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SOME ALGEBRAIC PROPERTIES OF DIFFERENTIAL OPERATORS

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ABSTRACT. First, we study the subskewfield of *rational* pseudodifferential operators over a differential field \mathcal{K} generated in the skewfield $\mathcal{K}((\partial^{-1}))$ of pseudodifferential operators over \mathcal{K} by the subalgebra $\mathcal{K}[\partial]$ of all differential operators. Second, we show that the Dieudonné determinant of a matrix pseudodifferential operator with coefficients in a differential subring \mathcal{A} of \mathcal{K} lies in the integral closure of \mathcal{A} in \mathcal{K} , and we give an example of a 2×2 matrix with entries in $\mathcal{A}[\partial]$ whose Dieudonné determinant does not lie in \mathcal{A} .

1. INTRODUCTION

Let \mathcal{K} be a differential field with derivation ∂ and let $\mathcal{K}[\partial]$ be the algebra of differential operators over \mathcal{K} . First, we recall the well known fact that the ring $\mathcal{K}[\partial]$ is left and right Euclidean, hence it satisfies the left and right Ore conditions. Consequently, we may consider its skewfield of fractions $\mathcal{K}(\partial)$, called the skewfield of rational pseudodifferential operators. It follows from the Ore Theorem (see e.g. [Art99]) that any rational pseudodifferential operator R can be represented as a right (resp. left) fraction AS^{-1} (resp. $S_1^{-1}A_1$), where $A, A_1, S, S_1 \in \mathcal{K}[\partial]$. We show that these fractions have a unique representations in “lowest terms”. Namely if S (resp. S_1) has minimal possible order and is monic, then any other right (resp. left) representation of R can be obtained by multiplying both A and S (resp. A_1 and S_1) on the right (resp. left) by a non-zero element of $\mathcal{K}[\partial]$ (Proposition 3.4(b)). Though this result is very simple and natural, we were not able to find it in the literature.

In early 50’s Leray [Ler53] introduced an important generalization of the characteristic matrix of a matrix pseudodifferential operator A . Using this generalization Hufford in [Huf65] developed a method to compute the Dieudonné determinant $\det_1 A$ (see Section 4 for its definition). Based on this method Sato and Kashiwara [SK75] and Miyake [Miy83] proved that $\det_1 A$ is holomorphic provided that A has holomorphic coefficients. In the present paper we extend this result and its proof from [Miy83] to the case

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when the algebra of holomorphic functions is replaced by an arbitrary differential domain \mathcal{A} (Theorem 4.9). Namely we show that if the coefficients of all entries of the matrix A lie in the domain \mathcal{A} , then its determinant $\det_1 A$ lies in the integral closure of \mathcal{A} in its differential field of fractions \mathcal{K} .

A simple example when $\det_1 A$ does not lie in \mathcal{A} itself is the following:

$$A = \begin{pmatrix} a\partial & b\partial + \beta \\ b\partial - \beta & d\partial \end{pmatrix},$$

where $\mathcal{A} = \mathbb{C}[a^{(n)}, b^{(n)}, d^{(n)} \mid n \in \mathbb{Z}_+]/\{ad - b^2\}$, $\partial x^{(n)} = x^{(n+1)}$ for $x = a, b, c$, $\beta \in \mathbb{C} \setminus \{0\}$, and $\{p\}$ denotes the differential ideal generated by $p \in \mathcal{A}$. Then $\det_1 A = \beta^2 - \beta a \left(\frac{b}{a}\right)'$ does not lie in the domain \mathcal{A} .

We derive from the proof of Theorem 4.9 that, for an integrally closed differential domain \mathcal{A} , a matrix pseudodifferential operator A is invertible in the ring $\text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$ if and only if $\det_1 A$ is an invertible element of \mathcal{A} (Theorem 4.16).

The above problems arose in the paper [DSK12] on non-local Hamiltonian structures.

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2. DIFFERENTIAL AND PSEUDODIFFERENTIAL OPERATORS OVER A DIFFERENTIAL FIELD

Let \mathcal{K} be a differential field of characteristic zero, with the derivation ∂ . For $a \in \mathcal{K}$, we denote $a' = \partial(a)$ and $a^{(n)} = \partial^n(a)$, for a non negative integer n . We denote by $\mathcal{C} \subset \mathcal{K}$ the subfield of *constants*, i.e. $\mathcal{C} = \{\alpha \in \mathcal{K} \mid \alpha' = 0\}$.

Recall that a *pseudodifferential operator* over \mathcal{K} is an expression of the form

$$(2.1) \quad A(\partial) = \sum_{n=-\infty}^N a_n \partial^n, \quad a_n \in \mathcal{K}, N \in \mathbb{Z}.$$

If $a_N \neq 0$, one says that $A(\partial)$ has *order* $\text{ord}(A) = N$, and a_N is called its *leading coefficient*. One also lets $\text{ord}(A) = -\infty$ for $A = 0$. Pseudodifferential operators form a unital associative algebra, denoted by $\mathcal{K}((\partial^{-1}))$, with product \circ defined by letting

$$(2.2) \quad \partial^n \circ a = \sum_{j \in \mathbb{Z}_+} \binom{n}{j} a^{(j)} \partial^{n-j}, \quad n \in \mathbb{Z}, a \in \mathcal{K}.$$

We will often omit \circ if no confusion may arise. Obviously, for non-zero $A, B \in \mathcal{K}((\partial^{-1}))$ we have $\text{ord}(AB) = \text{ord}(A) + \text{ord}(B)$, and the leading coefficient of AB is the product of their leading coefficients.

The algebra $\mathcal{K}((\partial^{-1}))$ is a skewfield extension of \mathcal{K} . Indeed, if $A(\partial) \in \mathcal{K}((\partial^{-1}))$ is a non-zero pseudodifferential operator of order N as in (2.1), its

inverse $A^{-1}(\partial) \in \mathcal{K}((\partial^{-1}))$ is computed as follows. We write

$$A(\partial) = a_N \left(1 + \sum_{n=-\infty}^{-1} a_N^{-1} a_{n+N} \partial^n \right) \partial^N,$$

and expanding by geometric progression, we get

$$(2.3) \quad A^{-1}(\partial) = \partial^{-N} \circ \sum_{k=0}^{\infty} \left(- \sum_{n=-\infty}^{-1} a_N^{-1} a_{n+N} \partial^n \right)^k \circ a_N^{-1},$$

which is well defined as a pseudodifferential operator in $\mathcal{K}((\partial^{-1}))$, since, by formula (2.2), the powers of ∂ are bounded above by $-N$, and the coefficient of each power of ∂ is a finite sum.

The *symbol* of the pseudodifferential operator $A(\partial)$ in (2.1) is the formal Laurent series $A(\lambda) = \sum_{n=-\infty}^N a_n \lambda^n \in \mathcal{K}((\lambda^{-1}))$, where λ is an indeterminate commuting with \mathcal{K} . We thus get a bijective map $\mathcal{K}((\partial^{-1})) \rightarrow \mathcal{K}((\lambda^{-1}))$ (which is not an algebra homomorphism). A closed formula for the associative product in $\mathcal{K}((\partial^{-1}))$ in terms of the corresponding symbols is the following:

$$(2.4) \quad (A \circ B)(\lambda) = A(\lambda + \partial)B(\lambda).$$

Here and further on, we always expand an expression as $(\lambda + \partial)^n$, $n \in \mathbb{Z}$, in non-negative powers of ∂ :

$$(2.5) \quad (\lambda + \partial)^n = \sum_{j=0}^{\infty} \binom{n}{j} \lambda^{n-j} \partial^j.$$

Therefore, the RHS of (2.4) means $\sum_{m,n=-\infty}^N \sum_{j=0}^{\infty} \binom{m}{j} a_m b_n^{(j)} \lambda^{m+n-j}$.

The algebra (over \mathcal{C}) $\mathcal{K}[\partial]$ of *differential operators* over \mathcal{K} consists of operators of the form

$$A(\partial) = \sum_{n=0}^N a_n \partial^n, \quad a_n \in \mathcal{K}, \quad N \in \mathbb{Z}_+.$$

It is a subalgebra of $\mathcal{K}((\partial^{-1}))$, and a bimodule over \mathcal{K} .

Proposition 2.1. *The algebra $\mathcal{K}[\partial]$ of differential operators over the differential field \mathcal{K} is right (respectively left) Euclidean, i.e. for every $A, B \in \mathcal{K}[\partial]$, with $A \neq 0$, there exist unique $Q, R \in \mathcal{K}[\partial]$ such that $B = AQ + R$ (resp. $B = QA + R$) and either $R = 0$ or $\text{ord}(R) < \text{ord}(A)$.*

Proof. First, we prove existence of Q and R . If $B = 0$, then we can take $Q = R = 0$. If $B \neq 0$ and $\text{ord}(B) < \text{ord}(A)$, we can take $Q = 0$ and $R = B$. For $B \neq 0$ and $\text{ord}(B) \geq \text{ord}(A)$, we proceed by induction on $\text{ord}(B)$. If $\text{ord}(B) = 0 (= \text{ord}(A))$, then A, B lie in \mathcal{K} , and we can take $Q = \frac{B}{A}$, $R = 0$. Finally, consider the case when $\text{ord}(A) = m \geq 0$ and $\text{ord}(B) = n \geq 1$, with $n \geq m$. Letting a_m be the leading coefficient of A and b_n be the leading coefficient of B , the differential operator $\tilde{B} = B - A \frac{b_n}{a_m} \partial^{n-m}$ has

order $\text{ord}(\tilde{B}) < n$. Hence, we can apply the inductive assumption to find \tilde{Q} and \tilde{R} in $\mathcal{K}[\partial]$ such that $\tilde{B} = A\tilde{Q} + \tilde{R}$ and $\text{ord}(\tilde{R}) < m$. Then, letting $Q = \frac{b_n}{a_m}\partial^{n-m} + \tilde{Q}$ and $R = \tilde{R}$, we get the desired decomposition. As for the uniqueness, if one has $AQ + R = AS + T$ with $\max(\text{deg}(R), \text{deg}(T)) < \text{deg}(A)$ then $A(Q - S) = T - R$. Comparing the orders of both sides, we conclude that $Q = S$, hence $R = T$. The proof of the left Euclidean condition is the same. \square

By the standard argument, we obtain the following:

Corollary 2.2. *Every non-zero right (resp. left) ideal \mathcal{I} of $\mathcal{K}[\partial]$ is principal: $\mathcal{I} = A\mathcal{K}[\partial]$ (resp. $\mathcal{I} = \mathcal{K}[\partial]A$), and it is generated by its element $A \in \mathcal{I}$ of minimal order (defined up to multiplication on the right (resp. left) by a non-zero element of \mathcal{K}).*

Remark 2.3. If \mathcal{A} is a differential domain, with field of fractions \mathcal{K} , then $\mathcal{K}((\partial^{-1}))$ is the skewfield of fractions of $\mathcal{A}((\partial^{-1}))$. Indeed, if $A \in \mathcal{A}((\partial^{-1}))$ is non-zero, then equation (2.3) provides its inverse in $\mathcal{K}((\partial^{-1}))$.

Remark 2.4. If \mathcal{A} is a (commutative) differential ring, and $A \in \mathcal{A}((\partial^{-1}))$ is an element whose leading coefficient is not a zero divisor, then equation (2.3) still makes sense, showing that A is invertible in $\mathcal{QA}((\partial^{-1}))$, where \mathcal{QA} is the ring of fractions of \mathcal{A} , obtained by inverting all non zero divisors of \mathcal{A} . In general, though, the leading coefficient of an invertible pseudodifferential operator $A \in \mathcal{A}((\partial^{-1}))$ is either invertible or a zero divisor. As an example of the latter case, let $\mathcal{A} = \mathbb{F} \oplus \mathbb{F}$, with ∂ acting as zero. Then $A(\partial) = (0, 1) + (1, 0)\partial$ is invertible and its inverse is $(0, 1) + (1, 0)\partial^{-1}$.

3. RATIONAL PSEUDODIFFERENTIAL OPERATORS

As before, let \mathcal{K} be a differential field with derivation ∂ .

Definition 3.1. The algebra $\mathcal{K}(\partial)$ of *rational pseudodifferential operators* over \mathcal{K} is the smallest subskewfield of $\mathcal{K}((\partial^{-1}))$ containing $\mathcal{K}[\partial]$.

We can describe the algebra $\mathcal{K}(\partial)$ of rational pseudodifferential operators more explicitly using the Ore theory (for a review of the Ore theory, in the case of an arbitrary non-commutative domain, see for example [Art99]).

Lemma 3.2. *The algebra $\mathcal{K}[\partial]$ satisfies the right (resp. left) Ore condition: for every $A, B \in \mathcal{K}[\partial]$, there exist $A_1, B_1 \in \mathcal{K}[\partial]$ not both zero such that*

$$(3.6) \quad AB_1 = BA_1 \quad \left(\text{resp. } B_1A = A_1B \right).$$

Proof. If $A = 0$, the statement is obvious since we can take $A_1 = 0$. If $A \neq 0$, we prove the claim by induction on $\text{ord}(A)$. Since $\mathcal{K}[\partial]$ is Euclidean, there exist $Q, R \in \mathcal{K}[\partial]$ such that $B = AQ + R$ and either $R = 0$ or $\text{ord}(R) < \text{ord}(A)$. If $R = 0$, then we can take $B_1 = Q$ and $A_1 = 1$ and we are done. If $R \neq 0$, then by inductive assumption there exist $A_1, R_1 \in \mathcal{K}[\partial]$, both non-zero, such that $RA_1 = AR_1$. Hence, $BA_1 = A(QA_1 + R_1)$, therefore

the claim holds with $B_1 = QA_1 + R_1$. The left Ore condition is proved in the same way. \square

Remark 3.3. By Lemma 3.2, if A, B are non-zero elements of $\mathcal{K}[\partial]$, then the right (resp. left) principal ideals generated by them have a non-zero intersection. By Corollary 2.2 the intersection is again a right (resp. left) principal ideal, generated by some element M , defined uniquely up to multiplication by a non-zero element of \mathcal{K} on the right (resp. left). This element $M \in \mathcal{K}[\partial]$ is the right (resp. left) *least common multiple* (lcm) of A and B .

Also, by Corollary 2.2 the sum of the right (resp. left) ideals generated by A and B is a right (resp. left) principal ideal, generated by some element D , defined uniquely up to multiplication by a non-zero element of \mathcal{K} on the right (resp. left). This element $D \in \mathcal{K}[\partial]$ is the right (resp. left) *greatest common divisor* (gcd) of A and B .

Proposition 3.4. (a) *The skewfield of rational pseudodifferential operators over \mathcal{K} is*

$$\mathcal{K}(\partial) = \{AS^{-1} \mid A, S \in \mathcal{K}[\partial], S \neq 0\} = \{S^{-1}A \mid A, S \in \mathcal{K}[\partial], S \neq 0\}.$$

In other words, every rational pseudodifferential operator $L \in \mathcal{K}(\partial)$ can be written as a right and a left fraction $L = AS^{-1} = S_1^{-1}A_1$ for some $A, A_1, S, S_1 \in \mathcal{A}[\partial]$ with $S, S_1 \neq 0$.

(b) *The decompositions $L = AS^{-1} = S_1^{-1}A_1$ of an element $L \in \mathcal{K}(\partial)$ are unique if we require that S and S_1 have minimal possible order and leading coefficient 1. Any other decomposition $L = \tilde{A}\tilde{S}^{-1}$ (resp. $L = \tilde{S}_1^{-1}\tilde{A}_1$), with $\tilde{A}, \tilde{A}_1, \tilde{S}\tilde{S}_1 \in \mathcal{K}[\partial]$, $\tilde{S}, \tilde{S}_1 \neq 0$, can be obtained from the minimal one by multiplying both A and S (resp. A_1 and S_1) by the same non-zero factor from $\mathcal{K}[\partial]$ on the right (resp. left).*

Proof. The proof of part (a) can be deduced from the general Ore Theorem. Since in this case we have explicit realizations of the sets of right and of left fractions of $\mathcal{A}[\partial]$ as subsets of $\mathcal{K}((\partial^{-1}))$, we can give a direct proof using the right and left Ore conditions.

In order to prove part (a) it suffices to show that the sets $\mathcal{S}_{\text{right}} = \{AS^{-1} \mid A, S \in \mathcal{K}[\partial], S \neq 0\}$ and $\mathcal{S}_{\text{left}} = \{S^{-1}A \mid A, S \in \mathcal{K}[\partial], S \neq 0\}$ are closed under addition and multiplication. Let $A, B, S, T \in \mathcal{K}[\partial]$, with $S, T \neq 0$. By the right Ore condition (3.6), there exist non-zero $S_1, T_1 \in \mathcal{F}[\partial]$ such that $ST_1 = TS_1$. Hence,

$$AS^{-1} + BT^{-1} = (AT_1 + BS_1)(S \circ T_1)^{-1},$$

proving that $\mathcal{S}_{\text{right}}$ is closed under addition. Again by the right Ore condition, there exist $S_1, B_1 \in \mathcal{A}[\partial]$, such that $S_1 \neq 0$ and $SB_1 = BS_1$. Hence,

$$AS^{-1} \circ BT^{-1} = (A \circ B_1)(T \circ S_1)^{-1},$$

proving that $\mathcal{S}_{\text{right}}$ is closed under multiplication. Similarly for the set $\mathcal{S}_{\text{left}}$ of left fractions.

For part (b), consider the set

$$\mathcal{I} = \{S \in \mathcal{K}[\partial] \setminus \{0\} \mid L = AS^{-1} \text{ for some } A \in \mathcal{K}[\partial]\} \cup \{0\}.$$

We claim that \mathcal{I} is a right ideal of $\mathcal{K}[\partial]$. First, if $L = AS^{-1}$ and $0 \neq T \in \mathcal{K}[\partial]$, then $L = AT(ST)^{-1}$, proving that $\mathcal{I}T \subset \mathcal{I}$. Moreover, if $L = A_1S_1^{-1} = A_2S_2^{-1}$, with $S_1 + S_2 \neq 0$, then $L = (A_1 + A_2)(S_1 + S_2)^{-1}$, proving that \mathcal{I} is closed under addition. By Corollary 2.2 there is a unique monic element S in \mathcal{I} of minimal order, and every other element of \mathcal{I} is obtained from S by a multiplication by a non-zero element of $\mathcal{K}[\partial]$. This proves part (b) for right fractions. The proof for left fractions is the same. \square

Remark 3.5. If \mathcal{A} is a differential domain and \mathcal{K} is its field of fractions, then we define $\mathcal{A}(\partial) = \mathcal{K}(\partial)$, and it is easy to see, clearing the denominators, that all its elements are of the form AS^{-1} (or $S^{-1}A$) for $A, S \in \mathcal{A}[\partial]$, $S \neq 0$.

Remark 3.6. If \mathcal{A} is a differential domain, we can ask whether the right (resp. left) Ore condition holds for any multiplicative subset $\mathcal{S} \subset \mathcal{A}[\partial]$: for every $A \in \mathcal{A}[\partial]$ and $S \in \mathcal{S}$, there exist $A_1 \in \mathcal{A}[\partial]$ and $S_1 \in \mathcal{S}$ such that $AS_1 = SA_1$ (resp. $S_1A = A_1S$). In fact, this is false, as the following example shows. Consider the algebra of differential polynomials in one variable, $\mathcal{A} = \mathbb{C}[u^{(n)}, n \in \mathbb{Z}_+]$, where $\partial(u^{(n)}) = u^{(n+1)}$, and let $\mathcal{S} \subset \mathcal{A}[\partial]$ be the multiplicative subset consisting of differential operators $A \in \mathcal{A}[\partial]$ with leading coefficient 1. Letting $A = u \in \mathcal{A}[\partial]$ and $S = \partial \in \mathcal{S}$, we find $A_1 = u^2$, $S_1 = u\partial + 2u' \in \mathcal{A}[\partial]$ such that $AS_1 = SA_1$, but it is not hard to prove that S_1 cannot be chosen with leading coefficient 1 (unless we allow to have the other coefficients in the field of fractions \mathcal{K}). This example provides an element $\partial^{-1} \circ u = u^2(u\partial + 2u')^{-1} \in \mathcal{A}((\partial^{-1}))$ which is a left fraction but not a right fraction (i.e. it is not of the form AS^{-1} with $A \in \mathcal{A}[\partial]$ and $S \in \mathcal{S}$).

Remark 3.7. If \mathcal{A} is a differential (commutative associative) ring possibly with zero divisors, we can define $\mathcal{A}(\partial)$ as the ring generated by $\mathcal{A}[\partial]$ and the inverses of all elements $S \in \mathcal{A}[\partial]$ which are invertible in $\mathcal{A}((\partial^{-1}))$. This contains both sets $\mathcal{S}_{\text{right}}$ and $\mathcal{S}_{\text{left}}$ of right and left fractions,

$$\begin{aligned} \mathcal{S}_{\text{right}} &= \{AS^{-1} \mid A \in \mathcal{A}[\partial], S \in \mathcal{A}[\partial] \cap \mathcal{A}((\partial^{-1}))^\times\}, \\ \mathcal{S}_{\text{left}} &= \{S^{-1}A \mid A, S \in \mathcal{A}[\partial], S \in \mathcal{A}[\partial] \cap \mathcal{A}((\partial^{-1}))^\times\}, \end{aligned}$$

but, in general, these two sets are not equal. An example when $\mathcal{S}_{\text{left}}$ and $\mathcal{S}_{\text{right}}$ are not equal was provided in Remark 3.6: $\partial^{-1} \circ u$ lies in $\mathcal{S}_{\text{left}}$ but not in $\mathcal{S}_{\text{right}}$. Note though that, when \mathcal{A} is a domain, this definition of $\mathcal{A}(\partial)$ is NOT the same as $\mathcal{K}(\partial)$ (which was the definition of $\mathcal{A}(\partial)$ given in Remark 3.5 in the case of domains).

4. MATRIX PSEUDODIFFERENTIAL OPERATORS

As in the previous sections, let \mathcal{K} be a differential field with derivation ∂ . We recall here some linear algebra over the skewfield $\mathcal{K}((\partial^{-1}))$ and,

in particular, the notion of the Dieudonné determinant (see [Art57] for an overview over an arbitrary skewfield).

Let \mathcal{D} be a subskewfield of $\mathcal{K}((\partial^{-1}))$. We are interested in the case when $\mathcal{D} = \mathcal{K}(\partial)$ or $\mathcal{K}((\partial^{-1}))$. An *elementary row operation* of the matrix pseudodifferential operator $A \in \text{Mat}_{m \times \ell} \mathcal{D}$ is either a permutation of two rows of it, or the operation $\mathcal{T}(i, j; P)$, where $1 \leq i \neq j \leq m$ and $P \in \mathcal{D}$, which replaces the j -th row by itself minus i -th row multiplied on the left by P . Using the usual Gauss elimination, we can get the analogues of standard linear algebra theorems for matrix pseudodifferential operators. In particular, any $m \times \ell$ matrix pseudodifferential operator A with entries in \mathcal{D} can be brought by elementary row operations to a row echelon form (over \mathcal{D}).

The *Dieudonné determinant* of $A \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$ has the form $\det A = c\lambda^d$, where $c \in \mathcal{K}$, λ is an indeterminate, and $d \in \mathbb{Z}$. It is defined by the following properties: $\det A$ changes sign if we permute two rows of A , and it is unchanged under any elementary row operation $\mathcal{T}(i, j; P)$ defined above, for arbitrary $i \neq j$ and a pseudodifferential operator $P \in \mathcal{K}((\partial^{-1}))$; furthermore, if A is upper triangular, with non-zero diagonal entries $A_{ii} \in \mathcal{K}((\partial^{-1}))$ of order n_i and leading coefficient $a_i \in \mathcal{K}$, then

$$\det A = (\det {}_1A)\lambda^{d(A)}, \quad \text{where } \det {}_1A = \prod_{i=1}^{\ell} a_i \quad \text{and} \quad d(A) = \sum_{i=1}^{\ell} n_i,$$

and, if one of the diagonal entries is zero, we let $\det A = 0$, $\det {}_1A = 0$ and $d(A) = -\infty$. It follows from the results in [Die43] that the Dieudonné determinant is well defined and $\det(AB) = (\det A)(\det B)$ for every $A, B \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$.

Remark 4.1. If $A \in \text{Mat}_{\ell \times \ell} \mathcal{K}[\partial]$ is a matrix differential operator, then it can be brought to an upper triangular form by elementary transformations involving only differential operators (see [DSK11, Lemma A.2.6]). Hence, if $\det A \neq 0$, then $d(A)$ is a non-negative integer.

Remark 4.2. Let $A \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$, and denote by A^* its adjoint matrix. Then $\det A = (-1)^{d(A)} \det A^*$. Indeed, if $A = ET$, where E is product of elementary matrices and T is upper triangular, then $A^* = T^*E^*$ and, clearly, $\det E^* = \det E$, while $\det T^* = (-1)^{d(A)} \det T$. However, in general $\det A$ and $\det A^T$ are not related to each other in an obvious way, as the following example shows: for $A = \begin{pmatrix} \partial & x\partial \\ 1 & x \end{pmatrix}$, we have $\det A = 1$, while $\det A^T = 0$. Note that, in this example, the matrix A is invertible in $\text{Mat}_{2 \times 2} \mathcal{K}((\partial^{-1}))$, its inverse being $A^{-1} = \begin{pmatrix} x & -x\partial + 1 \\ -1 & \partial \end{pmatrix}$, while A^T is not invertible.

Our main interest in the Dieudonné determinant is that it gives a way to characterize invertible matrix pseudodifferential operators. This is stated in the following:

- Proposition 4.3.** (a) *An element $A \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$ is invertible if and only if $\det A \neq 0$.*
(b) *Suppose that $A \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$ has $\det A \neq 0$. Then $A \in \text{Mat}_{\ell \times \ell}(\mathcal{D})$ if and only if $A^{-1} \in \text{Mat}_{\ell \times \ell}(\mathcal{D})$.*

Proof. As noted above, by performing elementary row operations, we can write $A = ET$, where E is product of elementary matrices, and T is an upper triangular matrix. Then A is invertible in the algebra $\text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$ if and only if T is invertible, and this happens if and only if all the diagonal entries $T_{11}, \dots, T_{\ell\ell}$ are non zero. On the other hand, we have $\det A = 0$ if one of the T_i 's is zero, while, otherwise, denoting by n_i the order of T_{ii} and by t_i its leading coefficient, we have $\det A = \pm t_1 \dots t_\ell \lambda^{n_1 + \dots + n_\ell} \neq 0$. This proves part (a). For part (b) it suffices to note that if A has entries in \mathcal{D} , so do E and T , and, therefore, A^{-1} . \square

Remark 4.4. Suppose \mathcal{A} is a differential domain with field of fractions \mathcal{K} . Let $A \in \text{Mat}_{\ell \times \ell} \mathcal{A}[\partial]$. By Proposition 4.3, if $\det A \neq 0$, then A^{-1} exists and it has entries in $\text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$. In fact, it is clear from the proof that its entries lie in $\tilde{\mathcal{A}}((\partial^{-1}))$, where $\tilde{\mathcal{A}}$ is an extension of \mathcal{A} obtained by adding inverses of finitely many non-zero elements.

Definition 4.5. Let $A = (A_{ij})_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$. The *total order* of A is defined as

$$\text{tord}(A) = \max_{\sigma \in S_\ell} \left(\sum_{i=1}^\ell \text{ord}(A_{i, \sigma(i)}) \right),$$

where S_ℓ denotes the group of permutations of $\{1, \dots, \ell\}$. Assuming $\det A \neq 0$, we define the *degeneracy degree* of A as

$$\text{dd}(A) = \text{tord}(A) - \text{d}(A),$$

(by Theorem 4.7(i) below, $\text{dd}(A)$ is a non-negative integer), and we say that A is *strongly non-degenerate* if $\text{dd}(A) = 0$.

Definition 4.6. Let $A = (A_{ij})_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$. A system of integers $(N_1, \dots, N_\ell, h_1, \dots, h_\ell)$ is called a *majorant* of A if

$$\text{ord}(A_{ij}) \leq N_j - h_i \quad \text{for every } i, j = 1, \dots, \ell.$$

The *characteristic matrix* $\bar{A}(\lambda) = (\bar{A}_{ij}(\lambda))_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell} \mathcal{K}[\lambda^{\pm 1}]$, associated to the majorant $\{N_i, h_i\}_{i=1}^\ell$, is defined by

$$\bar{A}_{ij}(\lambda) = a_{ij; N_j - h_i} \lambda^{N_j - h_i},$$

where $a_{ij; N_j - h_i}$ is the coefficient of $\partial^{N_j - h_i}$ in A_{ij} . Clearly,

$$(4.7) \quad \sum_{i=1}^n (N_i - h_i) \geq \text{tord}(A)$$

for every majorant $\{N_i, h_i\}_{i=1}^\ell$ of A . A majorant is called *optimal* if

$$\sum_{i=1}^n (N_i - h_i) = \text{tord}(A).$$

The following theorem follows from the results in [Huf65].

Theorem 4.7. *Let $A = (A_{ij})_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$ with $\det A \neq 0$. We have:*

- (i) $\text{dd}(A) \geq 0$;
- (ii) if $\det A \neq 0$, then there exists an optimal majorant of A ;
- (iii) if $\text{dd}(A) \geq 1$, then $\det(\bar{A}(\lambda)) = 0$ for any majorant;
- (iv) if $\text{dd}(A) = 0$, then $\det(\bar{A}(\lambda)) = 0$ for any majorant which is not optimal, and $\det(\bar{A}(\lambda)) = \det A$ for any majorant which is optimal.

In the special case when A is an $\ell \times \ell$ matrix pseudodifferential operator of order N with invertible leading coefficient $A_N \in \text{Mat}_{\ell \times \ell} \mathcal{K}$, we can take the (optimal) majorant $N_i = N, h_i = 0, i = 1, \dots, \ell$. The corresponding characteristic matrix is $\bar{A}(\lambda) = A_N \lambda^N$. We thus obtain the following

Corollary 4.8. *If A is an $\ell \times \ell$ matrix pseudodifferential operator of order N with invertible leading coefficient $A_N \in \text{Mat}_{\ell \times \ell} \mathcal{K}$, then $\det A = (\det A_N) \lambda^{N\ell}$.*

The proof of the following result is similar to that in [SK75] and [Miy83] in the case when \mathcal{A} is the algebra of holomorphic functions in a domain of the complex plain.

Theorem 4.9. *Let \mathcal{A} be a unital differential subring of the differential field \mathcal{K} , and let $\bar{\mathcal{A}}$ be its integral closure in \mathcal{K} . Then, for any $A \in \text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$ we have $\det_1 A \in \bar{\mathcal{A}}$.*

Let \mathcal{B} be a valuation ring in \mathcal{K} containing \mathcal{A} . By Proposition A.1, it suffices to prove that $\det_1 A \in \mathcal{B}$. This follows from the following two lemmas.

Lemma 4.10. *Suppose that $A \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$ has $\det A \neq 0$ and degeneracy degree $\text{dd}(A) \geq 1$. Then there exists a matrix $P \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$ such that $\det_1 P = 1$ and $\text{dd}(PA) \leq \text{dd}(A) - 1$.*

Proof. By Theorem 4.7(ii), there exists an optimal majorant $\{N_i, h_i\}_{i=1}^\ell$ for A . Since, by assumption, $\text{dd}(A) \geq 1$, by Theorem 4.7(iii), the characteristic matrix $\bar{A}(\lambda)$ associated to this optimal majorant is degenerate (and so is $\bar{A}(1) \in \text{Mat}_{\ell \times \ell} \mathcal{K}$). Let $(f_1, \dots, f_\ell) \in \mathcal{K}^\ell$ be a left eigenvector of $\bar{A}(1)$ with eigenvalue 0. Since \mathcal{B} is a valuation ring, condition (A_ℓ) in Proposition A.2 of the Appendix holds. Hence, after dividing all the entries f_j by a non-zero entry f_i , we may assume that $f_j \in \mathcal{B}$ for all $j = 1, \dots, \ell$ and $f_i = 1$ for some

i. Consider the following matrix

$$(4.8) \quad P = \begin{pmatrix} \partial^{h_1-h_\ell} & 0 & \dots & 0 & 0 \\ 0 & \partial^{h_2-h_\ell} & \dots & 0 & 0 \\ & \vdots & & \vdots & \\ f_1 \partial^{h_1-h_\ell} & f_2 \partial^{h_2-h_\ell} & \dots & f_{\ell-1} \partial^{h_{\ell-1}-h_\ell} & f_\ell \\ & \vdots & & \vdots & \\ 0 & 0 & \dots & \partial^{h_{\ell-1}-h_\ell} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \text{ row } i$$

Note that $P \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$, it is strongly non-degenerate (since the characteristic matrix associated to the majorant $\{h_j - h_\ell, 0\}_{j=1}^\ell$ is non-degenerate) and its Dieudonné determinant is

$$\det P = \lambda^{\sum_j (h_j - h_\ell)},$$

i.e. $\det_1 P = 1$ and $d(P) = \sum_{j=1}^\ell (h_j - h_\ell)$. We claim that the following is a majorant for the matrix PA :

$$(4.9) \quad \{N_j, h_\ell + \delta_{j,i}\}_{j=1}^\ell.$$

Indeed, for $j \neq i$ we have

$$\text{ord}(PA)_{jk} = \text{ord}(\partial^{h_j-h_\ell} A_{jk}) = h_j - h_\ell + \text{ord}(A_{jk}) \leq N_k - h_\ell.$$

For the entries in the i -th row of the matrix PA we have

$$(PA)_{ik} = \sum_{j=1}^\ell f_j \partial^{h_j-h_\ell} A_{jk},$$

which has order less than or equal to $\max_j \{h_j - h_\ell + \text{ord}(A_{jk})\} \leq N_k - h_\ell$, and the coefficient of $\partial^{N_k-h_\ell}$ in $(PA)_{ik}$ is

$$\sum_{j=1}^\ell f_j a_{jk; N_k-h_\ell} = \sum_{j=1}^\ell f_j \bar{A}(1)_{jk} = 0.$$

Hence,

$$\text{ord}(PA)_{ik} \leq N_k - (h_\ell + 1),$$

as we wanted. By (4.7) (applied to the majorant (4.9) of PA), we have

$$\begin{aligned} \text{tord}(PA) &\leq \sum_{j=1}^\ell (N_j - h_\ell - \delta_{j,i}) \\ &= \sum_{j=1}^\ell (N_j - h_j) + \sum_{j=1}^\ell (h_j - h_\ell) - 1 = \text{tord}(A) + d(P) - 1. \end{aligned}$$

The claim follows. \square

Lemma 4.11. *If $A \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$ has non-zero Dieudonné determinant, then there exists a matrix $P \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$ such that $\det_1 P = 1$ and PA is strongly non-degenerate.*

Proof. Recall that, by Theorem 4.7(i), $\text{dd}(A) \geq 0$. The claim follows from Lemma 4.10 by induction on $\text{dd}(A)$. \square

Proof of Theorem 4.9. By Lemma 4.11 there exists $P \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$ such that $\det_1 P = 1$ and PA is strongly non-degenerate. Therefore, if we fix an optimal majorant for the matrix PA and let $\overline{PA}(\lambda)$ be the corresponding characteristic matrix, we have $\det(PA) = \det(\overline{PA}(\lambda))$. On the one hand, we have that $\det_1(PA) = (\det_1 P)(\det_1 A) = \det_1 A$. On the other hand, $\overline{PA}(\lambda)$ has entries in $\mathcal{B}((\lambda^{-1}))$ (since both P and A have entries in $\mathcal{B}((\partial^{-1}))$), therefore its determinant lies in $\mathcal{B}((\lambda^{-1}))$, implying that $\det_1 A \in \mathcal{B}$. \square

Example 4.12. Consider an arbitrary 2×2 matrix differential operator of order 1 over a differential domain \mathcal{A} :

$$A = \begin{pmatrix} a\partial + \alpha & b\partial + \beta \\ c\partial + \gamma & d\partial + \delta \end{pmatrix},$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathcal{A}$. We may assume, without loss of generality, that $a \neq 0$. We denote,

$$\Delta_\lambda = \begin{vmatrix} a\lambda + \alpha & b\lambda + \beta \\ c\lambda + \gamma & d\lambda + \delta \end{vmatrix}.$$

It can be expanded as $\Delta_\lambda = \Delta_\infty \lambda^2 + \Delta'_0 \lambda + \Delta_0$, where

$$\Delta_\infty = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad \Delta'_0 = \begin{vmatrix} a & \beta \\ c & \delta \end{vmatrix} + \begin{vmatrix} \alpha & b \\ \gamma & d \end{vmatrix}, \quad \Delta_0 = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}.$$

There are the following three possibilities for $\det A$:

1. If $\Delta_\infty = ad - bc \neq 0$, the matrix A is strongly non-degenerate of total order 2. Its Dieudonné determinant is $\det A = \Delta_\infty \lambda^2$.
2. If $\Delta_\infty = 0$ and $\Delta'_0 \neq 0$, we have $\det A = \Delta'_0 \lambda$. In this case, the matrix A has total order 2 if $ad = bc \neq 0$, while it is strongly non-degenerate (of total order 1) if $ad = bc = 0$.
3. Finally, if $\Delta_\infty = \Delta'_0 = 0$, we have

$$\det A = \det_1 A = \Delta_0 - (\alpha c - \gamma a) \left(\frac{b}{a}\right)'.$$

For the total order of A there are several possibilities: if $ad = bc \neq 0$ then $\text{tord}(A) = 2$; if $ad = bc = 0$ and $(\alpha d, \delta a, \beta c, \gamma b) \neq (0, 0, 0, 0)$ then $\text{tord}(A) = 1$; finally if $ad = bc = \alpha d = \delta a = \beta c = \gamma b = 0$, and $(\alpha \delta, \beta \gamma) \neq (0, 0)$, then the matrix A is strongly non-degenerate of total order 0, unless $\det A = 0$. In general $\det A$ does not lie in the domain \mathcal{A} (see Example 4.13 below). On the other hand, from Theorem 4.9 we know that $\det A$ is solution of a monic polynomial equation with coefficients in \mathcal{A} . In fact, in this example we have

$$\begin{aligned} \det A + \det A^T &= 2\Delta_0 - (\alpha d' + \delta a' - \beta c' - \gamma b'), \\ (\det A)(\det A^T) &= \Delta_0^2 - \Delta_0(\alpha d' + \delta a' - \beta c' - \gamma b') \\ &\quad + (\beta \gamma - \alpha \delta)(b'c' - a'd') + \alpha \delta a'd' \in \mathcal{A}. \end{aligned}$$

Hence, $\det A$ is a root of the following quadratic polynomial with coefficients in \mathcal{A} :

$$x^2 - (\det A + \det A^T)x + (\det A)(\det A^T).$$

Example 4.13. In Example 4.12, if $\text{dd}(A) = \text{tord}(A) - \text{d}(A)$ is equal to 0 or 1, then $\det_1 A$ lies in the domain \mathcal{A} . On the other hand, if $\text{dd}(A) = 2$ (i.e. $\text{tord}(A) = 2$ and $\text{d}(A) = 0$), this is not necessarily the case. To see this, consider the algebra $\bar{\mathcal{A}} = \mathbb{C}[a, b, d]/(ad - b^2)$, and let $\mathcal{A} = \mathbb{C}[a^{(n)}, b^{(n)}, d^{(n)} \mid n \in \mathbb{Z}_+]/\{ad - b^2\}$, where $\{p\}$ denotes the differential ideal generated by p . Note that $\bar{\mathcal{A}}$ is a domain, hence, by Kolchin's Theorem [Kol73, Prop.IV.10], \mathcal{A} is a domain too. Consider the matrix A as in Example 4.12, with $c = b$ and $\alpha = \delta = 0, \beta = -\gamma \in \mathbb{C} \setminus \{0\}$. In this case we have $\Delta_\lambda = \Delta_0 = \beta^2 \in \mathbb{C}$, and $\det A = \beta^2 - \beta a \left(\frac{b}{a}\right)'$. Note that in this example the determinant of the transposed matrix is not the same as the determinant of A , in fact we have $\det(A^T) = \beta^2 + \beta a \left(\frac{b}{a}\right)'$. We now show that, with these choices, $\det A$ does not lie in the ring \mathcal{A} . Suppose, by contradiction, that $\det A \in \mathcal{A}$, hence $\frac{ba'}{a} \in \mathcal{A}$. Introduce the \mathbb{Z}_+ grading of \mathcal{A} by letting $\deg(a^{(n)}) = \deg(b^{(n)}) = \deg(d^{(n)}) = 1$. Then $\frac{ba'}{a}$ has degree 1, which means that we have an equality of the form

$$\frac{ba'}{a} = \sum_{n=0}^N (\alpha_n a^{(n)} + \beta_n b^{(n)} + \delta_n d^{(n)}),$$

with $N \in \mathbb{Z}_+$ and $\alpha_n, \beta_n, \delta_n \in \mathbb{C}$. We have an injective homomorphism of \mathbb{Z} graded algebras $\mathcal{A} \rightarrow \mathbb{C}[a^{-1}, a^{(n)}, b^{(n)} \mid n \in \mathbb{Z}_+]$ obtained by mapping $d^{(n)} \mapsto \left(\frac{b^2}{a}\right)^{(n)}$. By assumption, $\frac{ba'}{a} \in \mathbb{C}[a^{-1}, a^{(n)}, b^{(n)} \mid n \in \mathbb{Z}_+]$ is in the image of this map, and the above equality translates to the following equality in the ring $\mathbb{C}[a^{-1}, a^{(n)}, b^{(n)} \mid n \in \mathbb{Z}_+]$:

$$\frac{ba'}{a} = \sum_{n=0}^N \left(\alpha_n a^{(n)} + \beta_n b^{(n)} + \delta_n \left(\frac{b^2}{a}\right)^{(n)} \right).$$

Consider the quotient map $\mathbb{C}[a^{-1}, a^{(n)}, b^{(n)} \mid n \in \mathbb{Z}_+] \rightarrow \mathbb{C}[a^{-1}, a^{(n)}, b \mid n \in \mathbb{Z}_+]$, obtained by letting $b' = b'' = b^{(3)} = \dots = 0$. The above equation translates to the following identity in the ring $\mathbb{C}[a^{-1}, a^{(n)}, b \mid n \in \mathbb{Z}_+]$:

$$\frac{ba'}{a} = \sum_{n=0}^N \left(\alpha_n a^{(n)} + \beta_0 b + b^2 \delta_n \left(\frac{1}{a}\right)^{(n)} \right),$$

which implies, looking at the coefficients of b in both sides, that $\frac{a'}{a} = \beta_0 \in \mathbb{C}$ in the ring $\mathbb{C}[a^{-1}, a^{(n)} \mid n \in \mathbb{Z}_+]$, a contradiction.

Conjecture 4.14. If $\text{dd}(A) = 1$, then $\det_1 A$ lies in \mathcal{A} (and we know that if $\text{dd}(A) = 0$ then $\det_1 A \in \mathcal{A}$).

Remark 4.15. The proof of Lemma 4.10 provides an algorithm to compute $\det A$, inductively on the degeneracy degree $\text{dd}(A) = \text{tord}(A) - \text{d}(A)$.

Let $\{N_j, h_j\}_{j=1}^\ell$ be an optimal majorant for A . If $\text{dd}(A) = 0$ (i.e. A is strongly non-degenerate), then the characteristic matrix $\bar{A}(\lambda)$ associated to this majorant is non-degenerate, and in this case $\det A = \det \bar{A}(\lambda) = \det \bar{A}(1)\lambda^{\sum_j(N_j-h_j)}$. If $\bar{A}(\lambda)$ is not strongly non-degenerate, we take a left null vector (f_1, \dots, f_n) of the matrix $\bar{A}(1)$, normalized in such a way that $f_i = 1$ for some i . Then we multiply A on the left by the matrix P as in (4.8). The resulting matrix PA has determinant equal to $(\det A)\lambda^{\sum_j(h_j-h_\ell)}$, and its degeneracy degree is at most $\text{dd}(A) - 1$.

The algorithm just described is based on an optimal majorant $\{N_j, h_j\}_{j=1}^\ell$. We now give a constructive way of finding one. To do so we consider the matrix of orders: $M = (m_{ij})_{i,j=1}^\ell$, where $m_{ij} = \text{ord}(A_{ij})$. Note that a majorant of A only depends on M , and we define the notion of total order and majorant of M in the obvious way. For a permutation $\sigma \in S_\ell$, we let

$$m(\sigma) = \sum_{i=1}^\ell m_{i,\sigma(i)}.$$

Hence, by definition, $\text{tord}(M) = \max_{\sigma \in S_\ell} m(\sigma)$. We then define the following subset of $\{1, \dots, \ell\}^2$:

$$\Gamma(M) = \left\{ (i, j) \mid \sigma(i) = j \text{ for some } \sigma \in S_\ell \text{ such that } m(\sigma) = \text{tord}(M) \right\}.$$

If $\Gamma(M) = \{1, \dots, \ell\}^2$ then $\{N_j = m_{1j}, h_j = m_{11} - m_{j1}\}_{j=1}^\ell$ is an optimal majorant of M [Huf65, Cor.III.2]. For $i, j = 1, \dots, \ell$, we let

$$d_{ij}(M) = \text{tord}(M) - \max_{\sigma \in S_\ell \mid \sigma(i)=j} m(\sigma).$$

Note that, by definition, $\Gamma(M) = \{(i, j) \mid d_{ij}(M) = 0\}$. If $d_{i_1 j_1}(M) > 0$ for some (i_1, j_1) , define M_1 as the matrix obtained from M by replacing $m_{i_1 j_1}$ with $(m_{i_1 j_1} + d_{i_1 j_1}(M))$. It is not hard to prove that $\text{tord}(M_1) = \text{tord}(M)$, and $\sum_{i,j} d_{ij}(M_1) < \sum_{i,j} d_{ij}(M)$. By repeating this procedure n times, we get a new matrix M_n , with the same total order as M , whose entries are greater than or equal to the corresponding entries of M , and such that $\sum_{i,j} d_{ij}(M_n) = 0$ (i.e. $\Gamma(M_n) = \{1, \dots, \ell\}^2$). Hence, by the above result, we can find an optimal majorant of M_n , which is also an optimal majorant for M since $m_{ij} \leq (M_n)_{ij}$ for every $i, j = 1, \dots, \ell$, and $\text{tord}(M) = \text{tord}(M_n)$.

Theorem 4.16. *Let \mathcal{A} be a unital differential subring of the differential field \mathcal{K} , and assume that \mathcal{A} is integrally closed. Let $A \in \text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$ be a matrix with $\det A \neq 0$. Then A is invertible in $\text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$ if and only if $\det_1 A$ is invertible in \mathcal{A} .*

Lemma 4.17. *Let \mathcal{A} be an arbitrary unital differential subring of the differential field \mathcal{K} , and assume that the matrix $A \in \text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$ is strongly non-degenerate. Then A is invertible in $\text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$ if and only if $\det_1 A$ is invertible in \mathcal{A} .*

Proof. Let $\{N_i, h_i\}_{i=1}^\ell$ be an optimal majorant for the matrix A , and consider the new matrix

$$\tilde{A} = \begin{pmatrix} \partial^{h_1} & & 0 \\ & \ddots & \\ 0 & & \partial^{h_\ell} \end{pmatrix} A \begin{pmatrix} \partial^{-N_1} & & 0 \\ & \ddots & \\ 0 & & \partial^{-N_\ell} \end{pmatrix} \in \text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1})).$$

Since, by assumption, A is strongly non-degenerate, the matrix $\bar{A}(1) = (A_{ij; N_j - h_i})_{i,j=1}^\ell$ is non-degenerate. But this matrix is the leading coefficient of the matrix \tilde{A} . To conclude, we observe that, $\det_1 A = \det \bar{A}(1)$ is invertible in \mathcal{A} , if and only if the matrix $\bar{A}(1)$ is invertible in $\text{Mat}_{\ell \times \ell} \mathcal{A}$, which, by formula (2.3) (which works also in the matrix case), happens if and only if the matrix \tilde{A} is invertible in $\text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$, which, obviously, is the same as saying that the matrix A is invertible in $\text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$. \square

Proof of Theorem 4.16. Since \mathcal{A} is integrally closed, by Theorem 4.9 we have $\det_1 A \in \mathcal{A}$. Moreover, if the matrix A is invertible in $\text{Mat}_{\ell \times \ell} \mathcal{A}((\partial^{-1}))$, then also $\det_1 A^{-1} \in \mathcal{A}$, therefore $\det_1 A$ is an invertible element of \mathcal{A} . Conversely, assume that $\det_1 A$ is invertible in \mathcal{A} . Let $\mathcal{B} \subset \mathcal{K}$ be any valuation subring of \mathcal{K} containing \mathcal{A} , and let A^{-1} be the inverse of A in $\text{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$. By Lemma 4.11 there exists a matrix $P \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$ such that $\det_1 P = 1$ and PA is strongly non-degenerate. By assumption $\det_1(PA) = \det_1 A$ is invertible in \mathcal{A} (hence in \mathcal{B}), therefore, by Lemma 4.17, the matrix PA is invertible in $\text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$. On the other hand, P is product of matrices of the form (4.8), and each such factor is obviously invertible in $\text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$. Hence, $A^{-1} = (PA)^{-1}P \in \text{Mat}_{\ell \times \ell} \mathcal{B}((\partial^{-1}))$. Since this holds for every valuation ring $\mathcal{B} \subset \mathcal{K}$ containing \mathcal{A} , we obtain the claim. \square

APPENDIX A. VALUATION RINGS

Recall that a unital subring \mathcal{A} of a field \mathcal{K} is called a *valuation ring* if for any two non-zero elements $a, b \in \mathcal{A}$ either $\frac{a}{b} \in \mathcal{A}$ or $\frac{b}{a} \in \mathcal{A}$. Recall also that $\mathcal{A} \subset \mathcal{K}$ is called *integrally closed* in \mathcal{K} if the solutions (in \mathcal{K}) of every monic polynomial equation with coefficients in \mathcal{A} lie in \mathcal{A} . The *integral closure* of $\mathcal{A} \subset \mathcal{K}$ is the minimal subring of \mathcal{K} containing \mathcal{A} which is integrally closed. The following fact is well known (see e.g. [AMC69, Cor.5.22]):

Proposition A.1. *The integral closure of \mathcal{A} in \mathcal{K} is the intersection of all valuation rings of \mathcal{K} which contain \mathcal{A} .*

The above proposition, and the following result, are used in the proof of Theorem 4.9.

Proposition A.2. *Let $\ell \geq 2$ be a fixed integer. A unital subring \mathcal{A} of the field \mathcal{K} is a valuation ring if and only if the following condition holds:*

(A $_\ell$) *for every ℓ -tuple $(a_1, \dots, a_\ell) \in \mathcal{K}^\ell \setminus \{(0, \dots, 0)\}$ there exists i such that $a_i \neq 0$ and $\frac{a_1}{a_i}, \dots, \frac{a_\ell}{a_i} \in \mathcal{A}$.*

Proof. First, note that condition (A_2) is the same as the definition of valuation ring. Clearly, for $\ell \geq 3$, condition (A_ℓ) implies condition $(A_{\ell-1})$, by letting $a_\ell = 0$. It remains to prove that condition (A_2) implies condition (A_ℓ) for every ℓ .

Let $(a_1, \dots, a_\ell) \in \mathcal{K}^\ell \setminus \{(0, \dots, 0)\}$. If one of the entries is zero, the claim holds by assumption. Hence, we may assume that a_1, \dots, a_ℓ are all non-zero. Introduce a total order on the set $\{1, \dots, \ell\}$ by letting $j \leq i$ if $\frac{a_j}{a_i}$ lies in \mathcal{A} , and $j = i$ if both $\frac{a_j}{a_i}$ and $\frac{a_i}{a_j}$ lie in \mathcal{A} . The transitivity property of \leq follows from the fact that \mathcal{A} is a ring, and \leq is a total order thanks to the assumption (A_2) . Then, letting i be a maximal element, we get the desired result. \square

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