Asymptotic Hecke algebras and involutions

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Introduction and statement of results

0.1. In [**11**], a Hecke algebra module structure on a vector space spanned by the involutions in a Weyl group was defined and studied. In this paper this study is continued by relating it to the asymptotic Hecke algebra introduced in [**6**]. In particular we define a module over the asymptotic Hecke algebra which is spanned by the involutions in the Weyl group. We present a conjecture relating this module to equivariant vector bundles with respect to a group action on a finite set. This gives an explanation (not a proof) of a result of Kottwitz [**3**] in the case of classical Weyl groups, see 2.5. We also present a conjecture which realizes the module in [**11**] terms of an ideal in the Hecke algebra generated by a single element, see 3.4.

0.2. Let W be a Coxeter group with set of simple reflections S and with length function $l : W \to \mathbf{N}$.

Let $\underline{\mathcal{A}} = \mathbf{Z}[v, v^{-1}]$ where v be an indeterminate. We set $u = v^2$. Let $\mathcal{A}$ be the subring $\mathbf{Z}[u, u^{-1}]$ of $\underline{\mathcal{A}}$. Let $\mathcal{H}$ (resp. $\mathfrak{H}$) be the free $\underline{\mathcal{A}}$-module (resp. free $\mathcal{A}$-module) with basis $(\dot{T}_w)_{w\in W}$ (resp. $(T_w)_{w\in W}$). We regard $\mathcal{H}$ (resp. $\mathfrak{H}$) as an associative $\underline{\mathcal{A}}$-algebra (resp. $\mathcal{A}$-algebra) with multiplication defined by $\dot{T}_w\dot{T}_{w'} = \dot{T}_{ww'}$ if $l(ww') = l(w) + l(w')$, $(\dot{T}_s + 1)(\dot{T}_s - u) = 0$ if $s \in S$ (resp. $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$, $(T_s + 1)(T_s - u^2) = 0$ if $s \in S$). For $y, w \in W$ let $P_{y,w}$ be the polynomial defined in [**2**]. For $w \in W$ let $\dot{c}_w = v^{-l(w)} \sum_{y\in W; y\le w} P_{y,w}(u)\dot{T}_y \in \mathcal{H}$, $c_w = u^{-l(w)} \sum_{y\in W; y\le w} P_{y,w}(u^2)T_y \in \mathfrak{H}$, see [**2**]. Let $y \le_{LR} w$, $y \sim_{LR} w$, $y \sim_L w$ be the relations defined in [**2**]. We shall write $\preceq, \sim$ instead of $\le_{LR}, \sim_{LR}$. The equivalence classes in W under $\sim$ (resp. $\sim_L$) are called two-sided cells (resp. left cells).

For $x, y, z \in W$ we define $\dot{h}_{x,y,z} \in \underline{\mathcal{A}}, h_{x,y,z} \in \mathcal{A}$ by $\dot{c}_x\dot{c}_y = \sum_{z\in W} \dot{h}_{x,y,z}\dot{c}_z$, $c_x c_y = \sum_{z\in W} h_{x,y,z}c_z$. Note that $h_{x,y,z}$ is obtained from $\dot{h}_{x,y,z}$ by the substitution $v \mapsto u$.

0.3. In this subsection we assume that W is a Weyl group or an (irreducible) affine Weyl group. From the definitions we have:

(a) *if $\dot{h}_{x,y,z} \ne 0$ (or if $h_{x,y,z} \ne 0$) then $z \preceq x$ and $z \preceq y$.*

For $z \in W$ there is a unique $a(z) \in \mathbf{N}$ such that $\dot{h}_{x,y,z} \in v^{a(z)}\mathbf{Z}[v^{-1}]$ for all $x, y \in W$

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and $\dot{h}_{x,y,z} \notin v^{a(z)-1}\mathbf{Z}[v^{-1}]$ for some $x, y \in W$. (See [**5**].) Hence for $z \in W$ we have $h_{x,y,z} \in u^{a(z)}\mathbf{Z}[u^{-1}]$ for all $x, y \in W$ and $h_{x,y,z} \notin u^{a(z)-1}\mathbf{Z}[u^{-1}]$ for some $x, y \in W$. For $x, y, z \in W$ we have $\dot{h}_{x,y,z} = \gamma_{x,y,z^{-1}} v^{a(z)} \mod v^{a(z)-1}\mathbf{Z}[v^{-1}]$, $\gamma_{x,y,z^{-1}} \in \mathbf{Z}$; hence we have $h_{x,y,z} = \gamma_{x,y,z^{-1}} u^{a(z)} \mod u^{a(z)-1}\mathbf{Z}[u^{-1}]$.

(b) *If $x, y \in W$ satisfy $x \preceq y$ then $a(x) \geq a(y)$. Hence if $x \sim y$ then $a(x) = a(y)$.* (See [**5**].)

Let $\mathcal{D}$ be the set of *distinguished involutions* of W (a finite set); see [**6**, 2.2]).

Let J be the free abelian group with basis $(t_w)_{w \in W}$. For $x, y \in W$ we set $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z \in J$ (the sum is finite). This defines an associative ring structure on J with unit element $1 = \sum_{d \in \mathcal{D}} t_d$ (see [**6**, 2.3]).

0.4. Let $* : W \to W$ (or $w \mapsto w^*$) be an automorphism of W such that $S^* = S$, $*^2 = 1$. Let $\mathbf{I}_* = \{w \in W; w^* = w^{-1}\}$; if $* = 1$ this is the set of involutions in W. Let M be the free $\mathcal{A}$-module with basis $(a_w)_{w \in \mathbf{I}_*}$. Following [**11**] for any $s \in S$ we define an $\mathcal{A}$-linear map $T_s : M \to M$ by

$T_s a_w = u a_w + (u+1) a_{sw}$ if $sw = ws^* > w$;
$T_s a_w = (u^2 - u - 1) a_w + (u^2 - u) a_{sw}$ if $sw = ws^* < w$;
$T_s a_w = a_{sws^*}$ if $sw \neq ws^* > w$;
$T_s a_w = (u^2 - 1) a_w + u^2 a_{sws^*}$ if $sw \neq ws^* < w$.

The following result was proved in the setup of 0.3 in [**11**] and then in the general case in [**10**].

(a) *These linear maps define an $\mathfrak{H}$-module structure on M.*

Let $\underline{\mathfrak{H}} = \underline{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}$, $\underline{M} = \underline{\mathcal{A}} \otimes_{\mathcal{A}} M$. We regard $\mathfrak{H}$ as a subring of $\underline{\mathfrak{H}}$ and M as a subgroup of $\underline{M}$ by $\xi \mapsto 1 \otimes \xi$. Note that the $\mathfrak{H}$-module structure on M extends naturally to an $\underline{\mathfrak{H}}$-module structure on $\underline{M}$.

Let $(A_w)_{w \in \mathbf{I}_*}$ be the $\underline{\mathcal{A}}$-basis of $\underline{M}$ defined in [**11**, 0.3]. (More precisely, in [**11**, 0.3] only the case where W is a Weyl group and $* = 1$ is considered in detail; the other cases are briefly mentioned in [**11**, 7.1]. A definition, valid in all cases is given in [**10**, 0.3].)

0.5. In the remainder of this section we assume that W is as in 0.3. For $x \in W$, $w, w' \in \mathbf{I}_*$ we define $f_{x,w,w'} \in \underline{\mathcal{A}}$ by $c_x A_w = \sum_{w' \in \mathbf{I}_*} f_{x,w,w'} A_{w'}$. The following result is proved in 1.1:

(a) *For $x \in W$, $w, w' \in \mathbf{I}_*$ we have $f_{x,w,w'} = \beta_{x,w,w'} v^{2a(w')} \mod v^{2a(w')-1}\mathbf{Z}[v^{-1}]$ where $\beta_{x,w,w'} \in \mathbf{Z}$. Moreover, if $\beta_{x,w,w'} \neq 0$ then $x \sim w \sim w'$.*

Let $\mathcal{M}$ be the free abelian group with basis $(\tau_w)_{w \in \mathbf{I}_*}$. For $x \in W$, $w \in \mathbf{I}_*$ we set $t_x \tau_w = \sum_{w' \in \mathbf{I}_*} \beta_{x,w,w'} \tau_{w'}$. (The last sum is finite: if $\beta_{x,w,w'} \neq 0$ then $f_{x,w,w'} \neq 0$ and we use the fact that $c_x A_w$ is a well defined element of $\underline{M}$.) We have the following result.

0.6 Theorem. *The bilinear pairing $J \times \mathcal{M} \to \mathcal{M}$ defined by $t_x, \tau_w \mapsto t_x \tau_w$ is a (unital) J-module structure on $\mathcal{M}$.*

The proof is given in §1.

0.7 Notation. Let $\mathbf{C}$ be the field of complex numbers. For any abelian group A we set $\underline{\underline{A}} = \mathbf{C} \otimes A$.

1. Proof of Theorem 0.6

1.1. In this section we assume that W is as in 0.3. For any $x, w \in W$ we have $\dot{c}_x \dot{c}_w \dot{c}_{x^{*-1}} = \sum_{w' \in W} H_{x,w,w'} \dot{c}_{w'}$ where $H_{x,w,w'} \in \underline{\mathcal{A}}$ satisfies

(a) $H_{x,w,w'} = \sum_{y\in W} \dot{h}(x,w,y)\dot{h}(y,x^{*-1},w')$.
From the geometric description of the elements A_w in [**11**] one can deduce that:

(b) *if $x \in W$ and $w, w' \in \mathbf{I}_*$ then there exist elements $H^+_{x,w,w'}, H^-_{x,w,w'}$ of $\mathbf{N}[v,v^{-1}]$ such that $H_{x,w,w'} = H^+_{x,w,w'} + H^-_{x,w,w'}$ and $f_{x,w,w'} = H^+_{x,w,w'} - H^-_{x,w,w'}$.*
(This fact has been already used in [**11**, 5.1] in the case where W is finite and $* = 1$.) Let $n \in \mathbf{Z}$, $x \in W$ and $w, w' \in \mathbf{I}_*$; from (b) we deduce:

(c) *If the coefficient of v^n in $H_{x,w,w'}$ is 0 then the coefficient of v^n in $f_{x,w,w'}$ is 0.*

(d) *If the coefficient of v^n in $H_{x,w,w'}$ is 1 then the coefficient of v^n in $f_{x,w,w'}$ is ± 1.*
We can now prove 0.5(a). Setting $a_0 = a(w')$ we have

$$H_{x,w,w'} = \sum_{y\in W; w'\preceq y} \dot{h}(x,w,y)\dot{h}(y,x^{*-1},w') = \sum_{y\in W; a(y)\le a_0)} \dot{h}(x,w,y)\dot{h}(y,x^{*-1},w')$$
$$= \sum_{y\in W; a(y)\le a_0} (\gamma_{x,w,y^{-1}} v^{a(y)} + \text{ lin.comb.of } v^{a(y)-1}, v^{a(y)-2}, \dots)$$
$$\times (\gamma_{y,x^{*-1},w'^{-1}} v^{a_0} + \text{ lin.comb.of } v^{a_0-1}, v^{a_0-2}, \dots)$$
$$= \sum_{y\in W; a(y)=a_0} \gamma_{x,w,y^{-1}}\gamma_{y,x^{*-1},w'^{-1}})v^{2a_0} + \text{ lin.comb.of } v^{2a_0-1}, v^{2a_0-2}, \dots.$$

Using this and (c) we deduce that

$$f_{x,w,w'} = \beta_{x,w,w'} v^{2a_0} + \text{ lin.comb.of } v^{2a_0-1}, v^{2a_0-2}, \dots)$$

where $\beta_{x,w,w'} \in \mathbf{Z}$ and that if $\beta_{x,w,w'} \ne 0$ then $\gamma_{x,w,y^{-1}} \ne 0, \gamma_{y,x^{*-1},w'^{-1}} \ne 0$ for some $y \in W$. For such y we have $x \sim w \sim y^{-1}$, $y \sim x^* \sim w'^{-1}$, see [**6**, 1.9]. We see that 0.5(a) holds.

The proof above shows also:

(e) *if $\beta_{x,w,w'} \ne 0$ then for some $y \in W$ we have $\gamma_{x,w,y^{-1}} \ne 0, \gamma_{y,x^{*-1},w'^{-1}} \ne 0$.*
We show:

(f) *If $x \in W$ and $w, w' \in \mathbf{I}_*$ satisfy $f_{x,w,w'} \ne 0$ then $w' \preceq w$ and $w' \preceq x$.*
Using (c) we see that $H_{x,w,w'} \ne 0$ hence for some $y \in W$ we have $\dot{h}(x,w,y) \ne 0$ and $\dot{h}(y,x^{-1},w') \ne 0$. It follows that $y \preceq x, y \preceq w, w' \preceq y$ and (f) follows.

1.2. Let $x, y \in W, w \in \mathbf{I}_*$. We show that $(t_x t_y)\tau_w = t_x(t_y \tau_w)$ or equivalently that, for any $w' \in \mathbf{I}_*$,

(a) $\sum_{y'\in W} \gamma_{x,y,y'^{-1}}\beta_{y',w,w'} = \sum_{z\in\mathbf{I}_*} \beta_{x,z,w'}\beta_{y,w,z}$
From the equality $(c_x c_y)A_w = c_x(c_y A_w)$ in $\underline{M}$ we deduce that

(b) $\sum_{y'\in W} h_{x,y,y'} f_{y',w,w'} = \sum_{z\in\mathbf{I}_*} f_{x,z,w'} f_{y,w,z}$.
Let $a_0 = a(w')$. In (b), the sum over y' can be restricted to those y' such that $f_{y',w,w'} \ne 0$ hence (by 1.1(f)) such that $w' \preceq y'$ (hence $a(y') \le a_0$); the sum over z can be restricted to those z such that $f_{x,z,w'} \ne 0$ hence (by 1.1(f)) such that $w' \preceq z$ (hence $a(z) \le a_0$). Thus we have

$$\sum_{y'\in W; a(y')\le a_0} h_{x,y,y'} f_{y',w,w'} = \sum_{z\in\mathbf{I}_*; a(z)\le a_0} f_{x,z,w'} f_{y,w,z}.$$

Using 0.5(a) this can be written as follows

$$\sum_{y'\in W;a(y')\le a_0}(\gamma_{x,y,y'^{-1}}v^{2a(y')}+\text{ lin.comb.of }v^{a(y')-1},v^{a(y')-2},\dots)$$
$$\times(\beta_{y',w,w'}v^{2a_0}+\text{ lin.comb.of }v^{2a_0-1},v^{2a_0-2},\dots)$$
$$=\sum_{z\in\mathbf{I}_*;a(z)\le a_0}(\beta_{x,z,w'}v^{2a_0}+\text{ lin.comb.of }v^{2a_0-1},v^{2a_0-2},\dots)$$
$$\times(\beta_{y,w,z}v^{2a(z)}+\text{ lin.comb.of }v^{2a(z)-1},v^{2a(z)-2},\dots)$$

that is,

$$\sum_{y'\in W;a(y')=a_0}\gamma_{x,y,y'^{-1}}v^{2a_0}\beta_{y',w,w'}v^{2a_0}+\text{ lin.comb.of }v^{4a_0-1},v^{4a_0-2},$$
$$=\sum_{z\in\mathbf{I}_*;a(z)=a_0}\beta_{x,z,w'}v^{2a_0}\beta_{y,w,z}v^{2a_0)}+\text{ lin.comb.of }v^{4a_0-1},v^{4a_0-2},\dots.$$

Taking the coefficient of v^{4a_0} in both sides we obtain

$$\sum_{y'\in W;a(y')=a_0}\gamma_{x,y,y'^{-1}}\beta_{y',w,w'}=\sum_{z\in\mathbf{I}_*;a(z)=a_0}\beta_{x,z,w'}\beta_{y,w,z}.$$

Now, if $\gamma_{x,y,y'^{-1}}\neq 0$ then $a(y')=a_0$ and if $\beta_{x,z,w'}\neq 0$ then $a(z)=a_0$. Hence we deduce

$$\sum_{y'\in W}\gamma_{x,y,y'^{-1}}\beta_{y',w,w'}=\sum_{z\in\mathbf{I}_*}\beta_{x,z,w'}\beta_{y,w,z}.$$

This proves (a).

1.3. Let $w\in\mathbf{I}_*$. We show that $1\tau_w=\tau_w$ or equivalently that, for any $w'\in\mathbf{I}_*$,

(a) $\sum_{d\in\mathcal{D}}\beta_{d,w,w'}=\delta_{w,w'}$

Let d_0 be the unique element of $\mathcal{D}$ contained in the left cell of w^{-1} (see [**6**, 1.10]). If $\beta_{d,w,w'}\neq 0$ with $d\in\mathcal{D}$ then using 1.1(e) we can find $y\in W$ such that $\gamma_{d,w,y^{-1}}\neq 0,\gamma_{y,d^*,w'^{-1}}\neq 0$. (Note that $d^*\in\mathcal{D}$.) Using [**6**, 1.8,1.4,1.9,1.10] we deduce $\gamma_{w,y^{-1},d}\neq 0,\gamma_{w'^{-1},y,d^*}\neq 0$ and $y=w$, $y=w'$, $d=d_0$, $\gamma_{w,y^{-1},d}=\gamma_{w'^{-1},y,d^*}=1$. Thus $\sum_{d\in\mathcal{D}}\beta_{d,w,w'}=\beta_{d_0,w,w'}$ and

$$\sum_{y\in W}\gamma_{d_0,w,y^{-1}}\gamma_{y,d^*,w'^{-1}}=\gamma_{d_0,w,w^{-1}}\gamma_{w,d_0^*,w^{-1}}\delta_{w,w'}=\delta_{w,w'}.$$

Thus the coefficient of $v^{2a(w')}$ in $H_{d_0,w,w'}$ is $\delta_{w,w'}$. Using 1.1(c),(d) we deduce that the coefficient of $v^{2a(w')}$ in $f_{d_0,w,w'}$ is $\pm\delta_{w,w'}$ that is, $\beta_{d_0,w,w'}=\pm\delta_{w,w'}$. Thus

(b) $1\tau_w=\epsilon(w)\tau_w$

where $\epsilon(w)=\pm 1$. Applying $1=\sum_{d\in\mathcal{D}}t_d$ to both sides of (b) and using the identity $(11)\tau_w=1(1\tau_w)$ that is $1\tau_w=1(1\tau_w)$ we obtain $\epsilon(w)\tau_w=1(\epsilon(w)\tau_w)=\epsilon(w)^2\tau_w$ hence $\epsilon(w)^2=\epsilon(w)$. Since $\epsilon(w)=\pm 1$ it follows that $\epsilon(w)=1$. This completes the proof of (a). Theorem 0.6 is proved.

1.4. For any two-sided cell c of W let J_c (resp. $\mathcal{M}_c$) be the subgroup of J (resp. $\mathcal{M}$) generated by $\{t_x;x\in c\}$ (resp. $\{\tau_w;w\in c\cap\mathbf{I}_*\}$. Note that J_c is a subring of J with unit element $1_c=\sum_{d\in\mathcal{D}\cap c}\tau_d$ and $J=\oplus_c J_c$ (direct sum of rings). We have $\mathcal{M}=\oplus_c\mathcal{M}_c$. From the last sentence in 0.5(a) we see that $J_c\mathcal{M}_c\subset\mathcal{M}_c$ and $J_c\mathcal{M}_{c'}=0$ and for any two sided cells $c\neq c'$. It follows that the J-module structure on $\mathcal{M}$ restricts for any c as above to a (unital) J_c-module structure on $\mathcal{M}_c$.

1.5. For any left cell λ of W such that $\lambda = \lambda^*$ let $J_{\lambda\cap\lambda^{-1}}$ (resp. $\mathcal{M}_{\lambda\cap\lambda^{-1}}$) be the subgroup of J (resp. $\mathcal{M}$) generated by $\{t_x; x \in \lambda\cap\lambda^{-1}\}$ (resp. $\{\tau_w; w \in \lambda\cap\lambda^{-1}\cap\mathbf{I}_*\}$. Note that $J_{\lambda\cap\lambda^{-1}}$ is a subring of J with unit element t_d where d is the unique element of $\mathcal{D}\cap\lambda$. Since $\lambda = \lambda^*$ we have $d = d^*$. If $x \in \lambda\cap\lambda^{-1}$, $w \in \lambda\cap\lambda^{-1}\cap\mathbf{I}_*$, $w' \in \mathbf{I}_*$ are such that $\beta_{x,w,w'} \neq 0$ then $w' \in \lambda\cap\lambda^{-1}\cap\mathbf{I}_*$. (Indeed, as we have seen earlier, we have $\gamma_{x,w,y^{-1}} \neq 0$, $\gamma_{y,x^{*-1},w'^{-1}} \neq 0$ for some $y \in W$. For such y we have $x \sim_L w^{-1}$, $w \sim_L y$, $y^{-1} \sim_L x^{-1}$, $y \sim_L x^*$, $x^{*-1} \sim_L w'$, $w'^{-1} \sim_L y^{-1}$, see [**6**, 1.9]. Hence $y \in \lambda$, $y^{-1} \in \lambda$ $w'^{-1} \in \lambda$, $w' \in \lambda^* = \lambda$, so that $w' \in \lambda\cap\lambda^{-1}$, as required.) It follows that the J-module structure on $\mathcal{M}$ restricts for any λ as above to a $J_{\lambda\cap\lambda^{-1}}$-module structure on $\mathcal{M}_{\lambda\cap\lambda^{-1}}$. Now if $d' \in \mathcal{D}-\lambda$, $w \in \lambda\cap\lambda^{-1}\cap\mathbf{I}_*$, $w' \in \mathbf{I}_*$ then $\beta_{d',w,w'} = 0$ so that $t_{d'}\tau_w = 0$. (Indeed, assume that $\beta_{d',w,w'} \neq 0$. Then, as we have seen earlier we have $\gamma_{d',w,y^{-1}} \neq 0$ for some $y \in W$. We then have $d' \sim_L w^{-1}$, see [**6**, 1.9], hence $d' \in \lambda$, contradiction.) Since $1\tau_w = \tau_w$ it follows that $t_d\tau_w = t_w$. We see that the $J_{\lambda\cap\lambda^{-1}}$-module structure on $\mathcal{M}_{\lambda\cap\lambda^{-1}}$ is unital.

2. Γ-equivariant vector bundles

2.1. Let Vec be the category of finite dimensional vector spaces over $\mathbf{C}$.

Let Γ be a finite group and let X be a finite set with a given Γ-action (a Γ-set). A Γ-equivariant $\mathbf{C}$-vector bundle (or Γ-v.b.) V on X is just a collection of objects $V_x \in Vec$ $(x \in X)$ with a given representation of Γ on $\oplus_{x\in X}V_x$ such that $gV_x = V_{gx}$ for all $g \in \Gamma, x \in X$. We say that V_x is the fibre of V at x. Now $X \times X$ is a Γ-set for the diagonal Γ-action. Let $\mathcal{C}_0$ be the category whose objects are the Γ-v.b. on $X \times X$. For $V \in \mathcal{C}_0$ let $V_{x,y} \in Vec$ be the fibre of V at (x,y); for $g \in \Gamma$ let $\mathcal{T}_g : V_{x,y} \to V_{gx,gy}$ be the isomorphism given by the equivariant structure of V.

For $V, V' \in \mathcal{C}_0$ we define the convolution $V\bigstar V' \in \mathcal{C}_0$ by

$(V\bigstar V')_{x,y} = \oplus_{z\in X}V_{x,z} \otimes V'_{z,y}$

for all x, y in X with the obvious Γ-equivariant structure. For $V, V', V'' \in \mathcal{C}_0$ we have an obvious identification $(V\bigstar V')\bigstar V'' = V\bigstar(V'\bigstar V'')$. Let $\mathbf{C}_\delta \in \mathcal{C}_0$ be the Γ-v.b. given by $(\mathbf{C}_\delta)_{x,x} = \mathbf{C}$ for all $x \in X$ and $(\mathbf{C}_\delta)_{x,y} = 0$ for all $x \neq y$ in X (with the obvious Γ-equivariant structure). For $V \in \mathcal{C}_0$ we have obvious identifications $\mathbf{C}_\delta\bigstar V = V = V\bigstar\mathbf{C}_\delta$. Define $\sigma : X \times X \to X \times X$ by $\sigma(x,y) = (y,x)$. For $V \in \mathcal{C}_0$ we set $V^\sigma = \sigma^*V$ that is $V^\sigma_{x,y} = V_{y,x}$ for all x, y in X. For $V, V' \in \mathcal{C}_0$ we have an obvious identification $(V\bigstar V')^\sigma = V'^\sigma\bigstar V^\sigma$. Note that $\bigstar$ is compatible with direct sums in both the V and V' factor. Hence if $K(\mathcal{C}_0)$ is the Grothendieck group of $\mathcal{C}_0$ then $\bigstar$ induces an associative ring structure on $K(\mathcal{C}_0)$ with unit element defined by $\mathbf{C}_\delta$; moreover, $V \mapsto V^\sigma$ induces an antiautomorphism of the ring $K(\mathcal{C}_0)$. Thus $\underline{\underline{K}}(\mathcal{C}_0)$ is an associative $\mathbf{C}$-algebra with 1.

2.2. Let $\mathcal{C}$ be the category whose objects are pairs (U, κ) where $U \in \mathcal{C}_0$ and $\kappa : U \xrightarrow{\sim} U^\sigma$ is an isomorphism in $\mathcal{C}_0$ (that is a collection of isomorphisms $\kappa_{x,y}$: $U_{x,y} \to U_{y,x}$ for each $x, y \in X$ such that $\kappa_{gx,gy}\mathcal{T}_g = \mathcal{T}_g\kappa_{x,y}$ for all $g \in \Gamma, \xi, y \in X$); it is assumed that $\kappa_{y,x}\kappa_{x,y} = 1 : U_{x,y} \to U_{x,y}$ for all $x, y \in X$.

For $V \in \mathcal{C}_0$ we define an isomorphism $\zeta : V \oplus V^\sigma \to (V \oplus V^\sigma)^\sigma = V^\sigma \oplus V$ by $a \oplus b \mapsto b \oplus a$. We have $(V \oplus V^\sigma, \zeta) \in \mathcal{C}$ and $V \mapsto (V \oplus V^\sigma, \zeta)$ can be viewed as functor $\Theta : \mathcal{C}_0 \to \mathcal{C}$. Let $K(\mathcal{C})$ be the Grothendieck group of $\mathcal{C}$ and let $K'(\mathcal{C})$ be the subgroup of $K(\mathcal{C})$ generated by the elements of the form $\Theta(V)$ with $V \in \mathcal{C}_0$. Let $\bar{K}(\mathcal{C}) = K(\mathcal{C})/K'(\mathcal{C})$. (This definition of $\bar{K}(\mathcal{C})$ is a special case of a definition in [**9**, 11.1.5] which applies to a category with a periodic functor.) Note that if $(U,\kappa)\in\mathcal{C}$ then $(U,-\kappa)\in\mathcal{C}$ and $(U,\kappa)+(U,-\kappa)=0$ in $\bar{K}(\mathcal{C})$.

For $V\in\mathcal{C}_0,(U,\kappa)\in\mathcal{C}$ we define $V\circ(U,\kappa)\in\mathcal{C}$ by $V\circ(U,\kappa)=(V\bigstar U\bigstar V^\sigma,\kappa')$ where for x,y in X,

$\kappa'_{x,y}:\oplus_{z,z'\in X}V_{x,z}\otimes U_{z,z'}\otimes V_{y,z'}\to\oplus_{z',z\in X}V_{y,z'}\otimes U_{z',z}\otimes V_{x,z}$
maps $a\otimes b\otimes c$ (in the z,z' summand) to $c\otimes\kappa(b)\otimes a$ (in the z',z summand). Now let $V,V'\in\mathcal{C}_0$ and $(U,\kappa)\in\mathcal{C}$. We have canonically

$(V\oplus V')\circ(U,\kappa)=V\circ(U,\kappa)\oplus V'\circ(U,\kappa)\oplus\Theta(V\bigstar U\bigstar V'^\sigma)$.
Moreover, we have canonically $V\circ\Theta(V')=\Theta(V\bigstar V'\bigstar V^\sigma)$. For $V,V'\in\mathcal{C}_0$ and $(U,\kappa)\in\mathcal{C}$ we have an obvious identification $(V'\bigstar V)\circ(U,\kappa)=V'\circ(V\circ(U,\kappa))$. For $(U,\kappa)\in\mathcal{C}$ we have an obvious identification $\mathbf{C}_\delta\circ(U,\kappa)=(U,\kappa)$. We see that $\circ$ defines a (unital) $K(\mathcal{C}_0)$-module structure on $\bar{K}(\mathcal{C})$ (but not on $K(\mathcal{C})$). Hence $\underline{\underline{\bar{K}}}(\mathcal{C})$ is naturally a (unital) $\underline{\underline{K}}(\mathcal{C}_0)$-module.

2.3. Note that $K(\mathcal{C}_0)$ has a **Z**-basis consisting of the the isomorphism classes of indecomposable Γ-v.b. V on $X\times X$ (these are indexed by a Γ-orbit in $X\times X$ and an irreducible representation of the isotropy group of a point in that orbit). Moreover, $\bar{K}(\mathcal{C})$ has a signed **Z**-basis consisting of the classes of (V,κ) where V is an indecomposable Γ-v.b. on $X\times X$ satisfying $V\cong V^\sigma$ and κ is defined up to a sign (so the class of (V,κ) is defined up to a sign).

2.4. Let $\mathcal{C}_\Gamma$ be the category of Γ-v.b. on Γ viewed as a Γ-set under conjugation. An object Y of $\mathcal{C}_\Gamma$ is a collection of objects $Y_g\in Vec$ $(g\in\Gamma)$ with a given representation of Γ on $\oplus_{g\in\Gamma}Y_g$ such that $gY_{g'}=Y_{gg'g^{-1}}$ for all $g,g'\in\Gamma,x\in X$. For $Y,Y'\in\mathcal{C}_\Gamma$ we define the convolution $Y\bigstar Y'\in\mathcal{C}_\Gamma$ by

$$(Y\bigstar Y')_g=\oplus_{g_1,g_2\in\Gamma;g_1g_2=g}Y_{g_1}\otimes Y'_{g_2}$$

for all $g\in\Gamma$ with the obvious Γ-equivariant structure. This defines a structure of associative ring with 1 on the Grothendieck group $K(\mathcal{C}_\Gamma)$. The unit element is given by the Γ-v.b. whose fibre at $g=1$ is $\mathbf{C}$ and whose fibre at any other element is 0. Hence $\underline{\underline{K}}(\mathcal{C}_\Gamma)$ is an associative $\mathbf{C}$-algebra with 1; by [**8**, 2.2], it is commutative and semisimple.

With $X,\mathcal{C}_0,\mathcal{C}$ as in 2.1, for any $Y\in\mathcal{C}_G$, we define as in [**8**, 2.2(h)] an object $\Psi(Y)\in\mathcal{C}_0$ by $\Psi(Y)_{x,y}=\oplus_{g\in\Gamma;x=gy}Y_g$ (with the obvious equivariant structure). Now $Y\mapsto\Psi(Y)$ defines a ring homomorphism $K(\mathcal{C}_\Gamma)\to K(\mathcal{C}_0)$ and a $\mathbf{C}$-algebra homomorphism $\underline{\underline{K}}(\mathcal{C}_\Gamma)\to\underline{\underline{K}}(\mathcal{C}_0)$. By [**8**, 2.2],

(a) $\underline{\underline{K}}(\mathcal{C}_0)$ *is a semisimple* $\mathbf{C}$*-algebra and the image of the homomorphism* $\underline{\underline{K}}(\mathcal{C}_\Gamma)\to\underline{\underline{K}}(\mathcal{C}_0)$ *is exactly the centre of* $\underline{\underline{K}}(\mathcal{C}_0)$.
We see that the $K(\mathcal{C}_0)$-module structure on $\bar{K}(\mathcal{C})$ restricts to a $K(\mathcal{C}_\Gamma)$-module structure on $\bar{K}(\mathcal{C})$ in which the product of the class of $Y\in\mathcal{C}_\Gamma$ with the class of $(V,\kappa)\in\mathcal{C}$ is the class of $(V',\kappa')\in\mathcal{C}$ where

$$V'_{x,y}=\oplus_{g,g'\in\Gamma,z,z'\in X;x=gz,y=g'z'}Y_g\otimes V_{z,z'}\otimes Y_{g'}$$

that is,

(b) $$V'_{x,y}=\oplus_{g,g'\in\Gamma}Y_g\otimes V_{g^{-1}x,g'^{-1}y}\otimes Y_{g'}$$

and, for x,y in X,

$$\kappa'_{x,y}:\oplus_{g,g'\in\Gamma}Y_g\otimes V_{g^{-1}x,g'^{-1}y}\otimes Y_{g'}\to\oplus_{g',g\in\Gamma}Y_{g'}\otimes V_{g'^{-1}y,g^{-1}x}\otimes Y_g$$

maps $a \otimes b \otimes c$ (in the g, g' summand) to $c \otimes \kappa(b) \otimes a$ (in the g', g summand). It follows also that $\underline{\underline{\bar{K}}}(\mathcal{C})$ is naturally a $\underline{\underline{K}}(\mathcal{C}_\Gamma)$-module.

Now assume in addition that

(c) Γ is an elementary abelian 2-group

and that $Y \in \mathcal{C}_\Gamma$ is such that for some $g_0 \in \Gamma$, $Y|_{\Gamma - \{g_0\}}$ is zero and $\dim Y_{g_0} = 1$. Then (b) becomes

$$V'_{x,y} = Y_{g_0} \otimes V_{g_0 x, g_0 y} \otimes Y_{g_0}$$

Now $Y_{g_0} \otimes Y_{g_0}$ is isomorphic to $\mathbf{C}$ as a representation of Γ and $V_{g_0 x, g_0 y}$ is canonically isomorphic to $V_{x,y}$. We see that $(V', \kappa') = (V, \kappa)$. Thus Y acts as identity in the $K(\mathcal{C}_\Gamma)$-module structure of $\bar{K}(\mathcal{C})$. It follows that if Y is any object of $\mathcal{C}_\Gamma$ then Y acts in the $K(\mathcal{C}_\Gamma)$-module structure of $\bar{K}(\mathcal{C})$ as multiplication by $\nu(Y) = \sum_{g \in \Gamma} \dim Y_g$. Note that ν defines a ring homomorphism $K(\mathcal{C}_\Gamma) \to \mathbf{Z}$ and a $\mathbf{C}$-algebra homomorphism $\underline{\underline{K}}(\mathcal{C}_\Gamma) \to \mathbf{C}$ (taking 1 to 1). We see that:

(d) *If Γ is as in (c) then for any $\xi \in \underline{\underline{K}}(\mathcal{C}_\Gamma), \xi' \in \underline{\underline{\bar{K}}}(\mathcal{C})$ we have $\xi\xi' = \nu(\xi)\xi'$. In particular, the $\underline{\underline{K}}(\mathcal{C}_\Gamma)$-module $\underline{\underline{\bar{K}}}(\mathcal{C})$ is ν-isotypic.*

Using this and (a) we see that the first assertion in (e) below holds.

(e) *If Γ is as in (c) then the $\underline{\underline{K}}(\mathcal{C}_0)$-module $\underline{\underline{\bar{K}}}(\mathcal{C})$ is isotypic. Moreover* $\dim_{\mathbf{C}} \underline{\underline{\bar{K}}}(\mathcal{C})$ *is equal to $|\Gamma|$ times the number of Γ-orbits in X.*

We now prove the second assertion in (e). By 2.3, $\dim_{\mathbf{C}} \underline{\underline{\bar{K}}}(\mathcal{C})$ is equal to $n_{\Gamma,X}$, the number of indecomposable Γ-v.b. on $X \tau X$ (up to isomorphism) such that $V \cong V^\sigma$. For such V there exists a unique Γ-orbit $\mathcal{O}$ on X such that $V_{x,y} \neq 0$ implies $x \in \mathcal{O}$ and $y \in \mathcal{O}$. Hence $n_{\Gamma,X} = \sum_{\mathcal{O}} n_{\Gamma,\mathcal{O}}$ where $\mathcal{O}$ runs over the Γ-orbits in X. This reduces the proof to the case where X is a single Γ-orbit. Let H be the isotropy group in Γ of some point in X; this is independent of the choice of point since Γ is commutative. Now any $(x, y) \in X \times X$ is in the same Γ-orbit as (y, x). (Indeed, we can find $g \in \Gamma$ such that $y = gx$. Then (x, y) is in the same orbit as $(gx, gy) = (y, g^2 x) = (y, x)$ since $g^2 = 1$.) For a given Γ-orbit $\mathcal{O}'$ in $X \tau X$ the number of indecomposable Γ-v.b. on $X \times X$ (up to isomorphism) with support equal to $\mathcal{O}'$ is the number of characters of characters of the isotropy group of any point in the orbit which is H. Thus $n_{\Gamma,X}$ is equal to $|H|$ times the number of Γ-orbits in $X \times X$ that is to $|H| \times |\Gamma/H| = |\Gamma|$. This proves (e).

2.5. In this subsection we assume that W is an irreducible Weyl group and $* = 1$. Let c be a two-sided cell of W such that for $z \in c$ we have $a(z) \neq 11$ (if W is of type E_7) and $a(z) \neq 11, a(z) \neq 26$ (if W is of type E_8). Let Γ be the finite group associated to c in [**8**, 3.15]. For each left cell λ in c let Γ_λ be the subgroup of Γ associated to λ in [**8**]. Let $X = \sqcup_\lambda (\Gamma/\Gamma_\lambda)$ (λ runs over the left cells in c). Note that Γ acts naturally on X. For any λ as above let $\mathbf{C}_\lambda$ be the Γ-v.b. on $X \times X$ which is $\mathbf{C}$ at any point of form (x, x), $x \in \Gamma/G_\lambda$, and is zero at all other points. Let $\mathcal{C}_0$ be defined in terms of this X. The following statement was conjectured in [**8**, 3.15] and proved in [**1**]:

(a) *There exists a isomorphism $\phi : J_c \xrightarrow{\sim} K(\mathcal{C}_0)$ which carries the basis $\{t_x; x \in c\}$ onto the canonical basis 2.3 of $K(\mathcal{C}_0)$ and is such that for any left cell λ in c, $\phi(t_d)$ (where $\mathcal{D} \cap \lambda = \{d\}$) is the class of $\mathbf{C}_\lambda$.*

The isomorphism ϕ has the following property conjectured in [**7**, 10.5(b)] in a closely related situation.

(b) *Let $x \in c$ and let $\phi(t_x)$ be the class of the indecomposable Γ-v.b. V on $X\tau X$. Then $\phi(t_{x^{-1}})$ is the class of $(\check{V})^\sigma$ where $\check{V}$ is the dual Γ-v.b. to V.*
(As R. Bezrukavnikov pointed out to me, (b) follows immediately from (a).) Next we note the following property.

(c) $\check{V} \cong V$ for any Γ-v.b. on $X \times X$.
It is enough to show that for any $(x, y) \in X \times X$, the stabilizer of (x, y) in Γ (that is the intersection of a Γ-conjugate of Γ_λ with a Γ-conjugate of $\Gamma_{\lambda'}$ where λ, λ' are two left cells in c) is isomorphic to a Weyl group (hence its irreducible representations are selfdual). This can be verified from the explicit description of the subgroups Γ_λ in [**8**].

Using (b),(c) we see that ϕ has the following property.

(d) *Let $x \in c$ and let $\phi(t_x)$ be the class of the indecomposable Γ-v.b. V on $X\tau X$. Then $\phi(t_{x^{-1}})$ is the class of V^σ.*
I want to formulate a refinement of (a).

(e) *Conjecture. There exists an isomorphism of abelian groups $\psi : \mathcal{M}_c \to \bar{K}(\mathcal{C})$ ($\mathcal{C}$ is defined in terms of X) with the following properties*:

-if $w \in c \cap \mathbf{I}_*$ and $\phi(t_w) = V$ (an indecomposable Γ-v.b. such that $V \cong V^\sigma$, see (d)) then $\psi(\tau_w) = (V, \kappa)$ for a unique choice of $\kappa : V \xrightarrow{\sim} V^\sigma$;

-the J_c-module structure on $\mathcal{M}_c$ corresponds under ϕ and ψ to the $K(\mathcal{C}_0)$-module structure on $\bar{K}(\mathcal{C})$.
Now let $\underline{\underline{J}}_c = \mathbf{C} \otimes J_c$, $\underline{\underline{\mathcal{M}}}_c = \mathbf{C} \otimes \mathcal{M}_c$. Note that $\underline{\underline{J}}_c$ is a semisimple algebra, see [**8**, 1.2, 3.1(j)]. Assuming that (e) holds we deduce:

(f) *If Γ is as in 2.4(c) then the $\underline{\underline{J}}_c$-module $\underline{\underline{\mathcal{M}}}_c$ is isotypic. Moreover,* $\dim_{\mathbf{C}} \underline{\underline{\mathcal{M}}}_c$ *is equal to $|\Gamma|$ times the number of left cells contained in c.*
(Note that the number of Γ-orbits on X is equal to the number of left cells contained in c.)

Now if W is of classical type, then Γ is as in 2.4(c) and (f) gives an explanation for the known structure of the W-module obtained from M for $u = 1$ (a consequence of the results of Kottwitz [**3**]); this can be viewed as evidence for the conjecture (e). (In this case, the second assertion of (f) was already known in [**4**, 12.17].)

Here we use the following property which can be easily verified for any Weyl group.

(g) Let $\underline{M}_{\preceq c}$ (resp. $\underline{M}_{\preceq c-c}$) be the $\underline{\mathcal{A}}$ submodule of $\underline{M}$ spanned by $\{A_x; x \preceq y \text{ for some } y \in c\}$ (resp. $\{A_x; x \preceq y \text{ for some } y \in c, x \notin c\}$). The decomposition pattern of the (semisimple) $\underline{\underline{J}}_c$-module $\underline{\underline{\mathcal{M}}}_c$ is the same as the decomposition pattern of the (semisimple) $\mathbf{C}(v) \otimes_{\underline{\mathcal{A}}} \underline{\mathfrak{H}}$-module $\mathbf{C}(v) \otimes_{\underline{\mathcal{A}}} (\underline{M}_{\preceq c}/\underline{M}_{\preceq c-c})$; in particular if the first module is isotypic then so is the second module.
One can show, using results in [**6**, 2.8, 2.9], that this property also holds when W is replaced by an affine Weyl group and c by a finite two-sided cell in that affine Weyl group.

3. A conjectural realization of the $\mathfrak{H}$-module M

3.1. Let $\mathfrak{H}^\bullet = \mathbf{Q}(u) \otimes_{\mathcal{A}} \mathfrak{H}$ (an algebra over $\mathbf{Q}(u)$) and let $M^\bullet = \mathbf{Q}(u) \otimes_{\mathcal{A}} M$. We regard $\mathfrak{H}$ as a subset of $\mathfrak{H}^\bullet$ and M as a subset of $M^\bullet$ by $\xi \mapsto 1 \otimes \xi$. The $\mathfrak{H}$-module structure on M extends in an obvious way to an $\mathfrak{H}^\bullet$-module structure on $M^\bullet$. Let $\hat{\mathfrak{H}}$ be the vector space consisting of all formal (possibly infinite) sums $\sum_{x \in W} c_x T_x$ where $c_x \in \mathbf{Q}(u)$. We can view $\mathfrak{H}^\bullet$ as a subspace of $\hat{\mathfrak{H}}$ in an obvious way. The $\mathfrak{H}^\bullet$-module structure on $\mathfrak{H}^\bullet$ (left multiplication) extends in an obvious way to a $\mathfrak{H}^\bullet$-module structure on $\hat{\mathfrak{H}}$. We set

$$X_\emptyset = \sum_{x\in W; x^*=x} u^{-l(x)} T_x \in \hat{\mathfrak{H}}.$$

Let $\mathbf{M} = \mathfrak{H}^\bullet X_\emptyset$ be the $\mathfrak{H}^\bullet$-submodule of $\hat{\mathfrak{H}}$ generated by $X_\emptyset$. In this section we will give a conjectural realization of the $\mathfrak{H}^\bullet$-module $M^\bullet$ in terms of $\mathbf{M}$.

We write $S = \{s_i; i \in I\}$ where I is an indexing set. For any sequence $i_1, i_2, \dots, i_k$ in I we write $i_1 i_2 \dots i_k$ instead of $s_{i_1} s_{i_2} \dots s_{i_k} \in W$.

3.2. In this subsection we assume that W is of type A_2, $* = 1$ and $S = \{s_1, s_2\}$. We set

$X_\emptyset = (-u)^{-3} T_{121} + u^{-2} T_{12} + u^{-2} T_{21} + u^{-1} T_1 + u^{-1} T_2 + 1,$

$X_1 = (1+u)^{-1}(T_1 - u) X'_\emptyset = (u-1)(u^{-3} T_{121} + u^{-2} T_{12} + u^{-1} T_1),$

$X_2 = (1+u)^{-1}(T_2 - u) X'_\emptyset = (u-1)(u^{-3} T_{121} + u^{-2} T_{21} + u^{-1} T_2),$

$X_{121} = T_1 X_2 = T_2 X_1 = (u-1)((u^{-1} + u^{-2} - u^{-3}) T_{121} + u^{-1} T_{12} + u^{-1} T_{21}).$

Clearly, $X_\emptyset, X_1, X_2, X_{121}$ form a basis of $\mathbf{M}$. In the $\mathfrak{H}^\bullet$-module $M^\bullet$ we have

$a_1 = (u+1)^{-1}(T_1 - u) a_\emptyset$, $a_2 = (u+1)^{-1}(T_2 - u) a_\emptyset$, $a_{121} = T_1 a_2 = T_2 a_1$.

We see that we have a (unique) isomorphism of $\mathfrak{H}^\bullet$-modules $\mathbf{M} \xrightarrow{\sim} M^\bullet$ such that $X_\emptyset \mapsto a_\emptyset, X_1 \mapsto a_1, X_2 \mapsto a_2, X_{121} \mapsto a_{121}$.

3.3. In this subsection we assume that W is of type A_1, $* = 1$ and $S = \{s_1\}$. We set $X_\emptyset = u^{-1} T_1 + 1$, $X_1 = (u-1) u^{-1} T_1$. Clearly, $X_\emptyset, X_1$ form a basis of $\mathbf{M}$. In the $\mathfrak{H}^\bullet$-module $M^\bullet$ we have $a_1 = (u+1)^{-1}(T_1 - u) a_\emptyset$. We see that we have a (unique) isomorphism of $\mathfrak{H}^\bullet$-modules $\mathbf{M} \xrightarrow{\sim} M^\bullet$ such that $X_\emptyset \mapsto a_\emptyset, X_s \mapsto a_1$.

3.4. We return to the setup in 3.1. Based on the examples in 3.2, 3.3 we state:

(a) *Conjecture. There exists a unique isomorphism of* $\mathfrak{H}^\bullet$*-modules* $\eta : \mathbf{M} \xrightarrow{\sim} M^\bullet$ *such that* $X_\emptyset \mapsto a_\emptyset$.

By 3.3, 3.2, conjecture (a) is true when W is of type A_1, A_2 (wth $* = 1$). It can be shown that it is also true when W is a dihedral group (any $*$) or of type A_3 (any $*$).

Assuming that (a) holds we set $X_w = \eta^{-1}(a_w)$ for any $w \in \mathbf{I}_*$.

3.5. We describe below the elements X_w for various $w \in \mathbf{I}_*$ when W is of type A_3 and $* = 1$. We write $S = \{s_1, s_2, s_3\}$ $(s_1 s_3 = s_3 s_1)$.

$X_\emptyset$ has been described in 3.4;

$$X_1 = (u-1)(u^{-1} T_1 + u^{-2} T_{12} + u^{-2} T_{13} + u^{-3} T_{121} + u^{-3} T_{123} + u^{-3} T_{132} + u^{-4} T_{1213} + u^{-4} T_{1232} + u^{-4} T_{1321} + u^{-5} T_{13213} + u^{-5} T_{12132} + u^{-6} T_{121321});$$

$$X_3 = (u-1)(u^{-1} T_3 + u^{-2} T_{32} + u^{-2} T_{13} + u^{-3} T_{323} + u^{-3} T_{321} + u^{-3} T_{132} + u^{-4} T_{3231} + u^{-4} T_{3212} + u^{-4} T_{1323} + u^{-5} T_{13213} + u^{-5} T_{32312} + u^{-6} T_{121321});$$

$$X_2 = (u-1)(u^{-1} T_2 + u^{-2} T_{21} + u^{-2} T_{23} + u^{-3} T_{121} + u^{-3} T_{323} + u^{-3} T_{213} + u^{-4} T_{1213} + u^{-4} T_{3231} + u^{-4} T_{2132} + u^{-5} T_{32312} + u^{-5} T_{12132} + u^{-6} T_{121321});$$

$$X_{13} = (u-1)^2(u^{-2} T_{13} + u^{-3} T_{132} + u^{-4} T_{1321} + u^{-4} T_{1323} + u^{-5} T_{13213} + u^{-6} T_{121321});$$

$$X_{121} = (u-1)(u^{-1}T_{12} + u^{-1}T_{21} + (u^{-1}+u^{-2}-u^{-3})T_{121} + u^{-2}T_{123}$$
$$+ u^{-2}T_{213} + (u^{-2}+u^{-3}-u^{-4})T_{1213} + u^{-3}T_{1323} + u^{-3}T_{2132}$$
$$+ (u^{-3}+u^{-4}-u^{-5})T_{12132} + u^{-4}T_{13213} + u^{-4}T_{21321}$$
$$+ (u^{-4}+u^{-5}-u^{-6})T_{121321});$$

$$X_{323} = (u-1)(u^{-1}T_{32} + u^{-1}T_{23} + (u^{-1}+u^{-2}-u^{-3})T_{323} + u^{-2}T_{321}$$
$$+ u^{-2}T_{213} + (u^{-2}+u^{-3}-u^{-4})T_{3213} + u^{-3}T_{1321} + u^{-3}T_{2132}$$
$$+ (u^{-3}+u^{-4}-u^{-5})T_{32132} + u^{-4}T_{13213} + u^{-4}T_{21323}$$
$$+ (u^{-4}+u^{-5}-u^{-6})T_{121321});$$

$$X_{2132} = (u-1)^2(u^{-2}T_{213} + u^{-3}T_{2132} + u^{-4}T_{21321}$$
$$+ u^{-4}T_{21323} + u^{-4}T_{12321} + (u^{-4}+u^{-5}-u^{-6})T_{121321});$$

$$X_{13213} = (u-1)(u^{-1}T_{132} + u^{-1}T_{123} + u^{-1}T_{321} + (u^{-1}+u^{-2}-u^{-3})T_{1321}$$
$$+ u^{-2}T_{3213} + (u^{-1}+u^{-2}-u^{-3})T_{1323} + u^{-2}T_{1213} + u^{-2}T_{2132}$$
$$+ (u^{-2}+u^{-3}-u^{-4})T_{21323} + (u^{-2}+u^{-3}-u^{-4})T_{21321}$$
$$+ (2u^{-2}+u^{-3}-2u^{-4})T_{13213} + (u^{-2}+2u^{-3}-u^{-4}-2u^{-5}+u^{-6})T_{121321});$$

$$X_{213213} = (u-1)^2(u^{-2}T_{1213} + u^{-2}T_{2132} + u^{-2}T_{2321}$$
$$+ (u^{-2}+u^{-3}-u^{-4})T_{21323} + (u^{-2}+u^{-3}-u^{-4})T_{21321}$$
$$+ (u^{-2}-u^{-4})T_{13231} + (u^{-2}+u^{-3}-u^{-4}-u^{-5}+u^{-6})T_{121321}).$$

3.6. We describe below the elements X_w for various $w \in \mathbf{I}_*$ when W is an infinite dihedral group and $* = 1$. We write $S = \{s_1, s_2\}$. $X_\emptyset$ has been described in 3.4;

$$X_1 = (u-1)(u^{-1}T_1 + u^{-2}T_{12} + u^{-3}T_{121} + u^{-4}T_{1212} + \dots);$$

$$X_2 = (u-1)(u^{-1}T_2 + u^{-2}T_{21} + u^{-3}T_{212} + u^{-4}T_{2121} + \dots);$$

$$X_{121} = (u-1)(u^{-1}T_{12} + u^{-2}T_{121} + u^{-3}T_{1212} + u^{-4}T_{12121} + \dots);$$

$$X_{212} = (u-1)(u^{-1}T_{21} + u^{-2}T_{212} + u^{-3}T_{2121} + u^{-4}T_{21212} + \dots);$$

$$X_{12121} = (u-1)(u^{-1}T_{121} + u^{-2}T_{1212} + u^{-3}T_{12121} + u^{-4}T_{121212} + \dots);$$

$$X_{21212} = (u-1)(u^{-1}T_{212} + u^{-2}T_{2121} + u^{-3}T_{21212} + u^{-4}T_{212121} + \dots);$$

$$X_{1212121} = (u-1)(u^{-1}T_{1212} + u^{-2}T_{12121} + u^{-3}T_{121212} + u^{-4}T_{1212121} + \dots);$$

$$X_{2121212} = (u-1)(u^{-1}T_{2121} + u^{-2}T_{21212} + u^{-3}T_{212121} + u^{-4}T_{2121212} + \dots);$$

$$\dots$$

3.7. Assume that Conjecture 3.4(a) holds for $W, *$. From the examples above we see that it is likely that the elements X_w ($w \in \mathbf{I}_*$) are formal $\mathbf{Z}[u^{-1}]$-linear combinations of elements T_x ($x \in W$). In particular the specializations $(X_w)_{u^{-1}=0}$ are well defined $\mathbf{Z}$-linear combinations of T_x ($x \in W$). From the example above it appears that there is a well defined (surjective) function $\pi : W \to \mathbf{I}_*$ such that $(X_w)_{u^{-1}=0} = \sum_{x \in \pi^{-1}(w)} T_x$. We describe the sets $\pi^{-1}(w)$ in a few cases with $* = 1$.

If W is of type A_1 we have $\pi^{-1}(\emptyset) = \{\emptyset\}, \pi^{-1}(1) = \{1\}$.

If W is of type A_2 we have
$\pi^{-1}(\emptyset) = \{\emptyset\}, \pi^{-1}(1) = \{1\}, \pi^{-1}(2) = \{2\}, \pi^{-1}(121) = \{12, 21, 121\}$.

If W is of type A_3 we have
$\pi^{-1}(\emptyset) = \{\emptyset\}, \pi^{-1}(1) = \{1\}, \pi^{-1}(2) = \{2\}, \pi^{-1}(3) = \{3\}, \pi^{-1}(13) = \{13\}$,
$\pi^{-1}(121) = \{12, 21, 121\}, \pi^{-1}(323) = \{32, 23, 323\}, \pi^{-1}(2132) = \{213\}$,
$\pi^{-1}(13213) = \{132, 123, 321, 1321, 1323\}$,
$\pi^{-1}(121321) = \{1213, 2132, 2321, 21323, 21321, 13231, 121321\}$.

If W is infinite dihedral we have
$\pi^{-1}(\emptyset) = \{\emptyset\}, \pi^{-1}(1) = \{1\}, \pi^{-1}(2) = \{2\}, \pi^{-1}(121) = \{12\}, \pi^{-1}(212) = \{21\}$,
$\pi^{-1}(12121) = \{121\}, \pi^{-1}(21212) = \{212\}, \pi^{-1}(1212121) = \{1212\}$,
$\pi^{-1}(2121212) = \{2121\}, \dots$

In each of these examples π is given by the following inductive rule. We have $\pi(\emptyset) = \emptyset$. If $x \in W$ is of the form $x = s_i x'$ with $i \in I$, $x' \in W$, $l(x) > l(x')$ so that $\pi(x')$ can be assumed known, then

$\pi(x) = s_i \pi(x')$ if $s_i \pi(x') = \pi(x') s_i > \pi(x')$,
$\pi(x) = s_i \pi(x') s_i$ if $s_i \pi(x') \neq \pi(x') s_i > \pi(x')$,
$\pi(x) = \pi(x')$ if $\pi(x') s_i < \pi(x')$.

In each of the examples above the following holds: if $x \in W$, $w = \pi(x) \in \mathbf{I}_*$, then $l(w) = l(x) + l(x^{-1}w)$. We expect that these properties hold in general.

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