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Lie conformal superalgebras of type K*

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IRREDUCIBLE MODULES OVER FINITE SIMPLE LIE CONFORMAL SUPERALGEBRAS OF TYPE K

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ABSTRACT. We construct all finite irreducible modules over Lie conformal superalgebras of type K

1. INTRODUCTION

Lie conformal superalgebras encode the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory [6].

A complete classification of (linear) finite simple Lie conformal superalgebras was obtained in [5]. The list consists of current Lie conformal superalgebras $\text{Cur } \mathfrak{g}$, where \mathfrak{g} is a simple finite-dimensional Lie superalgebra, four series of “Virasoro like” Lie conformal superalgebras $W_n (n \geq 0)$, $S_{n,b}$ and $\tilde{S}_n (n \geq 2, b \in \mathbb{C})$, $K_n (n \geq 0, n \neq 4)$, K'_4 , and the exceptional Lie conformal superalgebra CK_6 .

All finite irreducible representations of the simple conformal superalgebras $\text{Cur } \mathfrak{g}$, $K_0 = \text{Vir}$ and K_1 were constructed in [2], and those of $S_{2,0}$, $W_1 = K_2$, K_3 , and K_4 in [3]. More recently, the problem has been solved for all Lie conformal superalgebras from the three series W_n , $S_{n,b}$, and \tilde{S}_n [1].

The construction in all cases relies on the observation that the representation theory of a Lie conformal superalgebra R is controlled by the representation theory of the associated (extended) *annihilation algebra* $\mathfrak{g} = (\text{Lie } R)_+$ [2], thereby reducing the problem to the construction of continuous irreducible modules with discrete topology over the linearly compact superalgebra \mathfrak{g} .

The construction of the latter modules consists of two parts. First one constructs a collection of continuous \mathfrak{g} -modules $\text{Ind}(F)$, associated to all finite-dimensional irreducible \mathfrak{g}_0 -modules F , where \mathfrak{g}_0 is a certain subalgebra of \mathfrak{g} ($= \mathfrak{gl}(1|n)$ or $\mathfrak{sl}(1|n)$ for the W and S series, and $= \mathfrak{cso}_n$ for the K_n series).

The irreducible \mathfrak{g} -modules $\text{Ind}(F)$ are called *non-degenerate*, and the second part of the problem consists of two parts: (A) classify the \mathfrak{g}_0 -modules F , for which the \mathfrak{g} -modules $\text{Ind}(F)$ are non-degenerate, and (B) construct explicitly the irreducible quotients of $\text{Ind}(F)$, called *degenerate* \mathfrak{g} -modules, for reducible $\text{Ind}(F)$.

Both problems have been solved for types W and S in [1], and it turned out, remarkably, that all degenerate modules occur as cokernels of the super de Rham complex, or their duals.

In the present paper we solve the problem for the Lie conformal superalgebras K_n with $n \geq 4$ (recall that for $0 \leq n \leq 4$ the problem has been solved in [2] and

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[3], though in [3] the construction for $n = 3$ and 4 is not very explicit). First, we construct the \mathfrak{g} -modules $\text{Ind}(F)$ (Theorem 4.1). Second, we find all F , for which $\text{Ind}(F)$ is reducible, and, furthermore, find all singular vectors (Theorem 5.1). Finally, in Section 6 we construct a *contact complex*, which is a certain reduction of the de Rham complex, and show (using Theorem 5.1) that the cokernels in the contact complex and their duals produce all degenerate \mathfrak{g} -modules (Corollary 6.6). As a result, we obtain an explicit construction of all finite irreducible K_n -modules for $n \geq 4$ (Theorem 7.1).

We should mention that the construction of our (super) contact complex mimics the beautiful Rumin's construction [10] for ordinary (non-super) contact manifolds.

The remaining cases, namely, the representation theory of K'_4 (the derived algebra of K_4) and of the exceptional Lie conformal superalgebra CK_6 , and the explicit construction of degenerate modules for K_3 , will be worked out in a subsequent publication.

2. FORMAL DISTRIBUTIONS, LIE CONFORMAL SUPERALGEBRAS AND THEIR MODULES

In this section we introduce the basic definitions and notations in order to have a self-contained work, see [6, 4, 1, 3]. Let \mathfrak{g} be a Lie superalgebra. A \mathfrak{g} -valued *formal distribution* in one indeterminate z is a formal power series

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a_n \in \mathfrak{g}.$$

The vector superspace of all formal distributions, $\mathfrak{g}[[z, z^{-1}]]$, has a natural structure of a $\mathbb{C}[\partial_z]$ -module. We define

$$\text{Res}_z a(z) = a_0.$$

Let $a(z), b(z)$ be two \mathfrak{g} -valued formal distributions. They are called *local* if

$$(z-w)^N [a(z), b(w)] = 0 \quad \text{for } N \gg 0.$$

Let \mathfrak{g} be a Lie superalgebra, a family \mathcal{F} of \mathfrak{g} -valued formal distributions is called a *local family* if all pairs of formal distributions from \mathcal{F} are local. Then, the pair $(\mathfrak{g}, \mathcal{F})$ is called a *formal distribution Lie superalgebra* if \mathcal{F} is a local family of \mathfrak{g} -valued formal distributions and \mathfrak{g} is spanned by the coefficients of all formal distributions in \mathcal{F} . We define the *formal δ -function* by

$$\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z}\right)^n.$$

Then it is easy to show ([6, Corollary 2.2]), that two local formal distributions are local if and only if the bracket can be represented as a finite sum of the form

$$[a(z), b(w)] = \sum_j [a(z)_{(j)} b(w)] \partial_w^j \delta(z-w)/j!,$$

where $[a(z)_{(j)} b(w)] = \text{Res}_z (z-w)^j [a(z), b(w)]$. This is called the *operator product expansion*. Then we obtain a family of operations (n) , $n \in \mathbb{Z}_+$, on the space of formal distributions. By taking the generating series of these operations, we define the λ -bracket:

$$[a_\lambda b] = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} [a_{(n)} b].$$

The properties of the λ -bracket motivate the following definition:

Definition 2.1. A Lie conformal superalgebra R is a left $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[\partial]$ -module endowed with a \mathbb{C} -linear map $R \otimes R \rightarrow \mathbb{C}[\lambda] \otimes R$, $a \otimes b \mapsto a_\lambda b$, called the λ -bracket, and satisfying the following axioms ($a, b, c \in R$),

$$\begin{aligned} \text{Conformal sesquilinearity} \quad & [\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b], \\ \text{Skew-symmetry} \quad & [a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial} a], \\ \text{Jacobi identity} \quad & [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]]. \end{aligned}$$

Here and further $p(a) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of a .

A Lie conformal superalgebra is called *finite* if it has finite rank as a $\mathbb{C}[\partial]$ -module. The notions of homomorphism, ideal and subalgebras of a Lie conformal superalgebra are defined in the usual way. A Lie conformal superalgebra R is *simple* if $[R_\lambda R] \neq 0$ and contains no ideals except for zero and itself.

Definition 2.2. A module M over a Lie conformal superalgebra R is a $\mathbb{Z}/2\mathbb{Z}$ -graded $\mathbb{C}[\partial]$ -module endowed with a \mathbb{C} -linear map $R \otimes M \rightarrow \mathbb{C}[\lambda] \otimes M$, $a \otimes v \mapsto a_\lambda v$, satisfying the following axioms ($a, b \in R$), $v \in M$,

$$\begin{aligned} (M1)_\lambda \quad & (\partial a)_\lambda^M v = [\partial^M, a_\lambda^M]v = -\lambda a_\lambda^M v, \\ (M2)_\lambda \quad & [a_\lambda^M, b_\mu^M]v = [a_\lambda b]_{\lambda+\mu}^M v. \end{aligned}$$

An R -module M is called *finite* if it is finitely generated over $\mathbb{C}[\partial]$. An R -module M is called *irreducible* if it contains no non-trivial submodule, where the notion of submodule is the usual one.

Given a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$ denote by $\bar{\mathcal{F}}$ the minimal subspace of $\mathfrak{g}[[z, z^{-1}]]$ which contains \mathcal{F} and is closed under all j -th products and invariant under ∂_z . Due to Dong's lemma [6], we know that $\bar{\mathcal{F}}$ is a local family as well. Then $\text{Conf}(\mathfrak{g}, \mathcal{F}) := \bar{\mathcal{F}}$ is the Lie conformal superalgebra associated to the formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$.

In order to give the (more or less) reverse functorial construction, we need the following: let $\tilde{R} = R[t, t^{-1}]$ with $\tilde{\partial} = \partial + \partial_t$ and define the bracket [6]:

$$[at^n, bt^m] = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} [a_j b] t^{m+n-j}. \quad (2.1)$$

Observe that $\tilde{\partial}\tilde{R}$ is an ideal of \tilde{R} with respect to this bracket. Now, consider $\text{Alg}R = \tilde{R}/\tilde{\partial}\tilde{R}$ with this bracket and let

$$\mathcal{R} = \left\{ \sum_{n \in \mathbb{Z}} (at^n) z^{-n-1} = a\delta(t-z) \mid a \in R \right\}.$$

Then $(\text{Alg}R, \mathcal{R})$ is a formal distribution Lie superalgebra. Note that Alg is a functor from the category of Lie conformal superalgebras to the category of formal distribution Lie superalgebras. On has [6]:

$$\text{Conf}(\text{Alg}R) = R, \quad \text{Alg}(\text{Conf}(\mathfrak{g}, \mathcal{F})) = (\text{Alg}\bar{\mathcal{F}}, \bar{\mathcal{F}}).$$

Note also that $(\text{Alg}R, \mathcal{R})$ is the *maximal formal distribution superalgebra* associated to the conformal superalgebra R , in the sense that all formal distribution Lie superalgebras $(\mathfrak{g}, \mathcal{F})$ with $\text{Conf}(\mathfrak{g}, \mathcal{F}) = R$ are quotients of $(\text{Alg}R, \mathcal{R})$ by irregular ideals (that is, an ideal I in \mathfrak{g} with no non-zero $b(z) \in \mathcal{R}$ such that $b_n \in I$). Such formal distribution Lie superalgebras are called *equivalent*.

We thus have an equivalence of categories of Lie conformal superalgebras and equivalence classes of formal distribution Lie superalgebras. So the study of formal distribution Lie superalgebras reduces to the study of Lie conformal superalgebras.

An important tool for the study of Lie conformal superalgebras and their modules is the (extended) annihilation superalgebra. The *annihilation superalgebra* of a Lie conformal superalgebra R is the subalgebra $\mathcal{A}(R)$ (also denoted by $\text{Alg}R_+$) of the Lie superalgebra $\text{Alg}R$ spanned by all elements at^n , where $a \in R, n \in \mathbb{Z}_+$. It is clear from (2.1) that this is a subalgebra, which is invariant with respect to the derivation $\partial = -\partial_t$ of $\text{Alg}R$. The *extended annihilation superalgebra* is defined as

$$\mathcal{A}(R)^e = (\text{Alg}R)^+ := \mathbb{C}\partial \ltimes (\text{Alg}R)_+.$$

Introducing the generating series

$$a_\lambda = \sum_{j \in \mathbb{Z}_+} \frac{\lambda^j}{j!} (at^j), \quad a \in R, \quad (2.2)$$

we obtain from (2.1):

$$[a_\lambda, b_\mu] = [a_\lambda b]_{\lambda+\mu}, \quad \partial(a_\lambda) = (\partial a)_\lambda = -\lambda a_\lambda. \quad (2.3)$$

Formula (2.3) implies the following important proposition relating modules over a Lie conformal superalgebra R to certain modules over the corresponding extended annihilation superalgebra $(\text{Alg}R)^+$.

Proposition 2.3. [2] *A module over a Lie conformal superalgebra R is the same as a module over the Lie superalgebra $(\text{Alg}R)^+$ satisfying the property*

$$a_\lambda m \in \mathbb{C}[\lambda] \otimes M \quad \text{for any } a \in R, m \in M. \quad (2.4)$$

(One just views the action of the generating series a_λ of $(\text{Alg}R)^+$ as the λ -action of $a \in R$).

The problem of classifying modules over a Lie conformal superalgebra R is thus reduced to the problem of classifying a class of modules over the Lie superalgebra $(\text{Alg}R)^+$.

Let \mathfrak{g} be a Lie superalgebra satisfying the following three conditions (cf. [3], p.911):

(L1) \mathfrak{g} is \mathbb{Z} -graded of finite depth $d \in \mathbb{N}$, i.e. $\mathfrak{g} = \bigoplus_{j \geq -d} \mathfrak{g}_j$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

(L2) There exists a semisimple element $z \in \mathfrak{g}_0$ such that its centralizer in \mathfrak{g} is contained in \mathfrak{g}_0 .

(L3) There exists an element $\partial \in \mathfrak{g}_{-d}$ such that $[\partial, g_i] = g_{i-d}$, for $i \geq 0$.

Some examples of Lie superalgebras satisfying (L1)-(L3) are provided by annihilation superalgebras of Lie conformal superalgebras.

If \mathfrak{g} is the annihilation superalgebra of a Lie conformal superalgebra, then the modules V over \mathfrak{g} that correspond to finite modules over the corresponding Lie conformal superalgebra satisfy the following conditions:

(1) For all $v \in V$ there exists an integer $j_0 \geq -d$ such that $\mathfrak{g}_j v = 0$, for all $j \geq j_0$.

(2) V is finitely generated over $\mathbb{C}[\partial]$.

Motivated by this, the \mathfrak{g} -modules satisfying these two properties are called *finite conformal modules*.

We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_{<0} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{>0}, \quad \text{with } g_{<0} = \bigoplus_{j<0} \mathfrak{g}_j, \quad g_{>0} = \bigoplus_{j>0} \mathfrak{g}_j. \quad (2.5)$$

Let $\mathfrak{g}_{\geq 0} = \bigoplus_{j \geq 0} \mathfrak{g}_j$. Given a $\mathfrak{g}_{\geq 0}$ -module F , we may consider the associated induced \mathfrak{g} -module

$$\text{Ind}(F) = \text{Ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} F = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F,$$

called the *generalized Verma module* associated to F . We shall identify $\text{Ind}(F)$ with $U(\mathfrak{g}_{<0}) \otimes F$ via the PBW theorem.

Let V be an \mathfrak{g} -module. The elements of the subspace

$$\text{Sing}(V) := \{v \in V \mid \mathfrak{g}_{>0}v = 0\}$$

are called *singular vectors*. For us the most important case is when $V = \text{Ind}(F)$. The $\mathfrak{g}_{\geq 0}$ -module F is canonically an $\mathfrak{g}_{\geq 0}$ -submodule of $\text{Ind}(F)$, and $\text{Sing}(F)$ is a subspace of $\text{Sing}(\text{Ind}(F))$, called the *subspace of trivial singular vectors*. Observe that $\text{Ind}(F) = F \oplus F_+$, where $F_+ = U_+(\mathfrak{g}_{<0}) \otimes F$ and $U_+(\mathfrak{g}_{<0})$ is the augmentation ideal of the algebra $U(\mathfrak{g}_{<0})$. Then non-zero elements of the space

$$\text{Sing}_+(\text{Ind}(F)) := \text{Sing}(\text{Ind}(F)) \cap F_+$$

are called *non-trivial singular vectors*. The following key result will be used in the rest of the paper, see [7, 3].

Theorem 2.4. *Let \mathfrak{g} be a Lie superalgebra that satisfies (L1)-(L3).*

(a) *If F is an irreducible finite-dimensional $\mathfrak{g}_{\geq 0}$ -module, then the subalgebra $\mathfrak{g}_{>0}$ acts trivially on F and $\text{Ind}(F)$ has a unique maximal submodule.*

(b) *Denote by $\text{Ir}(F)$ the quotient by the unique maximal submodule of $\text{Ind}(F)$. Then the map $F \mapsto \text{Ir}(F)$ defines a bijective correspondence between irreducible finite-dimensional \mathfrak{g}_0 -modules and irreducible finite conformal \mathfrak{g} -modules.*

(c) *A \mathfrak{g} -module $\text{Ind}(F)$ is irreducible if and only if the \mathfrak{g}_0 -module F is irreducible and $\text{Ind}(F)$ has no non-trivial singular vectors.*

In the following section we will describe the Lie conformal superalgebra K_n and its annihilation superalgebra $K(1, n)_+$. In the remaining sections we shall study the induced $K(1, n)_+$ -modules and its singular vectors in order to apply Theorem 2.4 to get the classification of irreducible finite modules over the Lie conformal algebra K_n .

3. LIE CONFORMAL ALGEBRA K_n AND ANNIHILATION LIE ALGEBRA $K(1, n)_+$

Let $\Lambda(n)$ be the Grassmann superalgebra in the n odd indeterminates $\xi_1, \xi_2, \dots, \xi_n$. Let t be an even indeterminate, $\Lambda(1, n) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$, and consider the superalgebra of derivations of the superalgebra $\Lambda(1, n)$:

$$W(1, n) = \left\{ a\partial_t + \sum_{i=1}^n a_i \partial_i \mid a, a_i \in \Lambda(1, n) \right\}, \quad (3.1)$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$ and $\partial_t = \frac{\partial}{\partial t}$. The contact superalgebra $K(1, n)$ is the subalgebra of $W(1, n)$ defined by

$$K(1, n) := \{D \in W(1, n) \mid D\omega = f_D\omega, \text{ for some } f_D \in \Lambda(1, n)\}, \quad (3.2)$$

where $\omega = dt - \sum_{i=1}^n \xi_i d\xi_i$ is the standard contact form, and the action of D on ω is the usual action of vector fields on differential forms.

The space $\Lambda(1, n)$ can be identified with the Lie superalgebra $K(1, n)$ via the map

$$f \mapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^n \left(\xi_i \partial_t f + \partial_i f \right) \left(\xi_i \partial_t + \partial_i \right),$$

the corresponding Lie bracket for elements $f, g \in \Lambda(1, n)$ being

$$[f, g] = \left(2f - \sum_{i=1}^n \xi_i \partial_i f \right) (\partial_t g) - (\partial_t f) \left(2g - \sum_{i=1}^n \xi_i \partial_i g \right) + (-1)^{p(f)} \sum_{i=1}^n (\partial_i f) (\partial_i g).$$

The Lie superalgebra $K(1, n)$ is a formal distribution Lie superalgebra with the following family of mutually local formal distributions

$$a(z) = \sum_{j \in \mathbb{Z}} (at^j) z^{-j-1}, \text{ for } a = \xi_{i_1} \dots \xi_{i_r} \in \Lambda(n).$$

The associated Lie conformal superalgebra K_n is identified with

$$K_n = \mathbb{C}[\partial] \otimes \Lambda(n), \quad (3.3)$$

the λ -bracket for $f = \xi_{i_1} \dots \xi_{i_r}, g = \xi_{j_1} \dots \xi_{j_s}$ being as follows [5]:

$$[f_\lambda g] = \left((r-2)\partial(fg) + (-1)^r \sum_{i=1}^n (\partial_i f) (\partial_i g) \right) + \lambda(r+s-4)fg. \quad (3.4)$$

The Lie conformal superalgebra K_n has rank 2^n over $\mathbb{C}[\partial]$. It is simple for $n \geq 0, n \neq 4$, and the derived algebra K'_4 is simple and has codimension 1 in K_4 .

The annihilation superalgebra is

$$\mathcal{A}(K_n) = K(1, n)_+ = \Lambda(1, n)_+ := \mathbb{C}[t] \otimes \Lambda(n), \quad (3.5)$$

and the extended annihilation superalgebra is

$$\mathcal{A}(K_n)^e = K(1, n)^+ = \mathbb{C}\partial \ltimes K(1, n)_+,$$

where ∂ acts on it as $-\text{ad}\partial_t$. Note that $\mathcal{A}(K_n)^e$ is isomorphic to the direct sum of $\mathcal{A}(K_n)$ and the trivial 1-dimensional Lie algebra $\mathbb{C}(\partial + \frac{1}{2})$.

The Lie superalgebra $K(1, n)$ is \mathbb{Z} -graded by putting

$$\deg(t^m \xi_{i_1} \dots \xi_{i_k}) = 2m + k - 2,$$

and it induces a gradation on $K(1, n)_+$ making it a \mathbb{Z} -graded Lie superalgebra of depth 2: $K(1, n)_+ = \bigoplus_{j \geq -2} (K(1, n)_+)_j$. It is easy to check that $K(1, n)_+$ satisfies conditions (L1)-(L3).

Observe that $K(1, n)_+$ is the subalgebra of

$$W(1, n)_+ = \{a\partial_t + \sum_{i=1}^n a_i \partial_i \mid a, a_i \in \Lambda(1, n)_+\}, \quad (3.6)$$

defined by (cf.(3.2))

$$K(1, n)_+ := \{D \in W(1, n)_+ \mid D\omega = f_D\omega, \text{ for some } f_D \in \Lambda(1, n)_+\}. \quad (3.7)$$

4. INDUCED MODULES

Using Theorem 2.4, the classification of finite irreducible K_n -modules can be reduced to the study of induced modules for $K(1, n)_+$. Observe that

$$\begin{aligned} (K(1, n)_+)_{-2} &= \langle \{\mathbf{1}\} \rangle, \\ (K(1, n)_+)_{-1} &= \langle \{\xi_i : 1 \leq i \leq n\} \rangle \\ (K(1, n)_+)_0 &= \langle \{t\} \cup \{\xi_i \xi_j : 1 \leq i < j \leq n\} \rangle \end{aligned} \quad (4.1)$$

We shall use the following notation for the basis elements of $(K(1, n)_+)_0$:

$$E_{00} = t, \quad F_{ij} = -\xi_i \xi_j. \quad (4.2)$$

Observe that $(K(1, n)_+)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(n) \simeq \mathfrak{cso}(n)$. Take

$$\partial := -\frac{1}{2}\mathbf{1} \quad (4.3)$$

as the element that satisfies (L3) in section 2.

For the rest of this work, \mathfrak{g} will be $K(1, n)_+$. Let F be a finite-dimensional irreducible \mathfrak{g}_0 -module, which we extend to a $\mathfrak{g}_{\geq 0}$ -module by letting \mathfrak{g}_j with $j > 0$ acting trivially. Then we shall identify, as above

$$\text{Ind}(F) \simeq \Lambda(1, n) \otimes F \simeq \mathbb{C}[\partial] \otimes \Lambda(n) \otimes F \quad (4.4)$$

as \mathbb{C} -vector spaces. In order to describe the action of \mathfrak{g} in $\text{Ind}(F)$ we introduce the following notation:

$$\begin{aligned} \xi_I &:= \xi_{i_1} \dots \xi_{i_k}, & \text{if } I &= \{i_1, \dots, i_k\}, \\ \partial_L \xi_I &:= \partial_{l_1} \dots \partial_{l_s} \xi_I & \text{if } L &= \{l_1, \dots, l_s\}, \\ \partial_f \xi_I &:= \partial_L \xi_I & \text{if } f &= \xi_L, \\ |f| &:= k & \text{if } f &= \xi_{i_1} \dots \xi_{i_k}. \end{aligned} \quad (4.5)$$

In the following theorem, we describe the \mathfrak{g} -action on $\text{Ind}(F)$ using the λ -action notation in (2.2), i.e.

$$f_\lambda(g \otimes v) = \sum_{j \geq 0} \frac{\lambda^j}{j!} (t^j f) \cdot (g \otimes v)$$

for $f, g \in \Lambda(n)$ and $v \in F$.

Theorem 4.1. *For any monomials $f, g \in \Lambda(n)$ and $v \in F$, where F is a \mathfrak{g}_0 -module, we have the following formula for the λ -action of $\mathfrak{g} = K(1, n)_+$ on $\text{Ind}(F)$:*

$$\begin{aligned} f_\lambda(g \otimes v) &= \\ &= (-1)^{p(f)} (|f| - 2) \partial(\partial_f g) \otimes v + \sum_{i=1}^n \partial_{(\partial_i f)} (\xi_i g) \otimes v + (-1)^{p(f)} \sum_{r < s} \partial_{(\partial_r \partial_s f)} g \otimes F_{rs} v \\ &+ \lambda \left[(-1)^{p(f)} (\partial_f g) \otimes E_{00} v + (-1)^{p(f)+p(g)} \sum_{i=1}^n (\partial_f(\partial_i g)) \xi_i \otimes v + \sum_{i \neq j} \partial_{(\partial_i f)} (\partial_j g) \otimes F_{ij} v \right] \\ &+ \lambda^2 (-1)^{p(f)} \sum_{i < j} \partial_f (\partial_i \partial_j g) \otimes F_{ij} v. \end{aligned}$$

The proof of this theorem will be done through several lemmas. Since this is quite technical, we have moved the proof into Appendix A.

In the last part of this section we shall prove an easier formula for the λ -action in the induced module. This is done by taking the Hodge dual of the basis (cf. [3], pp. 922 and observe the difference). More precisely, for a monomial $\xi_I \in \Lambda(n)$, we let $\overline{\xi_I}$ be its Hodge dual, i.e. the unique monomial in $\Lambda(n)$ such that $\xi_I \xi_I = \xi_1 \dots \xi_n$.

Lemma 4.2. *For any monomials elements $f = \xi_I, g = \xi_L$, we have*

$$\begin{aligned}\overline{\partial_i g} &= \overline{g} \xi_i = (-1)^{|\overline{g}|} \xi_i \overline{g}, \\ \overline{\partial_f(g)} &= (-1)^{\frac{|f|(|f|-1)}{2} + |f||\overline{g}|} f \overline{g}, \\ \overline{\xi_i g} &= -(-1)^{|\overline{g}|} \partial_i \overline{g}, \\ \overline{g \xi_i} &= -(-1)^n \partial_i \overline{g}.\end{aligned}$$

Proof. The proof is left to the reader. \square

The following theorem translates Theorem 4.1 in terms of the Hodge dual basis.

Theorem 4.3. *Let F be a $\mathfrak{g}_0 = \mathfrak{cso}(n)$ -module. Then the λ -action of $K(1, n)_+$ in $\text{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(n) \otimes F$, given by Theorem 4.1, is equivalent to the following one:*

$$\begin{aligned}f_\lambda(g \otimes v) &= (-1)^{\frac{|f|(|f|+1)}{2} + |f||g|} \times \\ &\times \left\{ (|f| - 2) \partial(fg) \otimes v - (-1)^{p(f)} \sum_{i=1}^n (\partial_i f)(\partial_i g) \otimes v - \sum_{r < s} (\partial_r \partial_s f) g \otimes F_{rs} v \right. \\ &+ \lambda \left[fg \otimes E_{00} v - (-1)^{p(f)} \sum_{i=1}^n \partial_i(f \xi_i g) \otimes v + (-1)^{p(f)} \sum_{i \neq j} (\partial_i f) \xi_j g \otimes F_{ij} v \right] \\ &\left. - \lambda^2 \sum_{i < j} f \xi_i \xi_j g \otimes F_{ij} v \right\}.\end{aligned}$$

Proof. By simple computations, using Lemma 4.2, it is easy to obtain the λ -action in the Hodge dual basis. That is, let T be the vector space automorphism of $\text{Ind}(F)$ given by $T(g \otimes v) = \overline{g} \otimes v$, then the theorem gives the formula for the composition $T \circ (f_\lambda \cdot) \circ T^{-1}$. For example, in order to "dualize" the second term in the λ -action in Theorem 4.1, we write $\overline{\partial_{(\partial_i f)} \xi_i g}$ in terms of f and \overline{g} , as follows:

$$\begin{aligned}\overline{\partial_{(\partial_i f)} \xi_i g} &= (-1)^{\frac{(|f|-1)(|f|-2)}{2} + (|f|-1)|\overline{\xi_i g}|} (\partial_i f) \overline{\xi_i g} \\ &= (-1)^{\frac{(|f|-1)(|f|-2)}{2} + (|f|-1)(|\overline{g}|-1) + 1 + |\overline{g}|} (\partial_i f)(\partial_i \overline{g}) \\ &= (-1)^{\frac{|f|(|f|+1)}{2} + |f||\overline{g}| + 1 + |f|} (\partial_i f)(\partial_i \overline{g}),\end{aligned}$$

obtaining the second summand in the formula, given by the theorem. By similar computations the proof follows. \square

5. SINGULAR VECTORS

By Theorem 2.4, the classification of irreducible finite modules over the Lie conformal superalgebra K_n reduces to the study of singular vectors in the induced

modules $\text{Ind}(F)$, where F is an irreducible finite-dimensional $\mathfrak{so}(n)$ -module. This section will be devoted to the classification of singular vectors.

When we discuss the highest weight of vectors and singular vectors, we always mean with respect to the upper Borel subalgebra in $K(1, n)_+$ generated by $(K(1, n)_+)_{>0}$ and the elements of the Borel subalgebra of $\mathfrak{so}(n)$ in $(K(1, n)_+)_0$. More precisely, recall (4.2), where we defined $F_{ij} = -\xi_i \xi_j \in (K(1, n)_+)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(n)$. Observe that F_{ij} corresponds to $E_{ij} - E_{ji} \in \mathfrak{so}(n)$, where E_{ij} are the elements of the standard basis of matrices. Consider the following (standard) notation (cf. [8], p.83):

Case $\mathfrak{g} = \mathfrak{so}(2m + 1, \mathbb{C})$: Here we take

$$H_j = i F_{2j-1, 2j}, \quad 1 \leq j \leq m, \quad (5.1)$$

a basis of a Cartan subalgebra \mathfrak{h} . Let $\varepsilon_j \in \mathfrak{h}^*$ be given by $\varepsilon_j(H_k) = \delta_{jk}$. Let

$$\Delta = \{\pm \varepsilon_i \pm \varepsilon_j \mid i \neq j\} \cup \{\pm \varepsilon_k\}$$

be the set of roots. The root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad \text{with } \mathfrak{g}_\alpha = \mathbb{C}E_\alpha$$

where, for $1 \leq l < j \leq m$ and $1 \leq k \leq m$,

$$\begin{aligned} E_{\varepsilon_l - \varepsilon_j} &= F_{2l-1, 2j-1} + F_{2l, 2j} + i(F_{2l-1, 2j} - F_{2l, 2j-1}), \\ E_{\varepsilon_l + \varepsilon_j} &= F_{2l-1, 2j-1} - F_{2l, 2j} - i(F_{2l-1, 2j} + F_{2l, 2j-1}), \\ E_{-(\varepsilon_l - \varepsilon_j)} &= F_{2l-1, 2j-1} + F_{2l, 2j} - i(F_{2l-1, 2j} - F_{2l, 2j-1}), \\ E_{-(\varepsilon_l + \varepsilon_j)} &= F_{2l-1, 2j-1} - F_{2l, 2j} + i(F_{2l-1, 2j} + F_{2l, 2j-1}), \\ E_{\pm \varepsilon_k} &= F_{2k-1, 2m+1} \mp iF_{2k, 2m+1}. \end{aligned} \quad (5.2)$$

Let $\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}$ and $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid i \leq j\} \cup \{\varepsilon_k\}$, be the simple and positive roots respectively. Consider

$$\begin{aligned} \alpha_{lj} &:= F_{2l-1, 2j-1} - iF_{2l, 2j-1} = \frac{1}{2}(E_{\varepsilon_l - \varepsilon_j} + E_{\varepsilon_l + \varepsilon_j}) \\ \beta_{lj} &:= F_{2l, 2j} + iF_{2l-1, 2j} = \frac{1}{2}(E_{\varepsilon_l - \varepsilon_j} - E_{\varepsilon_l + \varepsilon_j}) \end{aligned} \quad (5.3)$$

$$\gamma_k := E_{\varepsilon_k}.$$

Then,

$$B_{\mathfrak{so}(2m+1)} = \langle \{\alpha_{lj}, \beta_{lj}, \gamma_k \mid 1 \leq i < j \leq m, 1 \leq k \leq m\} \rangle. \quad (5.4)$$

Case $\mathfrak{g} = \mathfrak{so}(2m, \mathbb{C})$: Here we take

$$H_j = i F_{2j-1, 2j}, \quad 1 \leq j \leq m,$$

a basis of a Cartan subalgebra \mathfrak{h} , as with $\mathfrak{so}(2m + 1)$. In this case

$$\Delta = \{\pm \varepsilon_i \pm \varepsilon_j \mid i \neq j\}$$

is the set of roots. Let $\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\}$ and $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid i \leq j\}$, be the simple and positive roots respectively. Then,

$$B_{\mathfrak{so}(2m)} = \langle \{\alpha_{lj}, \beta_{lj} \mid 1 \leq i < j \leq m\} \rangle. \quad (5.5)$$

In order to write explicitly weights for vectors in $K(1, n)_+$ -modules, we will consider the basis for the Cartan subalgebra \mathfrak{h} in $(K(1, n)_+)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(n)$, introduce above:

$$E_{00}; H_1, \dots, H_m, m = [n/2],$$

and we shall write the weight of an eigenvector for the Cartan subalgebra \mathfrak{h} as an $m + 1$ -tuple for the corresponding eigenvalues of this basis:

$$\mu = (\mu_0; \mu_1, \dots, \mu_m). \quad (5.6)$$

Observe that a vector \vec{m} in the $K(1, n)_+$ -module $\text{Ind}(F)$ is a singular highest weight vector if and only if the following conditions are satisfied

- (S1) $\frac{d^2}{d\lambda^2} (f \lambda \vec{m}) = 0$ for all $f \in \Lambda(n)$,
- (S2) $\frac{d}{d\lambda} (f \lambda \vec{m})|_{\lambda=0} = 0$ for all $f = \xi_I$ with $|I| \geq 1$,
- (S3) $(f \lambda \vec{m})|_{\lambda=0} = 0$ for all $f = \xi_I$ with $|I| \geq 3$ or $f \in B_{\mathfrak{so}(n)}$.

In order to classify the finite irreducible K_n -modules we should solve the equations (S1-3) to obtain the singular vectors. The next theorem is the main result of this section and gives us the complete classification of singular vectors:

Theorem 5.1. *Let F be an irreducible finite-dimensional $\mathfrak{cso}(n)$ -module with highest weight μ .*

If $n \geq 4$, then $\vec{m} \in \text{Ind}(F)$ is a non-trivial singular highest weight vector if and only if \vec{m} is one of the following vectors (in the Hodge dual basis):

- (a) $\vec{m} = (\xi_{\{2\}^c} - i \xi_{\{1\}^c}) \otimes v_\mu$, where v_μ is a highest weight vector of the $\mathfrak{cso}(n)$ -module F and $\mu = (-k; k, 0, \dots, 0)$, with $k \in \mathbb{Z}_{>0}$,
- (b) $\vec{m} = \sum_{l=1}^m \left[(\xi_{\{2l\}^c} + i \xi_{\{2l-1\}^c}) \otimes w_l + (\xi_{\{2l\}^c} - i \xi_{\{2l-1\}^c}) \otimes \bar{w}_l \right] - \delta_{n, \text{odd}} i \xi_{\{2m+1\}^c} \otimes w_{m+1}$,

where $w_1 = v_\mu$ is a highest weight vector of the $\mathfrak{cso}(n)$ -module F with highest weight

$$\mu = (n + k - 2; k, 0, \dots, 0), \text{ for } k \in \mathbb{Z}_{\geq 0},$$

and all w_l, \bar{w}_l are non-zero and uniquely determined by v_μ .

If $n = 3$, then $\vec{m} \in \text{Ind}(F)$ is a non-trivial singular highest weight vector if and only if \vec{m} is one of the following vectors:

- (a) $\vec{m} = (\xi_{\{2\}^c} - i \xi_{\{1\}^c}) \otimes v_\mu$, where v_μ is a highest weight vector of the $\mathfrak{cso}(3)$ -module V and $\mu = (-k; k)$, with $k \in \frac{1}{2}\mathbb{Z}_{>0}$,
- (b) $\vec{m} = (\xi_{\{2\}^c} + i \xi_{\{1\}^c}) \otimes v_\mu + (\xi_{\{2\}^c} - i \xi_{\{1\}^c}) \otimes w_1 - i \xi_{\{3\}^c} \otimes w_2$,

where v_μ is a highest weight vector of the $\mathfrak{cso}(3)$ -module F with highest weight

$$\mu = (k + 1; k), \text{ for } k \in \frac{1}{2}\mathbb{Z}_{\geq 0} \text{ and } k \neq \frac{1}{2},$$

and all w_1, w_2 are non-zero and uniquely determined by v_μ .

- (c) $\vec{m} = \partial (\xi_* \otimes v_\mu) + i \xi_{\{1,2\}^c} \otimes v_\mu - 2\xi_{\{2,3\}^c} \otimes F_{2,3}v_\mu + 2 \xi_{\{1,3\}^c} \otimes F_{1,3}v_\mu$, where v_μ is a highest weight vector of the $\mathfrak{cso}(3)$ -module F with highest weight $\mu = (\frac{3}{2}; \frac{1}{2})$.

The proof of this theorem will be done through several lemmas. Since this is quite technical, we have moved the proof into appendix B.

Remark 5.2. (a) The explicit expression of all non-zero vectors w_l, \bar{w}_l in terms of v_μ that appear in the second family of singular vectors for all $n \geq 3$, are written in (9.68), (9.69), (9.70), (9.71) and (9.72).

(b) If $n=4$, the first family of singular vectors $\vec{m} = (\xi_{\{2\}^c} - i \xi_{\{1\}^c}) \otimes v_\mu$, where v_μ is a highest weight vector of the $\mathfrak{cso}(4)$ -module F and $\mu = (-k; k, 0)$, with $k \in \mathbb{Z}_{>0}$, corresponds to the family of singular vectors b_2 in Proposition 7.2(i) in [3]. Finally, the second family of singular vectors in Theorem 5.1(b), correspond to the family of singular vectors b_5 in Proposition 7.2(ii) in [3].

(c) If $n=3$, the singular vectors in the cases (a), (b) and (c) described in the previous theorem, correspond to the vectors a_2, a_4 and a_6 in Proposition 5.1 in [3], respectively. Observe that the families (a) and (b) described for $n \geq 4$ correspond to the families (a) and (b) for $n = 3$, but in the latter case the parameter k is one half a positive integer. Observe that the missing case $(k + 1; k)$ with $k = \frac{1}{2}$ in the family (b) is completed by the case (c).

6. MODULES OF DIFFERENTIAL FORMS, THE CONTACT COMPLEX AND IRREDUCIBLE INDUCED $K(1, n)_+$ -MODULES

Let us recall some standard notation from [1]. In order to define the differential forms one considers an odd variable dt and even variables $d\xi_1, \dots, d\xi_n$ and defines the differential forms to be the (super)commutative algebra freely generated by these variables over $\Lambda(1, n)_+ = \mathbb{C}[t] \otimes \Lambda(n)$, or

$$\Omega_+ = \Omega_{n,+} := \Lambda(1, n)_+[d\xi_1, \dots, d\xi_n] \otimes \Lambda(dt).$$

Generally speaking Ω_+ is just a polynomial (super)algebra over the variables

$$t, \xi_1, \dots, \xi_n, dt, d\xi_1, \dots, d\xi_n,$$

where the parity is

$$p(t) = 0, \quad p(\xi_i) = 1, \quad p(dt) = 1, \quad p(d\xi_i) = 0.$$

These are called (*polynomial*) *differential forms*, and we define the *Laurent differential forms* to be the same algebra over $\Lambda(1, n) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$:

$$\Omega = \Omega_n = \Omega(1, n) := \Lambda(1, n)[d\xi_1, \dots, d\xi_n] \otimes \Lambda[dt].$$

We would like to consider a fixed complementary subspace Ω_- to Ω_+ in Ω chosen as follows

$$\Omega_- = \Omega_{n,-} := t^{-1}\mathbb{C}[t^{-1}] \otimes \Lambda(n) \otimes \mathbb{C}[d\xi_1, \dots, d\xi_n] \otimes \Lambda[dt].$$

For the differential forms we need the usual differential degree that measure only the involvement of the differential variables $dt, d\xi_1, \dots, d\xi_n$, that is

$$\deg t = 0, \quad \deg \xi_i = 0, \quad \deg dt = 1, \quad \deg d\xi_i = 1,$$

which gives the *standard \mathbb{Z} -gradation* both of Ω and Ω_{\pm} . As usual, we denote by Ω^k, Ω_{\pm}^k the corresponding graded components, and if we need to take care of the dependance on n they will be denoted by Ω_n^k and $\Omega_{n,\pm}^k$, respectively.

We denote by Ω_c^k the subspace of differential forms with constant coefficients in Ω^k .

The operator d is defined on Ω as usual, as an odd derivation, such that $d(t) = dt, d(\xi_i) = d\xi_i, d(dt) = d(d\xi_i) = 0$. Observe that d maps both Ω_+ and Ω_- into themselves and that $d^2 = 0$.

As usual, we extend the natural action of $W(1, n)_+$ on $\Lambda(1, n)$ to the whole Ω by imposing the property that D (super)commutes with d . It is clear that Ω_+ and all the subspaces Ω^k are $W(1, n)_+$ -invariant. Hence Ω_{\pm}^k and Ω^k are $W(1, n)_+$ -modules, which are called the *natural representations* of $W(1, n)_+$ in differential forms.

We define the action of $W(1, n)_+$ on Ω_- via the isomorphism of Ω_- with the factor of Ω by Ω_+ . Practically this means that in order to compute $D(f)$, where $f \in \Omega_-$, we apply D to f and "disregard terms with non-negative powers of t ".

The operator d restricted to Ω_{\pm}^k defines an odd morphism between the corresponding representations. Clearly the image and the kernel of such a morphism are submodules in Ω_{\pm}^k . The second statement of the following result is Proposition 4.1(3) in [1]. Now, we complete the proof of this result.

Proposition 6.1. (a) *The maps $d : \Omega_+^l \rightarrow \Omega_+^{l+1}$ are morphisms of $W(1, n)_+$ -modules. The kernel of one of them is equal to the image of the next one and it is a non-trivial proper submodule in Ω_+^l .*

(b) *The dual maps $d^{\#} : (\Omega_+^{l+1})^{\#} \rightarrow (\Omega_+^l)^{\#}$ are morphisms of $W(1, n)_+$ -modules. The kernel of one of them is equal to the image of the next one and it is a non-trivial proper submodule in $(\Omega_+^l)^{\#}$.*

Proof. (a) Consider the homotopy operator $K : \Omega_{n,+} \rightarrow \Omega_{n,+}$ given by

$$K(d\xi_n \nu) = \xi_n \nu, \quad K(\nu) = 0 \quad \text{if } \nu \text{ does not involve } d\xi_n.$$

Let $\varepsilon : \Omega_{n,+} \rightarrow \Omega_{n,+}$ be defined by

$$\varepsilon(d\xi_n \nu) = \varepsilon(\xi_n \nu) = 0, \quad \varepsilon(\nu) = \nu \quad \text{if } \nu \text{ does not involve both } d\xi_n \text{ and } \xi_n.$$

One can check that $Kd + dK = \text{Id} - \varepsilon$. By standard argument, using this homotopy operator, the proof follows.

(b) Considering the dual maps $K : (\Omega_{n,+})^{\#} \rightarrow (\Omega_{n,+})^{\#}$ and $\varepsilon : (\Omega_{n,+})^{\#} \rightarrow (\Omega_{n,+})^{\#}$, we obtain $K^{\#}d^{\#} + d^{\#}K^{\#} = \text{Id} - \varepsilon^{\#}$.

Therefore, if $\alpha \in (\Omega_{n,+})^{\#}$ is a closed form, we get $\alpha = d^{\#}(K^{\#}\alpha) + \varepsilon^{\#}(\alpha)$, and $\varepsilon^{\#}(\alpha)$ is also a closed form. Observe that $(\varepsilon^{\#}\alpha)(\nu) = \alpha(\varepsilon(\nu)) = 0$ if ν involve $d\xi_n$ or ξ_n . Hence $\varepsilon^{\#}\alpha$ is essentially an element in $(\Omega_{n-1,+})^{\#}$, namely it is equal to an

element in $(\Omega_{n-1,+})^\#$ trivially extended in ν 's that involve $d\xi_n$ or ξ_n . It follows by induction on n that

$$\alpha = d^\# \alpha_1 + \alpha_0, \quad (6.1)$$

for some $\alpha_0, \alpha_1 \in (\Omega_{n,+})^\#$ and α_0 is a closed form that is a trivial extension of an element $\tilde{\alpha}_0 \in (\Omega_{0,+})^\#$. But $\Omega_{0,+} = \mathbb{C}[t] \otimes \wedge(dt) = \{p(t) + q(t) dt \mid p, q \in \mathbb{C}[t]\}$ and $\tilde{\alpha}_0 \in (\Omega_{0,+})^\#$ is closed iff $\tilde{\alpha}_0(q(t)dt) = 0$ for all $q \in \mathbb{C}[t]$. In general, it is easy to see that $\gamma \in (\Omega_{0,+})^\#$ is exact iff γ is closed (i.e. $\gamma(q(t)dt) = 0$) and $\gamma(1) = 0$. Therefore, using (6.1), we have $\alpha = d^\# \beta + \alpha_0(1) \mathbf{1}^*$, where $\mathbf{1}^*(c1) = c$ and zero everywhere else. Since $\mathbf{1}^* \in (\Omega_{n,+}^0)^\#$, we get the exactness of the sequence

$$\dots \xrightarrow{d^\#} (\Omega_{n,+}^2)^\# \xrightarrow{d^\#} (\Omega_{n,+}^1)^\# \xrightarrow{d^\#} (\Omega_{n,+}^0)^\#.$$

□

Recall that $K(1, n)_+$ is a subalgebra of $W(1, n)_+$, defined by (3.7). Hence Ω_+ and Ω_+^k are $K(1, n)_+$ -modules as well.

Observe that the differential of the standard contact form $\omega = dt - \sum_{i=1}^n \xi_i d\xi_i$ is $d\omega = -\sum_{i=1}^n (d\xi_i)^2$, and following Rumin's construction in [10], consider for $k \geq 2$

$$I^k = d\omega \wedge \Omega^{k-2} + \omega \wedge \Omega^{k-1} \subset \Omega^k, \quad (6.2)$$

$$I_+^k = d\omega \wedge \Omega_+^{k-2} + \omega \wedge \Omega_+^{k-1} \subset \Omega_+^k, \quad (6.3)$$

and $I^1 = \omega \wedge \Omega^0$, $I_+^1 = \omega \wedge \Omega_+^0$, $I^0 = 0 = I_+^0$. It is clear that $d(I^k) \subseteq I^{k+1}$ and $d(I_+^k) \subseteq I_+^{k+1}$, and using (3.7) it is easy to prove that I^k and I_+^k are $K(1, n)_+$ -submodules of Ω^k and Ω_+^k , respectively. Therefore we have the following *contact complex* of $K(1, n)_+$ -modules (we also denote by d the induced maps in the quotients):

$$0 \longrightarrow \mathbb{C} \xrightarrow{d} \Omega_+^0 \xrightarrow{d} \Omega_+^1/I_+^1 \xrightarrow{d} \Omega_+^2/I_+^2 \xrightarrow{d} \dots \quad (6.4)$$

Let $\mathbb{C}[d\xi_i]^l \subseteq \Omega_+^l$ be the subspace of homogeneous polynomials in $d\xi_1, \dots, d\xi_n$ of degree l . Using that the action of $\mathfrak{cso}(n) = \mathbb{C} E_{00} \oplus \mathfrak{so}(n) = (K(1, n)_+)_0$ in Ω_+^l is given by

$$E_{00} \longmapsto 2t \partial_t + \sum_{i=1}^n \xi_i \partial_i, \quad F_{ij} \longmapsto \xi_i \partial_j - \xi_j \partial_i, \quad (6.5)$$

it follows that $\mathbb{C}[d\xi_i]^l$ is a $\mathfrak{cso}(n)$ -invariant subspace. Now, consider $\Gamma^l = \pi(\mathbb{C}[d\xi_i]^l)$, where $\pi : \Omega_+^l \longrightarrow \Omega_+^l/I_+^l$, and take $\Theta^l = (\Gamma^l)^\#$. Here and further, we denote by $\#$ the restricted dual, that is the sum of the dual of all the graded components of the initial module, as in [1], section B1. Then, we have

Proposition 6.2. (1) *The $\mathfrak{cso}(n)$ -module $\Theta^l, l \geq 0$, is irreducible with highest weight $(-l; l, 0, \dots, 0)$.*

(2) *The $K(1, n)_+$ -module $(\Omega_+^l/I_+^l)^\#, l \geq 0$, contains Θ^l and this inclusion induces the isomorphism*

$$(\Omega_+^l/I_+^l)^\# = \text{Ind}(\Theta^l).$$

(3) The dual maps $d^\# : (\Omega_+^{l+1}/I_+^{l+1})^\# \rightarrow (\Omega_+^l/I_+^l)^\#$ are morphisms of $K(1, n)_+$ -modules. The kernel of one of them is equal to the image of the next one and it is a non-trivial proper submodule in $(\Omega_+^l/I_+^l)^\#$.

Proof. (1) Consider $\Gamma^l = \pi(S^l(d\xi_i))$, where $\pi : \Omega_+^l \rightarrow \Omega_+^l/I_+^l$. Observe that

$$\Gamma^l \simeq \mathbb{C}[d\xi_1, \dots, d\xi_n]^l / \mathbb{C}[d\xi_1, \dots, d\xi_n]^{l-2} \left(\sum (d\xi_i)^2 \right),$$

and it is well known that Γ^l are irreducible lowest weight $\mathfrak{cso}(n)$ -modules with lowest weight vector $(d\xi_1 + id\xi_2)^l$ whose weight is $(l; -l, 0, \dots, 0)$, see [8]. Therefore, $\Theta^l = (\Gamma^l)^\#$ are irreducible highest weight $\mathfrak{cso}(n)$ -modules with highest weight $(-l; l, 0, \dots, 0)$.

(2) By the definition of the restricted dual, it is the sum of the dual of all the graded components of the initial module. In our case Γ^l is the component of minimal degree in Ω_+^l/I_+^l , so Θ^l becomes the component of maximal degree in $(\Omega_+^l/I_+^l)^\#$. This implies that $\mathfrak{g}_{>0}$ acts trivially on Θ^l , so the morphism $\text{Ind } \Theta^l \rightarrow (\Omega_+^l/I_+^l)^\#$ is defined. Clearly Ω_+^l/I_+^l is isomorphic to

$$\Gamma^l \otimes \mathbb{C}[t, \xi_1, \dots, \xi_n],$$

so it is a cofree module. Then the module $(\Omega_+^l/I_+^l)^\#$ is a free $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$ -module and the morphism

$$\text{Ind}(\Theta^l) \rightarrow (\Omega_+^l/I_+^l)^\#$$

is therefore an isomorphism.

(3) The first part of this statement follows immediately from the fact that d commutes with the action of vector fields. It remains to prove that the kernel of one of them is equal to the image of the next one.

First, we shall prove the exactness of the sequence (6.4) except for level 1, where we have $\ker d = \text{Im } d + \mathbb{C}t\overline{dt}$. Let $\alpha \in \Omega_+^k$ such that $d\alpha \in I_+^{k+1}$. Then $d\alpha = \omega \wedge \beta + d\omega \wedge \gamma$, with $\beta \in \Omega_+^k$ and $\gamma \in \Omega_+^{k-1}$. Observe that $d(\alpha - \omega \wedge \gamma) = \omega \wedge (\beta - d\gamma)$, hence, by replacing α by another representative, we may assume that $\gamma = 0$. Since $0 = d^2\alpha = d(\omega \wedge \beta) = d\omega \wedge \beta - \omega \wedge d\beta$, then $d\omega \wedge d\alpha = d\omega \wedge (\omega \wedge \beta) = (sgn)\omega \wedge d\omega \wedge \beta = (sgn)\omega \wedge \omega \wedge d\beta = 0$. Therefore, $d\alpha \in \text{Ker}(d\omega \wedge \cdot) = 0$. But the differential complex (Ω_+^*, d) is exact by Proposition 6.1(a), proving the exactness of (6.4). By standard arguments, it is easy to see the exactness of the dual, finishing the proof. \square

Corollary 6.3. *The following $K(1, n)_+$ -modules are isomorphic*

$$\Omega_+^k/I_+^k = (\text{Ind}(\Gamma^k))^*.$$

Let us now study the $K(1, n)_+$ -modules Ω_-^k . Recall that we identified (via isomorphism) Ω_-^k with Ω^k/Ω_+^k . Let $\tilde{\pi} : \Omega^k \rightarrow \Omega^k/\Omega_+^k = \Omega_-^k$. Observe that $I_-^k = \tilde{\pi}(I^k)$ is a $K(1, n)_+$ -submodule of Ω_-^k , and $d(I_-^k) \subseteq I_-^{k+1}$. Let

$$\xi_* = \xi_1 \cdots \xi_n, \quad \text{and} \quad \Gamma_-^k = t^{-1}\xi_*\Omega_c^k \subset \Omega_-^k.$$

Proposition 6.4. *For $\mathfrak{g} = K(1, n)_+$, we have:*

(1) The $\mathfrak{cso}(n)$ -module Γ_-^k is an irreducible submodule of Ω_-^k with highest weight $(n + k - 2; k, 0, \dots, 0)$, for $k \geq 0$,

and $\mathfrak{g}_{>0}$ acts trivially on Γ_-^k .

(2) There is a \mathfrak{g} -module isomorphism $\Omega_-^k/I_-^k = \text{Ind}(\Gamma_-^k)$.

(3) The differential d gives us \mathfrak{g} -module morphisms on Ω_-^k/I_-^k , and the kernel and image of d are \mathfrak{g} -submodules in Ω_-^k/I_-^k .

(4) The kernel of d and image of d in Ω_-^k/I_-^k for $k \geq 2$ coincide, in Ω_-^1/I_-^1 we have $\text{Ker } d = \mathbb{C}(\overline{t^{-1}dt}) + \text{Im } d$, and in Ω_-^0 , we have $\text{Ker } d = 0$.

Proof. (1) First, a simple computation shows that $\mathfrak{g}_{>0}$ maps Γ_-^k to zero. Also, as a \mathfrak{g}_0 -module, Γ_-^k is isomorphic to the space of harmonic polynomials in $d\xi_1, \dots, d\xi_n$ of degree k multiplied by the 1-dimensional module $\langle t^{-1}\xi_* \rangle$. This permits us to see that its highest weight vectors are

$$\begin{aligned} \langle t^{-1}\xi_* \rangle & \quad \text{for } k = 0, \\ \langle t^{-1}\xi_*(d\xi_1 - id\xi_2)^k \rangle & \quad \text{for } k \geq 1. \end{aligned}$$

The values of the highest weights are easy to compute using (6.5).

(2) It is straightforward to see that Ω_-^0 is a free rank 1 $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$ -module. Now, the action of $\partial_0, \partial_1, \dots, \partial_n$ on Ω_-^k/I_-^k is coefficientwise, hence the fact that Ω_-^k/I_-^k is a free $\mathbb{C}[\partial_0, \partial_1, \dots, \partial_n]$ -module follows. This gives us the isomorphism $\Omega_-^k/I_-^k = \text{Ind}(\Gamma_-^k)$.

(3) It follows immediately from the fact that d commutes with the action of vector fields.

(4) Let $\alpha \in \Omega_-^k$ be such that $d\alpha \in I_-^{k+1}$. Then $d\alpha = \omega \wedge \beta + d\omega \wedge \gamma$, with $\beta \in \Omega_-^k$ and $\gamma \in \Omega_-^{k-1}$. Observe that $d(\alpha - \omega \wedge \gamma) = \omega \wedge (\beta - d\gamma)$, hence, by replacing α by another representative, we may assume that $\gamma = 0$. Since $0 = d^2\alpha = d(\omega \wedge \beta) = d\omega \wedge \beta - \omega \wedge d\beta$, then $d\omega \wedge d\alpha = d\omega \wedge (\omega \wedge \beta) = \omega \wedge \omega \wedge d\beta = 0$. Therefore, $d\alpha \in \text{Ker}(d\omega \wedge \cdot) = 0$. But the differential complex (Ω_-^\bullet, d) is exact except for $k = 1$ (see Proposition 4.3 in [1]), proving the statement. \square

In the last part of this section, we classify the irreducible induced $K(1, n)_+$ -modules. Let $\mathfrak{g} = K(1, n)_+$. Now, we have the following:

Theorem 6.5. *Let F_μ be an irreducible \mathfrak{g}_0 -module with highest weight μ .*

If $n \geq 4$, then the \mathfrak{g} -module $\text{Ind}(F_\mu)$ is an irreducible (finite conformal) module except for the following cases:

(a) $\mu = (-l; l, 0, \dots, 0)$, $l \geq 0$, $\text{Ind}(F_\mu) = (\Omega_+^l/I_+^l)^\#$, and $d^\#(\Omega_+^{l+1}/I_+^{l+1})^\#$ is the only non-trivial proper submodule.

(b) $\mu = (n+k-2; k, 0, \dots, 0)$, $k \geq 1$, and $\text{Ind}(F_\mu) = \Omega_-^k/I_-^k$. For $k \geq 2$ the image $d\Omega_-^{k-1}/I_-^{k-1}$ is the only non-trivial proper submodule. For $k = 1$, both $\text{Im}(d)$ and $\text{Ker}(d)$ are proper submodules, and $\text{Ker}(d)$ is a maximal submodule.

Proof. We know from Theorem 2.4 that in order for the \mathfrak{g} -module $\text{Ind}(F)$ to be reducible it has to have non-trivial singular vectors and the possible highest weights of F in this situation are listed in Theorem 5.1 above.

The fact that the induced modules are actually reducible in those cases is known because we have got nice realizations for these induced modules in Propositions 6.2 and 6.4 together with morphisms defined by $d, d^\#$, so kernels and images of these morphisms become submodules.

The subtle thing is to prove that a submodule is really a maximal one. We notice that in each case the factor is isomorphic to a submodule in another induced module so it is enough to show that the submodule is irreducible. This can be proved as follows, a submodule in the induced module is irreducible if it is generated by any highest singular vector that it contains. We see from our list of non-trivial singular vectors that there is at most one such a vector for each case and the images and kernels in question are exactly generated by those vectors, hence they are irreducible. \square

Corollary 6.6. *The theorem gives us a description of finite conformal irreducible $K(1, n)_+$ -modules for $n \geq 4$. Such a module is either $\text{Ind}(F)$ for an irreducible finite-dimensional \mathfrak{g}_0 -module F , where the highest weight of F does not belong to the types listed in (a), (b) of the theorem, or the factor of an induced module from (a), (b) by its submodule $\text{Ker}(d)$.*

7. FINITE IRREDUCIBLE K_n -MODULES

In the first part of this section, we follow Section E in [1]. In order to give an explicit construction and classification of all finite irreducible K_n -modules, we need the following definitions. Recall that $W(1, n)$ acts by derivations on the algebra of differential forms $\Omega = \Omega(1, n)$, and note that this is a conformal module by taking the family of formal distributions

$$E = \{\delta(z-t)\omega \text{ and } \delta(z-t)\omega dt \mid \omega \in \Omega(n)\}$$

Translating this and all other attributes of differential forms, like de Rham differential, etc. into the conformal algebra language, we have the following definitions.

Recall that given an algebra A , the associated current formal distribution algebra is $A[t, t^{-1}]$ with the local family $F = \{a(z) = \sum_{n \in \mathbb{Z}} (at^n)z^{-n-1} = a\delta(z-t)\}_{a \in A}$. The associated conformal algebra is $\text{Cur}A = \mathbb{C}[\partial] \otimes A$ with multiplication defined by $a_\lambda b = ab$ for $a, b \in A$ and extended using sesquilinearity. This is called the *current conformal algebra*, see [6] for details.

The conformal algebra of differential forms Ω_n is the current algebra over the commutative associative superalgebra $\Omega(n) + \Omega(n) dt$ with the obvious multiplication and parity, subject to the relation $(dt)^2 = 0$:

$$\Omega_n = \text{Cur}(\Omega(n) + \Omega(n) dt) = \mathbb{C}[\partial] \otimes (\Omega(n) + \Omega(n) dt).$$

The de Rham differential \tilde{d} of Ω_n (we use the tilde in order to distinguish it from the de Rham differential d on $\Omega(n)$) is a derivation of the conformal algebra Ω_n such that:

$$\tilde{d}(\omega_1 + \omega_2 dt) = d\omega_1 + d\omega_2 dt - (-1)^{p(\omega_1)} \partial(\omega_1 dt). \quad (7.1)$$

here and further $\omega_i \in \Omega(n)$.

The standard \mathbb{Z}_+ -gradation $\Omega(n) = \bigoplus_{j \in \mathbb{Z}_+} \Omega(n)^j$ of the superalgebra of differential forms by their degree induces a \mathbb{Z}_+ -gradation

$$\Omega_n = \bigoplus_{j \in \mathbb{Z}_+} \Omega_n^j, \quad \text{where } \Omega_n^j = \mathbb{C}[\partial] \otimes (\Omega(n)^j + \Omega(n)^{j-1} dt),$$

so that $\tilde{d}: \Omega_n^j \rightarrow \Omega_n^{j+1}$.

Let $\omega = dt - \sum_{i=1}^n \xi_i d\xi_i \in \Omega_n^1$. Observe that $\tilde{d}\omega = -\sum_{i=1}^n (d\xi_i)^2$. Now, we define, for $j \geq 2$,

$$\begin{aligned} I_n^j &= \mathbb{C}[\partial] \otimes (\omega \wedge \Omega(n)^{j-1} + d\omega \wedge \Omega(n)^{j-2} dt) \subset \Omega_n^j, \\ I_n^1 &= \mathbb{C}[\partial] \otimes (\omega \wedge \Omega(n)^0), \quad I^0 = 0. \end{aligned} \quad (7.2)$$

It is clear that $\tilde{d}(I_n^j) \subseteq I_n^{j+1}$, and it is easy to prove that I_n^j are K_n -submodules of Ω_n^j . Therefore, we get a Rumin conformal complex $(\Omega_n^j/I_n^j, \tilde{d})$, where we also denote by \tilde{d} the differential in the quotient.

Let V be a finite dimensional irreducible $\mathfrak{cs}\mathfrak{o}(n)$ -module, using the results of Section 2 and recalling that the annihilation algebra of K_n is $K(1, n)_+$, we have that the $K(1, n)_+$ -modules $\text{Ind}(V)$ studied in the previous section are K_n -modules with the λ -action given by Theorem 4.3. We denote by $\text{Tens}(V)$ the corresponding K_n -module.

Since the extended annihilation algebra $K(1, n)^+$ is a direct sum of $K(1, n)_+$ and a 1-dimensional Lie algebra $\mathbb{C}a$, any irreducible $K(1, n)^+$ -module is obtained from a $K(1, n)_+$ -module M by extending to $K(1, n)^+$, letting $a \mapsto -\alpha$, where $\alpha \in \mathbb{C}$. Translating into the conformal language (see Proposition 2.3), we see that all K_n -modules are obtained from conformal $K(1, n)_+$ -modules by taking for the action of ∂ the action of $-\partial_t + \alpha I$, $\alpha \in \mathbb{C}$. We denote by $\text{Tens}_\alpha V$ and $\Omega_{k, \alpha}$, $\alpha \in \mathbb{C}$, the K_n -modules obtained from $\text{Tens}V$ and Ω_k by replacing ∂ by $\partial + \alpha$ in the corresponding actions.

As in [1], we see that Theorem 6.5 and Corollary 6.6, along with Section 2 and Propositions 2.3, together with Propositions 2.6, 2.8 and 2.9 in [1], give us a complete description of finite irreducible K_n -modules, namely we obtain the following theorem.

Theorem 7.1. *The following is a complete list of non-trivial finite irreducible K_n -modules ($n \geq 4, \alpha \in \mathbb{C}$):*

- (1) $\text{Tens}_\alpha V$, where V is a finite-dimensional irreducible $\mathfrak{cs}\mathfrak{o}(n)$ -module with highest weight different from $(-k; k, 0, \dots, 0)$ and $(n+k-2; k, 0, \dots, 0)$ for $k = 1, 2, \dots$,
- (2) $\left(\Omega_n^k/I_n^k\right)_\alpha^* / \text{Ker } \tilde{d}^*$, $k = 1, 2, \dots$, and the same modules with reversed parity,
- (3) K_n -modules dual to (2), with $k > 1$.

Remark 7.2. (a) Using Proposition 6.4, we have that the kernel of \tilde{d} and the image of \tilde{d} coincide in Ω_n^k/I_n^k for $k \geq 2$. Now, since Ω_n^{k+2}/I_n^{k+2} is a free $\mathbb{C}[\partial]$ -module of finite rank and $(\Omega_n^{k+1}/I_n^{k+1})/\text{Im } \tilde{d} = (\Omega_n^{k+1}/I_n^{k+1})/\text{Ker } \tilde{d} \simeq \text{Im } \tilde{d} \subset \Omega_n^{k+2}/I_n^{k+2}$, we obtain that $(\Omega_n^{k+1}/I_n^{k+1})/\text{Im } \tilde{d}$ is a finitely generated free $\mathbb{C}[\partial]$ -module. Therefore, we can apply Proposition 2.6 in [1], and we have that

$$\left(\Omega_n^{k+1}/I_n^{k+1}\right)^* / \text{Ker } \tilde{d}^* \simeq \left((\Omega_n^k/I_n^k)/\text{Ker } \tilde{d}\right)^* \quad (7.3)$$

for $k \geq 1$.

(b) Since for a free finite rank module M over a Lie conformal superalgebra we have $M^{**} = M$, using (7.3), the K_n -modules in case (3) of Theorem 7.1 are isomorphic to $(\Omega_n^k/I_n^k)_\alpha / \text{Ker } \tilde{d}$, $k = 1, 2, \dots$

(c) Let V be a finite-dimensional (one dimensional in fact) irreducible $\mathfrak{cso}(n)$ -module with highest weight $(0; 0, \dots, 0)$. Observe that the module $\text{Tens } V$ has a maximal submodule of codimension 1 over \mathbb{C} . Hence, the irreducible quotient is the one dimensional (over \mathbb{C}) trivial K_n -module. Therefore, we excluded the case $k = 0$ in Theorem 7.1.a.2.

(d) Let V be a finite-dimensional irreducible $\mathfrak{cso}(n)$ -module with highest weight $(n - 2; 0, 0, \dots, 0)$. Observe that in case (3) in Theorem 7.1, we excluded $k = 1$, because in this case the dual corresponds to the module $\text{Tens}_\alpha V$, which is isomorphic to $\Omega_{0,\alpha}$ and it is an irreducible tensor module, therefore this module is included in case (1) of Theorem 7.1.

(e) The case $K_2 \simeq W_1$ was studied in full detail at the end of Section E in [1].

(f) The remaining cases K_3 , K'_4 and CK_6 will be worked out in a subsequent publication.

8. APPENDIX A

This appendix is devoted to the proof of Theorem 4.1, and it will be done through several lemmas.

Given $I \subseteq \{1, \dots, n\}$ we shall use the following notation:

$$\varepsilon_i = \varepsilon_i^I := \#\{j \in I : j < i\}.$$

It is easy to see the following useful formulas:

$$\partial_I \xi_I = (-1)^{\frac{|I|(|I|-1)}{2}}, \quad (8.1)$$

$$\partial_I (\xi_J \xi_K) = (-1)^{|I||J|} \xi_J \partial_I (\xi_K), \quad \text{if } J \cap I = \emptyset, \quad (8.2)$$

$$\partial_{I-\{i\}} \xi_I = (-1)^{\varepsilon_i + \frac{|I|(|I|-1)}{2}} \xi_i. \quad (8.3)$$

Without loss of generality, we shall assume all over the proofs that

$$f = \xi_I, \quad g = \xi_J \xi_K, \quad \text{with } J \cap I = \emptyset, \text{ and } K \subseteq I.$$

Lemma 8.1. *For any $m \geq 3$, $f, g \in \Lambda(n)$, we have $t^m f \cdot (g \otimes v) = 0$.*

Proof. Using that

$$[t^m \xi_I, \xi_r] = \begin{cases} -mt^{m-1} \xi_I \xi_r, & \text{if } r \notin I; \\ (-1)^I t^m \partial_r \xi_I, & \text{if } r \in I. \end{cases} \quad (8.4)$$

it is easy to see that

$$\begin{aligned} t^m f \cdot (g \otimes v) &= t^m \xi_I \cdot (\xi_J \xi_K \otimes v) \\ &= \sum_{i=0}^m \sum_{S \subseteq J, |S|=i} (sgn)_{i,S} \frac{m!}{(m-i)!} (\partial_S \xi_J) (t^{m-i} \xi_I \xi_S) \xi_K \otimes v \\ &= \sum_{i=0}^m \sum_{S \subseteq J, |S|=i} \sum_{L \subseteq K} (sgn)_{i,S,L} \frac{m!}{(m-i)!} (\partial_S \xi_J) (\partial_L \xi_K) (t^{m-i} \partial_L (\xi_I \xi_S)) \otimes v, \end{aligned} \quad (8.5)$$

for certain signs $(sgn)_{i,S}, (sgn)_{i,S,L}$ that are not needed explicitly yet. Now, observe that for $|S| = i$ and $L \subseteq K \subseteq I$ we have

$$\deg(t^{m-i} \partial_L(\xi_I \xi_S)) = 2(m-i) + |I| + |S| - |L| - 2 = 2m - i + |I| - |L| - 2 \geq m - 2.$$

Hence, using (8.5) and $m \geq 3$ we prove the lemma. \square

From Lemma 8.1, the λ -action has degree at most 2 in λ . Now, we study the λ^0 -term.

Lemma 8.2.

$$\xi_I \cdot (\xi_J \xi_K \otimes v) = \sum_{L \subseteq K} (-1)^{|I|(|J|+|K|) + \frac{|L|(|L|-1)}{2} - |L|(|K|-|L|)} \xi_J(\partial_L \xi_K)(\partial_L \xi_I) \otimes v. \quad (8.6)$$

Proof. Using (8.4), it is clear that

$$\xi_I \cdot (\xi_J \xi_K \otimes v) = (-1)^{|I||J|} \xi_J(\xi_I)(\xi_K) \otimes v.$$

Hence, we may suppose that $J = \emptyset$ and we shall apply induction on $|K|$. If $|K| = 0$ the statement is obvious. Now, consider $\xi_j \xi_K$, with $j < k_i$ for any $k_i \in K$. Observe that

$$\begin{aligned} \xi_I \cdot (\xi_j \xi_K \otimes v) &= (-1)^{|I|} \xi_j \xi_I \xi_K \otimes v + (-1)^{|I|} (\partial_j \xi_I) \xi_K \otimes v \\ &= \sum_{L \subseteq K} (-1)^{|I|+|I||K| + \frac{|L|(|L|-1)}{2} - |L|(|K|-|L|)} \xi_j(\partial_L \xi_K)(\partial_L \xi_I) \otimes v \\ &\quad + \sum_{L \subseteq K} (-1)^{|I|+(|I|-1)|K| + \frac{|L|(|L|-1)}{2} - |L|(|K|-|L|)} (\partial_L \xi_K)(\partial_L \partial_j \xi_I) \otimes v \end{aligned} \quad (8.7)$$

Now, using that

$$\begin{aligned} \partial_L(\xi_j \xi_K) &= (-1)^{|L|} \xi_j(\partial_L \xi_K), & \text{if } j \notin L \\ \partial_j \partial_L(\xi_j \xi_K) &= (-1)^{|L|} (\partial_L \xi_K), & \text{if } j \notin L \\ \partial_L \partial_j \xi_I &= (-1)^{|L|} \partial_j \partial_L \xi_I, \end{aligned}$$

equation (8.7) becomes

$$\begin{aligned} \xi_I \cdot (\xi_j \xi_K \otimes v) &= \\ &= \sum_{L \subseteq K} (-1)^{|I|(|K|+1) + \frac{|L|(|L|-1)}{2} - |L|(|K|-|L|) + |L|} (\partial_L(\xi_j \xi_K))(\partial_L \xi_I) \otimes v \\ &\quad + \sum_{L \cup \{j\} \subseteq K \cup \{j\}} (-1)^{|I|+(|I|-1)|K| + \frac{|L|(|L|-1)}{2} - |L|(|K|-|L|) + |L| + |L|} \\ &\hspace{15em} (\partial_j \partial_L)(\xi_j \xi_K)(\partial_j \partial_L \xi_I) \otimes v \\ &= \sum_{L \subseteq K \cup \{j\}} (-1)^{|I|(|K|+1) + \frac{|L|(|L|-1)}{2} - |L|(|K|+1-|L|)} (\partial_L)(\xi_j \xi_K)(\partial_L \xi_I) \otimes v \end{aligned}$$

finishing the proof. \square

The following lemma provides the λ^0 -term in the λ -action formula of Theorem 4.1.

Lemma 8.3. *For any monomials elements $f = \xi_I, g = \xi_L$ with $I \neq \emptyset$, we have*

$$f \cdot (g \otimes v) = (-1)^{p(f)}(|f| - 2)\partial(\partial_f g) \otimes v + \sum_{i=1}^n \partial_{(\partial_i f)}(\xi_i g) \otimes v \\ + (-1)^{p(f)} \sum_{i < j} \partial_{(\partial_i \partial_j f)} g \otimes F_{ij} v$$

Proof. Consider as before $f = \xi_I, g = \xi_J \xi_K$ with $J \cap I = \emptyset$ and $K \subseteq I$. Recall formula (8.6)

$$\xi_I \cdot (\xi_J \xi_K \otimes v) = \sum_{L \subseteq K} (-1)^{|I|(|J|+|K|) + \frac{|L|(|L|-1)}{2} - |L|(|K|-|L|)} \xi_J (\partial_L \xi_K) (\partial_L \xi_I) \otimes v.$$

Since $\partial_L \xi_I \in \mathfrak{g}_{>0}$ if $|I - L| > 2$, it is enough to consider the summands that appear in the cases $|I - L| = 0, 1, 2$.

Case $|I - L| = 0$:

This summand appear if and only if $K = I$, and it correspond to the single possible choice of $L = K$. Using (8.1), we get

$$\delta_{K,I} (-1)^{|I||J|+|I|^2 + \frac{|I|(|I|-1)}{2}} \xi_J \mathbf{1} \otimes v \quad (8.8)$$

and using (8.2) together with $\mathbf{1} = -2\partial$, it can be rewritten as

$$-2\partial (-1)^{p(f)} \partial_f(g) \otimes v \quad (8.9)$$

obtaining part of the first term of the statement of this lemma. Observe that the term $\partial_f(g)$ is non-zero iff $K = I$, therefore the expression (8.9) also contains the $\delta_{K,I}$ in (8.8). This kind of analysis will be repeatedly used.

Case $|I - L| = 1$:

This case is clearly divided in two subcases:

- (1-a) $K = I$ and $L = I - \{i\}$ moving $i \in I$, or
- (1-b) $K = I - \{k\}$, and L takes the single value K .

Let us compute each subcase separately.

Subcase (1-a): Recalling (8.6) and using (8.2), the summands in this subcase become

$$\text{terms(1-a)} = \delta_{K,I} \sum_{i \in I} (-1)^{|I||J|+|I|^2 + \frac{(|I|-1)(|I|-2)}{2} - (|I|-1)} \xi_J (\partial_{I-\{i\}} \xi_I) (\partial_{I-\{i\}} \xi_I) \otimes v \\ = -\delta_{K,I} \sum_{i \in I} (-1)^{|I||J| + \frac{(|I|-1)(|I|-2)}{2}} \xi_J \xi_i \xi_i \otimes v.$$

Now, observe that $0 \neq \xi_i \xi_i \otimes v \in \text{Ind}(V)$. Moreover, using that $\xi_i \xi_i + \xi_i \xi_i = [\xi_i, \xi_i] = -\mathbf{1} \in \mathfrak{g}_{-2}$, we obtain

$$\text{terms(1-a)} = -\delta_{K,I} (-1)^{|I||J| + \frac{(|I|-1)(|I|-2)}{2}} |I| \partial \xi_J \otimes v.$$

On the other hand, as in (8.9), if $K = I$ we have

$$\partial_f(g) = (-1)^{|I||J| + \frac{|I|(|I|-1)}{2}} \xi_J, \quad (8.10)$$

obtaining

$$\text{terms(1-a)} = (-1)^{p(f)} |f| \partial(\partial_f g) \otimes v,$$

getting the other part of the first term in the statement of this lemma.

Subcase (1-b): Recalling (8.6) and using (8.1) and (8.2)

$$\begin{aligned} \text{terms(1-b)} &= \delta_{K, I-\{k\}} (-1)^{|I||J| + |I|(|I|-1) + \frac{(|I|-1)(|I|-2)}{2}} \xi_J (\partial_{I-\{k\}} \xi_{I-\{k\}}) (\partial_{I-\{k\}} \xi_I) \\ &= \delta_{K, I-\{k\}} (-1)^{|I||J| + \varepsilon_k + \frac{|I|(|I|-1)}{2}} \xi_J \xi_k. \end{aligned}$$

On the other hand, observe that $\partial_{(\partial_j f)}(\xi_j g) \neq 0$ iff $j \notin K \cup J, j \in I$ and $I - \{j\} \subseteq \{j\} \cup K \cup J$, i.e. $K = I - \{j\}$. Hence, if $K = I - \{k\}$, then

$$\begin{aligned} \sum_{j=1}^n \partial_{(\partial_j f)}(\xi_j g) &= \partial_{(\partial_k f)}(\xi_k g) = (-1)^{\varepsilon_k} \partial_{I-\{k\}}(\xi_k \xi_J \xi_{I-\{k\}}) \\ &= (-1)^{\varepsilon_k + (|J|+1)(|I|-1) + \frac{(|I|-1)(|I|-2)}{2}} \xi_k \xi_J \\ &= (-1)^{\varepsilon_k + |J||I| + \frac{(|I|(|I|-1))}{2}} \xi_J \xi_k \end{aligned}$$

obtaining terms(1-b) and the second term of the statement of this lemma.

Case $|I - L| = 2$: It remains to see that this case produce the last term in the statement of this lemma. In order to prove it, observe that this case must be divided in the following subcases, depending on the relation between f and g , more precisely, depending on the relation between K and I , namely:

- (2-a) $K = I$, hence $L = I - \{i, j\}$ moving $i < j, i, j \in I$, or
- (2-b) $K = I - \{r\}$, hence $L = I - \{r, s\}$ moving $s \in I$ with $s \neq r$, or
- (2-c) $K = I - \{r, s\}$ with $r < s$, hence L takes the single value K .

Now, we must show that for each choice of K as in (2-a,b,c) the resulting sum over the corresponding subsets L 's is always equal to

$$(-1)^{p(f)} \sum_{i < j} \partial_{(\partial_i \partial_j f)} g \otimes F_{ij} v.$$

Using (8.6), it is clear that

$$\begin{aligned} \text{terms(2-a)} &= \sum_{i < j; i, j \in I} (-1)^{|I||J| + |I| + \frac{(|I|-2)(|I|-3)}{2} - (|I|-2)^2} \xi_J \xi_i \xi_j (\xi_i \xi_j) \otimes v \\ &= - \sum_{i < j; i, j \in I} (-1)^{|I||J| + \frac{|I|(|I|+1)}{2} + 1} \xi_J \xi_i \xi_j \otimes F_{ij} v. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{i < j} \partial_{(\partial_i \partial_j \xi_I)}(\xi_J \xi_I) \otimes F_{ij} v &= \sum_{i < j; i, j \in I} (-1)^{\varepsilon_i + \varepsilon_j} \partial_{I - \{i, j\}}(\xi_J \xi_I) \otimes F_{ij} v \\
&= \sum_{i < j; i, j \in I} (-1)^{\varepsilon_i + \varepsilon_j + (|I| - 2)|J|} \xi_J \partial_{I - \{i, j\}}(\xi_I) \otimes F_{ij} v \\
&= \sum_{i < j; i, j \in I} (-1)^{|I||J| + \frac{|I|(|I| - 1)}{2}} \xi_J \xi_i \xi_j \otimes F_{ij} v,
\end{aligned}$$

where in the last equality we are using the following formula that can be easily verified for $i < j$

$$\partial_{I - \{i, j\}}(\xi_I) = \begin{cases} (-1)^{\varepsilon_i + \varepsilon_j + \frac{|I|(|I| - 1)}{2}} \xi_i \xi_j, & \text{if } i < j; \\ (-1)^{\varepsilon_i + \varepsilon_j + \frac{|I|(|I| - 1)}{2}} \xi_j \xi_i, & \text{if } i > j. \end{cases} \quad (8.11)$$

Therefore, taking care of the sign of the last term in the statement, we proved that it corresponds to terms(2-a).

In order to study case (2-b), suppose that $K = I - \{r\}$. Then, using (8.6), terms(2-b) =

$$\begin{aligned}
&= \sum_{s \in I, s \neq r} (-1)^{|I||J| + |I|(|I| - 1) + \frac{(|I| - 2)(|I| - 3)}{2} - (|I| - 2)} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) (\partial_{I - \{r, s\}} \xi_I) \otimes v \\
&= \sum_{s \in I, s \neq r} (-1)^{|I||J| + \frac{(|I| - 1)(|I| - 2)}{2}} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) (\partial_{I - \{r, s\}} \xi_I) \otimes v.
\end{aligned}$$

Using (8.11) it become

$$\begin{aligned}
\text{terms(2-b)} &= - \sum_{s \in I, s < r} (-1)^{|I||J| + \frac{(|I| - 1)(|I| - 2)}{2} + \varepsilon_r + \varepsilon_s + \frac{|I|(|I| - 1)}{2}} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) \otimes F_{sr} v \\
&\quad - \sum_{s \in I, r < s} (-1)^{|I||J| + \frac{(|I| - 1)(|I| - 2)}{2} + \varepsilon_r + \varepsilon_s + \frac{|I|(|I| - 1)}{2}} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) \otimes F_{rs} v \\
&= - \sum_{s \in I, s < r} (-1)^{|I||J| + \varepsilon_r + \varepsilon_s + |I| + 1} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) \otimes F_{sr} v \\
&\quad - \sum_{s \in I, r < s} (-1)^{|I||J| + \varepsilon_r + \varepsilon_s + |I| + 1} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) \otimes F_{rs} v. \quad (8.12)
\end{aligned}$$

On the other hand, if $K = I - \{r\}$ we have

$$\begin{aligned}
\sum_{i < j} \partial_{(\partial_i \partial_j f)} g \otimes F_{ij} v &= \sum_{i < j; i, j \in I} (-1)^{\varepsilon_i + \varepsilon_j} \partial_{I - \{i, j\}}(\xi_J \xi_{I - \{r\}}) \otimes F_{ij} v \\
&= \sum_{s < r; s \in I} (-1)^{\varepsilon_r + \varepsilon_s + |I||J|} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) \otimes F_{sr} v \\
&\quad + \sum_{r < s; s \in I} (-1)^{\varepsilon_r + \varepsilon_s + |I||J|} \xi_J (\partial_{I - \{r, s\}} \xi_{I - \{r\}}) \otimes F_{rs} v.
\end{aligned}$$

Therefore, comparing the last equation with (8.12) and taking care of the sign in the last term of the statement, we prove that terms(2-b) correspond to it for $K = I - \{r\}$.

Finally, suppose that $K = I - \{r, s\}$ with $r < s$, then (2-c) or more precisely the sum in (8.6) over those L with $|I - L| = 2$ become

$$\begin{aligned} \text{terms(2-c)} &= (-1)^{|I||J|+|I|(|I|-2)+\frac{(|I|-2)(|I|-3)}{2}} \xi_J(\partial_{I-\{r,s\}} \xi_{I-\{r,s\}})(\partial_{I-\{r,s\}} \xi_I) \otimes v \\ &= -(-1)^{|I||J|+|I|+\frac{(|I|-2)(|I|-3)}{2}+\varepsilon_r+\varepsilon_s+\frac{|I|(|I|-1)}{2}} \xi_J(\partial_{I-\{r,s\}} \xi_{I-\{r,s\}}) \otimes F_{ij}v \\ &= -(-1)^{|I||J|+|I|+1+\varepsilon_r+\varepsilon_s} \xi_J(\partial_{I-\{r,s\}} \xi_{I-\{r,s\}}) \otimes F_{ij}v. \end{aligned} \quad (8.13)$$

On the other hand, if $K = I - \{r, s\}$ with $r < s$, we have

$$\begin{aligned} \sum_{i < j} \partial_{(\partial_i \partial_j f)} g \otimes F_{ij}v &= \sum_{i < j; i, j \in I} (-1)^{\varepsilon_i + \varepsilon_j} \partial_{I-\{i,j\}}(\xi_J \xi_{I-\{r,s\}}) \otimes F_{ij}v \\ &= (-1)^{\varepsilon_r + \varepsilon_s + |I||J|} \xi_J(\partial_{I-\{r,s\}} \xi_{I-\{r,s\}}) \otimes F_{rs}v \end{aligned}$$

Therefore, comparing the last equation with (8.13) and taking care of the sign in the last term of the statement, we prove that (2-c) correspond to it for $K = I - \{r, s\}$, finishing the proof. \square

The following lemma gives us the λ^1 -coefficient of the λ -action.

Lemma 8.4. *For any monomials elements $f = \xi_I, g = \xi_L$ with $I \neq \emptyset$, we have*

$$\begin{aligned} tf \cdot (g \otimes v) &= (-1)^{p(f)} (\partial_f g) \otimes E_{00}v \\ &+ (-1)^{p(f)+p(g)} \sum_{i=1}^n (\partial_f (\partial_i g)) \xi_i \otimes v + \sum_{i \neq j} \partial_{(\partial_i f)} (\partial_j g) \otimes F_{ij}v. \end{aligned}$$

Proof. We shall use the usual notation: $f = \xi_I, g = \xi_J \xi_K$ with $J \cap I = \emptyset$ and $K \subseteq I$. Using (8.4) and (8.5), it is easy to see that

$$t\xi_I \cdot (\xi_J \xi_K \otimes v) = (-1)^{|I||J|} \xi_J(t\xi_I) \xi_K \otimes v + \sum_{j=1}^n (-1)^{|I||J|-|I|+|J|} (\partial_j \xi_J) (\xi_I \xi_j) \xi_K \otimes v.$$

and in the second term we can apply the (0)-action formula given by Lemma 8.3, in the special case of $\tilde{f} = \xi_I \xi_j$ and $\tilde{g} = \xi_K$, hence

$$\begin{aligned} t\xi_I \cdot (\xi_J \xi_K \otimes v) &= (-1)^{|I||J|} \xi_J(t\xi_I) \xi_K \otimes v \\ &+ \sum_{i,j=1}^n (-1)^{|I||J|-|I|+|J|} (\partial_j \xi_J) (\partial_{(\partial_i \xi_I \xi_j)} (\xi_i \xi_K)) \otimes v \\ &+ (-1)^{|I|+1} \sum_{j=1}^n \sum_{r < s} (-1)^{|I||J|-|I|+|J|} (\partial_j \xi_J) (\partial_{(\partial_r \partial_s \xi_I \xi_j)} \xi_K) \otimes F_{rs}v. \end{aligned} \quad (8.14)$$

It remains to see that the three terms in the above equation correspond exactly to the terms in the statement. In order to do it, let us consider the first term of (8.14), and using (8.5), we obtain

$$\begin{aligned} (-1)^{|I||J|} \xi_J(t\xi_I) \xi_K \otimes v &= (-1)^{|I||J|} \xi_J \sum_{L \subseteq K} \widetilde{(sgn)}_L \xi_{K-L}(t\xi_{I-L}) \otimes v \\ &= \delta_{K,I} \widetilde{(sg)} (-1)^{|I||J|} \xi_J \otimes E_{00}v \end{aligned} \quad (8.15)$$

since $\deg(t\xi_{I-L}) = |I - L|$ has to be 0, i.e. we have only one summand that correspond to $L = I$ and we must have $K = I$. Observe that the term $L = I$

corresponds to take all the brackets against $\xi_{k_1}, \dots, \xi_{k_l}$, if $K = \{k_1, \dots, k_l\}$, hence it allows us to compute the sign in (8.15), obtaining

$$(-1)^{|I||J|} \xi_J(t\xi_I) \xi_K \otimes v = \delta_{K,I} (-1)^{|I||J| + \frac{|I|(|I|+1)}{2}} \xi_J \otimes E_{00}v = (-1)^{p(f)} \partial_f g,$$

where we used (8.10) to prove the last equality, getting the first term of the statement.

Now, let us consider the second term of (8.14) and observe on it the expressions $(\partial_j \xi_J)$ and $\partial_{(\partial_i \xi_I \xi_j)} \xi_i \xi_K$. In order to be non-zero, we must have $i = j$, and $j \in J$. Therefore,

$$\begin{aligned} & \sum_{i,j=1}^n (-1)^{|I||J| - |I| + |J|} (\partial_j \xi_J) (\partial_{(\partial_i \xi_I \xi_j)} (\xi_i \xi_K)) \otimes v = \\ & = \sum_{j \in J} (-1)^{|I||J| - |I| + |J|} (\partial_j \xi_J) (\partial_{(\partial_j \xi_I \xi_j)} (\xi_j \xi_K)) \otimes v \\ & = \sum_{j \in J} (-1)^{|I||J| + |J|} (\partial_j \xi_J) (\partial_I (\xi_j \xi_K)) \otimes v = \sum_{j \in J} (-1)^{|J| - |I|} \partial_I \left((\partial_j \xi_J) \xi_j \xi_K \right) \otimes v \\ & = \sum_{j \in J} (-1)^{|J| - |I| + |K|} \partial_I \left((\partial_j \xi_J) \xi_K \xi_j \right) \otimes v \\ & = \sum_{j \in J} (-1)^{|J| - |I| + |K|} \partial_I \left((\partial_j \xi_J \xi_K) \xi_j \right) \otimes v \\ & = \sum_{j \in J} (-1)^{|J| - |I| + |K|} (\partial_I (\partial_j \xi_J \xi_K)) \xi_j \otimes v = \sum_{j=1}^n (-1)^{p(f) + p(g)} (\partial_f (\partial_j g)) \xi_j \otimes v, \end{aligned}$$

proving that it corresponds to the second term of the statement of this lemma.

Finally, consider the last term in (8.14). Observe that the expression $\partial_{(\partial_r \partial_s \xi_I \xi_j)} \xi_K$ implies that $r = j$ or $s = j$. Therefore, this last term can be rewritten as follows:

$$\begin{aligned}
 & (-1)^{|I|+1} \sum_{j=1}^n \sum_{r<s} (-1)^{|I||J|-|I|+|J|} (\partial_j \xi_J) (\partial_{(\partial_r \partial_s \xi_I \xi_j)} \xi_K) \otimes F_{rs} v \\
 &= - \sum_{j=1}^n (-1)^{|I||J|+|J|} (\partial_j \xi_J) \left[\sum_{i<j} (\partial_{(\partial_i \partial_j \xi_I \xi_j)} \xi_K) \otimes F_{ij} v + \sum_{j<i} (\partial_{(\partial_j \partial_i \xi_I \xi_j)} \xi_K) \otimes F_{ji} v \right] \\
 &= - \sum_{j=1}^n (-1)^{|I||J|+|J|+|I|} (\partial_j \xi_J) \left[\sum_{i<j} (\partial_{(\partial_i \xi_I)} \xi_K) \otimes F_{ij} v - \sum_{j<i} (\partial_{(\partial_i \xi_I)} \xi_K) \otimes F_{ji} v \right] \\
 &= \sum_{j \in J} \left[\sum_{i<j, i \in I} \partial_{(\partial_i \xi_I)} ((\partial_j \xi_J) \xi_K) \otimes F_{ij} v - \sum_{j<i, i \in I} \partial_{(\partial_i \xi_I)} ((\partial_j \xi_J) \xi_K) \otimes F_{ji} v \right] \\
 &= \sum_{j \in J} \sum_{i \in I} \partial_{(\partial_i \xi_I)} ((\partial_j \xi_J) \xi_K) \otimes F_{ij} v \quad (\text{since } F_{ij} = -F_{ji}) \\
 &= \sum_{j \in J} \sum_{i \in I} \partial_{(\partial_i f)} (\partial_j g) \otimes F_{ij} v \\
 &= \sum_{i \neq j} \partial_{(\partial_i f)} (\partial_j g) \otimes F_{ij} v
 \end{aligned}$$

finishing the proof. \square

The following lemma gives us the λ^2 -coefficient of the λ -action.

Lemma 8.5. *For any monomials elements $f = \xi_I, g = \xi_L$ with $I \neq \emptyset$, we have*

$$\left(\frac{1}{2} t^2 f \right) \cdot (g \otimes v) = (-1)^{p(f)} \sum_{i<j} \partial_f (\partial_i \partial_j g) \otimes F_{ij} v.$$

Proof. Using (8.5), we have

$$\begin{aligned}
 t^2 \xi_I \cdot \xi_J \xi_K \otimes v &= \sum_{L \subseteq K} (sgn) \xi_J \xi_{K-L} (t^2 \xi_{I-L}) \otimes v \tag{8.16} \\
 &+ \sum_{j \in J} \sum_{L \subseteq K} 2(sgn) \xi_{J-\{j\}} \xi_{K-L} (t \xi_{I-L} \xi_j) \otimes v \\
 &+ \sum_{\{i,j\} \subseteq J, i<j} \sum_{L \subseteq K} 2(sgn) \xi_{J-\{i,j\}} \xi_{K-L} (\xi_{I-L} \xi_i \xi_j) \otimes v
 \end{aligned}$$

for certain signs that depend on the parameters. Now, observe that the first two terms in (8.16) are 0, because $\deg(t^2 \xi_{I-L}) \geq 2$ and $\deg(t \xi_{I-L} \xi_j) \geq 1$ since $j \notin I$. Using that $\deg(\xi_{I-L} \xi_i \xi_j) \geq 1$ for $|I-L| \geq 1$, the last term of (8.16) is non-zero only if $L = K = I$, therefore

$$t^2 \xi_I \cdot (\xi_J \xi_K \otimes v) = - \sum_{\{i,j\} \subseteq J, i<j} 2(sgn)_{i,j} \xi_{J-\{i,j\}} \otimes F_{ij} v. \tag{8.17}$$

It remains to compute the sign $(sgn)_{i,j}$ and rewrite (8.17) as in the statement of this lemma.

Suppose that $\xi_J = \xi_* \dots \xi_i \dots \xi_j \dots \xi_*$, then the term that appears in (8.17) is obtained (super) commuting the ξ 's, namely

$$\begin{aligned}
& t^2 \xi_I \cdot \xi_* \dots \xi_i \dots \xi_j \dots \xi_* \xi_K \otimes v = & (8.18) \\
& = \sum_{\{i,j\} \subseteq J, i < j} 2(-1)^{|I|\epsilon_i^J + 1 + (\epsilon_j^J - (\epsilon_i^J + 1))(|I| + 1)} \xi_* \dots \widehat{\xi}_i \dots (t \xi_I \xi_i) \xi_j \dots \xi_* \xi_K \otimes v \\
& = \sum_{\{i,j\} \subseteq J, i < j} 2(-1)^{|I|\epsilon_i^J + 1 + (\epsilon_j^J - (\epsilon_i^J + 1))(|I| + 1) + 1} \xi_* \dots \widehat{\xi}_i \dots \widehat{\xi}_j (\xi_I \xi_i \xi_j) \dots \xi_* \xi_K \otimes v \\
& = \sum_{\{i,j\} \subseteq J, i < j} 2(-1)^{|I|\epsilon_i^J + 1 + (\epsilon_j^J - (\epsilon_i^J + 1))(|I| + 1) + 1 + (|I| + 2)(|J| - (\epsilon_j^J + 1))} \times \\
& \quad \times \xi_* \dots \widehat{\xi}_i \dots \widehat{\xi}_j \dots \xi_* (\xi_I \xi_i \xi_j) \xi_K \otimes v \\
& = \sum_{\{i,j\} \subseteq J, i < j} 2(-1)^{|I||J| + \epsilon_i^J + \epsilon_j^J + 1} \xi_* \dots \widehat{\xi}_i \dots \widehat{\xi}_j \dots \xi_* (\xi_I \xi_i \xi_j) \xi_K \otimes v \\
& = -2(-1)^{|I||J|} \sum_{i < j} (\partial_i \partial_j \xi_J) (\xi_I \xi_i \xi_j) \xi_K \otimes v,
\end{aligned}$$

where we used in the last equality that $\xi_{J - \{i,j\}} = (-1)^{\epsilon_i^J + \epsilon_j^J} \partial_i \partial_j \xi_J$, and the term $\partial_i \partial_j \xi_J$ implicitly contains the condition $\{i, j\} \subseteq J$.

Now, in order to move through ξ_K , we may apply the (0)-action formula or make the direct computation recalling that the only surviving term corresponds to the case $L = K = I$ in (8.16), namely, it is non-zero if $K = I$ and we have to take all the brackets, that is, if $\xi_I = \xi_{i_1} \dots \xi_{i_s}$, then

$$\begin{aligned}
(\xi_I \xi_i \xi_j) \cdot (\xi_I \otimes v) &= (-1)^{|I|} (\xi_{i_2} \dots \xi_{i_s} \xi_i \xi_j) \xi_{i_2} \dots \xi_{i_s} \otimes v \\
&= (-1)^{|I| + (|I| - 1) + \dots + 1} \xi_i \xi_j \otimes v \\
&= (-1)^{\frac{|I|(|I| + 1)}{2}} \xi_i \xi_j \otimes v.
\end{aligned} \tag{8.19}$$

Now, inserting (8.19) into (8.18), we have

$$t^2 \xi_I \cdot (\xi_J \xi_K \otimes v) = 2\delta_{I,K} (-1)^{|I||J| + \frac{|I|(|I| + 1)}{2}} \sum_{i < j} (\partial_i \partial_j \xi_J) \otimes F_{ij} v. \tag{8.20}$$

On the other hand, if $f = \xi_I$ and $g = \xi_J \xi_K$, with $K \subseteq I$, $J \cap I = \emptyset$, then $\partial_f (\partial_i \partial_j g) \neq 0$ iff $K = I$ and $\{i, j\} \subseteq J$. Hence it capture the above conditions. Finally, observe that

$$\begin{aligned}
\partial_f (\partial_i \partial_j g) &= \partial_f (\partial_i \partial_j (\xi_J \xi_K)) = \partial_f (\partial_i [(\partial_j \xi_J) \xi_K + (-1)^{|J|} \xi_J (\partial_i \xi_K)]) \\
&= \partial_f ((\partial_i \partial_j \xi_J) \xi_K) = (-1)^{|I|(|J| - 2)} (\partial_i \partial_j \xi_J) (\partial_f \xi_K) \\
&= \delta_{I,K} (-1)^{|I||J| + \frac{|I|(|I| - 1)}{2}} (\partial_i \partial_j \xi_J), \quad (\text{by (8.1)})
\end{aligned} \tag{8.21}$$

replacing (8.21) in (8.20), we prove the lemma. \square

A simple computation shows that Theorem 4.1 also holds for $f = \xi_\emptyset$.

9. APPENDIX B

This appendix will be devoted to the proof of the classification of singular vectors in Theorem 5.1. First, we shall consider some technical results.

Let $\vec{m} \in \text{Ind}(V) = \mathbb{C}[\partial] \otimes \Lambda(n) \otimes V$ be a singular vector, then

$$\vec{m} = \sum_{k=0}^N \sum_I \partial^k (\xi_I \otimes v_{I,k}), \quad \text{with } v_{I,k} \in V.$$

In order to obtain the singular vectors, we need some reduction lemmas. In Lemmas 9.1-9.4, we prove that $N \leq 1$ and $|I| \geq n - 2$. In Lemma 9.5, the case $N = 1$ is discarded for $n \geq 4$, and in the case $n = 3$ we explicitly found the corresponding singular vector. Finally, the proof of Theorem 5.1 is completed at the end of this appendix.

Lemma 9.1. *If $\vec{m} \in \text{Ind}(V)$ is a singular vector, then the degree of \vec{m} in ∂ is at most 2.*

Proof. Using Theorem 4.3 for $f = 1$ and (S1), we have

$$\begin{aligned} 0 &= \frac{d^2}{d\lambda^2} (1_\lambda \vec{m}) = \sum_{k=2}^N \sum_I k(k-1)(\lambda + \partial)^{k-2} \left[(-2)\partial(\xi_I \otimes v_{I,k}) + \right. & (9.1) \\ &+ \lambda \left(\xi_I \otimes E_{00}v_{I,k} - n(1 - \delta_{|I|,n})\xi_I \otimes v_{I,k} \right) - \lambda^2 \sum_{i<j} \xi_i \xi_j \xi_I \otimes F_{ij}v_{I,k} \left. \right] \\ &+ \sum_{k=1}^N \sum_I 2k(\lambda + \partial)^{k-1} \left[\xi_I \otimes E_{00}v_{I,k} - n(1 - \delta_{|I|,n})\xi_I \otimes v_{I,k} + \right. \\ &\qquad\qquad\qquad \left. - 2\lambda \sum_{i<j} \xi_i \xi_j \xi_I \otimes F_{ij}v_{I,k} \right] \\ &- \sum_{k=0}^N \sum_I (\lambda + \partial)^k 2 \sum_{i<j} \xi_i \xi_j \xi_I \otimes F_{ij}v_{I,k} \left. \right]. \end{aligned}$$

Rewriting ∂ as $(\lambda + \partial) - \lambda$, we can consider (9.1) as a polynomial in $\lambda + \partial$ and λ . Then the terms in $(\lambda + \partial)^k \lambda^2$, gives us

$$0 = \sum_I \sum_{i<j} \xi_i \xi_j \xi_I \otimes F_{ij}v_{I,k} \quad \text{for all } k \geq 2. \quad (9.2)$$

Using it and considering the coefficient of $(\lambda + \partial)^l \lambda$ in (9.1) for $l \geq 1$, we have

$$0 = \sum_I \xi_I \otimes \left(E_{00}v_{I,k} - n(1 - \delta_{|I|,n})v_{I,k} + 2v_{I,k} \right), \quad \text{for all } k > 2.$$

Hence

$$E_{00}v_{I,k} - n(1 - \delta_{|I|,n})v_{I,k} = -2v_{I,k}, \quad \text{for all } k > 2. \quad (9.3)$$

Now, using (9.2), (9.3) and taking the coefficient of $(\lambda + \partial)^l$ in (9.1), for $l \geq 2$, we obtain

$$\begin{aligned} 0 &= \sum_I \left((-2)k(k-1) \xi_I \otimes v_{I,k} + 2k \xi_I \otimes (E_{00}v_{I,k} - n(1 - \delta_{|I|,n})v_{I,k}) \right) \\ &= \sum_I (-2)k(k+1) \xi_I \otimes v_{I,k}, \quad \text{for all } k > 2, \end{aligned}$$

getting $v_{I,k} = 0$ for all I and $k > 2$, finishing the proof. \square

From the previous Lemma, any singular vector have the form

$$\vec{m} = \partial^2 \left(\sum_I \xi_I \otimes v_{I,2} \right) + \partial \left(\sum_I \xi_I \otimes v_{I,1} \right) + \left(\sum_I \xi_I \otimes v_{I,0} \right).$$

Now, we shall introduce a very important notation. Observe that the formula for the action given by Theorem 4.3 have the form

$$f_\lambda(g \otimes v) = \partial a + b + \lambda B + \lambda^2 C = (\lambda + \partial) a + b + \lambda (B - a) + \lambda^2 C,$$

by taking the coefficients in ∂ and λ^j . Using it, we can write the λ -action for the singular vector \vec{m} of degree 2 in ∂ , as follows

$$\begin{aligned} f_\lambda \vec{m} &= \left[(\lambda + \partial) a_0 + b_0 + \lambda (B_0 - a_0) + \lambda^2 C_0 \right] \\ &\quad + (\lambda + \partial) \left[(\lambda + \partial) a_1 + b_1 + \lambda (B_1 - a_1) + \lambda^2 C_1 \right] \\ &\quad + (\lambda + \partial)^2 \left[(\lambda + \partial) a_2 + b_2 + \lambda (B_2 - a_2) + \lambda^2 C_2 \right]. \end{aligned}$$

For example,

$$C_2 = - \sum_I \sum_{i < j} (-1)^{\frac{|I|(|I|+1)}{2} + |f||I|} f \xi_i \xi_j \xi_I \otimes F_{ij} v_{I,2}.$$

Obviously, these coefficients depend also in f , and sometimes we shall write for example $a_2(f)$ to emphasize the dependance, but we will keep it implicit in the notation if no confusion may arise.

In order to study conditions (S1) – (S3), we need to compute

$$\begin{aligned} (f_\lambda \vec{m})' &= B_0 + 2 \lambda C_0 \\ &\quad + \left[(\lambda + \partial) a_1 + b_1 + \lambda (B_1 - a_1) + \lambda^2 C_1 \right] + (\lambda + \partial) (B_1 + 2\lambda C_1) \\ &\quad + 2(\lambda + \partial) \left[(\lambda + \partial) a_2 + b_2 + \lambda (B_2 - a_2) + \lambda^2 C_2 \right] + (\lambda + \partial)^2 (B_2 + 2\lambda C_2). \end{aligned}$$

and

$$\begin{aligned} (f_\lambda \vec{m})'' &= 2 C_0 + 2 B_1 + 4 \lambda C_1 + 2(\lambda + \partial) C_1 \\ &\quad + 2 \left[(\lambda + \partial) a_2 + b_2 + \lambda (B_2 - a_2) + \lambda^2 C_2 \right] \\ &\quad + 4(\lambda + \partial)(B_2 + 2\lambda C_2) + 2(\lambda + \partial)^2 C_2. \end{aligned}$$

Therefore, by taking coefficients in $(\lambda + \partial)^i \lambda^j$, conditions (S1)-(S3) translate into the following list:

- For all $f \in \Lambda(n)$:

$$\begin{aligned} 0 &= C_2 \\ C_1 &= a_2 = -B_2 \\ 0 &= C_0 + B_1 + b_2. \end{aligned} \tag{9.4}$$

- For $f = \xi_I$, with $|I| \geq 1$:

$$\begin{aligned} 0 &= a_2 = B_2 \\ 0 &= a_1 + B_1 + 2b_2 \\ 0 &= B_0 + b_1. \end{aligned} \tag{9.5}$$

- For $f = \xi_I$, with $|I| \geq 3$ or $f \in B_{\text{so}(n)}$:

$$\begin{aligned} a_1 &= -b_2 \\ a_0 &= -b_1 \\ 0 &= b_0. \end{aligned} \tag{9.6}$$

Lemma 9.2. *The following conditions hold in a singular vector:*

- (1) If $|I| \neq n$, $v_{I,2} = 0$.
- (2) If $|I| \leq n - 3$, $v_{I,1} = 0$.
- (3) If $|I| \leq n - 5$, $v_{I,0} = 0$.

Proof. (1) Using (9.5), we have $a_2 = 0$ if $f = \xi_J$ with $|J| \geq 1$, that is

$$0 = \sum_I (-1)^{\frac{|J|(|J|+1)}{2} + |J||I|} (|J| - 2)(\xi_J \xi_I \otimes v_{I,2}).$$

Now, suppose there exists I such that $v_{I,2} \neq 0$ with $|I| \leq n - 1$. Let I_0 be one set of minimal length with this property. Then

$$0 = a_2(f) = \sum_{|I| \geq |I_0|} (\text{sgn})_{I,f} (|f| - 2)(f \xi_I \otimes v_{I,2}).$$

Then take $f = \xi_{I_0^c}$ if $|I_0^c| \neq 2$ (where from now on A^c denote the complement of A in $\{1, \dots, n\}$), and take $f = \xi_{i_0}$ for a fixed $i_0 \notin I_0$ if $|I_0^c| = 2$. Then, we compute $a_2(f)$ with this choice of f , obtaining

$$0 = (\text{sgn})(|I_0^c| - 2) \xi_* \otimes v_{I_0,2}, \quad \text{if } |I_0^c| \neq 2;$$

and, if $|I_0^c| = 2$, we have

$$0 = (\text{sgn}) \xi_{i_0} \xi_{I_0} \otimes v_{I_0,2} + (\text{sgn}) \xi_* \otimes v_{I_0 \cup \{i_1\},2} + \sum_{i \in I_0} (\text{sgn}) \xi_{\{i\}^c} \otimes v_{I_0 \cup \{i_1\} \setminus \{i\},2}$$

where i_1 satisfies $J_0 \cup \{i_0, i_1\} = \{1, \dots, n\}$, and $\xi_* = \xi_1 \dots \xi_n$ as before. Hence $v_{I_0,2} = 0$, finishing the proof of (1).

(2) Using (1), observe that for $f = \xi_I$ with $|I| \geq 3$, we have

$$b_2(f) = \sum_{j=1}^n (\text{sgn})_{j,f} (\partial_j f) (\partial_j \xi_*) \otimes v_{*,2} - \sum_{r < s} (\partial_r \partial_s f) \xi_* \otimes F_{rs} v_{*,2} = 0.$$

Therefore, using (9.6), we get $a_1(\xi_I) = 0$ for $|I| \geq 3$.

Now, suppose there exist J such that $v_{J,1} \neq 0$ with $|J| \leq n-3$, and take J_0 with minimal length satisfying this property. Then, since $|J_0^c| \geq 3$, we have

$$0 = a_1(\xi_{J_0^c}) = \sum_{|J| \geq |J_0|} (\text{sgn})(|J_0^c| - 2) \xi_{J_0^c} \xi_J \otimes v_{J,1} = K \xi_* \otimes v_{J_0,1}, \quad K \neq 0$$

proving that $v_{J_0,1} = 0$.

(3) Since $v_{J,1} = 0$ for $|J| \leq n-3$ (by the previous proof), it is easy to see that $b_1(\xi_I) = 0$ if $|I| \geq 5$. Then, by (9.6), we have that $a_0(\xi_I) = 0$ if $|I| \geq 5$. Hence, $v_{J,0} = 0$ if $|J| \leq n-5$, finishing the proof. \square

After this lemma, we have that any singular vector have this form:

$$\vec{m} = \partial^2 \xi_* \otimes v_{*,2} + \partial \sum_{|I| \geq n-2} \xi_I \otimes v_{I,1} + \sum_{|I| \geq n-4} \xi_I \otimes v_{I,0}.$$

Now, we shall continue with more reduction lemmas:

Lemma 9.3. *If $n \geq 3$, then $v_{*,2} = 0$.*

Proof. Using (9.4), we have $a_2(f) = c_1(f)$ for any f . In particular, taking $f = 1$, we have on one hand

$$a_2(1) = -2 \xi_* \otimes v_{*,2},$$

and, on the other hand

$$c_1(1) = - \sum_{i < j} \sum_{|J| \geq n-2} \xi_i \xi_j \xi_J \otimes F_{ij} v_{J,1} = - \sum_{i < j} (-1)^{i+j-1} \xi_* \otimes F_{ij}(v_{\{i,j\}^c,1}),$$

since we must take $J = \{i, j\}^c$ and $\xi_i \xi_j \xi_{\{i,j\}^c} = (-1)^{i+j-1} \xi_*$. Therefore,

$$2 v_{*,2} = - \sum_{i < j} (-1)^{i+j} F_{ij}(v_{\{i,j\}^c,1}). \quad (9.7)$$

Now, we shall study condition $a_1 + B_1 + 2b_2 = 0$ for $|f| \geq 1$, and compare it with (9.7). Fix $f = \xi_{i_0}$, and observe that

$$b_2(f) = (-1)^{1+n} \partial_{i_0} \xi_* \otimes v_{*,2} = (-1)^{i_0+n} \xi_{\{i_0\}^c} \otimes v_{*,2}. \quad (9.8)$$

Then, from the last equation, we need to pick up the term with $\xi_{\{i_0\}^c}$ in $a_1(f)$ and $B_1(f)$. Since

$$a_1(f) = \sum_{|I| \geq n-2} (-1)^{|I|} \xi_{i_0} \xi_I \otimes v_{I,1},$$

then, $a_1(f)$ does not have terms without ξ_{i_0} . On the other hand

$$\begin{aligned} B_1(f) &= \sum_{|I| \geq n-2} (-1)^{|I|+1} \xi_{i_0} \xi_I \otimes E_{00} v_{I,1} + \sum_{i \neq i_0} \sum_{|I| \geq n-2} (-1)^{|I|+1} \partial_i (\xi_{i_0} \xi_i \xi_I) \otimes v_{I,1} \\ &\quad - \sum_{i \neq j} \sum_{|I| \geq n-2} (-1)^{|I|+1} (\partial_i \xi_{i_0}) \xi_j \xi_I \otimes F_{ij} v_{I,1}, \end{aligned} \quad (9.9)$$

hence, only the last summand of (9.9) have the term $\xi_{\{i_0\}^c}$, and this is possible only if $I = \{j, i_0\}^c$, $i = i_0$ and $j \neq i_0$, namely

$$\begin{aligned} (\text{term } \xi_{\{i_0\}^c} \text{ in } B_1(f)) &= - \sum_{j \neq i_0} (-1)^{n+1} \xi_j \xi_{\{j, i_0\}^c} \otimes F_{i_0 j}(v_{\{j, i_0\}^c, 1}) \\ &= - \sum_{j < i_0} (-1)^{n+j} \xi_{\{i_0\}^c} \otimes F_{i_0 j}(v_{\{j, i_0\}^c, 1}) - \sum_{j > i_0} (-1)^{n+1+j} \xi_{\{i_0\}^c} \otimes F_{i_0 j}(v_{\{j, i_0\}^c, 1}), \end{aligned} \quad (9.10)$$

where we used that $\xi_j \xi_{\{j, i_0\}^c} = (-1)^{j-2} \xi_{\{i_0\}^c}$ if $i_0 < j$, and $\xi_j \xi_{\{j, i_0\}^c} = (-1)^{j-1} \xi_{\{i_0\}^c}$ if $i_0 > j$. Comparing (9.10) with (9.8), we have

$$\begin{aligned} 2 v_{*,2} &= - \sum_{j < i_0} (-1)^{i_0+j+1} F_{i_0 j}(v_{\{j, i_0\}^c, 1}) - \sum_{i_0 < j} (-1)^{i_0+j} F_{i_0 j}(v_{\{j, i_0\}^c, 1}) \\ &= - \sum_{j < i_0} (-1)^{i_0+j} F_{j i_0}(v_{\{j, i_0\}^c, 1}) - \sum_{i_0 < j} (-1)^{i_0+j} F_{i_0 j}(v_{\{j, i_0\}^c, 1}), \end{aligned} \quad (9.11)$$

where we used in the last equality that $F_{i_0 j} = -F_{j i_0}$. Since (9.11) holds for all i_0 , we may take the sum over $i_0 = 1, \dots, n$ and compare it with (9.7), obtaining

$$2n v_{*,2} = 4 v_{*,2},$$

proving that $v_{*,2} = 0$ for all $n \geq 3$. \square

Lemma 9.4. *If $n \geq 3$, then any singular vector in $\text{Ind}(V)$ have this form:*

$$\vec{m} = \partial (\xi_* \otimes v_{*,1}) + \sum_{|I| \geq n-2} \xi_I \otimes v_{I,0}.$$

for certain $v_{*,1}, v_{I,0} \in V$.

Proof. Claim 1: For all $b \neq c$, $v_{\{b,c\}^c, 1} = 0$.

Proof of Claim 1: Combining (9.4) and (9.5), we have $a_1(f) = c_0(f)$ for all $|f| \geq 1$, since $b_2(f) = 0$ by the previous lemma. Let us fix $b \neq c$. We may suppose $b < c$. Consider $f = \xi_b \xi_c$, then (obviously) $a_1(f) = 0$. Hence

$$\begin{aligned} 0 = c_0(\xi_b \xi_c) &= - \sum_{i < j} \sum_{|I|=n-4} \xi_b \xi_c \xi_i \xi_j \xi_I \otimes F_{ij}(v_{I,0}) \\ &= - \sum_{i < j; i, j \neq b, c} \xi_b \xi_c \xi_i \xi_j \xi_{\{b, c, i, j\}^c} \otimes F_{ij}(v_{\{b, c, i, j\}^c, 0}) \end{aligned}$$

which may be rewritten as follows

$$0 = \sum_{i < j; i, j \neq b, c} \xi_i \xi_j \xi_{\{b, c, i, j\}^c} \otimes F_{ij}(v_{\{b, c, i, j\}^c, 0}). \quad (9.12)$$

On the other hand, fix $a \neq b, c$. By (9.4), $b_1(\xi_a) + B_0(\xi_a) = 0$. Observe that

$$\begin{aligned} b_1(\xi_a) &= \sum_{|I| \geq n-2} (-1)^{1+|I|} \partial_a \xi_I \otimes v_{I,1} \\ &= \sum_{j < k; j, k \neq a} (-1)^{n-1} (\partial_a \xi_{\{j, k\}^c}) \otimes v_{\{j, k\}^c, 1} \\ &\quad + \sum_{i \neq a} (-1)^n (\partial_a \xi_{\{i\}^c}) \otimes v_{\{i\}^c, 1} + (-1)^{n+1} \partial_a \xi_* \otimes v_{*,1}. \end{aligned} \quad (9.13)$$

Now,

$$\left(\text{term } \xi_{\{a,b,c\}^c} \text{ of } b_1(\xi_a) \right) = (-1)^{n-1} (\partial_a \xi_{\{b,c\}^c}) \otimes v_{\{b,c\}^c,1}. \quad (9.14)$$

Similarly,

$$\begin{aligned} B_0(\xi_a) &= \sum_{|I| \geq n-4} (-1)^{1+|I|} \xi_a \xi_I \otimes E_{00} v_{I,0} + \sum_i \sum_{|I| \geq n-4} (-1)^{1+|I|} \partial_i (\xi_a \xi_i \xi_I) \otimes v_{I,0} \\ &\quad - \sum_{|I| \geq n-4} \sum_{i \neq j} (-1)^{1+|I|} (\partial_i \xi_a) \xi_j \xi_I \otimes F_{ij}(v_{I,0}). \end{aligned} \quad (9.15)$$

Obviously the term $\xi_{\{a,b,c\}^c}$ will appear in (9.15) only in the last sum, for certain values of I , namely

$$\left(\text{term } \xi_{\{a,b,c\}^c} \text{ of } B_0(\xi_a) \right) = - \sum_{l \neq a,b,c} (-1)^{n-1} \xi_l \xi_{\{a,b,c,l\}^c} \otimes F_{a,l}(v_{\{a,b,c,l\}^c,0}). \quad (9.16)$$

Using (9.14), (9.16) and the fact that $0 = b_1(\xi_a) + B_0(\xi_a)$, we get

$$0 = H(a) := (\partial_a \xi_{\{b,c\}^c}) \otimes v_{\{b,c\}^c,1} - \sum_{l \neq a,b,c} \xi_l \xi_{\{a,b,c,l\}^c} \otimes F_{a,l}(v_{\{a,b,c,l\}^c,0}).$$

Now, moving a , we may take

$$\begin{aligned} 0 &= \sum_{a \neq b,c} \xi_a \cdot H(a) \\ &= \sum_{a \neq b,c} \xi_a (\partial_a \xi_{\{b,c\}^c}) \otimes v_{\{b,c\}^c,1} - \sum_{a \neq b,c} \sum_{l \neq a,b,c} \xi_a \xi_l \xi_{\{a,b,c,l\}^c} \otimes F_{a,l}(v_{\{a,b,c,l\}^c,0}) \\ &= \sum_{a \neq b,c} \xi_a (\partial_a \xi_{\{b,c\}^c}) \otimes v_{\{b,c\}^c,1} - \sum_{a < l; a, l \neq b,c} \xi_a \xi_l \xi_{\{a,b,c,l\}^c} \otimes F_{a,l}(v_{\{a,b,c,l\}^c,0}) \\ &\quad - \sum_{l < a; a, l \neq b,c} \xi_a \xi_l \xi_{\{a,b,c,l\}^c} \otimes F_{a,l}(v_{\{a,b,c,l\}^c,0}) \end{aligned}$$

and using that $\xi_a \xi_l = -\xi_l \xi_a$, $F_{a,l} = -F_{l,a}$ and $\xi_a (\partial_a \xi_{\{b,c\}^c}) = \xi_{\{b,c\}^c}$, the last equation becomes

$$\begin{aligned} 0 &= \sum_{a \neq b,c} \xi_{\{b,c\}^c} \otimes v_{\{b,c\}^c,1} - 2 \left(\sum_{a < l; a, l \neq b,c} \xi_a \xi_l \xi_{\{a,b,c,l\}^c} \otimes F_{a,l}(v_{\{a,b,c,l\}^c,0}) \right) \\ &= \sum_{a \neq b,c} \xi_{\{b,c\}^c} \otimes v_{\{b,c\}^c,1} \quad (\text{using (9.12)}) \\ &= (n-2) (\xi_{\{b,c\}^c} \otimes v_{\{b,c\}^c,1}), \end{aligned}$$

proving Claim 1 for $n \geq 3$.

Claim 2: For all b , $v_{\{b\}^c,1} = 0$.

Proof of Claim 2: The idea is similar to the proof of Claim 1, but taking other monomial terms.

Fix b . As in Claim 1, we have $a_1(f) = C_0(f)$ for all $|f| \geq 1$. In particular, since

$$C_0(\xi_b) = - \sum_{|I| \geq n-4} \sum_{i < j} (-1)^{|I|+1} \xi_b \xi_i \xi_j \xi_I \otimes F_{ij}(v_{I,0}),$$

we have

$$\left(\text{term } \xi_* \text{ of } C_0(\xi_b) \right) = - \sum_{i < j; i, j \neq b} (-1)^n \xi_b \xi_i \xi_j \xi_{\{b, i, j\}^c} \otimes F_{ij}(v_{\{b, i, j\}^c, 0}),$$

and it is easy to see that

$$\left(\text{term } \xi_* \text{ of } a_1(\xi_b) \right) = (-1)^{n-1} \xi_b \xi_{\{b\}^c} \otimes v_{\{b\}^c, 1}.$$

Therefore

$$\xi_{\{b\}^c} \otimes v_{\{b\}^c, 1} = \sum_{i < j; i, j \neq b} \xi_i \xi_j \xi_{\{b, i, j\}^c} \otimes F_{ij}(v_{\{b, i, j\}^c, 0}). \quad (9.17)$$

Now, take $a \neq b$. Using (9.13), we obtain

$$\left(\text{term } \xi_{\{a, b\}^c} \text{ of } b_1(\xi_a) \right) = (-1)^n (\partial_a \xi_{\{b\}^c}) \otimes v_{\{b\}^c, 1}. \quad (9.18)$$

Similarly, using (9.15)

$$\left(\text{term } \xi_{\{a, b\}^c} \text{ of } B_0(\xi_a) \right) = - \sum_{l \neq a, b} (-1)^{n-2} \xi_l \xi_{\{a, b, l\}^c} \otimes F_{a, l}(v_{\{a, b, l\}^c, 0}). \quad (9.19)$$

Using (9.18), (9.19) and the fact that $0 = b_1(\xi_a) + B_0(\xi_a)$, we get

$$0 = L(a) := \partial_a \xi_{\{b\}^c} \otimes v_{\{b\}^c, 1} - \sum_{l \neq a, b} \xi_l \xi_{\{a, b, l\}^c} \otimes F_{a, l}(v_{\{a, b, l\}^c, 0}).$$

Now, moving a , we may take

$$\begin{aligned} 0 &= \sum_{a \neq b} \xi_a \cdot L(a) \\ &= \sum_{a \neq b} \xi_a (\partial_a \xi_{\{b\}^c}) \otimes v_{\{b\}^c, 1} - \sum_{a \neq b} \sum_{l \neq a, b} \xi_a \xi_l \xi_{\{a, b, l\}^c} \otimes F_{a, l}(v_{\{a, b, l\}^c, 0}) \\ &= \sum_{a \neq b} \xi_{\{b\}^c} \otimes v_{\{b\}^c, 1} - \sum_{a < l; a, l \neq b} \xi_a \xi_l \xi_{\{a, b, l\}^c} \otimes F_{a, l}(v_{\{a, b, l\}^c, 0}) \\ &\quad - \sum_{l < a; a, l \neq b} \xi_a \xi_l \xi_{\{a, b, l\}^c} \otimes F_{a, l}(v_{\{a, b, l\}^c, 0}) \\ &= (n-1)(\xi_{\{b\}^c} \otimes v_{\{b\}^c, 1}) - 2 \left(\sum_{a < l; a, l \neq b} \xi_a \xi_l \xi_{\{a, b, l\}^c} \otimes F_{a, l}(v_{\{a, b, l\}^c, 0}) \right) \\ &= (n-3)(\xi_{\{b\}^c} \otimes v_{\{b\}^c, 1}) \quad (\text{using (9.17)}). \end{aligned}$$

Hence $v_{\{b\}^c, 1} = 0$ for all b and $n \geq 4$.

If $n = 3$, condition $a_1(\xi_b) = C_0(\xi_b)$ give us

$$v_{\{b\}^c, 1} = F_{ij}(v_{\emptyset, 0}), \quad (9.20)$$

where $\{b\}^c = \{i, j\}$ and $i < j$. On the other hand, by taking the term ξ_b of $b_0(\xi_1 \xi_2 \xi_3)$ and using that $b_0(\xi_1 \xi_2 \xi_3) = 0$, we obtain that $F_{ij}(v_{\emptyset, 0}) = 0$ for all $i < j$, which combined with (9.20) produce the desired result, finishing the proof of Claim 2.

Finally, in order to complete the proof of this lemma, we need to study the vectors $v_{I, 0}$. Since $v_{I, 1} = 0$ if $|I| \leq n-1$, it is clear that $b_1(f) = 0$ for $|f| \geq 3$.

Therefore, using condition (9.6), we have $a_0(f) = 0$ if $|f| \geq 3$, which immediately gives us that $v_{I,0} = 0$ if $|I| = n - 3$ or $n - 4$, completing the proof. \square

From the previous lemma, any singular vector have this form:

$$\vec{m} = \partial (\xi_* \otimes v_{*,1}) + \sum_{|I| \geq n-2} \xi_I \otimes v_{I,0}.$$

Using (4.1), (8.1) and (4.4), the \mathbb{Z} -gradation in $K(1, n)_+$, translates into a $\mathbb{Z}_{\leq 0}$ -gradation in $\text{Ind}(V)$:

$$\begin{aligned} \text{Ind}(V) &\simeq \Lambda(1, n) \otimes V \simeq \mathbb{C}[\partial] \otimes \Lambda(n) \otimes V \\ &\simeq \underbrace{\mathbb{C} 1 \otimes V}_{\text{deg } 0} \oplus \underbrace{\mathbb{C}^n \otimes V}_{\text{deg } -1} \oplus \underbrace{(\mathbb{C} \partial \otimes V \oplus \Lambda^2(\mathbb{C}^n) \otimes V)}_{\text{deg } -2} \oplus \dots \end{aligned}$$

Therefore, in the previous lemmas, we have proved that any singular vector must have degree -1 or -2.

Recall that in Theorem 4.3, we considered the Hodge dual of the natural bases in order to simplify the formula of the action. Hence, any singular vector must have one of the following forms:

1. $\vec{m} = \partial (\xi_* \otimes v_*) + \sum_{i < j} \xi_{\{i,j\}^c} \otimes v_{\{i,j\}}$.
2. $\vec{m} = \sum_i \xi_{\{i\}^c} \otimes v_i$.

The next lemma study the first one.

Lemma 9.5. *If $n > 3$, the first case is not possible. If $n = 3$, then*

$$\vec{m} = \partial (\xi_* \otimes v_\mu) + i \xi_{\{1,2\}^c} \otimes v_\mu - 2\xi_{\{2,3\}^c} \otimes F_{2,3}v_\mu + 2 \xi_{\{1,3\}^c} \otimes F_{1,3}v_\mu$$

is a singular vector, where v_μ is a highest weight vector of the $\mathfrak{so}(3)$ -module of highest weight $\mu = (\frac{3}{2}, \frac{1}{2})$.

Proof. From now on, we assume that $\vec{m} = \partial(\xi_* \otimes v_*) + \sum_{i < j} \xi_{\{i,j\}^c} \otimes v_{\{i,j\}}$. Observe that conditions (9.4), (9.5) and (9.6), clearly becomes

- (1) $b_1(f) = 0$, if $f \in B_{\mathfrak{so}(n)}$.
- (2) $b_1(f) + B_0(f) = 0$, if $|f| = 1$ or 2 .
- (3) $C_0(f) + B_1(f) = 0$, if $f = 1$.
- (4) $b_0(f) = 0$, if $|f| = 3, 4$ or $f \in B_{\mathfrak{so}(n)}$.

It is possible to see that they are equivalent to the following equations in terms of the vectors $v_{\{i,j\}}$ and v_* :

$$\underline{b_1(f) = 0, \text{ if } f \in B_{\mathfrak{so}(n)}} : \quad B_{\mathfrak{so}(n)} \cdot v_* = 0. \quad (9.21)$$

$$\underline{b_1(\xi_a) + B_0(\xi_a) = 0} : \quad (1 \leq a \leq n)$$

$$-v_* = \sum_{j < a} (-1)^{a+j} F_{ja}(v_{\{j,a\}}) + \sum_{a < j} (-1)^{a+j} F_{aj}(v_{\{a,j\}}), \quad (9.22)$$

for $a < b$:

$$0 = E_{00}(v_{\{a,b\}}) - v_{\{a,b\}} - \sum_{j < b, j \neq a} (-1)^{a+j} F_{aj}(v_{\{j,b\}}) + \sum_{b < j} (-1)^{a+j} F_{aj}(v_{\{b,j\}}) \quad (9.23)$$

and for $b < a$:

$$0 = E_{00}(v_{\{a,b\}}) - v_{\{a,b\}} + \sum_{j < b} (-1)^{a+j} F_{aj}(v_{\{j,b\}}) - \sum_{b < j, j \neq a} (-1)^{a+j} F_{aj}(v_{\{b,j\}}). \quad (9.24)$$

$$\underline{b_1(\xi_a \xi_b) + B_0(\xi_a \xi_b) = 0}: \quad (a < b)$$

$$\begin{aligned} 0 &= -F_{ab}(v_*) + (-1)^{a+b} E_{00}(v_{\{a,b\}}) \\ &\quad - \sum_{j < b, j \neq a} (-1)^{b+j} F_{aj}(v_{\{j,b\}}) + \sum_{b < j} (-1)^{b+j} F_{aj}(v_{\{b,j\}}) \\ &\quad + \sum_{j < a} (-1)^{a+j} F_{bj}(v_{\{a,j\}}) - \sum_{a < j, j \neq b} (-1)^{a+j} F_{bj}(v_{\{a,j\}}). \end{aligned} \quad (9.25)$$

$$\underline{C_0(1) + B_1(1) = 0}:$$

$$0 = E_{00}(v_*) + \sum_{i < j} (-1)^{i+j} F_{ij}(v_{\{i,j\}}). \quad (9.26)$$

Finally, in the case of condition $b_0(f) = 0$, if $|f| = 3, 4$ or $f \in B_{so(n)}$, we shall only need the following equations that are deduced from $b_0(\xi_a \xi_b \xi_c) = 0$, with $a < b < c$:

$$0 = (-1)^{b+c} v_{\{b,c\}} + (-1)^{a+c} F_{ab}(v_{\{a,c\}}) - (-1)^{a+b} F_{ac}(v_{\{a,b\}}) \quad (9.27)$$

$$0 = (-1)^{a+c} v_{\{a,c\}} - (-1)^{b+c} F_{ab}(v_{\{b,c\}}) + (-1)^{a+b} F_{bc}(v_{\{a,b\}}) \quad (9.28)$$

$$0 = (-1)^{a+b} v_{\{a,b\}} + (-1)^{b+c} F_{ac}(v_{\{b,c\}}) - (-1)^{a+c} F_{bc}(v_{\{a,c\}}). \quad (9.29)$$

Now, fix $a < b$, by taking a linear combination of (9.23) and (9.24), we obtain

$$\begin{aligned} 0 &= -2(-1)^{a+b} E_{00}(v_{\{a,b\}}) + 2(-1)^{a+b} v_{\{a,b\}} \\ &\quad + \sum_{j < b, j \neq a} (-1)^{b+j} F_{aj}(v_{\{j,b\}}) - \sum_{b < j} (-1)^{b+j} F_{aj}(v_{\{b,j\}}) \\ &\quad - \sum_{j < a} (-1)^{a+j} F_{bj}(v_{\{a,j\}}) + \sum_{a < j, j \neq b} (-1)^{a+j} F_{bj}(v_{\{a,j\}}) \end{aligned}$$

and, comparing the last four summands with (9.25), we obtain

$$F_{ab}(v_*) = (-1)^{a+b+1} E_{00}(v_{\{a,b\}}) + 2(-1)^{a+b} v_{\{a,b\}}. \quad (9.30)$$

On the other hand, observe that (9.25) can be rewritten as follows ($a < b$)

$$\begin{aligned}
0 &= -F_{ab}(v_*) + (-1)^{a+b} E_{00}(v_{\{a,b\}}) \\
&\quad - \sum_{j < a} [(-1)^{b+j} F_{aj}(v_{\{j,b\}}) - (-1)^{a+j} F_{bj}(v_{\{a,j\}})] \\
&\quad - \sum_{a < j < b} [(-1)^{b+j} F_{aj}(v_{\{j,b\}}) + (-1)^{a+j} F_{bj}(v_{\{a,j\}})] \\
&\quad - \sum_{b < j} [(-1)^{b+j+1} F_{aj}(v_{\{j,b\}}) + (-1)^{a+j} F_{bj}(v_{\{a,j\}})].
\end{aligned} \tag{9.31}$$

Therefore, inserting (9.27), (9.28) and (9.29) in the last three summands of (9.31), we have

$$F_{ab}(v_*) = (-1)^{a+b} E_{00}(v_{\{a,b\}}) - (n-2) (-1)^{a+b} v_{\{a,b\}}. \tag{9.32}$$

Hence, using (9.30) and (9.32), we get

$$E_{00}(v_{\{a,b\}}) = \frac{n}{2} v_{\{a,b\}}.$$

and

$$2 F_{ab}(v_*) = (n-4) (-1)^{a+b+1} v_{\{a,b\}}, \tag{9.33}$$

or, with some restrictions,

$$v_{\{a,b\}} = (-1)^{a+b+1} \frac{2}{(n-4)} F_{ab}(v_*) \quad n \neq 4. \tag{9.34}$$

Now, combining (5.3), (9.21) and (9.34), it is easy to prove the following identities ($1 \leq l < j \leq m$)

$$v_{\{2l,2j\}} = i v_{\{2l-1,2j\}} \tag{9.35}$$

$$v_{\{2l-1,2j-1\}} = -i v_{\{2l,2j-1\}} \tag{9.36}$$

$$v_{\{2l-1,2m+1\}} = -i v_{\{2l,2m+1\}} \tag{9.37}$$

Taking the sum over a in (9.22), and using (9.26), we get

$$n v_* = -2 \sum_{i < j} (-1)^{i+j} F_{ij}(v_{\{i,j\}}) = 2 E_{00}(v_*),$$

obtaining

$$E_{00}(v_*) = \frac{n}{2} v_*. \tag{9.38}$$

Let $\mu = (\frac{n}{2}; \mu_1, \dots, \mu_m)$ be the weight of the highest weight vector v_* (see (9.21) and (9.38)). Since $H_1 = iF_{12}$, then by (9.34), we have

$$v_{1,2} = -2i \frac{\mu_1}{(n-4)} v_*. \tag{9.39}$$

Now, considering (9.22) with $a = 1$, and using (9.35), (9.36), (9.37) and (9.39), we have:

$$\begin{aligned}
 v_* &= - \sum_{1 < j} (-1)^{1+j} F_{1j}(v_{\{1,j\}}) \\
 &= -i H_1(v_{1,2}) - \sum_{1 < l \leq m} F_{1,2l-1}(v_{\{1,2l-1\}}) + \sum_{1 < l \leq m} F_{1,2l}(v_{\{1,2l\}}) \\
 &\quad - \delta_{n,\text{odd}} F_{1,2m+1}(v_{\{1,2m+1\}}) \\
 &= -2 \frac{\mu_1}{(n-4)} H_1(v_*) + i \sum_{1 < l \leq m} F_{1,2l-1}(v_{\{2,2l-1\}}) - i \sum_{1 < l \leq m} F_{1,2l}(v_{\{2,2l\}}) \\
 &\quad + i \delta_{n,\text{odd}} F_{1,2m+1}(v_{\{2,2m+1\}}),
 \end{aligned}$$

that is

$$v_* = -2 \frac{\mu_1^2}{(n-4)} v_* + i \sum_{2 < j} (-1)^{1+j} F_{1,j}(v_{\{2,j\}}). \quad (9.40)$$

Considering (9.23) with $a = 1, b = 2$, and inserting (9.40) and (9.39) on it, it is easy to see that

$$0 = 2\mu_1^2 + (n-2)\mu_1 + (n-4), \quad (9.41)$$

obtaining $\mu_1 = -1$ or $\frac{4-n}{2}$, which is negative for $n \geq 5$ and it is impossible for the highest weight of an irreducible $\mathfrak{so}(n)$ -module, finishing the proof in this case.

If $n = 3$, observe that equations (9.34), (9.38), (9.39) and (9.41) hold in this case, obtaining the result of the statement of this lemma.

If $n = 4$, using (9.33) we have $v_{a,b} = 0$ for all $a < b$, obtaining a trivial singular vector and finishing the proof. \square

From now on, we assume that the singular vector has the form $\vec{m} = \sum_i \xi_{\{i\}^c} \otimes v_i$, and we shall use the following notation, for $n = 2m$ or $n = 2m + 1$:

$$\begin{aligned}
 \vec{m} &= \sum_{i=1}^n \xi_{\{i\}^c} \otimes v_i \\
 &= \sum_{l=1}^m \left[(\xi_{\{2l\}^c} + i\xi_{\{2l-1\}^c}) \otimes w_l + (\xi_{\{2l\}^c} - i\xi_{\{2l-1\}^c}) \otimes \bar{w}_l \right] - \\
 &\quad - \delta_{n,\text{odd}} i\xi_{\{2m+1\}^c} \otimes w_{m+1},
 \end{aligned} \quad (9.42)$$

that is, for $1 \leq l \leq m$

$$v_{2l} = w_l + \bar{w}_l, \quad v_{2l-1} = i(w_l - \bar{w}_l), \quad v_{2m+1} = iw_{m+1}. \quad (9.43)$$

Observe that conditions (9.4), (9.5) and (9.6), clearly reduce to

- (1) If $|f| = 1$, $B_0(f) = 0$.
- (2) If $|f| = 3$ or $f \in B_{\mathfrak{so}(n)}$, $b_0(f) = 0$.

After some lengthly computations, it is possible to see that they are equivalent to the following equations in terms of the vectors v_i, w_l, \bar{w}_k :

$B_0(\xi_a) = 0$:

$$0 = (-1)^a E_{00} v_a - \sum_{k \neq a} (-1)^k F_{ak} v_k. \quad (9.44)$$

$b_0(\xi_a \xi_b \xi_c) = 0$, with $a < b < c$:

$$0 = (-1)^c F_{ab}(v_c) - (-1)^b F_{ac}(v_b) + (-1)^a F_{bc}(v_a). \quad (9.45)$$

Recall the basis of the Borel subalgebra introduced in (5.4) and (5.5), and using that

$$\begin{aligned} \text{term with } \xi_{\{a\}^c} \text{ in } b_0(\xi_a \xi_b) &= (-1)^{a+b} v_b + F_{ab}(v_a), \\ \text{term with } \xi_{\{b\}^c} \text{ in } b_0(\xi_a \xi_b) &= -(-1)^{a+b} v_a + F_{ab}(v_b), \\ \text{term with } \xi_{\{l\}^c} \text{ in } b_0(\xi_a \xi_b) &= F_{ab}(v_l), \quad l \neq a, b, \end{aligned}$$

condition $b_0(f) = 0$ for $f \in B_{\mathfrak{so}(n)}$, becomes for $n = 2m$ or $n = 2m + 1$:

$b_0(\alpha_{ij}) = 0$, with $1 \leq i < j \leq m$:

$$\alpha_{ij}(w_i) = 0 \quad (9.46)$$

$$\alpha_{ij}(\bar{w}_i) = w_j - \bar{w}_j \quad (9.47)$$

$$\alpha_{ij}(w_j) = w_i = -\alpha_{ij}(\bar{w}_j) \quad (9.48)$$

$$\alpha_{ij}(w_k) = 0 = \alpha_{ij}(\bar{w}_k), \quad k \neq i, j. \quad (9.49)$$

$b_0(\beta_{lj}) = 0$, with $1 \leq l < j \leq m$:

$$\beta_{lj}(w_i) = 0 \quad (9.50)$$

$$\beta_{lj}(\bar{w}_i) = -(w_j + \bar{w}_j) \quad (9.51)$$

$$\beta_{lj}(w_j) = w_i = \beta_{lj}(\bar{w}_j) \quad (9.52)$$

$$\beta_{lj}(w_k) = 0 = \beta_{lj}(\bar{w}_k), \quad k \neq i, j. \quad (9.53)$$

$b_0(\gamma_k) = 0$, with $1 \leq k \leq m$, and $n = 2m + 1$, corresponds to:

$$\gamma_k(w_k) = 0 \quad (9.54)$$

$$\gamma_k(\bar{w}_k) = w_{m+1} \quad (9.55)$$

$$\gamma_k(w_{m+1}) = 2 w_k \quad (9.56)$$

$$\gamma_k(w_l) = 0 = \gamma_k(\bar{w}_l), \quad 1 \leq l \leq m, \quad l \neq k. \quad (9.57)$$

Now, we shall impose conditions (9.44)-(9.57) to get the final reduction. Recall notation (5.1) and (5.2):

Lemma 9.6. *If $n = 2m$ or $n = 2m + 1$, equation (9.44) is equivalent to the following identities ($1 \leq j \leq m$)*

$$\begin{aligned} 2(E_{00} + H_j)(\bar{w}_j) &= \sum_{1 \leq l < j} [E_{-(\varepsilon_l + \varepsilon_j)}(w_l) - E_{(\varepsilon_l - \varepsilon_j)}(\bar{w}_l)] \\ &+ \sum_{j < l \leq m} [E_{-(\varepsilon_j - \varepsilon_l)}(\bar{w}_l) - E_{-(\varepsilon_j + \varepsilon_l)}(w_l)] - \delta_{n, \text{odd}} E_{-\varepsilon_j}(w_{m+1}) \end{aligned} \quad (9.58)$$

and

$$\begin{aligned} 2(E_{00} - H_j)(w_j) &= \sum_{1 \leq l < j} [E_{(\varepsilon_l + \varepsilon_j)}(\bar{w}_l) - E_{-(\varepsilon_l - \varepsilon_j)}(w_l)] \\ &+ \sum_{j < l \leq m} [E_{(\varepsilon_j - \varepsilon_l)}(w_l) - E_{(\varepsilon_j + \varepsilon_l)}(\bar{w}_l)] + \delta_{n, \text{odd}} E_{\varepsilon_j}(w_{m+1}) \end{aligned} \quad (9.59)$$

and for $n = 2m + 1$ we we have the additional equation

$$E_{00}(w_{m+1}) = \sum_{1 \leq l \leq m} [E_{\varepsilon_l}(\bar{w}_l) - E_{-\varepsilon_l}(w_l)].$$

Proof. It follows by a straightforward computation, by considering a linear combination of equation (9.44) for the cases where a is $2j$ and $2j - 1$, and replacing the vectors v_i 's in terms of w_i 's and \bar{w}_i 's. The last equation follows from (9.44) for $a = 2m + 1$. \square

Proof of Theorem 5.1. Suppose that $n = 2m$ or $n = 2m + 1$, with $m \geq 2$. Case $n = 3$ will be considered at the end of this proof.

Using (9.52) and (9.56), we have that $w_i \neq 0$ implies $w_j \neq 0$ for all $i < j$. Now, we shall show that there are only two possible cases: $w_i \neq 0$ for all i , or $w_i = 0$ for all i . Suppose that $w_j \neq 0$ for some j , and let j_0 be the minimal index j such that $w_j \neq 0$. Then w_{j_0} is a highest weight vector, by using (9.48), (9.52) and (9.56). Now, suppose $1 < j_0 \leq m$. Using (9.43), we have that equation (9.45), for $a = 1$, $b = 2$ and $c = 2j_0$ with $j_0 > 1$, becomes

$$0 = F_{1,2}(w_{j_0} + \bar{w}_{j_0}) - F_{1,2j_0}(w_1 - \bar{w}_1) - i F_{2,2j_0}(w_1 - \bar{w}_1), \quad (9.60)$$

and for $a = 1$, $b = 2$ and $c = 2j_0 - 1$ with $j_0 > 1$, it becomes

$$0 = i F_{1,2}(w_{j_0} - \bar{w}_{j_0}) + F_{1,2j_0-1}(w_1 + \bar{w}_1) + i F_{2,2j_0-1}(w_1 - \bar{w}_1). \quad (9.61)$$

Now, taking the linear combination (9.61) + i (9.60), and using (9.48) together with (9.52), we have

$$H_1(w_{j_0}) = -w_{j_0}.$$

which is impossible for a highest weight vector. Similarly, if n is odd and $j_0 = m + 1$, by considering (9.43), we have that equation (9.45), with $a = 1$, $b = 2$ and $c = 2m + 1$, becomes $H_1(w_{m+1}) = -w_{m+1}$, getting a contradiction. Therefore, all w_i 's are zero or all of them are non-zero.

- If $w_i = 0$ for all i , then $\bar{w}_j \neq 0$ for some j . Let j_0 be the maximal one with this property. As before, observe that \bar{w}_{j_0} is annihilated by the Borel subalgebra by using (9.47), (9.49), (9.51), (9.55) and (9.57), hence \bar{w}_{j_0} is a highest weight vector. Now, we shall prove that $j_0 = 1$. Suppose that $j_0 > 1$, then taking the linear combination (9.61) - i (9.60), and using (9.48) together with (9.52), we have

$$H_1(\bar{w}_{j_0}) = -\bar{w}_{j_0}.$$

which is impossible for a highest weight vector. Therefore, in this case, the singular vector has the form

$$\vec{m} = (\xi_{\{2\}^c} - i \xi_{\{1\}^c}) \otimes \bar{w}_1$$

as in part (a) of the statement of this theorem. Recall that \bar{w}_1 is a highest weight vector of $V = V(\mu)$. It remains to find necessary and sufficient conditions on the highest weight μ in order to get a singular vector of this form. Observe that we

only have to impose (9.44) and (9.45). Using Lemma 9.6, we obtain that it reduces to the following conditions:

$$E_{00}(\bar{w}_1) = -H_1(\bar{w}_1), \quad \text{and} \quad H_j(\bar{w}_1) = 0 \text{ for } j > 1.$$

Hence $\mu = (-k; k, 0, \dots, 0)$, finishing part (a) of this theorem.

- If $w_i \neq 0$ for all i , we should obtain part (b) of the present theorem. Using (9.52), we have $\bar{w}_j \neq 0$ for all $1 < j \leq m$. It remains to prove that $\bar{w}_1 \neq 0$. If not, combining (9.47) and (9.51) we get a contradiction. Therefore, all w_i 's and \bar{w}_j 's are non-zero. Observe that w_1 is annihilated by the Borel subalgebra by using (9.46), (9.49), (9.50), (9.53), (9.54) and (9.57). Therefore w_1 is a highest weight vector of $V = V(\mu)$. It remains to find conditions on the highest weight μ in order to get a singular vector of this form, and we should also show that all w_i 's and \bar{w}_j 's are fully determined by w_1 .

Let us compute μ . Using (9.43), we have that equation (9.45), with $a = 1$, $b = 2j - 1$ and $c = 2j$ for $j > 1$, becomes

$$0 = F_{1,2j-1}(w_j + \bar{w}_j) + i F_{1,2j}(w_j - \bar{w}_j) - i F_{2j-1,2j}(w_1 - \bar{w}_1), \quad (9.62)$$

and for $a = 2$, $b = 2j - 1$ and $c = 2j$ for $j > 1$, it becomes

$$0 = F_{2,2j-1}(w_j + \bar{w}_j) + i F_{2,2j}(w_j - \bar{w}_j) + F_{2j-1,2j}(w_1 + \bar{w}_1). \quad (9.63)$$

Now, taking the linear combination (9.62) $- i$ (9.63), and using (9.48) together with (9.52), we have

$$H_j(w_1) = 0, \quad \text{for } j > 1.$$

We have to compute $E_{00}(w_1)$ and $H_1(w_1)$. Observe that equation (9.59), for $j = 1$, becomes

$$\begin{aligned} 2(E_{00} - H_1)(w_1) &= \sum_{1 < l \leq m} [E_{(\varepsilon_1 - \varepsilon_l)}(w_l) - E_{(\varepsilon_1 + \varepsilon_l)}(\bar{w}_l)] + \delta_{n,\text{odd}} E_{\varepsilon_1}(w_{m+1}) \\ &= \sum_{1 < l \leq m} [\alpha_{1l}(w_l - \bar{w}_l) + \beta_{1l}(w_l + \bar{w}_l)] + \delta_{n,\text{odd}} \gamma_1(w_{m+1}) \end{aligned}$$

and inserting (9.48), (9.52) and (9.56) in the previous equation, we get

$$E_{00}(w_1) = H_1(w_1) + 2(m-1) w_1 + \delta_{n,\text{odd}} w_1.$$

proving, as the statement of this theorem, that w_1 is a highest weight vector of the $\mathfrak{cs}\mathfrak{o}(n)$ -module V with highest weight

$$(\mu_1 + 2(m-1) + \delta_{n,\text{odd}}; \mu_1, 0, \dots, 0), \text{ for } \mu_1 \in \mathbb{Z}_{>0}. \quad (9.64)$$

Now, we shall show that all w_i 's and \bar{w}_j 's are fully determined by w_1 .

Using (9.43), we have that equation (9.45), with $a = 1$, $b = 2$ and $c = 2k - 1$ for $k > 1$, becomes

$$0 = H_1(w_k - \bar{w}_k) + F_{1,2k-1}(w_1 + \bar{w}_1) + i F_{2,2k-1}(w_1 - \bar{w}_1), \quad (9.65)$$

and for $a = 1$, $b = 2$ and $c = 2k$ for $k > 1$, it becomes

$$0 = i H_1(w_k + \bar{w}_k) + F_{1,2k}(w_1 + \bar{w}_1) + i F_{2,2k}(w_1 - \bar{w}_1). \quad (9.66)$$

Now, taking the linear combination (9.65) $- i$ (9.66), we have

$$0 = 2 H_1(w_k) + E_{-(\varepsilon_1 - \varepsilon_k)}(w_1) + (\alpha_{1k} - \beta_{1k})(\bar{w}_1),$$

and using (9.47) together with (9.51), we obtain

$$0 = 2 H_1(w_k) + E_{-(\varepsilon_1 - \varepsilon_k)}(w_1) + 2 w_k. \quad (9.67)$$

Now, by applying β_{1k} to (9.67) and using (9.52), it is possible to see that $H_1(w_k) = \mu_1 - 1$ where μ_1 was defined above as $H_1(w_1) = \mu_1 w_1$. Therefore, from (9.67), we get

$$w_k = -\frac{1}{2\mu_1} E_{-(\varepsilon_1 - \varepsilon_k)}(w_1), \quad k > 1. \quad (9.68)$$

In a similar way, by taking the linear combination (9.65) + i (9.66), we can deduce

$$\bar{w}_k = \frac{1}{2\mu_1} E_{-(\varepsilon_1 + \varepsilon_k)}(w_1), \quad k > 1. \quad (9.69)$$

In the odd case, taking (9.45) with $a = 1, b = 2$ and $c = 2m + 1$, it is possible to deduce by a similar computation that

$$w_{m+1} = -\frac{1}{\mu_1} E_{-\varepsilon_1}(w_1). \quad (9.70)$$

Considering (9.58) for $j = 1$, and using (9.68), (9.69) and (9.70), we have that

$$\begin{aligned} 2(E_{00} + H_1)(\bar{w}_1) &= \sum_{1 < l \leq m} [E_{-(\varepsilon_1 - \varepsilon_l)}(\bar{w}_l) - E_{-(\varepsilon_1 + \varepsilon_l)}(w_l)] - \delta_{n, \text{odd}} E_{-\varepsilon_1}(w_{m+1}) \\ &= \frac{1}{\mu_1} \left[\sum_{1 < l \leq m} E_{-(\varepsilon_1 + \varepsilon_l)} E_{-(\varepsilon_1 - \varepsilon_l)}(w_1) + \delta_{n, \text{odd}} E_{-\varepsilon_1} E_{-\varepsilon_1}(w_1) \right] \end{aligned}$$

Applying $E_{(\varepsilon_1 + \varepsilon_2)}$ to the previous equation, we obtain

$$(E_{00} + H_1)(\bar{w}_1) = (2(m - 2 + \mu_1) + \delta_{n, \text{odd}}) \bar{w}_1 = (n - 4 + 2\mu_1) \bar{w}_1.$$

Therefore, we have

$$\bar{w}_1 = C \left[\sum_{1 < l \leq m} E_{-(\varepsilon_1 + \varepsilon_l)} E_{-(\varepsilon_1 - \varepsilon_l)}(w_1) + \delta_{n, \text{odd}} E_{-\varepsilon_1} E_{-\varepsilon_1}(w_1) \right], \quad (9.71)$$

where $C = \frac{1}{2\mu_1(n-4+2\mu_1)}$. Hence, equations (9.68)-(9.71) show that all w_i 's and \bar{w}_j 's are fully determined by w_1 .

Conversely, using the expressions of w_i 's and \bar{w}_j 's, it is possible to prove, after some lengthy computations, that the vector \vec{m} in (9.42) satisfies equations (9.44)-(9.57), finishing the proof of this lemma for $n \geq 4$.

If $n = 3$, all the previous equations holds except for (9.71) for $\mu_1 = \frac{1}{2}$. More precisely, the same reduction holds and we have the first family of singular vectors $\vec{m} = (\xi_{\{2\}^c} - i \xi_{\{1\}^c}) \otimes \bar{w}_1$, where \bar{w}_1 is a highest weight vector of weight $(-k; k)$, but in this case $k \in \frac{1}{2}\mathbb{Z}_{>0}$. The second family, corresponding to $w_i \neq 0$ for all i , is also present. In this case, using (9.64), w_1 is a highest weight vector of the $\mathfrak{cs}\mathfrak{o}(3)$ -module V , with highest weight $(\mu_1 + 1; \mu_1)$, but in this case $\mu_1 \in \frac{1}{2}\mathbb{Z}_{>0}$. Now, using (9.70) and (9.71), we have the complete expression of the singular vector \vec{m} , that is (in this case $m = 1$)

$$w_2 = -\frac{1}{\mu_1} E_{-\varepsilon_1}(w_1), \quad \bar{w}_1 = \frac{1}{2\mu_1(2\mu_1 - 1)} \left(E_{-\varepsilon_1} E_{-\varepsilon_1}(w_1) \right), \quad (9.72)$$

but the last equation makes sense if $\mu_1 \neq \frac{1}{2}$. Observe that in the particular case $\mu_1 = \frac{1}{2}$, there is no singular vector of this form, because in this case the $\mathfrak{so}(3)$ -module V is the 2-dimensional module corresponding to the standard $\mathfrak{sl}(2) \simeq \mathfrak{so}(3)$ representation, and in this case \bar{w}_1 must be a linear combination of w_1 and w_2 , which is incompatible with (9.55) and (9.56). Finally observe that in the case $\mu_1 = \frac{1}{2}$, that is the weight $(\frac{3}{2}; \frac{1}{2})$ that we discarded, there is a singular vector, and it was found in Lemma 9.5, finishing the proof of Theorem 5.1. \square

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