Two-Dimensional Hybrid Particle-In-Cell Modeling of Hall Thrusters

by

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Submitted to the Department of Aeronautics and Astronautics
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Abstract

A two dimensional numerical simulation was written for the acceleration zone and plume of a Hall thruster, and the results were compared with experimental measurements. This code was constructed to help understand the operation of Hall thrusters, and to be used as a design tool for next-generation thrusters. The model assumes quasineutrality, Maxwellian electrons, and Bohm diffusion across the magnetic field lines. Heavy particles are simulated directly with a Particle-In-Cell (PIC) method, while electrons are modeled as a fluid continuum. A time-accurate electron energy equation is used to determine electron temperature, and a generalized Ohm's Law is used to determine the electric field strengths. Results indicate a strong correlation with experimental performance data. In particular, the simulation is able to accurately predict wall erosion rates, thrust, torque, power, and efficiency. Two-dimensional plasma distributions are similar to experiment, but do not match in all cases.

Thesis Supervisor: Manuel Martinez-Sanchez
Title: Professor
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Chapter 1

Introduction

1.1 Electric Propulsion

Electric space propulsion systems use electric power to accelerate propellant and provide thrust. Since electric propulsion systems do not necessarily rely on chemical energy, they can provide large amounts of kinetic energy to relatively small amounts of propellant, therefore achieving high specific impulses at low thrust. Several types of space missions are well suited for this type of system, especially long-duration missions with high velocity changes.

There are several types of electric propulsion systems, the main categories being electrothermal, electromagnetic, and electrostatic. Electrothermal thrusters, such as arcjets, use electric power to heat a working gas which is subsequently accelerated upon expansion. Electrostatic thrusters, such as ion engines, accelerate ionized propellant across an electric potential difference to achieve high exit velocities. Electromagnetic thrusters accelerate a body of ionized gas by the interaction of currents driven through the gas with magnetic fields established either by those fields or by other means [14]. Hall thrusters fall into this category. They use the interaction of electron currents and an imposed magnetic field to maintain a potential difference which accelerates ions.

1.2 Hall Thrusters

Hall thrusters ionize a working gas, usually Xenon, and accelerate it across a potential difference to produce thrust. Electrons are released at the cathode and stream toward the anode, which is held at high potential. The interaction of the electron motion with an
imposed radial magnetic field causes the electrons to drift in the $\vec{E} \times \vec{B}$ direction, which is azimuthal. Gas propellant (Argon or Xenon) is injected through holes in the cathode and ionized by collisions with high-energy electrons. The ions are accelerated toward the exit due to the electric field applied between the anode and the cathode. The Larmor radius of the ions is large, so they experience very little curvature and do not drift. In order to maintain charge neutrality, the cathode must also emit enough electrons to fully neutralize the exiting ion beam. Figure 1-1 shows the basic layout of a Hall thruster, and Figure 1-2 shows the engineering sketches of two models.

![Figure 1-1: Basic Hall thruster diagram – side view.](image)

Typical Hall thruster operating characteristics are given in Table 1.1. Hall Thrusters are comparable to ion engines in performance and specific impulse. Lifetime of production models is limited to approximately 4,000 hours due to accelerator wall erosion from ion impact [7].
Figure 1-2: Two hall thrusters. (a) Russian SPT-100 [6]. (b) Japanese Type II [16].

<table>
<thead>
<tr>
<th>Specific Impulse</th>
<th>1000 - 1800 sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Efficiency</td>
<td>15 - 50 %</td>
</tr>
<tr>
<td>Thrust</td>
<td>.01 - .1 N</td>
</tr>
<tr>
<td>Mass Flow Rate</td>
<td>1 - 5 mg/sec</td>
</tr>
<tr>
<td>Anode Current</td>
<td>1 - 5 A</td>
</tr>
<tr>
<td>Anode Potential</td>
<td>120 - 500 Volts</td>
</tr>
<tr>
<td>Magnetic Field</td>
<td>.01 - .1 Tesla</td>
</tr>
<tr>
<td>Service Life</td>
<td>1000 - 4000 hours</td>
</tr>
</tbody>
</table>

Table 1.1: Typical Hall thruster operating characteristics.
1.3 Status of Hall Thruster Research

Hall thrusters were first developed experimentally in the United States by Brown and Pinsley [8] and later by Janes and Lowder [15]. However, severe plasma oscillations led to the abandonment of U.S. Hall thruster efforts. Russian experiments began in the early 1960's with a concept thruster designed by A. I. Morozov. Since then, experimental work has continued in Russia, resulting in papers by Morozov [22, 23, 24], Bugrova [9, 10], Smirnov [25], Bishaev [4, 5], Esipchuk [12], and Zubkov [26] to name a few. In addition to ground tests, over fifty Hall thrusters were launched on Soviet Meteor satellites for station keeping.

In 1992, Lentz [19] used a one-dimensional numerical model to accurately predict the operating characteristics and plasma parameters in the acceleration zone of a Japanese Type II Hall Thruster. The assumptions of the Lentz model included quasineutrality, Bohm diffusion across the magnetic field, constant ratio of ionization energy loss to total electron energy loss, and fixed magnetic field.

1.4 Overview of This Research

Due to the success of the Lentz one-dimensional model, this research extends the model to two dimensions, using similar assumptions. The physical dimensions are considered an input to the numerical model, so any actual Hall thruster geometry may be used, as well as concept designs. A computational grid is mapped to physical space using nonuniform mapping techniques common in computational fluid mechanics. The magnetic field is pre-computed from the iron pole locations and solenoid strengths. Electrons are modeled as a Maxwellian Fluid, while the heavy species are treated with a modified Particle-In-Cell (modified PIC) methodology. Collisionality is limited to electron-neutral ionization and ions-neutral momentum exchange. The overall scheme may be called “hybrid-PIC” since both fluid and PIC methods are used self-consistently.

The analytical model is detailed in Chapter 1. Chapter 3 describes the numerical method. Chapter 4 shows the results and compares them to available experimental data. Conclusions and recommendations for future work are given in Chapter 5.
Chapter 2

Governing Equations

2.1 Magnetic Field

If the local current density is small, the induced magnetic field will be small compared to the externally imposed field, and $\vec{B}$ can be considered constant. A magnetic potential function, $\sigma$ can then be defined as,

$$\vec{B} = \nabla \sigma.$$  \hfill (2.1)

Using the conservation form of Gauss's law, the governing equation of a magnetic field in a vacuum can be written as Laplace's equation,

$$\nabla^2 \sigma = 0.$$ \hfill (2.2)

Or, in cylindrical coordinates,

$$\frac{\partial^2 \sigma}{\partial z^2} + \frac{\partial^2 \sigma}{\partial r^2} + \frac{1}{r} \frac{\partial \sigma}{\partial r} = 0.$$  \hfill (2.3)

Also, since $\nabla \cdot \vec{B} = 0$, it is possible to define a magnetic stream function whose gradient is everywhere orthogonal to $\vec{B}$. One such stream function is $\lambda$, given by,

$$\frac{\partial \lambda}{\partial z} = r \frac{\partial \sigma}{\partial r} = r B_r,$$ \hfill (2.4)

$$\frac{\partial \lambda}{\partial r} = -r \frac{\partial \sigma}{\partial z} = -r B_z.$$ \hfill (2.5)
If $\hat{n}$ is the distance normal to the magnetic field lines, as shown in Figure 2-1, then,

$$\frac{\partial}{\partial \hat{n}} = \frac{\partial \lambda}{\partial \hat{n}} \frac{\partial}{\partial \lambda} = -rB \frac{\partial}{\partial \lambda}. \quad (2.6)$$

2.2 Electron Equations

The electron equations consist of a generalized Ohm's law, a current conservation equation, and an electron temperature equation. Assuming a Maxwellian electron distribution, and given a particular ion field, these three equations are sufficient to yield electron velocity, space potential, and electron temperature as a function of time.

Different assumptions are made along magnetic field lines versus across them. Figure 2-1 shows the notation used to describe coordinate directions with respect to magnetic field lines.

![Figure 2-1: Hall thruster notational diagram: magnetic field lines and their associated curvilinear coordinates; current conventions.](image-url)
2.2.1 Electron Equilibrium Along Lines of Force

The diffusion coefficient of electrons along magnetic field lines is assumed to be much greater than the diffusion coefficient across them. Ignoring the magnetic pinch effect, a balance can be written between the pressure force and the electric force along magnetic field lines:

\[
\frac{\partial (n_e kT_e)}{\partial t} = \epsilon n_e \frac{\partial \phi}{\partial t},
\]

(2.7)

which, assuming constant electron temperature along magnetic field lines, gives a Boltzmann's equilibrium,

\[
\phi - \frac{kT_e}{e} \ln(n_e) = \phi^*(\hat{n}).
\]

(2.8)

Since both sides of Equation 2.8 can only vary across the magnetic field, we can write it in terms of the magnetic stream function, \( \lambda \):

\[
\phi - \frac{kT_e}{e} \ln(n_e) = \phi^*(\lambda).
\]

(2.9)

2.2.2 Ohm's Law Across Lines of Force

Electron diffusion in Hall thruster plasmas is determined by both the electric field and by pressure gradients. Considering only electron-ion collisions, a force balance on electrons may be written as,

\[
\nabla p_e = -\epsilon n_e E + m_e n_e \nu_{ei} (\bar{u}_i - \bar{u}_e),
\]

(2.10)

where \( \nu_{ei} \) is the electron-ion collision frequency. Defining electron mobility,

\[
\mu_e = \frac{e}{m_e \nu_{ei}},
\]

(2.11)

and a diffusion coefficient,

\[
D_e = \frac{kT_e}{m_e \nu_{ei}},
\]

(2.12)

The electron diffusion velocity may be written as,

\[
\bar{U}_e = \bar{u}_e - \bar{u}_i = -\mu_e E - D_e \frac{\nabla p_e}{p_e}.
\]

(2.13)
Using Equations 2.11 and 2.12,

\[ \bar{U}_e = -\mu_e \left( \bar{E} + kT_e \frac{\nabla p_e}{e p_e} \right). \]  

(2.14)

Using \( p = n_e kT_e \),

\[ \bar{U}_e = -\mu_e \left( \bar{E} + \frac{kT_e \nabla n_e}{en_e} + \frac{k \nabla T_e}{e} \right). \]  

(2.15)

Taking only the \( \hat{n} \) component,

\[ U_{e,\hat{n}} = -\mu_e \left( E_{\hat{n}} + \frac{kT_e \partial n_e}{en_e \partial n} + \frac{k dT_e}{e d \hat{n}} \right). \]  

(2.16)

Applying \( \bar{E} = -\nabla \phi \), and \( U_{e,\hat{n}} = u_{e,\hat{n}} - u_{i,\hat{n}} \),

\[ u_{e,\hat{n}} = \mu_e \left( \frac{\partial \phi}{\partial \hat{n}} - \frac{kT_e \partial n_e}{en_e \partial \hat{n}} - \frac{k dT_e}{e d \hat{n}} \right) + u_{i,\hat{n}}. \]  

(2.17)

### 2.2.3 Bohm Diffusion

It is shown [15] that classical \( 1/B^2 \) diffusion is not adequate to describe the high electron transport across lines of force in the presence of a strong magnetic field. An empirical fit, however, shows the diffusion coefficient to have a \( 1/B \) dependence [17] called Bohm diffusion:

\[ D_{Bohm} \approx \frac{kT_e}{16 e B}. \]  

(2.18)

Combining this with Equations 2.11 and 2.12 gives a Bohm mobility,

\[ \mu_e = \frac{1}{16 B}. \]  

(2.19)

Applying the Bohm mobility to Equation 2.17,

\[ u_{e,\hat{n}} = \frac{1}{16 B} \left( \frac{\partial \phi}{\partial \hat{n}} - \frac{kT_e \partial n_e}{en_e \partial \hat{n}} - \frac{k dT_e}{e d \hat{n}} \right) + u_{i,\hat{n}}. \]  

(2.20)

Differentiating Equation 2.9 and rearranging,

\[ \frac{\partial \phi}{\partial \hat{n}} = \frac{d \phi^*}{d \hat{n}} - \frac{kT_e \partial n_e}{en_e \partial \hat{n}} + \frac{k \ln(n_e) dT_e}{e d \hat{n}}. \]  

(2.21)
Substituting into Equation 2.20,

\[ u_{e,\hat{n}} = \frac{1}{16B} \frac{d\phi^*}{dn} + \frac{k}{16eB} (\ln(n_e) - 1) \frac{dT_e}{dn} + u_{i,\hat{n}}. \]  

(2.22)

Rewriting Equations 2.21 and 2.22 using Equation 2.6,

\[ \frac{\partial \phi}{\partial \hat{n}} = -rB \frac{d\phi^*}{d\lambda} + \frac{kT_e}{en_e} \frac{\partial n_e}{\partial \hat{n}} - \frac{rBk \ln(n_e)}{e} \frac{dT_e}{d\lambda}, \]  

(2.23)

\[ u_{e,\hat{n}} = -\frac{r}{16} \frac{d\phi^*}{d\lambda} - \frac{k}{16e} (\ln(n_e) - 1) \frac{dT_e}{d\lambda} + u_{i,\hat{n}}. \]  

(2.24)

### 2.2.4 Current Conservation

Since quasineutrality is imposed, no space charge can accumulate, and current must be conserved for the whole device. Figure 2-1 shows the components of current for a typical Hall thruster. The cathode emits enough electrons \( (I_a) \) to neutralize the ion beam \( (I_b) \) as well as enough to initiate ionization in the acceleration zone \( (I_d) \). The current emitted by the cathode must also be equal to the current collected by the anode, or the entire thruster will charge. A current conservation equation can be written as,

\[ I_a = I_e + I_i, \]  

(2.25)

where \( I_i \) is the ion beam current across the magnetic field at any location in the acceleration zone. In terms of integrals along each magnetic field line,

\[ I_a = 2\pi e \int_0^l n_e u_{e,\hat{n}} r ds - 2\pi e \int_0^l n_i u_{i,\hat{n}} r ds. \]  

(2.26)

Substituting Equation 2.24 into Equation 2.26,

\[ I_a = -2\pi e \int_0^l \frac{n_e}{16} \frac{d\phi^*}{dn} r^2 ds - 2\pi e \int_0^l \frac{n_e k}{16e} (\ln(n_e) - 1) \frac{dT_e}{dn} r^2 ds. \]  

(2.27)

Taking functions of \( \lambda \) outside the integrals,

\[ I_a = -\frac{2\pi e}{16} \frac{d\phi^*}{d\lambda} \int_0^l n_e r^2 ds - \frac{2\pi k}{16} \frac{dT_e}{d\lambda} \int_0^l n_e (\ln(n_e) - 1) r^2 ds. \]  

(2.28)
Rearranging,
\[
\frac{d\phi^*}{d\lambda} = -I_a - \frac{2\pi k}{16} \frac{d\rho_T}{d\lambda} \int_0^1 n_e (\ln(n_e) - 1) r^2 ds \cdot \frac{2\pi e}{16} \int_0^1 n_e r^2 ds.
\] (2.29)

2.2.5 Electron Energy

By taking the second and third moment of the Boltzmann Equation, fluid relations for conservation of electron momentum and energy can be obtained as,
\[
\frac{\partial(n_e m_e \bar{u}_e)}{\partial t} + \nabla \cdot (n_e m_e \bar{u}_e \bar{u}_e) + \nabla \cdot \bar{p}_e = \sum_r (\bar{M}_{re}),
\] (2.30)

and,
\[
\frac{\partial(\frac{3}{2} n_e k T_e)}{\partial t} + \nabla \cdot \left( \frac{3}{2} n_e \bar{u}_e k T_e + \bar{q}_e \right) + \bar{\cdot} (p_e: \nabla \bar{u}_e) \cdot \bar{u}_e = \sum_r (E_{re} - \bar{u}_e \cdot \bar{M}_{re}).
\] (2.31)

The term \( \bar{q}_e \) in Equation 2.31 is the thermal conduction vector,
\[
\bar{q}_e = -K_e \nabla T_e,
\] (2.32)

where \( K_e \) is the heat conduction coefficient. Considering only Bohm diffusion, this is shown to be \cite{19},
\[
K_e = \frac{k^2 n_e T_e}{2\pi e B}.
\] (2.33)

The right hand sides of Equations 2.30 and 2.31 represent the net momentum and energy transfer rates to the electrons, respectively. The formulation can be simplified by the following assumptions:

- Electrons have a Maxwellian velocity distribution
- The pressure dyad reduces to a scalar pressure term, \( n_e k T_e \)
- Energy sources can be lumped into a single term, \( S \)

Using these assumptions, and adding Equation 2.31 to the dot product of Equation 2.30 with \( \bar{u}_e \),
\[
\frac{\partial}{\partial t} \left( \frac{3}{2} n_e k T_e + n_e m_e \bar{u}_e^2 \right) + \nabla \cdot \left( \frac{5}{2} n_e \bar{u}_e k T_e + n_e m_e \bar{u}_e \bar{u}_e \frac{u_e^2}{2} + \bar{q}_e \right) = S.
\] (2.34)
It is found that, for Hall thrusters, the directed kinetic energy term is smaller than the random kinetic energy term. Therefore, Equation 2.34 reduces to,

$$\frac{\partial}{\partial t} \left( \frac{3}{2} n_e kT_e \right) + \nabla \cdot \left( \frac{5}{2} n_e \bar{u}_e kT_e + \bar{q}_e \right) = S.$$  \hspace{1cm} (2.35)

### 2.2.6 Source Terms

The electron energy source term, $S$ in Equation 2.35, consists of:

- Losses due to ionization
- Losses due to radiation
- Sources from the electric field

Considerable energy is lost to electronic excitation of neutral atoms. The net energy cost for producing a single ion, $\varphi$, can be expressed as the sum of the energy required for ionization, plus the energy lost to excitation of neutral atoms.

$$\varphi = E_i + \frac{\sum_j (Q_j(u_e)u_e)E_j}{(Q^+(u_e)u_e)},$$  \hspace{1cm} (2.36)

where $E_i$ is the first ionization energy, $Q_j(u_e)$ is the cross section for excitation from the ground state to energy level $E_j$, and $Q^+(u_e)$ is the cross section for ground state ionization. A complicated analytical expression for $\varphi$ is derived by Dugan et. al. [11]. The result can be fitted closely as,

$$\varphi' = Ae^{-\frac{B}{z}} + C,$$  \hspace{1cm} (2.37)

where $\varphi'$ and $z$ are normalized ion production cost and dimensionless electron kinetic temperature,

$$\varphi' = \frac{\varphi}{E_i},$$  \hspace{1cm} (2.38)

$$z = \frac{kT_e}{E_i}.$$  \hspace{1cm} (2.39)

The constants $A$, $B$, and $C$ are given in Table 2.1, and $\varphi'$ is plotted in Figure 2-2 versus dimensionless electron kinetic temperature, $z = \frac{kT_e}{E_i}$. Using the normalized ion production cost, the volumetric electron energy loss rate can then be given as,

$$S_1 = \dot{n}_e \varphi' E_i.$$  \hspace{1cm} (2.40)
<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argon</td>
<td>0.188</td>
<td>0.624</td>
<td>1.75</td>
</tr>
<tr>
<td>Xenon</td>
<td>0.254</td>
<td>0.677</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Table 2.1: Constants to fit the Dugan ion production cost model.

Figure 2.2: Normalized ion production cost, $\varphi' = \frac{\varphi}{\bar{E}_i}$, versus dimensionless electron kinetic temperature, $z = \frac{kT_e}{\bar{E}_i}$, for argon and xenon.
The electron energy source due to collisions with other species is given by,

\[
S_2 = \sum_r m_e n_e \nu_{er}^* \left[ (\bar{u}_r - \bar{u}_e)^2 + \frac{2}{m_r} \frac{3}{2} k (T'_r - T'_e) \right],
\]  

(2.41)

where \( \nu_{er}^* \) is the average momentum transfer collision frequency of electrons with species \( r \), \( \bar{u}_r \) is the mean velocity of species \( r \), and \( T'_r \) is the temperature associated with random velocities with respect to the mean velocity of species \( r \). The first term in Equation 2.41 is dominant. Considering only electron-ion interactions (via the random fluctuations that originate the finite resistivity in the Ohm's Law for electrons), Equation 2.41 may be written as,

\[
S_2 = m_e \nu_{ei}^* n_e (\bar{u}_i - \bar{u}_e)^2 = \frac{j^2}{\sigma},
\]  

(2.42)

where, \( j = e n_e (\bar{u}_i - \bar{u}_e) \) is the total current density, and \( \sigma \) is the electrical conductivity,

\[
\sigma_e = \frac{e^2 n_e}{m_e \nu_e^*} = en_e \mu_e.
\]  

(2.43)

For Bohm diffusion, Equation 2.19 may be used, giving,

\[
\sigma_e = \frac{en_e}{16B}.
\]  

(2.44)

and,

\[
S_2 = \frac{16 Bj^2}{en_e}.
\]  

(2.45)

All together,

\[
S = -S_1 + S_2.
\]  

(2.46)

Explicitly,

\[
S = -\dot{\mathbf{n}}_e \mathbf{v}' E_i + \frac{16 Bj^2}{en_e}.
\]  

(2.47)

2.2.7 Discussion of Alternative Forms of Electron Energy Sources

An vector form of the electron momentum balance may be written as,

\[
\vec{j} + \vec{j} \times \vec{\beta} = \sigma \left( \vec{E}' + \frac{\nabla p}{en_e} \right),
\]  

(2.48)
where,
\[ \beta = \frac{eB}{m_e \nu_e}, \quad (2.49) \]

and,
\[ \vec{B}' = \vec{B} + \vec{u}_i \times \vec{B}. \quad (2.50) \]

Using these relations, Equation 2.42 may be written as,
\[ S_2 = \vec{j}_e \cdot \vec{E} + e n_e \vec{u}_i \cdot (\vec{E} + \vec{u}_e \times \vec{B}) + (\vec{u}_i - \vec{u}_e) \cdot \nabla p_e. \quad (2.51) \]

An effective Hall parameter can be written for the case of Bohm diffusion as,
\[ \beta = \frac{1}{16}, \quad (2.52) \]

which implies that the azimuthal component of electron velocity is small compared to the axial component. Neglecting the azimuthal component in Equation 2.51,
\[ S_2 = \vec{j}_e \cdot \vec{E} + e n_e \vec{u}_i \cdot \vec{E} + (\vec{u}_i - \vec{u}_e) \cdot \nabla p_e. \quad (2.53) \]

This must be equivalent to,
\[ S_2 = m_e v_{i, e} n_e (\vec{u}_i - \vec{u}_{e, n})^2. \quad (2.54) \]

The form actually used in the simulation, however, is just the first term from Equation 2.51,
\[ S_2 = j_{e, n} \cdot \vec{E} = j_{e, n} E_n. \quad (2.55) \]

This assumes the second and third terms are small and cancelling, which is true for most regions in the Hall thruster. A more accurate approach is saved for the future.

### 2.3 Heavy Species

Heavy species are modeled as discrete particles with negligible temperature. Therefore, only conservation of mass and momentum are applicable. Recombination and charge exchange are found to be small and both are neglected. The ion-neutral collision cross section for argon and xenon are given in table 2.2. Using these values with experimental Hall thruster
<table>
<thead>
<tr>
<th></th>
<th>$Q_{i,n}[m^2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argon</td>
<td>1.40e-18</td>
</tr>
<tr>
<td>Xenon</td>
<td>2.15e-18</td>
</tr>
</tbody>
</table>

Table 2.2: Ion-neutral momentum exchange cross sections [20].

plasma densities, the mean free path for both ions and neutrals is found to be large in most regions. Therefore, ion-neutral momentum exchange is neglected.

2.3.1 Ions

The magnetic part of the Lorentz force is ignored, since the Larmor radius for ions is large. Therefore, the force on an ion is given by,

$$
\vec{F}_i = e\vec{E}.
$$

(2.56)

2.3.2 Neutrals

Neutrals, being uncharged, only experience velocity changes if they encounter walls. Their mean free path is large compared to the scale of the device.

2.4 Ionization Rate

The ionization rate, $\dot{n}_i$, is the rate at which ions are created per unit volume. In this model, it is assumed that only electron-neutral collisions can produce ions. Also, it is assumed that only one degree of ionization may exist. It may be important to relax this assumption, but that task is saved for the future. For now,

$$
\dot{n}_i = \dot{n}_e.
$$

(2.57)

The ion production rate is given by,

$$
\dot{n}_e = n_n \int_0^\infty f_ec_e^3 4\pi \sigma_i(c_e)dc_e,
$$

(2.58)
<table>
<thead>
<tr>
<th></th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( Q \text{[m}^3\text{Joule}^{-1}\text{s}^{-1]} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argon</td>
<td>0.82</td>
<td>1.00</td>
<td>( 2.77 \times 10^{-13} )</td>
</tr>
<tr>
<td>Xenon</td>
<td>1.00</td>
<td>0.80</td>
<td>( 4.13 \times 10^{-13} )</td>
</tr>
</tbody>
</table>

Table 2.3: Constants for the Drawin ionization model with Maxwellian electrons.

where \( \sigma_i(c_e) \) is the ionization cross section, and \( f_e \) is the Maxwellian electron velocity distribution function,

\[
f_e = n_e \left( \frac{m_e}{2\pi kT_e} \right)^{3/2} e^{-\frac{m_e c_e^2}{2 kT_e}}. \tag{2.59}
\]

The nonelastic ionization cross section is approximated according to Drawin [21] as a function of electron energy,

\[
\sigma_i = 2.66 \pi a_0^3 \beta_1 \left( \frac{e_i}{e_i} \right)^2 \xi g(u), \tag{2.60}
\]

where,

\[
g(u) = \frac{u - 1}{u^2} \ln(1.25 \beta_2 u). \tag{2.61}
\]

Here \( e_i^H \) is the ionization energy of hydrogen, \( e_i \) is the first ionization energy of the heavy species, \( \xi \) is the number of equivalent electrons in the outer shell, and \( \beta_1 \) and \( \beta_2 \) are adjustable constants of order unity. Comparing with empirical data, \( \beta_1 \) and \( \beta_2 \) can be found iteratively, and are given in Table 2.3. Substituting Equations 2.59 and 2.61 into Equation 2.58 [19],

\[
\dot{n}_e = n_e n_n Q \beta_1 \frac{I(\theta)}{\theta_s^2}, \tag{2.62}
\]

where,

\[
I(\theta) = \int_1^\infty e^{-\frac{u - 1}{u}} \ln(1.25 \beta_2 u) du. \tag{2.63}
\]

Here \( \theta = kT_e/E_i \), and \( Q \) is a constant shown in Table 2.3. Considering only random kinetic energy, the ionization rate may be written as,

\[
\dot{n}_e = n_e n_n \zeta(T_e), \tag{2.64}
\]

where, from Equation 2.62,

\[
\zeta(T_e) = Q \beta_1 \frac{I(\theta)}{\theta_s^2}. \tag{2.65}
\]

The quantity \( \zeta \) is plotted for argon and xenon versus electron temperature in Figure 2-3.
Figure 2-3: Ionization rate parameter, $\zeta \left[ m^3 \cdot s^{-1} \right]$, for argon and xenon.
2.5 Boundary Conditions

To determine the electron energy loss to the walls, it is assumed that the electron flux is equal to the ion flux normal to the wall, and that all ions recombine there with no secondary electron emission. Therefore, the electron energy flux to the wall is simply,

\[ E_w = 2kT_e \Gamma_{i,w} \tag{2.66} \]

All neutrals are assumed to deflect off the walls elastically, at random angles. Likewise, ions recombine at the wall to form neutrals with random direction and velocity magnitudes equal to the impinging ion’s.

The cathode and anode are assumed to be at a constant potential difference. Also, the electron temperature at the cathode is fixed, based on experimental data, to 2eV. The electron temperature at the anode is assumed to have zero slope. The inner and outer walls of the thruster are assumed to be electrically insulating.

Neutrals are assumed to enter the acceleration zone at the injector. The injector is modeled as an annular ring in the center of the back wall (anode) of the acceleration zone. The total area of the injector is set roughly to the injector area of the device being modeled. However, precise data is not available for the SPT, so the injector area is set to approximately 30

Neutrals are introduced with an axial velocity equal to the speed of sound \( \left( \sqrt{\frac{5kT_a}{3m_n}} \right) \) at the anode temperature (1000 °K for the SPT). The radial velocity component for each neutral is taken to be a random value between \(-|\vec{v}_z|\) and \(|\vec{v}_z|\), where \(|\vec{v}_z| = \sqrt{\frac{2kT_a}{\pi m_n}}\) is the mean directional velocity for a Maxwellian fluid.

2.6 Power Control

The thruster is assumed to have an ideal power supply which maintains constant discharge potential while allowing the discharge current to vary. However, if the power level exceeds a given point (1 kW for the SPT operating at 200 Volts), it goes into a power limiting mode which drops the discharge potential accordingly.
2.7 Performance Parameters

The axial thrust is given by a summation of the external forces over all species,

\[ F = \sum_r \ddot{z} \cdot \vec{F}_r. \]  \hfill (2.67)

The net axial electrostatic force on electrons returning to the anode is assumed balanced by the magnetic force, \( ev_B r \). Also, the collisional force between ions and neutrals cancels. Therefore, only the electrostatic force on ions is considered,

\[ F = F_i \bar{E}. \]  \hfill (2.68)

The specific impulse can be written as,

\[ I_{SP} = \frac{F}{mg}. \]  \hfill (2.69)

The overall thrust efficiency, \( \eta \), is given as the jet power divided by the input power, or,

\[ \eta = \frac{F^2}{2m \dot{m}_a \phi_a}, \]  \hfill (2.70)

where \( I_a \) is the anode current, and \( \phi_a \) is the anode potential. The overall efficiency can be broken down into three components [16],

\[ \eta = \eta_u \eta_a \eta_e, \]  \hfill (2.71)

where \( \eta_u \) is the propellant utilization efficiency, \( \eta_a \) is the acceleration efficiency, and \( \eta_e \) is the beam energy efficiency. These are given by,

\[ \eta_u = \frac{\dot{m}_i}{\dot{m}}, \]  \hfill (2.72)

\[ \eta_a = \frac{I_b}{I_a}, \]  \hfill (2.73)

\[ \eta_e = \frac{e}{2m_i \phi_a} \left( \frac{F}{I_b} \right)^2, \]  \hfill (2.74)

where \( \dot{m}_i \) is the ion beam mass flow rate, and \( I_b \) is the ion beam current.
Chapter 3

Numerical Method

3.1 Overall Scheme

A time-accurate solution to the governing equations can be achieved by separating the slow time scale (ion and neutral) motion from the fast time scale (electron) motion, and iterating successively, as shown in Figure 3-1. Individual ion and neutral atoms are simulated using a Particle-In-Cell method as described in Section 3.5. The electron motion is modeled as a fluid continuum with the equations derived in Section 2.2.

3.2 Grid Generation

In order to easily reference nodes in space, and to simplify boundary conditions, a nonuniform spatial grid is used. As shown in Figure 3-2, the nonuniform spatial grid is generated with \((z,r)\)-coordinates which contour the walls of the acceleration zone and parts of the plume. This grid maps directly to a uniform computational grid in \((\xi, \eta)\)-space, such that the inner boundary of the device is always at \(\eta = 0\), the left boundary (anode) is always at \(\xi = 0\), and the top boundary is always at \(\eta = \eta_{\text{max}}\). The right hand side of the grid contacts the downstream region, which is not modeled. For simplicity, the distance between nodes in computational space is set to one unit in each direction.

Rotational symmetry is assumed. Therefore, only a meridional section of the Hall thruster acceleration zone is modeled.

Grid spacing is determined by the timestep of the simulation. It is set to the smallest value which is much larger than the maximum distance traveled by a particle in one timestep.
Figure 3-1: Overview of the numerical method.
Figure 3-2: Spatial grid for the SPT-100 geometry.
The grid generation is accomplished by selecting the boundary nodes and solving an
elliptic partial differential equation iteratively for the interior node locations. This method,
commonly used in computational fluid dynamics, generates a smooth grid which automatically
contours the boundaries.

3.3 Magnetic Field Solution

The magnetic field is generated by specifying the geometry of the iron poles, assuming
infinite permeability of the iron poles, and solving Laplace’s equation on the regions exterior
to the poles. The coils are assumed to be perfect solenoids, so the problem reduces to that
of potential flow, with each pole piece set to a given magnetic potential.

The solution of Laplace’s equation is performed using the method of red-black ordering,
which works as follows. A uniform grid is divided into alternating “red” and “black” nodes,
where no red node is directly adjacent to any black node, as shown in Figure 3.3. During
each iteration, the following two steps are performed:

- Red nodes are updated using a central four-point stencil of black nodes.
- Black nodes are updated using a central four-point stencil of red nodes.

Boundary nodes in the iron core have a fixed potential and are not updated during iteration.

This method is simple to implement and accepts any geometry that can be represented
on a uniform grid. Convergence is slow, but the computation is considered pre-processing.
It is only performed when the thruster geometry changes. With a 400 × 400 grid, three
hours on a DEC 5000 is adequate for reasonable accuracy. Figure 4.5 shows results for the
SPT geometry.

Once σ is known, Equation 2.1 can be used to find the magnetic field,

\[ B_z = \frac{\partial \sigma}{\partial z}, \quad (3.1) \]
\[ B_r = \frac{\partial \sigma}{\partial r}. \quad (3.2) \]

Likewise, the magnetic stream function, λ may be found from Equation 2.5,

\[ \frac{\partial \lambda}{\partial z} = rB_r, \quad (3.3) \]
Figure 3-3: Graphical representation of the method of red-black ordering.
\[ \frac{\partial \lambda}{\partial r} = -rB_z. \] (3.4)

Equations 3.1 through 3.4 are discretized directly using central differences. Then, \( \sigma \) and \( \lambda \) are determined in one pass. Additionally, the magnetic field strength is found by,

\[ B = \sqrt{B_z^2 + B_r^2}. \] (3.5)

Once all of the relevant magnetic field parameters are determined for the geometry in question, the values are transferred to the nonuniform spatial grid (Figure 3-2) using bilinear interpolation. Finally, all magnetic field parameters are normalized by a constant to give a specified magnetic field strength at a control point in the acceleration zone. For the SPT-100, \( B = 0.018 \text{ Tesla} \) at the midpoint of the acceleration zone.

### 3.4 Integration of the Electron Equations

It is possible to combine the electron temperature equation with Ohm's Law and the electron current equation. The resulting expression is a function of slow time parameters and of the electron temperature. The slow time parameters are taken to be those related to heavy particle motion \( (n_e, n_n \text{ and } u_i) \), and those fixed geometrically \( (B, r) \). Holding the slow parameters constant, this equation is solved time-accurately for electron temperature using MacCormack's method. The space potential can then be found on the whole domain by using Boltzmann equilibrium and Ohm's Law.

#### 3.4.1 Quasi-One-Dimensional Electron Energy Equation

The electron energy equation (Equation 2.35) is,

\[ \frac{\partial}{\partial t} \left( \frac{3}{2} n_e kT_e \right) + \nabla \cdot \left( \frac{5}{2} n_e u_e kT_e + q_e \right) = -\dot{n}_e \varphi E_i + \dot{j}_e n E_A. \] (3.6)

However, it is assumed in Section 2.2 that electron temperature is constant along magnetic field lines. Also, the electrons are assumed to be in equilibrium along magnetic field lines, such that the mean tangential electron velocity, \( u_e, i \) is zero. Therefore, Equation 3.6 may
be written as,

$$\frac{\partial}{\partial t} \left( \frac{3}{2} n_e kT_e \right) + \frac{\partial (\frac{5}{2} n_e \bar{v}_e kT_e + q_e \hat{n})}{\partial \hat{n}} = -\hat{n}_e \varphi' E_i + j_{e,\hat{n}} E_{\hat{n}}. \quad (3.7)$$

Expanding the differentials and rearranging,

$$\frac{3}{2} n_e k \frac{\partial T_e}{\partial t} + 5 k_n e u_{e,\hat{n}} \frac{\partial T_e}{\partial \hat{n}} + \frac{3}{2} k_{T_e} \frac{\partial n_e}{\partial t} + 5 k_{T_e} \frac{\partial (n_e u_{e,\hat{n}})}{\partial \hat{n}} + \frac{\partial q_e \hat{n}}{\partial \hat{n}} = -\hat{n}_e \varphi' E_i + j_{e,\hat{n}} E_{\hat{n}}. \quad (3.8)$$

By electron and ion continuity, the third and fourth terms in Equation 3.8 can be written as,

$$\frac{3}{2} k_{T_e} \frac{\partial n_e}{\partial t} + 5 k_{T_e} \frac{\partial (n_e u_{e,\hat{n}})}{\partial \hat{n}} = \frac{3}{2} k_{T_e} \hat{n}_e + \nabla \cdot (n_e \bar{u}_i). \quad (3.9)$$

Therefore,

$$\frac{3}{2} n_e k \frac{\partial T_e}{\partial t} + 5 k_n e u_{e,\hat{n}} \frac{\partial T_e}{\partial \hat{n}} + \frac{3}{2} k_{T_e} \hat{n}_e + k_{T_e} \frac{\partial (n_e u_{i,\hat{n}})}{\partial \hat{n}} + \frac{\partial q_e \hat{n}}{\partial \hat{n}} = -\hat{n}_e \varphi' E_i + j_{e,\hat{n}} E_{\hat{n}}. \quad (3.10)$$

### 3.4.2 Expansion in Terms of Electron Temperature

During integration of the electron temperature equation, Equation 3.10, “slow” parameters such as $n_e$ and $\bar{u}_i$ are considered constant. Therefore, by writing it in terms of powers and derivatives of $T_e$, it may be approached with a standard discretization technique and solved iteratively. The terms in Equation 3.10 which are not yet explicitly in terms of $T_e$ are $\frac{\partial \phi}{\partial \hat{n}}$, $u_{e,\hat{n}}$, $\hat{n}_e$, and $\varphi'$. The potential gradient and electron velocity normal to magnetic field lines are given by Equations 2.23 and 2.24 as,

$$\frac{\partial \phi}{\partial \hat{n}} = -r B \frac{d \phi^*}{d \lambda} + \frac{k T_e}{e n_e} \frac{\partial n_e}{\partial \hat{n}} - \frac{r B k}{e} \ln (n_e) \frac{d T_e}{d \lambda}, \quad (3.11)$$

$$u_{e,\hat{n}} = -\frac{r}{16} \frac{d \phi^*}{d \lambda} - \frac{k T_e}{16 e} (\ln (n_e) - 1) \frac{d T_e}{d \lambda} + u_{i,\hat{n}}. \quad (3.12)$$

These can also be written as,

$$\frac{\partial \phi}{\partial \hat{n}} = F_1 \frac{d \phi^*}{d \lambda} + F_2 \frac{d T_e}{d \lambda} + F_3 T_e, \quad (3.13)$$

$$u_{e,\hat{n}} = E_1 \frac{d \phi^*}{d \lambda} + E_2 \frac{d T_e}{d \lambda} + E_3, \quad (3.14)$$
where,

\[ F_1 = -rB, \]  \hspace{1cm} (3.15) \\
\[ F_2 = -\frac{rBk\ln(n_e)}{e}, \]  \hspace{1cm} (3.16) \\
\[ F_3 = \frac{k}{en_e} \frac{\partial n_e}{\partial \hat{n}}, \]  \hspace{1cm} (3.17) \\
\[ E_1 = -\frac{r}{16}, \]  \hspace{1cm} (3.18) \\
\[ E_2 = -\frac{kr}{16e} (\ln(n_e) - 1), \]  \hspace{1cm} (3.19) \\
\[ E_3 = u_{i,\hat{n}}. \]  \hspace{1cm} (3.20)

Likewise, from Equation 2.29,

\[ \frac{d\phi^*}{d\lambda} = H_1 + H_2 \frac{dT_e}{d\lambda}, \]  \hspace{1cm} (3.21)

where,

\[ H_1 = \frac{-16I_a}{2\pi e \int_0^l n_\epsilon r^2 ds}, \]  \hspace{1cm} (3.22) \\
\[ H_2 = \frac{-k \int_0^l n_\epsilon r^2 (\ln n_\epsilon - 1) ds}{e \int_0^l n_\epsilon r^2 ds}. \]  \hspace{1cm} (3.23)

The heat flux term in Equation 3.10 is,

\[ \frac{\partial q_{e,\hat{n}}}{\partial \hat{n}} = \frac{\partial}{\partial \hat{n}} \left(-K e \frac{\partial T_e}{\partial \hat{n}}\right). \]  \hspace{1cm} (3.24)

This can be expanded as,

\[ \frac{\partial q_{e,\hat{n}}}{\partial \hat{n}} = K_1 T_e \frac{\partial^2 T_e}{\partial \lambda^2} + K_2 T_e \frac{\partial T_e}{\partial \lambda} + K_3 \left(\frac{\partial T_e}{\partial \lambda}\right)^2, \]  \hspace{1cm} (3.25)

where,

\[ K_1 = -\frac{k^2 n_\epsilon r^2 B}{2\pi e}, \]  \hspace{1cm} (3.26) \\
\[ K_2 = \frac{k^2}{2\pi e} \left(\frac{n_\epsilon}{B} \frac{\partial r B}{\partial \hat{n}} + rB \frac{\partial n_e}{\partial \hat{n}}\right), \]  \hspace{1cm} (3.27) \\
\[ K_3 = K_1. \]  \hspace{1cm} (3.28)
3.4.3 Linearization of Terms

Two terms in Equation 3.10 are very nonlinear in $T_e$ and must be linearized for stability in the numerical method. The ionization rate, $\dot{n}_e$, and the normalized ion production cost, $\varphi'$, are written as the first two terms of the Taylor series expansion about $T_{e,0}$,

\[\dot{n}_e = \dot{n}_e(T_{e,0}) + \left(\frac{\partial \dot{n}_e}{\partial T_e}\right)_{T_e = T_{e,0}} (T_e - T_{e,0}), \tag{3.29}\]

\[\varphi' = \varphi'(T_{e,0}) + \left(\frac{\partial \varphi'}{\partial T_e}\right)_{T_e = T_{e,0}} (T_e - T_{e,0}). \tag{3.30}\]

3.4.4 Expanded Form of the Electron Energy Equation

By applying the equations in Sections 3.4.2 and 3.4.3 to Equation 3.10 and grouping like terms, a consolidated electron energy equation may be written of the form,

\[\frac{\partial T_e}{\partial t} + I \frac{\partial T_e}{\partial \lambda} + J \left(\frac{\partial T_e}{\partial \lambda}\right)^2 + K \frac{\partial T_e}{\partial \lambda} T_e + LT_e + MT_e^2 + N + OT_e \frac{\partial^2 T_e}{\partial \lambda^2} = 0. \tag{3.31}\]

The expressions $I$, $J$, $K$, $L$, $M$, $N$ and $O$ contain only constants and “slow” parameters. Equation 3.31 may be integrated along lines of force as,

\[\int_0^t \left[\frac{\partial T_e}{\partial t} + I \frac{\partial T_e}{\partial \lambda} + J \left(\frac{\partial T_e}{\partial \lambda}\right)^2 + K \frac{\partial T_e}{\partial \lambda} T_e + LT_e + MT_e^2 + N + OT_e \frac{\partial^2 T_e}{\partial \lambda^2}\right] ds = 0. \tag{3.32}\]

Pulling the functions of $\lambda$ out of the integral gives,

\[\frac{\partial T_e}{\partial t} + I \frac{\partial T_e}{\partial \lambda} + J \left(\frac{\partial T_e}{\partial \lambda}\right)^2 + K \frac{\partial T_e}{\partial \lambda} T_e + LT_e + MT_e^2 + \tilde{N} + OT_e \frac{\partial^2 T_e}{\partial \lambda^2} = 0, \tag{3.33}\]

where,

\[\tilde{X} = \int_0^t X ds, \tag{3.34}\]

for $X = I, J, K, L, M, N$, and $O$.

3.4.5 MacCormack’s Method

The solution of Equation 3.33 is accomplished by using a modified MacCormack’s method [2]. MacCormack’s method is a predictor-corrector integration scheme for convective-diffusive equations. It is an explicit method with second order accuracy in time and space.

For application to Equation 3.33, a standard MacCormack’s method is used for the
convective term. However, to account for the other terms, discretized forms are included in both the predictor and corrector steps. The predictor is given by,

\[
P_j = T_{e,j} - \Delta t \left[ \frac{I_j}{\Delta \lambda} (T_{e,j+1}^n + T_{e,j}^n) + \frac{J_j}{(\Delta \lambda)^2} (T_{e,j+1}^n - T_{e,j}^n)^2 \right.
+ \frac{K_j T_{e,j}}{\Delta \lambda} (T_{e,j+1}^n - T_{e,j}^n) + \bar{L}_j T_{e,j}^n
+ \bar{M}_j (T_{e,j}^n)^2 + \bar{N}_j + \frac{O_j T_{e,j}}{(\Delta \lambda)^2} (T_{e,j+1}^n - 2T_{e,j}^n + T_{e,j-1}^n) \right].
\]

(3.35)

And the corrector is given by,

\[
T_{e,j}^{n+1} = \frac{1}{2} \left[ T_{e,j} + \frac{P_j - T_{e,j}}{\Delta \lambda} \left[ \frac{I_j}{\Delta \lambda} (P_j + P_{j-1}) + \frac{J_j}{(\Delta \lambda)^2} (P_j - P_{j-1})^2 \right.ight.
+ \frac{K_j P_j}{\Delta \lambda} (P_j - P_{j-1}) + \bar{L}_j P_j
+ \bar{M}_j (P_j)^2 + \bar{N}_j + \frac{O_j P_j}{(\Delta \lambda)^2} (P_{j+1} - 2P_j + P_{j-1}) \left. \right] \right].
\]

(3.36)

Here, \( j \) is the spatial index and \( n \) is the time index.

### 3.4.6 Timestep

For a general convective-diffusive equation,

\[
U_t + AU_x + BU_{xx} = 0,
\]

(3.37)

MacCormack’s method requires \([2]\),

\[
\Delta t \leq \frac{(\Delta x)^2}{|A| \Delta x + 2B},
\]

(3.38)

for stability based on empirical data. This does not apply directly to Equation 3.33 due to the additional terms. Therefore, the convective-diffusive criterion was used as a starting point for an iterative search for a stable timestep. This was found to be \(2 \times 10^{-10}\) seconds for \(\Delta \lambda = 2.8 \times 10^{-7} \ T \cdot m^2\).
3.4.7 Space Potential

Once the electron temperature is determined by integrating Equation 3.33, $\phi^*$ may be found by integrating Equation 3.21 using Simpson’s rule. Then, using Equation 2.8, $\phi$ may be found on the entire domain by,

$$\phi = \phi^*(\bar{n}) + \frac{kT_e}{e} \ln(n_e).$$  \hspace{1cm} (3.39)

3.5 Particle-In-Cell Method for Heavy Species

The standard particle-in-cell method [3] is a direct simulation technique which models a gas as discrete particles. Collisions are usually accounted for by a Monte Carlo technique, which computes a probability of collision for each particle for each timestep. Typically, both ions and electrons are modeled with PIC, and Poisson’s equation is solved to find the space potential.

In this simulation, heavy species are modeled time-accurately using a modified Particle-In-Cell (PIC) method. Although based on the standard PIC method, this method has several unique features which enhance performance for the specific problem of simulating a Hall thruster plasma with a nonuniform grid.

3.5.1 Standard PIC Method

To use the standard PIC method in two dimensions, the domain is divided into uniform cells, as shown in Figure 3-4. Then, some initial guess of particle locations and velocities is taken. After initialization, an iterative method begins. In each timestep,

- The particle equations of motion are integrated
- The field vectors are computed

A flow diagram is shown in Figure 3-5.

The particle densities and current densities are found at the nodes by weighting – usually first order area weighting. The electromagnetic fields are then computed at the nodes using these values.

The particle equations of motion are integrated independently, resulting in new positions and velocities. To apply the field vectors at the nodes to particles inside the cell, an inverse-
Figure 3-4: Uniform PIC grid for density, current, and field computations [3].

Figure 3-5: PIC computational cycle [3].
weighting scheme is used. Typically, a leap-frog integration scheme is used for the particle motion equations, so the position and velocity are offset by one half timestep.

3.5.2 Quasineutral Approximation – Hybrid PIC

With conventional PIC methods, both electrons and heavy species are modeled with discrete particles. However, since the Hall thruster plasma is assumed to be quasineutral, Poisson's equation is not used to find the space potential (except in its limiting form to impose \( n_e = n_i \); where \( n_i \) comes from the ion tracking). Instead, ions and neutrals are modeled with PIC, and the electrons are modeled as a fluid continuum. The space potential is a result of imposing Ohm's Law across magnetic field lines, and Boltzmann equilibrium along them, as described in Section 2.2. For this reason, the method is called "hybrid-PIC."

3.5.3 Particle Follower for Nonuniform Grids

Typically, PIC methods employ a uniform grid with dimensions normalized to one unit distance per cell. This simplifies the weighting scheme and the integration of the equations of motion. For nonuniform grids, such as the one used in this simulation, several modifications to the standard PIC method must be made. The mapping between physical space and computational space becomes nontrivial. The transformation from the physical domain \((z, r)\) to the computational domain, \((\xi, \eta)\) must be tabulated for each node as,

\[
\begin{pmatrix}
  d\xi \\
  d\eta
\end{pmatrix} =
\begin{bmatrix}
  \xi_z & \xi_r \\
  \eta_z & \eta_r
\end{bmatrix}
\begin{pmatrix}
  dz \\
  dr
\end{pmatrix},
\]

(3.40)

where the indices, \(j\) and \(k\), refer to nodes in the \(\xi\) and \(\eta\) direction, respectively.

The forces on the particles are known in terms of physical \((z, r)\) coordinates, and the equations of motion contain second derivatives of position in \(z, r\) coordinates. Recasting the equations of motion and the forces in terms of \(\xi, \eta\) coordinates poses many problems. Therefore, a novel algorithm was devised to perform integrations in terms of spatial coordinates, while maintaining computational coordinates for each particle.

The algorithm uses Newton's method to continually refine \(\xi\) and \(\eta\) based on the current \(z, r\) coordinates and a weighting function. A first order weighting function, \(f\), is used. It is
written in terms of the four grid nodes at the corners of the cell containing the particle:

\[ f_w(\xi_{\text{rel}}, \eta_{\text{rel}}) = f_1(1 - \xi_{\text{rel}})(1 - \eta_{\text{rel}}) + f_2\xi_{\text{rel}}(1 - \eta_{\text{rel}}) + f_3\xi_{\text{rel}}\eta_{\text{rel}} + f_4(1 - \xi_{\text{rel}})\eta_{\text{rel}}, \]

(3.41)

where \( \xi_{\text{rel}} \) and \( \eta_{\text{rel}} \) are distances, in computational space, from the particle to the lower left node (see Figure 3-6). Writing the error between \( z \) and \( r \), and the weighting function values for \( z \) and \( r \),

\[ \{F\} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} z(\xi_{\text{rel}}, \eta_{\text{rel}}) - z_w(\xi_{\text{rel}}, \eta_{\text{rel}}) \\ r(\xi_{\text{rel}}, \eta_{\text{rel}}) - r_w(\xi_{\text{rel}}, \eta_{\text{rel}}) \end{pmatrix}. \]

(3.42)

Newton’s method for nonlinear systems is given by the vector form of the first two terms of the Taylor series expansion,

\[ [J] \begin{pmatrix} \Delta \xi_{\text{rel}} \\ \Delta \eta_{\text{rel}} \end{pmatrix} = - \{F\}, \]

(3.43)

where \([J]\) is the Jacobian matrix,

\[ [J] := \begin{bmatrix} \frac{\partial F_1}{\partial \xi_{\text{rel}}} & \frac{\partial F_1}{\partial \eta_{\text{rel}}} \\ \frac{\partial F_2}{\partial \xi_{\text{rel}}} & \frac{\partial F_2}{\partial \eta_{\text{rel}}} \end{bmatrix}. \]

(3.44)

At each iteration of the PIC method, the error term, Equation 3.42 is computed for each particle. If it is less than some fraction of a grid cell dimension, a single Newton iteration is
performed to find $\Delta \xi$ and $\Delta \eta$.

3.5.4 Variable Particle Mass

PIC methods do not simulate the motion of every atom or molecule. They lump a large number of them into "superparticles" in order to reduce the number of computations. These superparticles generally all have the same mass.

In the Hall thruster simulation, however, this requirement has been eliminated. The Hall thruster simulation uses a constant number of ionizations per second per grid cell, in order to evenly distribute ions. This means that the masses of the ion superparticles created are not necessarily identical. This tends to distribute computing power evenly to all cells, but it is found that simulation of dynamic ionizing plasmas is greatly improved by this method. The even particle distribution insures that statistical significance is maintained for all cells. Otherwise, cells with low density would have correspondingly low particle numbers and bad statistics. This is particularly important when a region of low ion density experiences the beginnings of ionization instability.

In addition, this method allows neutral particles to have a mass different than ions. This is extremely important in the Hall thruster simulation, since the neutral density is orders of magnitude higher than the ion density in most regions. All of the computational power would be spent moving a multitude of slow neutrals. With variable mass, however, the mass of neutral superparticles may be adjusted to achieve a reasonable total number of particles without bogging down the CPU.

The simulation is run with approximately 60,000 ion particles and 30,000 neutral particles. This gives 100 and 50 ion and neutral particles per grid cell, respectively, for a $13 \times 50$ computational grid.

3.5.5 Ionization

Once per PIC timestep, the ionization rate is computed and new ions are created. The ionization rate is given by Equation 2.62. The integral is performed at each node using Simpson's rule.

Fixing the number of ionizations per grid cell per timestep, and knowing the number of
neutrals per cell, an ionization probability is computed for each neutral:

\[
P_i = \frac{\text{ionizations per cell}}{\text{neutrals per cell}}.
\]  

(3.45)

Ions are then created by comparing \( P_i \) with a random number.

This method is similar to Monte Carlo, except that the neutral is not necessarily removed after the ionization, since its mass may be different than the new ion’s. The new ion’s mass is determined by the local ionization rate:

\[
M_i = \frac{\Delta t \hat{\eta}_e (2\pi A_{\text{cell}}) m_i}{\text{ionizations per cell}},
\]  

(3.46)

where \( A_{\text{cell}} \) is the geometric area of the cell containing the new ion, and \( m_i \) is the mass of a single ion.

For mass conservation, some neutral mass must be removed. A fraction of each neutral’s mass is deducted, based on the local ionization rate:

\[
\Delta M_n = -\frac{M_{n,0} \Delta t \hat{\eta}_e}{n_n},
\]  

(3.47)

where \( M_{n,0} \) is the original mass of the neutral superparticle.

### 3.6 Boundary Conditions and Wall Interactions

The electron boundary conditions are handled by directly fixing the cathode temperature, and by continually updating the anode temperature with the zero-slope condition. For the electron energy loss to the walls, the energy flux is divided by the length of the magnetic field line and treated as a source term (power per unit volume).

At each PIC iteration, checks are made for ions and neutrals which have fallen outside the boundaries. Ions are eliminated, and neutrals are reflected back into the domain with random directions. Although all ions are recombined at the wall, neutral superparticles are created at the walls based on a recombination probability at that point. This allows the recombined neutrals to have a much higher mass than the recombined ions. Again, this prevents the accumulation of neutrals, and allows the user to adjust the number of neutrals created at the wall. The recombination probability is a function of the ion flux, such that
the total neutral flux is equal to the ion flux, when averaged over time.

3.7 Collisions

Collisions between ions and neutrals are handled by the simple effective collision force described in section 2.3. A constant ionization cross section is assumed. The force on ions is exactly opposite of the force on neutrals, so total momentum is conserved.

3.8 Convergence

Since the method is time-accurate, there is no guarantee that the system will converge to a steady state solution. Two types of instabilities may prevent convergence:

- Plasma instabilities due to convection and diffusion of electrons and ions
- Numerical "noise" due to the limitations of the PIC method in simulating a continuum of heavy particles

Nevertheless, a solution is considered complete when the fluctuations reach a regular frequency and amplitude, and have repeated several periods. Or, if there is no discernable pattern to the fluctuations, the parameters are averaged over a long time scale, and iteration is stopped when the averages reach a constant value. The time it takes for convergence is assumed to be, at the very minimum, the time for convection of slow neutrals along the length of the grid.
Chapter 4

Results and Discussion

4.1 Geometric Model

The geometry used in the Hall thruster simulation is that of Bishaev and Kim [4]. This appears to be an experimental predecessor to the SPT-100. A rough geometry of the acceleration zone is shown in Figure 4-1.

![Figure 4-1: Acceleration zone geometry of the Bishaev SPT [4]. The dotted line represents the axial variation of radial magnetic field, and the solid lines are lines of force.](image-url)
4.2 Experimental Results by Bishaev and Kim

Bishaev and Kim used diagnostic probes to measure plasma parameters in the acceleration zone of the Hall thruster [4]. The operating conditions are given in Table 4.1.

<table>
<thead>
<tr>
<th>Operating Parameters for the Bishaev/Kim SPT</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Propellant</td>
<td>Xenon</td>
</tr>
<tr>
<td>$\phi_d$</td>
<td>200 Volts</td>
</tr>
<tr>
<td>$\dot{m}$</td>
<td>3 mg/s</td>
</tr>
<tr>
<td>Vacuum Chamber Pressure</td>
<td>$2 \times 10^{-4}$ Torr</td>
</tr>
<tr>
<td>$B_{r,max}$</td>
<td>.018 Tesla</td>
</tr>
</tbody>
</table>

Table 4.1: Operating conditions for the Bishaev/Kim SPT experiments.

The two-dimensional results from their paper are given in Figures 4-2 through 4-4. Note that the ion current density diagram, Figure 4-4, has units which may be in terms of the probe area (not given).

4.3 Magnetic Field

The dimensions of the iron poles are taken from a second paper by Bishaev [5]. Figure 4-5 shows the geometry of the iron poles and the resulting contours of magnetic potential. The magnetic field strength in the center of the channel is normalized to 1.0 Tesla in Figure 4-5, but for subsequent computations, the magnetic field parameters are multiplied by the magnetic field strength. For the Bishaev/Kim SPT, the published value of .018 Tesla is assumed to be the maximum radial magnetic field strength in the center of the acceleration zone.

The magnetic field parameters are shown on the computational domain in Figures 4-6 through 4-8. These do not correlate exactly with the experimental magnetic field geometry shown in Figure 4-1. Although the radial magnetic field strength has been closely matched, the contours of magnetic flux are not exactly the same. This is due to differences in iron pole geometry, as well as to the infinite permeability and ideal solenoid assumptions used in the numerical model. The iron pole extension on the outer radius of the acceleration zone is not seen in all SPT papers. It is shown in the most recent papers, however.
Figure 4-2: Space potential, $\phi$, and plasma density, $n_e$, from experiments by Bishaev and Kim [4].
Figure 4-3: Electron temperature, $T_e$, and bulk ionization rate, $\dot{n}_e$, from experiments by Bishaev and Kim [4].
Figure 4-4: Directed ion currents from experiments by Bishaev and Kim [4].

Figure 4-5: Overall magnetic potential contours, $\sigma \, [T \cdot m]$, computed for the Bishaev/Kim SPT geometry. The contour numbers correspond to the values of $\sigma$ listed in the key.
Figure 4-6: Magnetic potential contours, $\sigma \ [T \cdot m]$, computed for the Bishaev/Kim SPT geometry. The contour numbers correspond to the values of $\sigma$ listed in the key.

Figure 4-7: Magnetic stream contours, $\lambda \ [T \cdot m^2]$, computed for the Bishaev/Kim SPT geometry. The contour numbers correspond to the values of $\lambda$ listed in the key.
Figure 4-8: Magnitude of the magnetic field, $B \ [T]$, computed for the Bishaev/Kim SPT geometry. The contour numbers correspond to the values of $B$ listed in the key.

4.4 Performance

Although the Bohm diffusion coefficient is generally written as,

$$D_{Bohm} \approx \frac{kT_e}{16eB},$$

(4.1)

it is known to vary by an order of magnitude depending upon the plasma parameters. Therefore, the simulation was run for three cases to isolate the best fit for the Hall thruster. A constant $K_{Bohm}$ was introduced as a multiplier,

$$D_{Bohm} \approx K_{Bohm} \frac{kT_e}{16eB},$$

(4.2)

and the simulation was run for $K_{Bohm} = 0.75$, 1.0, and 1.25. After convergence, the results were averaged in time. Results are shown in Table 4.2.

It is interesting to note that, for the model used, varying $K_{Bohm}$ is equivalent to inversely varying $B$. This is due to the form of the Bohm conductivity. $K_{Bohm}$ and $B$ always appear together as $\frac{K_{Bohm}}{B}$. 
<table>
<thead>
<tr>
<th></th>
<th>Experimental</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_{Bohm} = 0.75$</td>
<td>$K_{Bohm} = 1.00$</td>
</tr>
<tr>
<td>$I_a [A]$</td>
<td>3.15</td>
<td>0.39</td>
</tr>
<tr>
<td>$I_b [A]$</td>
<td>2.10</td>
<td>0.52</td>
</tr>
<tr>
<td>$I_{SP} [s]$</td>
<td>1530</td>
<td>200</td>
</tr>
<tr>
<td>$F [N]$</td>
<td>.045</td>
<td>.006</td>
</tr>
<tr>
<td>$\eta_u$</td>
<td>0.95</td>
<td>0.24</td>
</tr>
<tr>
<td>$\eta_a$</td>
<td>0.67</td>
<td>1.33</td>
</tr>
<tr>
<td>$\eta_e$</td>
<td>0.84</td>
<td>0.24</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.54</td>
<td>0.08</td>
</tr>
<tr>
<td>$I_{wall} [A]$</td>
<td>1.0</td>
<td>0.03</td>
</tr>
<tr>
<td>$\xi_e [\times 10^{-9} m/s]$</td>
<td>7.0</td>
<td>1.2</td>
</tr>
<tr>
<td>$Torque/(F \cdot r_{mean})$</td>
<td>-</td>
<td>.037</td>
</tr>
</tbody>
</table>

Table 4.2: Performance results from the numerical simulation. $\dot{m} = 3$ mg/s; $\phi = 200$ Volts.

The first case ($K_{Bohm} = 0.75$) shows the effect of “quenching” the device with more propellant than is possible to ionize. At low $K_{Bohm}$, the power input to the electrons is lower, and ionization rate is reduced. The higher neutral density acts as an energy sink, keeping the electron temperature low and reducing the ionization rate further. This is self-perpetuating, since the ion production cost increases at lower energies, as seen in Figure 2-2.

In the second case, the anode current matches the experimental value almost exactly. Also, $K_{Bohm} = 1.0$ is the most realistic case, despite the uncertainty of the empirical Bohm diffusion model. The efficiency of this model exactly the advertised efficiency of the SPT-100, which is 50%. The specific impulse is also a close match with experimental data. This is not surprising, since the discharge potential was fixed to the same value measured, and the simulation reached full propellant utilization.

The third case, $K_{Bohm} = 1.25$, represents the effect of reduced electron impedance. The higher electron mobility increases their heating rate. This moves the ionization region toward the anode, increasing the ion wall losses.

The Bohm diffusion mechanism is not well understood. It is an empirical fit with some uncertainty. Therefore, it is surprising that the second case fits the experimental performance data so well. Furthermore, it is curious that the correlation degrades so rapidly as $K_{Bohm}$ deviates from unity.
4.5 Two-Dimensional Plasma Parameters

For each of the three cases in Table 4.2, two dimensional results were obtained from the numerical simulation. Figures 4-9 through 4-13 show the time-averaged two-dimensional results for the case $K_{Bohm} = 1.0$.

Figure 4-9: Contours of space potential ($\phi [V/m]$). The contour numbers correspond to the values of $\phi$ listed in the key. $K_{Bohm} = 1.0$; $\dot{m} = 3$ mg/s; $\phi = 200$ Volts; $I_a = 3.1$ A.

Comparing Figure 4-9 with Figure 4-2, a significant difference can be seen. The potential at the exit of the acceleration zone is found to be 10 Volts, whereas the numerical model predicts it to be 118 Volts. This discrepancy may be due to the location of the cathode. The simulation cathode is assumed to be located about 2.3 cm downstream of the acceleration zone. For the experiments of Bishaev and Kim, it may have been located closer. A finite vacuum chamber back-pressure was thought to cause this, but it was ruled out by numerically modeling the neutral particle flux from the chamber. The effect was negligible, so chamber back-pressure was ignored. Another explanation may be some type of enhanced electron conductivity in or near the plume which is not being modeled.

The ion density in Figure 4-10 also matches poorly with the experimental results. The peak experimental value is about $7 \times 10^{17} \text{ m}^{-3}$ near the exit of the acceleration zone.
Figure 4-10: Contours of plasma density ($n_i \, [m^{-3}]$). The contour numbers correspond to the values of $n_i$ listed in the key. $K_{Bohm} = 1.0; \dot{m} = 3 \, mg/s; \phi = 200 \, Volts; I_a = 3.1 \, A$.

Figure 4-11: Contours of electron temperature ($T_e \, [^\circ K]$). The contour numbers correspond to the values of $T_e$ listed in the key. $K_{Bohm} = 1.0; \dot{m} = 3 \, mg/s; \phi = 200 \, Volts; I_a = 3.1 \, A$. 

55
Figure 4-12: Contours of ionization rate ($\dot{n}_i \ [m^{-3}s^{-1}]$). The contour numbers correspond to the values of $\dot{n}_i$ listed in the key. $K_{Bohm} = 1.0; \ \dot{m} = 3 \ \text{mg/s}; \ \phi = 200 \ \text{Volts}; \ I_a = 3.1 \ \text{A}$.

Figure 4-13: Vectors of ion number flux ($\vec{\dot{n}}_i \ [m^{-2}s^{-1}]$). The reference vector represents a flux of $14.8 \times 10^{20} m^2/s$. The contour numbers correspond to the values of $\vec{\dot{n}}_i$ listed in the key. $K_{Bohm} = 1.0; \ \dot{m} = 3 \ \text{mg/s}; \ \phi = 200 \ \text{Volts}; \ I_a = 3.1 \ \text{A}$.
Figure 4-14: Contours of neutral number density ($n_n \text{[m}^{-3}\text{]}$). $K_{Bohm} = 1.0; \dot{n} = 3 \text{ mg/s; } \phi = 200 \text{ Volts; } I_a = 3.1 \text{ A.}$

The simulation predicts it to be twice that and to occur closer to the anode.

The assumption of constant electron temperature along lines of force seems to hold partially from Figure 4-3. This may imply that the plasma is not entirely in equilibrium along the magnetic lines, and perhaps the Boltzmann relation is not completely applicable. Alternatively, perhaps they are in equilibrium, but an electron energy balance holds instead of Boltzmann's relation. This is a matter under investigation. Nevertheless, the peak electron temperature is predicted correctly.

Also, the temperature gradients are strong in the experimental measurements. This is not the case with the simulation. One reason for this discrepancy may be overprediction of the thermal diffusion coefficient.

Examining the ion flux measurements in Figure 4-4, most of the flux vectors leave the region of high plasma density, as expected, and travel along potential lines or to the walls. The simulation results (Figure 4-13) show a similar pattern of ion flux, with the vectors emanating from a high density region closer to the anode and closer to the inner wall.

The two-dimensional numerical results are consistent when taken as a set for the model assumed in Chapter 2. Electron temperature increases from the cathode to the acceleration
zone, since there is little inelastic collision loss in that region due to the low neutral density. Once inside the channel, the electrons enter a region of higher neutral density. Ionization peaks when the neutral density is increasing and the electron temperature decreases from inelastic losses.

The potential gradient is strongest in regions of increasing pressure gradient, in accordance with the diffusion model. The region of highest potential gradient is therefore at the entrance to the acceleration zone.

The quenching effect of reducing $K_{Bohm}$ (or increasing $B$) can be seen in Figures 4-15 through 4-19 for the case of $K_{Bohm} = 0.75$. This runaway effect is caused by neutral density increasing along the length of the channel. Inelastic electron energy losses increase until, at some point, electron temperature drops below a critical value, and ionization rate drops as well. The plasma becomes less conductive and the anode current drops, reducing the net ionization power of the device further.

Figure 4-15: Contours of space potential ($\phi$ [V/m]). The contour numbers correspond to the values of $\phi$ listed in the key. $K_{Bohm} = 0.75$; $\dot{m} = 3$ mg/s; $\phi = 200$ Volts; $I_a = 3.1$ A.
Figure 4-16: Contours of plasma density \((n_i \, [m^{-3}])\). The contour numbers correspond to the values of \(n_i\) listed in the key. \(K_{Bohm} = 0.75\); \(\dot{m} = 3 \, \text{mg/s}\); \(\phi = 200 \, \text{Volts}\); \(I_a = 3.1 \, \text{A}\).

Figure 4-17: Contours of electron temperature \((T_e \, [eV])\). The contour numbers correspond to the values of \(T_e\) listed in the key. \(K_{Bohm} = 0.75\); \(\dot{m} = 3 \, \text{mg/s}\); \(\phi = 200 \, \text{Volts}\); \(I_a = 3.1 \, \text{A}\).
Figure 4-18: Contours of ionization rate ($\dot{n}_i \ [m^3 s^{-1}]$). The contour numbers correspond to the values of $\dot{n}_i$ listed in the key. $K_{Bohm} = 0.75$; $\dot{m} = 3 \text{ mg/s}$; $\phi = 200 \text{ Volts}$; $I_a = 3.1 \text{ A}$.

Figure 4-19: Contours of neutral number density ($n_n \ [m^{-3}]$). $K_{Bohm} = 0.75$; $\dot{m} = 3 \text{ mg/s}$; $\phi = 200 \text{ Volts}$; $I_a = 3.1 \text{ A}$.
4.6 Torque

The torque on the device can be written as the sum of the azimuthal moments on the ions in each grid cell as,

\[ T = \sum 2\pi r^2 A_{cell} \tau_i \times \vec{B}. \] (4.3)

The torque is generally nondimensionalized by dividing by the thrust and the mean radius. The torque parameters for each case are given in Table 4.2.

4.7 Ion Wall Impingement

The potential gradients follow strongly the magnetic field lines, but some smaller gradients exist across magnetic field lines due to the Boltzmann equilibrium. These gradients play a large role in sending ion flux to the walls. Figure 4.13 shows the ion flux vectors.

From the numerical simulation, average wall impingement current is calculated to be 1.1 Amperes. According to Bishaev and Kim [4], ion loss to the walls is approximately 1.0 Ampere. Ion wall currents for the other cases are also listed in Table 4.2.

The SPT-100 insulator wall is known to sputter most near the acceleration zone exit [13]. Indeed, Figure 4.20 shows the mean ion velocities at the wall to be a maximum near the exit of the channel. At the inner wall, ion current density is \(48A/m^2\) normal to the wall. A comparison of wall sputtering rate can be made by using the relation,

\[ \xi_s = j_{i,z} S_v, \] (4.4)

where \(\xi_s\) is the wall reduction rate in \(m/s\), \(j_{i,z}\) is the normal component of the ion current density, and \(S_v\) is the volumetric sputtering coefficient. Abgaryan et. al. [1] measured the sputtering properties of borosil, the dielectric wall material used in SPTs. They found that \(S_v \approx 0.08 (mm)^3/C\) for ion energies of 33 eV at an 80 degree angle of incidence. Therefore,

\[ \xi_s \approx 4 \times 10^{-9} m/s, \] (4.5)

which is very close to the experimental value of wall sputtering for SPTs. At the inner insulator wall, the initial sputtering rate is \(7 \times 10^{-9} m/s\) from experiments by Garner et. al. [13]. The SPT may easily have twice the ion flux at that point, however, in which case the
calculated value would be very close. Nevertheless, the order of magnitude analysis agrees with experimental results.

The same analysis has been performed for the other two $K_{Bohm}$ cases, and the results are shown in table 4.2. For all three cases the maximum wall sputtering rate occurred at the inner wall of the acceleration zone exit.

Figure 4-20: Vectors of ion velocity ($\bar{u}_i [m/s]$). The reference vector represents a velocity magnitude of 16317 m/s. $K_{Bohm} = 1.0$; $\dot{m} = 3$ mg/s; $\phi = 200$ Volts; $I_a = 3.1$ A.

4.8 Plasma Oscillations

The Hall thruster simulation does not reach a steady state solution. Plasma parameters fluctuate continually, even after long convergence times. The two-dimensional results presented in Section 4.5 have been averaged over .5 ms. The overall operational parameters for this period are given in Figures 4-21 through Figure 4-23. It can be seen that these parameters fluctuate ±15%.

To better understand local oscillations in Hall thrusters, plasma parameters have been plotted on a short time scale (.05 ms) in Figures 4-24 through 4-28 for a particular node ($z = 1.9$ cm and $r = 3.6$ cm). From Figures 4-24 and 4-26, the plasma density and ionization rate
Figure 4-21: Long time history of anode current, $I_a$ [Amperes].

Figure 4-22: Long time history of ion beam current, $I_b$ [Amperes].
can be seen to be in phase. Also, there is a slight correlation between electron temperature and ionization rate.

The neutral density fluctuations in Figure 4-28 are lower in frequency and seem to be superimposed on the ionization rate. This low frequency is approximately 83 kHz. The higher frequency oscillations visible in the rest of the parameters occur at a frequency of approximately 300 kHz.

Both of these high frequency oscillations may be related to the time scale of ion and neutral passage across grid cells. At the measurement point, \( u_i = 500 \text{ m/s} \), and \( u_n = 300 \text{ m/s} \). For a grid spacing of 1.2 mm, this gives passage frequencies of 416 and 250 kHz.
Figure 4-24: Short time history of plasma density, $n_i \ [m^{-3}]$, at $z = 1.9 \ cm$ and $r = 3.6 \ cm$.

Figure 4-25: Short time history of electron temperature, $T_e \ [eV]$, at $z = 1.9 \ cm$ and $r = 3.6 \ cm$. 

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Figure 4-26: Short time history of bulk ionization rate, $\dot{n}_i \text{ [m}^{-3}\text{s}^{-1}]$, at $z = 1.9$ cm and $r = 3.6$ cm.

Figure 4-27: Short time history of axial electric field, $E_z \text{ [V/m]}$, at $z = 1.9$ cm and $r = 3.6$ cm.
Figure 4-28: Short time history of neutral number density, $n_n \ [m^{-3}]$, at $z = 1.9 \ cm$ and $r = 3.6 \ cm$. 
Chapter 5

Conclusions

The Hall thruster simulation predicted the performance parameters extremely accurately for the Bishaev/Kim SPT. Efficiency and specific impulse were accurate to within 8%. Anode current and beam current were accurate to within 5%. This implies that the numerical model may be very useful in predicting the performance of alternative Hall thruster geometries.

To test the Bohm diffusion assumption, three cases with different Bohm diffusion constants were tried. Surprisingly, the most accurate fit was achieved with a Bohm diffusivity equal to its traditionally quoted value of $\frac{kT_e}{16B}$, even though this is regarded as a rough approximation only.

The two-dimensional numerical results followed similar trends as the experimental values. However, the simulation predicted higher ionization rate near the anode. Also, the electron temperature rise outside the acceleration zone was not seen in experimental results.

Total wall impingement current was a close to the experimental value. Using the wall current density and an experimental sputtering yield, the wall erosion rate was calculated. Comparing this to experiments with the SPT-100, a close agreement was found.

Therefore, the numerical model was useful for predicting the performance parameters of the Bishaev/Kim SPT geometry. The two-dimensional results lacked close agreement with experiment, but were consistent with the numerical model. The usefulness of the simulation is in predicting the performance, wall erosion rate, and torque of prototype Hall thrusters, and as a tool for understanding the physics of the plasma acceleration process.

In the future, a better model of electron energy is desired. Particularly, the sources and
losses should be more carefully identified. Also, a good model of electron transport at the wall is needed. Finally, a detailed physical model of the anode would shed some light on the processes at work in that region.
Bibliography


