5}$$

The convergence also holds for Wigner Matrices for which the entries have a finite second moment, and the convergence happens almost surely. As shown in [6, 73, 67], the empirical distribution of the particles also converge for β-ensembles with mild assumptions on the potential V, but with a limit μ_V depending on V. In fact, μ_V is the unique minimizer of the energy functional

$$E(\mu) = \iint \left(\frac{V(x_1)+V(x_2)}{2} - \frac{\beta}{2}\log|x_1-x_2|\right) d\mu(x_1)d\mu(x_2) \tag{1.1.6}$$

on the space $\mathcal{M}_1(\mathbb{R})$ of probability measures on $\mathbb{R}$, and a large deviation principle holds with good rate function $E(\mu) - E(\mu_V)$ (see [6]). Moreover, the measure μ_V has compact support and is uniquely determined by the existence of a constant c_V such that:

$$\beta \int \log|x-y| d\mu_V(y) - V(x) \leq c_V\ ,$$

with equality almost everywhere on the support. For now, we assume the following

- V is analytic on a neighbourhood of the support.
- $\liminf \frac{V(x)}{\log|x|} = +\infty$.
- The function $\beta \int \log|x-y|d\mu_V(y) - V(x) - c_V$ does not vanish outside the support.

In that case the support of the equilibrium measure μ_V is an union of intervals $A = \cup_{i=0}^{g}[\alpha_g^-;\alpha_g^+]$ and has a density that can be written in the following form (see [34])

$$\frac{d\mu_V(x)}{dx} = S(x)\prod_{i=0}^{g}\sqrt{x-\alpha_g^-}\sqrt{\alpha_g^+ - x}\ , \tag{1.1.7}$$

where S is analytic on a neighbourhood of the support. From there we can distinguish several regimes, and the distinctions will be essential in what follows.

- We will say that we are in the *single cut regime* if the support of μ_V is connected (i.e $g = 0$), and that we are in the *multicut regime* otherwise.
- We will say that the regime is *non-critical* if $S > 0$ on the support, and that the regime is critical otherwise.

Notice that there are two kinds of criticality: S can vanish either in the interior of the support, or at the edges of the intervals. It is shown in [35] that in either cases, S vanishes like an even power.

Central Limit Theorems

From the previous paragraph, we see that for a continuous function f with compact support,

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) \longrightarrow \int f(x)d\mu_V(x). \tag{1.1.8}$$

A natural extension would be the study of the fluctuations of the linear statistics

$$\sum_{i=1}^{N} f(\lambda_i) - N\int f(x)d\mu_V(x) \tag{1.1.9}$$

Note that, due to the rigidity of the eigenvalues (see next paragraph), there is no $1/\sqrt{N}$ normalization. The fluctuations of linear statistics of the eigenvalues were shown to converge towards a Gaussian random variable for Wigner ensembles (see [4, 63, 76]), and β-ensembles in the non-critical onecut regime first in [51] and then in [16, 52]. However, it was shown in [77, 15] that the result does not hold for all test functions f in the multicut regime. Although the macroscopic limit μ_V is not universal, global fluctuations exhibit universality in the sense that the mean and variance of the limiting Gaussian do not depend on the potential.

In a similar vein, one can also ask about the fluctuations of the linear statistics at the mesoscopic scale $\alpha \in (0;1)$ around a fixed energy level E in the interior of the support of the equilibrium measure:

$$\sum_{i=1}^{N} f(N^{\alpha}(\lambda_i - E)) - N\int f(N^{\alpha}(x-E))d\mu_V(x) \tag{1.1.10}$$

Results in this direction were obtained in a variety of settings, for Gaussian Ensembles [22, 46], and for invariant ensembles [24, 55]. In many cases the results were shown at all scales $\alpha \in (0;1)$, often with the use of distribution specific properties. An early paper studying mesoscopic statistics for Wigner Matrices was [23], here the regime studied was $\alpha \in (0;1/8)$. The recent work [50] has pushed this to all scales.

Local Laws and Rigidity

The Law of Large Numbers determines the asymptotic number of particles in a fixed interval $[a;b]$. One could ask if the result still holds when the size of the interval scales with N; i.e can we prove for E fixed in the interior of the support of the equilibrium measure

$$\frac{1}{2N\eta}\#\{i\ ,\lambda_i \in [E-\eta;E+\eta]\} \longrightarrow \mu_V(E)\ , \tag{1.1.11}$$

where $N^{-1+\varepsilon} < \eta < N^{-\varepsilon}$ for some $\varepsilon > 0$.

Local Laws were proved first for Wigner Ensembles (see [39, 41], and [14] for a survey) and subsequently for β-ensembles in the single cut non-critical case in [20, 19, 18] with various assumptions on the potential (convex analytic, $V \in C^4$, and then V non-convex). In these articles, the authors also prove the rigidity of the particles with the same assumptions on the potential: for any $\zeta > 0$, there exists some constants $C, c > 0$ such that

$$\mathbb{P}^N_{V,\beta}\Big(\exists i,\ |\tilde{\lambda}_{i+1} - \tilde{\lambda}_i| \geq N^{-1+\zeta}\Big) \leq C\exp(-N^c) \ , \tag{1.1.12}$$

where $\tilde{\lambda}$ denote the reordered eigenvalues. Rigidity is proved by a multiscale analysis, by bootstraping concentration estimates through the use of loop equations, and was also proved more recently in the multicut case in [61].

Partition function

For Gaussian Ensembles, there is an exact formula for the partition function given by a Selberg integral (see [75])

$$Z^N_{G\beta E} = (2\pi)^{N/2}(N\beta/2)^{-\beta N^2/4+(\beta/4-1/2)N}\frac{\prod_{i=1}^N \Gamma(1+i\beta/2)}{\Gamma(1+\beta/2)^N} \tag{1.1.13}$$

For invariant ensembles, Partition functions can also be computed using orthgonal polynomials (see [65]). In the case $\beta \notin \{1, 2, 4\}$, orthogonal polynomial methods do not apply anymore. In [77], M. Shcherbina derives an asymptotic expansion of $\log Z^N_{V,\beta}$ with $o(1)$ precision. Using the loop equations (or Dyson-Schwinger equations), G.Borot and A. Guionnet derived in [16] a $1/N$ asymptotic expansion of the free energy at all orders for β-ensembles in the single cut non-critical case. The loop equations (see Proposition 4.2.1) are a family of exact equations that can be obtained by integration by parts and determine a relationship between the correlators. They were used to study the global fluctuations of the linear statistics first in [51]. Such a $1/N$ expansion does not hold anymore in the multicut regime, but the authors provide in [15] an alternative formulation with oscillating terms.

Eigenvalue gaps

In the Bulk

Wigner's original interest was to study the eigenvalue gaps. It was shown by M. Gaudin that for the GUE, if we fix an energy level $E \in (-2; 2)$ and define

$$S_N(\boldsymbol{\lambda}, s, E) = \frac{1}{l_N}\#\ \{i\ , \tilde{\lambda}_{i+1} - \tilde{\lambda}_i \leq \frac{s}{N\mu_{sc}(E)}, |\tilde{\lambda}_i - E| \leq \frac{l_N}{N\mu_{sc}(E)}\}\ , \tag{1.1.14}$$

where $l_N \longrightarrow \infty$, $l_N/N \longrightarrow 0$, and $\tilde{\lambda}$ correspond to the reordered eigenvalues then

$$\mathbb{E}^N_{GUE}\Big(S_N(\boldsymbol{\lambda}, s, E)\Big) \longrightarrow \int_0^s p(t)dt$$

where p is a probability distribution - named the Gaudin distribution. In [35], P. Deift et al. used the essential fact that for invariant Hermitian ensembles, the correlation functions can be written in terms of orthogonal polynomials and used Riemann-Hilbert methods to provide asymptotics for orthogonal polynomials, thus proving the averaged gap universality. One might want to look instead at the distribution of a single eigenvalue gap. In fact T. Tao showed in [79] that if $\varepsilon N < i < (1-\varepsilon)N$ for $\varepsilon > 0$ then

$$\mathbb{P}^N_{GUE}(N\mu_{sc}(\tilde{\lambda}_i)(\tilde{\lambda}_{i+1} - \tilde{\lambda}_i) \leq s) \longrightarrow \int_0^s p(t)dt \ . \tag{1.1.15}$$

As will be shown in Chapters 2 and 3, fixed eigenvalue gaps are universal for β-ensembles with regular one-cut potentials (see also [20]), and regular multicut potentials. Their convergence can be obtained using the translation invariance of the eigenvalue gaps as in [42].

At the Edge

Eigenvalues at the edge of the spectrum have a different behaviour than in the bulk. They typically scale as $N^{2/3}$ and C.Tracy and H.Widom showed (see [82]) that for the GUE

$$\mathbb{P}^N_{GUE}(N^{2/3}(\lambda_1 + 2) \leq s) \longrightarrow F(s) \ , \tag{1.1.16}$$

Where F is the so-called Tracy-Widom law.

Correlation functions

Universality at the microscopic levels can also be stated in terms of the correlation functions. We define the $k - point$ correlation function by

$$\rho^N_k(\lambda_1, \cdots \lambda_k) = \int \rho^N(\lambda_1, \ldots, \lambda_N) d\lambda_{k+1} \cdots d\lambda_N \ . \tag{1.1.17}$$

Here ρ^N denotes the density of the unordered eigenvalues for a Matrix Ensemble. Thus we have for all continous and compactly supported function f on $\mathbb{R}^k$

$$\mathbb{E}^N\Big(\sum_{i_1 \neq \cdots \neq i_k} f(\lambda_{i_1}, \cdots, \lambda_{i_k})\Big) = \frac{N!}{(N-k)!} \int f(\lambda_1, \cdots, \lambda_k) \rho^N_k(\lambda_1, \cdots \lambda_k) d\lambda_1 \cdots d\lambda_k \ . \tag{1.1.18}$$

With these notations, the Law of Large Numbers states that the one-point correlation function converges towards a probability density. At the microscopie level, it was shown by by F. Dyson, M. Gaudin, M. Mehta (see [65]) using orthogonal polynomials that for the GUE and $E \in (-2; 2)$ fixed

$$
\begin{aligned}
&\mathbb{E}^N_{GUE}\Big(\sum_{i_1\neq\cdots\neq i_k} f\big(N\mu_{sc}(E)(\lambda_{i_1}-E),\cdots,N\mu_{sc}(E)(\lambda_{i_k}-E)\big)\Big)\\
&=\frac{N!}{(N-k)!N^k}\frac{1}{\mu_{sc}(E)^k}\int f(\lambda_1,\cdots,\lambda_k)\rho^N_k(NE+\frac{\lambda_1}{\mu_{sc}(E)},\cdots,NE+\frac{\lambda_k}{\mu_{sc}(E)})d\lambda_1\cdots d\lambda_k\\
&\longrightarrow\int f(\lambda_1,\cdots,\lambda_k)\det(K(\lambda_i,\lambda_j)_{i,j\leq k})d\lambda_1\cdots d\lambda_k
\end{aligned}
\tag{1.1.19}
$$

where K is the sine kernel:

$$
K(x,y)=\frac{\sin(x-y)}{x-y}.
$$

This is proved using the fact that the eigenvalues of the GUE form a determinantal point process, i.e its correlation functions can be written explicitely in terms of a determinant $\det(K^N(\lambda_i,\lambda_j)_{i,j\leq k})$, where K^N can be written in terms of the Hermite orthogonal polynomials. For $\beta\neq 2$, it is not true anymore that the point process formed by the eigenvalues from a β-ensemble with Gaussian potential $\beta\lambda^2/4$ is determinantal, and one has to use alternate techniques to find the limit in law of this point process. In fact, A. Edelman and I. Dumutriu showed in [37] that we can realize these eigenvalues as the eigenvalues of a random symmetric tridiagonal matrix. Using this representation, B. Valkó and B. Virág showed in [84] that the point process $(N\mu_{sc}(E)(\lambda_n-E))_{n\leq N}$ converges in law with respect to the counting measure towards a point process called the $Sine_\beta$ point process (see also [85] for an alternative description). At the edge, and extending (1.1.16), it has also been shown by J. Ramírez, B. Rider and B. Virág in [72] that the eigenvalues are described by an operator called the Stochacstic Airy Operator and the k first recaled eigenvalues $(N^{2/3}(\lambda_1+2),\cdots,N^{2/3}(\lambda_k+2))$ converge in distribution to $(\Lambda_1,\cdots,\Lambda_k)$ where Λ_i is the i-th smallest eigenvalue of the stochastic Airy operator SAO_β.

Universality of the correlation functions (in the weak sense) in the bulk states that if we fix an energy level $E\in\text{supp}(\mu_V)$ such that $\mu_V(E)>0$, we have for all smooth compactly suppoerted function f

$$
\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}^N_{V,\beta}\Big(\sum_{i_1\neq\cdots\neq i_k} f\big(N\mu_V(E)(\lambda_{i_1}-E),\cdots,N\mu_V(E)(\lambda_{i_k}-E)\big)\Big)\\
&=\lim_{N\to\infty}\mathbb{E}^N_{G\beta E}\Big(\sum_{i_1\neq\cdots\neq i_k} f\big(N\mu_{sc}(0)\lambda_{i_1},\cdots,N\mu_{sc}(0)\lambda_{i_k}\big)\Big),
\end{aligned}
\tag{1.1.20}
$$

Note that the latter limit depends neither in E or V, and is described by the $Sine_\beta$ process. It is often easier to prove "averaged energy universality", where we consider the previous quantity averaged on a window of size $\eta=N^{-1+\varepsilon}$ with $0<\varepsilon<1$. Universality for β-ensembles at the edge usually refers to the convergence in law of the first rescaled eigenvalues of non-critical β-ensembles to those of the stochastic Airy operator SAO_β. A different limit is expected to appear for critical potentials, and the $N^{2/3}$ scaling would differ.

Universality was proved first for invariant ensembles in the case $\beta = 2$ using the Christoffel-Darboux formula for orthogonal polynomials (see [70] [69]), and subsequently for other classical values of $\beta = 1, 4$ in [31],[32], both in the bulk and at the edge of the spectrum. Universality of the correlation functions was proved more recently by P. Bourgade, L. Erdös and H.T. Yau for general non-critical β-ensembles with one-cut potentials in [20, 19, 18], with various assumptions on the potential (eventually $V \in C^4$), both in the bulk and at the edge. Universality of the correlation functions in the bulk was subsequently proved in the multicut case for analytic potential in [78], using change of variable methods. Similar statements hold for Wigner Ensembles (see [40, 43, 42, 81, 80]).

The main goal of these Thesis is to investigate some of the universal features of β-ensembles. More specifically we will focus on

- Universality of the eigenvalue gaps, in the bulk and at the edge. Chapter 2 treats the single cut case, whereas Chapter 3 deals with the multicut case.

- Central Limit Theorems at all mesoscopic scales in the one cut regime in Chapter 4.

- Central Limit Theorems at the macroscopic scale in the multicut and critical cases in Chapter 5.

Each Chapter corresponds to a different article. We briefly present here the content of each Chapter, and the main results they contain.

1.2 Results

The second Chapter corresponds to the article [11] written in collaboration with A. Figalli and A. Guionnet. In this Chapter, we consider the probability density (1.1.2), where V is a potential such that the equlibrium measure is single cut and non critical, and we want to investigate universality of the rescaled eigenvalue gaps. If we are given two potentials $V, W : \mathbb{R} \to \mathbb{R}$, we know from optimal transport theory (see [86]) that there is a transport map $T^N : \mathbb{R}^N \to \mathbb{R}^N$ that transports $\mathbb{P}^N_{V,\beta}$ to $\mathbb{P}^N_{V+W,\beta}$, i.e such that for all bounded measurable function $f : \mathbb{R}^N \to \mathbb{R}$

$$\int f \circ T^N(\lambda_1, \ldots, \lambda_N)\, \mathbb{P}^N_{V,\beta}(d\lambda_1, \ldots, d\lambda_N) = \int f(\lambda_1, \ldots, \lambda_N)\, \mathbb{P}^N_{V+W,\beta}(d\lambda_1, \ldots, d\lambda_N). \tag{1.2.21}$$

However it is not clear how does this map depend on N and how to deduce universality results from it. The main idea behind this Chapter is to find an approximate transport map that will satisfy (1.2.21) up to an error $(\log N)^3 \|f\|_\infty / N$ and for which we have an explicit expansion in N. This map is obtained as the flow of an approximate solution of a linearized Monge-Ampere equation, and can be written as $T^N = (T_1^N, \cdots, T_N^N)$ with $T_i^N(\boldsymbol{\lambda}) = T_0(\lambda_i) + T_1^{N,i}(\boldsymbol{\lambda})$ where T_0 is a transport map from μ_V to μ_{V+W} and T_1^N is of order $\frac{\log N}{N}$ (see Theorem 2.1.3).

Once the linearized Monge-Ampere equation is written, the construction of the main term T_0 involves the inversion of a linear operator Ξ acting on smooth functions f by

$$\Xi f(x) := -\beta \int \frac{f(x) - f(y)}{x - y} d\mu_V(y) + V'(x) f(x), \tag{1.2.22}$$

This operator is essential to the construction of the transport map, and to the analysis of the loop equations and will appear in all subsequent Chapters. One of the key features of Ξ is that it is invertible in the space of smooth functions if the measure μ_V is single cut (modulo a constant term). The error terms are then controlled by explicit bounds on the moments of linear statistics through the use of loop equations. From this expansion, we can deduce the universality of the eigenvalue gaps in the bulk, as well as universality at the edge by a Taylor expansion. More precisely, we prove for $V, W \in C^{31}(\mathbb{R})$

Theorem 1.2.1. *For a constant C, for all $m \in \mathbb{N}^*$, $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ Lipschitz and compactly supported in $[-M; M]$ we have*

1. *In the Bulk*

$$\Big| \int f\big(N(\lambda_{i+1} - \lambda_i), \cdots, N(\lambda_{i+m} - \lambda_i)\big) d\tilde{\mathbb{P}}^N_{V+W,\beta}$$
$$- \int f\big(N(T_0)'(\lambda_i)(\lambda_{i+1} - \lambda_i), \cdots, N(T_0)'(\lambda_i)(\lambda_{i+m} - \lambda_i)\big) d\tilde{\mathbb{P}}^N_{V,\beta} \Big|$$
$$\leq C \frac{(\log N)^3}{N} \|f\|_\infty + C(\sqrt{m} \frac{(\log N)^2}{N^{1/2}} + M \frac{(\log N)}{N^{1/2}} + \frac{M^2}{N}) \|\nabla f\|_\infty$$

2. *At the Edge*

$$\Big| \int f\big(N^{2/3}(\lambda_1 - \alpha_{\overline{V+W}}), \cdots, N^{2/3}(\lambda_m - \alpha_{\overline{V+W}})\big) d\tilde{\mathbb{P}}^N_{V+W,\beta}$$
$$- \int f\big(N^{2/3}(T_0)'(\alpha_{\bar{V}})(\lambda_1 - \alpha_{\bar{V}}), \cdots, N^{2/3}(T_0)'(\alpha_{\bar{V}})(\lambda_m - \alpha_{\bar{V}})\big) d\tilde{\mathbb{P}}^N_{V,\beta} \Big|$$
$$\leq C \frac{(\log N)^3}{N} \|f\|_\infty + C(\sqrt{m} \frac{(\log N)^2}{N^{5/6}} + \frac{\log N}{N^{1/3}} + \frac{M^2}{N^{4/3}}) \|\nabla f\|_\infty$$

where $\tilde{\mathbb{P}}^N_\beta$ denotes the measure on the ordered eigenvalues.

This establishes universality of the fixed eigenvalue gaps in the bulk and at the edge for β-ensembles in the single cut non critical regime.

The third Chapter extends the result to the multicut case and corresponds to [10]. In this Chapter only, the potential will be assumed to be analytic. The main obstruction in carrying the approach taken in the previous Chapter is that the operator Ξ defined in (1.2.22) is no longer invertible in the multicut case. To deal with this, one has to consider the auxiliary fixed filling fractions model, introduced in [15, 17]. The fixed filling fractions model correspond to the β-ensemble model in which the number of particles in each cut is conditionned to be fixed. One can then do a transport between the measure with fixed filling fractions, and a measure in which the interaction between the cuts has been removed. As the latter measure corresponds just to a product measure of β-ensembles in the single cut regime, one can use the results from the previous Chapter to study the local fluctuations. The sharp concentration estimates derived in [15] on the number of particles in each cut then allow us to extend the results obtained in the fixed filling fractions model to the original one. This establishes fixed eigenvalue gaps universality in the bulk, and universality at the edge (see Theorems 3.1.3, 3.1.4 and 3.1.5) in the multicut case.

In the two previous Chapter, we investigated the local fluctuations of the particles. In the two last Chapters, we focus on the fluctuations of the linear statistics at the mesoscopic and macroscopic scales, and prove Central Limit Theorems. The fourth Chapter describes the results obtained in the article [13] written with A. Lodhia about the Central Limit Theorem at mesoscopic scales for general β-ensembles, with single cut regular potential. More precisely the main point of this Chapter is to provide a proof of the following

Theorem 1.2.2. *Let $0 < \alpha < 1$, E a point in the bulk, $V \in C^7(\mathbb{R})$ and $f \in C^6(\mathbb{R})$ with compact support. Then, under $\mathbb{P}_V^N$*

$$\sum_{i=1}^{N} f\Big(N^\alpha(\lambda_i - E)\Big) - N \int f(N^\alpha(x-E))d\mu_V(x) \xrightarrow{\mathcal{M}} \mathcal{N}(0,\sigma_f^2) ,$$

where the convergence holds in moments (and thus in distribution), and

$$\sigma_f^2 = \frac{1}{2\beta\pi^2} \iint \left(\frac{f(x)-f(y)}{x-y}\right)^2 dxdy .$$

The main idea is to use the rigidity estimates proved in [19] to obtain a bound on the linear statistics with high probability, and use the loop equations described in Proposition 4.2.1 to compute recursively the moments of these linear statistics. The operator Ξ appears naturally in this context and it is necessary to invert it in order to compute the moments of order $n+2$ in terms of the moments of order n.

The recent article [12] written in collaboration with T. Leblé and S. Serfaty constitutes the final Chapter and investigates the global fluctuations of β-ensembles with general potential, including the multicut and critical cases. As in these cases the operator Ξ is not invertible one cannot use the loop equations to deduce a Central Limit Theorem in a straightforward manner as in the previous Chapter. A central observation, and trick that was already used in [51] is that we can express the Laplace transform of the linear statistics as a ratio of partition functions with different potentials: for a measurable bounded function ξ and $s \in \mathbb{R}$

$$\mathbb{E}_{V,\beta}^N\Big(\exp(s\sum \xi(\lambda_i)\Big) = \frac{Z_{V_s,\beta}^N}{Z_{V,\beta}^N} ,$$

where $V_s = V - \frac{s\xi}{N}$. The idea is then to use the splitting formula (5.2.21), and do a transport between the equilibrium measures μ_V and μ_{V_s} in order to compare the partition functions

$Z^N_{V_s,\beta}$ and $Z^N_{V,\beta}$. One issue that we have is that once again, doing an exact transport for all test functions ξ is only possible when Ξ is invertible, in the single cut non critical regime that is. However, in the multicut and critical regimes, we are able to do a transport between μ_V and an approximation $\tilde{\mu}_{V_s}$ of μ_{V_s} as soon as ξ lies in the image of Ξ. Using this approach we can prove the following theorem:

Theorem 1 (Central limit theorem for fluctuations of linear statistics). *Let ξ be a function in $C^{\mathsf{r}}(\mathbb{R})$, assume that H1-H3 (see Chapter 5) hold. We let*

$$\mathsf{k} = \max_{i=1,\ldots,\mathsf{m}} 2\mathsf{k}_i,$$

where the k_i's are as in (5.1.11), and assume that, p (resp. r) denoting the regularity of V (resp. ξ)

$$\mathsf{p} \geq (3\mathsf{k}+5), \quad \mathsf{r} \geq (2\mathsf{k}+3). \tag{1.2.23}$$

If $\mathsf{n} \geq 1$, assume that ξ satisfies the n following conditions

$$\int_{\Sigma_V} \frac{\xi(y)y^d}{\sigma(y)} dy = 0 \qquad \text{for } d = 0, \ldots, \mathsf{n}-1. \tag{1.2.24}$$

Moreover, if $\mathsf{m} \geq 1$, assume that for all $i = 1, \ldots, \mathsf{m}$

$$\int_{\Sigma_V} \frac{\xi(y) - R_{s_i,d}\xi(y)}{\sigma(y)(y-s_i)^d} dy = 0 \qquad \text{for } d = 1, \ldots, 2\mathsf{k}_i, \tag{1.2.25}$$

where $R_{x,d}\xi$ is the Taylor expansion of ξ to order $d-1$ around x given by

$$R_{x,d}\xi(y) = \xi(x) + (y-x)\xi'(x) + \cdots + \frac{(y-x)^{d-1}}{(d-1)!}\xi^{(d-1)}(x).$$

Then there exists a constant c_ξ and a function ψ of class C^2 in some open neighborhood U of Σ_V such that $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$ on U, and the fluctuation $\mathrm{Fluct}_N(\xi)$ *converges in law as $N \to \infty$ to a Gaussian distribution with mean*

$$m_\xi = \left(\frac{\beta}{2} - 1\right) \int \psi' \, d\mu_V,$$

and variance

$$v_\xi = -\int \psi\xi' d\mu_V.$$

This approach also yields a rate of convergence of the Laplace transform in the single cut regular cases.

1.3 Future Directions

We discuss some possible continuations of the work done in this thesis.

Universality in the critical case

A natural extension of the Chapters 3 and 4 would be to investigate whether universality also holds for critical potentials. We distinguish two cases: universality in the bulk and universality at the edge. In the bulk, the criticality should not play a role and the $Sine_\beta$ process should still be the limiting point process. At the edge, everything changes and one should expect different limits (see [28], [27] for the description of the possible limits in the case $\beta = 2$, and [53] Section 13 for general β).

Fluctuations of the linear statistics at the edge

In Chapter 5, instead of considering a point E in the bulk, we can consider E as one of the endpoints. We expect the same result to hold with a very similar proof, but with a different variance for the limiting Gaussian

$$\sigma_f^2 = \frac{1}{2\beta\pi^2} \int_0^\infty \int_0^\infty \left(\frac{f(x) - f(y)}{x - y} \right)^2 \frac{x+y}{\sqrt{xy}} dxdy.$$

The only (mild) challenge would be to extend the lemma 4.2.5.

CLT at the mesoscopic scale in the critical and multicut cases

We expect the approach taken in Chapter 5 to carry on to the mesoscopic scale. Some extra care is needed when we compare the measure μ_{V_s} and its approximation $\tilde{\mu}_{V_s}$, as these measure would also depend on N in the mesoscopic case, and the current estimate (5.3.32) would blow up.

Universality for higher dimensional Coulomb gases

Recently, a lot of work has been done to understand the local (see [25, 57, 58, 9]) and global (see [59, 7]) behaviour of higher dimensional Coulomb gases. Rigidity of the particles has been proved in two-dimensions, and a Central Limit Theorem on the fluctuations of the linear statistics as well. However, at this point, there is no statement or proof about universality in higher dimensions. It would be interesting to see if the transport methods would apply in that context.

Chapter 2

Transport maps for β-matrix models and Universality

This Chapter is based on the article [11] written with A. Figalli and A. Guionnet.

2.1 Introduction.

Given a potential $V : \mathbb{R} \to \mathbb{R}$ and $\beta > 0$, we consider the β-ensemble

$$\mathbb{P}_V^N(d\lambda_1, \dots, d\lambda_N) := \frac{1}{Z_V^N} \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} d\lambda_1 \dots d\lambda_N, \tag{2.1.1}$$

where $Z_V^N := \int \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} d\lambda_1 \dots d\lambda_N$.
We assume that V goes to infinity faster than $\beta \log|x|$ (that is $V(x)/\beta\log|x| \to +\infty$ as $|x| \to +\infty$) so that in particular Z_V^N is finite.

We will use μ_V to denote the equilibrium measure, which is obtained as limit of the spectral measure and is characterized as the unique minimizer (among probability measures) of

$$I_V(\mu) := \frac{1}{2}\int \Big(V(x) + V(y) - \beta \log|x-y|\Big) d\mu(x) d\mu(y)\,. \tag{2.1.2}$$

We assume hereafter that another smooth potential W is given so that $V + W$ goes to infinity faster than $\beta \log|x|$. We set $V_t := V + tW$, and we shall make the following assumption:

Hypothesis 2.1.1. *We assume that μ_{V_0} and μ_{V_1} have a connected support and are non-critical, that is, there exists a constant $\bar{c} > 0$ such that, for $t = 0, 1$,*

$$\frac{d\mu_{V_t}}{dx} = S_t(x)\sqrt{(x-a_t)(b_t-x)} \qquad \textit{with } S_t \ge \bar{c} \textit{ a.e. on } [a_t, b_t].$$

Finally, we assume that the eigenvalues stay in a neighborhood of the support $[a_t - \varepsilon, b_t + \varepsilon]$ with large enough $\mathbb{P}_V^N$-probability, that is with probability greater than $1 - C\,N^{-p}$ for some p large enough. By [15, Lemma 3.1], the latter is fulfilled as soon as:

Hypothesis 2.1.2. *For $t = 0, 1$,*

$$U_{V_t}(x) := V_t(x) - \beta \int d\mu_{V_t}(y) \log |x - y| \tag{2.1.3}$$

achieves its minimal value on $[a, b]^c$ at its boundary $\{a, b\}$

All these assumptions are verified for instance if V_t is uniformly convex for $t = 0, 1$.

The main goal of this Chapter is to build an approximate transport map between $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$: more precisely, we construct a map $T^N : \mathbb{R}^N \to \mathbb{R}^N$ such that, for any bounded measurable function χ,

$$\left| \int \chi \circ T^N \, d\mathbb{P}_V^N - \int \chi \, d\mathbb{P}_{V+W}^N \right| \le C \frac{(\log N)^3}{N} \|\chi\|_\infty \tag{2.1.4}$$

for some constant C independent of N, and which has a very precise expansion in the dimension (in the following result, $T_0 : \mathbb{R} \to \mathbb{R}$ is a smooth transport map of μ_V onto μ_{V+W}, see Section 2.4):

Theorem 2.1.3. *Assume that V, W are of class C^{31} and satisfy Hypotheses 2.1.1 and 2.1.2. Then there exists a map $T^N = (T^{N,1}, \ldots, T^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ which satisfies* (2.1.4) *and has the form*

$$T^{N,i}(\hat{\lambda}) = T_0(\lambda_i) + \frac{1}{N} T_1^{N,i}(\hat{\lambda}) \quad \forall i = 1, \ldots, N, \qquad \hat{\lambda} := (\lambda_1, \ldots, \lambda_N),$$

where $T_0 : \mathbb{R} \to \mathbb{R}$ and $T_1^{N,i} : \mathbb{R}^N \to \mathbb{R}$ are smooth and satisfy uniform (in N) regularity estimates. More precisely, T^N is of class C^{23} and we have the decomposition $T_1^{N,i} = X_1^{N,i} + \frac{1}{N} X_2^{N,i}$ where

$$\sup_{1 \le k \le N} \|X_1^{N,k}\|_{L^4(\mathbb{P}_V^N)} \le C \log N, \qquad \|X_2^N\|_{L^2(\mathbb{P}_V^N)} \le C N^{1/2} (\log N)^2, \tag{2.1.5}$$

for some constant $C > 0$ independent of N. In addition, with probability greater than $1 - N^{-N/C}$,

$$\max_{1 \le k, k' \le N} |X_1^{N,k}(\lambda) - X_1^{N,k'}(\lambda)| \le C \log N \sqrt{N} |\lambda_k - \lambda_{k'}|. \tag{2.1.6}$$

As we shall see in Section 2.5, this result implies universality as follows (compare with [19, Theorem 2.4]):

Theorem 2.1.4. *Assume $V, W \in C^{31}$, and let T_0 be as in Theorem 2.1.3 above. Denote $\tilde{P}_V^N$ the distribution of the increasingly ordered eigenvalues λ_i under $\mathbb{P}_V^N$. There exists a constant $\hat{C} > 0$, independent of N, such that the following two facts hold true:*

1. *Let $M \in (0,\infty)$ and $m \in \mathbb{N}$. For any Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$ supported inside $[-M,M]^m$,*

$$\Big| \int f\Big(N(\lambda_{i+1}-\lambda_i),\dots,N(\lambda_{i+m}-\lambda_i)\Big)\, d\tilde P^N_{V+W} - \int f\Big(T_0'(\lambda_i)N(\lambda_{i+1}-\lambda_i),\dots,T_0'(\lambda_i)N(\lambda_{i+m}-\lambda_i)\Big) d\tilde P^N_V \Big| \le \hat C\, \frac{(\log N)^3}{N}\|f\|_\infty + \hat C\left(\sqrt{m}\,\frac{(\log N)^2}{N^{1/2}} + M\,\frac{\log N}{N^{1/2}} + \frac{M^2}{N}\right)\|\nabla f\|_\infty.$$

2. *Let a_V (resp. a_{V+W}) denote the smallest point in the support of μ_V (resp. μ_{V+W}), so that* $\mathrm{supp}(\mu_V) \subset [a_V,\infty)$ *(resp.* $\mathrm{supp}(\mu_{V+W}) \subset [a_{V+W},\infty)$*). Let $M \in (0,\infty)$. Then, for any Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$ supported inside $[-M,M]^m$,*

$$\Big| \int f\Big(N^{2/3}(\lambda_1-a_{V+W}),\dots,N^{2/3}(\lambda_m-a_{V+W})\Big)\, d\tilde P^N_{V+W} - \int f\Big(N^{2/3}T_0'(a_V)\big(\lambda_1-a_V\big),\dots,N^{2/3}T_0'(a_V)\big(\lambda_m-a_V\big)\Big)\, d\tilde P^N_V \Big| \le \hat C\, \frac{(\log N)^3}{N}\|f\|_\infty + \hat C\left(\sqrt{m}\,\frac{(\log N)^2}{N^{5/6}} + M\,\frac{\log N}{N^{5/6}} + \frac{M^2}{N^{4/3}} + \frac{\log N}{N^{1/3}}\right)\|\nabla f\|_\infty.$$

The same bound holds around the largest point in the support of μ_V.

Remark 2.1.5. The condition that $V, W \in C^{31}$ in the theorem above is clearly non-optimal (compare with [20]). For instance, by using Stieltjes transform instead of Fourier transform in some of our estimates, we could reduce the regularity assumptions on V, W to C^{21} by a slightly more cumbersome proof. In addition, by using [19, Theorem 2.4] we could also weaken our regularity assumptions in Theorem 2.1.3, as we could use that result to estimate the error terms in Section 2.3.4. However, the main point of this hypothesis for us is to stress that we do not need to have analytic potentials, as often required in matrix models theory. Moreover, under this assumption we can provide self-contained and short proofs of Theorems 2.1.3 and 2.1.4.

Our strategy is very robust and flexible. For instance, although we shall not pursue this direction here, it is possible using this strategy to prove universality of the correlation functions (as was done in [20] and [78]) on an averaged window, using the same arguments as in [44], Lemma 4.1.

The Chapter is structured as follows: In Section 2.2 we describe the general strategy to construct our transport map as the flow of vector fields obtained by approximately solving a linearization of the Monge-Ampère equation (see (2.2.8)). As we shall explain there, this idea comes from optimal transport theory. In Section 2.3 we make an ansatz on the structure of an approximate solution to (2.2.8) and we show that our ansatz actually provides a smooth solution which enjoys very nice regularity estimates that are uniform as $N \to \infty$. In Section 2.4 we reconstruct the approximate transport map from $\mathbb{P}^N_V$ to $\mathbb{P}^N_{V+W}$ via a flow argument. The estimates proved in this section will be crucial in Section 2.5 to show universality.

2.2 Approximate Monge-Ampère equation

2.2.1 Propagating the hypotheses

The central idea of the Chapter is to build transport maps as flows, and in fact to build transport maps between $\mathbb{P}_V^N$ and $\mathbb{P}_{V_t}^N$ where $t \mapsto V_t$ is a smooth function so that $V_0 = V$, $V_1 = V + W$. In order to have a good interpolation between V and $V + W$, it will be convenient to assume that the support of the two equilibrium measures μ_V and μ_{V+W} (see (2.1.2)) are the same. This can always be done up to an affine transformation. Indeed, if $L : \mathbb{R} \to \mathbb{R}$ is the affine transformation which maps $[a_1, b_1]$ (the support of μ_{V_1}) onto $[a_0, b_0]$ (the support of μ_{V_0}), we first construct a transport map from $\mathbb{P}_V^N$ to $L_\sharp^{\otimes N}\mathbb{P}_{V+W}^N = \mathbb{P}_N^{V+\tilde{W}}$ where

$$\tilde{W} = V \circ L^{-1} + W \circ L^{-1} - V, \tag{2.2.7}$$

and then we simply compose our transport map with $(L^{-1})^{\otimes N}$ to get the desired map from $\mathbb{P}_V^N$ to $\mathbb{P}_{V+W}^N$. Hence, without loss of generality we will hereafter assume that μ_V and μ_{V+W} have the same support. We then consider the interpolation μ_{V_t} with $V_t = V + tW$, $t \in [0, 1]$. We have:

Lemma 2.2.1. *If Hypotheses 2.1.1 and 2.1.2 are fulfilled for* $t = 0, 1$, *then Hypothesis 2.1.1 is also fulfilled for all* $t \in [0, 1]$. *Moreover, we may assume without loss of generality that* V *goes to infinity as fast as we want up to modify* $\mathbb{P}_V^N$ *and* $\mathbb{P}_{V+W}^N$ *by a negligible error (in total variation).*

Proof. Let Σ denote the support of μ_V and μ_{V+W}. Following [16, Lemma 5.1], the measure μ_{V_t} is simply given by

$$\mu_{V_t} = (1-t)\mu_V + t\mu_{V+W}.$$

Indeed, μ_V is uniquely determined by the fact that there exists a constant c such that

$$\beta \int \log|x-y| d\mu_V(x) - V \le c$$

with equality on the support of μ_V, and this property extends to linear combinations. As a consequence the support of μ_{V_t} is Σ, and its density is bounded away from zero on Σ. This shows that Hypothesis 2.1.1 is fulfilled for all $t \in [0, 1]$.

Furthermore, we can modify $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$ outside an open neighborhood of Σ without changing the final result, as eigenvalues will quit this neighborhood only with very small probability under our assumption of non-criticality according to the large deviation estimates developped in [16] and culminating in [15] as follows:

$$\limsup_{N\to\infty} \frac{1}{N} \ln \mathbb{P}_N^V [\exists i \,:\, \lambda_i \in F] \le -\frac{\beta}{2} \inf_{x\in F} \tilde{U}_V(x),$$
$$\liminf_{N\to\infty} \frac{1}{N} \ln \mathbb{P}_N^V [\exists i \,:\, \lambda_i \in \Omega] \ge -\frac{\beta}{2} \inf_{x\in \Omega} \tilde{U}_V(x).$$

where $\tilde{U}_V := U_V - \inf U_V$, and U_V is defined as in (2.1.2). □

Thanks to the above lemma and the discussion immediately before it, we can assume that μ_V and μ_{V+W} have the same support, that W is bounded, and that V goes to infinity faster than x^p for some $p > 0$ large enough.

2.2.2 Monge-Ampère equation

Given the two probability densities $\mathbb{P}^N_{V_t}$ to $\mathbb{P}^N_{V_s}$ as in (2.1.1) with $0 \le t \le s \le 1$, by optimal transport theory it is well-known that there exists a (convex) function $\phi^N_{t,s}$ such that $\nabla\phi^N_{t,s}$ pushes forward $\mathbb{P}^N_{V_t}$ onto $\mathbb{P}^N_{V_s}$ and which satisfies the Monge-Ampère equation

$$\det(D^2\phi^N_{t,s}) = \frac{\rho_t}{\rho_s(\nabla\phi^N_{t,s})}, \qquad \rho_\tau := \frac{d\mathbb{P}^N_{V_\tau}}{d\lambda_1 \dots d\lambda_N}$$

(see for instance [86, Chapters 3 and 4] or the recent survey paper [29] for an account on optimal transport theory and its link to the Monge-Ampère equation).

Because $\phi_{t,t}(x) = |x|^2/2$ (since $\nabla\phi_{t,t}$ is the identity map), we can differentiate the above equation with respect to s and set $s = t$ to get

$$\Delta\psi^N_t = c^N_t - \beta\sum_{i<j}\frac{\partial_i\psi^N_t - \partial_j\psi^N_t}{\lambda_i - \lambda_j} + N\sum_i W(\lambda_i) + N\sum_i V_t'(\lambda_i)\partial_i\psi^N_t, \tag{2.2.8}$$

where $\psi^N_t := \partial_s\phi^N_{t,s}|_{s=t}$ and

$$c^N_t := -N\int\sum_i W(\lambda_i)\, d\mathbb{P}^N_{V_t} = \partial_t \log Z^N_{V_t}.$$

Although this is a formal argument, it suggests to us a way to construct maps $T^N_{0,t} : \mathbb{R}^N \to \mathbb{R}^N$ sending $\mathbb{P}^N_V$ onto $\mathbb{P}^N_{V_t}$: indeed, if $T^N_{0,t}$ sends $\mathbb{P}^N_V$ onto $\mathbb{P}^N_{V_t}$ then $\nabla\phi^N_{t,s} \circ T^N_{0,t}$ sends $\mathbb{P}^N_V$ onto $\mathbb{P}^N_{V_s}$. Hence, we may try to find $T^N_{0,s}$ of the form $T^N_{0,s} = \nabla\phi^N_{t,s} \circ T^N_{0,t} + o(s-t)$. By differentiating this relation with respect to s and setting $s = t$ we obtain $\partial_t T^N_{0,t} = \nabla\psi^N_t(T^N_{0,t})$.

Thus, to construct a transport map T^N from $\mathbb{P}^N_V$ onto $\mathbb{P}^N_{V+W}$ we could first try to find ψ^N_t by solving (2.2.8), and then construct T^N solving the ODE $\dot{X}^N_t = \nabla\psi^N_t(X^N_t)$ and setting $T^N := X^N_1$. We notice that, in general, T^N is not an optimal transport map for the quadratic cost.

Unfortunately, finding an exact solution of (2.2.8) enjoying "nice" regularity estimates that are uniform in N seems extremely difficult. So, instead, we make an ansatz on the structure of ψ^N_t (see (2.3.12) below): the idea is that at first order eigenvalues do not interact, then at order $1/N$ eigenvalues interact at most by pairs, and so on. As we shall see, in order to construct a function which enjoys nice regularity estimates and satisfies (2.2.8) up to a error that goes to zero as $N \to \infty$, it will be enough to stop the expansion at $1/N$. Actually, while the argument before provides us the right intuition, we notice that there is no need to assume that the vector field generating the flow X^N_t is a gradient, so we will consider general vector fields $\mathbf{Y}^N_t = (\mathbf{Y}^N_{1,t}, \dots, \mathbf{Y}^N_{N,t}) : \mathbb{R}^N \to \mathbb{R}^N$ that approximately solve

$$\mathrm{div}\mathbf{Y}^N_t = c^N_t - \beta\sum_{i<j}\frac{\mathbf{Y}^N_{i,t} - \mathbf{Y}^N_{j,t}}{\lambda_i - \lambda_j} + N\sum_i W(\lambda_i) + N\sum_i V_t'(\lambda_i)\mathbf{Y}^N_{i,t}, \tag{2.2.9}$$

We begin by checking that the flow of an approximate solution of (2.2.9) gives an approximate transport map.

2.2.3 Approximate Jacobian equation

Here we show that if a C^1 vector field $\mathbf{Y}_t^N$ approximately satisfies (2.2.9), then its flow

$$\dot{X}_t^N = \mathbf{Y}_t^N(X_t^N), \qquad X_0^N = \mathrm{Id},$$

produces almost a transport map.

More precisely, let $\mathbf{Y}_t^N : \mathbb{R}^N \to \mathbb{R}^N$ be a smooth vector field and denote

$$\mathcal{R}_t^N(\mathbf{Y}^N) := c_t^N - \beta \sum_{i<j} \frac{\mathbf{Y}_{i,t}^N - \mathbf{Y}_{j,t}^N}{\lambda_i - \lambda_j} + N \sum_i W(\lambda_i) + N \sum_i V_t'(\lambda_i) \mathbf{Y}_{i,t}^N - \mathrm{div} \mathbf{Y}_t^N.$$

Lemma 2.2.2. *Let $\chi : \mathbb{R}^N \to \mathbb{R}$ be a bounded measurable function, and let X_t^N be the flow of $\mathbf{Y}_t^N$. Then*

$$\left| \int \chi(X_t^N)\, d\mathbb{P}_V^N - \int \chi\, d\mathbb{P}_{V_t}^N \right| \leq \|\chi\|_\infty \int_0^t \|\mathcal{R}_s^N(\mathbf{Y}^N)\|_{L^1(\mathbb{P}_{V_s}^N)} ds.$$

Proof. Since $\mathbf{Y}_t^N \in C^1$, by Cauchy-Lipschitz Theorem its flow is a bi-Lipchitz homeomorphism.

If JX_t^N denotes the Jacobian of X_t^N and ρ_t the density of $\mathbb{P}_{V_t}^N$, by the change of variable formula it follows that

$$\int \chi\, d\mathbb{P}_{V_t}^N = \int \chi(X_t^N) JX_t^N \rho_t(X_t^N)\, dx$$

thus

$$\left| \int \chi(X_t^N)\, d\mathbb{P}_V^N - \int \chi\, d\mathbb{P}_{V_t}^N \right| \leq \|\chi\|_\infty \int |\rho_0 - JX_t^N \rho_t(X_t^N)|\, dx =: \|\chi\|_\infty A_t \tag{2.2.10}$$

Using that $\partial_t(JX_t^N) = \mathrm{div}\mathbf{Y}_t^N\, JX_t^N$ and that the derivative of the norm is smaller than the norm of the derivative, we get

$$\begin{aligned} |\partial_t A_t| &\leq \int \left| \partial_t \Big(JX_t^N \rho_t(X_t^N) \Big) \right| dx \\ &= \int |\mathrm{div}\mathbf{Y}_t^N\, JX_t^N\, \rho_t(X_t^N) + JX_t^N\, (\partial_t \rho_t)(X_t^N) + JX_t^N\, \nabla \rho_t(X_t^N) \cdot \partial_t X_t^N|\, dx \\ &= \int |\mathcal{R}_t^N(\mathbf{Y})|(X_t^N)\, JX_t^N\, \rho_t(X_t^N)\, dx \\ &= \int |\mathcal{R}_t^N(\mathbf{Y})|\, d\mathbb{P}_{V_t}^N. \end{aligned}$$

Integrating the above estimate in time completes the proof. □

By taking the supremum over all functions χ with $\|\chi\|_\infty \leq 1$, the lemma above gives:

Corollary 2.2.3. *Let X_t^N be the flow of $\mathbf{Y}_t^N$, and set $\hat{\mathbb{P}}_t^N := (X_t^N)_\# \mathbb{P}_V^N$ the image of $\mathbb{P}_V^N$ by X_t^N. Then*

$$\|\hat{\mathbb{P}}_t^N - \mathbb{P}_{V_t}^N\|_{TV} \leq \int_0^t \|\mathcal{R}_s^N(\mathbf{Y}^N)\|_{L^1(\mathbb{P}_{V_s}^N)} ds.$$

2.3 Constructing an approximate solution to (2.2.8)

Fix $t \in [0,1]$ and define the random measures

$$L_N := \frac{1}{N}\sum_i \delta_{\lambda_i} \quad \text{and} \quad M_N := \sum_i \delta_{\lambda_i} - N\mu_{V_t}. \tag{2.3.11}$$

As we explained in the previous section, a natural ansatz to find an approximate solution of (2.2.8) is given by

$$\psi_t^N(\lambda_1,\dots,\lambda_N) := \int \Big[\psi_{0,t}(x) + \frac{1}{N}\psi_{1,t}(x)\Big]\, dM_N(x) + \frac{1}{2N}\iint \psi_{2,t}(x,y)\, dM_N(x)\, dM_N(y), \tag{2.3.12}$$

where (without loss of generality) we assume that $\psi_{2,t}(x,y) = \psi_{2,t}(y,x)$.

Since we do not want to use gradient of functions but general vector fields (as this gives us more flexibility), in order to find an ansatz for an approximate solution of (2.2.9) we compute first the gradient of ψ:

$$\partial_i\psi_t^N = \psi_{0,t}'(\lambda_i) + \frac{1}{N}\psi_{1,t}'(\lambda_i) + \frac{1}{N}\xi_{1,t}^N(\lambda_i, M_N), \qquad \xi_{1,t}^N(x, M_N) := \int \partial_1\psi_{2,t}(x,y)\, dM_N(y).$$

This suggests us the following ansatz for the components of $\mathbf{Y}_t^N$:

$$\mathbf{Y}_{i,t}^N(\lambda_1,\dots,\lambda_N) := \mathbf{y}_{0,t}(\lambda_i) + \frac{1}{N}\mathbf{y}_{1,t}(\lambda_i) + \frac{1}{N}\boldsymbol{\xi}_t(\lambda_i, M_N), \quad \boldsymbol{\xi}_t(x, M_N) := \int \mathbf{z}_t(x,y)\, dM_N(y), \tag{2.3.13}$$

for some functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t} : \mathbb{R} \to \mathbb{R}$, $\mathbf{z}_t : \mathbb{R}^2 \to \mathbb{R}$.

Here and in the following, given a function of two variables ψ, we write $\psi \in C^{s,v}$ to denote that it is s times continuously differentiable with respect to the first variable and v times with respect to the second.

The aim of this section is to prove the following result:

Proposition 2.3.1. *Assume $V, W \in C^r$ with $r \geq 31$. Then, there exist $\mathbf{y}_{0,t} \in C^{r-2}$, $\mathbf{y}_{1,t} \in C^{r-8}$, and $\mathbf{z}_t \in C^{s,v}$ for $s+v \leq r-5$, such that*

$$\mathcal{R}_t^N := \left(c_t^N - \beta\sum_{i<j}\frac{\mathbf{Y}_{i,t}^N - \mathbf{Y}_{j,t}^N}{\lambda_i - \lambda_j} + N\sum_i W(\lambda_i) + N\sum_i V_t'(\lambda_i)\mathbf{Y}_{i,t}^N\right) - \operatorname{div}\mathbf{Y}_t^N$$

satisfies

$$\|\mathcal{R}_t^N\|_{L^1(\mathbb{P}_{V_t}^N)} \leq C\,\frac{(\log N)^3}{N}$$

for some positive constant C independent of $t \in [0,1]$.

The proof of this proposition is pretty involved, and will take the rest of the section.

2.3.1 Finding an equation for $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$.

Using (2.3.13) we compute

$$\mathrm{div}\mathbf{Y}_t^N = N\int \mathbf{y}_{0,t}'(x)\,dL_N(x) + \int \mathbf{y}_{1,t}'(\lambda)\,dL_N(x) + \int \partial_1\boldsymbol{\xi}_t(x, M_N)\,dL_N(x) + \boldsymbol{\eta}(L_N),$$

where, given a measure ν, we set

$$\boldsymbol{\eta}(\nu) := \int \partial_2\mathbf{z}_t(y,y)\,d\nu(y).$$

Therefore, recalling that $M_N = N(L_N - \mu_{V_t})$, we get

$$\begin{aligned}
\mathcal{R}_t^N = &-\frac{\beta N^2}{2}\iint \frac{\mathbf{y}_{0,t}(x) - \mathbf{y}_{0,t}(y)}{x-y}\,dL_N(x)\,dL_N(y) + N^2\int V_t'\,\mathbf{y}_{0,t}\,dL_N + N^2\int W\,dL_N\\
&-\frac{\beta N}{2}\iint \frac{\mathbf{y}_{1,t}(x) - \mathbf{y}_{1,t}(y)}{x-y}\,dL_N(x)\,dL_N(y) + N\int V_t'\,\mathbf{y}_{1,t}\,dL_N\\
&-\frac{\beta N}{2}\iint \frac{\boldsymbol{\xi}_t(x,M_N) - \boldsymbol{\xi}_t(y,M_N)}{x-y}\,dL_N(x)\,dL_N(y) + N\int V_t'(x)\,\boldsymbol{\xi}_t(x,M_N)\,dL_N(x)\\
&-\frac{1}{N}\boldsymbol{\eta}(M_N) - N\Big(1-\frac{\beta}{2}\Big)\int \mathbf{y}_{0,t}'\,dL_N\\
&-\Big(1-\frac{\beta}{2}\Big)\int \mathbf{y}_{1,t}'\,dL_N - \Big(1-\frac{\beta}{2}\Big)\int \partial_1\boldsymbol{\xi}_t(x,M_N)\,dL_N(x) - \boldsymbol{\eta}(\mu_{V_t}) + \tilde{c}_t^N,
\end{aligned}$$

where $\tilde{c}_t^N$ is a constant and we use the convention that, when we integrate a function of the form $\frac{f(x)-f(y)}{x-y}$ with respect to $L_N \otimes L_N$, the diagonal terms give $f'(x)$.

We now observe that L_N converges towards μ_{V_t} as $N \to \infty$ [67], see also [6, 3] for the corresponding large deviation principle, and the latter minimizes I_{V_t} (see (2.1.2)). Hence, considering $\mu_\varepsilon := (x + \varepsilon f)_\# \mu_{V_t}$ and writing that $I_{V_t}(\mu_\varepsilon) \geq I_{V_t}(\mu_{V_t})$, by taking the derivative with respect to ε at $\varepsilon = 0$ we get

$$\int V_t'(x) f(x)\,d\mu_{V_t}(x) = \frac{\beta}{2}\iint \frac{f(x)-f(y)}{x-y}\,d\mu_{V_t}(x)\,d\mu_{V_t}(y) \tag{2.3.14}$$

for all smooth bounded functions $f : \mathbb{R} \to \mathbb{R}$. Therefore we can recenter L_N by μ_{V_t} in the formula above: more precisely, if we set

$$\Xi f(x) := -\beta\int \frac{f(x)-f(y)}{x-y}\,d\mu_{V_t}(y) + V_t'(x) f(x), \tag{2.3.15}$$

then

$$\begin{aligned}
N^2\int V_t' f\,dL_N - \frac{\beta N^2}{2}\iint \frac{f(x)-f(y)}{x-y}\,dL_N(x)\,dL_N(y)&\\
= N\int \Xi f\,dM_N - \frac{\beta}{2}\iint \frac{f(x)-f(y)}{x-y}\,dM_N(x)\,dM_N(y)&
\end{aligned}$$

Applying this identity to $f = \mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \boldsymbol{\xi}_t(\cdot, M_N)$ and recalling the definition of $\boldsymbol{\xi}_t(\cdot, M_N)$ (see (2.3.13)), we find

$$\begin{aligned}\mathcal{R}_t^N &= N \int [\Xi \mathbf{y}_{0,t} + W]\, dM_N \\ &+ \int \left(\Xi \mathbf{y}_{1,t} + \left(\frac{\beta}{2} - 1\right)\left[\mathbf{y}_{0,t}' + \int \partial_1 \mathbf{z}_t(z, \cdot) d\mu_{V_t}(z)\right]\right) dM_N \\ &+ \iint dM_N(x)\, dM_N(y) \left(\Xi \mathbf{z}_t(\cdot, y)[x] - \frac{\beta}{2} \frac{\mathbf{y}_{0,t}(x) - \mathbf{y}_{0,t}(y)}{x - y}\right) + C_t^N + E_N,\end{aligned}$$

where

$$\Xi \mathbf{z}_t(\cdot, y)[x] = -\beta \int \frac{\mathbf{z}_t(x, y) - \mathbf{z}_t(\tilde{x}, y)}{x - \tilde{x}} d\mu_{V_t}(\tilde{x}) + V_t'(x) \mathbf{z}_t(x, y),$$

C_t^N is a deterministic term, and E_N is a remainder that we will prove to be negligible:

$$\begin{aligned}E_N := &-\frac{1}{N} \int \partial_2 \mathbf{z}_t(x, x)\, dM_N(x) - \frac{1}{N}\left(1 - \frac{\beta}{2}\right) \int \mathbf{y}_{1,t}'\, dM_N \\ &- \frac{1}{N}\left(1 - \frac{\beta}{2}\right) \iint \partial_1 \mathbf{z}_t(x, y)\, dM_N(x)\, dM_N(y) \\ &- \frac{\beta}{2N} \iint \frac{\mathbf{y}_{1,t}(x) - \mathbf{y}_{1,t}(y)}{x - y} dM_N(x)\, dM_N(y) \\ &- \frac{\beta}{2N} \iiint \frac{\mathbf{z}_t(x, y) - \mathbf{z}_t(\tilde{x}, y)}{x - \tilde{x}} dM_N(x)\, dM_N(y)\, dM_N(\tilde{x}).\end{aligned} \tag{2.3.16}$$

Hence, for $\mathcal{R}_t^N$ to be small we want to impose

$$\begin{aligned}\Xi \mathbf{y}_{0,t} &= -W + c, \\ \Xi \mathbf{z}_t(\cdot, y)[x] &= -\frac{\beta}{2} \frac{\mathbf{y}_{0,t}(x) - \mathbf{y}_{0,t}(y)}{x - y} + c(y), \\ \Xi \mathbf{y}_{1,t} &= -\left(\frac{\beta}{2} - 1\right)\left[\mathbf{y}_{0,t}' + \int \partial_1 \mathbf{z}_t(z, \cdot)\, d\mu_{V_t}(z)\right] + c',\end{aligned} \tag{2.3.17}$$

where c, c' are some constant to be fixed later, and $c(y)$ does not depend on x.

2.3.2 Inverting the operator Ξ.

We now prove a key lemma, that will allow us to find the desired functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$.

Lemma 2.3.2. *Given $V : \mathbb{R} \to \mathbb{R}$, assume that μ_V has support given by $[a, b]$ and that*

$$\frac{d\mu_V}{dx}(x) = S(x)\sqrt{(x - a)(b - x)}$$

with $S(x) \geq \bar{c} > 0$ a.e. on $[a, b]$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a C^k function and assume that V is of class C^p. Set

$$\Xi f(x) := -\beta \int \frac{f(x) - f(y)}{x - y} d\mu_V(x) + V'(x) f(x)$$

Then there exists a unique constant c_g such that the equation

$$\Xi f(x) = g(x) + c_g$$

has a solution of class $C^{(k-2)\wedge(p-3)}$. More precisely, for $j \le (k-2)\wedge(p-3)$ there is a finite constant C_j such that

$$\|f\|_{C^j(\mathbb{R})} \le C_j \|g\|_{C^{j+2}(\mathbb{R})}, \tag{2.3.18}$$

where, for a function h, $\|h\|_{C^j(\mathbb{R})} := \sum_{r=0}^{j} \|h^{(r)}\|_{L^\infty(\mathbb{R})}$.

Moreover f (and its derivatives) behaves like $(g(x)+c_g)/V'(x)$ (and its corresponding derivatives) when $|x| \to +\infty$.

This solution will be denoted by $\Xi^{-1}g$.

Note that $Lf(x) = \Xi f'(x)$ can be seen as the asymptotics of the infinitesimal generator of the Dyson Brownian motion taken in the set where the spectral measure approximates μ_V. This operator is central in our approach, as much as the Dyson Brownian motion is central to prove universality in e.g. [20, 19].

Proof. As a consequence of (2.3.14), we have

$$\beta\, PV \int \frac{1}{x-y}\, d\mu_V(y) = V'(x) \qquad \text{on the support of } \mu_V. \tag{2.3.19}$$

Therefore the equation $\Xi f(x) = g(x) + c_g$ on the support of μ_V amounts to

$$\beta\, PV \int \frac{f(y)}{x-y}\, d\mu_V(y) = g(x) + c_g \qquad \forall x \in [a,b]. \tag{2.3.20}$$

Let us write

$$d(x) := d\mu_V/dx = S(x)\sqrt{(x-a)(b-x)}$$

with S positive inside the support $[a,b]$. We claim that $S \in C^{p-3}([a,b])$.

Indeed, by (2.3.14) with $f(x) = (z-x)^{-1}$ for $z \in [a,b]^c$, we find that the Stieltjes transform $G(z) = \int (z-y)^{-1}\, d\mu_V(y)$ satisfies, for z outside $[a,b]$,

$$\frac{\beta}{2} G(z)^2 = G(z)V'(\mathcal{R}(z)) + F(z), \qquad \text{with } F(z) = \int \frac{V'(y) - V'(\mathcal{R}(z))}{z-y}\, d\mu_V(y)\,.$$

Solving this quadratic equation so that $G \to 0$ as $|z| \to \infty$ yields

$$G(z) = \frac{1}{\beta}\Big(V'(\mathcal{R}(z)) - \sqrt{[V'(\mathcal{R}(z))]^2 + 2\beta F(z)}\Big). \tag{2.3.21}$$

Notice that $V'(\mathcal{R}(z))^2 + 2\beta F(z)$ becomes real as z goes to the real axis, and negative inside $[a,b]$. Hence, since $-\pi^{-1}\mathcal{I}G(z)$ converges to the density of μ_V as z goes to the real axis (see e.g [3, Theorem 2.4.3]), we get

$$-S(x)^2(x-a)(b-x) = (\beta\pi)^{-2}\left[V'(x)^2 + 2\beta F(x)\right]. \tag{2.3.22}$$

This implies in particular that $\{a, b\}$ are the two points of the real line where $V'(x)^2+2\beta F(x)$ vanishes. Moreover $F(x) = -\int\int_0^1 V''(\alpha y + (1-\alpha)x)\, d\alpha\, d\mu_V(y)$ is of class C^{p-2} on $\mathbb{R}$ (recall that $V \in C^p$ by assumption), therefore $(V')^2 + 2\beta F \in C^{p-2}(\mathbb{R})$. Since we assumed that S does not vanish in $[a, b]$, from (2.3.22) we deduce that S is of class C^{p-3} on $[a, b]$.

To solve (2.3.20) we apply Tricomi's formula [83, formula 12, p.181] and we find that, for $x \in [a, b]$,

$$\beta f(x)\sqrt{(x-a)(b-x)}d(x) = PV\int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}(g(y)+c_g)dy + c_2 := h(x)$$

for some constant c_2, hence

$$\begin{aligned} h(x) &= \beta f(x)(x-a)(b-x)S(x) \\ &= PV\int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}(g(y)+c_g)dy + c_2 \\ &= PV\int_a^b \sqrt{(y-a)(b-y)}\frac{g(y)-g(x)}{y-x}dy + (g(x)+c_g)PV\int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}dy + c_2 \\ &= \int_a^b \sqrt{(y-a)(b-y)}\frac{g(y)-g(x)}{y-x}dy - \pi\left(x - \tfrac{a+b}{2}\right)(g(x)+c_g) + c_2, \end{aligned}$$

where we used that, for $x \in [a, b]$,

$$PV\int_a^b \frac{\sqrt{(y-a)(b-y)}}{y-x}dy = -\pi\left(x - \frac{a+b}{2}\right).$$

Set

$$h_0(x) = \int_a^b \sqrt{(y-a)(b-y)}\,\frac{g(y)-g(x)}{y-x}dy.$$

Then h_0 is of class C^{k-1} (recall that g is of class C^k). We next choose c_g and c_2 such that h vanishes at a and b (notice that this choice uniquely identifies c_g).

We note that $f \in C^{(k-2)\wedge(p-3)}([a, b])$. Moreover, we can bound its derivatives in terms of the derivatives of h_0, g and S: if we assume $j \le p-3$, we find that there exists a constant C_j, which depends only on the derivatives of S, such that

$$\|f^{(j)}\|_{L^\infty([a,b])} \le C_j \max_{p\le j}\left(\|h_0^{(p+1)}\|_{L^\infty([a,b])} + \|g^{(p+1)}\|_{L^\infty([a,b])}\right) \le C_j \max_{p\le j+2}\|g^{(p)}\|_{L^\infty([a,b])}.$$

Let us define

$$k(x) := \beta\, PV\int \frac{f(y)}{x-y}d\mu_V(y) - g(x) - c_g \qquad \forall x \in \mathbb{R}.$$

By (2.3.20) we see that $k \equiv 0$ on $[a, b]$. To ensure that $\Xi f = g + c_g$ also outside the support of μ_V we want

$$f(x)\left(\beta\, PV\int \frac{1}{x-y}\, d\mu_V(y) - V'(x)\right) = k(x) \qquad \forall x \in [a, b]^c.$$

Let us consider the function $\ell : \mathbb{R} \to \mathbb{R}$ defined as

$$\ell(x) := \beta\, PV\int \frac{1}{x-y}\, d\mu_V(y) - V'(x). \qquad (2.3.23)$$

Notice that, thanks to (2.3.21), $\ell(x) = \beta G(x) - V'(x) = -\beta\sqrt{[V'(x)]^2 + 2\beta F(x)}$. Hence, comparing this expression with (2.3.22) and recalling that $S \geq \bar{c} > 0$ in $[a,b]$, we deduce that $[V'(x)]^2 + 2\beta F(x)$ is smooth and has simple zeroes both at a and b, therefore $[V'(x)]^2 + 2\beta F(x) > 0$ in $[a-\varepsilon, b+\varepsilon]\backslash[a,b]$ for some $\varepsilon > 0$.

This shows that ℓ does not vanish in $[a-\varepsilon, b+\varepsilon]\backslash[a,b]$. Recalling that we can freely modify V outside $[a-\varepsilon, b+\varepsilon]$ (see proof of Lemma 2.2.1), we can actually assume that ℓ vanishes at $\{a,b\}$ and does not vanish in the whole $[a,b]^c$.

We claim that ℓ is Hölder 1/2 at the boundary points, and in fact is equivalent to a square root there. Indeed, it is immediate to check that ℓ is of class C^{p-1} except possibly at the boundary points $\{a,b\}$. Moreover

$$PV\int \frac{1}{x-y}\,d\mu_V(y) = S(a)\int_a^b \frac{1}{x-y}\sqrt{(y-a)(b-y)}\,dy$$
$$+\int_a^b \frac{y-a}{x-y}\left(\int_0^1 S'(\alpha a + (1-\alpha)y)d\alpha\right)\sqrt{(y-a)(b-y)}\,dy.$$

The first term can be computed exactly and we have, for some $c \neq 0$,

$$\int_a^b \frac{1}{x-y}\sqrt{(y-a)(b-y)}\,dy = c(b-a)\left(\frac{x-\frac{a+b}{2}}{b-a} - \sqrt{\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2 - \frac{1}{4}}\right) \qquad (2.3.24)$$

which is Hölder 1/2, and in fact behaves as a square root at the boundary points. On the other hand, since S is of class C^{p-3} on $[a,b]$ with $p \geq 4$, the second function is differentiable, with derivative at a given by

$$\int_a^b \frac{1}{a-y}\left(\int_0^1 S'(\alpha a + (1-\alpha)y)d\alpha\right)\sqrt{(y-a)(b-y)}\,dy,$$

which is a convergent integral. The claim follows.

Thus, for x outside the support of μ_V we can set

$$f(x) := \ell(x)^{-1}k(x).$$

With this choice $\Xi f = g + c_g$ and f is of class $C^{(k-2)\wedge(p-3)}$ on $\mathbb{R}\setminus\{a,b\}$.

We now want to show that f is of class $C^{(k-2)\wedge(p-3)}$ on the whole $\mathbb{R}$. For this we need to check the continuity of f and its derivatives at the boundary points, say at a (the case of b being similar). We take hereafter $r \leq (k-2)\wedge(p-3)$, so that f has r derivatives inside $[a,b]$ according to the above considerations.

Let us first deduce the continuity of f at a. We write, with $f(a^+) = \lim_{x\downarrow a} f(x)$,

$$k(x) = f(a^+)\ell(x) + k_1(x)$$

with

$$k_1(x) := \beta\left(PV\int \frac{f(y)}{x-y}d\mu_V(y) - PV\int \frac{f(a^+)}{x-y}d\mu_V(y)\right) + g(x) + c_g + f(a^+)V_t'(x).$$

Notice that since $f = \ell^{-1}k$ outside $[a,b]$, if we can show that $\ell^{-1}(x)k_1(x) \to 0$ as $x \uparrow a$ then we would get $f(a^-) = f(a^+)$, proving the desired continuity.

To prove it we first notice that k_1 vanishes at a (since both k and ℓ vanish inside $[a,b]$), hence

$$\begin{aligned} k_1(x) &= \beta\left(PV\int \frac{f(y)-f(a^+)}{x-y}d\mu_V(y) - PV\int \frac{f(y)-f(a^+)}{a-y}d\mu_V(y)\right) + \tilde{g}(x) - \tilde{g}(a) \\ &= \beta(a-x)\,PV\int \frac{f(y)-f(a^+)}{(x-y)(a-y)}d\mu_V(y) + \tilde{g}(x) - \tilde{g}(a), \end{aligned}$$

with $\tilde{g} := g + f(a^+)V' \in C^1$. Assume $1 \le (k-2)\wedge(p-3)$. Since f is of class C^1 inside $[a,b]$ we have $|f(y)-f(a^+)| \le C|y-a|$, from which we deduce that $|k_1(x)| \le C|x-a|$ for $x \le a$.

Hence $\ell^{-1}(x)k_1(x) \to 0$ as $x \uparrow a$ (recall that ℓ behaves as a square root near a), which proves that

$$\lim_{x\uparrow a} f(x) = \lim_{x\downarrow a} f(x)$$

and shows the continuity of f at a.

We now consider the next derivative: we write

$$k(x) = \Big[f(a) + f'(a^+)(x-a)\Big]\ell(x) + k_2(x)$$

with

$$\begin{aligned} k_2(x) :=\ & \beta(a-x)\,PV\int \frac{f(y)-f(a^+)-(y-a)f'(a^+)}{(x-y)(a-y)}\,d\mu_V(y) \\ & + \tilde{g}(x) - \tilde{g}(a) + f'(a^+)(x-a)V_t'(x). \end{aligned}$$

Since $k = \ell \equiv 0$ on $[a,b]$ we have $k_2(a) = k_2'(a^+) = k_2'(a^-) = 0$. Hence, since f is of class C^2 on $[a,b]$, we see that $|k_2(x)| \le C|x-a|^2$ for $x \le a$, therefore $k_2(x)/\ell(x)$ is of order $|x-a|^{3/2}$, thus

$$f(x) = f(a) + f'(a^+)(x-a) + O(|x-a|^{3/2}) \qquad \text{for } x \le a,$$

which shows that f has also a continuous derivative.

We obtain the continuity of the next derivatives similarly. Moreover, away from the boundary point the j-th derivative of f outside $[a,b]$ is of the same order than that of g/V', while near the boundary points it is governed by the derivatives of g nearby, therefore

$$\|f^{(j)}\|_{L^\infty([a,b]^c)} \le C_j' \max_{r\le j+2} \|g^{(r)}\|_{L^\infty(\mathbb{R})}\,. \tag{2.3.25}$$

Finally, it is clear that f behaves like $(g(x)+c_g)/V'(x)$ when x goes to infinity. □

2.3.3 Defining the functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$

To define the functions $\mathbf{y}_{0,t}, \mathbf{y}_{1,t}, \mathbf{z}_t$ according to (2.3.17), notice that Lemma 2.2.1 shows that the hypothesis of Lemma 2.3.2 are fulfilled. Hence, as a consequence of Lemma 2.3.2 we find the following result (recall that $\psi \in C^{s,v}$ means that ψ is s times continuously differentiable with respect to the first variable and v times with respect to the second).

Lemma 2.3.3. *Let $r \geq 7$. If $W, V' \in C^r$, we can choose $\mathbf{y}_{0,t}$ of class C^{r-2}, $\mathbf{z}_t \in C^{s,v}$ for $s+v \leq r-5$, and $\mathbf{y}_{1,t} \in C^{r-8}$. Moreover, these functions (and their derivatives) go to zero at infinity like $1/V'$ (and its corresponding derivatives).*

Proof. By Lemma 2.3.2 we have $\mathbf{y}_{0,t} = \Xi^{-1}W \in C^{r-2}$. For $\mathbf{z}_t$, we can rewrite

$$\begin{aligned}\Xi\mathbf{z}_t(\cdot, y)[x] &= -\frac{\beta}{2}\int_0^1 \mathbf{y}'_{0,t}(\alpha x + (1-\alpha)y)\, d\alpha + c(y) \\ &= -\frac{\beta}{2}\int_0^1 [\mathbf{y}'_{0,t}(\alpha x + (1-\alpha)y) + c_\alpha(y)]\, d\alpha\end{aligned}$$

where we choose $c_\alpha(y)$ to be the unique constant provided by Lemma 2.3.2 which ensures that $\Xi^{-1}[\mathbf{y}'_{0,t}(\alpha x + (1-\alpha)y) + c_\alpha(y)]$ is smooth. This gives that $c(y) = \int_0^1 c_\alpha(y) d\alpha$. Since Ξ^{-1} is a linear integral operator, we have

$$\mathbf{z}_t(x,y) = -\frac{\beta}{2}\int_0^1 \Xi^{-1}[\mathbf{y}'_{0,t}(\alpha \cdot + (1-\alpha)y)](x)\, d\alpha\,.$$

As the variable y is only a translation, it is not difficult to check that $\mathbf{z}_t \in C^{s,v}$ for any $s+v \leq r-5$. It follows that

$$-\left(\frac{\beta}{2}-1\right)\left[\mathbf{y}'_{0,t} + \int \partial_1 \mathbf{z}_t(z,\cdot)\, d\mu_{V_t}(z)\right] + c'$$

is of class C^{r-6} and therefore by Lemma 2.3.2 we can choose $\mathbf{y}_{1,t} \in C^{r-8}$, as desired.

The decay at infinity is finally again a consequence of Lemma 2.3.2. □

2.3.4 Getting rid of the random error term E_N

We show that the $L^1_{\mathbb{P}^N_{V_t}}$-norm of the error term E_N defined in (2.3.16) goes to zero. This could be easily derived from [20], but we here provide a self-contained proof. To this end, we first make some general consideration on the growth of variances.

Following e.g. [64, Theorem 1.6], up to assume that V_t goes sufficiently fast at infinity (which we did, see Lemma 2.2.1), it is easy to show that there exists a constant $\tau_0 > 0$ so that for all $\tau \geq \tau_0$,

$$\mathbb{P}^N_{V_t}\left(D(L_N, \mu_{V_t}) \geq \tau\sqrt{\frac{\log N}{N}}\right) \leq e^{-c\tau^2 N \log N}\,,$$

where D is the 1-Wasserstein distance

$$D(\mu,\nu) := \sup_{\|f'\|_\infty \leq 1}\left|\int f(d\mu - d\nu)\right|.$$

Since $M_N = N(L_N - \mu_{V_t})$ we get

$$D(L_N, \mu_{V_t}) = \frac{1}{N}\sup_{\|f'\|_\infty \leq 1}\left|\int f\, dM_N\right|, \tag{2.3.26}$$

hence for $\tau \geq \tau_0$

$$\mathbb{P}^N_{V_t}\left(\sup_{\|f\|_{\mathrm{Lip}}\leq 1}\left|\int f\,dM_N\right| \geq \tau\sqrt{N\log N}\right) \leq e^{-c\tau^2 N\log N}. \tag{2.3.27}$$

This already shows that, if f is sufficiently smooth, $\int f(x,y)dM_N(x)\,dM_N(y)$ is of order at most $N\log N$. More precisely

$$\int f(x,y)\,dM_N(x)\,dM_N(y) = \int \hat{f}(\zeta,\xi)\left(\int e^{i\zeta x}dM_N(x)\int e^{i\xi x}dM_N(x)\right)d\xi\,d\zeta,$$

so that with probability greater than $1-e^{-c\tau_0^2 N\log N}$ we have

$$\left|\int f(x,y)\,dM_N(x)\,dM_N(y)\right| \leq \tau_0^2\, N\log N\int |\hat{f}(\zeta,\xi)|\,|\zeta|\,|\xi|\,d\zeta\,d\xi\,. \tag{2.3.28}$$

To improve this estimate, we shall use loop equations as well as Lemma 2.3.2. Given a function g and a measure ν, we use the notation $\nu(g) := \int g\,d\nu$.

Lemma 2.3.4. *Let g be a smooth function. Then, if $\tilde{M}_N = NL_N - N\mathbb{E}_{V_t}[L_N]$, there exists a finite constant C such that*

$$\begin{aligned}
\sigma_N^{(1)}(g) &:= \left|\int M_N(g)\,d\mathbb{P}^N_{V_t}\right| \leq C\,m(g) =: B_N^1(g)\\
\sigma_N^{(2)}(g) &:= \int\left(\tilde{M}_N(g)\right)^2 d\mathbb{P}^N_{V_t} \leq C\Big(m(g)^2 + m(g)\|g\|_\infty + \|\Xi^{-1}g\|_\infty\|g'\|_\infty\Big) =: B_N^2(g)\\
\sigma_N^{(4)}(g) &:= \int\left(\tilde{M}_N(g)\right)^4 d\mathbb{P}^N_{V_t}\\
&\leq C\Big(\|\Xi^{-1}g\|_\infty\|g'\|_\infty\sigma_N^{(2)}(g) + \|g\|_\infty^3 m(g) + m(g)^2\sigma_N^{(2)}(g) + m(g)^4\Big) =: B_N^4(g),
\end{aligned}$$

where

$$m(g) := \left|1-\frac{\beta}{2}\right|\|(\Xi^{-1}g)'\|_\infty + \frac{\beta}{2}\log N\int |\hat{\Xi}^{-1}g|(\xi)\,|\xi|^3\,d\xi.$$

Proof. First observe that, by integration by parts, for any C^1 function f

$$\int\left(N\sum_i V'(\lambda_i)f(\lambda_i) - \beta\sum_{i<j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\right)d\mathbb{P}^N_{V_t} = \int\sum_i f'(\lambda_i)\,d\mathbb{P}^N_{V_t} \tag{2.3.29}$$

which we can rewrite as the first loop equation

$$\int M_N(\Xi f)\,d\mathbb{P}^N_{V_t} = \int\left[\left(1-\frac{\beta}{2}\right)\int f'dL_N + \frac{\beta}{2N}\int\frac{f(x)-f(y)}{x-y}dM_N(x)dM_N(y)\right]d\mathbb{P}^N_{V_t}\,. \tag{2.3.30}$$

We denote

$$F_N(g) := \left(1-\frac{\beta}{2}\right)\int(\Xi^{-1}g)'\,dL_N + \frac{\beta}{2N}\int\frac{\Xi^{-1}g(x)-\Xi^{-1}g(y)}{x-y}\,dM_N(x)\,dM_N(y)$$

so that taking $f := \Xi^{-1}g$ in (2.3.30) we deduce

$$\int M_N(g)\, d\mathbb{P}^N_{V_t} = \int F_N(g)\, d\mathbb{P}^N_{V_t}.$$

To bound the right hand side above, we notice that $\Xi^{-1}g$ goes to zero at infinity like $1/V'$ (see Lemma 2.3.2). Hence we can write its Fourier transform and get

$$\int \frac{\Xi^{-1}g(x) - \Xi^{-1}g(y)}{x-y}\, dM_N(x)\, dM_N(y) \\ = i \int d\xi\, \xi\, \hat{\Xi}^{-1}g(\xi) \int_0^1 d\alpha \int e^{i\alpha\xi x}\, dM_N(x) \int e^{i(1-\alpha)\xi y}\, dM_N(y),$$

so that we deduce (recall (2.3.26))

$$\sup_{D(L_N,\mu_{V_t}) \le \tau_0 \sqrt{\log N/N}} F_N(g) \le (1+\tau_0^2)\, m(g).$$

On the other hand, as the mass of M_N is always bounded by $2N$, we deduce that $F_N(g)$ is bounded everywhere by $Nm(g)$. Since the set $\{D(L_N, \mu_{V_t}) \ge \tau_0\sqrt{N \log N}\}$ has small probability (see (2.3.27)), we conclude that

$$\left| \int M_N(g)\, d\mathbb{P}^N_{V_t} \right| \le N e^{-c\tau_0^2 N \log N} m(g) + (1+\tau_0^2) m(g) \le C\, m(g), \tag{2.3.31}$$

which proves our first bound.

Before proving the next estimates, let us make a simple remark: using the definition of M_N and $\tilde{M}_N$ it is easy to check that, for any function g,

$$\left| M_N(g) - \tilde{M}_N(g) \right| = \left| \int M_N(g)\, d\mathbb{P}^N_{V_t} \right|. \tag{2.3.32}$$

To get estimates on the covariance we obtain the second loop equation by changing $V(x)$ into $V(x) + \delta\, g(x)$ in (2.3.29) and differentiating with respect to δ at $\delta = 0$. This gives

$$\int M_N(\Xi f) \tilde{M}_N(g)\, d\mathbb{P}^N_{V_t} = \int L_N(fg')\, d\mathbb{P}^N_{V_t} \\ + \int \left[\left(1 - \frac{\beta}{2}\right) \int f' dL_N + \frac{\beta}{2N} \int \frac{f(x)-f(y)}{x-y} dM_N(x) dM_N(y) \right] \tilde{M}_N(g)\, d\mathbb{P}^N_{V_t}. \tag{2.3.33}$$

We now notice that $M_N(\Xi f) - \tilde{M}_N(\Xi f)$ is deterministic and $\int \tilde{M}_N(g)\, d\mathbb{P}^N_{V_t} = 0$, hence the left hand side is equal to

$$\int \tilde{M}_N(\Xi f) \tilde{M}_N(g)\, d\mathbb{P}^N_{V_t}.$$

We take $f := \Xi^{-1}g$ and we argue similarly to above (that is, splitting the estimate depending whether $D(L_N, \mu_{V_t}) \ge \tau_0\sqrt{N \log N}$ or not, and use that $|\tilde{M}_N(g)| \le N\|g\|_\infty$) to deduce that

$\sigma_N^{(2)}(g) := \int |\tilde{M}_N(f)|^2 d\mathbb{P}_{V_t}^N$ satisfies

$$\begin{aligned}\sigma_N^{(2)}(g) &\le \|g'\Xi^{-1}g\|_\infty + \int |F_N(g)||\tilde{M}_N(g)|d\mathbb{P}_{V_t}^N \\ &\le \|\Xi^{-1}g\|_\infty \|g'\|_\infty + N^2 e^{-c\tau_0^2 N\log N}\|g\|_\infty m(g) + C\,m(g)\int |\tilde{M}_N(g)|d\mathbb{P}_{V_t}^N \\ &= \|\Xi^{-1}g\|_\infty \|g'\|_\infty + N^2 e^{-c\tau_0^2 N\log N} m(g)\|g\|_\infty + C\,m(g)\sigma_N^{(2)}(g)^{1/2}.\end{aligned} \tag{2.3.34}$$

Solving this quadratic inequality yields

$$\sigma_N^{(2)}(g) \le C\Big[m(g)^2 + m(g)\|g\|_\infty + \|\Xi^{-1}g\|_\infty\|g'\|_\infty\Big]$$

for some finite constant C.

We finally turn to the fourth moment. If we make an infinitesimal change of potential $V(x)$ into $V(x) + \delta_1\, g_2(x) + \delta_2\, g_3(x)$ and differentiate at $\delta_1 = \delta_2 = 0$ into (2.3.33) we get, denoting $g = g_1$,

$$\begin{aligned}&\int M_N(\Xi f)\tilde{M}_N(g_1)\tilde{M}_N(g_2)\tilde{M}_N(g_3)\, d\mathbb{P}_{V_t}^N = \int\Big[\sum_\sigma L_N(f g'_{\sigma(1)})\tilde{M}_N(g_{\sigma(2)})\tilde{M}_N(g_{\sigma(3)})\Big]\, d\mathbb{P}_{V_t}^N + \\ &\int\Big[\Big(1-\frac{\beta}{2}\Big)\int f'\, dL_N + \frac{\beta}{2N}\int \frac{f(x)-f(y)}{x-y}\, dM_N(x)\, dM_N(y)\Big] M_N(g_1)\tilde{M}_N(g_2)\tilde{M}_N(g_3)\, d\mathbb{P}_{V_t}^N,\end{aligned} \tag{2.3.35}$$

where we sum over the permutation σ of $\{1,2,3\}$. Taking $\Xi f = g_1 = g_2 = g_3 = g$, by (2.3.32), (2.3.31), and Cauchy-Schwarz inequality we get

$$\sigma_N^{(4)}(g) \le C\Big[\|g'\Xi^{-1}g\|_\infty \sigma_N^{(2)}(g) + \|g\|_\infty^3 m(g) + m(g)\sigma_N^{(4)}(g)^{3/4} + m(g)^2\sigma_N^{(2)}(g)\Big],$$

which implies

$$\sigma_N^{(4)}(g) \le C\Big[\|g'\Xi^{-1}g\|_\infty \sigma_N^{(2)}(g) + \|g\|_\infty^3 m(g) + m(g)^2\sigma_N^{(2)}(g) + m(g)^4\Big].$$

□

Applying the above result with $g = e^{i\lambda\cdot}$ we get the following:

Corollary 2.3.5. *Assume that $V', W \in C^r$ with $r \ge 8$. Then there exists a finite constant C such that, for all $\lambda \in \mathbb{R}$,*

$$\int |M_N(e^{i\lambda\cdot})|^2 d\mathbb{P}_{V_t}^N \le C[\log N(1+|\lambda|^7)]^2\,, \tag{2.3.36}$$

$$\int |M_N(e^{i\lambda\cdot})|^4 d\mathbb{P}_{V_t}^N \le C[\log N(1+|\lambda|^7)]^4\,. \tag{2.3.37}$$

Proof. In the case $g(x) = e^{i\lambda x}$ we estimate the norms of $\Xi^{-1}g$ by using Lemma 2.3.2, and we get a finite constant C such that

$$\|\Xi^{-1}g\|_\infty \le C|\lambda|^2, \qquad \|\Xi^{-1}g'\|_\infty \le C|\lambda|^3,$$

whereas, since $\Xi^{-1}g$ goes fast to zero at infinity (as $1/V'$), for $j \le r-3$ we have (see Lemma 2.3.2)

$$|\hat{\Xi}^{-1}g|(\xi) \le C\frac{\|\Xi^{-1}g\|_{C^j}}{1+|\xi|^j} \le C'\frac{\|g\|_{C^{j+2}}}{1+|\xi|^j} \le C'\frac{1+|\lambda|^{j+2}}{1+|\xi|^j}.$$

Hence, we deduce that there exists a finite constant C' such that

$$\begin{aligned}
m(g) &\le C\log N\left(|\lambda|^3 + 1 + \int d\xi \, \frac{1+|\lambda|^7}{1+|\xi|^5}|\xi|^3\right) = C'\log N\left(1+|\lambda|^7\right),\\
B_N^1(g) &\le C'\log N\left(1+|\lambda|^7\right),\\
B_N^2(g) &\le C'(\log N)^2\left(1+|\lambda|^7\right)^2,\\
B_N^4(g) &\le C'(\log N)^4\left(1+|\lambda|^7\right)^4.
\end{aligned}$$

Finally, for $k = 2, 4$, using (2.3.32) and (2.3.31) we have

$$\int |M_N(e^{i\lambda\cdot})|^k \, d\mathbb{P}_{V_t}^N \le 2^{k-1}\left(\int |\tilde{M}_N(e^{i\lambda\cdot})|^k \, d\mathbb{P}_{V_t}^N + \left(B_N^1(g)\right)^k\right)$$

from which the result follows. □

We can now estimate E_N.

The linear term can be handled in the same way as we shall do now for the quadratic and cubic terms (which are actually more delicate), so we just focus on them.

We have two quadratic terms in M_N which sum up into

$$E_N^1 = -\frac{1}{N}\left(1-\frac{\beta}{2}\right)\iint \partial_1\mathbf{z}_t(x,y)\, dM_N(x)\, dM_N(y) - \frac{\beta}{2N}\iint \frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y}\, dM_N(x)\, dM_N(y).$$

Writing

$$\frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y} = \int_0^1 \mathbf{y}_{1,t}'(\alpha x + (1-\alpha)y)\, d\alpha = \int_0^1 \left(\int \widehat{\mathbf{y}_{1,t}'}(\xi) e^{i(\alpha x + (1-\alpha)y)\xi} d\xi\right) d\alpha$$

we see that

$$\iint \frac{\mathbf{y}_{1,t}(x)-\mathbf{y}_{1,t}(y)}{x-y}\, dM_N(x)\, dM_N(y) = \int d\xi \, \widehat{\mathbf{y}_{1,t}'}(\xi) \int_0^1 d\alpha \, M_N(e^{i\alpha\xi\cdot}) M_N(e^{i(1-\alpha)\xi\cdot}),$$

so using (2.3.36) we get

$$\int |E_N^1|\, d\mathbb{P}_{V_t}^N \le C\,\frac{(\log N)^2}{N}\Big(\int d\xi\, |\hat{\mathbf{y}}_{1,t}|(\xi)\,|\xi|\,\big(1+|\xi|^7\big)^2 + \iint d\xi\, d\zeta\, |\hat{\mathbf{z}}_t|(\xi,\zeta)\,|\xi|\,\big(1+|\xi|^7\big)\big(1+|\zeta|^7\big)\Big).$$

It is easy to see that the right hand side is finite if $\mathbf{y}_{1,t}$ and $\mathbf{z}_t$ are smooth enough (recall that these functions and their derivatives decay fast at infinity). More precisely, to ensure that

$$|\hat{\mathbf{y}}_{1,t}|(\xi)\,|\xi|\,\big(1+|\xi|^7\big)^2 \le \frac{C}{1+|\xi|^2} \in L^1(\mathbb{R})$$

and

$$|\hat{\mathbf{z}}_t|(\xi,\zeta)\,|\xi|\,\big(1+|\xi|^7\big)\big(1+|\zeta|^7\big) \le \frac{C}{1+|\xi|^3+|\zeta|^3} \in L^1(\mathbb{R}^2),$$

we need $\mathbf{y}_{1,t} \in C^{17}$ and $\mathbf{z}_t \in C^{11,7} \cap C^{8,10}$, so (recalling Lemma 2.3.3) $V', W \in C^{25}$ is enough to guarantee that the right hand side is finite.

Using (2.3.36), (2.3.37), and Hölder inequality, we can similarly bound the expectation of the cubic term

$$\begin{aligned} E_N^2 &= \frac{\beta}{2N}\iiint \frac{\mathbf{z}_t(x,y)-\mathbf{z}_t(\tilde{x},y)}{x-\tilde{x}}\, dM_N(x)\, dM_N(y)\, dM_N(\tilde{x}) \\ &= i\frac{\beta}{2N}\iint d\xi\, d\zeta\, \widehat{\partial_1\mathbf{z}_t}(\xi,\zeta)\int_0^1 d\alpha\, M_N(e^{i\alpha\xi\cdot})M_N(e^{i(1-\alpha)\xi\cdot})M_N(e^{i\zeta\cdot}) \end{aligned}$$

to get

$$\int |E_N^2|\, d\mathbb{P}_{V_t}^N \le C\,\frac{(\log N)^3}{N}\iint d\xi\, d\zeta\, |\hat{\mathbf{z}}_t(\xi,\zeta)|\,|\xi|\,\big(1+|\xi|^7\big)^2\big(1+|\zeta|^7\big).$$

Again the right hand side is finite if $\mathbf{z}_t \in C^{18,7} \cap C^{15,10}$, which is ensured by Lemma 2.3.3 if V, W are of class C^{31}.

2.3.5 Control on the deterministic term C_t^N

By what we proved above we have

$$\int |\mathcal{R}_t^N - C_t^N|\, d\mathbb{P}_{V_t}^N \le C\,\frac{(\log N)^3}{N},$$

thus, in particular,

$$\big|C_t^N - \mathbb{E}[\mathcal{R}_t^N]\big| \le C\,\frac{(\log N)^3}{N}.$$

Notice now that, by construction,

$$\mathcal{R}_t^N = -\mathcal{L}\mathbf{Y}_t^N + N\sum_i W(\lambda_i) + c_t^N$$

with $c_t^N = -\mathbb{E}[N\sum_i W(\lambda_i)]$ and

$$\mathcal{L}\mathbf{Y} := \operatorname{div}\mathbf{Y} + \beta \sum_{i<j} \frac{\mathbf{Y}^i - \mathbf{Y}^j}{\lambda_i - \lambda_j} - N \sum_i V'(\lambda_i)\mathbf{Y}^i,$$

and an integration by parts shows that, under P_V^N, $\mathbb{E}[\mathcal{L}\mathbf{Y}] = 0$ for any vector field $\mathbf{Y}$. This implies that $\mathbb{E}[\mathcal{R}_t^N] = 0$, therefore $|C_t^N| \le C\frac{(\log N)^3}{N}$.

This concludes the proof of Proposition 2.3.1.

2.4 Reconstructing the transport map via the flow

In this section we study the properties of the flow generated by the vector field $\mathbf{Y}_t^N$ defined in (2.3.13). As we shall see, we will need to assume that $W, V \in C^r$ with $r \ge 15$.

We consider the flow of $\mathbf{Y}_t^N$ given by

$$X_t^N : \mathbb{R}^N \to \mathbb{R}^N, \qquad \dot{X}_t^N = \mathbf{Y}_t^N(X_t^N).$$

Recalling the form of $\mathbf{Y}_t^N$ (see (2.3.13)) it is natural to expect that we can give an expansion for X_t^N. More precisely, let us define the flow of $\mathbf{y}_{0,t}$,

$$X_{0,t} : \mathbb{R} \to \mathbb{R}, \qquad \dot{X}_{0,t} = \mathbf{y}_{0,t}(X_{0,t}), \quad X_{0,0}(\lambda) = \lambda, \tag{2.4.38}$$

and let $X_{1,t}^N = (X_{1,t}^{N,1}, \ldots, X_{1,t}^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ be the solution of the linear ODE

$$\begin{aligned} \dot{X}_{1,t}^{N,k}(\lambda_1, \ldots, \lambda_N) &= \mathbf{y}_{0,t}'(X_{0,t}(\lambda_k)) \cdot X_{1,t}^{N,k}(\lambda_1, \ldots, \lambda_N) + \mathbf{y}_{1,t}(X_{0,t}(\lambda_k)) \\ &\quad + \int \mathbf{z}_t(X_{0,t}(\lambda_k), y)\, dM_N^{X_{0,t}}(y) \\ &\quad + \frac{1}{N}\sum_{j=1}^N \partial_2 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big) \cdot X_{1,t}^{N,j}(\lambda_1, \ldots, \lambda_N) \end{aligned} \tag{2.4.39}$$

with the initial condition $X_{1,t}^N = 0$, and $M_N^{X_{0,t}}$ is defined as

$$\int f(y) dM_N^{X_{0,t}}(y) = \sum_{i=1}^N \Big[f(X_{0,t}(\lambda_i)) - \int f d\mu_{V_t} \Big] \qquad \forall f \in C_c(\mathbb{R}).$$

If we set

$$X_{0,t}^N(\lambda_1, \ldots, \lambda_N) := \Big(X_{0,t}(\lambda_1), \ldots, X_{0,t}(\lambda_N)\Big),$$

then the following result holds.

Lemma 2.4.1. *Assume that $W, V \in C^r$ with $r \ge 15$. Then the flow $X_t^N = (X_t^{N,1}, \ldots, X_t^{N,N}) : \mathbb{R}^N \to \mathbb{R}^N$ is of class C^{r-8} and the following properties hold: Let $X_{0,t}$ and $X_{1,t}^N$ be as in* (2.4.38) *and* (2.4.39) *above, and define $X_{2,t}^N : \mathbb{R}^N \to \mathbb{R}^N$ via the identity*

$$X_t^N = X_{0,t}^N + \frac{1}{N} X_{1,t}^N + \frac{1}{N^2} X_{2,t}^N.$$

Then

$$\sup_{1\le k\le N} \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)} \le C\log N, \qquad \|X_{2,t}^N\|_{L^2(\mathbb{P}_V^N)} \le CN^{1/2}(\log N)^2, \tag{2.4.40}$$

where

$$\|X_{i,t}^N\|_{L^2(\mathbb{P}_V^N)} = \left(\int |X_{i,t}^N|^2 d\mathbb{P}_V^N\right)^{1/2}, \qquad |X_{i,t}^N| := \sqrt{\sum_{j=1,\dots,N} |X_{i,t}^{N,j}|^2}, \quad i = 0,1,2.$$

In addition, there exists a constant $C > 0$ such that, with probability greater than $1 - N^{-N/C}$,

$$\max_{1\le k,k'\le N} |X_{1,t}^{N,k}(\lambda_1,\dots,\lambda_N) - X_{1,t}^{N,k'}(\lambda_1,\dots,\lambda_N)| \le C\,\log N\sqrt{N}|\lambda_k - \lambda_{k'}|. \tag{2.4.41}$$

Proof. Since $\mathbf{Y}_t^N \in C^{r-8}$ (see Lemma 2.3.3) it follows by Cauchy-Lipschitz theory that X_t^N is of class C^{r-8}.

Using the notation $\hat\lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{R}^N$ and

$$X_t^{N,k,\sigma}(\hat\lambda) := X_{0,t}(\lambda_k) + \sigma\frac{X_{1,t}^{N,k}}{N}(\hat\lambda) + \sigma\frac{X_{2,t}^{N,k}}{N^2}(\hat\lambda) = (1-\sigma)X_{0,t}(\lambda_k) + \sigma X_t^{N,k}(\hat\lambda)$$

and defining the measure $M_N^{X_t^{N,s}}$ as

$$\int f(y)\,dM_N^{X_t^{N,s}}(y) = \sum_{i=1}^N\left[f\Big((1-s)X_{0,t}(\lambda_i) + sX_t^{N,i}(\hat\lambda)\Big) - \int f\,d\mu_{V_t}\right] \qquad \forall f\in C_c(\mathbb{R}). \tag{2.4.42}$$

by a Taylor expansion we get an ODE for $X_{2,t}^N$:

$$\begin{aligned}
\dot X_{2,t}^{N,k}(\hat\lambda) &= \int_0^1 \mathbf{y}_{0,t}'\Big(X_t^{N,k,s}(\hat\lambda)\Big)\,ds\cdot X_{2,t}^{N,k}(\hat\lambda)\\
&+ N\int_0^1\Big[\mathbf{y}_{0,t}'\Big(X_t^{N,k,s}(\hat\lambda)\Big) - \mathbf{y}_{0,t}'\Big(X_{0,t}(\lambda_k)\Big)\Big]\,ds\cdot X_{1,t}^{N,k}(\hat\lambda)\\
&+ \int_0^1 \mathbf{y}_{1,t}'\Big(X_t^{N,k,s}(\hat\lambda)\Big)\,ds\cdot\left(X_{1,t}^{N,k}(\hat\lambda) + \frac{X_{2,t}^{N,k}(\hat\lambda)}{N}\right)\\
&+ \int_0^1\Big[\int \partial_1\mathbf{z}_t\Big(X_t^{N,k,s}(\hat\lambda), y\Big)\,dM_N^{X_t^{N,s}}(y)\\
&\qquad - \int \partial_1\mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\,dM_N^{X_{0,t}}(y)\Big]\,ds\cdot\left(X_{1,t}^{N,k}(\hat\lambda) + \frac{X_{2,t}^{N,k}(\hat\lambda)}{N}\right)\\
&+ \int \partial_1\mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\,dM_N^{X_{0,t}}(y)\cdot\left(X_{1,t}^{N,k}(\hat\lambda) + \frac{X_{2,t}^{N,k}(\hat\lambda)}{N}\right)\\
&+ \sum_{j=1}^N\int_0^1\Big[\partial_2\mathbf{z}_t\Big(X_t^{N,k,s}(\hat\lambda), X_t^{N,j,s}(\hat\lambda)\Big) - \partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big)\Big]\,ds\cdot X_{1,t}^{N,j}(\hat\lambda)\\
&+ \sum_{j=1}^N\int_0^1\Big[\partial_2\mathbf{z}_t\Big(X_t^{N,k,s}(\hat\lambda), X_t^{N,j,s}(\hat\lambda)\Big)\Big]\,ds\cdot\frac{X_{2,t}^{N,j}(\hat\lambda)}{N},
\end{aligned} \tag{2.4.43}$$

with the initial condition $X_{2,0}^{N,k} = 0$. Using that

$$\|\mathbf{y}_{0,t}\|_{C^{r-2}(\mathbb{R})} \le C$$

(see Lemma 2.3.3) we obtain

$$\|X_{0,t}\|_{C^{r-2}(\mathbb{R})} \le C. \tag{2.4.44}$$

We now start to control $X_{1,t}^N$. First, simply by using that M_N has mass bounded by $2N$ we obtain the rough bound $|X_{1,t}^{N,k}| \le C\,N$. Inserting this bound into (2.4.43) one easily obtain the bound $|X_{2,t}^{N,k}| \le C\,N^2$.

We now prove finer estimates. First, by (2.3.27) together with the fact that $X_{0,t}$ and $x \mapsto \mathbf{z}_t(y,x)$ are Lipschitz (uniformly in y), it follows that there exists a finite constant C such that, with probability greater than $1 - N^{-N/C}$,

$$\left\| \int \mathbf{z}_t(\cdot,\lambda)\, dM_N^{X_{0,t}}(\lambda) \right\|_\infty \le C\, \log N\sqrt{N}. \tag{2.4.45}$$

Hence it follows easily from (2.4.39) that

$$\max_k \|X_{1,t}^{N,k}\|_\infty \le C \log N\sqrt{N} \tag{2.4.46}$$

outside a set of probability bounded by $N^{-N/C}$.

In order to control $X_{2,t}^N$ we first estimate $X_{1,t}^N$ in $L^4(\mathbb{P}_V^N)$: using (2.4.39) again, we get

$$\frac{d}{dt}\left(\max_k \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)}\right) \\ \le C\left(\max_k \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)} + 1 + \left\|\int \mathbf{z}_t(X_{0,t}(\lambda_k),y)\, dM_N^{X_{0,t}}(y)\right\|_{L^4(\mathbb{P}_V^N)}\right). \tag{2.4.47}$$

To bound $X_{1,t}^N$ in $L^4(\mathbb{P}_V^N)$ and then to be able to estimate $X_{2,t}^N$ in $L^2(\mathbb{P}_V^N)$, we will use the following estimates:

Lemma 2.4.2. *For any $k = 1, \dots, N$,*

$$\left\|\int \mathbf{z}_t(X_{0,t}(\lambda_k),y)\, dM_N^{X_{0,t}}(y)\right\|_{L^4(\mathbb{P}_V^N)} \le C \log N, \tag{2.4.48}$$

$$\left\|\int \partial_1 \mathbf{z}_t\Big(X_{0,t}(\lambda_k),y\Big)\, dM_N^{X_{0,t}}(y)\right\|_{L^4(\mathbb{P}_V^N)} \le C \log N. \tag{2.4.49}$$

Proof. We write the Fourier decomposition of $\eta_t(x,y) := \mathbf{z}_t(X_{0,t}(x), X_{0,t}(y))$ to get

$$\int \eta_t(x,y)\, dM_N(y) = \int \hat{\eta}_t(x,\xi) \int e^{i\xi y}\, dM_N(y)\, d\xi\,.$$

Since $\mathbf{z}_t \in C^{u,v}$ for $u+v \le r-5$ and $X_{0,t} \in C^{r-2}$ (see (2.4.44)), we deduce that

$$|\hat{\eta}_t(x,\xi)| \le \frac{C}{1+|\xi|^{r-5}},$$

so that using (2.3.37) we get

$$\left\| \sup_x \left| \int \eta_t(x,y)\, dM_N(y) \right| \right\|_{L^4(\mathbb{P}_V^N)} \le \int \left\| \hat{\eta}_t(\cdot,\xi) \right\|_\infty \left\| \int e^{i\xi y} dM_N(y) \right\|_{L^4(\mathbb{P}_V^N)} d\xi$$

$$\le C \log N \int \left\| \hat{\eta}_t(\cdot,\xi) \right\|_\infty \left(1 + |\xi|^7\right) d\xi$$

$$\le C \log N,$$

provided $r > 13$. The same arguments work for $\partial_1 \mathbf{z}_t$ provided $r > 14$. Since by assumption $r \ge 15$, this concludes the proof. □

Inserting (2.4.48) into (2.4.47) we get

$$\|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)} \le C \log N \qquad \forall\, k = 1, \dots, N, \tag{2.4.50}$$

which proves the first part of (2.4.40).

We now bound the time derivative of the L^2 norm of $X_{2,t}^N$: using that M_N has mass bounded by $2N$, in (2.4.43) we can easily estimate

$$\left| N \int_0^1 \left[\mathbf{y}'_{0,t}\left(X_t^{N,k,s}(\hat{\lambda})\right) - \mathbf{y}'_{0,t}\left(X_{0,t}(\lambda_k)\right) \right] ds \cdot X_{1,t}^{N,k}(\hat{\lambda}) \right| \le C |X_{1,t}^{N,k}|^2 + \frac{C}{N} |X_{1,t}^{N,k}|\, |X_{2,t}^{N,k}|,$$

$$\int_0^1 \left| \int \partial_1 \mathbf{z}_t\left(X_t^{N,k,s}(\hat{\lambda}), y\right) dM_N^{X_t^{N,s}}(y) - \int \partial_1 \mathbf{z}_t\left(X_{0,t}(\lambda_k), y\right) dM_N^{X_{0,t}}(y) \right| ds$$

$$\le C |X_{1,t}^{N,k}| + \frac{C}{N} |X_{2,t}^{N,k}| + \frac{C}{N} \sum_j \left(|X_{1,t}^{N,j}| + \frac{1}{N} |X_{2,t}^{N,j}| \right),$$

$$\sum_{j=1}^N \int_0^1 \left| \partial_2 \mathbf{z}_t\left(X_t^{N,k,s}(\hat{\lambda}), X_t^{N,j,s}(\hat{\lambda})\right) - \partial_2 \mathbf{z}_t\left(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\right) \right| ds\, |X_{1,t}^{N,j}|$$

$$\le \frac{C}{N} \sum_j \left(|X_{1,t}^{N,j}|^2 + \frac{1}{N} |X_{2,t}^{N,j}|\, |X_{1,t}^{N,j}| \right),$$

hence

$$\begin{aligned}
\frac{d}{dt}\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} &= 2\int \sum_k X_{2,t}^{N,k}\cdot \dot{X}_{2,t}^{N,k}\, d\mathbb{P}_V^N \\
&\le C\int \sum_k |X_{2,t}^{N,k}|^2\, d\mathbb{P}_V^N + C\int \sum_k |X_{1,t}^{N,k}|^2 |X_{2,t}^{N,k}| d\mathbb{P}_V^N \\
&+ \frac{C}{N}\int \sum_k |X_{1,t}^{N,k}||X_{2,t}^{N,k}|^2 d\mathbb{P}_V^N + C\int \sum_k |X_{1,t}^{N,k}||X_{2,t}^{N,k}|\, d\mathbb{P}_V^N \\
&+ \frac{C}{N^2}\int \sum_k |X_{2,t}^{N,k}|^3\, d\mathbb{P}_V^N + \frac{C}{N}\int \sum_{k,j} |X_{1,t}^{N,j}|\,|X_{1,t}^{N,k}|\,|X_{2,t}^{N,k}|\, d\mathbb{P}_V^N \\
&+ \frac{C}{N^3}\int \sum_{k,j} |X_{2,t}^{N,k}|^2\,|X_{2,t}^{N,j}|\, d\mathbb{P}_V^N \\
&+ \sum_k \int X_{2,t}^{N,k}\cdot \int_0^1 \left[\int \partial_1 \mathbf{z}_t\Big(X_{0,t}(\lambda_k), y\Big)\, dM_N^{X_{0,t}}(y)\right] ds \cdot X_{1,t}^{N,k}\, d\mathbb{P}_V^N \\
&+ \frac{C}{N}\int \sum_{k,j} |X_{1,t}^{N,j}|^2\,|X_{2,t}^{N,k}|\, d\mathbb{P}_V^N + \frac{C}{N^2}\int \sum_{k,j} |X_{2,t}^{N,k}|\,|X_{2,t}^{N,j}|\,|X_{1,t}^{N,j}|\, d\mathbb{P}_V^N \\
&+ \frac{C}{N}\int \sum_{k,j} |X_{2,t}^{N,k}|\,|X_{2,t}^{N,j}|\, d\mathbb{P}_V^N.
\end{aligned}$$

Using the trivial bounds $|X_{1,t}^{N,k}| \le C\,N$ and $|X_{2,t}^{N,k}| \le C\,N^2$, (2.4.49), and elementary inequalities such as, for instance,

$$\sum_{k,j} |X_{1,t}^{N,j}|\,|X_{1,t}^{N,k}|\,|X_{2,t}^{N,k}| \le \sum_{k,j}\Big(|X_{1,t}^{N,j}|^4 + |X_{1,t}^{N,k}|^4 + |X_{2,t}^{N,k}|^2\Big),$$

we obtain

$$\begin{aligned}
\frac{d}{dt}\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} \le C\bigg(&\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + \int \sum_k |X_{1,t}^{N,k}|^4\, d\mathbb{P}_V^N \\
&+ \int \sum_k |X_{1,t}^{N,k}|^2\, d\mathbb{P}_V^N + \sum_k \log N\, \|X_{2,t}^{N,k}\|_{L^2(\mathbb{P}_V^N)} \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)}\bigg).
\end{aligned} \tag{2.4.51}$$

We now observe, by (2.4.50), that the last term is bounded by

$$\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + (\log N)^2 \sum_k \|X_{1,t}^{N,k}\|^2_{L^4(\mathbb{P}_V^N)} \le \|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + C\,N(\log N)^4.$$

Hence, using that $\|X_{1,t}^{N,k}\|_{L^2(\mathbb{P}_V^N)} \le \|X_{1,t}^{N,k}\|_{L^4(\mathbb{P}_V^N)}$ and (2.4.50) again, the right hand side of (2.4.51) can be bounded by $C\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} + C\,N(\log N)^4$, and a Gronwall argument gives

$$\|X_{2,t}^N\|^2_{L^2(\mathbb{P}_V^N)} \le C\,N(\log N)^4,$$

thus

$$\|X_{2,t}^N\|_{L^2(\mathbb{P}_V^N)} \le C\,N^{1/2}(\log N)^2,$$

concluding the proof of (2.4.40).

We now prove (2.4.41): using (2.4.39) we have

$$
\begin{aligned}
&|\dot X^{N,k}_{1,t}(\hat\lambda) - \dot X^{N,k'}_{1,t}(\hat\lambda)| \\
&\le |\mathbf{y}'_{0,t}(X_{0,t}(\lambda_k)) - \mathbf{y}'_{0,t}(X_{0,t}(\lambda_{k'}))|\,|X^{N,k}_{1,t}(\hat\lambda)| \\
&+ |\mathbf{y}'_{0,t}(X_{0,t}(\lambda_{k'}))|\,|X^{N,k}_{1,t}(\hat\lambda) - X^{N,k'}_{1,t}(\hat\lambda)| + |\mathbf{y}_{1,t}(X_{0,t}(\lambda_k)) - \mathbf{y}_{1,t}(X_{0,t}(\lambda_{k'}))| \\
&+ \left|\int \Big(\mathbf{z}_t(X_{0,t}(\lambda_k), y) - \mathbf{z}_t(X_{0,t}(\lambda_{k'}), y)\Big)\, dM_N^{X_{0,t}}(y)\right| \\
&+ \frac{1}{N}\sum_{j=1}^{N}\int_0^1 \left|\partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_k), X_{0,t}(\lambda_j)\Big) - \partial_2\mathbf{z}_t\Big(X_{0,t}(\lambda_{k'}), X_{0,t}(\lambda_j)\Big)\right| ds\, |X^{N,j}_{1,t}(\hat\lambda)|.
\end{aligned}
$$

Using that $|X_{0,t}(\lambda_k) - X_{0,t}(\lambda_{k'})| \le C|\lambda_k - \lambda_{k'}|$, the bound (2.4.46), the Lipschitz regularity of $\mathbf{y}'_{0,t}$, $\mathbf{y}_{1,t}$, $\mathbf{z}_t$, and $\partial_2\mathbf{z}_t$, and the fact that

$$
\left\|\int \partial_1\mathbf{z}_t(\cdot,\lambda)\, dM_N^{X_{0,t}}(\lambda)\right\|_\infty \le C\,\log N\sqrt{N}
$$

with probability greater than $1 - N^{-N/C}$ (see (2.3.27)), we get

$$
|\dot X^{N,k}_{1,t}(\hat\lambda) - \dot X^{N,k'}_{1,t}(\hat\lambda)| \le C|X^{N,k}_{1,t}(\hat\lambda) - X^{N,k'}_{1,t}(\hat\lambda)| + C\log N\sqrt{N}|\lambda_k - \lambda_{k'}|
$$

outside a set of probability less than $N^{-N/C}$, so (2.4.41) follows from Gronwall.

□

2.5 Transport and universality

In this section we prove Theorem 2.1.4 on universality using the regularity properties of the approximate transport maps obtained in the previous sections.

Proof of Theorem 2.1.4. Let us first remark that the map T_0 from Theorem 2.1.3 coincides with $X_{0,1}$, where $X_{0,t}$ is the flow defined in (2.4.38). Also, notice that $X_1^N : \mathbb{R}^N \to \mathbb{R}^N$ is an approximate transport of $\mathbb{P}_V^N$ onto $\mathbb{P}_{V+W}^N$ (see Lemma 2.2.2 and Proposition 2.3.1). Set $\hat X_1^N := X_{0,1}^N + \frac{1}{N}X_{1,1}^N$, with $X_{0,t}^N$ and $X_{1,t}^N$ as in Lemma 2.4.1. Since $X_1^N - \hat X_1^N = \frac{1}{N^2}X_{2,1}^N$, recalling (2.4.40) and using Hölder inequality to control the L^1 norm with the L^2 norm, we see that

$$
\begin{aligned}
\left|\int g(\hat X_1^N)\, d\mathbb{P}_V^N - \int g(X_1^N)\, d\mathbb{P}_V^N\right| &\le \|\nabla g\|_\infty \frac{1}{N^2}\int |X_{2,1}^N|\, d\mathbb{P}_V^N \\
&\le \|\nabla g\|_\infty \frac{1}{N^2}\|X_{2,1}^N\|_{L^2(\mathbb{P}_V)} \qquad (2.5.52)\\
&\le C\|\nabla g\|_\infty \frac{(\log N)^2}{N^{3/2}}.
\end{aligned}
$$

This implies that also $\hat X_1^N : \mathbb{R}^N \to \mathbb{R}^N$ is an approximate transport of $\mathbb{P}_V^N$ onto $\mathbb{P}_{V+W}^N$. In addition, we see that $\hat X_1^N$ preserves the order of the λ_i with large probability. Indeed, first of all $X_{0,t} : \mathbb{R} \to \mathbb{R}$ is the flow of $\mathbf{y}_{0,t}$ which is Lipschitz with some constant L, hence differentiating (2.4.38) we get

$$\frac{d}{dt}\left|X'_{0,t}\right| \leq \left|\mathbf{y}'_{0,t}(X_{0,t})\right| \left|X'_{0,t}\right| \leq L\left|X'_{0,t}\right|, \qquad X'_{0,0} = 1,$$

so Gronwall's inequality gives the bound

$$e^{-Lt} \leq \left|X'_{0,t}\right| \leq e^{Lt}$$

Since $X'_{0,0} = 1$, it follows by continuity that $X'_{0,t}$ must remain positive for all time and it satisfies

$$e^{-Lt} \leq X'_{0,t} \leq e^{Lt}, \tag{2.5.53}$$

from which we deduce that

$$e^{-Lt}\left(\lambda_j - \lambda_i\right) \leq X_{0,t}(\lambda_j) - X_{0,t}(\lambda_i) \leq e^{Lt}\left(\lambda_j - \lambda_i\right), \qquad \forall\, \lambda_i < \lambda_j.$$

In particular,

$$e^{-L}\left(\lambda_j - \lambda_i\right) \leq X_{0,1}(\lambda_j) - X_{0,1}(\lambda_i) \leq e^{L}\left(\lambda_j - \lambda_i\right).$$

Hence, using the notation $\hat{\lambda} = (\lambda_1, \ldots, \lambda_N)$, since

$$\left|\frac{1}{N}X_{1,t}^{N,j}(\hat{\lambda}) - \frac{1}{N}X_{1,t}^{N,i}(\hat{\lambda})\right| \leq C\frac{\log N}{\sqrt{N}}|\lambda_i - \lambda_j|$$

(see (2.4.41)) with probability greater than $1 - N^{-N/C}$ we get

$$\frac{1}{C}\left(\lambda_j - \lambda_i\right) \leq \hat{X}_1^{N,j}(\hat{\lambda}) - \hat{X}_1^{N,i}(\hat{\lambda}) \leq C\left(\lambda_j - \lambda_i\right)$$

with probability greater than $1 - N^{-N/C}$.

We now make the following observation: the ordered measures $\tilde{P}_V^N$ and $\tilde{P}_{V+W}^N$ are obtained as the image of $\mathbb{P}_V^N$ and $\mathbb{P}_{V+W}^N$ via the map $R : \mathbb{R}^N \to \mathbb{R}^N$ defined as

$$[R(x_1, \ldots, x_N)]_i := \min_{\sharp J = i} \max_{j \in J} x_j.$$

Notice that this map is 1-Lipschitz for the sup norm.

Hence, if g is a function of m-variables we have $\|\nabla(g \circ R)\|_\infty \leq \sqrt{m}\|\nabla g\|_\infty$, so by Lemma 2.2.2, Proposition 2.3.1, and (2.5.52), we get

$$\left|\int g \circ R(\hat{X}_1^N)\, d\mathbb{P}_V^N - \int g \circ R\, d\mathbb{P}_{V+W}^N\right| \leq C\frac{(\log N)^3}{N}\|g\|_\infty + C\sqrt{m}\,\frac{(\log N)^2}{N^{3/2}}\|\nabla g\|_\infty.$$

Since $\hat{X}_1^N$ preserves the order with probability greater than $1 - N^{-N/C}$, we can replace $g \circ R(N\hat{X}_1^N)$ with $g(N\hat{X}_1^N \circ R)$ up to a very small error bounded by $\|g\|_\infty N^{-N/C}$. Hence, since $R_\# \mathbb{P}_V^N = \tilde{P}_V^N$ and $R_\# \mathbb{P}_{V+W}^N = \tilde{P}_{V+W}^N$, we deduce that, for any Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$,

$$\left| \int f\Big(N(\lambda_{i+1} - \lambda_i), \ldots, N(\lambda_{i+m} - \lambda_i)\Big)\, d\tilde{P}_{V+W}^N \right.$$
$$\left. - \int f\Big(N\big(\hat{X}_1^{N,i+1}(\hat{\lambda}) - \hat{X}_1^{N,i}(\hat{\lambda})\big), \ldots, N\big(\hat{X}_1^{N,i+m}(\hat{\lambda}) - \hat{X}_1^{N,i}(\hat{\lambda})\big)\Big) d\tilde{P}_V^N \right|$$
$$\leq C\, \frac{(\log N)^3}{N} \|f\|_\infty + C\,\sqrt{m}\, \frac{(\log N)^2}{N^{1/2}} \|\nabla f\|_\infty.$$

Recalling that

$$\hat{X}_1^{N,j}(\hat{\lambda}) = X_{0,1}(\lambda_j) + \frac{1}{N} X_{1,1}^{N,j}(\hat{\lambda}),$$

we observe that, as $X_{0,1}$ is of class C^2,

$$X_{0,1}(\lambda_{i+k}) - X_{0,1}(\lambda_i) = X_{0,1}'(\lambda_i)\,(\lambda_{i+k} - \lambda_i) + O(|\lambda_{i+k} - \lambda_i|^2).$$

Also, by (2.4.41) we deduce that, out of a set of probability bounded by $N^{-N/C}$,

$$|X_{1,1}^{N,i+k}(\hat{\lambda}) - X_{1,1}^{N,i}(\hat{\lambda})| \leq C\, \log N \sqrt{N} |\lambda_{i+k} - \lambda_i|\,. \tag{2.5.54}$$

As $X_{0,1}'(\lambda_i) \geq e^{-L}$ (see (2.5.53)) we deduce

$$\frac{1}{N}\, |X_{1,1}^{N,i+k}(\hat{\lambda}) - X_{1,1}^{N,i}(\hat{\lambda})| \leq C\, |X_{0,1}'(\lambda_i)\,(\lambda_{i+k} - \lambda_i)|\, \frac{\log N}{N^{1/2}}$$

and

$$O(|\lambda_{i+k} - \lambda_i|^2) = O\Big(\big|X_{0,1}'(\lambda_i)\,(\lambda_{i+k} - \lambda_i)\big|^2\Big)$$

hence with probability greater than $1 - N^{-N/C}$ it holds

$$\hat{X}_1^{N,i+k}(\hat{\lambda}) - \hat{X}_1^{N,i}(\hat{\lambda}) = X_{0,t}'(\lambda_i)\,(\lambda_{i+k} - \lambda_i) \left[1 + O\Big(\frac{\log N}{N^{1/2}}\Big) + O\Big(\big|X_{0,1}'(\lambda_i)\,(\lambda_{i+k} - \lambda_i)\big|\Big) \right].$$

Since we assume f supported in $[-M, M]^m$, the domain of integration is restricted to $\hat{\lambda}$ such that $\{N X_{0,t}'(\lambda_i)\,(\lambda_i - \lambda_{i+k})\}_{1 \leq k \leq m}$ is bounded by $2M$ for N large enough, therefore

$$\hat{X}_1^{N,i+k}(\hat{\lambda}) - \hat{X}_1^{N,i}(\hat{\lambda}) = X_{0,t}'(\lambda_i)\,(\lambda_{i+k} - \lambda_i) + O\Big(2M \frac{\log N}{N^{3/2}}\Big) + O\Big(\frac{4M^2}{N^2}\Big),$$

from which the first bound follows easily.

For the second point we observe that $a_{V+W} = X_{0,1}(a_V)$ and, arguing as before,

$$\left| \int f\Big(N^{2/3}(\lambda_1 - a_{V+W}), \ldots, N^{2/3}(\lambda_m - a_{V+W})\Big)\, d\tilde{P}_{V+W}^N \right.$$
$$\left. - \int f\Big(N^{2/3}\big(\hat{X}_1^{N,1}(\hat{\lambda}) - X_{0,1}(a_V)\big), \ldots, N^{2/3}\big(\hat{X}_1^{N,m}(\hat{\lambda}) - X_{0,1}(a_V)\big)\Big)\, d\tilde{P}_V^N \right|$$
$$\leq C\, \frac{(\log N)^3}{N}\, \|f\|_\infty + C\,\sqrt{m} \frac{(\log N)^2}{N^{5/6}}\, \|\nabla f\|_\infty.$$

Since, by (2.4.40),

$$\begin{aligned}\hat{X}_1^{N,i}(\lambda) &= X_{0,1}(\lambda_i) + O_{L^4(\mathbb{P}_V^N)}\left(\frac{\log N}{N}\right) \\ &= X_{0,1}(a_V) + X_{0,1}'(a_V)\,(\lambda_i - a_V) + O\Big(|\lambda_i - a_V|^2\Big) + O_{L^4(\mathbb{P}_V^N)}\left(\frac{\log N}{N}\right),\end{aligned}$$

we conclude as in the first point. □

Chapter 3

Transport Maps for β-Matrix Models in the Multi-Cut Regime

This Chapter is based on the article [10].

3.1 Introduction

In the previous Chapter we construct an approximate transport maps with an accurate dependence in the dimension. The dependence in N allows to compare the local fluctuation of the eigenvalues under two different potentials. The potentials do not need to be analytic, but an important hypothesis made in this previous Chapter was the connectedness of the support of the limit of the spectral measure . Here, we assume that the potentials are analytic but remove the one-cut assumption and use the same methods to construct approximate transport maps in the case where the filling fractions of each cut is fixed. As a result, we obtain universality of fixed eigenvalue gaps at the edge and in the bulk . The plan of this Chapter is as follows: In the first section we intoduce some notations and state our main results. We reintroduce in section 2 a more general model discussed in [17] of β log-gases with Coulomb interaction and construct an approximate transport map between two measures from this model when the number of particles in each cut is fixed. We will see how this approximate transport can lead to universality results in the fixed filling fractions case, and conclude for the initial model in Section 4. The main results of this Chapter are Theorems 3.1.3, 3.1.4 and 3.1.5.

We consider the general β-matrix model. For a subset $\mathcal{A}$ of $\mathbb{R}$ union of disjoint (possibly semi-infinite or infinite) intervals and a potential $V : \mathcal{A} \longrightarrow \mathbb{R}$ and $\beta > 0$, we denote the measure on $\mathcal{A}^N$

$$\mathbb{P}^N_{V,\mathcal{A}}\left(d\lambda_1, \cdots, d\lambda_N\right) := \frac{1}{Z^N_{V,\mathcal{A}}} \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \exp\Big(-N \sum_{1\leq i\leq N} V(\lambda_i)\Big) \prod d\lambda_i \,, \tag{3.1.1}$$

with

$$Z^N_{V,\mathcal{A}} = \int_{\mathcal{A}^N} \prod_{1\leq i<j\leq N} |\lambda_i - \lambda_j|^\beta \exp\Big(- \sum_{1\leq i\leq N} V(\lambda_i)\Big) \prod d\lambda_i.$$

It is well known (see [3], [6] and [30]) that under $\mathbb{P}^N_{V,\mathcal{A}}$ the empirical measure of the eigenvalues converge towards an equilibrium measure:

Proposition 3.1.1. *Assume that $V : \mathcal{A} \longrightarrow \mathbb{R}$ is continuous and if $\infty \in \mathcal{A}$ assume that*

$$\liminf_{x\to\infty} \frac{V(x)}{\beta \log|x|} > 1.$$

then the energy defined by

$$E(\mu) = \int V(x)d\mu(x) - \frac{\beta}{2}\log|x_1 - x_2|d\mu(x_1)d\mu(x_2) \tag{3.1.2}$$

has a unique global minimum on the space $\mathcal{M}_1(\mathcal{A})$ of probability measures on $\mathcal{A}$.

Moreover, under $\mathbb{P}^N_{V,\mathcal{A}}$ the normalized empirical measure $L_N = N^{-1}\sum_{i=1}^N \delta_{\lambda_i}$ converges almost surely and in expectation towards the unique probability measure μ_V which minimizes the energy.
It has compact support A and it is uniquely determined by the existence of a constant C such that:

$$\beta \int_A \log|x-y|d\mu_V(y) - V(x) \leq C\ ,$$

with equality almost everywhere on the support. The support of μ_V is a union of intervals $A = \bigcup_{0\leq h\leq g} [\alpha_{h,-};\alpha_{h,+}]$ with $\alpha_{h,-} < \alpha_{h,+}$ and if V is analytic on a neighbourhood of A,

$$\frac{d\mu_V}{dx} = S(x)\prod_{h=0}^{g}\sqrt{|x-\alpha_{h,-}||x-\alpha_{h,+}|}\ ,$$

with S analytic on a neighbourhood of A.

We make the following assumptions:

Hypothesis 3.1.2.

- V is continuous and goes to infinity faster than $\beta\ \log|x|$ if $\mathcal{A}$ is semi-infinite.
- The support of μ_V is a union of $g+1$ intervals $A = \bigcup_{0\leq h\leq g} A_h$ with $A_h = [\alpha_{h,-};\alpha_{h,+}]$, $\alpha_{h,-} < \alpha_{h,+}$ and

$$\frac{d\mu_V}{dx} = \rho_V(x) = S(x)\prod_{h=0}^{g}\sqrt{|x-\alpha_{h,-}||x-\alpha_{h,+}|} \quad \textit{with } S>0 \textit{ on } [\alpha_{h,-};\alpha_{h,+}]. \tag{3.1.3}$$

- V extends to an holomorphic function on a open neighborhood U of A, $U = \bigcup_{0\leq h\leq g} U_h$, $A_h \subset U_h$ and U_h disjoints.
- The function $V(\cdot) - \beta\int_A \log|\cdot - y|d\mu_V(y)$ achieves its minimum on the support only.

The last hypothesis is useful to ensure a control of large deviations. Before stating the main theorems, we will introduce some notations.

Notations

- For all $0 \leq h \leq g$, $\varepsilon_{\star,h} = \mu_V(A_h)$ and $\boldsymbol{\varepsilon}_\star = (\varepsilon_{\star,0}, \cdots, \varepsilon_{\star,g})$.
- For all $0 \leq h \leq g$, $N_{\star,h} = N\varepsilon_{\star,h}$, $\boldsymbol{N}_\star = N\boldsymbol{\varepsilon}_\star$, and $\lfloor \boldsymbol{N}_\star \rfloor = (\lfloor N\varepsilon_{\star,0} \rfloor, \cdots, \lfloor N\varepsilon_{\star,g} \rfloor)$.
- For a configuration $\boldsymbol{\lambda} \in \mathbb{R}^N$, $N(\boldsymbol{\lambda})$ denotes the vector such that for all $0 \leq h \leq g$, $(N(\boldsymbol{\lambda}))_h$ is the number of eigenvalues in U_h.
- For an index i, we introduce the classical location $E_i^{V,N}$ of the $i-th$ eigenvalue by
$$\int_{-\infty}^{E_i^{V,N}} \rho_V(x)dx = \frac{i}{N}.$$
In the case where the fraction i/N exactly equals to the sum of the mass of the first cuts, we consider the smallest E satisfying the equality.
- For a configuration $\boldsymbol{\lambda} \in \mathbb{R}^N$, let $\lambda_{h,i}$ be the i-th smallest eigenvalue in U_h.
- For a vector $\mathbf{x} \in \mathbb{R}^{g+1}$ and $0 \leq h \leq g$, $[\mathbf{x}]_h = x_0 + \cdots + x_h$ and $[\mathbf{x}]_{-1} = 0$.
- For a vector $\mathbf{x} \in \mathbb{R}^{g+1}$, $0 \leq h \leq g$ and $i \in \mathbb{N}$ we write $i[h, \mathbf{x}] = i - [\mathbf{x}]_{h-1}$.
- For a signed measure ν and a function $f \in L^1(d|\nu|)$ we will write $\nu(f) = \int f d\nu$.

The main goal of this Chapter is to prove universality results in the bulk and at the edge in the multicut regime.
Fixed eigenvalue gaps have been proved to be universal for regular one-cut potentials (see [11], [20]), and their convergence can be obtained using the translation invariance of the eigenvalue gaps as in [42] (see also [79] for the case of the GUE). More precisely, if V is the Gaussian potential $G(\lambda) := \beta\frac{\lambda^2}{4}$ we have for i away from the edge

$$N\rho_V(E_i^{V,N})(\lambda_{i+1} - \lambda_i) \xrightarrow{\mathcal{L}} \mathcal{G}_\beta , \tag{3.1.4}$$

where $\mathcal{G}_\beta$ is some distribution (corresponding to the Gaudin distribution for $\beta = 2$).

Our first Theorem states that this result holds for any multi-cut potential satisfying Hypothesis 3.1.2.

Theorem 3.1.3. *Let $\beta > 0$ and assume that V satisfies Hypothesis 3.1.2.*

Let $i \leq N$ such that for some $\varepsilon > 0$ and $h \in [\![0; g]\!]$, $\varepsilon N < i - [\boldsymbol{N}_\star]_{h-1} < N_{\star,h} - \varepsilon N$. Then

$$N\rho_V(E_i^{V,N})(\lambda_{i+1} - \lambda_i) \xrightarrow{\mathcal{L}} \mathcal{G}_\beta.$$

We now state the results at the edge. Under a Gaussian potential and for general β, the behaviour of the eigenvalues at the edge is described by the Stochastic Airy Operator (We refer to [72]). Indeed, J. Ramírez, B. Rider and B. Virág have shown that the k first rescaled eigenvalues $(N^{2/3}(\lambda_1 + 2), \cdots, N^{2/3}(\lambda_k + 2))$ converge in distribution to $(\Lambda_1, \cdots, \Lambda_k)$ where Λ_i is the i-th smallest eigenvalue of the stochastic Airy operator SAO_β.

In the following result, Φ^h are smooth transport maps (defined later).

Theorem 3.1.4. *Assume that V satisfies Hypothesis 3.1.2. Let $\tilde{\mathbb{P}}^N_{V,\mathcal{A}}$ denote the distribution of the ordered eigenvalues under $\mathbb{P}^N_{V,\mathcal{A}}$.*

If for all $0 \leq h \leq g$ $f_h : \mathbb{R}^m \longrightarrow \mathbb{R}$ is Lipschitz and compactly supported we have:

$$\begin{aligned}
&\lim_{N\to\infty} \int \prod_{0\leq h\leq g} f_h\Big(N^{2/3}(\lambda_{h,1} - \alpha_{h,-}), \cdots, N^{2/3}(\lambda_{h,m} - \alpha_{h,-})\Big) d\tilde{\mathbb{P}}^N_{V,\mathcal{A}} \\
&= \prod_{0\leq h\leq g} \mathbb{E}_{SAO_\beta}\, f_h(\Phi^h(-2)\Lambda_1, \cdots, \Phi^h(-2)\Lambda_m).
\end{aligned}$$

It would also be interesting to study the behaviour of the $i - th$ eigenvalue where $i = [\lfloor \boldsymbol{N}_\star \rfloor]_{h-1} + 1$. This eigenvalue would be typically located at the right edge of the h-th cut or the left edge of the $h+1$-st cut. The following theorem gives the limiting distribution of such eigenvalues. We will use the following fact proved by G.Borot and A.Guionnet in [15]: along the subsequences such that $\boldsymbol{N}_\star \mod \mathbb{Z}^{g+1} \longrightarrow \kappa$ where $\kappa \in [0;1[^{g+1}$ and under $\mathbb{P}^N_{V,\mathcal{A}}$, the vector $N(\boldsymbol{\lambda}) - \lfloor \boldsymbol{N}_\star \rfloor$ converges towards a random discrete Gaussian vector (not necessarily centered).

Theorem 3.1.5. *Let $0 \leq h \leq g$, $i = [\lfloor \boldsymbol{N}_\star \rfloor]_{h-1} + 1$ and $\Delta_h(\boldsymbol{\lambda}) = [\lfloor \boldsymbol{N}_\star \rfloor]_{h-1} - [N(\boldsymbol{\lambda})]_{h-1}$. Define*

$$\xi_h(\boldsymbol{\lambda}) = \mathbf{1}_{\Delta_h(\boldsymbol{\lambda})\geq 0}\, \alpha_h^- + \mathbf{1}_{\Delta_h(\boldsymbol{\lambda})<0}\, \alpha_{h-1}^+ \ ,$$

where the expression above simplifies to α_0^- for $h = 0$. Then along the subsequences $\boldsymbol{N}_\star \mod \mathbb{Z}^{g+1} \longrightarrow \kappa$ and under $\tilde{\mathbb{P}}^N_{V,\mathcal{A}}$

$$\begin{aligned}
\xi_h &\xrightarrow{\ \mathcal{L}\ } \mathbf{1}_{\Delta_{h,\kappa}\geq 0}\, \alpha_h^- + \mathbf{1}_{\Delta_{h,\kappa}<0}\, \alpha_{h-1}^+ \ , \\
N^{2/3}(\lambda_i - \xi_h) &\xrightarrow{\ \mathcal{L}\ } \mathbf{1}_{\Delta_{h,\kappa}\geq 0}\, \Lambda_{\Delta_{h,\kappa}+1}\, \Phi^h(-2) + \mathbf{1}_{\Delta_{h,\kappa}<0}\, \Lambda_{-\Delta_{h,\kappa}}\, \Phi^{h-1}(2) \ ,
\end{aligned}$$

where $(\Lambda_i)_i$ denote the eigenvalues of SAO_β, Φ^h is a transport map introduced later and $\Delta_{h,\kappa}$ is a discrete Gaussian random variable independent from Λ if $1 \leq h \leq g$, and equals to 0 if $h = 0$.

We could state a similar result about the joint distribution of k consecutive eigenvalues as well. We note also that using the transport methods of this Chapter, and adapting the methods presented in [44] (notably Lemma 4.1 and the proof of Corollary 2.8), we could prove universality of the correlation functions in the bulk. This would require rigidity estimate for the fixed filling fractions model introduced in the next section , which was done in [61]. As this universality result has already been proved in [78], we do not continue in this direction.

In order to study the fluctuations of the eigenvalues we place ourselves in the setting of the fixed filling fraction model introduced in [15], in which the number of eigenvalues in each cut is fixed. The idea is to construct an approximate transport between our original measure, and a measure in which the interaction inbetween different cuts has been removed. This measure can then be written as a product measure and we can use the results proved for the one cut regime in [11]. We will construct this map in the second section and show universality in the fixed filling fractions models in Section 3. We will deduce from it the proofs of Theorems 3.1.3, 3.1.4 and 3.1.5 in the fourth section.

3.2 Fixed Filling Fractions

3.2.1 Introducing the model

We consider a slightly different model with a more general type of interaction between the particles and in which the number of particles in each cut is fixed. We will refer to [17] for the known results in this setting. For each $0 \leq h \leq g$, let $B_h = [\beta_{h,-};\beta_{h,+}]$ be a small enlargement of $A_h = [\alpha_{h,-};\alpha_{h;+}]$ included in U_h and $B = \bigcup_{0\leq h\leq g} B_h$. It is well known (see for instance [16]) that under our Hypothesis, the eigenvalues will leave B with an exponentially small probability and we can thus study the behaviour of the eigenvalues under $\mathbb{P}^N_{V,B}$ instead of $\mathbb{P}^N_{V,\mathcal{A}}$ without loss of generality.

We fix $\mathbf{N} = (N_0,\cdots,N_g) \in \mathbb{N}^{g+1}$ such that $\sum_{h=0}^g N_h = N$ and we want to consider a model in which the number of particles in each B_h is fixed equal to N_h. Let $\boldsymbol{\varepsilon} = \mathbf{N}/N \in]0;1[^{g+1}$ and for $T : B\times B \longrightarrow \mathbb{R}$ consider the probability measure on $\mathbf{B} = \prod_{h=0}^g (B_h)^{N_h}$:

$$\begin{aligned}\mathbb{P}^{N,\varepsilon}_{T,B}(\mathbf{d\lambda}) := \frac{1}{Z^{N,\varepsilon}_{T,B}} \prod_{h=0}^{g}\prod_{1\leq i<j\leq N_h} |\lambda_{h,i}-\lambda_{h,j}|^\beta \exp\left(\frac{1}{2}\sum_{0\leq h,h'\leq g}\sum_{\substack{1\leq i\leq N_h\\ 1\leq j\leq N_{h'}}} T(\lambda_{h,i},\lambda_{h',j})\right)\\ \prod_{0\leq h<h'\leq g}\prod_{\substack{1\leq i\leq N_h\\ 1\leq j\leq N_{h'}}} |\lambda_{h,i}-\lambda_{h',j}|^\beta \prod_{h=0}^{g}\prod_{i=1}^{N_h} \mathbf{1}_{B_h}(\lambda_{h,i})d\lambda_{h,i}.\end{aligned} \tag{3.2.5}$$

Note that with $T(\lambda_1,\lambda_2) = -(V(\lambda_1)+V(\lambda_2))$ and without the location constraints, we are in the same setting as in the previous section.

As in the original model, we can prove the following result (see [17]):

Proposition 3.2.1. *Assume that $T : B\times B \longrightarrow \mathbb{R}$ is continuous.*

Assume also that the energy defined by

$$E(\mu) = -\frac{1}{2}\int T(x_1,x_2) + \beta\log|x_1-x_2|d\mu(x_1)d\mu(x_2) \tag{3.2.6}$$

has a unique global minimum on the space $\mathcal{M}^\varepsilon_1(B)$ of probability measures on B satisfying $\mu[B_h] = \varepsilon_h$.

Then under $\mathbb{P}^{N,\varepsilon}_{T,B}$ the normalized empirical measure $L_N = N^{-1}\sum_{h=0}^g\sum_{i=1}^{N_h}\delta_{\lambda_{h,i}}$ converges almost surely and in expectation towards the unique probability measure μ^ε_T which minimizes *the energy.*

Moreover it has compact support A_T^ε and it is uniquely determined by the existence of constants $C_{\varepsilon,h}$ such that:

$$\beta \int_B \log|x-y| d\mu_T^\varepsilon(y) + \int_B T(x,y) d\mu_T^\varepsilon(y) \leq C_{\varepsilon,h} \quad on \quad B_h \tag{3.2.7}$$

with equality almost everywhere on the support. The support of μ_T^ε is a union of $l+1$ intervals $A_T^\varepsilon = \bigcup_{0\leq h\leq l} [\alpha_{h,-}^{T,\varepsilon}; \alpha_{h,+}^{T,\varepsilon}]$ with $\alpha_{h,-}^{T,\varepsilon} < \alpha_{h,+}^{T,\varepsilon}$, $l \geq g$ and if T is analytic on a neighbourhood of A_T^ε ,

$$\frac{d\mu_T^\varepsilon}{dx} = S_T^\varepsilon(x) \prod_{h=0}^{l} \sqrt{|x-\alpha_{h,-}^{T,\varepsilon}||x-\alpha_{h,+}^{T,\varepsilon}|} \, ,$$

with S_T^ε analytic on a neighbourhood of A_T^ε.

We point out the fact that the previous theorem is also valid in the unconstrained case. In that case, we denote by μ_T the equilibrium measure. Let $\varepsilon_{\star,T} = (\mu_T(B_h))_{0\leq h\leq g}$. Then it is obvious that $\mu_T^{\varepsilon_{\star,T}} = \mu_T$. It is shown in [17] that we have the following:

Lemma 3.2.2. *If T extends to an analytic function on a neighbourhood of B and the energy definied in (3.2.6) has a unique minimizer over $\mathcal{M}_1(B)$ then for ε close enough from $\varepsilon_\star$, the energy has a unique minimizer over $\mathcal{M}_1^\varepsilon(B)$ and the number of cuts of the support of μ_T^ε and μ_T are the same. Moreover, $\alpha_{h,-}^{T,\varepsilon}$, $\alpha_{h,+}^{T,\varepsilon}$ and S_T^ε are smooth functions of ε (for the L^∞ norm on B).*

They also prove a control of large deviations of the largest eigenvalue under $\mathbb{P}_{T,B}^{N,\varepsilon}$.

We define the effective potential as

$$\tilde{T}^\varepsilon(x) = \beta \int_B \log|x-y| d\mu_T^\varepsilon(y) + \int_B T(x,y) d\mu_T^\varepsilon(y) - C_{\varepsilon,h} \quad on \quad B_h. \tag{3.2.8}$$

Lemma 3.2.3. *Let T satisfy the conditions of the previous theorem. Then for any closed $F \subset B \setminus A_T^\varepsilon$ and open $O \subset B \setminus A_T^\varepsilon$ we have*

$$\begin{cases} \limsup \frac{1}{N} \log \mathbb{P}_{T,B}^{N,\varepsilon}(\exists i \ \lambda_i \in F) \leq \sup_{x\in F} \tilde{T}^\varepsilon(x). \\ \liminf \frac{1}{N} \log \mathbb{P}_{T,B}^{N,\varepsilon}(\exists i \ \lambda_i \in O) \geq \sup_{x\in O} \tilde{T}^\varepsilon(x). \end{cases}$$

We consider a potential V on $\mathcal{A}$ satifying Hypothethis 3.1.2 and the potentials $T_0(x,y) = -(V(x)+V(y))$ and $T_1(x,y) = -(\tilde{V}^\varepsilon(x) + \tilde{V}^\varepsilon(y) + W(x,y))$ where

$$W(x,y) = \begin{cases} \beta \log(x-y) & if \ x \in U_h \ , \ y \in U_{h'} \ \ h > h' \\ \beta \log(y-x) & if \ x \in U_h \ , \ y \in U_{h'} \ \ h < h' \\ 0 & if \ x \in U_h \ , \ y \in U_h \end{cases}$$

and

$$\tilde{V}^\varepsilon(x) = V(x) - \int W(x,y) d\mu_V^\varepsilon(y).$$

The key point is that $d\mathbb{P}^{N,\varepsilon}_{T_1,B}$ is a product measure as the interaction between cuts has been removed. Moreover, we can check by the characterization (3.2.7) that

$$\mu_V^\varepsilon = \mu_{T_0}^\varepsilon = \mu_{T_1}^\varepsilon.$$

We now consider

$$T_t = (1-t)T_0 + t\, T_1 \quad , \quad t \in [0;1]. \tag{3.2.9}$$

Still by (3.2.7) we can check that for all $t \in [0;1]$ we have:

$$\mu_{T_t}^\varepsilon = (1-t)\mu_{T_0}^\varepsilon + t\mu_{T_1}^\varepsilon = \mu_V^\varepsilon.$$

Remark 3.2.4. *Note that, by Lemma 2.2, for ε in a small neighbourhood of $\varepsilon_\star$ (that we will denote $\tilde{\mathcal{E}}$) the support A^ε of $\mu_{T_t}^\varepsilon = \mu_V^\varepsilon$ has $g+1$ cut and we can write*

$$d\mu_{T_t}^\varepsilon = d\mu_V^\varepsilon = S^\varepsilon(x) \prod_{h=0}^{g} \sqrt{|x - \alpha_{h,-}^\varepsilon||x - \alpha_{h,+}^\varepsilon|}dx \ , \tag{3.2.10}$$

with S^ε positive on A^ε.

Remark 3.2.5. *Note also that by the last point of Hypothesis 3.1.2 and by Lemma 2.2, if we fix a closed interval $F \subset B \setminus A$, then for ε close enough to $\varepsilon_\star$ and all $t \in [0;1]$, $\tilde{T}_t^\varepsilon < 0$ on F.*

The goal is to build first an approximate transport map between the measures $d\mathbb{P}^{N,\varepsilon}_{T_t,B}$ for a fixed ε in $\tilde{\mathcal{E}}$ i.e find a map $X_1^{N,\varepsilon}$ that satisfies for all $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ bounded measurable function

$$|\int f(X_1^{N,\varepsilon})\, d\mathbb{P}^{N,\varepsilon}_{V,B} - \int f\, d\mathbb{P}^{N,\varepsilon}_{T_1,B}| \leq C\,\|f\|_\infty \frac{(\log N)^3}{N}. \tag{3.2.11}$$

We will see that we can build a transport map depending smoothly on ε and show universality in the fixed filling model. We will then use this result to prove universality in the original model.

Proposition 3.2.6. *Assume that V satisfies Hypothesis 3.1.2, and that T_t is as defined previously. Let $\mathbf{N} = (N_0, \cdots, N_g)$ such that $\varepsilon = \mathbf{N}/N$ is in $\tilde{\mathcal{E}}$ and $\tilde{\mathbb{P}}^{N,\varepsilon}_{T,B}$ denote the distribution of the ordered eigenvalues under $\mathbb{P}^{N,\varepsilon}_{T,B}$. Then for a constant C independent of ε and N, and if for all $0 \leq h \leq g$ $f_h : \mathbb{R}^m \longrightarrow \mathbb{R}$ is Lipschitz supported inside $[-M,M]^m$ we have:*

1. *Eigenvalue gaps in the Bulk*

$$\Big| \int \prod_{0\le h\le g} f_h\Big(N(\lambda_{h,i_h+1}-\lambda_{h,i_h}),\cdots,N(\lambda_{h,i_h+m}-\lambda_{h,i_h})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B} - \int \prod_{0\le h\le g} f_h\Big(N(\lambda_{h,i_h+1}-\lambda_{h,i_h}),\cdots,N(\lambda_{h,i_h+m}-\lambda_{h,i_h})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{T_1,B}\Big| \le C\frac{(\log N)^3}{N}\|f\|_\infty + C(\sqrt{m}\frac{(\log N)^2}{N^{1/2}} + M\frac{(\log N)}{N^{1/2}})\|\nabla f\|_\infty .$$

2. *Eigenvalue gaps at the Edge*

$$\Big| \int \prod_{0\le h\le g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha^\varepsilon_{h,-}),\cdots,N^{2/3}(\lambda_{h,m}-\alpha^\varepsilon_{h,-})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B} - \int \prod_{0\le h\le g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha^\varepsilon_{h,-}),\cdots,N^{2/3}(\lambda_{h,m}-\alpha^\varepsilon_{h,-})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{T_1,B}\Big| \le C\frac{(\log N)^3}{N}\|f\|_\infty + C(\sqrt{m}\frac{(\log N)^2}{N^{5/6}} + \frac{\log N}{N^{1/3}})\|\nabla f\|_\infty$$

where we defined $f : \mathbb{R}^{m(g+1)} \longrightarrow \mathbb{R}$ *by* $f(\boldsymbol{x}_0,\cdots,\boldsymbol{x}_g) = \prod_{0\le h\le g} f_h(\boldsymbol{x}_h)$.

We deduce the following corollary from the results obtained in the one-cut regime in [11], and from the fact that $\tilde{\mathbb{P}}^{N,\varepsilon}_{T_1,B}$ is a product measure.

Corollary 3.2.7. *Assume the same hypothesis as in the precedent proposition. We write* $\mu^\varepsilon_V = \sum_{0\le h\le g}\varepsilon_h\, \mu^{\varepsilon,h}_V$ *where* $\mu^{\varepsilon,h}_V$ *has connected support . For some transport maps* $\Phi^{\varepsilon,h}$ *from* μ_G *to* $\mu^{\varepsilon,h}_V$,

1. *Eigenvalue gaps in the Bulk*

$$\Big| \int \prod_{0\le h\le g} f_h\Big(N(\lambda_{h,i_h+1}-\lambda_{h,i_h}),\cdots,N(\lambda_{h,i_h+m}-\lambda_{h,i_h})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B} - \prod_{0\le h\le g}\left(\int f_h\Big(N(\Phi^{\varepsilon,h})'(\lambda_{i_h})(\lambda_{i_h+1}-\lambda_{i_h}),\cdots,N(\Phi^{\varepsilon,h})'(\lambda_{i_h})(\lambda_{i_h+m}-\lambda_{i_h})\Big) d\tilde{\mathbb{P}}^{N_h}_G\right)\Big| \le C\frac{(\log N)^3}{N}\|f\|_\infty + C(\sqrt{m}\frac{(\log N)^2}{N^{1/2}} + M\frac{(\log N)}{N^{1/2}} + \frac{M^2}{N})\|\nabla f\|_\infty$$

2. *Eigenvalue gaps at the Edge*

$$\Big| \int \prod_{0\le h\le g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha^{\varepsilon}_{h,-}),\cdots,N^{2/3}(\lambda_{h,m}-\alpha^{\varepsilon}_{h,-})\Big)d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B}$$
$$-\prod_{0\le h\le g}\left(\int f_h\Big(N^{2/3}(\Phi^{\varepsilon,h})'(-2)(\lambda_1+2),\cdots,N^{2/3}(\Phi^{\varepsilon,h})'(-2)(\lambda_m+2)\Big)d\tilde{\mathbb{P}}^{N_h}_{G}\right)\Big|$$
$$\le C\frac{(\log N)^3}{N}\|f\|_\infty + C(\sqrt{m}\frac{(\log N)^2}{N^{5/6}}+\frac{\log N}{N^{1/3}}+\frac{M^2}{N^{4/3}})\|\nabla f\|_\infty .$$

The proof of the theorem will be similar to what has already been done in the one-cut case, one major difference being the inversion of the operator Ξ introduced in the previous Chapter.

3.2.2 Approximate Monge Ampère Equation

The analysis done in the one-cut regime suggests to look at the transport as the flow of an approximate solution to the Monge Ampère equation $\mathbf{Y}^{N,\varepsilon}_t = (\mathbf{Y}^{N,\varepsilon}_{0,t},\cdots,\mathbf{Y}^{N,\varepsilon}_{g,t}) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ where $\mathbf{Y}^{N,\varepsilon}_{h,t} : \mathbb{R}^N \longrightarrow \mathbb{R}^{N_h}$ solves the following equation:

$$\begin{aligned}\operatorname{div}(\mathbf{Y}^{N,\varepsilon}_t) &= c^{N,\varepsilon}_t - \beta\sum_{h=0}^{g}\sum_{1\le i<j\le N_h}\frac{\mathbf{Y}^{N,\varepsilon}_{h,i,t}-\mathbf{Y}^{N,\varepsilon}_{h,j,t}}{\lambda_{h,i}-\lambda_{h,j}} - \beta\sum_{0\le h<h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}\frac{\mathbf{Y}^{N,\varepsilon}_{h,i,t}-\mathbf{Y}^{N,\varepsilon}_{h',j,t}}{\lambda_{h,i}-\lambda_{h',j}}\\ &-\sum_{0\le h,h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}(\partial_1 T(\lambda_{h,i},\lambda_{h',j})\mathbf{Y}^{N,\varepsilon}_{h,i,t}-\frac{1}{2}W(\lambda_{h,i},\lambda_{h',j})) - N\sum_{0\le h\le g}\sum_{1\le i\le N_h}\int W(\lambda_{h,i},z)d\mu^{\varepsilon}_V(z)\end{aligned} \tag{3.2.12}$$

where

$$\begin{aligned}c^{N,\varepsilon}_t &= \int\left(N\sum_{0\le h\le g}\sum_{1\le i\le N_h}\int W(\lambda_{h,i},z)d\mu^{\varepsilon}_V(z) - \frac{1}{2}\sum_{0\le h,h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}W(\lambda_{h,i},\lambda_{h',j})\right)d\mathbb{P}^{N,\varepsilon}_{T_t,B}(\boldsymbol{\lambda})\\ &= \partial_t\log(Z^{N,\varepsilon}_{V_t,B}).\end{aligned}$$

Let $\mathcal{R}^{N,\varepsilon}_t(\mathbf{Y}^{N,\varepsilon})$ the error term defined as

$$\begin{aligned}\mathcal{R}^{N,\varepsilon}_t(\mathbf{Y}^{N,\varepsilon}) &= \beta\sum_{h=0}^{g}\sum_{1\le i<j\le N_h}\frac{\mathbf{Y}^{N,\varepsilon}_{h,i,t}-\mathbf{Y}^{N,\varepsilon}_{h,j,t}}{\lambda_{h,i}-\lambda_{h,j}} + \beta\sum_{0\le h<h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}\frac{\mathbf{Y}^{N,\varepsilon}_{h,i,t}-\mathbf{Y}^{N,\varepsilon}_{h',j,t}}{\lambda_{h,i}-\lambda_{h',j}}\\ &+\sum_{0\le h,h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}(\partial_1 T(\lambda_{h,i},\lambda_{h',j})\mathbf{Y}^{N,\varepsilon}_{h,i,t}-\frac{1}{2}W(\lambda_{h,i},\lambda_{h',j})) + N\sum_{0\le h\le g}\sum_{1\le i\le N_h}\int W(\lambda_{h,i},z)d\mu^{\varepsilon}_V(z)\\ &+\operatorname{div}(\mathbf{Y}^{N,\varepsilon}_t) - c^{N,\varepsilon}_t.\end{aligned} \tag{3.2.13}$$

We have the following stability lemma

Lemma 3.2.8. *Let $\mathbf{Y}_t^{N,\varepsilon} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a smooth vector field and let $X^{N,\varepsilon}$ be its flow:*

$$\dot{X}_t^{N,\varepsilon} = \mathbf{Y}_t^{N,\varepsilon}(X_t^{N,\varepsilon}) \qquad X_0^{N,\varepsilon} = Id. \tag{3.2.14}$$

Assume that $\mathbf{Y}_t^{N,\varepsilon}$ vanishes on the boundary of $\mathbf{B}$.
Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a bounded measurable function. Then

$$\left| \int f(X_t^{N,\varepsilon})\ d\mathbb{P}_{V,B}^{N,\varepsilon} - \int f\ d\mathbb{P}_{T_t,B}^{N,\varepsilon} \right| \leq \|f\|_\infty \int_0^t \left\| \mathcal{R}_s^{N,\varepsilon}(\mathbf{Y}^{N,\varepsilon}) \right\|_{L^1(\mathbb{P}_{T_s,B}^{N,\varepsilon})} ds.$$

Proof. Let

$$\rho_t(\boldsymbol{\lambda}) := \frac{1}{Z_{T,B}^{N,\varepsilon}} \prod_{h=0}^{g} \prod_{1\leq i<j\leq N_h} |\lambda_{h,i} - \lambda_{h,j}|^\beta \exp\left(-\frac{1}{2} \sum_{0\leq h,h'\leq g} \sum_{\substack{1\leq i\leq N_h \\ 1\leq j\leq N_{h'}}} T_t(\lambda_{i,h}, \lambda_{j,h'})\right)$$
$$\prod_{0\leq h<h'\leq g} \prod_{\substack{1\leq i\leq N_h \\ 1\leq j\leq N_{h'}}} |\lambda_{h,i} - \lambda_{h',j}|^\beta$$

and $JX_t^{N,\varepsilon}$ denote the Jacobian of $X_t^{N,\varepsilon}$. As $\mathbf{Y}_t^{N,\varepsilon}$ vanishes on the boundary of $\mathbf{B}$, $X_t^{N,\varepsilon}(\mathbf{B}) = \mathbf{B}$. By the change of variable formula we have

$$\begin{aligned} \int f\ d\mathbb{P}_{T_t,B}^{N,\varepsilon} &= \int_{\mathbf{B}} f(\boldsymbol{\lambda})\rho_t(\boldsymbol{\lambda}) \boldsymbol{d\lambda} = \int_{X_t^{N,\varepsilon}(\mathbf{B})} f(\boldsymbol{\lambda})\rho_t(\boldsymbol{\lambda}) \boldsymbol{d\lambda} \\ &= \int_{\mathbf{B}} f(X_t^{N,\varepsilon})\rho_t(X_t^{N,\varepsilon}) JX_t^{N,\varepsilon} \boldsymbol{d\lambda} \end{aligned}$$

Thus we have

$$\left| \int f(X_t^{N,\varepsilon})\ d\mathbb{P}_{T_0,B}^{N,\varepsilon} - \int f\ d\mathbb{P}_{T_t,B}^{N,\varepsilon} \right| \leq \|f\|_\infty \int_{\mathbf{B}} |\rho_0(\boldsymbol{\lambda}) - \rho_t(X_t^{N,\varepsilon}) JX_t^{N,\varepsilon}| \boldsymbol{d\lambda}.$$

Let

$$\Delta_t = \partial_t \int_{\mathbf{B}} |\rho_0(\boldsymbol{\lambda}) - \rho_t(X_t^{N,\varepsilon}) JX_t^{N,\varepsilon}| \boldsymbol{d\lambda}.$$

Using $\partial_t(JX_t^{N,\varepsilon}) = \operatorname{div}(\mathbf{Y}_t^{N,\varepsilon}) JX_t^{N,\varepsilon}$ we have

$$\begin{aligned} \Delta_t &\leq \int_{\mathbf{B}} |\partial_t \left(JX_t^{N,\varepsilon} \rho_t(X_t^{N,\varepsilon}) \right)| \boldsymbol{d\lambda} \\ &= \int_{\mathbf{B}} |\operatorname{div}(\mathbf{Y}_t^{N,\varepsilon}) JX_t^{N,\varepsilon} \rho_t(X_t^{N,\varepsilon}) + JX_t^{N,\varepsilon} (\partial_t \rho_t)(X_t^{N,\varepsilon}) + JX_t^{N,\varepsilon} \nabla \rho_t(X_t^{N,\varepsilon}) \dot{X}_t^{N,\varepsilon}| \boldsymbol{d\lambda} \\ &= \int |\mathcal{R}_t^{N,\varepsilon}(\mathbf{Y}^{N,\varepsilon})| d\mathbb{P}_{T_t,B}^{N,\varepsilon} \end{aligned}$$

and this gives the lemma.

□

3.2.3 Constructing an Approximate Solution

The construction of the approximate solution will be very similar to what was done in the previous Chapter.

We fix $t \in [0;1]$, $\mathbf{N} = (N_0, \cdots, N_g) \in \mathbb{N}^{g+1}$ such that $\sum_{h=0}^{g} N_h = N$ and set $\boldsymbol{\varepsilon} = \mathbf{N}/N \in]0;1[^{g+1}$.

Let

$$L_N = \frac{1}{N}\sum_{h,i}\delta_{\lambda_{h,i}} \quad , \quad M_N = \sum_{h,i}\delta_{\lambda_{h,i}} - N\mu_V^{\varepsilon}.$$

We look for a map $\mathbf{Y}_t^{N,\varepsilon} = (\mathbf{Y}_{0,1,t}^{N,\varepsilon}, \cdots, \mathbf{Y}_{g,N_g,t}^{N,\varepsilon}) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ approximately solving (3.2.12). As in the one-cut regime, we make the following ansatz:

$$\mathbf{Y}_{h,i,t}^{N,\varepsilon}(\boldsymbol{\lambda}) = \frac{1}{N}\mathbf{y}_{1,t}^{\varepsilon}(\lambda_{h,i}) + \frac{1}{N}\xi_t^{\varepsilon}(\lambda_{h,i}, M_N) \ , \ \xi_t^{\varepsilon}(x, M_N) = \int \mathbf{z}_t^{\varepsilon}(x,y)dM_N(y) \qquad (3.2.15)$$

for some functions $\mathbf{y}_{1,t}^{\varepsilon} : \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathbf{z}_t^{\varepsilon} : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Proposition 3.2.9. *Let V satisfy Hypothesis 3.1.2 and T_t is as in (3.2.9) . Then there are $\mathbf{y}_{1,t}^{\varepsilon}$ in $C^\infty(\mathbb{R})$ and $\mathbf{z}_t^{\varepsilon}$ in $C^\infty(\mathbb{R}^2)$ such that for a constant C, for all $t \in [0;1]$ and $\boldsymbol{\varepsilon} \in \tilde{\mathcal{E}}$:*

$$\left\|\mathcal{R}_t^{N,\varepsilon}(\mathbf{Y}^{N,\varepsilon})\right\|_{L^1(\mathbb{P}_{T_t,B}^{N,\varepsilon})} \leq C\frac{(\log N)^3}{N}.$$

Using the substitution (3.2.15), we have to find equations for $\mathbf{y}_{1,t}^{\varepsilon}$ and $\mathbf{z}_t^{\varepsilon}$. To simplify the notations,we will write $\mathcal{R}$ instead of $\mathcal{R}_t^{N,\varepsilon}(\mathbf{Y}^{N,\varepsilon})$. We obtain:

$$\begin{aligned}
\mathcal{R} = &-\frac{N^2}{2}\iint WdL_NdL_N + N^2\int WdL_Nd\mu_V^{\varepsilon} \\
&+ \frac{\beta N}{2}\iint \frac{\mathbf{y}_{1,t}^{\varepsilon}(x) - \mathbf{y}_{1,t}^{\varepsilon}(y)}{x-y}dL_N(x)dL_N(y) + N\int \partial_1 T_t(x,y)\mathbf{y}_{1,t}^{\varepsilon}(x)dL_N(x)dL_N(y) \\
&+ \frac{\beta N}{2}\iint \frac{\xi_t^{\varepsilon}(x,M_N) - \xi_t^{\varepsilon}(y,M_N)}{x-y}dL_N(x)dL_N(y) + N\int \partial_1 T_t(x,y)\xi_t^{\varepsilon}(x,M_N)dL_N(x)dL_N(y) \\
&+ \frac{1}{N}\eta(M_N) + \left(1-\frac{\beta}{2}\right)\int \mathbf{y}_{1,t}^{\varepsilon}{}'dL_N + \left(1-\frac{\beta}{2}\right)\int \partial_1\xi_t^{\varepsilon}(x,M_N)dL_N(x) + \tilde{c}_t^N
\end{aligned}$$

where $\tilde{c}_t^N$ is a constant and for any measure ν we set

$$\eta(\nu) = \int \partial_2 \mathbf{z}_t^{\varepsilon}(y,y)d\nu(y).$$

We use equilibrium relations to recenter L_N by μ_V^{ε}. Consider f a bounded measurable function on B and $\mu_{V,\delta}^{\varepsilon} = (x + \delta f(x))\#\mu_V^{\varepsilon}$. Then as for δ small enough $\mu_{V,\delta}^{\varepsilon}(B_h) = \varepsilon_h$

for all $0 \leq h \leq g$, we have $E(\mu^{\varepsilon}_{V,\delta}) \geq E(\mu^{\varepsilon}_V)$ where we defined the energy in (3.1.2). By differentiating at $\delta = 0$ we obtain

$$\frac{\beta}{2}\iint \frac{f(x)-f(y)}{x-y} d\mu^{\varepsilon}_V(x)d\mu^{\varepsilon}_V(y) + \int \partial_1 T_t(x,y) f(x) d\mu^{\varepsilon}_V(x)d\mu^{\varepsilon}_V(y) = 0. \tag{3.2.16}$$

Thus, if we define the operator Ξ acting on smooth functions $f : B \longrightarrow \mathbb{R}$ by

$$\Xi f(x) = \int \left[\beta\, \frac{f(x)-f(y)}{x-y} + \partial_1 T_t(x,y) f(x) + \partial_2 T_t(x,y) f(y)\right] d\mu^{\varepsilon}_V(y) ,$$

we obtain

$$\begin{aligned}
&\frac{\beta}{2}\iint \frac{f(x)-f(y)}{x-y} dL_N(x)dL_N(y) + \int \partial_1 T_t(x,y) f(x) dL_N(x) dL_N(y) \\
&= \frac{1}{N}\int \Xi f dM_N + \frac{1}{N^2}\left[\frac{\beta}{2}\iint \frac{f(x)-f(y)}{x-y} dM_N(x)dM_N(y) + \iint \partial_1 T_t(x,y) f(x) dM_N(x) dM_N(y)\right] .
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
\mathcal{R} &= \int \left[\Xi \mathbf{y}^{\varepsilon}_{1,t} + \left(1-\frac{\beta}{2}\right)\int \partial_1 \mathbf{z}^{\varepsilon}_t(z,\cdot) d\mu^{\varepsilon}_V(z)\right] dM_N \\
&+ \iint \left[\Xi \mathbf{z}^{\varepsilon}_t(\cdot,y)[x] - \frac{1}{2}W(x,y)\right] dM_N(x)dM_N(y) + C^{N,\varepsilon}_t + E
\end{aligned}$$

with

$$\Xi \mathbf{z}^{\varepsilon}_t(\cdot,y)[x] = \int \left[\beta \frac{\mathbf{z}^{\varepsilon}_t(x,y) - \mathbf{z}^{\varepsilon}_t(z,y)}{x-z} + \partial_1 T_t(x,z)\mathbf{z}^{\varepsilon}_t(x,y) + \partial_2 T_t(x,z)\mathbf{z}^{\varepsilon}_t(z,y)\right] d\mu^{\varepsilon}_V(z)$$

where $C^{N,\varepsilon}_t$ is deterministic and E is an error term:

$$\begin{aligned}
E &= \frac{1}{N}\int \partial_2 \mathbf{z}^{\varepsilon}_t(x,x) dM_N(x) + \frac{1}{N}\left(1-\frac{\beta}{2}\right)\int \mathbf{y}^{\varepsilon\,\prime}_{1,t} dM_N \\
&+ \frac{1}{N}\left(1-\frac{\beta}{2}\right)\iint \partial_1 \mathbf{z}^{\varepsilon}_t(x,y) dM_N(x)dM_N(y) \\
&+ \frac{1}{N}\iint \left[\frac{\beta}{2}\frac{\mathbf{y}^{\varepsilon}_{1,t}(x) - \mathbf{y}^{\varepsilon}_{1,t}(y)}{x-y} + \partial_1 T_t(x,y)\mathbf{y}^{\varepsilon}_{1,t}(x)\right] dM_N(x)dM_N(y) \\
&+ \frac{1}{N}\iiint \left[\frac{\beta}{2}\frac{\mathbf{z}^{\varepsilon}_t(x,y) - \mathbf{z}^{\varepsilon}_t(z,y)}{x-z} + \partial_1 T_t(x,z)\mathbf{z}^{\varepsilon}_t(x,y)\right] dM_N(x)dM_N(y)dM_N(z)
\end{aligned} \tag{3.2.17}$$

To make $\mathcal{R}$ small we need

$$\begin{cases}
\Xi \mathbf{z}^{\varepsilon}_t(\cdot,y)[x] = \frac{1}{2}W(x,y) + \kappa_1(x,y) , \\
\Xi \mathbf{y}^{\varepsilon}_{1,t} = \left(\frac{\beta}{2}-1\right)\int \partial_1 \mathbf{z}^{\varepsilon}_t(z,\cdot) d\mu^{\varepsilon}_V(z) + \kappa_2 ,
\end{cases}$$

where κ_2 and $\kappa_1(\cdot, y)$ are functions on B constant on each B_h.

The following lemma shows how to invert Ξ and will give us the desired functions. We will denote by $\mathcal{O}(U)$ the set of holomorphic functions on U.

Lemma 3.2.10. *Let V satisfy Hypothesis 3.1.2 , T_t as in (3.2.9) and $\varepsilon = \mathbf{N}/N$ in $\tilde{\mathcal{E}}$. The support of μ_V^ε is a union of $g+1$ intervals $A^\varepsilon = \bigcup_{0 \le h \le g} [\alpha_{h,-}^\varepsilon; \alpha_{h,+}^\varepsilon]$ with $\alpha_{h,-}^\varepsilon < \alpha_{h,+}^\varepsilon$ and,*

$$\frac{d\mu_V^\varepsilon}{dx} = S(x) \prod \sqrt{|x - \alpha_{h,-}^\varepsilon||x - \alpha_{h,+}^\varepsilon|}$$

with S positive on A^ε.
Let $k \in \mathcal{O}(U)$ and set for $f \in \mathcal{O}(U)$

$$\Xi f(x) = \int \left[\beta \frac{f(x) - f(y)}{x - y} + \partial_1 T_t(x, y) f(x) + \partial_2 T_t(x, y) f(y) \right] d\mu_{T_t}^\varepsilon(y) \qquad \forall x \in U.$$

Then there exists a unique function κ_k on U constant on each U_h such that the equation

$$\Xi f = k + \kappa_k$$

has a solution in $\mathcal{O}(U)$. Moreover, for all $x \in U_h$

$$f(x) = -\frac{1}{2\beta\pi^2 \sigma(x) \sigma_h(x) S(x)} \left[\oint \frac{i\sigma_h(\xi)(k(\xi) + c_h^1)}{(\xi - x)} d\xi + c_h^2 \right] , \qquad (3.2.18)$$

where the contour surrounds x and A_h^ε in U_h and

$$\sigma^2(x) = \prod_{h'} \left(x - \alpha_{h',-}^\varepsilon\right) \left(x - \alpha_{h',+}^\varepsilon\right)$$
$$\sigma(x) \underset{x \to \infty}{\sim} x^{g+1}$$
$$\sigma_h^2(x) = \left(x - \alpha_{h,-}^\varepsilon\right) \left(x - \alpha_{h,+}^\varepsilon\right)$$
$$\sigma_h(x) \underset{x \to \infty}{\sim} x$$

and the constants c_h^1 and c_h^2 are chosen in a way such that the expression under the bracket vanishes at $x = \alpha_{h,-}^\varepsilon$ and $x = \alpha_{h,+}^\varepsilon$ for each h (see the following Lemma).

Moreover f satisfies for all j

$$\|f\|_{C^j(B)} \le C_j \|k\|_{C^{j+2}(B)} \qquad (3.2.19)$$

for some constants C_j. We will denote f by $\Xi^{-1} k$.

Before proving this lemma we need another lemma

Lemma 3.2.11. *Let $V \in \mathcal{O}(U)$ and μ_V^ϵ as in the previous lemma.*

Then for all $0 \leq h \leq g$ the linear operator

$$\Theta_h := \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(c^1, c^2) \longrightarrow \left(c^1 \oint \frac{\sigma_h(\xi)}{(\xi - \alpha_{h,-}^\epsilon)} d\xi + c^2, c^1 \oint \frac{\sigma_h(\xi)}{(\xi - \alpha_{h,+}^\epsilon)} d\xi + c^2\right)$$

is invertible and Θ_h^{-1} is analytic.

Proof. This comes easily from the fact that

$$\int_{\alpha_{h,-}^\epsilon}^{\alpha_{h,+}^\epsilon} \frac{\sqrt{(y - \alpha_{h,-}^\epsilon)(\alpha_{h,+}^\epsilon - y)}}{y - \alpha_{h,-}^\epsilon} dy = \pi \, \frac{\alpha_{h,+}^\epsilon - \alpha_{h,-}^\epsilon}{2}$$
$$\int_{\alpha_{h,-}^\epsilon}^{\alpha_{h,+}^\epsilon} \frac{\sqrt{(y - \alpha_{h,-}^\epsilon)(\alpha_{h,+}^\epsilon - y)}}{y - \alpha_{h,+}^\epsilon} dy = \pi \, \frac{\alpha_{h,-}^\epsilon - \alpha_{h,+}^\epsilon}{2}$$

□

Proof of Lemma 3.2.10. By the identity (3.2.16) with $f(x) = (z - x)^{-1}$ and z outside the support, we obtain that the Stieltjes transform $G(z) = \int \frac{1}{z-y} d\mu_V^\epsilon(y)$ satisfies

$$\frac{\beta}{2} G(z)^2 + G(z) \int \partial_1 T_t(z, y) d\mu_V^\epsilon(y) + F(z) = 0 \quad \text{with } F(z) = \iint \frac{\partial_1 T_t(\tilde{y}, y) - \partial_1 T_t(z, y)}{\tilde{y} - z} d\mu_V^\epsilon(\tilde{y}) d\mu_V^\epsilon(y)$$

and this gives

$$\beta G(z) + \int \partial_1 T_t(z, y) d\mu_V^\epsilon(y) = -\sqrt{\left(\int \partial_1 T_t(z, y) d\mu_V^\epsilon(y)\right)^2 - 2\beta F(z)}.$$

As $-\pi^{-1}\mathcal{I}G(z)$ converges towards the density of μ_V^ϵ as z goes to the real axis (see for instance [3], Section 2.4 for the basic properties of the Stieltjes transform) and the quantity under the square root converges to a real number, this number has to be negative on the support (otherwise the density would vanish) and thus for $x \in A^\epsilon$

$$\frac{d\mu_V^\epsilon}{dx} = \frac{1}{\beta\pi} \sqrt{2\beta F(x) - \left(\int \partial_1 T_t(x, y) d\mu_V^\epsilon(y)\right)^2} \, .$$

Noticing that σ becomes purely imaginary when z converges towards the support, we may write

$$\beta G(z) + \int \partial_1 T_t(z, y) d\mu_V^\epsilon(y) = \beta\pi \tilde{S}(z)\sigma(z) \tag{3.2.20}$$

where $\tilde{S}$ is an analytic extension of S in U (we can assume $\tilde{S}$ non zero on U by possibly shrinking U). We will keep writing S for $\tilde{S}$.

For f analytic in $U \setminus A^\epsilon$ and $z \in U \setminus A^\epsilon$ let

$$\tilde{\Xi} f(z) = \frac{i}{2} \oint \left(\frac{\beta f(\xi)}{z-\xi} - \partial_2 T_t(z,\xi) f(\xi) \right) S(\xi)\sigma(\xi) d\xi$$

where the contour surrounds z and each A_h^ϵ. Then $\tilde{\Xi} f \in \mathcal{O}(U \setminus A^\epsilon)$ and, noticing that $-iS(x+i\delta)\sigma(x+i\delta) \underset{\delta \to 0^+}{\longrightarrow} \frac{d\mu_V^\epsilon}{dx}$, we have

$$\begin{aligned}
\Xi f(z) &= -\int \left(\beta \frac{f(y)}{z-y} - \partial_2 T_t(z,y) f(y) \right) d\mu_V^\epsilon(y) + f(z) \left(\int \partial_1 T_t(z,y) d\mu_V^\epsilon(y) + \beta \int \frac{d\mu_V^\epsilon(y)}{z-y} \right) \\
&= \frac{i}{2} \oint \left(\frac{\beta f(\xi)}{z-\xi} - \partial_2 T_t(z,\xi) f(\xi) \right) S(\xi)\sigma(\xi) d\xi + \beta\pi f(z) S(z) \sigma(z) \\
&= \tilde{\Xi} f(z)
\end{aligned} \tag{3.2.21}$$

where the contour surrounds each A_h^ϵ (but not z), and we used Cauchy's formula and (3.2.20). If furthermore $f \in \mathcal{O}(U)$, by continuity this formula extends to $z \in U$.

Let $k \in \mathcal{O}(U)$. We want to show that the function defined on each U_h by

$$f(z) = -\frac{1}{2\beta\pi^2 \sigma(z) \sigma_h(z) S(z)} \left[\oint \frac{i\sigma_h(\xi)(k(\xi) + c_h^1)}{(\xi - z)} d\xi + c_h^2 \right]$$

where the contour surronds A_h^ϵ and lays in U_h, and c_h^1 and c_h^2 are defined as in the statement of the lemma, is a solution of $\Xi f = k + \kappa_k$ in $\mathcal{O}(U)$. The fact that $f \in \mathcal{O}(U)$ is clear (the function is meromorphic and the poles are removable by construction of c^1 and c^2). Thus, by previous remark, it suffices to prove that $\tilde{\Xi} f = k + \kappa_k$.

By (3.2.9) We have

$$\tilde{\Xi} f = (1-t)\, \tilde{\Xi}_0 f + t\, \tilde{\Xi}_1 f + c_t \tag{3.2.22}$$

where c_t is a function constant on each U_h depending on t and

$$\begin{cases}
\tilde{\Xi}_0 f(z) = \dfrac{\beta i}{2} \displaystyle\oint \frac{f(\xi)\sigma(\xi)S(\xi)}{z-\xi} d\xi \\
\tilde{\Xi}_1 f(z) = \dfrac{\beta i}{2} \displaystyle\oint \frac{f(\xi)\sigma(\xi)S(\xi)}{z-\xi} d\xi
\end{cases}$$

where the first contour surronds z and each $A_{h'}^\epsilon$ whereas the second one surrounds z and A_h^ϵ when $z \in U_h$.

Let f_0 and f_1 be the functions analytic in $U \setminus A^\epsilon$ defined on each $U_h \setminus A_h^\epsilon$ by

$$\begin{aligned}
f_0(z) &= -\frac{1}{2\beta\pi^2 \sigma(z)\sigma_h(z) S(z)} \oint \frac{i\sigma_h(\xi)(k(\xi)+c_h^1)}{(\xi - z)} d\xi \\
f_1(z) &= -\frac{c_h^2}{2\beta\pi^2 \sigma(z)\sigma_h(z) S(z)}
\end{aligned}$$

So that $f = f_0 + f_1$

$$\begin{aligned}\tilde{\Xi}_0(f_0)(z) &= \frac{\beta i}{2}\oint_C \frac{f_0(\xi)S(\xi)\sigma(\xi)}{z-\xi}d\xi \\ &= -\frac{\beta i}{2}\sum_h \oint_{C_h} \frac{1}{z-\xi}\frac{1}{2\beta\pi^2\sigma(\xi)\sigma_h(\xi)S(\xi)}\left(\oint_{C'_h}\frac{i\sigma_h(\eta)(k(\eta)+c_h^1)}{(\eta-\xi)}d\eta\right)S(\xi)\sigma(\xi)d\xi \\ &= \frac{1}{4\pi^2}\sum_h \oint_{C_h}\oint_{C'_h}\frac{\sigma_h(\eta)(k(\eta)+c_h^1)}{(z-\xi)(\eta-\xi)\sigma_h(\xi)}d\eta d\xi\end{aligned}$$

where C_h surrounds z and A_h^ϵ (integral in ξ) , and C'_h surrounds C_h (integral in η).

Cauchy formula gives

$$\oint_{C'_h}\frac{\sigma_h(\eta)(k(\eta)+c_h^1)}{(\eta-\xi)}d\eta = 2i\pi(k(\xi)+c_h^1)\sigma_h(\xi) + \oint_{C''_h}\frac{\sigma_h(\eta)(k(\eta)+c_h^1)}{(\eta-\xi)}d\eta$$

with C_h surrounding C''_h. Thus:

$$\tilde{\Xi}_0(f_0)(z) = \frac{1}{4\pi^2}\sum_h \oint_{C_h}\oint_{C''_h}\frac{\sigma_h(\eta)(k(\eta)+c_h^1)}{(z-\xi)(\eta-\xi)\sigma_h(\xi)}d\eta d\xi + \frac{1}{4\pi^2}\sum_h\oint_{C_h}\frac{2i\pi(k(\xi)+c_h^1)}{z-\xi}d\xi.$$

Letting each C_h go to infinity, we see that the first integral goes to zero and using Cauchy formula again we see that the second term equals $k(z)+c^1$.

We now prove $\tilde{\Xi}_0(f_1) = 0$.

$$\tilde{\Xi}_0(f_1)(z) = -\frac{i}{4\pi^2}\sum_h\oint_{C_h}\frac{c_h^2 S(\xi)\sigma(\xi)}{\sigma(\xi)\sigma_h(\xi)S(\xi)(z-\xi)}d\xi = -\frac{i}{4\pi^2}\sum_h\oint_{C_h}\frac{c_h^2}{\sigma_h(\xi)(z-\xi)}d\xi = 0$$

where we let the contours go to infinity.

By the exact same reasoning, we show that $\tilde{\Xi}_1(f_0) = k + c^1$ and $\tilde{\Xi}_1(f_1) = 0$.

By setting $\kappa_k = c_t + c_h^1$ on each U_h we have the desired result. The unicity of κ_k is implied by the previous lemma. Formula (3.2.19) can be easily deduced by (3.2.18).

□

Remark 3.2.12. *By Lemma 3.2.11 and (3.2.18) , if k defined on $U \times U$ is analytic in each variable then f defined on $U \times U$ and solution of*

$$\Xi f(\cdot, y) = k(x,y) + \kappa_k(x,y)\ \forall y \in U\ ,$$

with $\kappa(.,y)$ constant on each U_h is analytic in each variable.

We can now construct our approximate solution of the Monge-Ampère equation. As we want the domain $\mathbf{B}$ to be fixed by the flow of this approximate solution, we would like to choose

$\mathbf{y}^{\epsilon}_{1,t}$ and $\mathbf{z}^{\epsilon}_{t}$ vanishing at the boundaries of B (and $B \times B$). Fix $\delta > 0$ small and denote $B^{\delta} = \bigcup_{0 \le h \le g} [\beta_{h,-} + \delta; \beta_{h,+} - \delta]$.
For a function $f : B \longrightarrow \mathbb{R}$ let $\Upsilon(f)$ be the multiplication of f by a smooth plateau function equal to 1 on B^{δ} and 0 outside B. If we are given a function $k \in \mathcal{O}(U)$ and $f \in \mathcal{O}(U)$ satisfying $\Xi(f) = k + \kappa_k$, then :

- $\Upsilon(f) = f$ on B^{δ}.
- $\Upsilon(f)$ is C^{∞} and has compact support in B (and can thus be extended by 0 to $\mathbb{R}$).
- $\Xi(\Upsilon(f)) = k + \kappa_k$ on B^{δ} (By definition of Ξ and the fact that f and $\Upsilon(f)$ coincide on B^{δ}).
- $\|\Upsilon(f)\|_{C^j(\mathbb{R})} \le C_j \|k\|_{C^{j+2}(B)}$ for some constants C_j.

Note that by Remark 3.2.5, possibly by shrinking $\tilde{\mathcal{E}}$ we can assume $\tilde{T}_t^{\epsilon} < 0$ outside B^{δ}. Thus for N large enough and a constant $\eta > 0$

$$\mathbb{P}^{N,\epsilon}_{T_t,B}(\exists i \ \lambda_i \notin B^{\delta}) \le \exp(-N\eta). \tag{3.2.23}$$

Moreover

$$\int \left(\int |k - \Xi(\Upsilon(f))| dM_N\right) d\mathbb{P}^{N,\epsilon}_{T_t,B} \le \int \left(\int |k - \Xi f| dM_N\right) d\mathbb{P}^{N,\epsilon}_{T_t,B} + \int \left(\int |\Xi f - \Xi(\Upsilon(f))| dM_N\right) d\mathbb{P}^{N,\epsilon}_{T_t,B}. \tag{3.2.24}$$

The fist term on the right hand side is 0 as κ_k is constant on each B_h and the second term is exponentially small by the large deviation estimate.

We first choose

$$\begin{cases} \hat{\mathbf{z}}^{\epsilon}_t(\cdot, y) = \frac{1}{2}\, \Xi^{-1}\left(W(\cdot, y)\right) \ \forall y \in B \\ \hat{\mathbf{y}}^{\epsilon}_{1,t} = \left(\frac{\beta}{2} - 1\right) \Xi^{-1}\left(\int \partial_1 \hat{\mathbf{z}}^{\epsilon}_t(z, \cdot) d\mu^{\epsilon}_V(z)\right) \end{cases}$$

and then

$$\begin{cases} \mathbf{z}^{\epsilon}_t(\cdot, y) = \Upsilon(\hat{\mathbf{z}}^{\epsilon}_t(\cdot, y)) \ \forall y \in B \\ \mathbf{y}^{\epsilon}_{1,t} = \Upsilon(\hat{\mathbf{y}}^{\epsilon}_{1,t}) \end{cases}$$

With this choice of function and by inequality (3.2.24) we have that

$$\mathcal{R} = E + C^{N,\epsilon}_t + o\left(\frac{1}{N}\right). \tag{3.2.25}$$

We now have to control the error term E. To do so we will use a direct consequence of the concentration result proved in Corollary 3.5 of [17] (adapted from a result from [64]):

Proposition 3.2.13. *Let V satisfy Hypothesis 3.1.2 and T_t is as in (3.2.9). Then there exist constants c, c' and s_0 such that for N large enough, $s \geq s_0\sqrt{\frac{\log N}{N}}$, and for any $\varepsilon = \mathbf{N}/N \in \tilde{\mathcal{E}}$, $t \in [0;1]$ we have*

$$\mathbb{P}^{N,\varepsilon}_{T_t,B}\left(\sup_{\substack{\phi \in C^1_c(B)\\ \|\phi'\|_\infty \leq 1}} \left|\int \phi(x) d(L_N - \mu^\varepsilon_V)(x)\right| \geq s\right) \leq \exp(-cN^2s^2) + \exp(-c'N^2). \tag{3.2.26}$$

In order to control the error term we will make use of the following three loop equations. We recall that $M_N = N(L_N - N\mu^\varepsilon_V)$ and we will denote $\tilde{M}_N = N(L_N - \mathbb{E}^{N,\varepsilon}_{T_t,B}[L_N])$.

Lemma 3.2.14. *Let $f \in C^1(B)$ such that for all $0 \leq h \leq g$, $f(\beta_{h,-}) = f(\beta_{h,+}) = 0$. Then*

$$\mathbb{E}^{N,\varepsilon}_{T_t,B}\left(M_N(\Xi f) + \left(1 - \frac{\beta}{2}\right) L_N(f') + \frac{1}{N}\left[\iint \left(\frac{\beta}{2}\frac{f(x) - f(y)}{x - y} + \partial_1 T_t(x,y) f(x)\right) dM_N(x) dM_N(y)\right]\right) = 0 \tag{3.2.27}$$

If k_1 is also in $C^1(B)$ then

$$\begin{aligned}\mathbb{E}^{N,\varepsilon}_{T_t,B}\Bigg(& L_N(fk_1') + M_N(\Xi f)\tilde{M}_N(k_1) + \left(1 - \frac{\beta}{2}\right) L_N(f')\tilde{M}_N(k_1) \\ &+ \frac{1}{N}\left[\iint \left(\frac{\beta}{2}\frac{f(x) - f(y)}{x - y} + \partial_1 T_t(x,y) f(x)\right) dM_N(x) dM_N(y)\right] \tilde{M}_N(k_1)\Bigg) = 0.\end{aligned} \tag{3.2.28}$$

If k_2 and k_3 are also in $C^1(B)$ then

$$\begin{aligned}\mathbb{E}^{N,\varepsilon}_{T_t,B}\Bigg(& \sum_\sigma L_N(fk'_{\sigma(1)})\tilde{M}_N(k_{\sigma(2)})\tilde{M}_N(k_{\sigma(3)}) + M_N(\Xi f)\tilde{M}_N(k_1)\tilde{M}_N(k_2)\tilde{M}_N(k_3) \\ &+ \frac{1}{N}\left[\iint \left(\frac{\beta}{2}\frac{f(x) - f(y)}{x - y} + \partial_1 T_t(x,y) f(x)\right) dM_N(x) dM_N(y)\right] \tilde{M}_N(k_1)\tilde{M}_N(k_2)\tilde{M}_N(k_3) \\ &+ \left(1 - \frac{\beta}{2}\right) L_N(f')\tilde{M}_N(k_1)\tilde{M}_N(k_2)\tilde{M}_N(k_3)\Bigg) = 0.\end{aligned} \tag{3.2.29}$$

where the sum ranges over the permutations of $\mathfrak{S}_3$

Proof. Using integration by parts we show

$$\mathbb{E}^{N,\varepsilon}_{T_t,B}\left(\iint \left(\frac{\beta}{2}\frac{f(x) - f(y)}{x - y} + \partial_1 T_t(x,y) f(x)\right) dL_N(x) dL_N(y) + \frac{1}{N}\left(1 - \frac{\beta}{2}\right) L_N(f')\right) = 0 \tag{3.2.30}$$

we deduce the first loop equation by using the definition of Ξ.

The second loop equation is obtained by replacing in (3.2.30) $T_t(x,y)$ by $T_t(x,y) - \delta_1(k_1(x) + k_1(y))$ and differentiating at $\delta = 0$.

The third one is obtained by replacing in (3.2.30) $T_t(x,y)$ by $T_t(x,y) - \delta_1(k_1(x) + k_1(y)) - \delta_2(k_2(x) + k_2(y)) - \delta_3(k_3(x) + k_3(y))$ and differentiating at $\delta_1 = \delta_2 = \delta_3 = 0$.

□

We will now put in use these loop equations and the concentration result of Proposition 3.2.13 to obtain some estimates.

Lemma 3.2.15. *Let k be an analytic function on U. Then for some constant C:*

$$\left| \mathbb{E}^{N,\varepsilon}_{T_t,B} \left(M_N(k) \right) \right| \le C \log N \;\; \|k\|_{C^6(B)} \,.$$
$$\mathbb{E}^{N,\varepsilon}_{T_t,B} \left(M_N(k)^2 \right) \le C (\log N)^2 \;\; \|k\|^2_{C^6(B)} \,.$$
$$\mathbb{E}^{N,\varepsilon}_{T_t,B} \left(M_N(k)^4 \right) \le C (\log N)^4 \;\; \|k\|^4_{C^6(B)} \,.$$

Proof. We apply (3.2.27) to $f = \Upsilon(\Xi^{-1}k)$. Using (3.2.24) we obtain

$$\mathbb{E}^{N,\varepsilon}_{T_t,B} \Bigg(M_N(k) + \frac{1}{N} \left[\frac{\beta}{2} \iint \left(\frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x-y} + \partial_1 T_t(x,y) \Upsilon(\Xi^{-1}k)(x) \right) dM_N(x) dM_N(y) \right]$$
$$+ \left(1 - \frac{\beta}{2}\right) L_N((\Upsilon(\Xi^{-1}k))') \Bigg) = O\left(N \|k\|_{L^\infty(B)} \exp(-N\eta) \right).$$

Let

$$\Lambda(k) = \frac{1}{N} \left[\frac{\beta}{2} \iint \left(\frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x-y} + \partial_1 T_t(x,y) \Upsilon(\Xi^{-1}k)(x) \right) dM_N(x) dM_N(y) \right]$$
$$+ \left(1 - \frac{\beta}{2}\right) L_N((\Upsilon(\Xi^{-1}k))').$$

Denoting by $\mathcal{F}$ the fourier transform operator (for functions of either one or several variables) we have

$$\iint \frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x-y} dM_N(x) dM_N(y)$$
$$= i \int \left(\int_0^1 d\alpha \int e^{i\alpha\xi x} dM_N(x) \int e^{i(1-\alpha)\xi y} dM_N(y) \right) \mathcal{F}(\Upsilon(\Xi^{-1}k))(\xi) \xi d\xi$$

and

$$\iint \partial_1 T_t(x,y) \Upsilon(\Xi^{-1}k)(x) dM_N(x) dM_N(y)$$
$$= \int \left(\int e^{i\xi x} dM_N(x) \int e^{i\zeta y} dM_N(y) \right) \mathcal{F}(\partial_1 T_t \; \Upsilon(\Xi^{-1}k))(\xi, \zeta) d\xi d\zeta.$$

Now on the set $\Omega = \left\{ \sup_{\substack{\phi \in C^1_c(B) \\ \|\phi'\|_\infty \le 1}} \left| \int \phi(x) d(L_N - \mu^\varepsilon_V)(x) \right| \le s_0 \sqrt{\frac{\log N}{N}} \right\}$ we have

$$\left| \int e^{i\xi x} dM_N(x) \right| \le \left| \int \Upsilon(e^{i\xi\cdot})(x) dM_N(x) \right| + 2Ne^{-N\eta}$$
$$\le C(1 + |\xi|) \sqrt{N \log N} + 2Ne^{-N\eta}$$

consequently, on this set

$$\left| \iint \frac{\Upsilon(\Xi^{-1}k)(x) - \Upsilon(\Xi^{-1}k)(y)}{x-y} dM_N(x)dM_N(y) \right|$$
$$\leq C(N\log N) \int |\mathcal{F}(\Upsilon(\Xi^{-1}k))(\xi)|(1+|\xi|)^3 d\xi + O(Ne^{-N\eta}).$$

The integral is bounded by the norm $\mathcal{H}^4(\mathbb{R})$ of $\Upsilon(\Xi^{-1}k)$ and we have:

$$\left\|\Upsilon(\Xi^{-1}k)\right\|_{\mathcal{H}^4(\mathbb{R})} \leq C\Big(\left\|\Upsilon(\Xi^{-1}k)\right\|_{\mathcal{L}^2(\mathbb{R})} + \left\|\left(\Upsilon(\Xi^{-1}k)\right)^{(4)}\right\|_{\mathcal{L}^2(\mathbb{R})} \Big).$$

As $\Upsilon(\Xi^{-1}k)$ has its support in B, the $\mathcal{L}^2(\mathbb{R})$ norm can be in turn controlled by the $\mathcal{L}^\infty(\mathbb{R})$ norm and we can use (3.2.19). Similarly on Ω we have

$$\left| \iint \partial_1 T_t(x,y)\Upsilon(\Xi^{-1}k)(x)dM_N(x)dM_N(y) \right| \leq C(N\log N)\left\|k\right\|_{C^6(B)} + O(Ne^{-N\eta})$$

Note that here the constant depends on T_t but we can make it uniform in t and $\varepsilon \in \tilde{\mathcal{E}}$.
On Ω^c we can use the trivial bound

$$\left| \int e^{i\xi x} dM_N(x) \right| \leq 2N$$

to prove that $|\Lambda(k)|$ is bounded everywhere by $CN\left\|k\right\|_{C^6(B)}$. By using Proposition 3.2.13 we obtain

$$|\mathbb{E}^{N,\varepsilon}_{T_t,B}(\Lambda(k))| \leq C\Big((\log N)\left\|k\right\|_{C^6(B)} + Ne^{-cs_0^2 N\log N}\left\|k\right\|_{C^6(B)} \Big)$$

and we can conclude the proof of the first inequality.
To prove the second inequality, using (3.2.28) and (3.2.24) we have

$$\mathbb{E}^{N,\varepsilon}_{T_t,B}\Big(M_N(k)\tilde{M}_N(k)\Big) = -\mathbb{E}^{N,\varepsilon}_{T_t,B}\Big(\Lambda(k)\tilde{M}_N(k) + L_N(k'\,\Upsilon(\Xi^{-1}k))\Big) + O(N^2\left\|k\right\|^2_{L^\infty(B)} e^{-N\eta}).$$

By splitting on Ω and Ω^c we see that

$$\left|\mathbb{E}^{N,\varepsilon}_{T_t,B}\Big(M_N(k)\tilde{M}_N(k)\Big)\right| \leq C\Big(\log N\left\|k\right\|_{C^6(B)}\mathbb{E}^{N,\varepsilon}_{T_t,B}\Big(|\tilde{M}_N(k)|\Big) + \left\|k\right\|^2_{C^6(B)}\Big(1 + N^2e^{-cs_0^2N\log N} + N^2e^{-N\eta}\Big)\Big)$$

We notice that $M_N(k) - \tilde{M}_N(k) = \mathbb{E}^{N,\varepsilon}_{T_t,B}(M_N(k))$ is deterministic and that $\mathbb{E}^{N,\varepsilon}_{T_t,B}(\tilde{M}_N(k))$ vanishes. The term on the left is thus equal to $\left|\mathbb{E}^{N,\varepsilon}_{T_t,B}\Big(\tilde{M}_N(k)^2\Big)\right|$ and we obtain

$$\mathbb{E}^{N,\varepsilon}_{T_t,B}\Big(\tilde{M}_N(k)^2\Big) \leq C\Big(\log N\left\|k\right\|_{C^6(B)}\sqrt{\mathbb{E}^{N,\varepsilon}_{T_t,B}\Big(\tilde{M}_N(k)^2\Big)} + \left\|k\right\|^2_{C^6(B)}\Big(1+N^2e^{-cs_0^2N\log N}+N^2e^{-N\eta}\Big)\Big).$$

Elementary manipulations show that this implies that $\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\tilde{M}_N(k)^2\right) \le C(\log N)^2 \|k\|^2_{C^6(B)}$ with a different constant.

Writing

$$\begin{aligned}\mathbb{E}^{N,\epsilon}_{T_t,B}\left(M_N(k)^2\right) &\le 2\,\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\tilde{M}_N(k)^2\right) + 2\,\mathbb{E}^{N,\epsilon}_{T_t,B}\left((\tilde{M}_N(k)-M_N(k))^2\right)\\ &= 2\,\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\tilde{M}_N(k)^2\right) + 2\,\mathbb{E}^{N,\epsilon}_{T_t,B}(M_N(k))^2\end{aligned}$$

and using the first inequality yields to the second one.

Finally, to prove the last inequality, (3.2.29) gives :

$$\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\tilde{M}_N(k)^4\right) \le C\Big(\log N\,\|k\|_{C^6(B)}\,\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\tilde{M}_N(k)^4\right)^{\frac{3}{4}} + \|k\|^4_{C^6(B)}\left((\log N)^2 + N^4 e^{-cs_0^2 N\log N} + N^4 e^{-N\eta}\right)\Big)$$

which shows $\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\tilde{M}_N(k)^4\right) \le C(\log N)^4\,\|k\|^4_{C^6(B)}$. We conclude by using the identity

$$\mathbb{E}^{N,\epsilon}_{T_t,B}\left(M_N(k)^4\right) \le 8\,\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\tilde{M}_N(k)^4\right) + 8\,\mathbb{E}^{N,\epsilon}_{T_t,B}\left((\tilde{M}_N(k)-M_N(k))^4\right).$$

□

We will need a last lemma to estimate the error E.

Lemma 3.2.16. *There exists a constant C such that for $\phi \in C^\infty(\mathbb{R})$ (resp. $\psi \in C^\infty(\mathbb{R}^2)$, $\chi \in C^\infty(\mathbb{R}^3)$) of compact support in B (resp. B^2, B^3) we have*

$$\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\left|\int \phi(x)dM_N(x)\right|\right) \le C\,\|\phi\|_{C^6(B)}\log N$$

$$\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\left|\iint \frac{\phi(x)-\phi(y)}{x-y}dM_N(x)dM_N(y)\right|\right) \le C\,\|\phi\|_{C^{14}(B)}\log N^2$$

$$\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\left|\iint \psi(x,y)dM_N(x)dM_N(y)\right|\right) \le C\,\|\psi\|_{C^{14}(B^2)}\log N^2$$

$$\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\left|\iiint \chi(x,y,z)dM_N(x)dM_N(y)dM_N(z)\right|\right) \le C\,\|\chi\|_{C^{21}(B^3)}\log N^3$$

$$\mathbb{E}^{N,\epsilon}_{T_t,B}\left(\left|\iiint \frac{\psi(x,y)-\psi(z,y)}{x-z}dM_N(x)dM_N(y)dM_N(z)\right|\right) \le C\,\|\psi\|_{C^{21}(B^2)}\log N^3$$

Proof. We will prove the last inequality as the other ones are simpler and can be proved the same way.

$$\begin{aligned}&\iiint \frac{\psi(x,y)-\psi(z,y)}{x-z}dM_N(x)dM_N(y)dM_N(z)\\ &\qquad = i\iint\left(\int_0^1 d\alpha M_N(e^{i\alpha\xi\cdot})M_N(e^{i(1-\alpha)\xi\cdot})M_N(e^{i\zeta\cdot})\right)\mathcal{F}(\partial_1\psi)(\xi,\zeta)d\xi d\zeta\end{aligned}$$

and by using Hölder inequality we obtain

$$\begin{aligned}
&\mathbb{E}^{N,\epsilon}_{T_t,B}\Big(\Big|\iiint \frac{\psi(x,y)-\psi(z,y)}{x-z}dM_N(x)dM_N(y)dM_N(z)\Big|\Big)\\
&\le \iint \Big(\mathbb{E}^{N,\epsilon}_{T_t,B}\Big(M_N(e^{i\alpha\xi\cdot})^4\Big)^{\frac14}\mathbb{E}^{N,\epsilon}_{T_t,B}\Big(M_N(e^{i(1-\alpha)\xi\cdot})^4\Big)^{\frac14}\mathbb{E}^{N,\epsilon}_{T_t,B}\Big(M_N(e^{i\zeta\cdot})^4\Big)^{\frac14}\Big)|\xi|\mathcal{F}(\psi)(\xi,\zeta)d\xi d\zeta\\
&\le C(\log N)^3\iint (1+|\xi|^6)^2(1+|\zeta|^6)|\xi||\mathcal{F}(\psi)(\xi,\zeta)|d\xi d\zeta
\end{aligned}$$

where we used the last identity of Lemma 3.2.15. The last term is controled by the $H^{21}(\mathbb{R}^2)$ norm of ψ and we have

$$\|\psi\|_{\mathcal{H}^{21}(\mathbb{R}^2)} \le C\Big(\|\psi\|_{\mathcal{L}^2(\mathbb{R}^2)} + \sup_{|\beta|\le 21}\left\|\partial^\beta\psi\right\|_{\mathcal{L}^2(\mathbb{R}^2)}\Big) \le C\,\|\psi\|_{\mathcal{C}^{21}(B^2)}\,.$$

□

A direct application of this lemma shows that $\mathbb{E}^{N,\epsilon}_{T_t,B}(\left|E\right|) \le C\frac{(\log N)^3}{N}$, and we could prove similarly using higher order loop equations that for all integer $k\ge 1$

$$\Big(\mathbb{E}^{N,\epsilon}_{T_t,B}(\left|E\right|^{2k})\Big)^{1/2k} \le C_k\frac{(\log N)^3}{N}. \tag{3.2.31}$$

In order to prove Propostion 3.2.9 it remains to control the deterministic term $C^{N,\epsilon}_t$. Let

$$\begin{aligned}
\boldsymbol{\mathcal{L}}(\mathbf{Y}) =&\beta\sum_{h=0}^{g}\sum_{1\le i<j\le N_h}\frac{\mathbf{Y}_{h,i}-\mathbf{Y}_{h,j}}{\lambda_{h,i}-\lambda_{h,j}} + \beta\sum_{0\le h<h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}\frac{\mathbf{Y}_{h,i}-\mathbf{Y}_{h',j}}{\lambda_{h,i}-\lambda_{h',j}}\\
&+\sum_{0\le h,h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}(\partial_1 T(\lambda_{h,i},\lambda_{h',j})\mathbf{Y}_{h,i,t}) + \operatorname{div}(\mathbf{Y}).
\end{aligned}$$

Integration by part shows that any vector field $\mathbf{Y}$ that vanishes on the boundary of $\mathbf{B}$ satisfies $\mathbb{E}^{N,\epsilon}_{T_t,B}(\boldsymbol{\mathcal{L}}(\mathbf{Y}))=0$. Thus

$$\begin{aligned}
\mathbb{E}^{N,\epsilon}_{T_t,B}\big(\mathcal{R}^{N,\epsilon}_t(\mathbf{Y}^{N,\epsilon})\big) = \mathbb{E}^{N,\epsilon}_{T_t,B}\Big(&-\frac12\sum_{0\le h,h'\le g}\sum_{\substack{1\le i\le N_h\\ 1\le j\le N_{h'}}}W(\lambda_{h,i},\lambda_{h',j}) + N\sum_{0\le h\le g}\sum_{1\le i\le N_h}\int W(\lambda_{h,i},z)d\mu^\epsilon_V(z)\\
&+\boldsymbol{\mathcal{L}}(\mathbf{Y}^{N,\epsilon}_t) - c^{N,\epsilon}_t\Big) = 0\,,
\end{aligned}$$

and by (3.2.25)

$$|C^{N,\epsilon}_t| = \left|\mathbb{E}^{N,\epsilon}_{T_t,B}\big(\mathcal{R}^{N,\epsilon}_t(\mathbf{Y}^{N,\epsilon}) - E\big)\right| + o\Big(\frac1N\Big) \le C\frac{(\log N)^3}{N}\,.$$

3.2.4 Obtaining the Transport map via the flow

In this section we will discuss the properties of the transport map given by the flow of the approximate solution $\mathbf{Y}^{N,\varepsilon}$ of the Monge-Ampère equation. As the equilibrium measures of the initial potential and the target potential are the same, this map is equal to the identity at the first order. The smaller order are then given by the expansion (3.2.15) of $\mathbf{Y}^{N,\varepsilon}$.

Lemma 3.2.17. *Let V satisfy Hypothesis 3.1.2 , T_t is as in (3.2.9) and $\varepsilon = \mathbf{N}/N \in \tilde{\mathcal{E}}$. Then the flow $X_t^{N,\varepsilon}$ can be written*

$$X_t^{N,\varepsilon} = Id + \frac{1}{N} X_t^{N,\varepsilon,1} + \frac{1}{N^2} X_t^{N,\varepsilon,2} \tag{3.2.32}$$

where $X_t^{N,\varepsilon,1}$ and $X_t^{N,\varepsilon,2}$ are in $C^\infty(\mathbb{R}^N)$ supported in $\mathbf{B}$, and for some constant $C > 0$

$$\sup_{\substack{0 \le h \le g \\ 1 \le i \le N_h}} \left\| X_{h,i,t}^{N,\varepsilon,1} \right\|_{L^4(\mathbb{P}_{V,B}^{N,\varepsilon})} \le C \log N \quad , \quad \left\| X_t^{N,\varepsilon,2} \right\|_{L^2(\mathbb{P}_{V,B}^{N,\varepsilon})} \le C\sqrt{N} (\log N)^2 \tag{3.2.33}$$

and with probability greater than $1 - N^{-\frac{N}{C}}$

$$\sup_{\substack{0 \le h \le g \\ 1 \le i,j \le N_h}} \left| X_{h,i,t}^{N,\varepsilon,1}(\boldsymbol{\lambda}) - X_{h,j,t}^{N,\varepsilon,1}(\boldsymbol{\lambda}) \right| \le C\sqrt{N} \log N |\lambda_{h,i} - \lambda_{h,j}| \tag{3.2.34}$$

$$\sup_{\substack{0 \le h \le g \\ 1 \le i,j \le N_h}} \left| X_{h,i,t}^{N,\varepsilon,2}(\boldsymbol{\lambda}) - X_{h,j,t}^{N,\varepsilon,2}(\boldsymbol{\lambda}) \right| \le CN\sqrt{N} \log N |\lambda_{h,i} - \lambda_{h,j}| \tag{3.2.35}$$

$$\sup_{\substack{0 \le h \le g \\ 1 \le i \le N_h}} \left\| X_{h,i,t}^{N,\varepsilon,1} \right\|_\infty \le C\sqrt{N} \log N \quad , \quad \sup_{\substack{0 \le h \le g \\ 1 \le i \le N_h}} \left\| X_{h,i,t}^{N,\varepsilon,2} \right\|_\infty \le CN\sqrt{N} \log N \tag{3.2.36}$$

Proof. The expansion (3.2.15) suggests to define $X_t^{N,\varepsilon,1} = (X_{0,1,t}^{N,\varepsilon,1}, \cdots, X_{g,N_g,t}^{N,\varepsilon,1})$ as the solution of the linear ODE

$$\dot{X}_{h,i,t}^{N,\varepsilon,1}(\boldsymbol{\lambda}) = \mathbf{y}_{1,t}^{\varepsilon}(\lambda_{h,i}) + \int \mathbf{z}_t^{\varepsilon}(\lambda_{h,i}, y) dM_N(y) + \frac{1}{N} \sum_{0 \le h' \le g} \sum_{1 \le j \le N_{h'}} \partial_2 \mathbf{z}_t^{\varepsilon}(\lambda_{h,i}, \lambda_{h',j}) X_{h',j,t}^{N,\varepsilon,1}(\boldsymbol{\lambda}) \tag{3.2.37}$$

with initial condition $X_t^{N,\varepsilon,1} = 0$. We then define $X_t^{N,\varepsilon,2}$ through the identity (3.2.32). Using the fact that $\mathbf{y}_{1,t}^{\varepsilon}$ and $\mathbf{z}_t^{\varepsilon}$ have compact support and are thus bounded, along with equation (3.2.37), we obtain:

$$\frac{d}{dt}\Big(\sup_{\substack{0 \le h \le g \\ 1 \le i \le N_h}} \left\| X_{h,i,t}^{N,\varepsilon,1} \right\|_{L^4(\mathbb{P}_{V,B}^{N,\varepsilon})} \Big) \le C \Big(1 + \sup_{\substack{0 \le h \le g \\ 1 \le i \le N_h}} \left\| X_{h,i,t}^{N,\varepsilon,1} \right\|_{L^4(\mathbb{P}_{V,B}^{N,\varepsilon})} + \sup_{\substack{0 \le h \le g \\ 1 \le i \le N_h}} \left\| \int \mathbf{z}_t^{\varepsilon}(\lambda_{h,i}, y) dM_N(y) \right\|_{L^4(\mathbb{P}_{V,B}^{N,\varepsilon})} \Big).$$

As in Lemma 3.2.16, we can prove that the last term is of order $\log N$. Using Grönwall's Lemma, this proves

$$\sup_{\substack{0\le h\le g\\ 1\le i\le N_h}} \left\|X^{N,\varepsilon,1}_{h,i,t}\right\|_{L^4(\mathbb{P}^{N,\varepsilon}_{V,B})} \le C\log N. \tag{3.2.38}$$

Furthermore, Proposition 3.2.13 shows that for some constant C, with probability greater than $1-N^{-\frac{N}{C}}$ we have

$$\left\|\int \partial_1 \mathbf{z}^\varepsilon_t(\cdot,y)dM_N(y)\right\|_\infty \le C\sqrt{N}\log N$$

and similarly, this proves (3.2.34). We now have to bound the norm of $X^{N,\varepsilon,2}_t$. For $s\in[0;1]$ let

$$X^{s,N,\varepsilon}_t = Id + \frac{s}{N}X^{N,\varepsilon,1}_t + \frac{s}{N^2}X^{N,\varepsilon,2}_t = (1-s)Id + sX^{N,\varepsilon}_t$$

and define the measure $M_N^{X^{s,N,\varepsilon}_t}$ by

$$\int f(y)dM_N^{X^{s,N,\varepsilon}_t}(y) = \sum_{0\le h\le g}\sum_{1\le i\le N_h} f(X^{s,N,\varepsilon}_{h,i,t}(\boldsymbol{\lambda})) - N\int f d\mu^\varepsilon_V.$$

Then a Taylor expansion gives us an ODE for $X^{N,\varepsilon,2}_t$

$$\begin{aligned}
\dot{X}^{N,\varepsilon,2}_{h,i,t}(\boldsymbol{\lambda}) &= \int_0^1 (\mathbf{y}^\varepsilon_{1,t})'\big(X^{s,N,\varepsilon}_{h,i,t}(\boldsymbol{\lambda})\big)ds \ \left(X^{N,\varepsilon,1}_{h,i,t}(\boldsymbol{\lambda}) + \frac{1}{N}X^{N,\varepsilon,2}_{h,i,t}(\boldsymbol{\lambda})\right)\\
&+ \int_0^1 \left[\int \partial_1\mathbf{z}^\varepsilon_t\big(X^{s,N,\varepsilon}_{h,i,t}(\boldsymbol{\lambda}),y\big)dM_N^{X^{s,N,\varepsilon}_t} - \int \partial_1\mathbf{z}^\varepsilon_t\big(\lambda_{h,i},y\big)dM_N(y)\right]ds \ \left(X^{N,\varepsilon,1}_{h,i,t}(\boldsymbol{\lambda}) + \frac{1}{N}X^{N,\varepsilon,2}_{h,i,t}(\boldsymbol{\lambda}\right.\\
&+ \int \partial_1\mathbf{z}^\varepsilon_t\big(\lambda_{h,i},y\big)dM_N(y) \ \left(X^{N,\varepsilon,1}_{h,i,t}(\boldsymbol{\lambda}) + \frac{1}{N}X^{N,\varepsilon,2}_{h,i,t}(\boldsymbol{\lambda})\right)\\
&+ \sum_{0\le h'\le g}\sum_{1\le j\le N_{h'}} \int_0^1 \left[\partial_2\mathbf{z}^\varepsilon_t\big(X^{s,N,\varepsilon}_{h,i,t}(\boldsymbol{\lambda}),X^{s,N,\varepsilon}_{h',j,t}(\boldsymbol{\lambda})\big) - \partial_2\mathbf{z}^\varepsilon_t\big(\lambda_{h,i},\lambda_{h',j}\big)\right]ds \ X^{N,\varepsilon,1}_{h',j,t}(\boldsymbol{\lambda})\\
&+ \sum_{0\le h'\le g}\sum_{1\le j\le N_{h'}} \int_0^1 \left[\partial_2\mathbf{z}^\varepsilon_t\big(X^{s,N,\varepsilon}_{h,i,t}(\boldsymbol{\lambda}),X^{s,N,\varepsilon}_{h',j,t}(\boldsymbol{\lambda})\big)\right]ds \ \frac{X^{N,\varepsilon,2}_{h',j,t}(\boldsymbol{\lambda})}{N}.
\end{aligned} \tag{3.2.39}$$

We then use the bounds

$$\begin{aligned}
&\int_0^1 \left|\int \partial_1\mathbf{z}^\varepsilon_t\big(X^{s,N,\varepsilon}_{h,i,t}(\boldsymbol{\lambda}),y\big)dM_N^{X^{s,N,\varepsilon}_t} - \int \partial_1\mathbf{z}^\varepsilon_t\big(\lambda_{h,i},y\big)dM_N(y)\right|ds\\
&\qquad\qquad \le C|X^{N,\varepsilon,1}_{h,i,t}| + \frac{C}{N}|X^{N,\varepsilon,2}_{h,i,t}| + \frac{C}{N}\sum_{h',j}\left(|X^{N,\varepsilon,1}_{h',j,t}| + \frac{1}{N}|X^{N,\varepsilon,2}_{h',j,t}|\right),\\
&\sum_{h',j}\int_0^1 \left|\partial_2\mathbf{z}^\varepsilon_t\big(X^{s,N,\varepsilon}_{h,i,t}(\boldsymbol{\lambda}),X^{s,N,\varepsilon}_{h',j,t}(\boldsymbol{\lambda})\big) - \partial_2\mathbf{z}^\varepsilon_t\big(\lambda_{h,i},\lambda_{h',j}\big)\right|ds\ |X^{N,\varepsilon,1}_{h',j,t}(\boldsymbol{\lambda})|\\
&\qquad\qquad \le \frac{C}{N}\sum_{h',j}\left(|X^{N,\varepsilon,1}_{h',j,t}|^2 + \frac{1}{N}|X^{N,\varepsilon,2}_{h',j,t}||X^{N,\varepsilon,1}_{h',j,t}|\right)
\end{aligned}$$

to obtain

$$
\begin{aligned}
\frac{d}{dt}\left\|X_t^{N,\varepsilon,2}\right\|^2_{L^2(\mathbb{P}^{N,\varepsilon}_{V,B})} &= 2\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}\dot X^{N,\varepsilon,2}_{h,i,t}\ X^{N,\varepsilon,2}_{h,i,t}\Bigg)\\
&\le C\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}|X^{N,\varepsilon,1}_{h,i,t}|\ |X^{N,\varepsilon,2}_{h,i,t}|\Bigg) + \frac{C}{N}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}|X^{N,\varepsilon,2}_{h,i,t}|^2\Bigg)\\
&+ C\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}|X^{N,\varepsilon,1}_{h,i,t}|^2\ |X^{N,\varepsilon,2}_{h,i,t}|\Bigg) + \frac{C}{N}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}|X^{N,\varepsilon,1}_{h,i,t}|\ |X^{N,\varepsilon,2}_{h,i,t}|^2\Bigg)\\
&+ \frac{C}{N^2}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}|X^{N,\varepsilon,2}_{h,i,t}|^3\Bigg)\\
&+ \frac{C}{N}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,1}_{h,i,t}|\ |X^{N,\varepsilon,1}_{h',j,t}|\ |X^{N,\varepsilon,2}_{h,i,t}|\Bigg) + \frac{C}{N^2}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,1}_{h,i,t}|\ |X^{N,\varepsilon,2}_{h',j,t}|\ |X^{N,\varepsilon,2}_{h,i,t}|\Bigg)\\
&+ \frac{C}{N^2}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,2}_{h,i,t}|^2\ |X^{N,\varepsilon,1}_{h',j,t}|\Bigg) + \frac{C}{N^3}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,2}_{h,i,t}|^2\ |X^{N,\varepsilon,2}_{h',j,t}|\Bigg)\\
&+ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}\left|\int \partial_1\mathbf{z}^\varepsilon_t\big(\lambda_{h,i},y\big)dM_N(y)\right||X^{N,\varepsilon,1}_{h,i,t}|\ |X^{N,\varepsilon,2}_{h,i,t}|\Bigg) + C\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{h,i}|X^{N,\varepsilon,2}_{h,i,t}|^2\Bigg)\\
&+ \frac{C}{N}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,2}_{h,i,t}|\ |X^{N,\varepsilon,1}_{h',j,t}|^2\Bigg) + \frac{C}{N^2}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,2}_{h,i,t}|\ |X^{N,\varepsilon,2}_{h',j,t}|\ |X^{N,\varepsilon,1}_{h',j,t}|\Bigg)\\
&+ \frac{C}{N}\ \mathbb{E}^{N,\varepsilon}_{V,B}\Bigg(\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,2}_{h,i,t}|\ |X^{N,\varepsilon,2}_{h',j,t}|\Bigg).
\end{aligned}
\tag{3.2.40}
$$

Using the bounds $\left\|\int \partial_1\mathbf{z}^\varepsilon_t\big(\lambda_{h,i},y\big)dM_N(y)\right\|_{L^4(\mathbb{P}^{N,\varepsilon}_{V,B})} \le C\log N$ (see Lemma 3.2.16), $|X^{N,\varepsilon,1}_{h,i,t}| \le C\ N$, $|X^{N,\varepsilon,2}_{h,i,t}| \le CN^2$ and inequalities such as

$$
\sum_{h,i}|X^{N,\varepsilon,1}_{h,i,t}|\ |X^{N,\varepsilon,2}_{h,i,t}| \le \frac{1}{2}\Bigg(\sum_{h,i}\Big((X^{N,\varepsilon,1}_{h,i,t})^2 + (X^{N,\varepsilon,2}_{h,i,t})^2\Big)\Bigg)
$$

$$
\sum_{\substack{h,i\\h',j}}|X^{N,\varepsilon,1}_{h,i,t}|\ |X^{N,\varepsilon,1}_{h',j,t}|\ |X^{N,\varepsilon,2}_{h,i,t}| \le \Bigg(\sum_{\substack{h,i\\h',j}}\Big((X^{N,\varepsilon,1}_{h,i,t})^4 + (X^{N,\varepsilon,1}_{h,i,t})^4 + (X^{N,\varepsilon,2}_{h',j,t})^2\Big)\Bigg)
$$

along with (3.2.38) and Hölder inequality, we get

$$
\frac{d}{dt}\left\|X_t^{N,\varepsilon,2}\right\|^2_{L^2(\mathbb{P}^{N,\varepsilon}_{V,B})} \le C\Big(\left\|X_t^{N,\varepsilon,2}\right\|^2_{L^2(\mathbb{P}^{N,\varepsilon}_{V,B})} + N(\log N)^4\Big). \tag{3.2.41}
$$

Using Grönwall's Lemma, we can conclude the proof. The bounds (3.2.35) and (3.2.36) are proven the same way. □

Remark 3.2.18. *Using (3.2.36), (3.2.37) and (3.2.40) we see that we have in fact for all integer $k \geq 1$*

$$\sup_{\substack{0\leq h\leq g\\ 1\leq i\leq N_h}} \left\|X^{N,\varepsilon,1}_{h,i,t}\right\|_{L^{2k}(\mathbb{P}^{N,\varepsilon}_{V,B})} \leq C_k \log N \quad , \quad \sup_{\substack{0\leq h\leq g\\ 1\leq i\leq N_h}} \left\|X^{N,\varepsilon,2}_{h,i,t}\right\|_{L^{2k}(\mathbb{P}^{N,\varepsilon}_{V,B})} \leq C_k\sqrt{N}(\log N)^2 \quad .$$

3.3 From Transport to Universality

In this section we will prove Proposition 3.2.6 and Corollary 3.2.7. We prove the results in the bulk as the proof is almost identical at the edge.

Proof of Proposition 3.2.6. Note that by Lemma 3.2.8 and by our construction of $\mathbf{Y}^{N,\varepsilon}_t$, $X^{N,\varepsilon}_1$ is an approximate transport map from $\mathbb{P}^{N,\varepsilon}_{V,B}$ to $\mathbb{P}^{N,\varepsilon}_{T_1,B}$ in the sense that it satisfies (3.2.11). Now, keeping our notations from the previous section, set $\hat{X}^{N,\varepsilon} = Id + \frac{1}{N}X^{N,\varepsilon,1}_1$. Then for all $f \in C^1(\mathbb{R})$

$$\begin{aligned}\left|\int f(\hat{X}^{N,\varepsilon})d\mathbb{P}^{N,\varepsilon}_{V,B} - \int f(X^{N,\varepsilon}_1)d\mathbb{P}^{N,\varepsilon}_{V,B}\right| &\leq \frac{\|\nabla f\|_\infty}{N^2}\int |X^{N,\varepsilon,2}_1|d\mathbb{P}^{N,\varepsilon}_{V,B}\\ &\leq \frac{\|\nabla f\|_\infty}{N^2}\left\|X^{N,\varepsilon,2}_1\right\|_{L^2(\mathbb{P}^{N,\varepsilon}_{V,B})}\\ &\leq \|\nabla f\|_\infty \frac{(\log N)^2}{N^{\frac{3}{2}}}\end{aligned}$$

and thus

$$\left|\int f(\hat{X}^{N,\varepsilon})d\mathbb{P}^{N,\varepsilon}_{V,B} - \int f d\mathbb{P}^{N,\varepsilon}_{T_1,B}\right| \leq C\frac{(\log N)^3}{N}\|f\|_\infty + \|\nabla f\|_\infty\frac{(\log N)^2}{N^{\frac{3}{2}}}.$$

Now for all $0 \leq h \leq g$ let $R^h : B_h^{N_h} \longrightarrow B_h^{N_h}$ the ordering map (i.e the map satisfying for all $(\lambda_1, \cdots, \lambda_{N_h}) \in B_h^{N_h}$ $R^{h,i}(\lambda_1, \cdots, \lambda_{N_h}) \leq R^{h,j}(\lambda_1, \cdots, \lambda_{N_h})$ if $i < j$ and $\{\lambda_1, \cdots, \lambda_{Nh}\} = \{R^{h,1}(\lambda_1, \cdots, \lambda_{N_h}), \cdots, R^{h,N_h}(\lambda_1, \cdots, \lambda_{N_h})\}$, so that if $R(\boldsymbol{\lambda}) = (R^0(\lambda_{0,1}, \cdots, \lambda_{0,N_0}), \cdots, R^g(\lambda_{g,1}, \cdots, \lambda_g$ we have $R\sharp d\mathbb{P}^{N,\varepsilon}_B = d\tilde{\mathbb{P}}^{N,\varepsilon}_B$.

Then if f_h is a function of m variables, we have $\left\|\nabla(f_h \circ R^h)\right\|_\infty \leq \sqrt{m}\,\|\nabla f_h\|_\infty$.

It is clear from (3.2.34) that $\hat{X}^{N,\varepsilon}$ preserves the order of the eigenvalues with probability greater than $1 - N^{-\frac{N}{C}}$. Thus, if we define $f : \mathbb{R}^{m(g+1)} \longrightarrow \mathbb{R}$ by $f(\boldsymbol{x}_0, \cdots, \boldsymbol{x}_g) = \prod_{0\leq h\leq g} f_h(\boldsymbol{x}_h)$ where $f_h : \mathbb{R}^m \longrightarrow \mathbb{R}$ we obtain

$$\Big| \int \prod_{0\le h\le g} f_h\Big(N(\lambda_{h,i_h+1}-\lambda_{h,i_h}),\cdots,N(\lambda_{h,i_h+m}-\lambda_{h,i_h})\Big)d\tilde{\mathbb{P}}^{N,\varepsilon}_{T_1,B}$$
$$-\int \prod_{0\le h\le g} f_h\Big(N(\hat{X}^{N,\varepsilon}_{h,i_h+1}(\boldsymbol{\lambda})-\hat{X}^{N,\varepsilon}_{h,i_h}(\boldsymbol{\lambda})),\cdots,N(\hat{X}^{N,\varepsilon}_{h,i_h+m}(\boldsymbol{\lambda})-\hat{X}^{N,\varepsilon}_{h,i_h}(\boldsymbol{\lambda}))\Big)d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B}\Big| \tag{3.3.42}$$
$$\le C\Big(\frac{(\log N)^3}{N}\|f\|_\infty+\|\nabla f\|_\infty\sqrt{m}\,\frac{(\log N)^2}{N^{\frac{1}{2}}}\Big).$$

Now, using (3.2.34) we notice that with probability greater than $1-N^{-\frac{N}{C}}$, for all $1\le k\le m$ and $0\le h\le g$

$$\hat{X}^{N,\varepsilon}_{h,i_h+k}(\boldsymbol{\lambda})-\hat{X}^{N,\varepsilon}_{h,i_h}(\boldsymbol{\lambda})=\lambda_{h,i+k}-\lambda_{h,i_h}+(\lambda_{h,i_h+k}-\lambda_{h,i_h})O(\frac{\log N}{\sqrt{N}}).$$

As f_h has compact support in $[-M,M]^m$, $(\lambda_{h,i_h+k}-\lambda_{h,i_h})$ remains bounded by $\frac{2M}{N}$ and

$$\hat{X}^{N,\varepsilon}_{h,i+k}(\boldsymbol{\lambda})-\hat{X}^{N,\varepsilon}_{h,i_h}(\boldsymbol{\lambda})=\lambda_{h,i+k}-\lambda_{h,i_h}+O(\frac{M\log N}{N\sqrt{N}}),$$

we easily deduce the first part of Proposition 3.2.6 .

□

Proof of Corollary 3.2.7. Noticing that $d\mathbb{P}^{N,\varepsilon}_{T_1,B}$ is a product measure we can write

$$\int \prod_{0\le h\le g} f_h\Big(N(\lambda_{h,i_h+1}-\lambda_{h,i_h}),\cdots,N(\lambda_{h,i_h+m}-\lambda_{h,i_h})\Big)d\mathbb{P}^{N,\varepsilon}_{T_1,B}=\frac{1}{Z^{N,\varepsilon}_{T_1,B}}\int \prod_{0\le h\le g}\prod_{1\le i\le N_h}1_{B_h}(\lambda_{h,i})d\lambda_{h,i}$$
$$\Big[f_h\Big(N(\lambda_{h,i_h+1}-\lambda_{h,i_h}),\cdots,N(\lambda_{h,i_h+m}-\lambda_{h,i_h})\Big)\prod_{1\le i<j\le N_h}|\lambda_{h,i}-\lambda_{h,j}|^\beta\exp\Big(-N\sum_{1\le i\le N_h}\tilde{V}^\varepsilon(\lambda_{h,i})\Big)\Big]$$
$$=\prod_{0\le h\le g}\int f_h\Big(N(\lambda_{h,i_h+1}-\lambda_{h,i_h}),\cdots,N(\lambda_{h,i_h+m}-\lambda_{h,i_h})\Big)d\mathbb{P}^{N_h}_{\tilde{V}^\varepsilon/\varepsilon_h,B_h}$$

We notice using (3.2.7) that

$$\mu_{\tilde{V}^\varepsilon/\varepsilon_h,B_h}=\mu^{\varepsilon,h}_V.$$

We conclude using Theorem 2.1.4. □

3.4 Universality in the initial model

To derive universality in the initial model, we expand the expectation of the quantity we want to compute in terms of the filling fractions, and we make use of Corollary 3.2.7.

First, we notice that for all $0\le h\le g$ the map $\Phi^{\varepsilon,h}$ is smooth in $\varepsilon\in\tilde{\mathcal{E}}$ and we have a bound

$$(\Phi^{\varepsilon,h})'(\lambda_{h,i})=(\Phi^{\varepsilon_\star,h})'(\lambda_{h,i})+O(|\varepsilon-\varepsilon_\star|)\qquad \textit{uniformely in } \lambda_{h,i}\in B \tag{3.4.43}$$

Indeed, it is shown in [11] (4.1) that our transport map $\Phi^{\varepsilon,h}$ is equal to X_1^ε where X_t^ε solves the ordinary diffential equation

$$\dot{X}_t^\varepsilon = y_t^\varepsilon(X_t^\varepsilon) \quad , \quad X_0^\varepsilon = Id$$

and y_t^ε is given by inverting Ξ. By formula (3.2.18) and Lemma 3.2.2, we see that y_t^ε is regular in ε, and from the standard theory of ordinary differential equations, so is Φ^ε.

We will use the following result proved in section 8.2, equations (8.18) and (8.19) of [15].

Lemma 3.4.1. *Along the subsequences such that $\boldsymbol{N}_\star$ mod $\mathbb{Z}^{g+1} \longrightarrow \kappa$ where $\kappa \in [0;1[^{g+1}$ and under $\mathbb{P}^N_{V,\mathcal{B}}$, the vector $\lfloor \boldsymbol{N}_\star \rfloor - N(\boldsymbol{\lambda})$ converges towards a random discrete Gaussian vector $\Delta_{h,\kappa}$. In particular*

$$\mathbb{P}^N_{V,B}\Big(|\mathrm{N}(\boldsymbol{\lambda}) - \lfloor \boldsymbol{N}_\star \rfloor| \geq K\Big) = O\Big(\exp(-K^2)\Big).$$

Note that the limit is not necessarily centered, and although the result is proved for $\boldsymbol{N}_\star - N(\boldsymbol{\lambda})$, it obviously also holds for $\lfloor \boldsymbol{N}_\star \rfloor - N(\boldsymbol{\lambda})$ since we are only considering subsequences such that $\boldsymbol{N}_\star - \lfloor \boldsymbol{N}_\star \rfloor \longrightarrow \kappa$. We will also need the following result, which can be proved using the previous result or Lemma 3.2.16

$$\sum_{\mathbf{N}=(N_0,\cdots,N_g)} \frac{N!}{\prod N_h!} \frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} |\varepsilon - \varepsilon_\star| = \mathbb{E}^N_{V,B}\Big(\sum_{0\leq h\leq g} |L_N(B_h) - \mu_V(B_h)|\Big) \leq C\frac{\log N}{N}. \quad (3.4.44)$$

We now provide a proof of Theorem 3.1.3. Let f be a function of compact support and i such as in the hypothesis of the theorem. Using Corollary 3.2.7 we have

$$\begin{aligned}
&\int f\Big(N\rho_V(E_i^{V,N})(\lambda_{i+1} - \lambda_i)\Big) d\mathbb{P}^N_{V,B} \\
&= \sum_{\mathbf{N}=(N_0,\cdots,N_g)} \int f\Big(N\rho_V(E_i^{V,N})(\lambda_{i+1} - \lambda_i)\Big) \mathbf{1}_{\mathrm{N}(\boldsymbol{\lambda})=\mathbf{N}}\, d\tilde{\mathbb{P}}^N_{V,B} \\
&= \sum_{\mathbf{N}=(N_0,\cdots,N_g)} \frac{N!}{\prod N_h!} \frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int f\Big(N\rho_V(E_i^{V,N})(\lambda_{h,i+1[h,\mathbf{N}]} - \lambda_{h,i[h,\mathbf{N}]})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B} \\
&= \sum_{\substack{\mathbf{N}=(N_0,\cdots,N_g) \\ |\mathrm{N}(\boldsymbol{\lambda})-\lfloor \boldsymbol{N}_\star \rfloor|\leq K}} \frac{N!}{\prod N_h!} \frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int f\Big(N\rho_V(E_i^{V,N})(\lambda_{h,i+1[h,\mathbf{N}]} - \lambda_{h,i[h,\mathbf{N}]})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B} \\
&\quad + O\Big(\|f\|_\infty \exp(-K^2)\Big) \\
&= \sum_{\substack{\mathbf{N}=(N_0,\cdots,N_g) \\ |\mathrm{N}(\boldsymbol{\lambda})-\lfloor \boldsymbol{N}_\star \rfloor|\leq K}} \frac{N!}{\prod N_h!} \frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int f\Big(N(\Phi^{\varepsilon,h})'(\lambda_{i[h,\mathbf{N}]})\rho_V(E_i^{V,N})(\lambda_{i+1[h,\mathbf{N}]} - \lambda_{i[h,\mathbf{N}]})\Big) d\tilde{\mathbb{P}}^{N_h}_G \\
&\quad + O\Big((\exp(-K^2) + \frac{(\log N)^3}{N}) \|f\|_\infty + (\sqrt{m}\frac{(\log N)^2}{N^{1/2}} + M\frac{(\log N)}{N^{1/2}} + \frac{M^2}{N}) \|\nabla f\|_\infty\Big).
\end{aligned}$$

If we manage to replace the term $N(\Phi^{\varepsilon,h})'(\lambda_{i[h,\mathbf{N}]})\rho_V(E_i^{V,N})$ by $N_h\ \rho_G(E_{i[h,\mathbf{N}]}^{G,N_h})$ then, using the convergence (3.1.4) we can conclude.

By (3.4.43) we can replace $(\Phi^{\varepsilon,h})'(\lambda_{i[h,\mathbf{N}]})$ by $(\Phi^{\varepsilon_\star,h})'(\lambda_{i[h,\mathbf{N}]})$ in the last equation and obtain an error of order K/N. Now, using that $\Phi^{\varepsilon_\star,h}$ is a transport from μ_G to $\mu_V^{\varepsilon_\star,h}$ we see that

$$(\Phi^{\varepsilon_\star,h})'(\lambda_{i[h,\mathbf{N}]}) = \frac{\rho_G(\lambda_{i[h,\mathbf{N}]})}{\rho_V^{\varepsilon_\star,h}(\Phi^{\varepsilon_\star,h}(\lambda_{i[h,\mathbf{N}]}))},$$

$$\int_{-\infty}^{\Phi^{\varepsilon_\star,h}(E_{i[h,\mathbf{N}]}^{G,N_h})} \rho_V^{\varepsilon_\star,h}(x)dx = \int_{-\infty}^{E_i^{V,N}} \rho_V^{\varepsilon_\star,h}(x)dx + O(K/N).$$

Thus $\Phi^{\varepsilon_\star,h}(E_{i[h,\mathbf{N}]}^{G,N_h}) = E_i^{V,N} + O(K/N)$ and using $\rho_V = \varepsilon_{\star,h}\ \rho_V^{\varepsilon_\star,h}$ on A_h we see that

$$N(\Phi^{\varepsilon_\star,h})'(\lambda_{i[h,\mathbf{N}]})\rho_V(E_i^{V,N}) = N_{\star,h}\ \rho_G(\lambda_{i[h,\mathbf{N}]})\ \frac{\rho_V^{\varepsilon_\star,h}(E_i^{V,N})}{\rho_V^{\varepsilon_\star,h}(\Phi^{\varepsilon_\star,h}(\lambda_{i[h,\mathbf{N}]}))}.$$

We can replace $\lambda_{i[h,\mathbf{N}]}$ by $E_{i[h,\mathbf{N}]}^{G,N_h}$ in the right hand side with an error term $o(N)$ with high probability under $\mathbb{P}_G^{N_h}$ using a very rough rigidity estimate that can be proved for instance using Proposition 3.2.13. As $(\Phi^{\varepsilon_\star,h})'$ is bounded by below and f is compact we notice that $N(\lambda_{i+1[h,\mathbf{N}]} - \lambda_{i[h,\mathbf{N}]})$ is of order 1 and we can conclude.

We can now proceed with the proof of Theorem 3.1.4. To simplify the notations, we will do the proof when $m = 1$ but the proof for general m is identical.

$$\begin{aligned}
&\int \prod_{h=0}^{g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha_{0,-})\Big)d\tilde{\mathbb{P}}^N_{V,B}\\
&= \sum_{\mathbf{N}=(N_0,\cdots,N_g)} \int \prod_{h=0}^{g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha_{0,-})\Big)1_{\mathbf{N}(\boldsymbol{\lambda})=\mathbf{N}}\, d\tilde{\mathbb{P}}^N_{V,B}\\
&= \sum_{\mathbf{N}=(N_0,\cdots,N_g)} \frac{N!}{\prod N_h!}\frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int \prod_{h=0}^{g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha_{0,-})\Big)d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B}\\
&= \sum_{\mathbf{N}=(N_0,\cdots,N_g)} \frac{N!}{\prod N_h!}\frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int \prod_{h=0}^{g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha^{\varepsilon}_{0,-})\Big)d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B}\\
&\quad+O\Big(\sum_{\mathbf{N}=(N_0,\cdots,N_g)} \frac{N!}{\prod N_h!}\frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \|\nabla f\|_\infty N^{2/3}|\varepsilon-\varepsilon_\star|\Big)\\
&= \sum_{\substack{\mathbf{N}=(N_0,\cdots,N_g)\\ |\mathbf{N}(\boldsymbol{\lambda})-\lfloor \boldsymbol{N}_\star\rfloor|\le K}} \frac{N!}{\prod N_h!}\frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int \prod_{h=0}^{g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha^{\varepsilon}_{h,-})\Big)d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B}\\
&\quad+O\Big(\frac{\log N}{N^{1/3}}\|\nabla f\|_\infty+\|f\|_\infty\exp(-K^2)\Big)\\
&= \sum_{\substack{\mathbf{N}=(N_0,\cdots,N_g)\\ |\mathbf{N}(\boldsymbol{\lambda})-\lfloor \boldsymbol{N}_\star\rfloor|\le K}} \frac{N!}{\prod N_h!}\frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \prod_{h=0}^{g}\int f_h\Big(N^{2/3}(\Phi^{\varepsilon,h})'(-2)(\lambda_1+2)\Big)d\tilde{\mathbb{P}}^{N_h}_{G}\\
&\quad+O\Big((\exp(-K^2)+\frac{(\log N)^3}{N})\|f\|_\infty+(\sqrt{m}\frac{(\log N)^2}{N^{5/6}}+\frac{\log N}{N^{1/3}}+\frac{M^2}{N^{4/3}})\|\nabla f\|_\infty\Big).
\end{aligned}$$

Using the fact that $(\Phi^{\varepsilon,h})'$ is bounded by below on B and that f_h is supported in $[-M;M]$ we obtain that $|\lambda_1+2|$ remains bounded by $\frac{CM}{N^{2/3}}$. Using (3.4.43) we get

$$\begin{aligned}
f_h\Big(N^{2/3}(\Phi^{\varepsilon,h})'(-2)(\lambda_1+2)\Big) =&f_h\Big(N^{2/3}(\Phi^{\varepsilon_\star,0})'(\alpha_{G,-})(\lambda_1-\alpha_{G,-})\Big)\\
&+O(M\|\nabla f\|_\infty|\varepsilon-\varepsilon_\star|).
\end{aligned}$$

This equation, along with (3.4.44), shows that

$$\begin{aligned}
&\int \prod_{h=0}^{g} f_h\Big(N^{2/3}(\lambda_{h,1}-\alpha_{h,-})\Big)d\tilde{\mathbb{P}}^N_{V,B}\\
&= \sum_{\substack{\mathbf{N}=(N_0,\cdots,N_g)\\ |\mathbf{N}(\boldsymbol{\lambda})-\lfloor \boldsymbol{N}_\star\rfloor|\le K}} \frac{N!}{\prod N_h!}\frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \prod_{h=0}^{g}\int f_h\Big(N^{2/3}(\Phi^{\varepsilon,h})'(-2)(\lambda_1+2)\Big)d\tilde{\mathbb{P}}^{N_h}_{G}\\
&+O\Big((\exp(-K^2)+\frac{(\log N)^3}{N})\|f\|_\infty+(\sqrt{m}\frac{(\log N)^2}{N^{5/6}}+\frac{\log N}{N^{1/3}}+\frac{M^2}{N^{4/3}})\|\nabla f\|_\infty\Big).
\end{aligned}$$

As Theorem 1.1 of [72] ensures the convergence of the expectation, we can conclude.

We now come to the proof of Theorem 3.1.5. Let $0 \le h \le g$, $i = [\lfloor \boldsymbol{N}_\star \rfloor]_{h-1} + 1$ and $\Delta_h(\boldsymbol{\lambda}) = [\lfloor \boldsymbol{N}_\star \rfloor]_{h-1} - [N(\boldsymbol{\lambda})]_{h-1}$. As before we obtain

$$\int f\Big(N^{2/3}(\lambda_i - \xi_h)\Big) d\tilde{\mathbb{P}}^N_{V,B} = \sum_{\substack{\mathbf{N}=(N_0,\cdots,N_g)\\ \Delta_h(\boldsymbol{\lambda})\ge 0}} \frac{N!}{\prod N_h!} \frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int f\Big(N^{2/3}(\lambda_{h,i[h,\mathbf{N}]} - \alpha_{h,-})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B}$$
$$+ \sum_{\substack{\mathbf{N}=(N_0,\cdots,N_g)\\ \Delta_h(\boldsymbol{\lambda})< 0}} \frac{N!}{\prod N_h!} \frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int f\Big(N^{2/3}(\lambda_{h-1,i[h-1,\mathbf{N}]} - \alpha_{h-1,+})\Big) d\tilde{\mathbb{P}}^{N,\varepsilon}_{V,B}.$$

We focus on the first term. Applying Corollary 3.2.7 we see that this term equals to

$$\sum_{\substack{|\mathbf{N}(\boldsymbol{\lambda}) - \lfloor \boldsymbol{N}_\star \rfloor| \le K\\ \Delta_h(\boldsymbol{\lambda})\ge 0}} \frac{N!}{\prod N_h!} \frac{Z^{N,\varepsilon}_{V,B}}{Z^N_{V,B}} \int f\Big(N^{2/3}(\Phi^{\varepsilon,h})'(-2)(\lambda_{i[h,\mathbf{N}]} + 2)\Big) d\tilde{\mathbb{P}}^{N_h}_{G}.$$

Noticing that $i[h,\mathbf{N}] = [\lfloor \boldsymbol{N}_\star \rfloor]_{h-1} - [\mathbf{N}]_{h-1} + 1$, we deduce the theorem from Lemma 3.4.1.

Chapter 4

Mesoscopic central limit theorem for general β-ensembles

This Chapter is based on the article [13] written with A. Lodhia.

4.1 Introduction

In this Chapter, we study the linear fluctuations of the eigenvalues of general β-ensembles at the mesoscopic scale; we prove that for $\alpha \in (0; 1)$ fixed, f a smooth function (whose regularity and decay at infinity will be specified later), and E a fixed point in the bulk of the spectrum

$$\sum_{i=1}^{N} f\big(N^\alpha(\lambda_i - E)\big) - N \int f(N^\alpha(x - E)) d\mu_V(x)$$

converges towards a Gaussian random variable. At the macroscopic level (i.e when $\alpha = 0$), it is known that the eigenvalues satisfy a central limit theorem and the re-centered linear statistics of the eigenvalues converge towards a Gaussian random variable. This was first proved in [51] for polynomial potentials satisfying the one-cut assumption. In [16], the authors derived a full expansion of the free energy in the one-cut regime from which they deduce the central limit theorem for analytic potentials. The multi-cut regime is more complicated and in this setting, the central limit theorem does not hold anymore for all test functions (see [15, 77]). Similar results have also been obtained for the eigenvalues of Random Matrices from different ensembles (see [4, 63, 76]).
Interest in mesoscopic linear statistics has surged in recent years. Extending the results to one dimensional β-ensembles is a natural step. We prove convergence at all mesoscopic scales. The proof of the main Theorem relies on the analysis of the loop equations (see Section 2.1) from which we can deduce a recurrence relationship between moments, and the rigidity results from [20, 19] to control the linear statistics. Similar results have been obtained before in [21, Theorem 5.4]. There, the authors showed the mesoscopic CLT in the case of a quadratic potential, for small α (see Remark 5.5).
In Section 1, we introduce the model and recall some background results and Section 2 will be dedicated to the proof of Theorem 4.1.4.

4.1.1 Definitions and Background

We consider the general β-matrix model. For a potential $V : \mathbb{R} \longrightarrow \mathbb{R}$ and $\beta > 0$, we denote the measure on $\mathbb{R}^N$

$$\mathbb{P}_V^N (d\lambda_1, \cdots, d\lambda_N) := \frac{1}{Z_V^N} \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum V(\lambda_i)} \prod d\lambda_i , \tag{4.1.1}$$

with

$$Z_V^N = \int \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum V(\lambda_i)} \prod d\lambda_i .$$

From the previous Chapters we know that under $\mathbb{P}_V^N$ the empirical measure of the eigenvalues converge towards an equilibrium measure.

4.1.2 Results

Hypothesis 4.1.1. *For what proceeds, we assume the following*

- *V is continuous and goes to infinity faster than β $\log|x|$.*
- *The support of μ_V is a connected interval $A = [a; b]$ and*

$$\frac{d\mu_V}{dx} = \rho_V(x) = S(x)\sqrt{(b-x)(x-a)} \quad \text{with } S > 0 \text{ on } [a;b].$$

- *The function $V(\cdot) - \beta \int \log|\cdot - y| d\mu_V(y)$ achieves its minimum on the support only.*

Remark 4.1.2. *The second and third assumptions are typically known as the one-cut and off-criticality assumptions. In the case where the support of the equilibrium measure is no longer connected, the macroscopic central limit theorem does not hold anymore in generality (see [15, 77, 12], and the next Chapter).*

Remark 4.1.3. *If the previous assumptions are fulfilled, and $V \in C^p(\mathbb{R})$ then $S \in C^{p-3}(\mathbb{R})$ (see Chapter 2).*

Theorem 4.1.4. *Let $0 < \alpha < 1$, E a point in the bulk $(a;b)$, $V \in C^7(\mathbb{R})$ and $f \in C^6(\mathbb{R})$ with compact support. Then, under $\mathbb{P}_V^N$*

$$\sum_{i=1}^N f\big(N^\alpha(\lambda_i - E)\big) - N \int f(N^\alpha(x-E)) d\mu_V(x) \xrightarrow{\mathcal{M}} \mathcal{N}(0, \sigma_f^2) ,$$

where the convergence holds in moments (and thus in distribution), and

$$\sigma_f^2 = \frac{1}{2\beta\pi^2} \iint \left(\frac{f(x) - f(y)}{x-y}\right)^2 dxdy .$$

Note that, as in the macroscopic central limit theorem, the variance is universal in the potential with a multiplicative factor proportional to $1/\beta$. Interestingly and in contrast with the macroscopic scale, the limit is always centered.
The proof relies on an explicit computation of the moments of the linear statistics. We will use two tools: optimal rigidity for the eigenvalues of β-ensembles to provide a bound on the linear statistics (as in [20, 19]) and the loop equations at all orders to derive a recurrence relationship between the moments.

4.2 Proof of Theorem 4.1.4

For what follows, set

$$L_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}, \quad M_N = \sum_{i=1}^{N}\delta_{\lambda_i} - N\mu_V.$$

and for a measure ν and an integrable function h set

$$\nu(h) = \int h d\nu \quad \text{and} \quad \tilde{\nu}(h) = \int h d\nu - \mathbb{E}_V^N\Big(\int h d\nu\Big), \tag{4.2.2}$$

when ν is random and where $\mathbb{E}_V^N$ is the expectation with respect to $\mathbb{P}_V^N$. Further f will be any function as in Theorem 4.1.4, and

$$f_N(x) := f(N^\alpha(x-E)).$$

Finally, for any function $g \in C^p(\mathbb{R})$, let

$$\|g\|_{C^p(\mathbb{R})} := \sum_{l=0}^{p}\sup_{x\in\mathbb{R}}|g^{(l)}(x)|,$$

when it exists.

4.2.1 Loop Equations

To prove the convergence, we use the loop equations at all orders. Loop equations have been used previously to derive recurrence relationships between correlators and derive a full expansion of the free energy for β-ensembles in [77, 16, 15] (from which the authors also derive a macroscopic central limit theorem). The first loop equation was used to prove the central limit theorem at the macroscopic scale in [51] and used subsequently in [21]. Here, rather than using the first loop equation to control the Stieltjes transform as in [51] and [21], we rely on the analysis of the loop equations at all orders to compute directly the moments.

Proposition 4.2.1. *Let $h, h_1, h_2, \cdots$ be a sequence of bounded functions in $C^1(\mathbb{R})$. Define*

$$F_1^N(h) := \frac{N\beta}{2}\iint \frac{h(x)-h(y)}{x-y}dL_N(x)dL_N(y) - NL_N(hV') + \Big(1-\frac{\beta}{2}\Big)L_N(h') \tag{4.2.3}$$

and for all $k \geq 1$

$$F_{k+1}^N(h, h_1, \cdots, h_k) := F_k^N(h, h_1, \cdots, h_{k-1})\tilde{M}_N(h_k) + \left(\prod_{l=1}^{k-1}\tilde{M}_N(h_l)\right)L_N(hh_k') \tag{4.2.4}$$

where the product is equal to 1 when $k=1$ and $\tilde{M}_N$ was defined by the convention eq. (4.2.2). Then we have for all $k \geq 1$

$$\mathbb{E}_V^N\Big(F_k^N(h, h_1, \cdots, h_{k-1})\Big) = 0, \tag{4.2.5}$$

which is called the loop equation of order k.

Proof. The first loop equation (4.2.3) is derived by integration by parts (see also [51] eq. (2.18) for a proof using a change of variables). More precisely, for a fixed index l, integration by parts with respect to λ_l yields the equality:

$$\mathbb{E}_V^N\Big(h'(\lambda_l)\Big) = -\mathbb{E}_V^N\left(h(\lambda_l)\left(\beta \sum_{\substack{1\le i\le N\\ i\neq l}} \frac{1}{\lambda_l-\lambda_i} - NV'(\lambda_l)\right)\right).$$

Summing over l we get by symmetry

$$\mathbb{E}_V^N\left(\frac{\beta}{2}\sum_{l=1}^N \sum_{\substack{1\le i\le N\\ i\neq l}} \frac{h(\lambda_l)-h(\lambda_i)}{\lambda_l-\lambda_i} - N\sum_{l=1}^N V'(\lambda_l)h(\lambda_l) + \sum_{l=1}^N h'(\lambda_l)\right) = 0$$

Writing the sums in term of L_N and taking the diagonal terms to be equal to $h'(\lambda_l)$ gives eq. (4.2.5) for $k=1$.
To derive the loop equation at order $k+1$ from the one at order k, replace V by $V-\delta h_k$ and notice that for any functional F that is independent of δ,

$$\left.\frac{\partial\, \mathbb{E}_{V-\delta h_k}^N(F)}{\partial\delta}\right|_{\delta=0} = N\,\mathbb{E}_V^N\Big(F\tilde{M}_N(h_k)\Big).$$

Also observe that the loop equation eq. (4.2.5) is now

$$\mathbb{E}_{V-\delta h_k}^N\Big(F_k^N(h,h_1,\cdots,h_{k-1})\Big) + \delta N \mathbb{E}_{V-\delta h_k}^N\left(\left(\prod_{l=1}^{k-1}\tilde{M}_N(h_l)\right)L_N(hh_k')\right) = 0,$$

by induction and the definitions given in eqns. (4.2.3) and (4.2.4). Differentiating both sides with respect to δ and setting $\delta=0$ yields the loop equation at order $k+1$. □

It will be easier to compute the moments of $M_N(f_N)$ by re-centering the first loop equation — that is, we wish to replace L_N by $L_N-\mu_V$. To that end, define the operator Ξ acting on smooth functions $h:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$\Xi h(x) := \beta\int \frac{h(x)-h(y)}{x-y}d\mu_V(y) - V'(x)h(x)\ .$$

This operator is once again central to our approach. We now use

$$\frac{\beta}{2}\iint \frac{h(x)-h(y)}{x-y}d\mu_V(x)d\mu_V(y) = \int V'(x)h(x)d\mu_V(x)\ , \tag{4.2.6}$$

to get

$$\frac{\beta}{2}\iint \frac{h(x)-h(y)}{x-y}dL_N(x)dL_N(y) - L_N(hV') =$$
$$\frac{1}{N}M_N(\Xi h) + \frac{\beta}{2N^2}\iint \frac{h(x)-h(y)}{x-y}dM_N(x)dM_N(y).$$

Consequently, we can write

$$F_1^N(h) = M_N(\Xi h) + \Big(1 - \frac{\beta}{2}\Big)L_N(h') + \frac{1}{N}\left[\frac{\beta}{2}\iint \frac{h(x)-h(y)}{x-y}dM_N(x)dM_N(y)\right]. \quad (4.2.7)$$

One of the key features of the operator Ξ is that, under the one-cut and non-critical assumptions, it is invertible (modulo constants) in the space of smooth functions. More precisely, we recall the following Lemma proved in Chapter 2 (written in a slightly different form):

Lemma 4.2.2. *Inversion of* Ξ
Assume that $V \in C^p(\mathbb{R})$ *and satisfies Hypothesis 4.1.1. Let* $[a;b]$ *denote the support of* μ_V *and set*

$$\frac{d\mu_V}{dx} = S(x)\sqrt{(b-x)(x-a)} = S(x)\sigma(x),$$

where $S > 0$ *on* $[a;b]$.
Then for any $k \in C^r(\mathbb{R})$ *there exists a unique constant* c_k *and* $h \in C^{(r-2)\wedge(p-3)}(\mathbb{R})$ *such that*

$$\Xi(h) = k + c_k\ .$$

Moreover the inverse is given by the following formulas:

- $\forall x \in \mathrm{supp}(\mu_V)$

$$h(x) = -\frac{1}{\beta\pi^2 S(x)}\Big(\int_a^b \frac{k(y)-k(x)}{\sigma(y)(y-x)}dy\Big) \quad (4.2.8)$$

- $\forall x \notin \mathrm{supp}(\mu_V)$

$$h(x) = \frac{\beta\int \frac{h(y)}{x-y}d\mu_V(y) + k(x) + c_k}{\beta\int \frac{1}{x-y}d\mu_V(y) - V'(x)}\ . \quad (4.2.9)$$

And $c_k = -\beta\int \frac{h(y)}{a-y}d\mu_V(y) - k(a)$. *Note that the definition (4.2.9) is proper since* h *has been defined on the support.*
We shall denote this inverse by $\Xi^{-1}k$.

Remark 4.2.3. *For* f *and* V *as in Theorem 4.1.4,* $p = 7$ *and* $r = 6$ *so* $\Xi^{-1}f_N \in C^4(\mathbb{R})$.

Remark 4.2.4. *The denominator* $\beta\int \frac{1}{x-y}d\mu_V(y) - V'(x)$ *is identically null on* $\mathrm{supp}\,\mu_V$ *and behaves like a square root at the edges. Since by the last point of Hypothesis 4.1.1 we can modify freely the potential outside any neighborhood of the support (see for instance the large deviation estimates Section 2.1 of [16]), we may assume that it does not vanish outside* μ_V.

In order to bound the linear statistics we use the following lemma to bound $\Xi^{-1}(f_N)$ and its derivatives.

Lemma 4.2.5. *Let* $\mathrm{supp}\, f \subset [-M, M]$ *for some constant* $M > 0$. *For each* $p \in \{0,1,2,3\}$, *there is a constant* $C > 0$ *such that*

$$\left\|\Xi^{-1}(f_N)\right\|_{C^p(\mathbb{R})} \leq CN^{p\alpha}\log N, \quad (4.2.10)$$

Moreover, there is a constant C such that whenever $x \in \operatorname{supp}\mu_V$ and $N^\alpha|x-E| \geq M+1$

$$\left|\Xi^{-1}(f_N)^{(p)}(x)\right| \leq \frac{C}{N^\alpha(x-E)^{p+1}}, \tag{4.2.11}$$

and when $x \notin \operatorname{supp}\mu_V$

$$\left|\Xi^{-1}(f_N)^{(p)}(x)\right| \leq \frac{C\log N}{N^\alpha}. \tag{4.2.12}$$

Proof. We start by proving (4.2.10) on the support. For $x \in \operatorname{supp}\mu_V$ we use

$$\Xi^{-1}(f_N)(x) = -\frac{N^\alpha}{\beta\pi^2 S(x)}\int_a^b \frac{1}{\sigma(y)}\int_0^1 f'\Big(N^\alpha t(x-E) + N^\alpha(1-t)(y-E)\Big)dtdy$$

so that

$$\Xi^{-1}(f_N)^{(p)}(x) = -\frac{1}{\beta\pi^2}\sum_{l=0}^{p}\left\{\binom{p}{l}\left(\frac{1}{S}\right)^{(p-l)}(x)\right.$$
$$\left.\times\int_a^b \frac{N^{(l+1)\alpha}}{\sigma(y)}\int_0^1 t^l f^{(l+1)}\Big(N^\alpha t(x-E) + N^\alpha(1-t)(y-E)\Big)dtdy\right\}.$$

Let $A(x) = \Big\{(t,y) \in [0;1]\times[a;b]\ ,\ N^\alpha|t(x-E)+(1-t)(y-E)| \leq M\Big\}$. We have

$$\int_0^1 \mathbb{1}_{A(x)}(t,y)dt \leq \frac{2M}{N^\alpha|x-y|}\wedge 1 \tag{4.2.13}$$

and thus

$$\int_a^b \frac{N^{(l+1)\alpha}}{\sigma(y)}\int_0^1 |f^{(l+1)}\Big(N^\alpha t(x-E) + N^\alpha(1-t)(y-E)\Big)|dtdy \ \leq C\log N N^{l\alpha},$$

and this proves (4.2.10).
We now proceed with the proof of (4.2.11). First, let $x \in \operatorname{supp}\mu_V$ such that $N^\alpha|x-E| \geq M+1$. The inversion formula (4.2.8) writes

$$\begin{aligned}\Xi^{-1}(f_N)(x) &= -\frac{1}{\beta\pi^2 S(x)}\int_a^b \frac{f(N^\alpha(y-E))}{\sigma(y)(y-x)}dy \\ &= -\frac{1}{\beta\pi^2 S(x)}\int_{-M}^{M}\frac{f(u)}{\sigma(E+\frac{u}{N^\alpha})(u-N^\alpha(x-E))}du.\end{aligned} \tag{4.2.14}$$

and we can conclude in this setting by differentiating under the integral. Moreover we see that $\Xi^{-1}(f_N)$ is in fact of class C^5 on $\operatorname{supp}\mu_V$ and similar bounds holds for $p \in \{4,5\}$.
We now prove the bounds for $x \notin \operatorname{supp}\mu_V$. Let ψ_N be an arbitrary extension of $\Xi^{-1}(f_N)|_{\operatorname{supp}\mu_V}$ in $C^5(\mathbb{R})$, bounded by C/N^α outside the support (and its five first derivatives as well). This

is possible by what we just proved and a Taylor expansion. Using (4.2.9) we notice that

$$\begin{aligned}\Xi^{-1}(f_N)(x) &= \frac{\beta \int \frac{\psi_N(y)}{x-y} d\mu_V(y) + c_{f_N}}{\beta \int \frac{1}{x-y} d\mu_V(y) - V'(x)} \\ &= \frac{-\beta \int \frac{\psi_N(x)-\psi_N(y)}{x-y} d\mu_V(y) + \beta\psi_N(x) \int \frac{d\mu_V(y)}{x-y} + c_{f_N}}{\beta \int \frac{1}{x-y} d\mu_V(y) - V'(x)} \\ &= \psi_N(x) - \frac{\Xi(\psi_N)(x) - c_{f_N}}{\beta \int \frac{1}{x-y} d\mu_V(y) - V'(x)} .\end{aligned} \tag{4.2.15}$$

Since f has compact support we may write $\Xi(\psi_N) - c_{f_N} = \Xi(\psi_N) - c_{f_N} - f_N$ on $[a; a+\varepsilon]$ and $[b-\varepsilon; b]$ for ε small enough. Furthermore this quantity vanishes identically on these intervals by definition of ψ_N. In particular, $\Xi(\psi_N) - c_{f_N}$ and its four first derivatives vanish at the edges. By definition, and using the previous bounds we also get that

$$\begin{aligned}|c_{f_N}| &= \left| \beta \int \frac{\psi_N(y)}{a-y} d\mu_V(y) \right| \\ &\leq C \log N \int_{|y-E| \leq 2M/N^\alpha} \frac{d\mu_V(y)}{y-a} + \frac{C}{N^\alpha} \int_{|y-E| \geq 2M/N^\alpha} \frac{d\mu_V(y)}{(y-a)|y-E|} \\ &\leq C \frac{\log N}{N^\alpha}\end{aligned}$$

On the other hand, for $p \in [\![0; 4]\!]$ and $x \notin \operatorname{supp} \mu_V$,

$$\begin{aligned}\Xi(\psi_N)^{(p)}(x) =& \beta p! \int \frac{\psi_N(y) - \psi_N(x) - \cdots - \psi_N^{(p)}(x)(y-x)^p/p!}{(y-x)^{p+1}} d\mu_V(y) \\ &- (V'\psi_N)^{(p)}(x) .\end{aligned}$$

By doing a similar splitting, and bounding the fifth derivative of ψ_N uniformly away from E, we obtain the same bound $C \log N/N^\alpha$ on $\Xi(\psi_N)^{(p)}$ outside the support. By Remark 4.2.4 and (4.2.15), we conclude that we can bound the C^3 norm of $\Xi^{-1}(f_N)$ by $C \log N/N^\alpha$ outside the support. □

4.2.2 Sketch of the Proof

We have developed the tools we need to prove Theorem 4.1.4. In order to motivate the technical estimates in the following section, we now sketch the proof by computing the first moments. The full proof of the theorem will be given in Section 2.4. Consider a function f satisfying the hypothesis of Theorem 4.1.4. Applying (4.2.7) to $\Xi^{-1}(f_N)$ yields

$$\begin{aligned}F_1^N\left(\Xi^{-1}(f_N)\right) =& M_N(f_N) + \left(1 - \frac{\beta}{2}\right) L_N\left((\Xi^{-1} f_N)'\right) \\ &+ \frac{1}{N}\left[\frac{\beta}{2} \iint \frac{\Xi^{-1} f_N(x) - \Xi^{-1} f_N(y)}{x-y} dM_N(x) dM_N(y)\right],\end{aligned}$$

If the central limit theorem holds, we expect terms of the type $M_N(h)$ where h is fixed to be almost of constant order, and this an easy consequence of the rigidity estimates from [19] (stated as Theorem 4.2.6 below). Due to the dependency in N of f_N (and its inverse under Ξ), a little care must be taken for these estimates to yield a bound on the last term in the right handside, and this is precisely the point of Lemma 4.2.9. Similarly, we have

$$L_N\left((\Xi^{-1}f_N)'\right) = \mu_V\left((\Xi^{-1}f_N)'\right) + \frac{1}{N}M_N\left((\Xi^{-1}f_N)'\right) ,$$

and Lemma 4.2.8 shows the last term in the right handside is a small error term. Thus admitting the results of the next section, we would get with high probability and for ε_N small

$$F_1^N\left(\Xi^{-1}(f_N)\right) = M_N(f_N) + \left(1-\frac{\beta}{2}\right)\mu_V\left((\Xi^{-1}f_N)'\right) + \varepsilon_N .$$

By the first loop equation from Proposition 4.2.1, the expectation of F_1^N is zero and this shows that the first moment

$$\mathbb{E}_V^N\Big(M_N(f_N)\Big) = -\left(1-\frac{\beta}{2}\right)\mu_V\left((\Xi^{-1}f_N)'\right) + o(1)$$

The term on the right handside is deterministic and is shown to decrease towards zero in Lemma 4.2.10. Thus the first moment converges to 0.
In order to exhibit all the terms we will need to control, we proceed with the computation of the second moment. By definition

$$F_2^N\left(\Xi^{-1}(f_N), f_N\right) = F_1^N\left(\Xi^{-1}(f_N)\right)\tilde{M}_N(f_N) + L_N\left(\Xi^{-1}(f_N){f_N}'\right) ,$$

which we can write (with now an ε_N incorporating the deterministic mean converging to zero)

$$F_2^N\left(\Xi^{-1}(f_N), f_N\right) = M_N(f_N)\tilde{M}_N(f_N) + \varepsilon_N\ \tilde{M}_N(f_N) + L_N\left(\Xi^{-1}(f_N)f_N'\right) ,$$

Lemma 4.2.8 ensures that $\varepsilon_N\tilde{M}_N(f_N)$ remains small, and that the term in the right handside of the decomposition

$$L_N\left(\Xi^{-1}(f_N)f_N'\right) = \mu_V\left(\Xi^{-1}(f_N)f_N'\right) + \frac{1}{N}M_N\left(\Xi^{-1}(f_N)f_N'\right) ,$$

is also controlled. Consequently, using the second loop equation we see that

$$\mathbb{E}_V^N\left(M_N(f_N)^2\right) = -\mu_V\left(\Xi^{-1}(f_N)f_N'\right) + o(1) \tag{4.2.16}$$

The limit of the term appearing on the right handside is then computed in Lemma 4.2.10, equation (4.2.34). The following moments are computed similarly (see section 2.4).
In the following section, we establish all the bounds we need for the proof of Theorem 4.1.4. The previous steps will then be made rigorous in the last section.

4.2.3 Control of the linear statistics

We now make use of the strong rigidity estimates proved in [19] (Theorem 2.4) to control the linear statistics. We recall the result here

Theorem 4.2.6. *Let γ_i the quantile defined by*

$$\int_a^{\gamma_i} d\mu_V(x) = \frac{i}{N}. \tag{4.2.17}$$

Then, under Hypothesis 4.1.1 and for all $\xi > 0$ there exists constants $c > 0$ such that for N large enough

$$\mathbb{P}_V^N\left(|\lambda_i - \gamma_i| \geq N^{-2/3+\xi}\, \hat{\imath}^{-1/3}\right) \leq e^{-N^c},$$

where $\hat{\imath} = i \wedge (N + 1 - i)$.

We will use the following lemma quite heavily in what proceeds.

Lemma 4.2.7. *Let γ_i and $\hat{\imath}$ be as in Theorem 4.2.6. Let λ_i, $i \in [\![1, N]\!]$, be a configuration of points such that $|\lambda_i - \gamma_i| \leq N^{-2/3+\xi}\, \hat{\imath}^{-1/3}$ for $0 < \xi < (1 - \alpha) \wedge \frac{2}{3}$, and let $M > 1$ be a constant. Define the pairwise disjoint sets:*

$$J_1 := \{i \in [\![1; N]\!],\ |N^\alpha(\gamma_i - E)| \leq 2M\}, \tag{4.2.18}$$

$$J_2 := \left\{i \in J_1^c,\ |(\gamma_i - E)| \leq \frac{1}{2}(E - a) \wedge (b - E)\right\}, \tag{4.2.19}$$

$$J_3 := J_1^c \cap J_2^c. \tag{4.2.20}$$

The following statements hold:

1. *For all $i \in J_1 \cup J_2$, $\hat{\imath} \geq CN$, for some $C > 0$ that depend only on μ_V. For all such i, $|\gamma_i - \gamma_{i+1}| \leq \frac{C}{N}$ for a constant $C > 0$.*
2. *Uniformly in all $i \in J_1^c = J_2 \cup J_3$, $x \in [\gamma_i, \gamma_{i+1}]$ and all $t \in [0; 1]$,*

$$|N^\alpha t(\lambda_i - x) + N^\alpha(x - E)| > M + 1, \tag{4.2.21}$$

   for N large enough.
3. *The cardinality of J_1 is of order $CN^{1-\alpha}$, where again, $C > 0$ depends only on μ_V in a neighborhood of E.*

Proof. The first part of statement 1 holds by the observation that for $i \in J_1 \cup J_2$, γ_i is in the bulk, so

$$0 < c \leq \int_a^{\gamma_i} d\mu_V(x) = \frac{i}{N} \leq C < 1$$

for constants C, $c > 0$ depending only on μ_V. For the second part of statement 1, the density of μ_V is bounded below uniformly in $i \in J_1 \cup J_2$, so

$$c|\gamma_i - \gamma_{i+1}| \leq \int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) = \frac{1}{N}.$$

Statement 2 can be seen as follows: let $i \in J_2$ and consider first $x = \gamma_i$. On this set $\hat{\imath} \geq CN$ by 1, so uniformly in such i, $N^\alpha|\lambda_i - \gamma_i| \leq CN^{\alpha-1+\xi}$, which goes to zero, while $N^\alpha|\gamma_i - E| > 2M$. On the other hand, for $i \in J_3$, we have $N^\alpha|\gamma_i - E| > \frac{1}{2}N^\alpha(E-a) \wedge (b-E)$, which goes to infinity faster than $N^\alpha|\lambda_i - \gamma_i| \leq N^{\alpha-\frac{2}{3}+\xi}$, by our choice of ξ. When we substitute γ_i by x, the same argument holds because $N^\alpha|x-\gamma_i| \leq N^\alpha|\gamma_i - \gamma_{i+1}|$, which is of order $N^{\alpha-1}$ on J_2 (as we showed in statement 1) and bounded by $CN^{\alpha-\frac{2}{3}}$ on J_3.
Statement 3 follows by the observation that on the set $x \in [a,b]$ such that $|x-E| \leq \frac{2M}{N^\alpha}$ the density of μ_V is bounded uniformly above and below, so

$$\frac{c}{N^\alpha} \leq \int_{|x-E|\leq \frac{2M}{N^\alpha}} d\mu_V(x) = \sum_{i\in J_1} \int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) + O\left(\frac{1}{N}\right) \leq \frac{C}{N^\alpha},$$

giving the required result. □

The rigidity of eigenvalues, Theorem 4.2.6, along with the previous Lemma leads to the following estimates

Lemma 4.2.8. *For all $0 < \xi < (1-\alpha) \wedge \frac{2}{3}$ there exists constants C, $c > 0$ such that for N large enough we have the concentration bounds*

$$\mathbb{P}_V^N\Big(|M_N(f_N)| \geq CN^\xi \, \|f\|_{C^1(\mathbb{R})}\Big) \leq e^{-N^c} \,, \tag{4.2.22}$$

$$\mathbb{P}_V^N\Big(|M_N(\Xi^{-1}(f_N)')| \geq CN^{\alpha+\xi} \, \|f\|_{C^1(\mathbb{R})}\Big) \leq e^{-N^c} \,, \tag{4.2.23}$$

$$\mathbb{P}_V^N\Big(|M_N(\Xi^{-1}(f_N)f_N')| \geq CN^{\alpha+\xi} \, \|f\|_{C^1(\mathbb{R})}\Big) \leq e^{-N^c} \,. \tag{4.2.24}$$

Proof. Let $M > 1$ such that $\operatorname{supp} f \subset [-M, M]$ and fix $0 < \xi < (1-\alpha)\wedge\frac{2}{3}$. For the remainder of the proof, we may assume that we are on the event $\Omega := \Big\{\forall i \,,\; |\lambda_i - \gamma_i| \leq N^{-2/3+\xi}\, \hat{\imath}^{-1/3}\Big\}$. This follows from the fact that, for example,

$$\mathbb{P}_N^V\left(|M_N(f_N)| \geq CN^\xi\|f\|_{C^1(\mathbb{R})}\right) \leq \mathbb{P}_N^V\left(\Big\{|M_N(f_N)| \geq CN^\xi\|f\|_{C^1(\mathbb{R})}\Big\} \cap \Omega\right) + \mathbb{P}_N^V(\Omega^c),$$

and by Theorem 4.2.6, we may bound $\mathbb{P}_N^V(\Omega^c)$ by e^{-N^c} for some constant $c > 0$, and N large enough. On Ω, as the λ_i satisfy the conditions of Lemma 4.2.7 we will utilize the sets J_1, J_2, and J_3 as defined there.
We begin by controlling (4.2.22). We have that

$$\begin{aligned} |M_N(f_N)| &= \left|\sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - N\mu_V(f_N)\right| \\ &= \left|\sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - N\int_{\gamma_i}^{\gamma_{i+1}} f(N^\alpha(x-E))d\mu_V(x)\right| \\ &\leq N\sum_{i=1}^N \int_{\gamma_i}^{\gamma_{i+1}} |f(N^\alpha(\lambda_i - E)) - f(N^\alpha(x-E))|\, d\mu_V(x) \\ &\leq N^{1+\alpha}\sum_{i\in J_1}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 |\lambda_i - x|\,|f'(N^\alpha t(\lambda_i - E) + N^\alpha(1-t)(x-E))|\,dt d\mu_V(x) \,, \end{aligned} \tag{4.2.25}$$

where we used eq. (4.2.21). Using Lemma 4.2.7 item (3) and the definition of Ω we obtain

$$|M_N(f_N)| \leq N^{1+\alpha}|J_1|N^{\xi-1}\|f\|_{C^1(\mathbb{R})}\int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) \leq CN^{\xi}\|f\|_{C^1(\mathbb{R})}.$$

This proves (4.2.22). We now proceed with the proof of (4.2.23).

$$\begin{aligned}\left|M_N(\Xi^{-1}(f_N)')\right| &= \left|\sum_{i=1}^N\left(\Xi^{-1}(f_N)'(\lambda_i) - N\int_{\gamma_i}^{\gamma_{i+1}}\Xi^{-1}(f_N)'(x)d\mu_V(x)\right)\right| \\ &\leq N\sum_{i=1}^N\int_{\gamma_i}^{\gamma_{i+1}}\left|\Xi^{-1}(f_N)'(\lambda_i) - \Xi^{-1}(f_N)'(x)\right|d\mu_V(x) \\ &\leq N\sum_{i=1}^N\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 |\lambda_i - x|\left|\Xi^{-1}(f_N)^{(2)}(t(\lambda_i - x)+x)\right| dt\, d\mu_V(x),\end{aligned}$$

Recall from the proof of Lemma 4.2.7 that uniformly in $i \in J_2$ and $x \in [\gamma_i, \gamma_{i+1}]$, $|\gamma_i - E| \geq \frac{2M}{N^\alpha}$ while $|x - \gamma_i| \leq \frac{C}{N}$. For what follows, as $|\lambda_i - x| \leq CN^{-1+\xi}$ for N large enough we can replace $|t(\lambda_i - x) + (\gamma_i - E)|$ by $|\gamma_i - E|$ uniformly in $t \in [0;1]$. Likewise, uniformly in $i \in J_3$, $x \in [\gamma_i, \gamma_{i+1}]$ and $t \in [0;1]$ we can bound below $|t(\lambda_i - x) + (x - E)|$ by a constant.
For $i \in J_2$, by the observations in the previous paragraph, along with Lemma 4.2.7 2, Lemma 4.2.5 (4.2.11) and Lemma 4.2.7 1,

$$\begin{aligned}&N\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 |\lambda_i - x|\left|\Xi^{-1}(f_N)^{(2)}(t(\lambda_i - x)+x)\right| dt d\mu_V(x) \\ &\qquad \leq N\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 \frac{C|\lambda_i - x|}{N^\alpha(|t(\lambda_i - x) + x - E|^3)}dtd\mu_V(x) \leq \sum_{i\in J_2}\frac{CN^{\xi-1-\alpha}}{(\gamma_i - E)^3},\end{aligned}$$

The same reasoning for $i \in J_3$ using also (4.2.12) yields

$$\begin{aligned}&N\sum_{i\in J_3}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 |\lambda_i - x|\left|\Xi^{-1}(f_N)^{(2)}(t(\lambda_i - x)+x)\right| dt\, d\mu_V(x) \\ &\qquad\qquad \leq \log N\sum_{i\in J_3} CN^{\xi-\alpha-\frac{2}{3}}\hat{\imath}^{-\frac{1}{3}}.\end{aligned}$$

For $i \in J_1$, by Lemma 4.2.5 eq. (4.2.10) and Lemma 4.2.7 1,

$$\begin{aligned}&N\sum_{i\in J_1}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 |\lambda_i - x|\left|\Xi^{-1}(f_N)^{(2)}(t(\lambda_i - x)+x)\right| dt\, d\mu_V(x) \\ &\qquad \leq N\sum_{i\in J_1}\int_{\gamma_i}^{\gamma_{i+1}} CN^{2\alpha}\log N|\lambda_i - x|d\mu_V(x) \leq \sum_{i\in J_1} CN^{2\alpha+\xi-1}\log N.\end{aligned}$$

It follows that

$$\begin{aligned}\left|M_N(\Xi^{-1}(f_N)')\right| &\leq \sum_{i\in J_1} CN^{2\alpha+\xi-1}\log N + \sum_{i\in J_2}\frac{CN^{\xi-1-\alpha}}{(\gamma_i - E)^3} + \log N\sum_{i\in J_3} CN^{\xi-\alpha-\frac{2}{3}}\hat{\imath}^{-\frac{1}{3}} \\ &\leq CN^{\alpha+\xi}\log N + CN^{\xi+\alpha} \leq CN^{\alpha+\xi}\log N,\end{aligned}$$

where we have used $|J_1| \leq CN^{1-\alpha}$ and the following estimates:

$$\sum_{i\in J_2} \frac{N^{\xi-\alpha-1}}{(\gamma_i - E)^3} \leq CN^{\xi-\alpha}\left(\int_a^{E-\frac{2M}{N^\alpha}} \frac{dx}{(x-E)^3} + \int_{E+\frac{2M}{N^\alpha}}^{b} \frac{dx}{(x-E)^3}\right) \leq CN^{\xi+\alpha},$$

$$CN^{\xi-\alpha-\frac{2}{3}} \sum_{i\in J_3} \hat{\imath}^{-\frac{1}{3}} \leq CN^{\xi-\alpha} \times \frac{1}{N}\sum_{i=1}^{N}\left(\frac{i}{N}\right)^{-\frac{1}{3}} \leq CN^{\xi-\alpha}.$$

This proves (4.2.23). The bound (4.2.24) is obtained in a similar way and we omit the details. □

For convenience we introduce the following notation: for a sequence of random variable $(X_N)_{N\in\mathbb{N}}$ we write $X_N = \omega(1)$ if there exists constants c, C and $\delta > 0$ such that the bound $|X_N| \leq \frac{C}{N^\delta}$ holds with probability greater than $1 - e^{-N^c}$.

Lemma 4.2.9. *We have*

$$\frac{1}{N}\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} dM_N(x)dM_N(y) = \omega(1)\ . \tag{4.2.26}$$

Proof. The proof will be similar to the proof of Lemma 4.2.8. As in Lemma 4.2.8 we may restrict our attention to the event $\Omega = \{\forall i : |\lambda_i - \gamma_i| \leq N^{-\frac{2}{3}+\xi}\hat{\imath}^{-\frac{1}{3}}\}$ by applying Theorem 4.2.6. Further, we use again the sets J_1, J_2 and J_3 defined in Lemma 4.2.7.

The general idea will be that we can use the uniform bounds (4.2.10) for particles close to the bulk point E (corresponding to the indices in J_1), and control the number of such particles. In the intermediary regime we will use the bounds (4.2.12) or the explicit formula (4.2.14). On the other hand, for the particles far away from E (corresponding to J_3) we can use the uniform decay of $\Xi^{-1}f_N$ and its derivative by (4.2.11) and (4.2.12).

Define for $j \in \{1,2,3\}$:

$$M_N^{(j)} = \sum_{i\in J_j}\left(\delta_{\lambda_i} - N\mathbb{1}_{[\gamma_i,\gamma_{i+1}]}\mu_V\right)$$

so that $M_N = M_N^{(1)} + M_N^{(2)} + M_N^{(3)}$. We can write

$$\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} dM_N(x)dM_N(y)$$
$$= \sum_{1\leq j_1,j_2\leq 3} \iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} dM_N^{(j_1)}(x)dM_N^{(j_2)}(y)$$

Integrating repeatedly for each (j_1, j_2), and using that $N\mu_V([\gamma_i,\gamma_{i+1}]) = 1$ for all indices i yields:

$$\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} dM_N^{(j_1)}(x)dM_N^{(j_2)}(y) =$$
$$N^2 \sum_{\substack{i_1\in J_{j_1}\\ i_2\in J_{j_2}}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} d\mu_V(x_1)\int_{\gamma_{i_2}}^{\gamma_{i_2+1}} d\mu_V(x_2)\int_T du\,dv\,dt\Big\{(\lambda_{i_1} - x_1)(\lambda_{i_2} - x_2)t(1-t)$$
$$\times\, \Xi^{-1}(f_N)^{(3)}\big(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + u(\lambda_{i_2} - x_2) + t(x_1 - x_2) + x_2\big)\Big\} \tag{4.2.27}$$

where $T = [0;1]^3$. We will bound (4.2.27) for each pair (j_1, j_2).
For $(j_1, j_2) = (1,1)$. Recall by Lemma 4.2.7 3 that $|J_1| \leq CN^{1-\alpha}$, and from the proof of Lemma 4.2.7, uniformly in $i \in J_1$ $|\lambda_i - x| \leq CN^{\xi-1}$ whenever $x \in [\gamma_i, \gamma_{i+1}]$. We use (4.2.27), Lemma 4.2.5 eq. (4.2.10) to obtain the upper bound

$$\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} dM_N^{(1)}(x) dM_N^{(1)}(y) \leq$$
$$N^2 \sum_{\substack{i_1 \in J_1 \\ i_2 \in J_1}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} N^{3\alpha} \log N |\lambda_{i_1} - x_1||\lambda_{i_2} - x_2| \, d\mu_V(x_1) \, d\mu_V(x_2) \leq CN^{2\xi+\alpha} \log N,$$

which is $\omega(1)$ when divided by N.
For $(j_1, j_2) = (3,3)$, we do as in the previous case. Using (4.2.12) instead and the fact that uniformly uniformly in $i \in J_3$, $|\lambda_i - x| \leq CN^{-\frac{2}{3}+\xi} \hat{i}^{-\frac{1}{3}}$,

$$\iint \frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} dM_N^{(3)}(x) dM_N^{(3)}(y) \leq$$
$$N^2 \sum_{\substack{i_1 \in J_3 \\ i_2 \in J_3}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} \frac{\log N}{N^\alpha} |\lambda_{i_1} - x_1||\lambda_{i_2} - x_2| \, d\mu_V(x_1) \, d\mu_V(x_2) \leq CN^{2\xi-\alpha} \log N,$$

which is $\omega(1)$ when divided by N.
For $(j_1, j_2) = (2,2)$. We remark that the strategy is not as straightforward as the case $i \in J_2$ in the proof of Lemma 4.2.8 eq. (4.2.23). This is because the term $t(x_1 - x_2) + x_2$ appearing as an argument in (4.2.27) may enter a neighborhood of E depending on the indices $i_1, i_2 \in J_2$ and we may not use the bound Lemma 4.2.5 eq. (4.2.11) uniformly in i_1, $i_2 \in J_2$. Some care is needed also because M_N is a signed measure so $|M_N(g)|$ need not be bounded by $M_N(|g|)$. It will be convenient to use directly eq. (4.2.14) from the proof of Lemma 4.2.5 (this can be done as J_2 corresponds to indices i such that γ_i is located outside the support of f). We can write for $x, y \in \{z \in \operatorname{supp} \mu_V, N^\alpha |z - E| > M + 1\}$

$$\begin{aligned}
&\frac{\Xi^{-1}(f_N)(x) - \Xi^{-1}(f_N)(y)}{x-y} \\
&= \frac{1}{\beta\pi^2} \int_{-M}^{M} \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})(x-y)} \left(\frac{1}{S(y)(u - N^\alpha(y-E))} - \frac{1}{S(x)(u - N^\alpha(x-E))} \right) du \\
&= \frac{1}{\beta\pi^2} \int_{-M}^{M} \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})} \left\{ \frac{S(x) - S(y)}{(x-y)} \frac{1}{S(x)S(y)(u - N^\alpha(y-E))} \right. \\
&\qquad \left. + \frac{N^\alpha}{S(x)(u - N^\alpha(x-E))(u - N^\alpha(y-E))} \right\} du. \qquad (4.2.28)
\end{aligned}$$

When we integrate the term on the third line of (4.2.28) against $M_N^{(2)} \otimes M_N^{(2)}$, we obtain

$$\int_{-M}^{M} \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})} \left\{ \int M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)} dt \right) \frac{1}{(u - N^\alpha(y-E))} dM_N^{(2)}(y) \right\} du. \tag{4.2.29}$$

Define the function

$$g(y) := M_N^{(2)}\left(\int_0^1 \frac{S'(t(\cdot - y)+y)}{S(\cdot)S(y)}dt\right).$$

First, $g(y)$ is bounded for any $y \in [a;b]$:

$$\begin{aligned}
&\left|M_N^{(2)}\left(\int_0^1 \frac{S'(t(\cdot - y)+y)}{S(\cdot)S(y)}dt\right)\right| \\
&\quad = \left|\frac{N}{S(y)}\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 \left(\frac{S'(t(\lambda_i - y)+y)}{S(\lambda_i)} - \frac{S'(t(x-y)+y)}{S(x)}\right) dt\, d\mu_V(x)\right| \\
&\quad \leq \left|\frac{N}{S(y)}\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 \frac{S'(t(\lambda_i - y)+y) - S'(t(x-y)+y)}{S(\lambda_i)} dt d\mu_V(x)\right| \\
&\qquad + \left|\frac{N}{S(y)}\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}}\int_0^1 \frac{S(x)-S(\lambda_i)}{S(x)S(\lambda_i)} S'(t(x-y)+y)\, dt\, d\mu_V(x)\right| \leq CN^{\xi},
\end{aligned}$$

where in the final line we used S and S' are smooth on $[a;b]$ (and therefore uniformly Lipschitz), $S > 0$ in a neighborhood of $[a;b]$, further $|x - \lambda_i| \leq CN^{\xi-1}$, and $|J_2| \leq CN$. Moreover, $g(y)$ is uniformly Lipschitz in $[a;b]$ with constant CN^{ξ}, since:

$$\begin{aligned}
&M_N^{(2)}\left(\int_0^1 \frac{S'(t(\cdot - y)+y)}{S(\cdot)S(y)} - \frac{S'(t(\cdot - z)+z)}{S(\cdot)S(z)}dt\right) = \\
&\qquad (z-y)M_N^{(2)}\left(\int_0^1\int_0^1 \frac{tS''(ut(z-y)+t(\cdot - z)+y)}{S(\cdot)S(y)}dtdu\right) \\
&\qquad\qquad + \frac{S(z)-S(y)}{S(z)S(y)} M_N^{(2)}\left(\int_0^1 \frac{S'(t(\cdot - z)+z)}{S(\cdot)}dt\right)
\end{aligned}$$

and both terms appearing in $M_N^{(2)}$ above are of the same form as g so they are bounded by CN^{ξ}. Returning to (4.2.29), we may bound

$$\begin{aligned}
&\left|M_N^{(2)}\left(\frac{g(y)}{u - N^{\alpha}(y-E)}\right)\right| \\
&\quad = \left|N\sum_{i\in J_2}\int_{\gamma_i}^{\gamma_{i+1}} \frac{g(\lambda_i)-g(x)}{(u - N^{\alpha}(\lambda_i - E))} + \frac{N^{\alpha}(\lambda_i - x)g(x)}{(u - N^{\alpha}(\lambda_i - E))(u - N^{\alpha}(x-E))} d\mu_V(x)\right| \\
&\quad \leq \int_{[a;b]\cap\{|x-E|\geq \frac{2M}{N^{\alpha}}\}} \frac{CN^{2\xi}}{|u - N^{\alpha}(x-E)|} + \frac{CN^{2\xi+\alpha}}{(u - N^{\alpha}(x-E))^2}dx \\
&\qquad \leq CN^{2\xi-\alpha}\log N + CN^{2\xi},
\end{aligned}$$

uniformly in $u \in [-M;M]$. Thus (4.2.29) is bounded by $CN^{2\xi}$ as f is bounded.
The remaining term in (4.2.28) is

$$N^{\alpha}\int_{-M}^{M} \frac{f(u)}{\sigma(E+\frac{u}{N^{\alpha}})} M_N^{(2)}\left(\frac{1}{S(\cdot)(u - N^{\alpha}(\cdot - E))}\right) M_N^{(2)}\left(\frac{1}{u - N^{\alpha}(\cdot - E)}\right) du. \tag{4.2.30}$$

Repeating our argument in the previous paragraph gives:

$$\left|M_N^{(2)}\left(\frac{1}{S(\cdot)(u-N^\alpha(\cdot-E))}\right)\right| \leq CN^{\xi-\alpha}\log N + CN^{\xi},$$
$$\left|M_N^{(2)}\left(\frac{1}{u-N^\alpha(\cdot-E)}\right)\right| \leq CN^{\xi},$$

where in the first inequality we use $1/S$ is uniformly bounded and uniformly Lipschitz on $[a;b]$. Inserting the bounds into (4.2.30) gives an upper bound of $CN^{2\xi+\alpha}$, as f is bounded. Altogether

$$\left|\frac{\Xi^{-1}(f_N)(x)-\Xi^{-1}(f_N)(y)}{x-y}dM_N^{(2)}(x)dM_N^{(2)}(y)\right| \leq CN^{2\xi+\alpha},$$

which is $\omega(1)$ when divided by N.

For $(j_1,j_2)=(1,2)$. By the bounds $|\lambda_{i_j}-\gamma_{i_j}|\leq CN^{\xi-1}$, $|\gamma_{i_j}-x_j|\leq \frac{C}{N}$ for $x_j\in[\gamma_{i_j};\gamma_{i_j+1}]$, whenever

$$N^\alpha|tv(\lambda_{i_1}-x_1)+ut(x_2-\lambda_{i_2})+t(x_1-x_2)+u(\lambda_{i_2}-x_2)+x_2-E|\geq M+1, \tag{4.2.31}$$

we have

$$N^\alpha(|t(\gamma_{j_1}-\gamma_{j_2})+(\gamma_{j_2}-E)|+CN^{\xi-1})\geq M+1,$$

and

$$\frac{1}{|tv(\lambda_{i_1}-x_1)+ut(x_2-\lambda_{i_2})+t(x_1-x_2)+u(\lambda_{i_2}-x_2)+x_2-E|} \leq \frac{C}{|t(\gamma_{i_1}-\gamma_{i_2})+(\gamma_{i_2}-E)|},$$

where the constant C only depends on M. Therefore, whenever (4.2.31) is satisfied, applying Lemma 4.2.5 eq. (4.2.11) yields

$$\left|\Xi^{-1}(f_N)^{(3)}\left(tv(\lambda_{i_1}-x_1)+ut(x_2-\lambda_{i_2})+t(x_1-x_2)+u(\lambda_{i_2}-x_2)+x_2\right)\right| \leq \frac{C}{N^\alpha(t(\gamma_{i_1}-\gamma_{i_2})+\gamma_{i_2}-E)^4}. \tag{4.2.32}$$

Now fix $t\in(0,1)$ and define the sets

$$K_t^1 := \left\{j\in J_2\ ,\ t\left(E-\frac{2M}{N^\alpha}-\gamma_j\right)+\gamma_j-E\geq\frac{2M}{N^\alpha}\right\},$$
$$K_t^2 := \left\{j\in J_2\ ,\ t\left(E+\frac{2M}{N^\alpha}-\gamma_j\right)+\gamma_j-E\leq-\frac{2M}{N^\alpha}\right\},$$
$$K_t := K_t^1\cup K_t^2.$$

By construction, if $i_2\in K_t^1$ then

$$|t(\gamma_{i_1}-\gamma_{i_2})+(\gamma_{i_2}-E)|\geq\frac{2M}{N^\alpha}$$

uniformly in $i_1 \in J_1$. Thus for such $i_2 \in K_t^1$, (4.2.31) is satisfied for N sufficiently large. The same statement holds for K_t^2.
We now proceed to bound (4.2.27) for $j_1 = 1$ and $j_2 = 2$ by splitting J_2 into the regions K_t^1, K_t^2 and $J_2 \backslash K_t$. We start with K_t^1 (the argument for K_t^2 is identical). Our observations from the previous paragraph along with (4.2.32) gives:

$$\int_T du\, dv\, dt \left| N^2 \sum_{\substack{i_1 \in J_1 \\ i_2 \in K_t^1}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} d\mu_V(x_2) \Big\{ (\lambda_{i_1} - x_1)(\lambda_{i_2} - x_2) t(1-t) \right.$$
$$\left. \times\, \Xi^{-1}(f_N)^{(3)} \big(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + u(\lambda_{i_2} - x_2) + t(x_1 - x_2) + x_2 \big) \Big\} \right|$$
$$\leq \int_0^1 \sum_{\substack{i_1 \in J_1 \\ i_2 \in K_t^1}} \frac{CN^{2\xi-2-\alpha} t(1-t)}{(t(\gamma_{i_1} - \gamma_{i_2}) + (\gamma_{i_2} - E))^4}\, dt \leq \int_0^1 \sum_{i_2 \in K_t^1} \frac{CN^{2\xi-1-2\alpha} t(1-t)}{\left((1-t)(\gamma_{i_2} - E) - \frac{t2M}{N^\alpha} \right)^4}\, dt$$

where in the final line we used $|J_1| \leq CN^{1-\alpha}$ from Lemma 4.2.7 eq. (3). Next, note that

$$\frac{1}{N} \sum_{i_2 \in K_t^1} \frac{1}{((1-t)(\gamma_{i_2} - E) - \frac{t2M}{N^\alpha})^4}$$
$$\leq C \int_{E + \frac{2M}{N^\alpha}\left(\frac{1+t}{1-t}\right)}^{E + \frac{1}{2}(E-a) \wedge (b-E)} \frac{dx}{((1-t)(x-E) - \frac{t2M}{N^\alpha})^4} \leq \frac{CN^{3\alpha}}{1-t},$$

since, by definition of K_t^1, $\gamma_{i_2} \geq E + \frac{2M}{N^\alpha}\left(\frac{1+t}{1-t}\right)$. We conclude,

$$\int_0^1 \sum_{i_2 \in K_t^1} \frac{CN^{2\xi-1-2\alpha} t(1-t)}{\left((1-t)(\gamma_{i_2} - E) - \frac{t2M}{N^\alpha} \right)^4}\, dt \leq CN^{2\xi+\alpha}.$$

We continue with $J_2 \backslash K_t$. By the same argument as in Lemma 4.2.7 3 $|J_2 \backslash K_t| \leq \frac{CN^{1-\alpha}}{1-t}$ where the constant C does not depend on t, we use this in addition with Lemma 4.2.5 eq. (4.2.11), $|J_1| \leq CN^{1-\alpha}$, and $|\lambda_{i_j} - x_j| \leq CN^{\xi-1}$ to obtain the bound

$$\int_T du\, dv\, dt \left| N^2 \sum_{\substack{i_1 \in J_1 \\ i_2 \in J_2 \backslash K_t}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} d\mu_V(x_2) \Big\{ (\lambda_{i_1} - x_1)(\lambda_{i_2} - x_2) t(1-t) \right.$$
$$\left. \times\, \Xi^{-1}(f_N)^{(3)} \big(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + u(\lambda_{i_2} - x_2) + t(x_1 - x_2) + x_2 \big) \Big\} \right|$$
$$\leq C \int_0^1 N^{3\alpha} \log N \times N^{2\xi-2} \times N^{2-2\alpha} t\, dt \leq CN^{\alpha+2\xi} \log N.$$

Combining the bounds we have obtained gives

$$\left| \iint \frac{\Xi^{-1} f_N(x) - \Xi^{-1} f_N(y)}{x - y} dM_N^{(1)}(x) dM_N^{(2)}(y) \right| \leq CN^{\alpha+2\xi} \log N,$$

which is $\omega(1)$ when divided by N for ξ small enough.
For $j_1 = 1$ or 2 and $j_2 = 3$. the proof is similar and we omit the details. □

Using Lemma 4.2.8 we also prove the following bounds:

Lemma 4.2.10. *The following estimates hold:*

$$L_N\Big(\Xi^{-1}(f_N)'\Big) = \omega(1) , \tag{4.2.33}$$

$$L_N\Big(\Xi^{-1}(f_N)f_N'\Big) + \sigma_f^2 = \omega(1) , \tag{4.2.34}$$

Proof. For both (4.2.33) and (4.2.34), we use

$$L_N\Big(\Xi^{-1}(f_N)'\Big) = \frac{M_N(\Xi^{-1}(f_N)')}{N} + \mu_V(\Xi^{-1}(f_N)'),$$
$$L_N\left(\Xi^{-1}(f_N)f_N'\right) = \frac{M_N\left(\Xi^{-1}(f_N)f_N'\right)}{N} + \mu_V\left(\Xi^{-1}(f_N)f_N'\right),$$

Lemma 4.2.8 implies that the first term in both equations are $\omega(1)$ so (4.2.33) and (4.2.34) simplify to deterministic statements about the speed of convergence of the integrals against μ_V above.

To show (4.2.33), integration by parts yields:

$$\int (\Xi^{-1}f_N)'(x)d\mu_V(x) = -\int_a^b (\Xi^{-1}f_N)(x)(S'(x)\sigma(x) + S(x)\sigma'(x))dx.$$

Inserting the formula for $\Xi^{-1}f_N$ we obtain

$$\left|\int (\Xi^{-1}f_N)'(x)d\mu_V(x)\right| \leq \frac{1}{\beta\pi^2}\int_a^b\int_a^b \left|\frac{f_N(x)-f_N(y)}{y-x}\right| \left(\left|\frac{S'(x)\sigma(x)}{S(x)\sigma(y)}\right| + \left|\frac{\sigma'(x)}{\sigma(y)}\right|\right) dxdy .$$

Recall that S is bounded below on $[a,b]$, S' is bounded above on $[a,b]$, further, up to a constant, $\frac{\sigma'(x)}{\sigma(y)}$ can be bounded above by $(\sigma(x)\sigma(y))^{-1}$. We define the sets

$$A_N := [N^\alpha(a-E); N^\alpha(b-E)],$$
$$B_N := \left[\frac{1}{2}N^\alpha(a-E); \frac{1}{2}N^\alpha(b-E)\right].$$

By the observations above, and the change of variable $u = N^\alpha(x-E)$ and $v = N^\alpha(y-E)$ we get

$$\left|\int (\Xi^{-1}f_N)'(x)d\mu_V(x)\right|$$
$$\leq \frac{C}{N^\alpha}\iint_{A_N^2}\left|\frac{f(u)-f(v)}{u-v}\right|\left(\frac{\sigma(E+\frac{u}{N^\alpha})}{\sigma(E+\frac{v}{N^\alpha})} + \frac{1}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right) dudv. \tag{4.2.35}$$

For large enough N, on the set $(u,v)\in(A_N\backslash B_N)^2$, the function $|f(u)-f(v)|$ is always zero, thus the integral on the right above can be divided into integrals over the sets:

$$(A_N\times A_N)\cap(A_N\backslash B_N\times A_N\backslash B_N)^c = B_N\times B_N\cup B_N\times(A_N\backslash B_N)\cup(A_N\backslash B_N)\times B_N. \tag{4.2.36}$$

We bound the integral in (4.2.35) over each set in (4.2.36). We begin with the first set in (4.2.36). For $(u,v) \in B_N \times B_N$, $\sigma(E+\frac{u}{N^\alpha})$ and $\sigma(E+\frac{v}{N^\alpha})$ are uniformly bounded above and below. Therefore, the integral in (4.2.35) can be bounded in this region by

$$\iint_{B_N^2} \left|\frac{f(u)-f(v)}{u-v}\right| dudv$$
$$= \iint_{[-M;M]^2} \left|\frac{f(u)-f(v)}{u-v}\right| dudv + 2\int_{-M}^{M}\int_{B_N\cap\{|u|\geq M\}} \left|\frac{f(v)}{u-v}\right| du\, dv,$$

the integral over $[-M;M]^2$ exists by the differentiability of f, while:

$$\int_{-M}^{M}\int_{B_N\cap\{|u|\geq M\}} \left|\frac{f(v)}{u-v}\right| du\, dv \;\leq\; C\int_{-M}^{M} |f(v)| \log[N|v+M||v-M|]\, dv \;\leq\; C\log N,$$

for N large enough.

For the second set in (4.2.35), observe that for $(u,v) \in B_N \times (A_N \backslash B_N)$, $f(v)$ is 0 for N sufficiently large, and $\sigma(E+\frac{u}{N^\alpha})$ is bounded uniformly above and below while $f(u)$ is 0 outside $[-M;M]$. This implies that the integral in (4.2.35) can be bounded in this region by

$$\int_{A_N\backslash B_N}\int_{-M}^{M} \left|\frac{f(u)}{u-v}\right| \left(\frac{\sigma(E+\frac{u}{N^\alpha})}{\sigma(E+\frac{v}{N^\alpha})} + \frac{1}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right) dudv$$
$$\leq \frac{C\|f\|_{C(\mathbb{R})}}{N^\alpha}\int_{A_N\backslash B_N} \frac{1}{\sigma(E+\frac{v}{N^\alpha})} dv \leq C\ ,$$

where in the final line we used $|u-v| \geq cN^\alpha$ for $u \in [-M;M]$, $v \in A_N \backslash B_N$.

We can do similarly for the third set in (4.2.35) and putting together these bounds on the right hand side of (4.2.35) gives

$$\left|\int (\Xi^{-1} f_N)'(x) d\mu_V(x)\right| \leq \frac{C\log N}{N^\alpha},$$

which is $\omega(1)$ as claimed.

We continue with (4.2.34). Recall that we reduced this problem to computing the limit of $\mu_V(\Xi^{-1}(f_N)f_N')$. Using the inversion formula we see that

$$\int \Xi^{-1} f_N(x) f_N'(x) d\mu_V(x) = -\frac{1}{\beta\pi^2}\int_a^b\int_a^b \frac{\sigma(x) f_N'(x)(f_N(x)-f_N(y))}{\sigma(y)(x-y)} dxdy$$

Observe that

$$\frac{1}{2}\partial_x (f_N(x)-f_N(y))^2 = f_N'(x)(f_N(x)-f_N(y)),$$
$$\partial_x\left(\frac{\sigma(x)}{x-y}\right) = \frac{-\frac{1}{2}(a+b)(x+y)+ab+xy}{\sigma(x)(x-y)^2}.$$

Therefore, integration by parts yields

$$\int \Xi^{-1} f_N(x) f_N'(x) d\mu_V(x) = -\frac{1}{2\beta\pi^2}\int_a^b\int_a^b \frac{\sigma(x)\partial_x(f_N(x)-f_N(y))^2}{\sigma(y)(x-y)}dxdy$$

$$= \frac{1}{2\beta\pi^2}\int_a^b\int_a^b \left(\frac{f_N(x)-f_N(y)}{x-y}\right)^2\left(\frac{ab+xy-\frac{1}{2}(a+b)(x+y)}{\sigma(x)\sigma(y)}\right)dxdy,$$

By changing variables again to $(u,v) = (N^\alpha(x-E), N^\alpha(y-E))$ and observing that

$$ab + xy - \frac{1}{2}(a+b)(x+y) = -\sigma(E)^2 + \frac{u+v}{N^\alpha}\left(\frac{a+b}{2}+E\right) + \frac{uv}{N^{2\alpha}},$$

we obtain

$$\int \Xi^{-1} f_N(x) f_N'(x) d\mu_V(x)$$

$$= -\frac{1}{2\beta\pi^2}\iint_{A_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2\left(\frac{\sigma(E)^2-\frac{u+v}{N^\alpha}(\frac{a+b}{2}+E)-\frac{uv}{N^{2\alpha}}}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right)dudv. \quad (4.2.37)$$

As before, $(f(u)-f(v))^2$ is zero for all $(u,v)\in(A_N\backslash B_N)^2$ for large enough N, therefore we split the above integral into the regions defined in (4.2.36).
Notice that uniformly in $u\in B_N$

$$\frac{1}{\sigma(E+\frac{u}{N^\alpha})} = \frac{1}{\sigma(E)} + O\left(\frac{|u|}{N^\alpha}\right),$$

and further notice $(u+v)/N^\alpha$ and $uv/N^{2\alpha}$ are bounded uniformly by constants in the entire region $A_N\times A_N$.
Consequently the integral (4.2.37) over the region $B_N\times B_N$ is:

$$\iint_{B_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2\left(\frac{\sigma(E)^2-\frac{u+v}{N^\alpha}\left(\frac{a+b}{2}+E\right)-\frac{uv}{N^{2\alpha}}}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right)dudv$$

$$= \iint_{B_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2 dudv + O\left(\frac{1}{N^\alpha}\iint_{B_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2(|u|+|v|)dudv\right). \quad (4.2.38)$$

The first term of (4.2.38) is equal to,

$$\frac{1}{2\beta\pi^2}\iint\left(\frac{f(u)-f(v)}{u-v}\right)^2 dudv + O\left(\frac{1}{N^\alpha}\right)$$

while the second term in (4.2.38) can be written as

$$\iint_{B_N^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2(|u|+|v|)dudv = \iint_{[-M;M]^2}\left(\frac{f(u)-f(v)}{u-v}\right)^2(|u|+|v|)dudv$$

$$+2\int_{-M}^{M}\int_{B_N\cap\{|u|\geq M\}}\left(\frac{f(v)}{u-v}\right)^2(|u|+|v|)dudv,$$

the integral over $[-M;M]^2$ is finite by differentiability of f while the second is bounded by

$$\int_{-M}^{M}\int_{B_N\cap\{|u|\geq M\}}|f(v)|^2\left(\frac{1}{|u-v|}+\frac{2|v|}{|u-v|^2}\right)dudv$$
$$\leq C\int_{-M}^{M}|f(v)|^2\left(\frac{1}{|v-M|}+\frac{1}{|M+v|}+\log[N|v-M||v+M|]\right)\leq C\log N$$

since supp $f\subset[-M,M]$.
In the region $(u,v)\in B_N\times(A_N\backslash B_N)$, $\sigma(E+\frac{u}{N^\alpha})$ is bounded above and below while, for N large enough $f(v)=0$, thus the integral over $B_N\times(A_N\backslash B_N)$ is bounded above by

$$\int_{A_N\backslash B_N}\int_{B_N}\left(\frac{f(u)-f(v)}{u-v}\right)^2\left(\frac{1}{\sigma(E+\frac{u}{N^\alpha})\sigma(E+\frac{v}{N^\alpha})}\right)dudv$$
$$\leq\int_{A_N\backslash B_N}\int_{-M}^{M}\left(\frac{f(u)}{u-v}\right)^2\frac{1}{\sigma(E+\frac{v}{N^\alpha})}dudv\leq\frac{C}{N^{2\alpha}}\int_{A_N\backslash B_N}\frac{1}{\sigma(E+\frac{v}{N^\alpha})}dv\leq\frac{C}{N^\alpha},$$

where in the second line we used $|u-v|\geq cN^\alpha$ for $u\in[-M;M]$ and $v\in A_N\backslash B$. By symmetry of the integrand in (4.2.37) this argument extends to the region $(u,v)\in(A_N\backslash B_N)\times B_N$. Altogether, our bounds show

$$\int\Xi^{-1}f_N(x)f_N'(x)d\mu_V(x)=-\frac{1}{2\beta\pi^2}\iint\left(\frac{f(x)-f(y)}{x-y}\right)^2dxdy+O\left(\frac{\log N}{N^\alpha}\right),$$

which shows (4.2.34). □

4.2.4 Proof of Theorem 4.1.4

We proceed with the proof of Theorem 4.1.4. As we did in the sketch of the proof, (4.2.7) applied to $h=\Xi^{-1}(f_N)$ yields

$$F_1^N\left(\Xi^{-1}(f_N)\right)=M_N(f_N)+\left(1-\frac{\beta}{2}\right)L_N\left((\Xi^{-1}f_N)'\right)$$
$$+\frac{1}{N}\left[\frac{\beta}{2}\iint\frac{\Xi^{-1}f_N(x)-\Xi^{-1}f_N(y)}{x-y}dM_N(x)dM_N(y)\right].$$

Combining Lemma 4.2.9 eq 4.2.26 and Lemma 4.2.10 eq. (4.2.33) we can bound the two terms on the right hand side to get

$$F_1^N(\Xi^{-1}(f_N))=M_N(f_N)+\omega(1). \tag{4.2.39}$$

We consider an event A_1 of probability higher than $1-e^{-N^c}$ on which

$$\left|F_1^N(\Xi^{-1}(f_N))-M_N(f_N)\right|\leq\frac{C}{N^\delta}, \tag{4.2.40}$$

for some positive constants c, C and δ. Using the first loop equation from Proposition 4.2.1, and the trivial deterministic bounds

$$M_N(f_N) = O\Big(N\|f\|_\infty\Big) \ , \ F_1^N(\Xi^{-1}(f_N)) = O\Big(N(\|f\|_\infty + \|\Xi^{-1}(f_N)\|_{C^1(\mathbb{R})})\Big) = O(N^3) \ ,$$

we obtain

$$\begin{aligned} 0 = \mathbb{E}_V^N\Big(F_1^N(\Xi^{-1}(f_N))\Big) &= \mathbb{E}_V^N\Big(F_1^N(\Xi^{-1}(f_N))\mathbb{1}_{A_1}\Big) + \mathbb{E}_V^N\Big(F_1^N(\Xi^{-1}(f_N))\mathbb{1}_{A_1^c}\Big) \\ &= \mathbb{E}_V^N\Big(M_N(f_N)\mathbb{1}_{A_1}\Big) + o(1) + O\Big(N^3\mathbb{P}_V^N\big(A_1^c\big)\Big) \\ &= \mathbb{E}_V^N\Big(M_N(f_N)\Big) + o(1) \ , \end{aligned} \tag{4.2.41}$$

and thus

$$\mathbb{E}_V^N\Big(M_N(f_N)\Big) = o(1). \tag{4.2.42}$$

We now show recursively that

$$F_k^N(\Xi^{-1}(f_N), f_N, \cdots, f_N) = \tilde{M}_N(f_N)^k - (k-1)\sigma_f^2\ \tilde{M}_N(f_N)^{k-2} + \omega(1). \tag{4.2.43}$$

Here, the set on which the bound holds might vary from one k to another but each bound has probability greater than $1 - e^{-N^{c_k}}$ for each fixed k.
The bound holds for $k = 1$, by (4.2.39). Now, assume this holds for $k \geq 1$. On a set of probability greater than $1 - e^{-N^{c_{k+1}}}$ we have by the induction hypothesis, Lemma 4.2.8 eq. (4.2.22) and Lemma 4.2.10 eq. (4.2.34), for some $\delta > 0$ and a constant C

$$\Big|F_k^N(\Xi^{-1}(f_N), f_N, \cdots, f_N) - \tilde{M}_N(f_N)^k + (k-1)\sigma_f^2\ \tilde{M}_N(f_N)^{k-2}\Big| \leq \frac{C}{N^\delta},$$

$$\Big|L_N\Big(\Xi^{-1}(f_N)f_N'\Big) + \sigma_f^2\Big| \leq \frac{C}{N^\delta} \ ,$$

$$\Big|M_N(f_N)\Big| \leq N^{\delta/2k} \ .$$

On this set, using the definition of F_{k+1}^N from Proposition 4.2.1,

$$\begin{aligned} F_{k+1}^N(\Xi^{-1}(f_N), f_N, \cdots, f_N) &= F_k^N(\Xi^{-1}(f_N), f_N, \cdots, f_N)\tilde{M}_N(f_N) \\ &\quad + \tilde{M}_N(f_N)^{k-1} L_N(\Xi^{-1}(f_N)f_N') \\ &= \left(\tilde{M}_N(f_N)^k - (k-1)\sigma_f^2\ \tilde{M}_N(f_N)^{k-2} + O\Big(\frac{1}{N^\delta}\Big)\right)\tilde{M}_N(f_N) \\ &\quad + \tilde{M}_N(f_N)^{k-1}\left(-\sigma_f^2 + O\Big(\frac{1}{N^\delta}\Big)\right) \\ &= \tilde{M}_N(f_N)^{k+1} - k\ \sigma_f^2\ \tilde{M}_N(f_N)^{k-1} + O\Big(\frac{1}{N^{\delta/2}}\Big) \end{aligned}$$

and this proves the induction. Using the fact that F_k is bounded polynomially and deterministically as before, we see that for any $k \geq 1$

$$\mathbb{E}_V^N\Big(M_N(f_N)^{k+1}\Big) = \sigma_f^2\, k\, \mathbb{E}_V^N\Big(M_N(f_N)^{k-1}\Big) + o(1). \tag{4.2.44}$$

Coupled with (4.2.42), the computation of the moments is then straightforward and we obtain for all $k \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}_V^N\Big(M_N(f_N)^{2k}\Big) &= \sigma_f^{2k}\,\frac{(2k)!}{2^k k!} + o(1),\\ \mathbb{E}_V^N\Big(M_N(f_N)^{2k+1}\Big) &= o(1).\end{aligned} \tag{4.2.45}$$

This concludes the proof of Theorem 4.1.4.

4.2.5 A few Remarks

The result of Theorem 4.1.4 naturally extends to the joint law of the fluctuations of finite families of test functions. More precisely, for any fixed k, if $f^1, \cdots, f^k$ satisfy the hypothesis of the theorem then $(M_N(f_N^1), \cdots, M_N(f_N^k))$ converges in distribution towards a centered Gaussian vector with covariance matrix

$$\Sigma_{i,j} = \frac{1}{2\beta\pi^2}\iint \left(\frac{f^i(x) - f^i(x)}{x-y}\right)\left(\frac{f^j(x) - f^j(x)}{x-y}\right) dxdy\ .$$

We would also like to point out that a similar proof should also yield the macroscopic central limit Theorem already shown in [51, 15, 77] (one-cut and off-critical cases) with appropriate decay conditions on f. Indeed, in the macroscopic case we get uniform bounds on $\Xi^{-1}f$ and its derivatives instead of the bounds obtained in Lemma 4.2.5. The major issue when dealing with the multicut and critical cases is that the operator Ξ is not invertible (as an operator acting on smooth functions). When dealing with functions that lie in the image of Ξ and with additional regularity assumptions, one can show using transport methods similar to [59] that the central limit Theorem do hold at the macroscopic scale. This is the object of the next Chapter.

Another interesting direction would be to study the fluctuations at the edge (i.e, $E = a$ or b). We expect the same result to hold with covariance matrix(if for instance $E = a$) equal to

$$\Sigma_{i,j} = \frac{1}{2\beta\pi^2}\int_0^\infty\int_0^\infty \left(\frac{f^i(x) - f^i(y)}{x-y}\right)\left(\frac{f^j(x) - f^j(y)}{x-y}\right)\frac{x+y}{\sqrt{xy}}dxdy.$$

Additional technical estimates as in Lemma 4.2.5 would be needed to reproduce the proof in the edge case. These estimates are not straightforward because of the singular behaviour of $\Xi^{-1}(f_N)$ at the edges and this is the object of a future work. However the covariance (in the case $k = 1$) would still be given by taking the limit of $-\mu_V\left(\Xi^{-1}(f_N)f_N'\right)$ as in Lemma 4.2.10, which yields the above formula.

Chapter 5

CLT for Fluctuations of β-ensembles with general potential

This Chapter is based on the article [12] written with T.Leblé and S.Serfaty. In this Chapter we adopt some slightly different notations.

5.1 Introduction

Let $\beta > 0$ be fixed. For $N \geq 1$, we are interested in the N-point canonical Gibbs measure for a one-dimensional log-gas at the *inverse temperature* β, defined by

$$d\mathbb{P}^V_{N,\beta}(\vec{X}_N) = \frac{1}{Z^V_{N,\beta}} \exp\left(-\frac{\beta}{2}\mathcal{H}^V_N(\vec{X}_N)\right) d\vec{X}_N, \tag{5.1.1}$$

where $\vec{X}_N = (x_1, \dots, x_N)$ is an N-tuple of points in $\mathbb{R}$, and $\mathcal{H}^V_N(\vec{X}_N)$, defined by

$$\mathcal{H}^V_N(\vec{X}_N) := \sum_{1\leq i\neq j\leq N} -\log|x_i - x_j| + \sum_{i=1}^N NV(x_i), \tag{5.1.2}$$

is the energy of the system in the state $\vec{X}_N$, given by the sum of the pairwise repulsive logarithmic interaction between all particles plus the effect on each particle of an external field or confining potential NV whose intensity is proportional to N. We will use $d\vec{X}_N$ to denote the Lebesgue measure on $\mathbb{R}^N$. The constant $Z^V_{N,\beta}$ in the definition (5.1.1) is the normalizing constant, called the *partition function*, and is equal to

$$Z^V_{N,\beta} := \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2}\mathcal{H}^V_N(\vec{X}_N)\right) d\vec{X}_N.$$

Such systems of particles with logarithmic repulsive interaction on the line have been extensively studied, in particular because of their connection with random matrix theory, see [45] for a survey.

Under mild assumptions on V, it is known that the empirical measure of the particles converges almost surely to some deterministic probability measure on $\mathbb{R}$ called the *equilibrium* *measure* μ_V, with no simple expression in terms of V. For any $N \geq 1$, let us define the *fluctuation measure*

$$\mathrm{fluct}_N := \sum_{i=1}^{N} \delta_{x_i} - N\mu_V, \tag{5.1.3}$$

which is a random signed measure. For any test function ξ regular enough we define the *fluctuations of the linear statistics associated to* ξ as the random real variable

$$\mathrm{Fluct}_N(\xi) := \int_{\mathbb{R}} \xi \, d\mathrm{fluct}_N. \tag{5.1.4}$$

The goal of this Chapter is to prove a Central Limit Theorem (CLT) for $\mathrm{Fluct}_N(\xi)$, under some regularity assumptions on V and ξ.

5.1.1 Assumptions

(H1) - Regularity and growth of V The potential V is in $C^{\mathsf{p}}(\mathbb{R})$ and satisfies the growth condition

$$\liminf_{|x|\to\infty} \frac{V(x)}{2\log|x|} > 1. \tag{5.1.5}$$

It is well-known, see e.g. [73], that if V satisfies H1 with $\mathsf{p} \geq 0$, then the *logarithmic potential energy* functional defined on the space of probability measures by

$$\mathcal{I}_V(\mu) = \int_{\mathbb{R}\times\mathbb{R}} -\log|x-y| \, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}} V(x)\, d\mu(x) \tag{5.1.6}$$

has a unique global minimizer μ_V, the aforementioned *equilibrium measure.* This measure has a compact support that we will denote by Σ_V, and μ_V is characterized by the fact that there exists a constant c_V such that the function ζ_V defined by

$$\zeta_V(x) := \int -\log|x-y| d\mu_V(y) + \frac{V(x)}{2} - c_V \tag{5.1.7}$$

satisfies the Euler-Lagrange conditions

$$\zeta_V \geq 0 \text{ in } \mathbb{R}, \quad \zeta_V = 0 \text{ on } \Sigma_V. \tag{5.1.8}$$

We will work under two additional assumptions: one deals with the possible form of μ_V and the other one is a non-criticality hypothesis concerning ζ_V.

(H2) - Form of the equilibrium measure The support Σ_V of μ_V is a finite union of $\mathsf{n}+1$ non-degenerate intervals

$$\Sigma_V = \bigcup_{0\leq l\leq \mathsf{n}} [\alpha_{l,-}; \alpha_{l,+}], \text{ with } \alpha_{l,-} < \alpha_{l,+}.$$

The points $\alpha_{l,\pm}$ are called the *endpoints* of the support Σ_V. For x in Σ_V, we let

$$\sigma(x) := \prod_{l=0}^{\mathsf{n}} \sqrt{|x-\alpha_{l,-}||x-\alpha_{l,+}|}. \tag{5.1.9}$$

We assume that the equilibrium measure has a density with respect to the Lebesgue measure on Σ_V given by

$$\mu_V(x) = S(x)\sigma(x), \tag{5.1.10}$$

where S can be written as

$$S(x) = S_0(x)\prod_{i=1}^{\mathsf{m}}(x-s_i)^{2\mathsf{k}_i}, \quad S_0 > 0 \text{ on } \Sigma_V, \tag{5.1.11}$$

where $\mathsf{m} \geq 0$, all the points s_i, called *singular points*[1], belong to Σ_V and the k_i are natural integers.

(H3) - Non-criticality of ζ_V The function ζ_V is positive on $\mathbb{R} \setminus \Sigma_V$.

We introduce the operator Ξ_V, which acts on C^1 functions by

$$\Xi_V[\psi] := -\frac{1}{2}\psi V' + \int \frac{\psi(\cdot) - \psi(y)}{\cdot - y}\, d\mu_V(y). \tag{5.1.12}$$

5.1.2 Main result

Theorem 2 (Central limit theorem for fluctuations of linear statistics). *Let ξ be a function in $C^{\mathsf{r}}(\mathbb{R})$, assume that H1-H3 hold. We let*

$$\mathsf{k} = \max_{i=1,\dots,\mathsf{m}} 2\mathsf{k}_i,$$

where the k_i's are as in (5.1.11), *and assume that, p (resp. r) denoting the regularity of V (resp. ξ)*

$$\mathsf{p} \geq (3\mathsf{k}+5), \quad \mathsf{r} \geq (2\mathsf{k}+3). \tag{5.1.13}$$

If $\mathsf{n} \geq 1$, assume that ξ satisfies the n following conditions

$$\int_{\Sigma_V} \frac{\xi(y)y^d}{\sigma(y)}dy = 0 \qquad \textit{for } d = 0, \dots, \mathsf{n}-1. \tag{5.1.14}$$

Moreover, if $\mathsf{m} \geq 1$, assume that for all $i = 1, \dots, \mathsf{m}$

$$\int_{\Sigma_V} \frac{\xi(y) - R_{s_i,d}\xi(y)}{\sigma(y)(y-s_i)^d}dy = 0 \qquad \textit{for } d = 1, \dots, 2\mathsf{k}_i, \tag{5.1.15}$$

where $R_{x,d}\xi$ is the Taylor expansion of ξ to order $d-1$ around x given by

$$R_{x,d}\xi(y) = \xi(x) + (y-x)\xi'(x) + \cdots + \frac{(y-x)^{d-1}}{(d-1)!}\xi^{(d-1)}(x).$$

Then there exists a constant c_ξ and a function ψ of class C^2 in some open neighborhood U of Σ_V such that $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$ on U, and the fluctuation $\mathrm{Fluct}_N(\xi)$ *converges in law as $N \to \infty$ to a Gaussian distribution with mean*

$$m_\xi = \left(1 - \frac{2}{\beta}\right)\int \psi' \, d\mu_V,$$

[1] Let us emphasize that a singular point s_i can be equal to an endpoint $\alpha_{l,\pm}$.

and variance

$$v_\xi = -\frac{2}{\beta}\int \psi\xi' d\mu_V.$$

It is proven in (5.6.122) that the variance v_ξ has the equivalent expression

$$v_\xi := \frac{2}{\beta}\left(\iint \left(\frac{\psi(x)-\psi(y)}{x-y}\right)^2 d\mu_V(x)d\mu_V(y) + \int V''\psi^2 d\mu_V\right). \tag{5.1.16}$$

Let us note that ψ, hence also m_ξ and v_ξ, can be explicitly written in terms of ξ.

5.1.3 Comments on the assumptions

The growth condition (5.1.5) is standard and expresses the fact that the logarithmic repulsion is beaten at long distance by the confinement, thus ensuring that μ_V has a compact support. Together with the non-criticality assumption H3 on ζ_V, it implies that the particles of the log-gas effectively stay within some neighborhood of Σ_V, up to very rare events.

The case $\mathsf{n} = 0$, where the support has a single connected component, is called *one-cut*, whereas $\mathsf{n} \geq 1$ is a *multi-cut* situation. If $\mathsf{m} \geq 1$, we are in a *critical case.*

The relationship between V and μ_V is complicated in general, and we mention some examples where μ_V is known to satisfy our assumptions.

- If V is real-analytic, then the assumptions are satisfied with n finite, m finite and S analytic on Σ_V, see [34, Theorem 1.38], [35, Sec.1].
- If V is real-analytic, then for a "generic" V the assumptions are satisfied with n finite, $\mathsf{m} = 0$ and S analytic on Σ_V, see [54].
- If V is uniformly convex and smooth, then the assumptions are satisfied with $\mathsf{n} = 0$, $\mathsf{m} = 0$, and S smooth on Σ_V, see e.g. [67, Example 1].
- Examples of multi-cut, non-critical situations with $\mathsf{n} = 0, 1, 2$ and $\mathsf{m} = 0$, are mentioned in [67, Examples 3-4].
- An example of criticality at the edge of the support is given by $V(x) = \frac{1}{20}x^4 - \frac{4}{15}x^3 + \frac{1}{5}x^2 + \frac{8}{5}x$, for which the equilibrium measure, as computed in [27, Example 1.2], is given by
$$\mu_V(x) = \frac{1}{10\pi}\sqrt{|x-(-2)||x-2|}(x-2)^2 \mathbf{1}_{[-2,2]}(x).$$
- An example of criticality in the bulk of the support is given by $V(x) = \frac{x^4}{4} - x^2$, for which the equilibrium measure, as computed in [28], is
$$\mu_V(x) = \frac{1}{2\pi}\sqrt{|x-(-2)||x-2|}(x-0)^2 \mathbf{1}_{[-2,2]}(x).$$

Following the terminology used in the literature [35, 54, 28], we may say that our assumptions allow the existence of singular points of type II (the density vanishes in the bulk) and III (the density vanishes at the edge faster than a square root). Assumption H3 rules out the possibility of singular points of type I, also called "birth of a new cut", for which the behavior might be quite different, see [26, 66].

5.1.4 Existing literature, strategy and perspectives

Connection to previous results

The CLT for fluctuations of linear statistics in the context of β-ensembles was proven in the pioneering paper [51] for polynomial potentials in the case $\mathsf{n} = 0, \mathsf{m} = 0$, and generalized in [77] to real-analytic potentials in the possibly multi-cut, non-critical cases ($\mathsf{n} \geq 0, \mathsf{m} = 0$), where a set of n necessary and sufficient conditions on a given test function in order to satisfy the CLT is derived. If these conditions are not fulfilled, the fluctuations are shown to exhibit oscillatory behaviour. Such results are also a by-product of the all-orders expansion of the partition function obtained in [16] ($\mathsf{n} = 0, \mathsf{m} = 0$) and [15] ($\mathsf{n} \geq 0, \mathsf{m} = 0$). A CLT for the fluctuations of linear statistics for test functions living at mesoscopic scales was recently obtained in [13]. Finally, a new proof of the CLT in the one-cut non-critical case was very recently given in [56]. It is based on Stein's method and provides a rate of convergence in Wasserstein distance.

Motivation and strategy

Our goal is twofold: on the one hand, we provide a simple proof of the CLT using a change of variables argument, retrieving the results cited above. On the other hand, our method allows to substantially relax the assumptions on V, in particular for the first time we are able to treat critical situations where $\mathsf{m} \geq 1$.

Our method, which is adapted from the one introduced in [59] for two-dimensional log-gases, can be summarized as follows

1. We prove the CLT by showing that the Laplace transform of the fluctuations converges to the Laplace transform of the correct Gaussian law. This idea is already present in [51] and many further works. Computing the Laplace transform of $\mathrm{Fluct}_N(\xi)$ leads to working with a new potential $V + t\xi$ (with t small), and thus to considering the associated perturbed equilibrium measure.
2. Following [59], our method then consists in finding a change of variables (or a transport map) that pushes μ_V onto the perturbed equilibrium measure. In fact we do not exactly achieve this, but rather we construct a transport map $I + t\psi$, which is a perturbation of identity, and consider the *approximate* perturbed equilibrium measure $(I + t\psi)\#\mu_V$. The map ψ is found by inverting the operator (5.1.12), which is well-known in this context, it appears e.g. in [16, 15, 77, 11]. A CLT will hold if the function ξ is (up to constants) in the image of Ξ_V, leading to the conditions (5.1.14)–(5.1.15). The change of variables approach for one-dimensional log-gases was already used e.g. in [78, 11], see also [48, 49] which deal with the non-commutative context.
3. The proof then leverages on the expansion of $\log Z^V_{N,\beta}$ up to order N proven in [58], valid in the multi-cut and critical case, and whose dependency in V is explicit enough. This step replaces the a priori bound on the commutators used e.g. in [16].

More comments and perspectives

Using the Cramer-Wold theorem, the result of Theorem 2 extends readily to any finite family of test functions satisfying the conditions ((5.1.14), (5.1.15)): the joint law of their fluctuations converges to a Gaussian vector, using the bilinear form associated to (5.1.16) in order to determine the covariance.

In the multi-cut case, the CLT results of [77] or [15] are stated under n necessary and sufficient conditions on the test function, and the non-Gaussian nature of the fluctuations if these conditions are not satisfied is explicitly described. In the critical cases, we only state sufficient conditions (5.1.15) under which the CLT holds. It would be interesting to prove that these conditions are necessary, and to characterize the behavior of the fluctuations for functions which do not satisfy (5.1.15).

Finally, we expect Theorem 2 to hold also at mesoscopic scales.

5.1.5 The one-cut noncritical case

In the case $\mathsf{n} = 0$ and $\mathsf{m} = 0$, following the transport approach, we can obtain the convergence of the Laplace transform of fluctuations with an explicit rate, under the assumption that ξ is very regular (we have not tried to optimize in the regularity):

Theorem 3 (Rate of convergence in the one-cut noncritical case). *Under the assumptions of Theorem 2, if in addition* $\mathsf{n} = 0$, $\mathsf{m} = 0$, $\mathsf{p} \geq 6$ *and* $\mathsf{r} \geq 17$, *then we also have*

$$\left|\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left(\exp(s\mathrm{Fluct}_N(\xi)\right) - sm_\xi - s^2 v_\xi\right| \leq C\left(\frac{s}{N}\|\xi\|_{C^{17}(\mathbb{R})} + \frac{s^3}{N}\|\xi\|_{C^2(\mathbb{R})} + \frac{s^4}{N^2}\|\xi\|^4_{C^3(\mathbb{R})}\right), \quad (5.1.17)$$

where the constant C depends only on V.

The assumed regularity on ξ allows to avoid using the result of [58] on the expansion of $\log Z^V_{N,\beta}$. Our transport approach also provides a functional relation on the expectation of fluctuations which allows by a boostrap procedure to recover an expansion of $\log Z^V_{N,\beta}$ (relative to a reference potential) to arbitrary powers of $1/N$ in very regular cases, i.e the result of [16] but without the analyticity assumption. All these results are presented in Appendix 5.5.

5.1.6 Some notation

We denote by $P.V.$ the principal value of an integral having a singularity at x_0, i.e.

$$P.V. \int f = \lim_{\varepsilon \to 0} \int_{-\infty}^{x_0 - \varepsilon} f + \int_{x_0+\varepsilon}^{+\infty} f. \quad (5.1.18)$$

If Φ is a C^1-diffeomorphism and μ a probability measure, we denote by $\Phi\#\mu$ the push-forward of μ by Φ, which is by definition such that for $A \subset \mathbb{R}$ Borel,

$$(\Phi\#\mu)(A) := \mu(\Phi^{-1}(A)).$$

If $A \subset \mathbb{R}$ we denote by $\mathring{A}$ its interior.

For $k \geq 0$, and U some bounded domain in $\mathbb{R}$, we endow the spaces $C^k(U)$ with the usual norm

$$\|\psi\|_{C^k(U)} := \sum_{j=0}^{k} \sup_{x\in U} |\psi^{(j)}(x)|.$$

If z is a complex number, we denote by $\mathcal{R}(z)$ (resp. $\mathcal{I}(z)$) its real (resp. imaginary) part.

For any probability measure μ on $\mathbb{R}$ we denote by h^μ the logarithmic potential generated by μ, defined as the map

$$x \in \mathbb{R}^2 \mapsto h^\mu(x) = \int -\log|x-y|d\mu(y). \tag{5.1.19}$$

5.2 Expressing the Laplace transform of the fluctuations

We start by the standard approach of reexpressing the Laplace transform of the fluctuations in terms the ratio of partition functions of a perturbed log-gas by that of the original one. This is combined with the energy splitting formula of [74] that separates fixed leading order terms from variable next order ones.

5.2.1 The next-order energy

For any probability measure μ, let us define,

$$F_N(\vec{X}_N,\mu) = -\iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log|x-y| \Big(\sum_{i=1}^N \delta_{x_i} - \mu\Big)(x)\Big(\sum_{i=1}^N \delta_{x_i} - \mu\Big)(y), \tag{5.2.20}$$

where $\triangle$ denotes the diagonal in $\mathbb{R}\times\mathbb{R}$.

We have the following splitting formula for the energy, as introduced in [74] (we recall the proof in Section 5.6.1).

Lemma 5.2.1. *For any $\vec{X}_N \in \mathbb{R}^N$, it holds that*

$$\mathcal{H}_N^V(\vec{X}_N) = N^2\mathcal{I}_V(\mu_V) + 2N\sum_{i=1}^N \zeta_V(x_i) + F_N(\vec{X}_N,\mu_V)\ . \tag{5.2.21}$$

Using this splitting formula (5.2.21), we may re-write $\mathbb{P}_{N,\beta}^V$ as

$$d\mathbb{P}_{N,\beta}^V(\vec{X}_N) = \frac{1}{K_{N,\beta}(\mu_V,\zeta_V)} \exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N,\mu_V) + 2N\sum_{i=1}^N \zeta_V(x_i)\right)\right) d\vec{X}_N, \tag{5.2.22}$$

with a next-order partition function $K_{N,\beta}(\mu_V,\zeta_V)$ defined by

$$K_{N,\beta}(\mu_V,\zeta_V) := \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N,\mu_V) + 2N\sum_{i=1}^N \zeta_V(x_i)\right)\right) d\vec{X}_N. \tag{5.2.23}$$

We extend this notation to $K_{N,\beta}(\mu,\zeta)$ where μ is a probability density and ζ is a confinement potential.

5.2.2 Perturbed potential and equilibrium measure

Let ξ be in $C^0(\mathbb{R})$ with compact support.

Definition 1. *For any $t \in \mathbb{R}$, we define*

- *The perturbed potential V_t as $V_t := V + t\xi$.*
- *The perturbed equilibrium measure μ_t as the equilibrium measure associated to V_t. Since ξ has compact support, V_t satisfies the growth assumption* (5.1.5) *and thus μ_t is well-defined. In particular, μ_0 coincides with μ_V.*
- *The next-order confinement term $\zeta_t := \zeta_{V_t}$, as in* (5.1.7).
- *The next-order energy $F_N(\vec{X}_N, \mu_t)$ as in* (5.2.20).
- *The next-order partition function $K_{N,\beta}(\mu_t, \zeta_t)$ as in* (5.2.23).

5.2.3 The Laplace transform of fluctuations as ratio of partition functions

Lemma 5.2.2. *For any $s \in \mathbb{R}$ we have, letting $t := \frac{-2s}{\beta N}$,*

$$\mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left[\exp\left(s\mathrm{Fluct}_N(\xi)\right)\right] = \frac{K_{N,\beta}(\mu_t,\zeta_t)}{K_{N,\beta}(\mu_0,\zeta_0)} \exp\left(-\frac{\beta}{2}N^2\left(\mathcal{I}_{V_t}(\mu_t) - \mathcal{I}_V(\mu_0) - t\int \xi d\mu_0\right)\right). \tag{5.2.24}$$

Proof. First, we notice that, for any s in $\mathbb{R}$

$$\mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left[\exp(s\mathrm{Fluct}_N(\xi))\right] = \frac{Z^{V_t}_{N,\beta}}{Z^V_{N,\beta}} \exp\left(-Ns\int \xi \, d\mu_0\right) \tag{5.2.25}$$

Using the splitting formula (5.2.21) and the definition of $K_{N,\beta}$ as in (5.2.23) we see that for any t

$$K_{N,\beta}(\mu_t,\zeta_t) = Z^{V_t}_{N,\beta} \exp\left(\frac{\beta}{2}N^2\mathcal{I}_{V_t}(\mu_t)\right), \tag{5.2.26}$$

thus combining (5.2.25) and (5.2.26), with $t := \frac{-2s}{\beta N}$ we obtain (5.2.24). □

5.2.4 Comparison of partition functions

If μ is a probability density, we denote by $\mathrm{Ent}(\mu)$ the entropy function given by $\mathrm{Ent}(\mu) := \int_{\mathbb{R}} \mu \log \mu$. The following asymptotic expansion is proven [58, Corollary 1.5] (cf. [58, Remark 4.3]) and valid in a general multi-cut critical situation.

Lemma 5.2.3. *Let μ be a probability density on $\mathbb{R}$. Assume that μ has the form* (5.1.10), (5.1.11) *with S_0 in $C^2(\Sigma)$, and that ζ is some Lipschitz function on $\mathbb{R}$ satisfying*

$$\zeta = 0 \text{ on } \Sigma, \quad \zeta > 0 \text{ on } \mathbb{R}\setminus\Sigma, \quad \int_{\mathbb{R}} e^{-\beta N\zeta(x)}dx < \infty \text{ for } N \text{ large enough.}$$

Then, with the notation of (5.2.23) *and for some C_β depending only on β, we have*

$$\log K_{N,\beta}(\mu,\zeta) = \frac{\beta}{2}N\log N + C_\beta N - N\left(1-\frac{\beta}{2}\right)\mathrm{Ent}(\mu) + No_N(1). \tag{5.2.27}$$

5.2.5 Additional bounds

Exponential moments of the next-order energy

Lemma 5.2.4. *We have, for some constant C depending on β and V*

$$\left|\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left[\exp\left(\frac{\beta}{4}\left(F_N(\vec{X}_N,\mu_V)+N\log N\right)\right)\right]\right| \le CN. \tag{5.2.28}$$

Proof. This follows e.g. from [74, Theorem 6], but we can also deduce it from Lemma 5.2.3. We may write

$$\mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left[\exp\left(\frac{\beta}{4}F_N(\vec{X}_N,\mu_V)\right)\right] = \frac{1}{K_{N,\beta}(\mu_V,\zeta_V)}\int \exp\left(-\frac{\beta}{4}\left(F_N(\vec{X}_N,\mu_V)-2N\sum_{i=1}^N 2\zeta_V(x_i)\right)\right)d\vec{X}_N = \frac{K_{N,\frac{\beta}{2}}(\mu_V,2\zeta_V)}{K_{N,\beta}(\mu_V,\zeta_V)}.$$

Taking the log and using (5.2.27) to expand both terms up to order N yields the result. □

The next-order energy controls the fluctuations

The following result is a consequence of the analysis of [74, 71], we give the proof in Section 5.6.2 for completeness. It shows that F_N controls fluct_N. Here $|\mathrm{Supp}\,\xi|$ denotes the diameter of the support of ξ.

Proposition 5.2.5. *If ξ is compactly supported and Lipschitz, we have, for some universal constant C*

$$\left|\int \xi\, d\mathrm{fluct}_N\right| \le C|\mathrm{Supp}\,\xi|^{\frac{1}{2}}\|\nabla\xi\|_{L^\infty}\left(F_N(\vec{X}_N,\mu_V)+N\log N+C(\|\mu_V\|_{L^\infty}+1)N\right)^{1/2}. \tag{5.2.29}$$

Confinement bound

We will also need the following bound on the confinement. The proof is very simple and identical to the proof of Lemma 3.3 of [59].

Lemma 5.2.6. *For any fixed open neighborhood U of Σ,*

$$\mathbb{P}^V_{N,\beta}\left(\vec{X}_N\in U^N\right)\ge 1-\exp(-cN)$$

where $c>0$ depends on U and β.

Lemma 5.2.6 is the only place where we use the non-degeneracy assumption H3 on the next-order confinement term ζ_V.

5.3 Inverting the operator and defining the approximate transport

The goal of this section is to find transport maps ϕ_t for t small enough such that the transported measure $\phi_t \# \mu_0$ approximates the equilibrium measures μ_t. Since the equilibrium measures are characterized by (5.1.7) with equality on the support, it is natural to seek ϕ_t such that the quantity

$$\int -\log|\phi_t(x) - \phi_t(y)| d\mu_0(y) + \frac{1}{2} V_t(\phi_t(x))$$

is close to a constant.

5.3.1 Preliminaries

Lemma 5.3.1. *We have the following*

- *The non-vanishing function S_0 in (5.1.11) is in $C^{\mathsf{p}-3-2\mathsf{k}}(\Sigma_V)$.*
- *There exists an open neighborhood U of Σ_V and a non-vanishing function M in $C^{\mathsf{p}-3-2\mathsf{k}}(U \setminus \mathring{\Sigma}_V)$ such that*

$$\zeta_V'(x) = M(x)\sigma(x) \prod_{i=1}^{m} (x - s_i)^{2\mathsf{k}_i}. \tag{5.3.30}$$

In particular, (5.3.30) quantifies how fast ζ_V' vanishes near an endpoint of the support. We postpone the proof to Section 5.6.3.

5.3.2 The approximate equilibrium measure equation

In the following, we let

- U be an open neighborhood of Σ_V such that (5.3.30) holds.
- B be the open ball of radius $\frac{1}{2}$ in $C^2(U)$.

We define a map $\mathcal{F}$ from $[-1, 1] \times B$ to $C^1(U)$ by setting $\phi := \mathrm{Id} + \psi$ and

$$\mathcal{F}(t, \psi) := \int -\log|\phi(\cdot) - \phi(y)| d\mu_V(y) + \frac{1}{2} V_t \circ \phi(\cdot), \tag{5.3.31}$$

Lemma 5.3.2. *The map $\mathcal{F}$ takes values in $C^1(U)$ and has continuous partial derivatives in both variables. Moreover there exists C depending only on V such that for all (t, ψ) in $[-1, 1] \times B$ we have*

$$\left\| \mathcal{F}(t, \psi) - \mathcal{F}(0, 0) - \frac{t}{2}\xi + \Xi_V[\psi] \right\|_{C^1(U)} \le Ct^2 \|\psi\|^2_{C^2(U)}. \tag{5.3.32}$$

The proof is postponed to Section 5.6.4.

5.3.3 Inverting the operator

Lemma 5.3.3. *Let ψ be defined by*

$$\psi(x) = -\frac{1}{2\pi^2 S(x)}\left(\int_\Sigma \frac{\xi(y)-\xi(x)}{\sigma(y)(y-x)}dy\right) \quad \textit{for } x \textit{ in } \Sigma_V, \tag{5.3.33}$$

$$\psi(x) = \frac{\displaystyle\int \frac{\psi(y)}{x-y}d\mu_V(y) + \frac{\xi}{2} + c_\xi}{\displaystyle\int \frac{1}{x-y}d\mu_V(y) - \frac{1}{2}V'(x)} \quad \textit{for } x \in U\backslash\Sigma_V, \tag{5.3.34}$$

then ψ is in $C^{\mathsf{l}}(U)$ with $\mathsf{l} = (\mathsf{p} - 3 - 3\mathsf{k}) \wedge (\mathsf{r} - 1 - 2\mathsf{k})$ and

$$\|\psi\|_{C^{\mathsf{l}}(U)} \le C\|\xi\|_{C^{\mathsf{r}}(\mathbb{R})} \tag{5.3.35}$$

for some constant C depending only on V, and there exists a constant c_ξ such that

$$\Xi_V[\psi] = \frac{\xi}{2} + c_\xi \textit{ in } U,$$

with Ξ_V as in (5.1.12).

The proof of Lemma 5.3.3 is postponed to Section 5.6.5. We may extend ψ to $\mathbb{R}$ in such a way that ψ is in $C^{\mathsf{l}}(\mathbb{R})$ with compact support.

5.3.4 Approximate transport and equilibrium measure

We let ψ be the function defined in Lemma 5.3.3, and c_ξ be such that

$$\Xi_V[\psi] = \frac{\xi}{2} + c_\xi \text{ on } U.$$

Definition 2. *For $t \in [-t_{\max}, t_{\max}]$, where $t_{\max} = \left(2\|\psi\|_{C^1(U)}\right)^{-1}$,*

- *We let ψ_t be given by $\psi_t := t\psi$.*
- *We let ϕ_t be the approximate transport, defined by $\phi_t := \mathrm{Id} + \psi_t$.*
- *We let $\tilde{\mu}_t$ be the approximate equilibrium measure, defined by $\tilde{\mu}_t := \phi_t \# \mu_V$.*
- *We let $\tilde{\zeta}_t$ be the approximate confining term $\tilde{\zeta}_t := \zeta_V \circ \phi_t^{-1}$*
- *We let $\mathbb{P}^{(t)}_{N,\beta}$ be the probability measure*

$$d\mathbb{P}^{(t)}_{N,\beta}(\vec{X}_N) = \frac{1}{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)} \exp\left(-\frac{\beta}{2}\left(F_N(\vec{X}_N, \tilde{\mu}_t) + 2N\sum_{i=1}^N \tilde{\zeta}_t(x_i)\right)\right) d\vec{X}_N, \tag{5.3.36}$$

where $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$ is as in (5.2.23).

Finally, we let τ_t be defined by

$$\tau_t := \mathcal{F}(t, \psi_t) - \mathcal{F}(0, 0) - \tilde{c}_t. \tag{5.3.37}$$

This quantifies how close $\tilde{\mu}_t$ is from satisfying the Euler-Lagrange equation for V_t and thus how well $\tilde{\mu}_t$ approximates the real equilibrium measure μ_t. We also define the extension $\hat{\tau}_t$ of $\tau_t \circ \phi_t^{-1}$ to $\mathbb{R}^2$ by

$$\hat{\tau}_t(x,y) = \chi(x,y)\ \tau_t \circ \phi_t^{-1}(x), \tag{5.3.38}$$

where χ is equal to one in a fixed neighborhood of $\mathrm{supp}(\mu_V)$ *included in U and is in $C_c^\infty(\mathbb{R}^2)$.*

Lemma 5.3.4. *The following holds*

- *The map ψ_t satisfies*
$$\Xi_V[\psi_t] = \frac{t}{2}\xi + \tilde{c}_t, \ \text{for } \tilde{c}_t := tc_\xi.$$
- *The map ϕ_t is a C^1-diffeomorphism which coincides with the identity outside a compact support independent of $t \in [-t_{\max}, t_{\max}]$.*
- *The error τ_t is a $O(t^2)$, more precisely*
$$\|\tau_t\|_{C^1(U)} \le Ct^2\|\psi\|^2_{C^2(U)} \tag{5.3.39}$$
$$\|\hat{\tau}_t\|_{C^1(\mathbb{R}^2)} \le Ct^2\|\psi\|^2_{C^2(U)}. \tag{5.3.40}$$
- *On $\phi_t(\Sigma_V)$, we have*
$$\tilde{\zeta}_t = h^{\tilde{\mu}_t} + \frac{V_t}{2} - \tilde{c}_t - c_V - \tau_t \circ \phi_t^{-1}. \tag{5.3.41}$$

Proof. The first two points are straightforward, the bound (5.3.39) follows from combining (5.3.32) with the conclusions of Lemma 5.3.2, and then (5.3.40) is an easy consequence.

For (5.3.41), let us first recall that

$$\mathcal{F}(t,\psi_t) = \int -\log|\phi_t(\cdot) - \phi_t(y)|d\mu_0(y) + \frac{1}{2}V_t \circ \phi_t,$$

which, with the notation of (5.1.19), yields

$$\mathcal{F}(t,\psi_t) = h^{\tilde{\mu}_t} \circ \phi_t + \frac{1}{2}V_t \circ \phi_t.$$

On the other hand, by definition of τ_t as in (5.3.37), we have

$$\mathcal{F}(t,\psi_t) = \mathcal{F}(0,0) + \tilde{c}_t + \tau_t.$$

Finally, we know that, on Σ_V

$$\mathcal{F}(0,0) = \zeta_V + c_V.$$

We thus see that

$$\zeta_V + c_V + \tilde{c}_t + \tau_t = h^{\tilde{\mu}_t} \circ \phi_t + \frac{1}{2}V_t \circ \phi_t.$$

Since, by definition, $\tilde{\zeta}_t = \zeta_V \circ \phi_t^{-1}$, we get (5.3.41). □

5.4 Study of the Laplace transform

The next goal is to compare the partition functions associated to μ_t and $\mu_0 = \mu_V$. We split the comparison into two steps: first, we compare $K_{N,\beta}(\mu_t, \zeta_t)$ with $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$ using the bounds, obtained in the previous section, showing that $\tilde{\mu}_t$ is a good approximation to $\overline{\mu}_t$, and then we compare $K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)$ and $K_{N,\beta}(\mu_0, \zeta_0)$ using the transport ϕ_t, as in [59].

5.4.1 Energy comparison: from μ_t to $\tilde{\mu}_t$

Lemma 5.4.1. *We have*

$$\int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 \leq C t^4 \|\psi\|^4_{C^2(U)}, \tag{5.4.42}$$

$$\int_{\mathbb{R}} \zeta_t d\tilde{\mu}_t + \int_{\mathbb{R}} \tilde{\zeta}_t d\mu_t \leq C t^4 \|\psi\|^4_{C^2(U)}, \tag{5.4.43}$$

where C is universal.

Proof. For t small enough, $\phi_t(U)$ contains some fixed open neighborhood of Σ_V, which itself contains the support of μ_t. Integrating by parts we thus get

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla h^{\mu_t - \tilde{\mu}_t}|^2 &= \int h^{\mu_t - \tilde{\mu}_t} d(\mu_t - \tilde{\mu}_t)\\ &= \int (\zeta_t - \tilde{\zeta}_t - \tau_t \circ \phi_t^{-1})\, d(\mu_t - \tilde{\mu}_t)\\ &= -\int \zeta_t d\tilde{\mu}_t - \int \tilde{\zeta}_t d\mu_t - \int_{\mathbb{R}} \tau_t \circ \phi_t^{-1} d(\mu_t - \tilde{\mu}_t)\\ &\leq -\int \tau_t \circ \phi_t^{-1} d(\mu_t - \tilde{\mu}_t). \end{aligned} \tag{5.4.44}$$

In the first equality, we have re-written h^{μ_t} and $h^{\tilde{\mu}_t}$ using the confining terms ζ_t and $\tilde{\zeta}_t$, see (5.1.7) and (5.3.41), discarding the constants which disappear when integrated against $d(\mu_t - \tilde{\mu}_t)$. In the second equality, we have used the fact that ζ_t vanishes on the support of μ_t and $\tilde{\zeta}_t$ on the support of $\tilde{\mu}_t$. Finally, the last inequality is due to the fact that ζ_t and $\tilde{\zeta}_t$ are nonnegative on $\mathbb{R}$. Using (5.3.38) and (5.3.40), we may thus write

$$\begin{aligned}\frac{1}{2\pi}\|\nabla h^{\mu_t - \tilde{\mu}_t}\|^2_{L^2(\mathbb{R}^2)} \leq \left| \int_{\mathbb{R}^2} \tau_t \circ \phi_t^{-1} d(\mu_t - \tilde{\mu}_t)\delta_{\mathbb{R}} \right| &\leq \|\nabla \hat{\tau}_t\|_{L^2(\mathbb{R}^2)} \|\nabla h^{\mu_t - \tilde{\mu}_t}\|_{L^2(\mathbb{R}^2)}\\ &\leq C t^2 \|\psi\|^2_{C^2(U)} \|\nabla h^{\mu_t - \tilde{\mu}_t}\|_{L^2(\mathbb{R}^2)},\end{aligned}$$

which proves (5.4.42). Coming back to (5.4.44), we also obtain

$$0 \leq -\int \zeta_t d\tilde{\mu}_t - \int \tilde{\zeta}_t d\mu_t + O\left(t^4 \|\psi\|^4_{C^2(U)}\right),$$

which in turn implies (5.4.43). □

Lemma 5.4.2 (Energy comparison : from μ_t to $\tilde{\mu}_t$). *For any $\vec{X}_N \in (\phi_t(U))^N$, we have*

$$\left|\left(F_N(\vec{X}_N,\mu_t)+2N\sum_{i=1}^N \zeta_t(x_i)\right)-\left(F_N(\vec{X}_N,\tilde{\mu}_t)+2N\sum_{i=1}^N \tilde{\zeta}_t(x_i)\right)\right| \\ \le C\left(Nt^2\|\psi\|^2_{C^2(U)}(F_N(\vec{X}_N,\tilde{\mu}_t)+N\log N)^{1/2}+N^2t^4\|\psi\|^4_{C^2(U)}\right). \quad (5.4.45)$$

Proof. By the definition (5.2.20) of the next-order energy, we may write

$$\begin{aligned} F_N(\vec{X}_N,\mu_t)-F_N(\vec{X}_N,\tilde{\mu}_t) &= N^2\int_{\mathbb{R}\times\mathbb{R}} -\log|x-y|\,d(\tilde{\mu}_t-\mu_t)(x)d(\tilde{\mu}_t-\mu_t)(y) \\ &\quad + 2N\int_{\mathbb{R}\times\mathbb{R}} -\log|x-y|d(\tilde{\mu}_t-\mu_t)(x)\left(\sum_{i=1}^N \delta_{x_i}-N\tilde{\mu}_t\right)(y) \\ &= N^2\int_{\mathbb{R}^2}|\nabla h^{\mu_t-\tilde{\mu}_t}|^2+2N\int_{\mathbb{R}} h^{\tilde{\mu}_t-\mu_t}\left(\sum_{i=1}^N\delta_{x_i}-N\tilde{\mu}_t\right). \quad (5.4.46)\end{aligned}$$

On the other hand, using that $\tilde{\zeta}_t$ vanishes on the support of $\tilde{\mu}_t$, we get

$$\begin{aligned}\sum_{i=1}^N\left(\zeta_t(x_i)-\tilde{\zeta}_t(x_i)\right) &= N\int_{\mathbb{R}}(\zeta_t-\tilde{\zeta}_t)d\tilde{\mu}_t+\int_{\mathbb{R}}(\zeta_t-\tilde{\zeta}_t)\left(\sum_{i=1}^N\delta_{x_i}-N\tilde{\mu}_t\right) \\ &= N\int_{\mathbb{R}}\zeta_t d\tilde{\mu}_t+\int_{\mathbb{R}}(\zeta_t-\tilde{\zeta}_t)\left(\sum_{i=1}^N\delta_{x_i}-N\tilde{\mu}_t\right). \quad (5.4.47)\end{aligned}$$

Combining (5.4.46) and (5.4.47), we obtain

$$\begin{aligned}&\left(F_N(\vec{X}_N,\mu_t)+2N\sum_{i=1}^N\zeta_t(x_i)\right)-\left(F_N(\vec{X}_N,\tilde{\mu}_t)+2N\sum_{i=1}^N\tilde{\zeta}_t(x_i)\right) \\ &\quad = N^2\int_{\mathbb{R}^2}|\nabla h^{\mu_t-\tilde{\mu}_t}|^2+2N^2\int_{\mathbb{R}}\zeta_t d\tilde{\mu}_t+2N\int_{\mathbb{R}}(h^{\tilde{\mu}_t-\mu_t}+\zeta_t-\tilde{\zeta}_t)\left(\sum_{i=1}^N\delta_{x_i}-N\tilde{\mu}_t\right).\end{aligned}$$

From (5.1.7), (5.3.41) (see also the notation (5.1.19)), we have

$$h^{\tilde{\mu}_t-\mu_t}+\zeta_t-\tilde{\zeta}_t=\tau_t\circ\phi_t^{-1}+\text{constant},$$

hence we find

$$\begin{aligned}&\left(F_N(\vec{X}_N,\mu_t)+2N\sum_{i=1}^N\zeta_t(x_i)\right)-\left(F_N(\vec{X}_N,\tilde{\mu}_t)+2N\sum_{i=1}^N\tilde{\zeta}_t(x_i)\right) \\ &\quad = N^2\int_{\mathbb{R}^2}|\nabla h^{\mu_t-\tilde{\mu}_t}|^2+2N^2\int_{\mathbb{R}}\zeta_t d\tilde{\mu}_t+2N\int\tau_t\circ\phi_t^{-1}\left(\sum_{i=1}^N\delta_{x_i}-N\tilde{\mu}_t\right). \quad (5.4.48)\end{aligned}$$

By the results of Lemma 5.4.1, the first two terms in the right-hand side of (5.4.48) are $O(N^2t^4)$, while the last term is bounded, using (5.3.39) and Proposition 5.2.5, by

$$N\int\tau_t\circ\phi_t^{-1}\left(\sum_{i=1}^N\delta_{x_i}-N\tilde{\mu}_t\right)=O\left(Nt^2(F_N(\vec{X}_N,\tilde{\mu}_t)+N\log N)^{1/2}\right),$$

which concludes the proof. □

Lemma 5.4.3. *We have, for any fixed $s \in \mathbb{R}$, with $t = \frac{-2s}{\beta N}$*

$$\left|\log \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_t, \zeta_t)}\right| \le CNt^2\sqrt{N}\|\psi\|^2_{C^2(U)} + Ct^4N^2\|\psi\|^4_{C^2(U)}$$
$$= O\left(s^2N^{-1/2}\|\psi\|^2_{C^2} + s^4N^{-2}\|\psi\|^4_{C^2}\right). \quad (5.4.49)$$

Proof. By definition of the next-order partition functions we may write

$$\frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_t, \zeta_t)} = \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2}\left(\left(F_N(\vec{X}_N, \mu_t) + 2N\sum_{i=1}^N \zeta_t(x_i)\right)\right.\right.$$
$$\left.\left.- \left(F_N(\vec{X}_N, \tilde{\mu}_t) + 2N\sum_{i=1}^N \tilde{\zeta}_t(x_i)\right)\right)\right) d\vec{X}_N.$$

The result follows from combining (5.2.28) and (5.4.45), and using Lemma 5.2.6 to argue that the particles $\vec{X}_N$ may be assumed to all belong to the neighborhood U for t small enough, except for an event of exponentially small probability. □

5.4.2 Energy comparison: from $\tilde{\mu}_t$ to μ_0

Let us define

$$\mathrm{fluct}_N^{(t)} = \sum_{i=1}^N \delta_{x_i} - N\tilde{\mu}_t \qquad \mathrm{Fluct}_N^{(t)}(\xi) = \int \xi \, d\mathrm{fluct}_N^{(t)}.$$

For any ψ, let us define the following quantity (that may be called *anisotropy* by analogy with [59])

$$\mathsf{A}^{(t)}[\vec{X}_N, \psi] = \iint_{\mathbb{R}\times\mathbb{R}} \frac{\psi(x) - \psi(y)}{x - y} \, d\mathrm{fluct}_N^{(t)}(x)\, d\mathrm{fluct}_N^{(t)}(y). \quad (5.4.50)$$

Lemma 5.4.4. *Assume $\psi \in C^2(\mathbb{R})$. For any $\vec{X}_N \in U^N$, letting $\Phi_t(\vec{X}_N) = (\phi_t(x_1), \cdots, \phi_t(x_N))$, we have*

$$\left|F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) - F_N(\vec{X}_N, \mu_0) - \sum_{i=1}^N \log \phi_t'(x_i) + \frac{t}{2}\mathsf{A}^{(0)}[\vec{X}_N, \psi]\right|$$
$$\le Ct^2\left(F_N(\vec{X}_N, \mu_0) + N\log N\right). \quad (5.4.51)$$

Proof. Since by definition $\tilde{\mu}_t = \phi_t \# \mu_0$ we may write

$$F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) - F_N(\vec{X}_N, \mu_0)$$
$$= -\iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log|x-y| \left(\sum_{i=1}^N \delta_{\phi_t(x_i)} - N\tilde{\mu}_t\right)(x)\left(\sum_{i=1}^N \delta_{\phi_t(x_i)} - N\tilde{\mu}_t\right)(y)$$
$$+ \iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log|x-y| d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y)$$
$$= -\iint_{\mathbb{R}\times\mathbb{R}\setminus\triangle} \log\frac{|\phi_t(x) - \phi_t(y)|}{|x-y|} d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y)$$
$$= -\iint_{\mathbb{R}\times\mathbb{R}} \log\frac{|\phi_t(x) - \phi_t(y)|}{|x-y|} d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y) + \sum_{i=1}^N \log \phi_t'(x_i).$$

Using that by definition $\phi_t = \mathrm{Id} + t\psi$ where ψ is in $C^2_c(\mathbb{R})$, we get by the chain rule

$$\log \frac{|\phi_t(x) - \phi_t(y)|}{|x-y|} = t\, \frac{\psi(x) - \psi(y)}{x-y} + t^2\, \varepsilon_t(x,y),$$

with $\|\varepsilon_t\|_{C^2(\mathbb{R}\times\mathbb{R})}$ uniformly bounded in t. Applying Proposition 5.2.5 twice, we get that

$$\left| \iint \varepsilon_t(x,y) d\mathrm{fluct}_N(x) d\mathrm{fluct}_N(y) \right| \le Ct^2 \left(F_N(\vec{X}_N, \mu_0) + N \log N \right),$$

which yields the result. □

5.4.3 Comparison of partition functions I: using the transport

In this section and the following one, we will write A instead of $\mathsf{A}^{(0)}[\vec{X}_N, \psi]$

Proposition 5.4.5. *We have, for any t small enough*

$$\frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} = \exp\left(N \left(1 - \frac{\beta}{2}\right) (\mathrm{Ent}(\mu_0) - \mathrm{Ent}(\tilde{\mu}_t)) \right)$$
$$\mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}} \left(\exp\left(\frac{\beta}{2} t \mathsf{A} + t^2 \mathsf{Error}_1(\vec{X}_N) + t \mathsf{Error}_2(\vec{X}_N) \right) \right), \quad (5.4.52)$$

with error terms bounded by

$$|\mathsf{Error}_1(\vec{X}_N)| \le C \left(F_N(\vec{X}_N, \mu_0) + N \log N \right), \quad (5.4.53)$$
$$|\mathsf{Error}_2(\vec{X}_N)| \le C \left(F_N(\vec{X}_N, \mu_0) + N \log N \right)^{1/2}. \quad (5.4.54)$$

Proof. By a change of variables and in view of (5.4.51), we may write

$$K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \int \exp\left(-\frac{\beta}{2} \left(F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) + 2N \sum_{i=1}^N \tilde{\zeta}_t \circ \phi_t(x_i) \right) + \sum_{i=1}^N \log \phi_t'(x_i) \right) d\vec{X}_N$$
$$= \int \exp\left(-\frac{\beta}{2} \left(F_N(\Phi_t(\vec{X}_N), \tilde{\mu}_t) + 2N \sum_{i=1}^N \zeta_0(x_i) \right) + \sum_{i=1}^N \log \phi_t'(x_i) \right) d\vec{X}_N, \quad (5.4.55)$$

since $\zeta_0 = \tilde{\zeta}_t \circ \phi_t$ by definition. Using Lemma 5.4.4 we may write

$$\frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} = \frac{1}{K_{N,\beta}(\mu_0, \zeta_0)} \int_{\mathbb{R}^N} \exp\left(-\frac{\beta}{2} \left(F_N(\vec{X}_N, \mu_0) + 2N \sum_{i=1}^N \zeta(x_i) \right) \right.$$
$$\left. + \left(1 - \frac{\beta}{2}\right) \sum_{i=1}^N \log \phi_t'(x_i) + \frac{\beta}{2} t\, \mathsf{A} + t^2 \mathsf{Error}_1(\vec{X}_N) \right) d\vec{X}_N$$
$$= \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}} \left(\exp\left(\left(1 - \frac{\beta}{2}\right) \sum_{i=1}^N \log \phi_t'(x_i) + \frac{\beta}{2} t\, \mathsf{A} + t^2 \mathsf{Error}_1(\vec{X}_N) \right) \right), \quad (5.4.56)$$

where the Error_1 term is bounded as in (5.4.53). On the other hand, since ϕ_t is regular enough, using Proposition 5.2.5 we may write

$$\sum_{i=1}^{N} \log \phi_t'(x_i) = N \int_{\mathbb{R}} \log \phi_t' \, d\mu_0 + t\mathsf{Error}_2(\vec{X}_N)$$

with an Error_2 term as in (5.4.54). Finally, since by definition $\phi_t \# \mu_0 = \tilde{\mu}_t$ we may observe that $\phi_t' = \frac{\mu_0}{\tilde{\mu}_t \circ \phi_t}$ and thus

$$\int_{\mathbb{R}} \log \phi_t' \, d\mu_0 = \int_{\mathbb{R}} \log \mu_0 \, d\mu_0 - \int_{\mathbb{R}} \log \mu_t \circ \phi_t \, d\mu_0 = \mathrm{Ent}(\mu_0) - \mathrm{Ent}(\tilde{\mu}_t). \tag{5.4.57}$$

This yields (5.4.52). □

5.4.4 Comparison of partition functions II: the anisotropy is small

Proposition 5.4.6. *For any s, we have*

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{-s}{N} \mathsf{A} \right) \right) = o_N(1). \tag{5.4.58}$$

Proof. Applying Cauchy-Schwarz to (5.4.52) we may write

$$\begin{aligned}
&\mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{\beta}{4} t\mathsf{A} \right) \right)^2 \\
&\quad \le \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{\beta}{2} t\mathsf{A} + t^2 \mathsf{Error}_1 + t\mathsf{Error}_2 \right) \right) \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(-t^2 \mathsf{Error}_1 - t\mathsf{Error}_2 \right) \right) \\
&\le \frac{K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t)}{K_{N,\beta}(\mu_0, \zeta_0)} \exp\left(\left(1 - \frac{\beta}{2} \right) N \left(\mathrm{Ent}(\tilde{\mu}_t) - \mathrm{Ent}(\mu_0) \right) \right) \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp(-t^2 \mathsf{Error}_1 - t\mathsf{Error}_2) \right).
\end{aligned} \tag{5.4.59}$$

In view of (5.2.28) we get, for t small enough,

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(t\mathsf{Error}_1)) \le CtN, \qquad \log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(t\mathsf{Error}_2)) \le CtN. \tag{5.4.60}$$

Inserting (5.2.27) into (5.4.59) we obtain that for t small enough,

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{\beta}{4} t\mathsf{A} \right) \right) \le C(Nt^2 + N^{1/2} t) + N\delta_N, \tag{5.4.61}$$

for some sequence $\{\delta_N\}_N$ with $\lim_{N\to\infty} \delta_N = 0$. Applying this to $t = 4\varepsilon/\beta$ with ε small and using Hölder's inequality, we deduce

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} \left(\exp\left(\frac{-s}{N} \mathsf{A} \right) \right) \le \frac{|s|}{N\varepsilon} \log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(\varepsilon \mathsf{A})) \le C|s|\varepsilon + \frac{|s|}{\varepsilon} \delta_N.$$

In particular, choosing $\varepsilon = \sqrt{\delta_N}$, we get (5.4.58). □

5.4.5 Conclusion: proof of Theorem 2

Proof. Combining (5.4.52) for $t = -\frac{2s}{\beta N}$ (where s is independent of N) and (5.4.58) we find

$$\log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = \left(1 - \frac{\beta}{2}\right) N \left(\mathrm{Ent}(\mu_0) - \mathrm{Ent}(\tilde{\mu}_{\frac{-2s}{\beta N}})\right) + o_N(1). \tag{5.4.62}$$

Using again (5.4.57) and $\phi_t' = 1 + t\psi'$, we may rewrite this as

$$\log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = -\left(1 - \frac{\beta}{2}\right) \frac{2s}{\beta} \int \psi' d\mu_0 + o_N(1). \tag{5.4.63}$$

Combining (5.4.49) and (5.4.63) and sending N to $+\infty$ we obtain,

$$\log \frac{K_{N,\beta}(\tilde{\mu}_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0)} = -\left(1 - \frac{\beta}{2}\right) \frac{2s}{\beta} \int \psi' d\mu_0 + o_N(1), \tag{5.4.64}$$

with an error $o_N(1)$ uniform for s in a compact set of $\mathbb{R}$.

To conclude, we need the following relation, whose proof is given in Section 5.6.6.

Lemma 5.4.7.

$$\mathcal{I}_{V_t}(\mu_t) - \mathcal{I}_V(\mu_0) = t \int \xi d\mu_0 + \frac{t^2}{2} \int \xi' \psi d\mu_0 + O(t^3 \|\xi\|_{C^2(U)} + t^4 \|\psi\|^4_{C^2(U)}), \tag{5.4.65}$$

where the O only depends on V.

Combining (5.2.24) with (5.4.64) and (5.4.65) we obtain,

$$\log \mathbf{E}_{\mathbb{P}^V_{N,\beta}} (\exp(s\mathrm{Fluct}_N(\xi)) = -\left(1 - \frac{\beta}{2}\right) \frac{2s}{\beta} \int \psi' d\mu_V - \frac{s^2}{\beta} \int_{\mathbb{R}} \xi' \psi \, d\mu_V + o_N(1),$$

with an error $o_N(1)$ uniform for s in a compact set of $\mathbb{R}$.

Thus the Laplace transform of $\mathrm{Fluct}_N(\xi)$ converges (uniformly on compact sets) to that of a Gaussian of mean m_ξ and variance v_ξ, which implies convergence in law and proves the main theorem. □

5.5 The one-cut regular case

In the one-cut noncritical case, every regular enough function is in the range of the operator Ξ, so that the map ψ can always be built. This allows to bootstrap the approach used for proving Theorem 2. In this appendix, we expand on how we can proceed in this simpler setting without refering to the result of [58] but assuming more regularity of ξ, and retrieve the findings of [16] (but without assuming analyticity), as well as a rate of convergence for the Laplace transform of the fluctuations.

5.5.1 The bootstrap argument

Let us first explain the main computational point for the bootstrap argument: by (5.4.55) and in view of Lemma 5.4.4, we may write

$$\frac{d}{dt}_{|t=0} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(0)}[\vec{X}_N, \psi] + (1-\frac{\beta}{2})\frac{d}{dt}_{|t=0}\sum_{i=1}^{N}\log\phi_t'(x_i)\right]. \quad (5.5.66)$$

Differentiating (5.2.24) with respect to t and using Lemma 5.4.7 we thus obtain

$$\frac{-\beta N}{2}\mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}[\mathrm{Fluct}_N^{(0)}(\xi)] = \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(0)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)\frac{d}{dt}_{|t=0}\sum_{i=1}^{N}\log\phi_t'(x_i)\right].$$

This is true as well for all $t \in [-t_{\max}, t_{\max}]$, i.e.

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}[\mathrm{Fluct}_N^{(t)}(\xi)] = -\frac{2}{\beta N}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)\frac{d}{dt}\sum_{i=1}^{N}\log\phi_t'(x_i)\right]. \quad (5.5.67)$$

We may in addition write that

$$\frac{d}{dt}\sum_{i=1}^{N}\log\phi_t'(x_i) = N\int\frac{d}{dt}\log\phi_t'\,d\tilde{\mu}_t + \mathrm{Fluct}_N^{(t)}\left(\frac{d}{dt}\log\phi_t'\right) \quad (5.5.68)$$

so that

$$\begin{aligned}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}[\mathrm{Fluct}_N^{(t)}(\xi)] &= -\frac{2}{\beta}\left(1-\frac{\beta}{2}\right)\int\frac{d}{dt}\log\phi_t'\,d\tilde{\mu}_t \\ &\quad - \frac{2}{\beta N}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1-\frac{\beta}{2}\right)\mathrm{Fluct}_N^{(t)}\left(\frac{d}{dt}\log\phi_t'\right)\right]. \quad (5.5.69)\end{aligned}$$

This provides a functional equation which gives the expectation of the fluctuation in terms of a constant term plus a lower order expectation of another fluctuation and the A term (which itself can be written as a fluctuation, as noted below), allowing to expand it in powers of $1/N$ recursively.

5.5.2 Improved control on the fluctuations

Lemma 5.5.1. *Under the assumptions of Theorem 2 and assuming in addition*

$$\mathsf{p} \geq 3\mathsf{k}+6 \qquad \mathsf{r} \geq 2\mathsf{k}+4 \quad (5.5.70)$$

we have for any t in $(-t_{\max}, t_{\max})$ and[2] s in $\mathbb{R}$

$$\begin{aligned}&\log\mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left[\exp\left(s\mathrm{Fluct}_N^{(t)}(\xi)\right)\right] \\ &\leq C\left(s\|\xi\|_{C^{2\mathsf{k}+4}(U)} + s^2\|\xi\|^2_{C^{2\mathsf{k}+3}(U)} + \frac{s^3}{N}\|\xi\|_{C^2(U)} + \frac{s^4}{N^2}\|\xi\|_{C^{2\mathsf{k}+3}(U)} + \frac{s^4}{N^2}\|\xi\|^4_{C^{2\mathsf{k}+3}(U)}\right)\end{aligned} \quad (5.5.71)$$

where C depends only on V.

[2] In this statement, s and t are not related.

Proof. Note that in view of Lemma 5.3.3, the assumption (5.5.70) ensures that the transport map ψ is in $C^3(U)$. By (5.4.55) and in view of Lemma 5.4.4, we may write

$$\frac{d}{dt}_{|t=0} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}} \left[-\frac{\beta}{2} \mathsf{A}^{(0)}[\vec{X}_N, \psi] + (1 - \frac{\beta}{2}) \frac{d}{dt}_{|t=0} \sum_{i=1}^N \log \phi_t'(x_i) \right]. \quad (5.5.72)$$

Similarly, we have for all t,

$$\frac{d}{dt} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) = \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[-\frac{\beta}{2} \mathsf{A}^{(t)}[\vec{X}_N, \psi] + (1 - \frac{\beta}{2}) \frac{d}{dt} \sum_{i=1}^N \log \phi_t'(x_i) \right]. \quad (5.5.73)$$

Indeed, V_t has the same regularity as V and $\tilde{\mu}_t$ the same as μ_0.

Next, we express the anisotropy term as a fluctuation, by writing

$$\mathsf{A}^{(t)}[\vec{X}_N, \psi] = \int g(x) d\mathrm{fluct}_N^{(t)}(x), \quad (5.5.74)$$

where we let

$$g(x) := \int \hat{\psi}(x,y) d\mathrm{fluct}_N^{(t)}(y), \quad \hat{\psi}(x,y) := \frac{\psi(x) - \psi(y)}{x - y}. \quad (5.5.75)$$

It is clear that

$$\|\hat{\psi}\|_{C^2(U \times U)} \le \|\psi\|_{C^3(U)}. \quad (5.5.76)$$

Using Proposition 5.2.5 twice, we can thus write

$$\|\nabla g\|_{L^\infty} \le \left| \int \nabla_x \hat{\psi}(x,y) d\mathrm{fluct}_N^{(t)}(y) \right| \le C \|\nabla_x \nabla_y \hat{\psi}\|_{L^\infty} \left(F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right)^{\frac{1}{2}}$$

and

$$|\mathsf{A}^{(t)}[\vec{X}_N, \psi]| = \left| \int g(x) d\mathrm{fluct}_N^{(t)}(x) \right| \le C \|\nabla g\|_{L^\infty} \left(F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right)^{\frac{1}{2}}$$
$$\le C \|\hat{\psi}\|_{C^2(U \times U)} \left(F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right).$$

In view of (5.2.28) and (5.5.76), we deduce that

$$\left| \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[-\frac{\beta}{2} \mathsf{A}^{(t)}[\vec{X}_N, \psi] \right] \right| \le CN \|\psi\|_{C^3(U)}. \quad (5.5.77)$$

For the term $\log \phi_t'$ we use (5.5.68) and in view of Proposition 5.2.5, since $\phi_t = \mathrm{Id} + t\psi$ is regular enough, we may write

$$\left| \int \frac{d}{dt} \log \phi_t' \, d\mathrm{fluct}_N^{(t)} \right| \le C \|\psi\|_{C^2(U)} \left(F_N(\vec{X}_N, \tilde{\mu}_t) + N \log N + CN \right)^{\frac{1}{2}}. \quad (5.5.78)$$

We conclude from (5.5.73), using again (5.2.28) that

$$\left| \frac{d}{dt} \log K_{N,\beta}(\tilde{\mu}_t, \tilde{\zeta}_t) \right| \le CN \|\psi\|_{C^3(U)}. \quad (5.5.79)$$

Integrating this relation between 0 and $-\frac{2s}{\beta N}$, and combining with (5.4.49), we find that, for $t = \frac{-2s}{\beta N}$,

$$\left|\log \frac{K_{N,\beta}(\mu_t, \zeta_t)}{K_{N,\beta}(\mu_0, \zeta_0)}\right| \leq Cs\|\psi\|_{C^3(U)}. \tag{5.5.80}$$

Inserting this, (5.4.49) and (5.4.65) into (5.2.24), we deduce that

$$\begin{aligned}&\left|\log \mathbf{E}_{\mathbb{P}^{(0)}_{N,\beta}}\left[\exp(s\mathrm{Fluct}_N(\xi))\right]\right| \\ &\quad \leq C\left(s\|\psi\|_{C^3(U)} + s^2\|\psi\|_{C^0(U)}\|\xi\|_{C^1(U)} + \frac{s^3}{N}\|\xi\|_{C^2(U)} + \frac{s^4}{N^2}\|\psi\|_{C^2(U)}\right. \\ &\qquad\qquad \left. + \frac{s^2}{\sqrt{N}}\|\psi\|^2_{C^2(U)} + \frac{s^4}{N^2}\|\psi\|^4_{C^2(U)}\right). \end{aligned} \tag{5.5.81}$$

In view of (5.3.35), it yields the result for the expectation under $\mathbb{P}^{(0)}_{N,\beta}$, and then this can be generalized from $\mathbb{P}^{(0)}_{N,\beta}$ to $\mathbb{P}^{(t)}_{N,\beta}$ for t in $(-t_{\max}, t_{\max})$ because $\tilde{\mu}_t$ has the same regularity as μ_0. □

Assuming from now on that $\mathsf{n} = 0$ and $\mathsf{m} = 0$ (so that every regular function is in the range of Ξ, up to a constant) we can upgrade this control of exponential moments into the control of a weak norm of $\mathrm{Fluct}_N^{(t)}$. Here we use the Sobolev spaces $H^\alpha(\mathbb{R})$.

Lemma 5.5.2. *Under the same assumptions, for $\alpha \geq 8$ we have*

$$\left|\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}_N^{(t)}\|^2_{H^{-\alpha}}\right]\right| \leq C, \tag{5.5.82}$$

where C depends only on V.

Proof. The proof is inspired by [5], in particular we start from [5, Prop. D.1] which states that

$$\|u\|^2_{H^{-\alpha}(\mathbb{R})} \leq C\int_0^1 r^{\alpha-1}\|u * \Phi(r,\cdot)\|^2_{L^2(\mathbb{R})}\,dr \tag{5.5.83}$$

where $\Phi(r,\cdot)$ is the standard heat kernel, i.e. $\Phi(r,x) = \frac{1}{\sqrt{4\pi r}}e^{-\frac{|x|^2}{4r}}$. It follows that

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}_N^{(t)}\|^2_{H^{-\alpha}(\mathbb{R})}\|\right] \leq C\int_0^1 r^{\alpha-1}\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}_N^{(t)} * \Phi(r,\cdot)\|^2_{L^2(\mathbb{R})}\right]dr. \tag{5.5.84}$$

On the other hand we may easily check that, letting $\xi_{x,r} := \Phi(r, x-\cdot)$, we have

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}_N^{(t)} * \Phi(r,\cdot)\|^2_{L^2(\mathbb{R})}\right] = \int \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\left(\mathrm{Fluct}_N^{(t)}(\xi_{x,r})\right)^2\right]dx. \tag{5.5.85}$$

Applying the result of Lemma 5.5.1 to $\xi_{x,r}$ gives us a control on the second moment of $\mathrm{Fluct}_N^{(t)}[\xi_{x,r}]$ of the form

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[(\mathrm{Fluct}_N^{(t)}(\xi_{x,r}))^2\right] \leq C\left(\|\xi_{x,r}\|_{C^4(U)} + \|\xi_{x,r}\|^2_{C^3(U)}\right).$$

Inserting into (5.5.84) and (5.5.85), we are led to

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\|\mathrm{fluct}^{(t)}_N\|^2_{H^{-\alpha}(\mathbb{R})}\right] \leq C\int_0^1\int r^{\alpha-1}\left(\|\xi_{x,r}\|_{C^4(U)}+\|\xi_{x,r}\|^2_{C^3(U)}\right)dx\,dr.$$

Since U is bounded, we may check that this right-hand side can be bounded by $C\int_0^1 r^{\alpha-1}r^{-7}\,dr$, which converges if $\alpha > 7$. □

5.5.3 Proof of Theorem 3

For any test function $\phi(x,y)$ we may write

$$\int \phi(x,y)d\mathrm{fluct}^{(t)}_N(x)\,d\mathrm{fluct}^{(t)}_N(y) \leq \|\phi\|_{C^{2\alpha}(U\times U)}\|\mathrm{fluct}^{(t)}_N\|^2_{H^{-\alpha}(\mathbb{R})}$$

and so by the result of Lemma 5.5.2, we find

$$\left|\mathbf{E}_{\mathbb{P}_{N,\beta}}\left(\int \phi(x,y)d\mathrm{fluct}^{(t)}_N(x)\,d\mathrm{fluct}^{(t)}_N(y)\right)\right| \leq C\|\phi\|_{C^{2\alpha}(U\times U)}. \tag{5.5.86}$$

We may now bootstrap the result of Lemma 5.5.1 by returning to (5.5.74) and, using (5.5.86), writing that

$$\left|\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathsf{A}^{(t)}[\vec{X}_N,\psi]\right]\right| \leq C\|\psi\|_{C^{2\alpha+1}(U)}. \tag{5.5.87}$$

On the other hand, by differentiating (5.5.71) applied with $\xi = \frac{d}{dt}\log\phi'_t$, we have

$$\left|\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\int \frac{d}{dt}\log\phi'_t d\mathrm{fluct}^{(t)}_N\right]\right| \leq C\|\psi\|_{C^5(U)} \tag{5.5.88}$$

Inserting (5.4.57) and (5.5.87) and (5.5.88), (5.5.68) into (5.5.73), and integrating between 0 and $t=-2s/N\beta$, we obtain

$$\log\frac{K_{N,\beta}(\tilde\mu_t,\tilde\zeta_t)}{K_{N,\beta}(\mu_0,\zeta_0)} = \left(1-\frac{\beta}{2}\right)N\left(\mathrm{Ent}(\tilde\mu_t)-\mathrm{Ent}(\mu_0)\right)+\frac{s}{N}O(\|\xi\|_{C^{2\alpha+1}(U)}).$$

Using again (5.4.57) and $\phi'_t = 1+t\psi'$, we may rewrite this as

$$\log\frac{K_{N,\beta}(\tilde\mu_{\frac{-2s}{\beta N}},\tilde\zeta_{\frac{-2s}{\beta N}})}{K_{N,\beta}(\mu_0,\zeta_0)} = -\left(1-\frac{\beta}{2}\right)\frac{2s}{\beta}\int\psi' d\mu_0 + O(\frac{s}{N}\|\xi\|_{C^{2\alpha+1}(U)})$$

Combining this with (5.4.49), (5.2.24) with (5.4.64) and (5.4.65) we obtain

$$\left|\log\mathbf{E}_{\mathbb{P}^V_{N,\beta}}\left(\exp(s\mathrm{Fluct}_N(\xi))\right)+\left(1-\frac{\beta}{2}\right)\frac{2s}{\beta}\int\psi' d\mu_V+\frac{s^2}{\beta}\int_{\mathbb{R}}\xi'\psi\,d\mu_V\right| \leq C\left(\frac{s}{N}\|\xi\|_{C^{2\alpha+1}}+\frac{s^3}{N}\|\xi\|_{C^2}+\frac{s^4}{N^2}\|\xi\|^4_{C^3}\right). \tag{5.5.89}$$

with C depending only on V. This proves Theorem 3.

5.5.4 Iteration and expansion of the partition function to arbitrary order

Let V, W be two C^∞ potentials, such that the associated equilibrium measures μ_V, μ_W satisfy our assumptions with $\mathsf{n}=0, \mathsf{m}=0$. In this section, we explain how to iterate the procedure described above to obtain a relative expansion of the partition function, namely an expansion of $\log Z^W_{N,\beta} - \log Z^V_{N,\beta}$ to any order of $1/N$. Up to applying an affine transformation to one of the gases, whose effect on the partition function is easy to compute, we may assume that μ_V and μ_W have the same support Σ, which is a line segment.

Since V, W are C^∞ and μ_V, μ_W have the same support and a density of the same form (5.1.10) which is C^∞ on the interior of Σ, the optimal transportation map (or monotone rearrangement) ϕ from μ_V to μ_W is C^∞ on Σ and can be extended as a C^∞ function with compact support on $\mathbb{R}$. We let $\psi := \phi - \mathrm{Id}$, which is smooth, and for $t \in [0,1]$ the map $\phi_t := \mathrm{Id} + t\psi$ is a C^∞-diffeomorphism, by the properties of optimal transport. We let $\tilde{\mu}_t := \phi_t \# \mu_V$ as before.

We can integrate (5.5.73) to obtain

$$\begin{aligned}
&\log \frac{K_{N,\beta}(\mu_W, \zeta_W)}{K_{N,\beta}(\mu_V, \zeta_V)} \\
&= \int_0^1 \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[-\frac{\beta}{2} \mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) N \int \frac{d}{dt} \log \phi'_t \, d\tilde{\mu}_t + \left(1 - \frac{\beta}{2}\right) \int \frac{d}{dt} \log \phi'_t d\mathrm{fluct}^{(t)}_N \right] dt \\
&\qquad = N \left(1 - \frac{\beta}{2}\right) (\mathrm{Ent}(\mu_W) - \mathrm{Ent}(\mu_V)) \\
&\qquad + \int_0^1 \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[-\frac{\beta}{2} \mathsf{A}^{(t)}[\vec{X}_N, \psi] + \left(1 - \frac{\beta}{2}\right) \mathrm{Fluct}_N \left[\int \frac{d}{dt} \log \phi'_t d\mathrm{fluct}^{(t)}_N \right] \right] dt.
\end{aligned}$$

The integral on the right-hand side is of order 1, and we claim that the terms in the integral can actually be computed and expanded up to an error $O(1/N)$ using the previous lemma. This is clear for the term $\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[\mathrm{Fluct}^{(t)}_N (\frac{d}{dt} \log \phi'_t)\right]$ which can be computed up to an error $O(1/N)$ by the result of Theorem 3. The term $\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[-\frac{\beta}{2} \mathsf{A}^{(t)}[\vec{X}_N, \psi]\right]$ can on the other hand be deduced from the knowledge of the covariance structure of the fluctuations. Let $\mathcal{F}$ denote the Fourier transform. In view of (5.5.74), using the identity

$$\frac{\psi(x) - \psi(y)}{x - y} = \int_0^1 \psi'(sx + (1-s)y) ds$$

and the Fourier inversion formula we may write

$$\begin{aligned}
\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[\mathsf{A}^{(t)}[\vec{X}_N, \psi]\right] &= \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[\iint_{\mathbb{R}\times\mathbb{R}} \int_0^1 \psi'(sx + (1-s)y) ds \, d\mathrm{fluct}^{(t)}_N(x) d\mathrm{fluct}^{(t)}_N(y) \right] \\
&= \int \int_0^1 \lambda \mathcal{F}(\psi)(\lambda) \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}} \left[\mathrm{Fluct}^{(t)}_N (e^{is\lambda\cdot}) \mathrm{Fluct}^{(t)}_N (e^{i(1-s)\lambda\cdot})\right] ds \, d\lambda. \quad (5.5.90)
\end{aligned}$$

On the other hand, let $\varphi_{s,\lambda}$ be the map associated to $e^{is\lambda\cdot}$ by Lemma 5.3.3. Separating the real part and the imaginary part we may use the results of the previous subsection to $e^{is\lambda\cdot}$

and obtain

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{is\lambda\cdot})\right] = \left(1-\frac{2}{\beta}\right)\int \varphi'_{s,\lambda}d\tilde{\mu}_t + O(\frac{1}{N}) .$$

By polarization of the expression for the variance (see (5.1.16)) and linearity

$$\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{is\lambda\cdot})\mathrm{Fluct}_N^{(t)}(e^{i(1-s)\lambda\cdot})\right] = \mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{is\lambda\cdot})\right]\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[\mathrm{Fluct}_N^{(t)}(e^{i(1-s)\lambda\cdot})\right]$$
$$+\frac{2}{\beta}\Big(\iint\left(\frac{\varphi_{s,\lambda}(u)-\varphi_{s,\lambda}(v)}{u-v}\right)\left(\frac{\varphi_{(1-s),\lambda}(u)-\varphi_{(1-s),\lambda}(v)}{u-v}\right)d\tilde{\mu}_t(u)d\tilde{\mu}_t(v)$$
$$+\int V_t''\varphi_{s,\lambda}\varphi_{(1-s),\lambda}d\tilde{\mu}_t\Big) + O(\frac{1}{N}).$$

Letting $N \to \infty$, we may then find the expansion up to $O(1/N)$ of $\mathbf{E}_{\mathbb{P}^{(t)}_{N,\beta}}\left[-\frac{\beta}{2}\mathsf{A}^{(t)}[\vec{X}_N,\psi]\right]$. Inserting it into the integral gives a relative expansion to order $1/N$ of the (logarithm of the) partition function $\log K_{N,\beta}$. This procedure can then be iterated to yield a relative expansion to arbitrary order of $1/N$ as desired.

5.6 Auxiliary proofs

5.6.1 Proof of Lemma 5.2.1

Proof. Denoting $\triangle$ the diagonal in $\mathbb{R}\times\mathbb{R}$ we may write

$$\mathcal{H}_N^V(\vec{X}_N) = \sum_{i\neq j} -\log|x_i - x_j| + N\sum_{i=1}^N V(x_i)$$
$$= \iint_{\triangle^c} -\log|x-y|\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x)\Big(\sum_{i=1}^N \delta_{x_i}\Big)(y) + N\int_{\mathbb{R}} V(x)\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x).$$

Writing $\sum_{i=1}^N \delta_{x_i}$ as $N\mu_V + \mathrm{fluct}_N$ we get

$$\mathcal{H}_N^V(\vec{X}_N) = N^2\iint_{\triangle^c} -\log|x-y|d\mu_V(x)d\mu_V(y) + N^2\int_{\mathbb{R}} Vd\mu_V$$
$$+2N\iint_{\triangle^c} -\log|x-y|d\mu_V(x)d\mathrm{fluct}_N(y) + N\int_{\mathbb{R}} Vd\mathrm{fluct}_N$$
$$+\iint_{\triangle^c} -\log|x-y|d\mathrm{fluct}_N(x)d\mathrm{fluct}_N(y). \quad (5.6.91)$$

We now recall that ζ_V was defined in (5.1.7), and that $\zeta_V = 0$ in Σ_V. With the help of this we may rewrite the medium line in the right-hand side of (5.6.91) as

$$2N \iint_{\triangle^c} -\log|x-y| d\mu_V(x) d\text{fluct}_N(y) + N\int_{\mathbb{R}} V d\text{fluct}_N$$
$$= 2N \int_{\mathbb{R}} \left(-\log|\cdot| * d\mu_V)(x) + \frac{V}{2}\right) d\text{fluct}_N = 2N \int_{\mathbb{R}} (\zeta_V + c) d\text{fluct}_N$$
$$= 2N \int_{\mathbb{R}} \zeta_V d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big) = 2N \sum_{i=1}^N \zeta_V(x_i).$$

The last equalities are due to the facts that ζ_V vanishes on the support of μ_V and that fluct_N has a total mass 0 since μ_V is a probability measure. We may also notice that since μ_V is absolutely continuous with respect to the Lebesgue measure, we may include the diagonal back into the domain of integration. By that same argument, one may recognize in the first line of the right-hand side of (5.6.91) the quantity $N^2\mathcal{I}_V(\mu_V)$. □

5.6.2 Proof of Proposition 5.2.5

We follow the energy approach introduced in [74, 71], which views the energy as a Coulomb interaction in the plane, after embedding the real line in the plane. We view $\mathbb{R}$ as identified with $\mathbb{R} \times \{0\} \subset \mathbb{R}^2 = \{(x,y), x \in \mathbb{R}, y \in \mathbb{R}\}$. Let us denote by $\delta_{\mathbb{R}}$ the uniform measure on $\mathbb{R} \times \{0\}$, i.e. such that for any smooth $\varphi(x,y)$ (with $x \in \mathbb{R}, y \in \mathbb{R}$) we have

$$\int_{\mathbb{R}^2} \varphi \delta_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(x,0)\, dx.$$

Given $(x_1, \dots, x_N)$ in $\mathbb{R}^N$, we identify them with the points $(x_1, 0), \dots, (x_N, 0)$ in $\mathbb{R}^2$. For a fixed $\vec{X}_N$ and a given probability density μ we introduce the electric potential H_N^μ by

$$H_N^\mu = (-\log|\cdot|) * \left(\sum_{i=1}^N \delta_{(x_i,0)} - N\mu\delta_{\mathbb{R}}\right). \tag{5.6.92}$$

Next, we define versions of this potential which are truncated hence regular near the point charges. For that let $\delta_x^{(\eta)}$ denote the uniform measure of mass 1 on $\partial B(x,\eta)$ (where B denotes an Euclidean ball in $\mathbb{R}^2$). We define $H_{N,\eta}^\mu$ in $\mathbb{R}^2$ by

$$H_{N,\eta}^\mu = (-\log|\cdot|) * \left(\sum_{i=1}^N \delta_{(x_i,0)}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right). \tag{5.6.93}$$

These potentials make sense as functions in $\mathbb{R}^2$ and are harmonic outside of the real axis. Moreover, $H_{N,\eta}^\mu$ solves

$$-\Delta H_{N,\eta}^\mu = 2\pi \left(\sum_{i=1}^N \delta_{(x_i,0)}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right). \tag{5.6.94}$$

Lemma 5.6.1. *For any probability density μ, $\vec{X}_N$ in $\mathbb{R}^N$ and η in $(0,1)$, we have*

$$F_N(\vec{X}_N, \mu) \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla H_{N,\eta}^\mu|^2 + N\log\eta - 2N^2\|\mu\|_{L^\infty}\eta. \tag{5.6.95}$$

Proof. First we notice that $\int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2$ is a convergent integral and that

$$\int_{\mathbb{R}^2} |\nabla H_{N,\vec{\eta}}|^2 = 2\pi \iint -\log|x-y| d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(y). \quad (5.6.96)$$

Indeed, we may choose R large enough so that all the points of $\vec{X}_N$ are contained in the ball $B_R = B(0,R)$. By Green's formula and (5.6.94), we have

$$\int_{B_R} |\nabla H_{N,\eta}|^2 = \int_{\partial B_R} H_{N,\eta} \frac{\partial H_N}{\partial \nu} + 2\pi \int_{B_R} H_{N,\eta} \left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right). \quad (5.6.97)$$

In view of the decay of H_N and ∇H_N, the boundary integral tends to 0 as $R \to \infty$, and so we may write

$$\int_{\mathbb{R}^2} |\nabla H_{N,\eta}|^2 = 2\pi \int_{\mathbb{R}^2} H_{N,\eta} \left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\right)$$

and thus (5.6.96) holds. We may next write

$$\begin{aligned}
&\iint -\log|x-y| d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(x) d\left(\sum_{i=1}^N \delta_{x_i}^{(\eta)} - N\mu\delta_{\mathbb{R}}\right)(y) \\
&\qquad - \iint_{\triangle^c} -\log|x-y| \, d\mathrm{fluct}_N(x) \, d\mathrm{fluct}_N(y) \\
&= -\sum_{i=1}^N \log\eta + \sum_{i\neq j} \iint -\log|x-y| \left(\delta_{x_i}^{(\eta)}\delta_{x_j}^{(\eta)} - \delta_{x_i}\delta_{x_j}\right) + 2N \sum_{i=1}^N \iint -\log|x-y| \left(\delta_{x_i} - \delta_{x_i}^{(\eta)}\right)\mu.
\end{aligned} \quad (5.6.98)$$

Let us now observe that $\int -\log|x-y|\delta_{x_i}^{(\eta)}(y)$, the potential generated by $\delta_{x_i}^{(\eta)}$ is equal to $\int -\log|x-y|\delta_{x_i}$ outside of $B(x_i,\eta)$, and smaller otherwise. Since its Laplacian is $-2\pi\delta_{x_i}^{(\eta)}$, a negative measure, this is also a superharmonic function, so by the maximum principle, its value at a point x_j is larger or equal to its average on a sphere centered at x_j. Moreover, outside $B(x_i,\eta)$ it is a harmonic function, so its values are equal to its averages. We deduce from these considerations, and reversing the roles of i and j, that for each $i \neq j$,

$$-\int \log|x-y|\delta_{x_i}^{(\eta)}\delta_{x_j}^{(\eta)} \leq -\int \log|x-y|\delta_{x_i}\delta_{x_j}^{(\eta)} \leq -\int \log|x-y|\delta_{x_i}\delta_{x_j}.$$

We may also obviously write

$$\int -\log|x-y|\delta_{x_i}\delta_{x_j} - \int -\log|x-y|\delta_{x_i}^{(\eta)}\delta_{x_j}^{(\eta)} \leq -\log|x_i - x_j| \mathbf{1}_{|x_i - x_j| \leq 2\eta}.$$

We conclude that the second term in the right-hand side of (5.6.98) is nonpositive, equal to 0 if all the balls are disjoint, and bounded below by $\sum_{i\neq j} \log|x_i - x_j| \mathbf{1}_{|x_i-x_j|\leq 2\eta}$. Finally, by the above considerations, since $\int -\log|x-y|\delta_{x_i}^{(\eta)}$ coincides with $\int -\log|x-y|\delta_{x_i}$ outside $B(x_i,\eta)$, we may rewrite the last term in the right-hand side of (5.6.98) as

$$2N \sum_{i=1}^N \int_{B(x_i,\eta)} (-\log|x-x_i| + \log\eta)) d\mu\delta_{\mathbb{R}}.$$

But we have that

$$\int_{B(0,\eta)} (-\log|x| + \log\eta)\delta_{\mathbb{R}} = \eta \tag{5.6.99}$$

so if $\mu \in L^\infty$, this last term is bounded by $2\|\mu\|_{L^\infty} N^2\eta$. Combining with all the above results yields the proof. □

Proof of Proposition 5.2.5. We now apply Lemma 5.6.1 for μ_V with $\eta = \frac{1}{2N}$. We obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla H^{\mu}_{N,\eta}|^2 \le F_N(\vec{X}_N, \mu_V) + N\log N + C(\|\mu_V\|_{L^\infty} + 1)N. \tag{5.6.100}$$

Let ξ be a smooth compactly supported test function in $\mathbb{R}$. We may extend it to a smooth compactly supported test function in $\mathbb{R}^2$ coinciding with $\xi(x)$ for any (x,y) such that $|y| \le 1$ and equal to 0 for $|y| \ge 2$. Letting $\#I$ denote the number of balls $B(x_i, \eta)$ intersecting the support of ξ, we have

$$\begin{aligned}\left|\int \left(\mathrm{fluct}_N - \left(\sum_{i=1}^N \delta^{(\eta)}_{x_i} - N\mu_V\right)\right)\xi\right| &= \left|\int \left(\sum_{i=1}^N (\delta_{x_i} - \delta^{(\eta)}_{x_i})\right)\xi\right| \\ &\le \#I\eta\|\nabla\xi\|_{L^\infty} = \frac{1}{2}\frac{\#I}{N}\|\nabla\xi\|_{L^\infty}. \end{aligned}\tag{5.6.101}$$

But in view of (5.6.94), we also have

$$\begin{aligned}\left|\int \left(\sum_{i=1}^N \delta^{(\eta)}_{x_i} - N\mu_V\right)\xi\right| &= \frac{1}{2\pi}\left|\int_{\mathbb{R}^2} \nabla H^{\mu_V}_{N,\eta}\cdot\nabla\xi\right| \\ &\le |\mathrm{Supp}\,\xi|^{\frac{1}{2}}\|\nabla\xi\|_{L^\infty}\|\nabla H^{\mu_V}_{N,\eta}\|_{L^2(\mathrm{Supp}\,\xi)}. \end{aligned}\tag{5.6.102}$$

Combining (5.6.100), (5.6.101) and (5.6.102), we obtain

$$\begin{aligned}&\left|\int \xi\,\mathrm{fluct}_N\right| \\ &\le C\|\nabla\xi\|_{L^\infty}\left(\frac{\#I}{N} + |\mathrm{Supp}\,\xi|^{\frac{1}{2}}\left(F_N(\vec{X}_N, \mu_V) + N\log N + C(\|\mu_V\|_{L^\infty} + 1)N\right)^{\frac{1}{2}}\right). \end{aligned}\tag{5.6.103}$$

Bounding $\#I$ by N yields the result. □

5.6.3 Proof of Lemma 5.3.1

Proof. Since μ_V minimizes the logarithmic potential energy (5.1.6), for any bounded continuous function h we have

$$\iint \frac{h(x) - h(y)}{x - y} d\mu_V(x)d\mu_V(y) = \int V'(x)h(x)d\mu_V(x). \tag{5.6.104}$$

Of course, an identity like (5.6.104) extends to complex-valued functions, and applying it to $h = \frac{1}{z-\cdot}$ for some fixed $z \in \mathbb{C}\setminus\Sigma_V$ leads to

$$G(z)^2 - G(z)V'(\mathcal{R}(z)) + L(z) = 0, \tag{5.6.105}$$

where G is the usual Stieltjes transform of μ_V

$$G(z) = \int \frac{1}{z-y} d\mu_V(y), \tag{5.6.106}$$

and L is defined by

$$L(z) = \int \frac{V'(\mathcal{R}(z)) - V'(y)}{z-y} d\mu_V(y). \tag{5.6.107}$$

Solving (5.6.105) for G yields

$$G(z) = \frac{1}{2}\left(V'(\mathcal{R}(z)) - \sqrt{V'(\mathcal{R}(z))^2 - 4L(z)}\right). \tag{5.6.108}$$

As is well-known, $-\frac{1}{\pi}\mathcal{I}(G(x+i\varepsilon))$ converges towards the density $\mu_V(x)$ as $\varepsilon \to 0^+$, hence we have for x in Σ_V

$$\mu_V(x)^2 = S(x)^2\sigma^2(x) = -\frac{1}{(2\pi)^2}(V'(x)^2 - 4L(x)). \tag{5.6.109}$$

This proves that μ_V has regularity $C^{\mathsf{p}-2}$ at any point where it does not vanish. Assuming the form (5.1.11) for S, we also deduce that the function S_0 has regularity at least $C^{\mathsf{p}-3-2\mathsf{k}}$ on Σ_V.

Applying (5.6.108) on $\mathbb{R} \setminus \Sigma$, we obtain

$$\frac{1}{2}V'(x) - \int \frac{1}{x-y} d\mu_V(y) = \frac{1}{2}\sqrt{V'(x)^2 - 4L(x)},$$

and the left-hand side is equal to $\zeta'(x)$.

Using (5.1.11), (5.6.109) and the fact that V is regular, we may find a neighborhood U small enough such that ζ' does not vanish on $U \setminus \Sigma_V$ and on which we can write ζ' as in (5.3.30). □

5.6.4 Proof of Lemma 5.3.2

Proof. We first prove that the image of F is indeed contained in $C^1(U)$.

For $(t,\psi) = (0,0)$, we have indeed $\mathcal{F}(0,0) = \zeta_V + c$ and ζ_V is in $C^1(\mathbb{R})$ by the regularity assumptions on V. We may also write

$$\mathcal{F}(t,\psi) = \mathcal{F}(0,0) - \int \log\frac{|\phi(\cdot) - \phi(y)|}{|\cdot - y|} d\mu_V(y) + \frac{1}{2}(V_t \circ \phi - V \circ \phi),$$

and since $\|\psi\|_{C^2(U)} \leq 1/2$, the second and third terms are also in $C^1(U)$.

Next, we compute the partial derivatives of $\mathcal{F}$ at a fixed point $(t_0, \psi_0) \in [-1,1] \times B$. It is easy to see that

$$\left.\frac{\partial \mathcal{F}}{\partial t}\right|_{(t_0,\psi_0)} = \frac{1}{2}\xi \circ \phi_0,$$

and the map $(t_0,\psi_0) \mapsto \xi \circ \phi_0$ is indeed continuous.

The Fréchet derivative of F with respect to the second variable can be computed as follows

$$\begin{aligned}\mathcal{F}(t_0,\psi_0+\psi_1) &= -\int \log\left|\left(\phi_0(\cdot)-\phi_0(y)\right)+\left(\psi_1(\cdot)-\psi_1(y)\right)\right|d\mu_V(y)+\frac{1}{2}V_{t_0}\circ(\phi_0+\psi_1)\\ &= \mathcal{F}(t_0,\psi_0)-\int\log\left|1+\frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)}\right|d\mu_V(y)+\frac{1}{2}\left(V_{t_0}\circ(\phi_0+\psi_1)-V_{t_0}\circ\phi_0\right)\\ &= \mathcal{F}(t_0,\psi_0)-\int\frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)}d\mu_V(y)+\frac{1}{2}\psi_1V'_{t_0}\circ\phi_0+\varepsilon_{t_0,\psi_0}(\psi_1)\ ,\end{aligned}$$

where $\varepsilon_{t_0,\psi_0}(\psi_1)$ is given by

$$\begin{aligned}\varepsilon_{t_0,\psi_0}(\psi_1) = &-\int\left[\log\left|1+\frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)}\right|-\frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)}\right]d\mu_V(y)\\ &+\frac{1}{2}\left(V_{t_0}\circ(\phi_0+\psi_1)-V_{t_0}\circ\phi_0-\psi_1V'_{t_0}\circ\phi_0\right)\ .\end{aligned}$$

By differentiating inside the integral we get the bound

$$\|\varepsilon_{t_0,\psi_0}(\psi_1)\|_{C^1(U)}\leq C(t_0,\psi_0)\|\psi_1\|^2_{C^2(U)},$$

with a constant depending on V. It implies that

$$\left.\frac{\partial\mathcal{F}}{\partial\psi}\right|_{(t_0,\psi_0)}[\psi_1]=-\int\frac{\psi_1(\cdot)-\psi_1(y)}{\phi_0(\cdot)-\phi_0(y)}d\mu_V(y)+\frac{1}{2}\psi_1V'_{t_0}\circ\phi_0\ ,$$

and we can check that this expression is also continuous in (t_0,ψ_0). In particular, we may observe that

$$\left.\frac{\partial\mathcal{F}}{\partial\psi}\right|_{(0,0)}[\psi]=-\Xi_V[\psi]. \tag{5.6.110}$$

Finally, we prove the bound (5.3.32). For any fixed $(t,\psi)\in[-1,1]\times B$, we write

$$\mathcal{F}(t,\psi)-\mathcal{F}(0,0)=\int_0^1\frac{d\mathcal{F}(st,s\psi)}{ds}ds=\int_0^1\left(t\left.\frac{\partial\mathcal{F}}{\partial t}\right|_{(st,s\psi)}+\left.\frac{\partial\mathcal{F}}{\partial\psi}\right|_{(st,s\psi)}[\psi]\right)ds\ ,$$

we get

$$\begin{aligned}\|\mathcal{F}(t,\psi)-\mathcal{F}(0,0)-\frac{t}{2}\xi+\Xi_V[\psi]\|_{C^1(U)}\leq\int_0^1&\left(\frac{t}{2}\|\xi\circ\phi_s-\xi\|_{C^1(U)}\right.\\ &\left.+\left\|\left.\frac{\partial\mathcal{F}}{\partial\psi}\right|_{(st,s\psi)}[\psi]-\left.\frac{\partial\mathcal{F}}{\partial\psi}\right|_{(0,0)}[\psi]\right\|_{C^1(U)}\right)ds,\end{aligned} \tag{5.6.111}$$

with $\phi_s=\mathrm{Id}+s\psi$. It is straightforward to check that

$$\|\xi\circ\phi_s-\xi\|_{C^1(U)}\leq C\|\xi\|_{C^2(U)}\|\psi\|_{C^1(U)}\ .$$

To control the second term inside the integral we write

$$\frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(st,s\psi)}[\psi] - \frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(0,0)}[\psi]$$
$$= -\int \left(\frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)} - \frac{\psi(\cdot)-\psi(y)}{\cdot - y} \right) d\mu_V(y) + \frac{1}{2}\left(V'_{st}\circ\phi_s - V'\right)\psi$$

and we obtain

$$\left\| \frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(st,s\psi)}[\psi] - \frac{\partial \mathcal{F}}{\partial \psi}\Big|_{(0,0)}[\psi] \right\|_{C^1(U)}$$
$$\leq \int \left\| \frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)} - \frac{\psi(\cdot)-\psi(y)}{\cdot - y} \right\|_{C^1(U)} d\mu_V(y)$$
$$+ \left\| \left(V'_{st}\circ\phi_s - V'\right)\psi \right\|_{C^1(U)}$$

We now use that

$$\begin{aligned}
\left\| \left(\frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)} - \frac{\psi(\cdot)-\psi(y)}{\cdot - y} \right) \right\|_{C^1(U)} &= \left\| \left(\frac{\psi(\cdot)-\psi(y)}{\cdot - y} \right) \left(\frac{\cdot - y}{\phi_s(\cdot)-\phi_s(y)} - 1 \right) \right\|_{C^1(U)} \\
&\leq C\|\psi\|_{C^2(U)} \left\| \frac{\cdot - y}{\phi_s(\cdot)-\phi_s(y)} - 1 \right\|_{C^1(U)} \\
&= Cs\|\psi\|_{C^2(U)} \left\| \frac{\psi(\cdot)-\psi(y)}{\phi_s(\cdot)-\phi_s(y)} \right\|_{C^1(U)} \\
&\leq C\|\psi\|^2_{C^2(U)} \left\| \frac{\cdot - y}{\phi_s(\cdot)-\phi_s(y)} \right\|_{C^1(U)} \\
&\leq C\|\psi\|^2_{C^2(U)} .
\end{aligned}$$

In the second and the fourth line, we used Leibniz formula. In the last line we used that $s(\psi(\cdot)-\psi(y))/(\cdot - y)$ is uniformely bounded by $1/2$ in $C^2(U)$ so its composition with the function $x \to 1/(1+x)$ is bounded in $C^2(U)$. We conclude by checking that

$$\|\left(V'_{st}\circ\phi_s - V'\right)\psi\|_{C^1(U)} \leq C\Big(\|V\|_{C^3(U)}\|\psi\|_{C^1(U)} + t\|\psi\|_{C^2(U)}\Big)\|\psi\|_{C^0(U)} .$$

□

5.6.5 Proof of Lemma 5.3.3

Proof. First, we solve the equation $\Xi_V[\psi] = \frac{1}{2}\xi + c_\xi$ in $\mathring{\Sigma}_V$, where Ξ_V is operator defined in (5.1.12). For x in $\mathring{\Sigma}_V$, we have the following Schwinger-Dyson equation

$$\frac{V'(x)}{2} = P.V. \int \frac{1}{x-y} d\mu_V(y). \tag{5.6.112}$$

In particular, for x in $\mathring{\Sigma}_V$, it implies

$$\Xi_V[\psi](x) := P.V. \int_{\Sigma_V} \frac{\psi(y)}{y-x} \mu_V(y) dy, \tag{5.6.113}$$

and we might thus try to solve

$$P.V. \int_{\Sigma_V} \frac{\psi(y)}{y-x} \mu_V(y) dy = \frac{1}{2}\xi + c_\xi. \tag{5.6.114}$$

Equation (5.6.114) is a singular integral equation, we refer to [68, Chap. 10-11-12] for a detailed treatment. In particular, it is known that if the conditions (5.1.14) are satisfied, then there exists a solution ψ_0 to

$$P.V. \int_{\Sigma_V} \frac{\psi_0(y)}{y-x} dy = \frac{1}{2}\xi + c_\xi \text{ on } \mathring{\Sigma}_V, \tag{5.6.115}$$

which is explicitly given by the formula

$$\psi_0(x) = -\frac{\sigma(x)}{2\pi^2} P.V. \int_{\Sigma_V} \frac{\xi(y)}{\sigma(y)(y-x)} dy. \tag{5.6.116}$$

Since we have, for x in $\mathring{\Sigma}_V$

$$P.V. \int_{\Sigma_V} \frac{1}{\sigma(y)(y-x)} dy = 0,$$

we may re-write (5.6.116) as

$$\psi_0(x) = -\frac{\sigma(x)}{2\pi^2} \int_{\Sigma_V} \frac{\xi(y)-\xi(x)}{\sigma(y)(y-x)} dy \text{ on } \mathring{\Sigma}_V, \tag{5.6.117}$$

where the integral is now a definite Riemann integral. From (5.6.117) we deduce that the map $\frac{\psi_0}{\sigma}$ is of class C^{r-1} in $\mathring{\Sigma}_V$ and extends readily to a C^{r-1} function on Σ_V.

For $d = 0, \dots, r-1$ and for $x \in \Sigma_V$, we compute that

$$\left(\frac{\psi_0}{\sigma}\right)^{(d)}(x) = -\frac{d!}{2\pi^2} \int_{\Sigma_V} \frac{\xi(y) - R_{s_i,d+1}\xi(y)}{\sigma(y)(y-s_i)^{d+1}} dy.$$

In particular, if conditions (5.1.15) hold, in view of Lemma 5.3.1 the map

$$\psi(x) := \frac{\psi_0(x)}{S(x)\sigma(x)}$$

extends to a function of class $(p-3-2k) \wedge (r-1-k)$, hence C^2 on Σ_V, and in view of (5.6.115) it satisfies $\Xi_V[\psi] = \frac{\xi}{2} + c_\xi$ on Σ_V.

Now, we define ψ outside Σ_V. By definition, for x outside Σ_V, the equation

$$\Xi_V[\psi](x) = \frac{1}{2}\xi(x) + c_\xi$$

can be written as

$$\psi(x)\int\frac{1}{x-y}\,d\mu_V(y)-\int\frac{\psi(y)}{x-y}\,d\mu_V(y)-\frac{1}{2}\psi(x)V'(x)=\frac{1}{2}\xi(x)+c_\xi,$$

and thus the choice (5.3.34) ensures that $\Xi_V[\psi]=\frac{1}{2}\xi+c_\xi$. Moreover, ψ is clearly of class $C^{\mathsf{r}\wedge(\mathsf{p}-1)}$ on $\mathbb{R}\setminus\Sigma_V$. It remains to check that ψ has the desired regularity at the endpoints of Σ_V. For a given endpoint α we consider $\tilde{\psi}$ the Taylor development of order $l:=(\mathsf{p}-3-2\mathsf{k})\wedge(\mathsf{r}-1-\mathsf{k})$ at α of ψ. We can write (5.3.34) as

$$\begin{aligned}\frac{\int\frac{\psi(y)}{x-y}d\mu_V(y)+\frac{\xi(x)}{2}+c_\xi}{\int\frac{1}{x-y}d\mu_V(y)-\frac{1}{2}V'(x)}&=\frac{-\int\frac{\tilde{\psi}(x)-\tilde{\psi}(y)}{x-y}d\mu_V(y)+\tilde{\psi}(x)\int\frac{1}{x-y}d\mu_V(y)+\frac{\xi(x)}{2}+c_\xi}{\int\frac{1}{x-y}d\mu_V(y)-\frac{1}{2}V'(x)}\\&=\tilde{\psi}(x)+\frac{\frac{\xi(x)}{2}+c_\xi-\Xi_V[\tilde{\psi}](x)}{\int\frac{1}{x-y}d\mu_V(y)-\frac{1}{2}V'(x)}\,.\end{aligned}$$

As $\Xi_V[\psi]=\frac{\xi}{2}+c_\xi$ on Σ_V, the numerator on the right hand side of the last equation and its first l derivatives vanish at α. From Lemma (5.3.1) we conclude that ψ is of class $l-\mathsf{k}=(\mathsf{p}-3-3\mathsf{k})\wedge(\mathsf{r}-1-2\mathsf{k})$ at α, hence C^2 from (5.1.13).

□

5.6.6 Proof of Lemma 5.4.7

Using definition (5.1.6) we can write $\mathcal{I}_{V_t}(\mu_t)$ in the following form

$$\mathcal{I}_{V_t}(\mu_t)=\int h^{\mu_t}\,d\mu_t+\int V_t\,d\mu_t.$$

To prove Lemma 5.4.7, we introduce the auxiliary quantity

$$\mathcal{I}(\tilde{\mu}_t):=\int h^{\tilde{\mu}_t}\,d\tilde{\mu}_t+\int V_t\,d\tilde{\mu}_t,$$

and we first prove that $\mathcal{I}(\tilde{\mu}_t)$ is close to $\mathcal{I}_{V_t}(\mu_t)$.

Claim 1. *We have*

$$\mathcal{I}_{V_t}(\mu_t)=\mathcal{I}(\tilde{\mu}_t)+O\left(t^4\|\psi\|^4_{C^2(U)}\right). \tag{5.6.118}$$

Proof. Let us write

$$\begin{aligned}\mathcal{I}_{V_t}(\mu_t)&=\int h^{\mu_t}\,d\mu_t+\int V_t\,d\mu_t\\&=\int h^{\tilde{\mu}_t}\,d\tilde{\mu}_t+\int(h^{\mu_t}+h^{\tilde{\mu}_t})d(\mu_t-\tilde{\mu}_t)+\int V_t\,d\mu_t.\end{aligned}\tag{5.6.119}$$

We have used the fact that, integrating by parts twice,

$$\int h^{\mu_t}d\tilde{\mu}_t=\int h^{\tilde{\mu}_t}d\mu_t.$$

We have, using the definition of $\zeta_t, \tilde{\zeta}_t$ and (5.3.39)

$$\int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) = \int \left(\zeta_t - \frac{1}{2}V_t - c_t + \tilde{\zeta}_t - \frac{1}{2}V_t - \tilde{c}_t + O(t^2\|\psi\|^2_{C^2(U)})\right) d(\mu_t - \tilde{\mu}_t).$$

In view of (5.4.42), (5.4.43), we thus get

$$\int (h^{\mu_t} + h^{\tilde{\mu}_t}) d(\mu_t - \tilde{\mu}_t) + \int V_t \, d\mu_t = O(t^4\|\psi\|^4_{C^2(U)}) + \int V_t d\tilde{\mu}_t. \tag{5.6.120}$$

Combining (5.6.119) and (5.6.120) yields the result. □

We may now compare $\mathcal{I}(\tilde{\mu}_t)$ and $\mathcal{I}_V(\mu_V)$ using the transport map.

Claim 2. *We have*

$$\mathcal{I}(\tilde{\mu}_t) = \mathcal{I}_V(\mu_V) + t\int \xi d\mu_V + \frac{t^2}{2}\left(\iint \left(\frac{\psi(x) - \psi(y)}{x - y}\right)^2 d\mu_V(x)d\mu_V(y) + \int V''\psi^2 d\mu_V + 2\int \xi'\psi d\mu_V\right) + O(t^3\|\xi\|_{C^2(U)}). \tag{5.6.121}$$

Proof. We may write

$$\mathcal{I}(\tilde{\mu}_t) = -\int \log|\phi_t(x) - \phi_t(y)| d\mu_0(x)d\mu_0(y) + \int V \circ \phi_t \, d\mu_0 + t\int \xi \circ \phi_t \, d\mu_0$$
$$= \int h^{\mu_0} \, d\mu_0 - \iint \log\left|1 + t\frac{\psi(x) - \psi(y)}{x - y}\right| d\mu_0(x)d\mu_0(y) + \int V \circ \phi_t \, d\mu_0 + t\int \xi \circ \phi_t \, d\mu_0.$$

By a Taylor expansion, we obtain

$$\mathcal{I}(\tilde{\mu}_t) = \mathcal{I}_V(\mu_0) - t\iint \frac{\psi(x) - \psi(y)}{x - y} d\mu_0(x)d\mu_0(y) + \frac{t^2}{2}\iint \left(\frac{\psi(x) - \psi(y)}{x - y}\right)^2 d\mu_0(x)d\mu_0(y)$$
$$+ t\int V'\psi \, d\mu_V + \frac{t^2}{2}\int V''\psi^2 d\mu_V + t\int \xi d\mu_V + t^2\int \xi'\psi d\mu_0 + O(t^3\|\xi\|_{C^2(\mathbb{R})}).$$

Let us recall that by definition $\mu_0 = \mu_V$. By (5.6.104) we have

$$\iint \frac{\psi(x) - \psi(y)}{x - y} d\mu_V(x)d\mu_V(y) = \int V'\psi \, d\mu_V,$$

hence we obtain (5.6.121). □

To conclude the proof of Lemma 5.4.7 it remains to prove the following identity.

Claim 3.

$$\int \xi'\psi d\mu_V = -\iint \left(\frac{\psi(x) - \psi(y)}{x - y}\right)^2 d\mu_V(x)d\mu_V(y) - \int V''\psi^2 d\mu_V. \tag{5.6.122}$$

Proof. By definition of ψ we have

$$\frac{1}{2}(\xi + c_\xi) = \int \frac{\psi(x) - \psi(y)}{x - y} d\mu_V(y) - \frac{1}{2}\psi V' ,$$

and thus

$$\xi' = 2\int \frac{\psi(y) - \psi(x) - \psi'(x)(y-x)}{(x-y)^2} d\mu_V(y) - \psi' V' - \psi V'' .$$

Integrating both sides against $\psi\mu_V$ yields

$$\int \xi'\psi d\mu_V = 2\iint \frac{(\psi(y) - \psi(x) - \psi'(x)(y-x))\psi(x)}{(x-y)^2} d\mu_V(y) d\mu_V(x) - \int \psi\psi' V' d\mu_V - \int V''\psi^2 d\mu_V.$$

Using (5.6.104) for the second term we obtain

$$\int \xi'\psi d\mu_V = 2\iint \frac{(\psi(y) - \psi(x) - \psi'(x)(y-x))\psi(x)}{(y-x)^2} d\mu_V(y) d\mu_V(x) - \iint \frac{\psi\psi'(y) - \psi\psi'(x)}{y-x} d\mu_V(x) d\mu_V(y) - \int V''\psi^2 d\mu_V.$$

We may then combine the first two terms in the right-hand side to obtain (5.6.122). □

Bibliography

[1] S. Albeverio, L. Pastur, and M. Shcherbina. On the 1/n expansion for some unitary invariant ensembles of random matrices. *Communications in Mathematical Physics*, 224(1):271–305, 2001.

[2] J. Ambjørn and Y. M. Makeenko. Properties of loop equations for the hermitian matrix model and for two-dimensional quantum gravity. *Modern Physics Letters A*, 5(22):1753–1763, 1990.

[3] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices.* Cambridge University Press, 2010.

[4] G. W. Anderson and O. Zeitouni. A CLT for a band matrix model. *Probability Theory and Related Fields*, 134(2):283–338, 2006.

[5] S. Armstrong, T. Kuusi, and J.-C. Mourrat. Quantitative stochastic homogenization and large-scale regularity. *https://arxiv.org/abs/1705.05300*, 2017.

[6] G. B. Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probability theory and related fields*, 108(4):517–542, 1997.

[7] R. Bauerschmidt, P. Bourgade, M. Nikula, and H.-T. Yau. The two-dimensional coulomb plasma: quasi-free approximation and central limit theorem. *arXiv preprint arXiv:1609.08582*, 2016.

[8] R. Bauerschmidt, P. Bourgade, M. Nikula, and H.-T. Yau. The two-dimensional coulomb plasma: quasi-free approximation and central limit theorem. *arXiv preprint arXiv:1609.08582*, 2016.

[9] R. Bauerschmidt, P. Bourgade, M. Nikula, and H.-T. Yau. Local density for two-dimensional one-component plasma. *Communications in Mathematical Physics*, 356(1):189–230, 2017.

[10] F. Bekerman. Transport maps for β-matrix models in the multi-cut regime. *Random Matrices: Theory and Applications*, 2017.

[11] F. Bekerman, A. Figalli, and A. Guionnet. Transport maps for β-matrix models and universality. *Communications in Mathematical Physics*, 338(2):589–619, 2013.

[12] F. Bekerman, T. Leblé, and S. Serfaty. CLT for fluctuations of β-ensembles with general potential. *arXiv preprint arXiv:1706.09663*, 2017.

[13] F. Bekerman and A. Lodhia. Mesoscopic central limit theorem for general β-ensembles. *arXiv preprint arXiv:1605.05206*, 2016.

[14] F. Benaych-Georges and A. Knowles. Lectures on the local semicircle law for wigner matrices. *arXiv preprint arXiv:1601.04055*, 2016.

[15] Borot and Guionnet. Asymptotic expansion of β matrix models in the multi-cut regime. *arXiv preprint arXiv:1303.1045*, 2013.

[16] G. Borot and A. Guionnet. Asymptotic expansion of β matrix models in the one-cut regime. *Communications in Mathematical Physics*, 317(2):447–483, 2013.

[17] G. Borot, A. Guionnet, and K. K. Kozlowski. Large-N asymptotic expansion for mean field models with coulomb gas interaction. *International Mathematics Research Notices*, page rnu260, 2015.

[18] P. Bourgade, L. Erdős, and H.-T. Yau. Bulk universality of general β-ensembles with non-convex potential. *Journal of Mathematical Physics*, 53(9):095221, 2012.

[19] P. Bourgade, L. Erdös, and H.-T. Yau. Edge universality of beta ensembles. *Communications in Mathematical Physics*, 332(1):261–353, 2014.

[20] P. Bourgade, L. Erdős, and H.-T. Yau. Universality of general β-ensembles. *Duke Mathematical Journal*, 163(6):1127–1190, 2014.

[21] P. Bourgade, L. Erdős, H.-T. Yau, and J. Yin. Fixed energy universality for generalized wigner matrices. *Communications on Pure and Applied Mathematics*, 69(10):1815–1881, 2016.

[22] A. Boutet de Monvel and A. Khorunzhy. Asymptotic distribution of smoothed eigenvalue density. I. Gaussian random matrices. *Random Oper. Stochastic Equations*, 7(1):1–22, 1999.

[23] A. Boutet de Monvel and A. Khorunzhy. Asymptotic distribution of smoothed eigenvalue density. II. Wigner random matrices. *Random Oper. Stochastic Equations*, 7(2):149–168, 1999.

[24] J. Breuer and M. Duits. Universality of mesoscopic fluctuations for orthogonal polynomial ensembles. *Comm. Math. Phys.*, 342(2):491–531, 2016.

[25] S. Chatterjee. Rigidity of the three-dimensional hierarchical coulomb gas. *arXiv preprint arXiv:1708.01965*, 2017.

[26] T. Claeys. Birth of a cut in unitary random matrix ensembles. *International Mathematics Research Notices*, 2008:rnm166, 2008.

[27] T. Claeys, I. Krasovsky, and A. Its. Higher-order analogues of the Tracy-Widom distribution and the Painlevé ii hierarchy. *Communications on pure and applied mathematics*, 63(3):362–412, 2010.

[28] T. Claeys and A. Kuijlaars. Universality of the double scaling limit in random matrix models. *Communications on Pure and Applied Mathematics*, 59(11):1573–1603, 2006.

[29] G. De Philippis and A. Figalli. The monge–ampère equation and its link to optimal transportation. *Bulletin of the American Mathematical Society*, 51(4):527–580, 2014.

[30] P. Deift. Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. *Courant Lecture Notes in Mathematics , New York University Courant Institute of Mathematical Sciences,vol. 3, New York*, 1999.

[31] P. Deift and D. Gioev. Universality at the edge of the spectrum for unitary, orthogonal, and symplectic ensembles of random matrices. *Communications on pure and applied mathematics*, 60(6):867–910, 2007.

[32] P. Deift and D. Gioev. Universality in random matrix theory for orthogonal and symplectic ensembles. *International Mathematics Research Papers*, 2007:rpm004, 2007.

[33] P. Deift and D. Gioev. *Random matrix theory: invariant ensembles and universality*, volume 18. American Mathematical Soc., 2009.

[34] P. Deift, T. Kriecherbauer, and K. T.-R. McLaughlin. New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *Journal of approximation theory*, 95(3):388–475, 1998.

[35] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Communications on Pure and Applied Mathematics*, 52(11):1335–1425, 1999.

[36] M. Duits and A. Kuijlaars. Painlevé i asymptotics for orthogonal polynomials with respect to a varying quartic weight. *Nonlinearity*, 19(10):2211, 2006.

[37] I. Dumitriu and A. Edelman. Matrix models for beta ensembles. *Journal of Mathematical Physics*, 43(11):5830–5847, 2002.

[38] F. J. Dyson. Statistical theory of the energy levels of complex systems. i. *Journal of Mathematical Physics*, 3, 1962.

[39] L. Erdős, B. Schlein, and H.-T. Yau. Local semicircle law and complete delocalization for wigner random matrices. *Communications in Mathematical Physics*, 287(2):641–655, 2009.

[40] L. Erdős, B. Schlein, and H.-T. Yau. Universality of random matrices and local relaxation flow. *Inventiones mathematicae*, 185(1):75–119, 2011.

[41] L. Erdős, B. Schlein, H.-T. Yau, et al. Semicircle law on short scales and delocalization of eigenvectors for wigner random matrices. *The Annals of Probability*, 37(3):815–852, 2009.

[42] L. Erdos and H.-T. Yau. Gap universality of generalized wigner and beta-ensembles. *arXiv preprint arXiv:1211.3786*, 2012.

[43] L. Erdős, H.-T. Yau, and J. Yin. Bulk universality for generalized wigner matrices. *Probability Theory and Related Fields*, pages 1–67, 2012.

[44] A. Figalli and A. Guionnet. Universality in several-matrix models via approximate transport maps. *arXiv preprint arXiv:1407.2759*, 2014.

[45] P. Forrester. *Log-gases and random matrices.* Princeton University Press, 2010.

[46] Y. V. Fyodorov, B. A. Khoruzhenko, and N. J. Simm. Fractional Brownian motion with Hurst index $H = 0$ and the Gaussian unitary ensemble. *Ann. Probab.*, 44(4):2980–3031, 2016.

[47] J. Ginibre. Statistical ensembles of complex, quaternion, and real matrices. *Journal of Mathematical Physics*, 6(3):440–449, 1965.

[48] A. Guionnet and E. Maurel-Segala. Second order asymptotics for matrix models. *The Annals of Probability*, 35(6):2160–2212, 2007.

[49] A. Guionnet and D. Shlyakhtenko. Free monotone transport. *Inventiones mathematicae*, 197(3):613–661, 2014.

[50] Y. He, A. Knowles, et al. Mesoscopic eigenvalue statistics of wigner matrices. *The Annals of Applied Probability*, 27(3):1510–1550, 2017.

[51] K. Johansson. On fluctuations of eigenvalues of random Hermitian matrices. *Duke Mathematical Journal*, 91(1):151–204, 1998.

[52] T. Kriecherbauer and M. Shcherbina. Fluctuations of eigenvalues of matrix models and their applications. *arXiv preprint arXiv:1003.6121*, 2010.

[53] M. Krishnapur, B. Rider, and B. Virág. Universality of the stochastic Airy operator. *Communications on Pure and Applied Mathematics*, 2015.

[54] A. Kuijlaars and K. McLaughlin. Generic behavior of the density of states in random matrix theory and equilibrium problems in the presence of real analytic external fields. *Communications on Pure and Applied Mathematics*, 53(6):736–785, 2000.

[55] G. Lambert. Mesoscopic fluctuations for unitary invariant ensembles. *arXiv preprint arXiv:1510.03641*, 2015.

[56] G. Lambert, M. Ledoux, and C. Webb. Stein's method for normal approximation of linear statistics of beta-ensembles. *https://arxiv.org/abs/1706.10251*, 06 2017.

[57] T. Leblé. Local microscopic behavior for 2d coulomb gases. *Probability Theory and Related Fields*, pages 1–46, 2015.

[58] T. Leblé and S. Serfaty. Large deviation principle for empirical fields of Log and Riesz gases. *To appear in Inventiones Mathematicae*, 2015.

[59] T. Leblé and S. Serfaty. Fluctuations of two-dimensional Coulomb gases. *arXiv preprint arXiv:1609.08088*, 2016.

[60] E. Levin and D. S. Lubinsky. Universality limits in the bulk for varying measures. *Advances in Mathematics*, 219(3):743–779, 2008.

[61] Y. Li. Rigidity of eigenvalues for β ensemble in multi-cut regime. *arXiv preprint arXiv:1611.06603*, 2016.

[62] A. Lodhia and N. Simm. Mesoscopic linear statistics of Wigner matrices. *arXiv preprint arXiv:1503.03533*, 2015.

[63] A. Lytova and L. Pastur. Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. *The Annals of Probability*, 37(5):1778–1840, 2009.

[64] M. Maïda and É. Maurel-Segala. Free transport-entropy inequalities for non-convex potentials and application to concentration for random matrices. *Probability Theory and Related Fields*, 159(1-2):329–356, 2014.

[65] M. Mehta. Random matrices. *Pure and Applied Mathematics, vol. 142, Elsevier/Academic, Amsterdam*, 2004.

[66] M. Mo. The Riemann–Hilbert approach to double scaling limit of random matrix eigenvalues near the "birth of a cut" transition. *International Mathematics Research Notices*, 2008.

[67] A. B. Monvel, L. Pastur, and M. Shcherbina. On the statistical mechanics approach in the random matrix theory: integrated density of states. *Journal of Statistical Physics*, 79(3-4):585–611, 1995.

[68] N. I. Muskhelishvili. *Singular integral equations.* Dover Publications, Inc., New York, 1992.

[69] L. Pastur and M. Shcherbina. Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles. *Journal of statistical physics*, 86(1):109–147, 1997.

[70] L. Pastur and M. Shcherbina. Bulk universality and related properties of Hermitian matrix models. *Journal of Statistical Physics*, 130(2):205–250, 2008.

[71] M. Petrache and S. Serfaty. Next order asymptotics and renormalized energy for riesz interactions. *Journal of the Institute of Mathematics of Jussieu*, pages 1–69, 2014.

[72] J. Ramírez, B. Rider, and B. Virág. Beta ensembles, stochastic Airy spectrum, and a diffusion. *Journal of the American Mathematical Society*, 24(4):919–944, 2011.

[73] E. Saff and V. Totik. *Logarithmic potentials with external fields*, volume 316. Springer Science & Business Media, 2013.

[74] É. Sandier and S. Serfaty. 1D log gases and the renormalized energy: crystallization at vanishing temperature. *Probability Theory and Related Fields*, 162(3-4):795–846, 2015.

[75] A. Selberg. Remarks on a multiple integral. *Norsk Mat. Tidsskr.*, 26:71–78, 1944.

[76] M. Shcherbina. Central limit theorem for linear eigenvalue statistics of the Wigner and sample covariance random matrices. 7(2):176–192, 2011.

[77] M. Shcherbina. Fluctuations of linear eigenvalue statistics of β matrix models in the multi-cut regime. *Journal of Statistical Physics*, 151(6):1004–1034, 2013.

[78] M. Shcherbina. Change of variables as a method to study general β-models: bulk universality. *Journal of Mathematical Physics*, 55(4):043504, 2014.

[79] T. Tao. The asymptotic distribution of a single eigenvalue gap of a Wigner matrix. *Probability Theory and Related Fields*, 157(1-2):81–106, 2013.

[80] T. Tao and V. Vu. Random matrices: Universality of local eigenvalue statistics up to the edge. *Communications in Mathematical Physics*, 298(2):549–572, 2010.

[81] T. Tao and V. Vu. Random matrices: universality of local eigenvalue statistics. *Acta mathematica*, 206(1):127–204, 2011.

[82] C. A. Tracy and H. Widom. Level-spacing distributions and the airy kernel. *Communications in Mathematical Physics*, 159(1):151–174, 1994.

[83] Tricomi. *Integral equations*. Dover Publications, Inc., New York, 1985. Reprint of the 1957 original.

[84] B. Valkó and B. Virág. Continuum limits of random matrices and the Brownian carousel. *Inventiones mathematicae*, 177(3):463–508, 2009.

[85] B. Valkó and B. Virág. The sine β operator. *Inventiones mathematicae*, 209(1):275–327, 2017.

[86] C. Villani. *Topics in optimal transportation*. Number 58. American Mathematical Soc., 2003.

[87] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions i. In *The Collected Works of Eugene Paul Wigner*, pages 524–540. Springer, 1993.