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SPECIAL REPRESENTATIONS OF WEYL GROUPS: A POSITIVITY PROPERTY

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Introduction

Let W be an irreducible Weyl group with length function $l : W \to \mathbf{N}$ and let $S = \{s \in W; l(s) = 1\}$. Let $\mathrm{Irr}W$ be a set of representatives for the isomorphism classes of irreducible representations of W (over $\mathbf{C}$). In [L1] a certain subset of $\mathrm{Irr}W$ was defined. The representations in this subset were later called *special representations*; they play a key role in the classification of unipotent representations of a reductive group over a finite field $\mathbf{F}_q$ for which W is the Weyl group. (The definition of special representations is reviewed in 3.1.)

It will be convenient to replace irreducible representations of W with the corresponding simple modules of the asymptotic Hecke algebra $\mathbf{J}$ (see [L6, 18.3]) associated to W via the canonical isomorphism $\psi : \mathbf{C}[W] \xrightarrow{\sim} \mathbf{J}$ (see 3.1); let E_∞ be the simple $\mathbf{J}$-module corresponding to $E \in \mathrm{Irr}W$ under ψ.

In this paper we show that a special representation E of W is characterized by the following positivity property of E_∞: there exists a $\mathbf{C}$-basis of E_∞ such that any element t_u in the standard basis of $\mathbf{J}$ acts in this basis through a matrix with all entries in $\mathbf{R}_{\geq 0}$.

The fact that for a special representation E, E_∞ has the positivity property above was pointed out (in the case where W is of classical type) in [L9]. In this paper I will recall the argument of [L9] (see 3.3) and I give two other proofs which apply for any W. One of these proofs (see 4.4) is based on the interpretation [L3], [BFO], of $\mathbf{J}$ (or its part attached to a fixed two-sided cell) in terms of G-equivariant vector bundles on $X \times X$ where X is a finite set with an action of a finite group G. Another proof (see Section 2) is based on the use of Perron's theorem for matrices with all entries in $\mathbf{R}_{>0}$. (Previously, Perron's theorem has been in used in the context of canonical bases in quantum groups in the study [L5] of total positivity and, very recently, in the context of the canonical basis [KL1] of $\mathbf{C}[W]$, in [KM]; in both cases the positivity properties of the appropriate canonical bases were used). We also show that the Hecke algebra representation corresponding to a special representation E can be realized essentially by a W-graph (in the sense

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of [KL1]) in which all labels are natural numbers. Some of our results admit also an extension to the case of affine Weyl groups (see Section 5).

1. Statement of the main theorem

1.1. Let v be an indeterminate and let $\mathcal{A}=\mathbf{Z}[v,v^{-1}]$. Let $\mathcal{H}$ be the Hecke algebra of W that is, the associative $\mathcal{A}$-algebra with 1 with an $\mathcal{A}$-basis $\{T_w; w\in W\}$ (where $T_1=1$) and with multiplication such that $T_wT_{w'}=T_{ww'}$ if $l(ww')=l(w)+l(w')$ and $(T_s+1)(T_s-v^2)=0$ if $s\in S$. Let $\{c_w; w\in W\}$ be the $\mathcal{A}$-basis of $\mathcal{H}$ denoted by $\{C'_w; w\in W\}$ in [KL1] (with $q=v^2$); see also [L6, 5.2]. For example, if $s\in S$, we have $c_s=v^{-1}T_s+v^{-1}$. The left cells and two-sided cells of W are the equivalence classes for the relations $\sim_L$ and $\sim_{LR}$ on W defined in [KL1], see also [L6, 8.1]; we shall write $\sim$ instead of $\sim_L$. For x,y in W we have $c_xc_y=\sum_{z\in W}h_{x,y,z}c_z$ where $h_{x,y,z}\in\mathbf{N}[v,v^{-1}]$. As in [L6, 13.6], for $z\in W$ we define $a(z)\in\mathbf{N}$ by $h_{x,y,z}\in v^{a(z)}\mathbf{Z}[v^{-1}]$ for all x,y in W and $h_{x,y,z}\notin v^{a(z)-1}\mathbf{Z}[v^{-1}]$ for some x,y in W. (For example, $a(1)=0$ and $a(s)=1$ if $s\in S$.) For x,y,z in W we have $h_{x,y,z}=\gamma_{x,y,z^{-1}}v^{a(z)} \mod v^{a(z)-1}\mathbf{Z}[v^{-1}]$ where $\gamma_{x,y,z^{-1}}\in\mathbf{N}$ is well defined. Let $\mathbf{J}$ be the $\mathbf{C}$-vector space with basis $\{t_w; w\in W\}$. For x,y in W we set $t_xt_y=\sum_{z\in W}\gamma_{x,y,z^{-1}}t_z\in\mathbf{J}$. This defines a structure of associative $\mathbf{C}$-algebra on $\mathbf{J}$ with unit element of the form $\sum_{d\in\mathcal{D}}t_d$ where $\mathcal{D}$ is a certain subset of the set of involutions in W, see [L6, 18.3]. For any subset X of W let $\mathbf{J}_X$ be the subspace of $\mathbf{J}$ with basis $\{t_w; w\in X\}$; let $\mathbf{J}_X^+$ be the set of elements of the form $\sum_{w\in X}f_wt_w\in\mathbf{J}_X$ with $f_w\in\mathbf{R}_{>0}$ for all $w\in X$. We have $\mathbf{J}=\oplus_{\mathbf{c}}\mathbf{J}_{\mathbf{c}}$ where $\mathbf{c}$ runs over the two-sided cells of W. Each $\mathbf{J}_{\mathbf{c}}$ is a subalgebra of $\mathbf{J}$ with unit element $\sum_{d\in\mathcal{D}_{\mathbf{c}}}t_d$ where $\mathcal{D}_{\mathbf{c}}=\mathbf{c}\cap\mathcal{D}$; moreover, $\mathbf{J}_{\mathbf{c}}\mathbf{J}_{\mathbf{c}'}=0$ if $\mathbf{c}\ne\mathbf{c}'$.

Until the end of Section 4 we fix a two-sided cell $\mathbf{c}$.

Let L be the set of left cells that are contained in $\mathbf{c}$. We have

$$\mathbf{c}=\sqcup_{\Gamma\in L}\Gamma=\sqcup_{\Gamma,\Gamma' \text{ in } L}(\Gamma\cap\Gamma'^{-1});$$

moreover, $\Gamma\cap\Gamma'^{-1}\ne\emptyset$ for any Γ,Γ' in L. It follows that

$$\mathbf{J}_{\mathbf{c}}=\oplus_{\Gamma\in L}\mathbf{J}_\Gamma=\oplus_{\Gamma,\Gamma' \text{ in } L}\mathbf{J}_{\Gamma\cap\Gamma'^{-1}};$$

moreover, $\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}\ne 0$. Note that for $\Gamma\in L$, $\mathbf{J}_\Gamma$ is a left ideal of $\mathbf{J}_{\mathbf{c}}$.

A line $\mathcal{L}$ in $\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$ is said to be *positive* if $\mathcal{L}^+:=\mathcal{L}\cap\mathbf{J}^+_{\Gamma\cap\Gamma'^{-1}}\ne\emptyset$; in this case, $\mathcal{L}^+$ consists of all $\mathbf{R}_{>0}$-multiples of a single nonzero vector. We now state our main result.

Theorem 1.2. *(a) Let $\Gamma\in L$. There is a unique left ideal M_Γ of $\mathbf{J}_{\mathbf{c}}$ such that property (♡) below holds:*

(♡) $M_\Gamma=\oplus_{\Gamma'\in L}M_{\Gamma,\Gamma'}$ where for any $\Gamma'\in L$, $M_{\Gamma,\Gamma'}:=M_\Gamma\cap\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$ is a positive line.

(b) Let $\Gamma\in L,\Gamma'\in L,u\in\mathbf{c}$. We have $u\in\tilde\Gamma\cap\tilde\Gamma'^{-1}$ for well-defined $\tilde\Gamma,\tilde\Gamma'$ in L. If $\tilde\Gamma\ne\Gamma'$, then $t_uM_{\Gamma,\Gamma'}=0$. If $\tilde\Gamma=\Gamma'$, then $t_uM_{\Gamma,\Gamma'}=M_{\Gamma,\tilde\Gamma'}$ and $t_uM^+_{\Gamma,\Gamma'}=M^+_{\Gamma,\tilde\Gamma'}$.

(c) The $\mathbf{J_c}$-module M_Γ in (a) is simple. Its isomorphism class is independent of $\Gamma \in L$.

(d) The subspace $\mathbf{I} = \oplus_{\Gamma \in L} M_\Gamma$ of $\mathbf{J_c}$ is a simple two-sided ideal of $\mathbf{J_c}$.

(e) Let $\Gamma, \Gamma', \tilde{\Gamma}, \tilde{\Gamma}'$ be in L. If $\Gamma \ne \tilde{\Gamma}'$ then $M_{\Gamma,\Gamma'} M_{\tilde{\Gamma},\tilde{\Gamma}'} = 0$. If $\Gamma = \tilde{\Gamma}'$, then multiplication in $\mathbf{J_c}$ defines an isomorphism $M_{\Gamma,\Gamma'} \otimes M_{\tilde{\Gamma},\Gamma} \xrightarrow{\sim} M_{\tilde{\Gamma},\Gamma'}$ and a surjective map $M^+_{\Gamma,\Gamma'} \times M^+_{\tilde{\Gamma},\Gamma} \to M^+_{\tilde{\Gamma},\Gamma'}$.

(f) Let Γ, Γ' be in L. The antiautomorphism $\theta : \mathbf{J_c} \to \mathbf{J_c}$ given by $t_x \mapsto t_{x^{-1}}$ for all $x \in \mathbf{c}$ maps $M_{\Gamma,\Gamma'}$ onto $M_{\Gamma',\Gamma}$ and $M^+_{\Gamma,\Gamma'}$ onto $M^+_{\Gamma',\Gamma}$.

The proof is given in Section 2.

1.3. As a consequence of Theorem 1.2, the simple $\mathbf{J_c}$-module M_Γ admits a $\mathbf{C}$-basis $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$ with the following property:

(i) *If $u \in \mathbf{c}$ and $\Gamma' \in L$, then $t_u \tilde{e}_{\Gamma'}$ is an $\mathbf{R}_{\ge 0}$-linear combination of elements $\tilde{e}_{\Gamma''}$ with $\Gamma'' \in L$; more precisely, if $u \in \tilde{\Gamma} \cap \tilde{\Gamma}'^{-1}$ with $\tilde{\Gamma}, \tilde{\Gamma}'$ in L, then*

$t_u \tilde{e}_{\Gamma'} = \lambda_{u,\Gamma',\tilde{\Gamma}'} \tilde{e}_{\tilde{\Gamma}'}$

with $\lambda_{u,\Gamma',\tilde{\Gamma}'} \in \mathbf{R}_{>0}$ if $\tilde{\Gamma} = \Gamma'$ and $\lambda_{u,\Gamma',\tilde{\Gamma}'} = 0$ if $\tilde{\Gamma} \ne \Gamma'$.

Indeed, we can take for $\tilde{e}_{\Gamma'}$ any element of $M^+_{\Gamma,\Gamma'}$ and we use 1.2(b).

1.4. Let $\le$ be the standard partial order on W. By [KL1], to any $y \ne w$ in W one can attach a number $\mu(y,w) \in \mathbf{Z}$ such that for any $s \in S$ and any $w \in W$ with $sw > w$ we have $c_s c_w = \sum_{y \in W; sy < y} \mu(y,w) c_y$. By [KL2] we have $\mu(y,w) \in \mathbf{N}$.

1.5. Let $\underline{\mathcal{H}} = \mathbf{C}(v) \otimes_{\mathcal{A}} \mathcal{H}$ where we use the obvious imbedding $\mathcal{A} \to \mathbf{C}(v)$; we denote $1 \otimes c_w$ again by c_w. Let $\underline{\mathbf{J}} = \mathbf{C}(v) \otimes_{\mathbf{C}} \mathbf{J}$ where we use the obvious imbedding $\mathbf{C} \to \mathbf{C}(v)$. We have a homomorphism of $\mathbf{C}(v)$-algebras (with 1) $\Psi : \underline{\mathcal{H}} \to \underline{\mathbf{J}}$ given by

$$\Psi(c_x) = \sum_{d \in \mathcal{D}, z \in W, d \sim z} h_{x,d,z} t_z$$

for all $x \in W$, see [L6, 18.9]. (Note that Ψ is in fact the composition of a homomorphism in *loc.cit.* with an automorphism of $\underline{\mathcal{H}}$.)

1.6. For any $\Gamma \in L$ let S_Γ be the set of all $t \in S$ such that $rt < r$ for some (or equivalently any) $r \in \Gamma$.

We fix $\Gamma \in L$. Let $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$ be a $\mathbf{C}$-basis of M_Γ as in 1.3; we use the notation of 1.3. We shall view $\mathbf{C}(v) \otimes M_\Gamma$ as an $\underline{\mathcal{H}}$-module via Ψ. Let $s \in S$ and let $\Gamma' \in L$; let δ be the unique element in $\Gamma' \cap \mathcal{D}$. We show:

(a) *If $s \in S_{\Gamma'}$, then $\Psi(T_s) \tilde{e}_{\Gamma'} = v^2 \tilde{e}_{\Gamma'}$.*

(b) *If $s \notin S_{\Gamma'}$, then*

$$\Psi(T_s) \tilde{e}_{\Gamma'} = -\tilde{e}_{\Gamma'} + \sum_{\tilde{\Gamma} \in L; s \in S_{\tilde{\Gamma}}} f_{\tilde{\Gamma},\Gamma'} v^{-1} \tilde{e}_{\tilde{\Gamma}}$$

where

$$f_{\tilde{\Gamma},\Gamma'} = \sum_{u \in \Gamma' \cap \tilde{\Gamma}^{-1}} \mu(u,\delta) \lambda_{u,\Gamma',\tilde{\Gamma}} \in \mathbf{R}_{\ge 0}.$$

By definition, we have

$$\Psi(c_s)\tilde e_{\Gamma'}=\sum_{d\in\mathcal{D},u\in c,d\sim u}h_{s,d,u}t_u\tilde e_{\Gamma'}=\sum_{\tilde\Gamma\in L}\sum_{d\in\mathcal{D},u\in\Gamma'\cap\tilde\Gamma^{-1},d\in\Gamma'}h_{s,d,u}\lambda_{u,\Gamma',\tilde\Gamma}\tilde e_{\tilde\Gamma}.$$

Since in the last sum we have $d\in\Gamma'$ we see that we can assume that $d=\delta$. Thus we have

$$\Psi(c_s)\tilde e_{\Gamma'}=\sum_{\tilde\Gamma\in L}\sum_{u\in\Gamma'\cap\tilde\Gamma^{-1}}h_{s,\delta,u}\lambda_{u,\Gamma',\tilde\Gamma}\tilde e_{\tilde\Gamma}.$$

If $s\delta<\delta$ (that is, $s\in S_{\Gamma'}$) we have $c_sc_\delta=(v+v^{-1})c_\delta$ hence $h_{s,\delta,u}$ is $(v+v^{-1})$ for $u=\delta$ and is 0 for $u\ne\delta$; hence in this case

$$\Psi(c_s)\tilde e_{\Gamma'}=(v+v^{-1})\tilde e_{\Gamma'};$$

(we use that $\lambda_{\delta,\Gamma',\Gamma'}=1$.)

We now assume that $s\delta>\delta$ (that is, $s\notin S_{\Gamma'}$). In this case, $h_{s,\delta,u}$ is $\mu_{u,\delta}$ if $su<u$ and is 0 if $su>u$ (see 1.4); hence

$$\Psi(c_s)\tilde e_{\Gamma'}=\sum_{\tilde\Gamma\in L}\sum_{u\in\Gamma'\cap\tilde\Gamma^{-1};su<u}\mu(u,\delta)\lambda_{u,\Gamma',\tilde\Gamma}\tilde e_{\tilde\Gamma}$$
$$=\sum_{\tilde\Gamma\in L;s\in S_{\tilde\Gamma}}\sum_{u\in\Gamma'\cap\tilde\Gamma^{-1}}\mu(u,\delta)\lambda_{u,\Gamma',\tilde\Gamma}\tilde e_{\tilde\Gamma}.$$

Now (a),(b) follow.

Note that (a),(b) show that in the $\underline{\mathcal{H}}$-module $\mathbf{C}(v)\otimes M_\Gamma$ the generators T_s act with respect to the basis $\{\tilde e_{\Gamma'};\Gamma'\in L\}$ essentially by formulas which are those in a W-graph (in the sense of [KL1]) in which all labels are in $\mathbf{R}_{\ge0}$.

1.7. In Section 4 we will give another proof of the existence part of 1.2(a) which also shows that $\tilde e_{\Gamma'}$ in 1.3 can be chosen so that

(i) each $\tilde e_{\Gamma'}$ is a $\mathbf{Z}_{>0}$-linear combination of elements in $\{t_x;x\in\Gamma\cap\Gamma'^{-1}\}$,

(ii) $\lambda_{u,\Gamma',\Gamma'_1}\in\mathbf{Z}_{\ge0}$ (notation of 1.3).

In particular, with this choice of $\tilde e_{\Gamma'}$, the constants $f_{\tilde\Gamma,\Gamma'}$ in the "W-graph formulas" in 1.6 are in $\mathbf{Z}_{\ge0}$.

2. Proof of Theorem 1.2

2.1. From [L6,§15] we see that, for x,y,u in W we have:

(a) $$\gamma_{x,y,u}=\gamma_{y,u,x}=\gamma_{u,x,y},$$

(b) $$\gamma_{x,y,u}\ne0\implies x\sim y^{-1},y\sim u^{-1},u\sim x^{-1}.$$

By [L6, 18.4(a)]:

(c) *for y,z in W we have $y\sim z$ if and only if $t_yt_{z^{-1}}\ne0$.*

2.2. Let $\Gamma, \Gamma', \tilde{\Gamma}, \tilde{\Gamma}'$ be in L. From 2.1(b) we deduce:

(a) $$\text{If } \Gamma \neq \tilde{\Gamma}', \text{ then } \mathbf{J}_{\Gamma\cap\Gamma'^{-1}}\mathbf{J}_{\tilde{\Gamma}\cap\tilde{\Gamma}'^{-1}} = 0,$$

(b) $$\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}\mathbf{J}_{\tilde{\Gamma}\cap\Gamma^{-1}} \subset \mathbf{J}_{\tilde{\Gamma}\cap\Gamma'^{-1}}.$$

We show:

(c) $$\text{If } u \in \Gamma\cap\Gamma'^{-1}, \text{ then } t_u\mathbf{J}^+_{\tilde{\Gamma}\cap\Gamma^{-1}} \subset \mathbf{J}^+_{\tilde{\Gamma}\cap\Gamma'^{-1}}.$$

Let $\xi = \sum_{y\in\tilde{\Gamma}\cap\Gamma^{-1}} f_y t_y \in \mathbf{J}_{\tilde{\Gamma}\cap\Gamma^{-1}}$ with $f_y \in \mathbf{R}_{>0}$ for all y. We must show that $t_u\xi \in \mathbf{J}^+_{\tilde{\Gamma}\cap\Gamma'^{-1}}$; it is enough to show that for any $z \in \tilde{\Gamma}\cap\Gamma'^{-1}$ there exists $y \in \tilde{\Gamma}\cap\Gamma^{-1}$ such that $\gamma_{u,y,z^{-1}} \neq 0$ or that there exists $y \in W$ such that $\gamma_{z^{-1},u,y} \neq 0$ (see 2.1(a)); such y is automatically in $\tilde{\Gamma}\cap\Gamma^{-1}$. Hence it is enough to show that for any $z \in \tilde{\Gamma}\cap\Gamma'^{-1}$ we have $t_{z^{-1}}t_u \neq 0$. This holds since $z^{-1} \sim u^{-1}$ (see 2.1(c)).

From (c) we deduce

(d) $$\mathbf{J}^+_{\Gamma\cap\Gamma'^{-1}}\mathbf{J}^+_{\tilde{\Gamma}\cap\Gamma^{-1}} \subset \mathbf{J}^+_{\tilde{\Gamma}\cap\Gamma'^{-1}}.$$

2.3. Let $\Gamma \in L$. For any $\Gamma' \in L$ we define a $\mathbf{C}$-linear map $T_{\Gamma'} : \mathbf{J}_{\Gamma\cap\Gamma'^{-1}} \to \mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$ by

$$T_{\Gamma'}(t_x) = \sum_{y\in\Gamma\cap\Gamma^{-1}} t_x t_y = \sum_{y\in\Gamma\cap\Gamma^{-1}, z\in\Gamma} \gamma_{x,y,z^{-1}} t_z.$$

We show:

(a) *the matrix representing $T_{\Gamma'}$ with respect to the basis $\{t_w; w \in \Gamma\cap\Gamma'^{-1}\}$ has all entries in $\mathbf{R}_{>0}$.*

An equivalent statement is: for any x, z in $\Gamma\cap\Gamma'^{-1}$, the sum $\sum_{y\in\Gamma\cap\Gamma^{-1}} \gamma_{x,y,z^{-1}}$ is > 0. Since $\gamma_{x,y,z^{-1}} \in \mathbf{N}$ for all y, it is enough to show that for some $y \in \Gamma\cap\Gamma^{-1}$ we have $\gamma_{x,y,z^{-1}} \neq 0$ or equivalently (see 2.1(a)) that for some $y \in W$ we have $\gamma_{z^{-1},x,y} \neq 0$ (we then have automatically $y \in \Gamma\cap\Gamma^{-1}$). Thus, it is enough to show that $t_{z^{-1}}t_x \neq 0$. This follows from 2.1(c) since $z^{-1} \sim x^{-1}$.

Applying Perron's theorem [Pe] to the matrix in (a) we see that there is a unique $T_{\Gamma'}$-stable positive line $\mathcal{L}_{\Gamma,\Gamma'}$ in $\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$ (the "Perron line").

Now let $u \in \mathbf{c}$; we have, $u \in \tilde{\Gamma}\cap\tilde{\Gamma}'^{-1}$ with $\tilde{\Gamma}, \tilde{\Gamma}'$ in L. From 2.2(a), 2.2(d), we deduce

(b) If $\tilde{\Gamma} \neq \Gamma'$, then $t_u\mathbf{J}_{\Gamma\cap\Gamma'^{-1}} = 0$ hence $t_u\mathcal{L}_{\Gamma,\Gamma'} = 0$;

(c) if $\tilde{\Gamma} = \Gamma'$, then $t_u\mathcal{L}^+_{\Gamma,\Gamma'} \subset \mathbf{J}^+_{\Gamma\cap\tilde{\Gamma}'^{-1}}$, hence $t_u\mathcal{L}_{\Gamma,\Gamma'}$ is a positive line in $\mathbf{J}_{\Gamma\cap\tilde{\Gamma}'^{-1}}$.

In the setup of (c), we have $t_u(T_{\Gamma'}(\xi)) = T_{\tilde{\Gamma}'}(t_u\xi)$ for any $\xi \in \mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$. It follows that $t_u\mathcal{L}_{\Gamma,\Gamma'}$ is a $T_{\tilde{\Gamma}'}$-stable line in $\mathbf{J}_{\Gamma\cap\tilde{\Gamma}'^{-1}}$. Thus,

(d) if $\tilde{\Gamma} = \Gamma'$, then $t_u\mathcal{L}_{\Gamma,\Gamma'} = \mathcal{L}_{\Gamma,\tilde{\Gamma}'}$.

We set $\mathcal{M}_\Gamma = \oplus_{\Gamma'\in L}\mathcal{L}_{\Gamma,\Gamma'}$. From (b),(d) we see that $\mathcal{M}_\Gamma$ is a $\mathbf{J}_\mathbf{c}$-submodule of $\mathbf{J}_\Gamma$.

We now see that the existence part of 1.2(a) is proved: we can take $M_\Gamma = \mathcal{M}_\Gamma$.

2.4. Let $\Gamma\in L$ and let M_Γ be any $\mathbf{J_c}$-submodule of $\mathbf{J}_\Gamma$ for which property ($\heartsuit$) in 1.2(a) holds. We show that 1.2(b) holds for M_Γ. Let $u\in\tilde\Gamma\cap\tilde\Gamma'^{-1}$ be as in 1.2(b) and let $\Gamma'\in L$. If $\tilde\Gamma\ne\Gamma'$, then $t_u\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}=0$ hence $t_uM_{\Gamma,\Gamma'}=0$. Now assume that $\tilde\Gamma=\Gamma'$. By 2.2(c), left multiplication by t_u maps $\mathbf{J}^+_{\Gamma\cap\Gamma'^{-1}}$ into $\mathbf{J}^+_{\Gamma\cap\tilde\Gamma'^{-1}}$ hence it maps any positive line in $\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$ onto a positive line in $\mathbf{J}_{\Gamma\cap\tilde\Gamma'^{-1}}$. In particular, it maps $M_{\Gamma,\Gamma'}$ onto a line in $\mathbf{J}_{\Gamma\cap\tilde\Gamma'^{-1}}$, which, being also contained in M_Γ, must be equal to $M_{\Gamma,\tilde\Gamma'}$; moreover, it maps $M^+_{\Gamma,\Gamma'}$ into $\mathbf{J}^+_{\Gamma\cap\tilde\Gamma'^{-1}}$ hence onto $M^+_{\Gamma,\tilde\Gamma'}$. Thus, 1.2(b) holds for M_Γ.

We now choose a basis $\{\tilde e_{\Gamma'};\Gamma'\in L\}$ of M_Γ such that $\tilde e_{\Gamma'}\in M^+_{\Gamma,\Gamma'}$ for any $\Gamma'\in L$; then for any $u\in\mathbf{c}$, the matrix of the t_u-action on M_Γ in this basis has entries in $\mathbf{R}_{\ge0}$. Thus,

(a) $\mathrm{tr}(t_u,M_\Gamma)\in\mathbf{R}_{\ge0}$ for all $u\in\mathbf{c}$.

We show:

(b)*the* $\mathbf{C}$*-linear map* $\nu:\mathbf{J_c}\to\mathrm{End}_{\mathbf{C}}(M_\Gamma)$ *given by the* $\mathbf{J_c}$*-module structure on* M_Γ *is surjective.*

It is enough to show that for any $\Gamma',\tilde\Gamma'$ in L there exists $u\in\mathbf{c}$ such that $\nu(t_u)$ carries the line $M_{\Gamma,\Gamma'}$ onto the line $M_{\Gamma,\tilde\Gamma'}$ and carries the line $M_{\Gamma,\Gamma''}$ (where $\Gamma''\in L$, $\Gamma''\ne\Gamma'$) to zero. Note that any $u\in\Gamma'\cap\tilde\Gamma'^{-1}$ has the required properties. This proves (b).

It follows that the $\mathbf{J_c}$-module M_Γ is simple. We show:

(c) *Assume that M' is any simple* $\mathbf{J_c}$*-module such that* $\mathrm{tr}(t_u,M')\in\mathbf{R}_{\ge0}$ *for all* $u\in\mathbf{c}$*. Then M' is isomorphic to* M_Γ.

Assume that this is not so. We use the orthogonality formula

$$\sum_{u\in\mathbf{c}}\mathrm{tr}(t_u,M_\Gamma)\mathrm{tr}(t_{u^{-1}},M')=0,$$

which is a special case of [L6, 19.2(e)] (taking into account [L6, 20.1(b)] and using that $u\mapsto u^{-1}$ maps $\mathbf{c}$ into $\mathbf{c}$). Since each term in the last sum is in $\mathbf{R}_{\ge0}$, it follows that each term in the last sum is 0. In particular, we have $\mathrm{tr}(t_d,M_\Gamma)\mathrm{tr}(t_d,M')=0$ for any $d\in\mathcal{D}_\mathbf{c}$. We show that for any $d\in\mathcal{D}_\mathbf{c}$ we have $\mathrm{tr}(t_d,M_\Gamma)\in\mathbf{R}_{>0}$. Using the basis of M_Γ employed in the proof of (a), it is enough to show that some diagonal entry of the matrix of the t_d-action in this basis is $\ne0$ (all entries are in $\mathbf{R}_{\ge0}$). We have $d\in\Gamma'$ for a unique $\Gamma'\in L$; then $t_dM^+_{\Gamma,\Gamma'}=M^+_{\Gamma,\Gamma'}$ and the desired property holds.

From $\mathrm{tr}(t_d,M_\Gamma)\mathrm{tr}(t_d,M')=0$ and $\mathrm{tr}(t_d,M_\Gamma)\in\mathbf{R}_{>0}$ we deduce that $\mathrm{tr}(t_d,M')=0$ for any $d\in\mathcal{D}_\mathbf{c}$. Since $\sum_{d\in\mathcal{D}_\mathbf{c}}t_d$ is the unit element $1_\mathbf{c}$ of $\mathbf{J_c}$, it follows that $\mathrm{tr}(1_\mathbf{c},M')=0$. This is a contradiction. This proves (c).

Let I be the simple ideal of $\mathbf{J_c}$ such that $I\mathcal{M}_\Gamma\ne0$. It is a $\mathbf{C}$-vector space of dimension N^2 where N is the number of elements in L. If $\tilde\Gamma\in L$, then $\mathcal{M}_{\tilde\Gamma}$ is a simple $\mathbf{J_c}$-module such that $\mathrm{tr}(t_u,\mathcal{M}_{\tilde\Gamma})\in\mathbf{R}_{\ge0}$ for all $u\in c$ (we use (a) with M_Γ replaced by $\mathcal{M}_{\tilde\Gamma}$); hence, by (c), we have $\mathcal{M}_{\tilde\Gamma}\cong M_\Gamma$ as $\mathbf{J_c}$-modules. In particular, the isomorphism class of $\mathcal{M}_{\tilde{\Gamma}}$ is independent of $\tilde{\Gamma}$. We see that the (necessarily direct) sum $\sum_{\tilde{\Gamma}\in L}\mathcal{M}_{\tilde{\Gamma}}$ is contained in I and has dimension N^2 hence it is equal to I; we also see that $M_\Gamma \subset I$ and, taking intersections with $\mathbf{J}_\Gamma$, we see that $M_\Gamma \subset \mathcal{M}_\Gamma$, hence $M_\Gamma = \mathcal{M}_\Gamma$ (since $\dim M_\Gamma = \dim \mathcal{M}_\Gamma = N$). We now see that the uniqueness part of 1.2(a) is proved. Note that 1.2(b),1.2(c),1.2(d) are also proved and we have $\mathbf{I} = I$.

2.5. We prove 1.2(e). In the setup of (e), if $\Gamma \neq \tilde{\Gamma}'$ then, using 2.2(a), we have $M_{\Gamma,\Gamma'}M_{\tilde{\Gamma},\tilde{\Gamma}'} = 0$. Assume now that $\Gamma = \tilde{\Gamma}'$. Using 2.2(d), we see that $M^+_{\Gamma,\Gamma'}M^+_{\tilde{\Gamma},\Gamma}$ is contained in $\mathbf{J}^+_{\tilde{\Gamma}\cap\Gamma'^{-1}}$; it is also contained in $\mathbf{I}$ (since $\mathbf{I}$ is closed under multiplication), hence it is contained in $\mathbf{I}\cap\mathbf{J}^+_{\tilde{\Gamma}\cap\Gamma'^{-1}} = M^+_{\tilde{\Gamma},\Gamma'}$. Thus, multiplication restricts to a map $M^+_{\Gamma,\Gamma'} \times M^+_{\tilde{\Gamma},\Gamma} \to M^+_{\tilde{\Gamma},\Gamma'}$. This map is necessarily surjective since $M^+_{\tilde{\Gamma},\Gamma'}$ is a single orbit of $\mathbf{R}_{>0}$ under scalar multiplication. This implies that the linear map between lines $M_{\Gamma,\Gamma'} \otimes M_{\tilde{\Gamma},\Gamma} \to M_{\tilde{\Gamma},\Gamma'}$ is an isomorphism. This proves 1.2(e).

2.6. We prove 1.2(f). Any element $\xi \in \mathbf{J_c}$ defines a linear map ${}^t(\theta(\xi)) : M^*_\Gamma \to M^*_\Gamma$ where M^*_Γ denotes the dual space and ^{t} denotes the transpose. This defines a $\mathbf{J_c}$-module structure on M^*_Γ such that for any $x \in \mathbf{c}$ we have $\mathrm{tr}(t_x, M^*_\Gamma) = \mathrm{tr}(t_{x^{-1}}, M_\Gamma)$. By the argument in [L6, 20.13(a)], $\mathrm{tr}(t_{x^{-1}}, M_\Gamma)$ is the complex conjugate of $\mathrm{tr}(t_x, M_\Gamma)$. But the last trace is a real number, so that $\mathrm{tr}(t_x, M^*_\Gamma) = \mathrm{tr}(t_x, M_\Gamma)$. It follows that $M^*_\Gamma \cong M_\Gamma$ as $\mathbf{J_c}$-modules. From the definitions, the simple two-sided ideal $\mathbf{I}'$ of $\mathbf{J_c}$ such that $\mathbf{I}'M^*_\Gamma \neq 0$ satisfies $\mathbf{I}' = \theta(\mathbf{I})$. It follows that $\theta(\mathbf{I}) = \mathbf{I}$. Since $\mathbf{I} = \oplus_{\tilde{\Gamma},\tilde{\Gamma}' \text{ in } L}M_{\tilde{\Gamma},\tilde{\Gamma}'}$ and $\theta(\mathbf{J}_{\Gamma,\Gamma'}) = \mathbf{J}_{\Gamma',\Gamma}$, it follows that

$$\theta(M_{\Gamma,\Gamma'}) \subset \mathbf{J}_{\Gamma',\Gamma} \cap \oplus_{\tilde{\Gamma},\tilde{\Gamma}' \text{ in } L}M_{\tilde{\Gamma},\tilde{\Gamma}'} = M_{\Gamma',\Gamma}.$$

Since θ is a vector space isomorphism, it follows that $\theta(M_{\Gamma,\Gamma'}) = M_{\Gamma',\Gamma}$. Note that $\theta(\mathbf{J}^+_{\Gamma,\Gamma'}) = \mathbf{J}^+_{\Gamma',\Gamma}$; hence

$$\theta(M^+_{\Gamma,\Gamma'}) \subset \mathbf{J}^+_{\Gamma',\Gamma} \cap M_{\Gamma',\Gamma} = M^+_{\Gamma',\Gamma}.$$

This forces the equality $\theta(M^+_{\Gamma,\Gamma'}) = M^+_{\Gamma',\Gamma}$ (since $M^+_{\Gamma',\Gamma}$ is a single orbit of $\mathbf{R}_{>0}$ under scalar multiplication). This proves 1.2(f). Theorem 1.2 is proved.

2.7. After an earlier version of this paper was posted, P. Etingof told me that the line $M_{\Gamma,\Gamma'}$ in 1.2 is the same as the line associated in [EGNO, 3.4.4] to the right $\mathbf{J}_{\Gamma\cap\Gamma^{-1}}$-module $\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$ (viewed as a based module over a based ring) that is, the unique positive line $\mathfrak{L}$ in $\mathbf{J}_{\Gamma\cap\Gamma'^{-1}}$ such that $\mathfrak{L}$ is a right $\mathbf{J}_{\Gamma\cap\Gamma^{-1}}$-submodule. (The discussion in *loc.cit.* concerns left (instead of right) indecomposable based modules over a fusion ring.) Indeed, from the definitions we see that $\mathfrak{L}$ must be the same as $\mathcal{L}_{\Gamma,\Gamma'}$ in 2.3, hence the same as $M_{\Gamma,\Gamma'}$.

3. Special representations

3.1. When $\mathcal{H}$ is tensored with $\mathbf{C}$ (using the ring homomorphism $\mathcal{A} \to \mathbf{C}$, $v \mapsto 1$), then it becomes $\mathbf{C}[W]$, the group algebra of W. (For $w \in W$ we have $1 \otimes T_w = w \in$

$\mathbf{C}[W]$; we denote $1 \otimes c_w$ again by c_w.) We have a homomorphism of $\mathbf{C}$-algebras (with 1) $\psi : \mathbf{C}[W] \to \mathbf{J}$ given by

$$\psi(c_x) = \sum_{d \in \mathcal{D}, z \in W, d \sim z} h_{x,d,z}|_{v=1} t_z$$

for all $x \in W$, see [L6, 18.9]; this is an isomorphism, see [L6, 20.1]. For example, if $W = \{1, s\}$ is of type A_1 we have $\psi(c_1) = t_1 + t_s$, $\psi(c_s) = 2t_s$; hence $\psi(1) = t_1 + t_s$, $\psi(s) = -t_1 + t_s$.

For each $E \in \mathrm{Irr}W$ let E_∞ be the simple $\mathbf{J}$-module corresponding to E under ψ and let $\mathbf{c}_E$ be the unique two-sided cell of W such that $\mathbf{J}_{\mathbf{c}_E} E \neq 0$. (Note that $E = E_\infty$ as $\mathbf{C}$-vector spaces.) Let $\mathrm{Irr}^{\mathbf{c}} W = \{E \in \mathrm{Irr}W; \mathbf{c}_E = \mathbf{c}\}$ and let $a' = a(xw_0)$ for any $x \in \mathbf{c}$, where w_0 is the longest element of W.

For any $k \in \mathbf{N}$ let $\mathfrak{S}^k$ be the k-th symmetric power of the reflection representation of W, viewed as a representation of W in an obvious way. For $E \in \mathrm{Irr}W$ let b_E be the smallest integer $k \geq 0$ such that E is a constituent of $\mathfrak{S}^k$. Now for any $E \in \mathrm{Irr}^{\mathbf{c}} W$ we have $b_E \geq a'$ and there is a unique $E \in \mathrm{Irr}^{\mathbf{c}} W$ such that $b_E = a'$; this E is denoted by $E^{\mathbf{c}}$ and is called the *special representation* associated to $\mathbf{c}$. (This is a reformulation of the definition of special representations given in [L1].)

Theorem 3.2. *In the setup of Theorem 1.2, for any $\Gamma \in L$, we have $M_\Gamma \cong E^{\mathbf{c}}_\infty$ as $\mathbf{J}_{\mathbf{c}}$-modules.*

We give two proofs; one is contained in 3.3,3.4,3.5. The other is given in 3.4,3.6.

3.3. In this subsection we assume that W is of type A, B or D.

Let $\Gamma \in L$. For any $\Gamma' \in L$ we set

$$\epsilon_{\Gamma'} = \sum_{z \in \Gamma \cap \Gamma'^{-1}} t_z \in \mathbf{J}^+_{\Gamma \cap \Gamma'^{-1}}.$$

By [L9, 4.8(b)], $\{\epsilon_{\Gamma'}; \Gamma' \in L\}$ is a $\mathbf{C}$-basis of the unique $\mathbf{J}$-submodule of $\mathbf{J}_\Gamma$ isomorphic to $E^{\mathbf{c}}_\infty$. By the uniqueness part of 1.2(a) this $\mathbf{J}$-submodule of $\mathbf{J}_\Gamma$ (viewed as a $\mathbf{J}_{\mathbf{c}}$-module) must be the same as M_Γ in 1.2(a). We see that in this case, M_Γ is isomorphic to $E^{\mathbf{c}}_\infty$ and $M_{\Gamma,\Gamma'}$ is the line spanned by $\epsilon_{\Gamma'}$. In particular, 3.2 holds in our case.

3.4. In this subsection we assume that $\mathbf{c}$ is such that $\mathrm{Irr}^{\mathbf{c}} W$ consists of exactly 2 irreducible representations. In this case, W is of type E_7 (resp. E_8) and the 2 irreducible representations in $\mathrm{Irr}^{\mathbf{c}} W$ have degree 512 (resp. 4096). Let $\Gamma \in L$ and let $d \in \mathcal{D} \cap \Gamma$. The $\mathbf{C}$-linear map $r : \mathbf{J}_\Gamma \to \mathbf{J}_\Gamma$ given by left multiplication by $(-1)^{l(d)} \psi(w_0)$ is in fact $\mathbf{J}$-linear (since w_0 is central in W) and $r(t_x) = t_{x^*}$ for any $x \in \Gamma$, where $x \mapsto x^*$ is a certain fixed point free involution of Γ, see [L7]. Then $\mathbf{J}^1_\Gamma = \{\xi \in \mathbf{J}_\Gamma; r(\xi) = \xi\}$ is a simple $\mathbf{J}_{\mathbf{c}}$-submodule of $\mathbf{J}_\Gamma$ with $\mathbf{C}$-basis $\{t_x + t_{x^*}; x \in \Gamma_1\}$ where Γ_1 is a set of representatives for the orbits of $x \mapsto x^*$ on Γ. Note that, if $x \in \Gamma$, then $\{x, x^*\}$ is the intersection of Γ with the inverse of a left cell $\Gamma' \in L$; hence $t_x + t_{x^*} \in \mathbf{J}^+_{\Gamma\cap\Gamma'^{-1}}$. By the uniqueness part of 1.2(a), we must have $M_\Gamma = \mathbf{J}^1_\Gamma$ and that for any $\Gamma' \in L$, $M_{\Gamma,\Gamma'}$ is the line spanned by $\sum_{x\in\Gamma\cap\Gamma'^{-1}} t_x$.

Now let $E \in \mathrm{Irr}^{\mathbf{c}}W$ be such that $E_\infty = \mathbf{J}^1_\Gamma$. From the definitions we have $\mathrm{tr}((-1)^{l(d)}w_0, E) = |\Gamma_1| = \dim E$. Hence, if $\epsilon = \pm 1$ is the scalar by which w_0 acts on E, then $(-1)^{l(d)}\epsilon = 1$. We have $l(d) = a(d) \mod 2$; hence $\epsilon = (-1)^{a(d)}$. But this equality characterizes the special representation in $\mathrm{Irr}^{\mathbf{c}}W$ (the special representation satisfies it, the nonspecial representation doesn't satisfy it). We see that $M_\Gamma = \mathbf{J}^1_\Gamma \cong E^{\mathbf{c}}_\infty$. In particular, 3.2 holds in our case.

3.5. In this subsection we assume that W is of exceptional type, but that $\mathbf{c}$ is not as in 3.4. In this case, $E^{\mathbf{c}}$ is the only representation in $\mathrm{Irr}^{\mathbf{c}}W$ of dimension equal to $|L|$; since M_Γ (in 1.2(a)) has dimension equal to $|L|$, it follows that $M_\Gamma = \mathbf{J}^1_\Gamma \cong E^c_\infty$. In particular, 3.2 holds in our case. This completes the proof of Theorem 3.2.

3.6. In this subsection we give a second proof of Theorem 3.2 assuming that $\mathbf{c}$ is not as in 3.4. Let $a = a(w)$ for any $w \in \mathbf{c}$. Let $\Gamma \in L$. Let $X = \sum_{w\in W} v^{-l(w)}T_w \in \underline{\mathcal{H}}$. We can view $\mathbf{C}(v) \otimes \mathbf{J}_\Gamma$ and $\mathbf{C}(v) \otimes M_\Gamma$ as $\underline{\mathcal{H}}$-modules via Ψ in 1.5. By [L9, 4.6], for any $x \in \Gamma$ we have

$$Xt_x = v^a \sum_{z\in\mathbf{c}} t_z t_x \mod \sum_{i<a} v^i \mathbf{J}_\Gamma.$$

By an argument as in the proof of 2.2(c), we see that $\sum_{z\in\mathbf{c}} t_z t_x \in \mathbf{J}^+_\Gamma$. It follows that if $\Gamma' \in L$ and $\xi \in M_{\Gamma,\Gamma'}$ then

$$X\xi = v^a \xi' \mod \sum_{i<a} v^i \mathbf{J}_\Gamma$$

where $\xi' \in \mathbf{J}^+_\Gamma$. In particular, we have $X\xi \neq 0$. Thus, $X(\mathbf{C}(v) \otimes M_\Gamma) \neq 0$. Using this and Theorem 4.2 in [L9] we deduce that the simple $\underline{\mathcal{H}}$-module $\mathbf{C}(v) \otimes M_\Gamma$ is a constituent of the "involution module" M in [L9, 0.1] (with $\mathbf{Q}(u)$ replaced by $\mathbf{C}(v)$). According to [L8] if a simple $\underline{\mathcal{H}}$-module appears in $\mathbf{C}(v) \otimes \mathbf{J}_{\Gamma'}$ for every $\Gamma' \in L$ and it appears in M, then that $\underline{\mathcal{H}}$-module corresponds to $E^{\mathbf{c}}$. We deduce that $M_\Gamma \cong E^{\mathbf{c}}_\infty$. This completes the second proof of Theorem 3.2, assuming that $\mathbf{c}$ is not as in 3.4.

4. Equivariant vector bundles

4.1. In this section we fix a reductive, not necessarily connected algebraic group G over $\mathbf{C}$ acting on a finite set X. Let $G\backslash X$ be the set of G-orbits on X. Representations of reductive groups over $\mathbf{C}$ are always assumed to be of finite dimension over $\mathbf{C}$ and algebraic. For $x \in X$ let $G_x = \{g \in G; gx = x\}$. Now G acts diagonally on $X \times X$ and we can consider the Grothendieck group $K_G(X \times X)$ of G-equivariant complex vector bundles (G-eq.v.b.) on $X \times X$. This is an (associative) ring with

1 under convolution, denoted by $*$ (see [L3, 2.2], [L4, 10.2]). For a G-eq.v.b. V on $X \times X$ we denote by $V_{x,y}$ the fibre of V at $(x,y) \in X \times X$. Let B be the set of pairs (Ω, ρ) where Ω is a G-orbit Ω in $X \times X$ and ρ is an irreducible representation of G_Ω (the isotropy group of a point $(x,y) \in \Omega$). For any $(\Omega,\rho) \in B$ we denote by $V^{\Omega,\rho}$ the G-eq.v.b. on $X \times X$ such that $V_{\Omega,\rho}|_{X \times X - \Omega} = 0$ and the action of G_Ω on $V^{\Omega,\rho}_{x,y}$ is equivalent to ρ. Now $\{V^{\Omega,\rho}; (\Omega,\rho) \in B\}$ is a $\mathbf{Z}$-basis of $K_G(X \times X)$. Let $\mathbf{K}_G(X \times X) = \mathbf{C} \otimes K_G(X \times X)$, viewed as a $\mathbf{C}$-algebra.

4.2. In this subsection we assume that G is finite. Let ω, ω' be in $G\backslash X$. Let $V^{\omega,\omega'}$ be the G-eq.v.b. on $X \times X$ such that $V^{\omega,\omega'}_{a,b} = \mathbf{C}[G]$ if $(a,b) \in \omega \times \omega'$, $V^{\omega,\omega'}_{a,b} = 0$ if $(a,b) \notin \omega \times \omega'$. (Here $\mathbf{C}[G]$ is the left regular representation of G.) The G-action $g : V^{\omega,\omega'}_{a,b} \to V^{\omega,\omega'}_{ga,gb}$ is left translation by g on $\mathbf{C}[G]$ (if $(a,b) \in \omega \times \omega'$) and is 0 if $(a,b) \notin \omega \times \omega'$. We show:

(a) *Let $(Om,\rho) \in B$; we have $\Omega \subset \omega_1 \times \omega_1'$ where ω_1, ω_1' are in $G\backslash X$. Then $U' := V^{\Omega,\rho} * V^{\omega,\omega'}$ is isomorphic to a direct sum of copies of the single G-eq.v.b. $V^{\omega_1,\omega'}$. More precisely, if $\omega_1' \ne \omega$, we have $U' = 0$; if $\omega_1' = \omega$, we have $U' = V_{\omega_1,\omega'}^{\oplus(\dim\rho|\Omega||\omega_1|^{-1})}$.*

For $(a,b) \in X \times X$ we have

$$U'_{a,b} = \oplus_{z \in \omega; (a,z) \in \Omega} V^{\Omega,\rho}_{a,z} \otimes \mathbf{C}[G] \text{ if } b \in \omega',$$
$$U'_{a,b} = 0 \text{ if } b \notin \omega'.$$

Thus the support of U' is contained in $\omega_1 \times \omega'$ and $U' = 0$ unless $\omega_1' = \omega$. We now assume that $\omega_1' = \omega$ and $(a,b) \in \omega_1 \times \omega'$. Then $U'_{a,b} = \oplus_{z;(a,z)\in\Omega} V^{\Omega,\rho}_{a,z} \otimes \mathbf{C}[G]$. We have $\dim U'_{a,b} = d|G||\Omega|/|\omega_1|$. We show:

(b) *as a $G_a \cap G_b$-module, $U'_{a,b}$ is a multiple of the regular representation.*

Let $\sigma_1, \dots, \sigma_k$ be the various $G_a \cap G_b$-orbits contained in ω'. We have $U'_{a,b} = \oplus_{i=1}^k R_i$, where $R_i = \oplus_{z \in \sigma_i} V^{\Omega,\rho}_{a,z} \otimes \mathbf{C}[G]$. We pick $z_i \in \sigma_i$. Now

$$R_i = \operatorname{ind}_{G_a \cap G_b \cap G_{z_i}}^{G_a \cap G_b} (A \otimes B)$$

where

$$A = \operatorname{res}^{G_a \cap G_{z_i}}_{G_a \cap G_b \cap G_{z_i}} (V^{\Omega,\rho}_{a,z_i}), \quad B = \operatorname{res}^{G_{z_i} \cap G_b}_{G_a \cap G_b \cap G_{z_i}} (\mathbf{C}[G]).$$

It is enough to show that R_i is a multiple of the regular representation of $G_a \cap G_b$. Since R_i is induced, it is enough to show that $A \otimes B$ is a multiple of the regular representation of $G_a \cap G_b \cap G_{z_i}$. It is also enough to show that B is a multiple of the regular representation of $G_a \cap G_b \cap G_{z_i}$. This follows from the fact that $\mathbf{C}[G]$ is a multiple of the regular representation of $G_{z_i} \cap G_b$. This proves (b). Now (a) follows.

Note that in $\mathbf{K}_G(X \times X)$ we have

(c) $$V^{\omega,\omega'} = \sum_{(\Omega,\rho) \in B; \Omega \subset \omega \times \omega'} \dim\rho|\Omega| V^{\Omega,\rho}.$$

4.3. We now drop the assumption (in 4.2) that G is finite. Let $\hat{\mathbf{K}}_G(X\times X)$ be the $\mathbf{C}$-vector space consisting of formal (possibly infinite) linear combinations $\sum_{(\Omega,\rho)\in B} f_{\Omega,\rho}V^{\Omega\rho}$ where $f_{\Omega,\rho}\in\mathbf{C}$. The left $\mathbf{K}_G(X\times X)$-module structure on $\mathbf{K}_G(X\times X)$ given by left multiplication extends naturally to a left $\mathbf{K}_G(X\times X)$-module structure on $\hat{\mathbf{K}}_G(X\times X)$. If ω,ω' are as in $G\backslash X$ then we can define $V^{\omega,\omega'}\in\hat{\mathbf{K}}_G(X\times X)$ by the sum 4.2(c) (which is now a possibly infinite sum); we set $\bar{V}^{\omega,\omega'}=|\omega|^{-1}V^{\omega,\omega'}$ so that

$$\bar{V}^{\omega,\omega'}=\sum_{(\Omega,\rho)\in B;\Omega\subset\omega\times\omega'}\dim\rho|\Omega||\omega|^{-1}V^{\Omega,\rho}. \tag{a}$$

Now formula 4.2(a) extends to the present case as follows. Let $(Om,\rho)\in B$; we have $\Omega\subset\omega_1\times\omega'_1$ where ω_1,ω'_1 are G-orbits in X. Then

$$V^{\Omega,\rho}V^{\omega,\omega'}=NV^{\omega_1,\omega'} \tag{b}$$

where $N\in\mathbf{Z}$ is 0 if $\omega'_1\neq\omega$ and $N=\dim\rho|\Omega||\omega_1|^{-1}$ if $\omega'_1=\omega$. Hence

$$V^{\Omega,\rho}\bar{V}^{\omega,\omega'}=N'\bar{V}^{\omega_1,\omega'} \tag{c}$$

where $N'\in\mathbf{Z}$ is 0 if $\omega'_1\neq\omega$ and $N'=\dim\rho|\Omega||\omega|^{-1}$ if $\omega'_1=\omega$.

Let $\omega'\in G\backslash X$. Let $\tilde{R}_{\omega'}$ be the subspace of $\hat{\mathbf{K}}_H(X\times X)$ consisting of formal (possibly infinite) linear combinations $\sum_{(\Omega,\rho)\in B;pr_2\Omega=\omega'} f_{\Omega,\rho}V^{\Omega,\rho}$ with $f_{\Omega,\rho}\in\mathbf{C}$. Here $pr_2:X\times X\to X$ is the second projection. Note that $\tilde{R}_{\omega'}$ is a $\mathbf{K}_G(X\times X)$-submodule of $\hat{\mathbf{K}}_H(X\times X)$.

Let $R_{\omega'}$ be the subspace of $\hat{\mathbf{K}}_G(X\times X)$ with basis formed by the elements $\bar{V}^{\omega,\omega'}$ for various $\omega\in G\backslash X$. Using (c) we see that $R_{\omega'}$ is a (simple) $\mathbf{K}_G(X\times X)$-submodule of $\hat{\mathbf{K}}_G(X\times X)$; we have $R_{\omega'}\subset\tilde{R}_{\omega'}$. Using (c) we see also that if $\omega''\in G\backslash X$ then $\bar{V}^{\omega,\omega'}\mapsto\bar{V}^{\omega,\omega''}$ defines an isomorphism of $\mathbf{K}_G(X\times X)$-modules $R_{\omega'}\xrightarrow{\sim}R_{\omega''}$. Hence the isomorphism class of the $\mathbf{K}_G(X\times X)$-module $R_{\omega'}$ is independent of the choice of ω'.

4.4. We now assume that G is the finite group associated to $\mathbf{c}$ in [L2] and that X is the finite G-set $\oplus_{\Gamma\in L}G/H_\Gamma$ where H_Γ is the subgroup of G defined in [L3, 3.8]. In this case we have $\hat{\mathbf{K}}_G(X\times X)=\mathbf{K}_G(X\times X)$. By a conjecture in [L3, 3.15], proved in [BFO], there exists an isomorphism of $\mathbf{C}$-algebras $\chi:\mathbf{K}_G(X\times X)\xrightarrow{\sim}\mathbf{J_c}$ carrying the basis $(V^{\Omega,\rho})$ of $\mathbf{K}_G(X\times X)$ onto the basis $\{t_x;x\in\mathbf{c}\}$ of $\mathbf{J_c}$. Under χ, the left ideal $\mathbf{J}_\Gamma$ of $\mathbf{J_c}$ (for $\Gamma\in L$) corresponds to the left ideal $\tilde{R}_{\omega'}$ of $\mathbf{K}_G(X\times X)$ where $\omega'\in G\backslash X$ corresponds to Γ, and the basis $\{t_x;x\in\Gamma\}$ of $\mathbf{J}_\Gamma$ corresponds to the intersection of the basis $(V^{\Omega,\rho})$ of $\mathbf{K}_G(X\times X)$ with $\tilde{R}_{\omega'}$. The basis of $R_{\omega'}$ formed by the elements $\bar{V}^{\omega,\omega'}$ corresponds to a family of elements $\{e_{\Gamma'};\Gamma'\in L\}$ in $\mathbf{J}_\Gamma$.

From 4.3(a) we see that $e_{\Gamma'} \in \mathbf{J}^+_{\Gamma\cap\Gamma'^{-1}}$ for any $\Gamma' \in L$ (in fact the coefficients of the various $t_x, x \in \Gamma \cap \Gamma'^{-1}$ are in $\mathbf{Z}_{>0}$) and from 4.3(c) we see that for $u \in \mathbf{c}$ the product $t_u e_{\Gamma'}$ is a $\mathbf{Z}_{\geq 0}$ multiple of an element $e_{\Gamma''}$. We see that the $\mathbf{C}$-subspace of $\mathbf{J}_\Gamma$ spanned by $\{e_{\Gamma'}; \Gamma' \in L\}$ satisfies property (♡) in 1.2(a) hence it is equal to M_Γ. This provides another proof in our case for the existence part of 1.2(a), with the additional integrality properties in 1.7.

5. Final remarks

5.1. Theorem 1.2 and its proof remain valid if W is replaced by an affine Weyl group (with $\mathbf{c}$ assumed to be finite) or by a finite Coxeter group; in the last case we use the positivity property of $h_{x,y,z}$ established in [EW]. In these cases, the simple $\mathbf{J_c}$-module given by Theorem 1.2 will be called the special $\mathbf{J_c}$-module.

5.2. Assume now that W is an (irreducible) affine Weyl group and that $\mathbf{c}$ is a not necessarily finite two-sided cell of W. We denote again by L the set of left cells of W that are contained in $\mathbf{c}$; this is a finite set. Then the $\mathbf{C}$-algebra $\mathbf{J_c}$ with its basis $\{t_x; x \in \mathbf{c}\}$ is defined. Let $\hat{J}J_{\mathbf{c}}$ be the set of formal (possibly infinite) linear combinations $\sum_{u\in\mathbf{c}} f_u t_u$ where $f_u \in \mathbf{C}$. This is naturally a left $\mathbf{J_c}$-module. For any subset X of $\mathbf{c}$ let $\hat{\mathbf{J}}_X$ be the set of all $\sum_{u\in\mathbf{c}} f_u t_u \in \hat{\mathbf{J}}_{\mathbf{c}}$ such that $f_u = 0$ for $u \in \mathbf{c}-X$. If $\Gamma \in L$, then $\hat{\mathbf{J}}_\Gamma$ is a $\mathbf{J_c}$-submodule of $\hat{\mathbf{J}}_{\mathbf{c}}$. We have $\hat{\mathbf{J}}_\Gamma = \oplus_{\Gamma'\in L}\hat{\mathbf{J}}_{\Gamma\cap\Gamma'^{-1}}$.

According to a conjecture in [L4, 10.5], proved in [BFO], we can find G, X as in 4.1 and an isomorphism of $\mathbf{C}$-algebras $\chi : \mathbf{K}_G(X \times X) \xrightarrow{\sim} \mathbf{J_c}$ carrying the basis $\{V^{\Omega,\rho}; (\Omega, \rho) \in B\}$ of $\mathbf{K}_G(X \times X)$ onto the basis $\{t_x; x \in \mathbf{c}\}$ of $\mathbf{J_c}$. This extends in an obvious way to an isomorphism $\hat{\chi} : \hat{\mathbf{K}}_G(X \times X) \xrightarrow{\sim} \hat{\mathbf{J}}_{\mathbf{c}}$ under which the left $\mathbf{K}_G(X \times X)$-module structure on $\hat{\mathbf{K}}_G(X \times X)$ corresponds to the left $\mathbf{J_c}$-module structure on $\hat{\mathbf{J}}_{\mathbf{c}}$. If $\Gamma \in L$, there is a unique $\omega' \in G\backslash X$ (the set of G-orbits in X) such that $\hat{\chi}$ carries $R_{\omega'}$ (see 4.3) onto a (simple) $\mathbf{J_c}$-submodule M_Γ of $\hat{\mathbf{J}}_\Gamma$ whose isomorphism class is independent of Γ; we say that this is the special $\mathbf{J_c}$-module. The $\mathbf{J_c}$-module M_Γ admits a basis $\{e_{\Gamma'}; \Gamma' \in L\}$ in which any t_u (with $u \in \mathbf{c}$) acts by a matrix with all entries in $\mathbf{Z}_{\geq 0}$, namely the basis corresponding to the basis $\{\bar{V}^{\omega,\omega'}; \omega \in G\backslash X\}$ of $R_{\omega'}$.

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