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Recently a remarkable map between 4-dimensional superconformal field theories and vertex algebras has been constructed [BLLPRV15]. This has lead to new insights in the theory of characters of vertex algebras. In particular it was observed that in some cases these characters decompose in nice products [XYY16], [Y16].

The purpose of this note is to explain the latter phenomena. Namely, we point out that it is immediate by our character formula [KW88], [KW89] that in the case of a *boundary level* the characters of admissible representations of affine Kac-Moody algebras and the corresponding W-algebras decompose in products in terms of the Jacobi form $\vartheta_{11}(\tau, z)$.

We would like to thank Wenbin Yan for drawing our attention to this question.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Delta \subset \mathfrak{h}^*$ be the set of roots. Let $Q = \mathbb{Z}\Delta$ be the root lattice and let $Q^* = \{h \in \mathfrak{h} \mid \alpha(h) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ be the dual lattice. Let $\Delta_+ \subset \Delta$ be a subset of positive roots, let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple roots and let ρ be half of the sum of positive roots. Let W be the Weyl group. Let $(\,.\,|\,.\,)$ be the invariant symmetric bilinear form on $\mathfrak{g}$, normalized by the condition $(\alpha|\alpha) = 2$ for a long root α, and let $h^\vee$ be the dual Coxeter number ($= \frac{1}{2}$ eigenvalue of the Casimir operator on $\mathfrak{g}$). We shall identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the form $(\,.\,|\,.\,)$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$ be the associated to $\mathfrak{g}$ affine Kac-Moody algebra (see [K90] for details), let $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d$ be its Cartan subalgebra. We extend the symmetric bilinear form $(\,.\,|\,.\,)$ from $\mathfrak{h}$ to $\widehat{\mathfrak{h}}$ by letting $(\mathfrak{h}|\mathbb{C}K + \mathbb{C}d) = 0, (K|K) = 0, (d|d) = 0, (d|K) = 1$, and we identify $\widehat{\mathfrak{h}}^*$ with $\widehat{\mathfrak{h}}$ using this form. Then d is identified with the 0^{th} fundamental weight $\Lambda_0 \in \widehat{\mathfrak{h}}^*$, such that $\Lambda_0|_{\mathfrak{g}[t,t^{-1}]+\mathbb{C}d} = 0, \Lambda_0(K) = 1$, and K is identified with the imaginary root $\delta \in \widehat{\mathfrak{h}}^*$. Then the set of real roots of $\widehat{\mathfrak{g}}$ is $\hat{\Delta}^{\rm re} = \{\alpha + n\delta |\, \alpha \in \Delta, n \in \mathbb{Z}\}$ and the subset of positive real roots is $\hat{\Delta}^{\rm re}_+ = \Delta_+ \cup \{\alpha + n\delta |\, \alpha \in \Delta, n \in \mathbb{Z}_{\geq 1}\}$. Let $\hat{\rho} = h^\vee \Lambda_0 + \rho$. Let

$$\hat{\Pi}_u = \{u\delta - \theta, \alpha_1, \ldots, \alpha_\ell\},$$

where $\theta \in \Delta_+$ is the highest root, so that $\hat{\Pi}_1$ is the set of simple roots of $\widehat{\mathfrak{g}}$. For $\alpha \in \hat{\Delta}^{\rm re}$ one lets $\alpha^\vee = 2\alpha/(\alpha|\alpha)$. Finally, for $\beta \in Q^*$ define the translation $t_\beta \in \mathrm{End}\,\widehat{\mathfrak{h}}^*$ by

$$t_\beta(\lambda) = \lambda + \lambda(K)\beta - ((\lambda|\beta) + \frac{1}{2}\lambda(K)|\beta|^2)\delta.$$

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Given $\Lambda \in \widehat{\mathfrak{h}}^*$ let $\hat{\Delta}^\Lambda = \{\alpha \in \hat{\Delta}^{\mathrm{re}} \,|\, (\Lambda|\alpha^\vee) \in \mathbb{Z}\}$. Then Λ is called an *admissible* weight if the following two properties hold

(i) $(\Lambda + \widehat{\rho}|\alpha^\vee) \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \hat{\Delta}_+$,

(ii) $\mathbb{Q}\hat{\Delta}^\Lambda = \mathbb{Q}\hat{\Delta}$.

If instead of (ii) a stronger condition holds:

(ii)′ $\varphi(\hat{\Delta}^\Lambda) = \hat{\Delta}$ for a linear isomorphism $\varphi : \widehat{\mathfrak{h}}^* \to \widehat{\mathfrak{h}}^*$,

then Λ is called a *principal* admissible weight. In [KW89] the classification and character formulas for admissible weights is reduced to that for principal admissible weights. The latter are described by the following proposition.

Proposition 1. *[KW89] Let Λ be a principal admissible weight and let $k = \Lambda(K)$ be its level. Then*

(a) k is a rational number with denominator $u \in \mathbb{Z}_{\geq 1}$, such that

$$k + h^\vee \geq \frac{h^\vee}{u} \text{ and } \gcd(u, h^\vee) = \gcd(u, r^\vee) = 1, \tag{1}$$

where $r^\vee = 1$ for $\mathfrak{g}$ of type A-D-E, $= 2$ for $\mathfrak{g}$ of type B, C, F, and $= 3$ for $\mathfrak{g} = G_2$.

(b) All principal admissible weights are of the form

$$\Lambda = (t_\beta y).(\Lambda^0 - (u-1)(k + h^\vee)\Lambda_0), \tag{2}$$

where $\beta \in Q^*, y \in W$ are such that $(t_\beta y)\hat{\Pi}_u \subset \hat{\Delta}_+, \Lambda^0$ is an integrable weight of level $u(k + h^\vee) - h^\vee$, and dot denotes the shifted action: $w.\Lambda = w(\Lambda + \widehat{\rho}) - \widehat{\rho}$.

(c) For $\mathfrak{g} = s\ell_N$ all admissible weights are principal admissible.

Recall that the normalized character of an irreducible highest weight $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ of level $k \neq -h^\vee$ is defined by

$$\mathrm{ch}_\Lambda(\tau, z, t) = q^{m_\Lambda} \mathrm{tr}_{\,L(\Lambda)} e^{2\pi i h}$$

where

$$h = -\tau d + z + tK,\ z \in \mathfrak{h},\ \tau, t \in \mathbb{C},\ \mathrm{Im}\,\tau > 0,\ q = e^{2\pi i \tau}, \tag{3}$$

and $m_\Lambda = \frac{|\Lambda + \widehat{\rho}|^2}{2(k+h^\vee)} - \frac{\dim \mathfrak{g}}{24}$ (the normalization factor q^{m_Λ} "improves" the modular invariance of the character).

In [KW89] the characters of the $\widehat{\mathfrak{g}}$-modules $L(\Lambda)$ for arbitrary admissible Λ were computed, see Theorem 3.1, or formula (3.3) there for another version in case of a principal admissible Λ. In order to write down the latter formula, recall the normalized affine denominator for $\widehat{\mathfrak{g}}$:

$$\hat{R}(h) = q^{\frac{\dim \mathfrak{g}}{24}} e^{\widehat{\rho}(h)} \prod_{n=1}^{\infty} (1 - q^n)^\ell \prod_{\alpha \in \Delta_+} (1 - e^{\alpha(z)} q^n)(1 - e^{-\alpha(z)} q^{n-1}).$$

In coordinates (3) this becomes:

$$\hat{R}(\tau,z,t)=(-i)^{|\Delta_+|}e^{2\pi i h^\vee t}\eta(\tau)^{\frac{1}{2}(3\ell-\dim\mathfrak{g})}\prod_{\alpha\in\Delta_+}\vartheta_{11}(\tau,\alpha(z)),\tag{4}$$

where

$$\vartheta_{11}(\tau,z)=-iq^{\frac{1}{12}}e^{-\pi i z}\eta(\tau)\prod_{n=1}^{\infty}(1-e^{-2\pi i z}q^n)(1-e^{2\pi i z}q^{n-1})$$

is one of the standard Jacobi forms ϑ_{ab}, $a,b=0$ or 1 (see e.g., Appendix to [KW14]), and $\eta(\tau)$ is the Dedekind eta function.

For a principal admissible Λ, given by (2), formula (3.3) from [KW89] becomes in coordinates (3):

$$(\hat{R}\mathrm{ch}_\Lambda)(\tau,z,t)=(\hat{R}\mathrm{ch}_{\Lambda^0})\left(u\tau,y^{-1}(z+\tau\beta),\frac{1}{u}(t+(z|\beta)+\frac{\tau|\beta|^2}{2})\right).\tag{5}$$

It follows from (5) that if $\Lambda^0=0$ in (2) (so that $\mathrm{ch}_{\Lambda^0}=1$), which is equivalent to

$$k+h^\vee=\frac{h^\vee}{u}\ \text{and}\ \gcd(u,h^\vee)=\gcd(u,r^\vee)=1,\tag{6}$$

the (normalized) character ch_Λ turns into a product. The level k, defined by (6), is naturally called the *boundary principal admissible* level in [KRW03], see formula (3.5) there. We obtain from Proposition 1, (4) and (5)

Proposition 2. *(a) All boundary principal admissible weights are of level k, given by (6), and are of the form*

$$\Lambda=(t_\beta y).(k\Lambda_0),\tag{7}$$

where $\beta\in Q^*, y\in W$ are such that $(t_\beta y)\hat{\Pi}_u\subset\hat{\Delta}_+$. In particular, $k\Lambda_0$ is a principal admissible weight of level (6).

(b) If Λ is of the form (7), then

$$\mathrm{ch}_\Lambda(\tau,z,t)=e^{2\pi i(kt+\frac{h^\vee}{u}(z|\beta))}q^{\frac{h^\vee}{2u}|\beta|^2}\left(\frac{\eta(u\tau)}{\eta(\tau)}\right)^{\frac{1}{2}(3\ell-\dim\mathfrak{g})}\prod_{\alpha\in\Delta_+}\frac{\vartheta_{11}(u\tau,y(\alpha)(z+\tau\beta))}{\vartheta_{11}(\tau,\alpha(z))}.$$

Remark 1. For the vacuum module $L(k\Lambda_0)$ of the boundary principal admissible level k the character formula from Proposition 2(b) becomes

$$\mathrm{ch}_{k\Lambda_0}(\tau,z,t)=e^{2\pi i k t}\left(\frac{\eta(u\tau)}{\eta(\tau)}\right)^{\frac{1}{2}(3\ell-\dim\mathfrak{g})}\prod_{\alpha\in\Delta_+}\frac{\vartheta_{11}(u\tau,\alpha(z))}{\vartheta_{11}(\tau,\alpha(z))}.$$

Example 1. Let $\mathfrak{g}=s\ell_2$, so that $h^\vee=2$. Then the boundary levels are $k=\frac{2}{u}-2$, where u is a positive odd integer, and all admissible weights are

$$\Lambda_{k,j}:=t_{-\frac{j}{2}\alpha_1}.(k\Lambda_0)=(k+\frac{2j}{u})\Lambda_0-\frac{2j}{u}\Lambda_1,\ j=0,1,\dots,u-1,$$

and the character formula from Proposition 2(b) becomes:

$$\mathrm{ch}_{\Lambda_{u,j}} = e^{2\pi i(kt - \frac{j}{u}z)} q^{\frac{j^2}{2u}} \frac{\vartheta_{11}(u\tau, z - j\tau)}{\vartheta_{11}(\tau, z)}. \tag{8}$$

For $u = 3$ and 5 some of these formulas were conjectured in [Y16].

Example 2. Let $\mathfrak{g} = s\ell_N$, so that $h^\vee = N$, let $N > 1$ be odd, and let $u = 2$. Then the boundary admissible level is $k = -\frac{N}{2}$, and the boundary admissible weights of the form $t_\beta.(k\Lambda_0)$ are:

$$\Lambda_{N,p} = -\frac{N}{2}\Lambda_p,\ p = 0, 1, \ldots, , N-1,$$

where Λ_p are the fundamental weights of $\widehat{\mathfrak{g}}$. Letting $z = \sum_{i=1}^{N-1} z_i \bar{\Lambda}_i$, where $\bar{\Lambda}_i$ are the fundamental weights of $\mathfrak{g}$, the character formula from Proposition 2 (b) becomes:

$$\mathrm{ch}_{\Lambda_{N,p}}(\tau, z, t) = i^{p(N-p)} e^{-\pi i N t} \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-\frac{(N-1)(N-2)}{2}}$$
$$\times \frac{\prod\limits_{\substack{1 \le i \le j < p \\ \text{or } p < i \le j < N}} \vartheta_{11}(2\tau, z_i + \ldots + z_j) \prod\limits_{1 \le i \le p \le j < N} \vartheta_{01}(2\tau, z_i + \ldots + z_j)}{\prod\limits_{1 \le i \le j < N} \vartheta_{11}(\tau, z_i + \ldots + z_j)},$$

where

$$\vartheta_{01}(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i z} q^{n - \frac{1}{2}})(1 - e^{-2\pi i z} q^{n - \frac{1}{2}}).$$

This follows from Proposition 2(b) by applying to ϑ_{11} an elliptic transformation (see e.g. [KW14], Appendix). In particular

$$\mathrm{ch}_{-\frac{N}{2}\Lambda_0} = e^{-\pi i N t} \left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-\frac{(N-1)(N-2)}{2}} \prod_{1 \le i \le j < N} \frac{\vartheta_{11}(2\tau, z_i + \ldots + z_j)}{\vartheta_{11}(\tau, z_i + \ldots + z_j)}.$$

The latter formula was conjectured in [XYY16].

Remark 2. For principal admissible weights $\Lambda = (t_\beta y).(k\Lambda_0)$ and $(t_{\beta'} y').(k\Lambda_0)$ of boundary level $k = \frac{h^\vee}{u} - h^\vee$ the S-transformation matrix $(a(\Lambda, \Lambda'))$, given by [KW89], Theorem 3.6, simplifies to

$$a(\Lambda, \Lambda') = |Q/uh^\vee Q^*|^{-\frac{1}{2}} \varepsilon(yy') \prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi i u(\rho|\alpha)}{h^\vee} e^{-2\pi i \left((\rho|\beta + \beta') + \frac{h^\vee(\beta|\beta')}{u}\right)}.$$

Remark 3. If $\mathfrak{g} = s\ell_2$ and k is as in Example 1, then

$$a(\Lambda_{k,j}, \Lambda_{k,j'}) = (-1)^{j+j'} e^{-\frac{2\pi i j j'}{u}} \frac{1}{\sqrt{u}} \sin \frac{u\pi}{2}.$$

One can compute fusion coefficients by Verlinde's formula:

$$N_{\Lambda_{k,j_1}, \Lambda_{k,j_2}, \Lambda_{k,j_3}} = (-1)^{j_1 + j_2 + j_3} \text{ if } j_1 + j_2 + j_3 \in u\mathbb{Z}, \text{ and } = 0 \text{ otherwise}.$$

Example 3. Let $\mathfrak{g} = sl_3$, so that $h^\vee = 3$, and let u be a positive integer, coprime to 3. Then all (principal) admissible weights have level $k = \frac{3}{u} - 3$ and are of the form (7), where

$$\beta = -(-1)^p (k_1 \bar{\Lambda}_1 + k_2 \bar{\Lambda}_2),\ y = r_\theta^p,\ p = 0 \,\text{or}\, 1,\ k_i \in \mathbb{Z}, k_i \geq \delta_{p,1},\ k_1 + k_2 \leq u - \delta_{p,0}.$$

Denote this weight by $\Lambda^{(p)}_{u;k_1,k_2} = (t_\beta y).(k\Lambda_0)$. Using Remark 2, one computes the fusion coefficients by Verlinde's formula:

$$N_{\Lambda^{(p)}_{u;k_1,k_2} \Lambda^{(p')}_{u;k'_1,k'_2} \Lambda^{(p'')}_{u;k''_1,k''_2}} = (-1)^{p+p'+p''} \,\text{if}\, (-1)^p k_i + (-1)^{p'} k'_i + (-1)^{p''} k''_i \in u\mathbb{Z} \,\text{for}\, i = 1, 2,$$

and $= 0$ otherwise.

Remark 4. If Λ is an arbitrary admissible weight, then $\hat{\Delta}^\Lambda$ decomposes in a disjoint union of several affine root systems. Then Λ has *boundary level* if restrictions of it to each of them has boundary level, and formula (3.4) from [KW89] shows that ch_Λ decomposes in a product of the corresponding boundary level characters. Note also that all the above holds also for twisted affine Kac-Moody algebras [KW89].

Remark 5. The product character formula for boundary level affine Kac-Moody superalgebras holds as well, see [GK15], formula (2).

Recall that to any $s\ell_2$-triple $\{f, x, e\}$ in $\mathfrak{g}$, where $[x, f] = -f,\ [x, e] = e$, one associates a W-algebra $W^k(g, f)$, obtained from the vacuum $\widehat{\mathfrak{g}}$-module of level k by quantum Hamiltonian reduction, so that any $\widehat{\mathfrak{g}}$-module $L(\Lambda)$ of level k produces either an irreducible $W^k(g, f)$-module $H(\Lambda)$ or zero. The characters of $L(\Lambda)$ and $H(\Lambda)$ are related by the following simple formula ([KRW03] or [KW14]):

$$\left(\overset{W}{R}\mathrm{ch}_{H(\Lambda)}\right)(\tau, z) = \left(\hat{R}\mathrm{ch}_\Lambda\right)(\tau, -\tau x + z, \frac{\tau}{2}(x|x)). \tag{9}$$

Here $z \in \mathfrak{h}^f$, the centralizer of f in $\mathfrak{h}$, and

$$\overset{W}{R}(\tau, z) = \eta(\tau)^{\frac{3}{2}l - \frac{1}{2}\dim(\mathfrak{g}_0 + \mathfrak{g}_{1/2})} \prod_{\alpha \in \Delta^0_+} \vartheta_{11}(\tau, \alpha(z)) \left(\prod_{\alpha \in \Delta_{1/2}} \vartheta_{01}(\tau, \alpha(z)) \right)^{1/2}, \tag{10}$$

where $\mathfrak{g} = \oplus_j \mathfrak{g}_j$ is the eigenspace decomposition for ad x, $\Delta_j \subset \Delta$ is the set of roots of root spaces in $\mathfrak{g}_j$ and $\Delta^0_+ = \Delta_+ \cap \Delta_0$ (we assume that $\Delta_j \subset \Delta_+$ for $j > 0$). If k is a boundary level (6), we obtain from Proposition 2(b) and formulas (9), (10) the following character formula for $H(\Lambda)$ if Λ is a principal admissible weight (7) ($z \in \mathfrak{h}^f$):

$$\begin{aligned} \mathrm{ch}_{H(\Lambda)}(\tau, z) &= (-i)^{|\Delta_+|} q^{\frac{h^\vee}{2u}|\beta - x|^2} e^{\frac{2\pi i h^\vee}{u}(\beta|z)} \\ &\times \frac{\eta(u\tau)^{\frac{3}{2}\ell - \frac{1}{2}\dim\mathfrak{g}}}{\eta(\tau)^{\frac{3}{2}\ell - \frac{1}{2}\dim(\mathfrak{g}_0 + \mathfrak{g}_{1/2})}} \frac{\prod_{\alpha \in \Delta_+} \vartheta_{11}(u\tau, y(\alpha)(z + \tau\beta - \tau x))}{\prod_{\alpha \in \Delta^0_+} \vartheta_{11}(\tau, \alpha(z)) \left(\prod_{\alpha \in \Delta_{1/2}} \vartheta_{01}(\tau, \alpha(z)) \right)^{1/2}}. \end{aligned} \tag{11}$$

Remark 6. A formula, similar to Proposition 2(b) and to formula (11), holds if $\mathfrak{g}$ is a basic Lie superalgebra; one has to replace the character by the supercharacter, dim by sdim, and the factor ϑ_{ab}, corresponding to a root α, by its inverse if this root is odd. Also, the character is obtained from the supercharacter by replacing ϑ_{ab} by $\vartheta_{a,b+1 \bmod 2}$ if the root α is odd.

Remark 7. An example of (11) is the minimal series representations of the Virasoro algebra with central charge $c = 1 - \frac{3(u-2)^2}{u}$, obtained by the quantum Hamiltonian reduction from the boundary admissible $\hat{sl}_2$-modules from Example 1. For $j = u - 1$ one gets 0, for $u = 3$ and $j = 0, 1$ one gets the trivial representation, but for all other j and $u \geq 5$ the characters are the product sides of the Gordon generalizations of the Rogers-Ramanujan idenities (the latter correspond to $u = 5$). Another example is the minimal series representations of the $N = 2$ superconformal algebras, see [KRW03], Section 7.

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