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THE CANONICAL BASIS OF THE QUANTUM ADJOINT REPRESENTATION

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Introduction

0.1. According to Drinfeld and Jimbo, the universal enveloping algebra of a simple split Lie algebra $\mathfrak{g}$ over $\mathbf{Q}$ admits a remarkable deformation $\mathbf{U}$ (as a Hopf algebra over $\mathbf{Q}(v)$, where v is an indeterminate) called a quantized enveloping algebra. Moreover, the irreducible finite dimensional $\mathfrak{g}$-modules admit quantum deformation to become simple $\mathbf{U}$-modules. In [L3], I found that these quantum deformations admit canonical bases with very favourable properties (at least when $\mathfrak{g}$ is of type A, D or E) which give also rise by specialization to canonical bases of the corresponding simple $\mathfrak{g}$-modules. (Later, Kashiwara [Ka] found another approach to the canonical bases.) In this paper we are interested in the canonical basis of the quantum deformation Λ of the adjoint representation of $\mathfrak{g}$. Before the introduction of the canonical bases, in [L1], [L2], I found a basis of Λ in which the generators E_i, F_i of $\mathbf{U}$ act through matrices whose entries are polynomials in $\mathbf{N}[v]$. By specialization, this gives rise to a basis of the adjoint representation of $\mathfrak{g}$ in which the Chevalley generators e_i, f_i of $\mathfrak{g}$ act through matrices whose entries are natural numbers, in contrast with the more traditional treatments where a multitude of signs appear.

In this paper (Section 1) I will prove that the basis of Λ from [L1], [L2] coincides with the canonical basis of Λ. I thank Meinolf Geck for suggesting that I should write down this proof. As an application (Section 2), I will give a definition of the Chevalley group over a field k associated to $\mathfrak{g}$ which seems to be simpler than Chevalley's original definition [Ch].

0.2. Let I be a finite set with a given $\mathbf{Z}$-valued symmetric bilinear form $y, y' \mapsto y \cdot y'$ on $Y = \mathbf{Z}[I]$ such that the symmetric matrix $(i \cdot j)_{i,j \in I}$ is positive definite and such that $i \cdot i/2 \in \{1, 2, 3, \dots\}$ for all $i \in I$, $i \cdot i/2 = 1$ for some $i \in I$ and $2\frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \dots\}$ for all $i, j \in I$. In the terminology of [L4, 1.1.1, 2.1.3], this is a *Cartan datum* of finite type. We shall assume that our Cartan datum is irreducible (see [L4, 2.1.3]). Let e be the maximum value of $i \cdot i/2$ for $i \in I$. We

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have $e \in \{1,2,3\}$. Let $I^1 = \{i \in I; i \cdot i/2 = 1\}$, $I^e = \{i \in I; i \cdot i/2 = e\}$. If $e = 1$ we have clearly $I^1 = I^e = I$; if $e > 1$, we have $I = I^1 \sqcup I^e$.

Let $X = \mathrm{Hom}(Y, \mathbf{Z})$ and let $\langle , \rangle : Y \times X \to \mathbf{Z}$ be the obvious pairing. For $j \in I$ we define $j' \in X$ by $\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i}$ for all $i \in I$. Let v be an indeterminate. For $i \in I$ we set $v_i = v^{i \cdot i/2}$; for $n \in \mathbf{Z}$ we set $[n]_i = \frac{v_i^n - v_i^{-n}}{v_i - v_i^{-1}}$; for $n \in \mathbf{N}$ we set $[n]_i^! = \prod_{s=1}^n [s]_i$.

Note that when $i \in I^1$ we have $v_i = v$ and we write $[n]$ instead of $[n]_i$.

0.3. Following Drinfeld and Jimbo we define $\mathbf{U}$ to be the associative $\mathbf{Q}(v)$-algebra with generators E_i, F_i $(i \in I)$, K_y $(y \in Y)$ and relations

$$K_y K_{y'} = K_{y+y'} \text{ for } y, y' \text{ in } Y,$$

$$K_i E_j = v^{\langle i, j' \rangle} E_j K_i \text{ for } i, j \text{ in } I,$$

$$K_i F_j = v^{-\langle i, j' \rangle} F_j K_i \text{ for } i, j \text{ in } I,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i^{i \cdot i/2} - K_i^{-i \cdot i/2}}{v_i - v_i^{-1}},$$

$$\sum_{p, p' \in \mathbf{N}; p+p' = 1 - \langle i, j' \rangle} (-1)^{p'} \frac{[p+p']_i^!}{[p]_i^! [p']_i^!} E_i^p E_j E_i^{p'} = 0 \text{ for } i \neq j \text{ in } I,$$

$$\sum_{p, p' \in \mathbf{N}; p+p' = 1 - \langle i, j' \rangle} (-1)^{p'} \frac{[p+p']_i^!}{[p]_i^! [p']_i^!} F_i^p F_j F_i^{p'} = 0 \text{ for } i \neq j \text{ in } I.$$

For $i \in I, s \in \mathbf{N}$ we set $E_i^{(s)} = ([s]_i^!)^{-1} E_i^s$, $F_i^{(s)} = ([s]_i^!)^{-1} F_i^s$.

By [L4, 3.1.12], there is a unique $\mathbf{Q}$-algebra isomorphism $^-: \mathbf{U} \to \mathbf{U}$ such that $\bar{E}_i = E_i, \bar{F}_i = F_i$ for $i \in I$, $\bar{K}_y = K_{-y}$ for $y \in Y$ and $\overline{v^n u} = v^{-n} \bar{u}$ for all $u \in \mathbf{U}$, $n \in \mathbf{Z}$.

0.4. Let W be the (finite) subgroup of $Aut(X)$ generated by the involutions $s_i : \lambda \mapsto \lambda - \langle i, \lambda \rangle i'$ of X $(i \in I)$. Let R be the smallest W-stable subset of X that contains $\{i'; i \in I\}$. This is a finite set. Let $R^+ = \{\alpha \in R; \alpha \in \sum_i \mathbf{N} i'\}$, $R^- = -R^+$. Let R^1 (resp. R^e be the smallest W-stable subset of X that contains I^1 (resp. I^e). Then R^1, R^e are W-orbits. If $e = 1$ we have $R = R^1 = R^e$; if $e > 1$ we have $R = R^1 \sqcup R^e$.

For $i \in I$ and $\alpha \in R$ let $p_{i,\alpha}$ be the largest integer ≥ 0 such that $\alpha, \alpha + i', \alpha + 2i', \ldots, \alpha + p_{i,\alpha} i'$ belong to R and let $q_{i,\alpha}$ be the largest integer ≥ 0 such that $\alpha, \alpha - i', \alpha - 2i', \ldots, \alpha - q_{i,\alpha} i'$ belong to R. Then:

(a) $\langle i, \alpha \rangle = q_{i,\alpha} - p_{i,\alpha}$ and $p_{i,\alpha} + q_{i,\alpha} \leq 3$.

(b) *If $p_{i,\alpha} + q_{i,\alpha} > 1$, then we must have $p_{i,\alpha} + q_{i,\alpha} = e$, $i \in I^1$; moreover, $\alpha - q_{i,\alpha} i' \in R^e$, $\alpha + p_{i,\alpha} i' \in R^e$ and $\alpha + ki' \in R^1$ for $-q_{i,\alpha} < k < p_{i,\alpha}$.*

(c) *If* $p_{i,\alpha}+q_{i,\alpha}=1$, *then either both* $\alpha-q_{i,\alpha}i'$, $\alpha+p_{i,\alpha}i'$ *belong to* R^e *or both belong to* R^1.
We define $h:R^+\to\mathbf{N}$ by $h(\alpha)=\sum_{i\in I}n_i$ where $\alpha=\sum_{i\in I}n_ii'$ with $n_i\in\mathbf{N}$. There is a unique $\alpha_0\in R^+$ such that $h(\alpha_0)$ is maximum. We then have $p_{i,\alpha_0}=0$ for all $i\in I$; it follows that $\langle i,\alpha_0\rangle\geq 0$ for any $i\in I$. We have $\alpha_0\in R^e$.

0.5. The $\mathbf{U}$-module $\Lambda:=\Lambda_{\alpha_0}$ (see [L4, 3.5.6]) is well defined; it is simple, see [L4, 6.2.3], and finite dimensional, see [L4, 6.3.4]. Let $\eta=\eta_{\alpha_0}\in\Lambda$ be as in [L4, 3.5.7]. We have a direct sum decomposition (as a vector space) $\Lambda=\oplus_{\lambda\in X}\Lambda^\lambda$ where $\Lambda^\lambda=\{x\in\Lambda;K_yx=v^{\langle y,\lambda\rangle}x\quad\forall y\in Y\}$. Note that for $i\in I,\lambda\in X$ we have $E_iX^\lambda\subset X^{\lambda+i'}$, $F_iX^\lambda\subset X^{\lambda-i'}$. Moreover, we have $\dim\Lambda^\alpha=1$ if $\alpha\in R$, $\dim\Lambda^0=\sharp(I)$ and $\Lambda^\lambda=0$ if $\lambda\notin R\cup\{0\}$.

Let $\mathbf{B}$ be the canonical basis of Λ defined in [L4, 14.4.11]. We now state the following result in which $||$ denotes absolute value.

Theorem 0.6. *(a)* Λ *has a unique* $\mathbf{Q}(v)$*-basis* $\mathfrak{E}=\{X_\alpha;\alpha\in R\}\sqcup\{t_i;i\in I\}$ *such that (i)-(iii) below hold.*
(i) $X_{\alpha_0}=\eta$;
(ii) for $\alpha\in R$ *we have* $X_\alpha\in\Lambda^\alpha$; *for* $i\in I$ *we have* $t_i\in\Lambda^0$;
(iii) for any $i\in I$ *the linear maps* $E_i:\Lambda\to\Lambda$, $F_i:\Lambda\to\Lambda$, *are given by*

$$E_iX_\alpha=[q_{i,\alpha}+1]_iX_{\alpha+i'}\ \text{ if } \alpha\in R,p_{i,\alpha}>0,$$

$$E_iX_{-i'}=t_i,$$

$$E_iX_\alpha=0\ \text{ if } \alpha\in R,p_{i,\alpha}=0,\alpha\neq -i',$$

$$E_it_j=[|\langle j,i'\rangle|]_jX_{i'},\ \text{ if } j\in I,$$

$$F_iX_\alpha=[p_{i,\alpha}+1]_iX_{\alpha-i'}\ \text{ if } \alpha\in R,q_{i,\alpha}>0,$$

$$F_iX_{i'}=t_i,$$

$$F_iX_\alpha=0\ \text{ if } \alpha\in R,q_{i,\alpha}=0,\alpha\neq i',$$

$$F_it_j=[|\langle j,i'\rangle|]_jX_{-i'}\ \text{ if } j\in I.$$

(b) We have $\mathfrak{E}=\mathbf{B}$.

Note that the uniqueness of $\mathfrak{E}$ in (a) is straightforward. The existence of $\mathfrak{E}$ is proved in [L1] under the assumption that $e=1$ and is stated in [L2] without assumption on e. We shall note use these results here. Instead, in 1.15 we shall give a new proof (based on results in [L4]) of the existence of $\mathfrak{E}$ at the same time as proving (b).

1. Proof of Theorem 0.6

1.1. For any $\lambda \in X$, $\mathbf{B} \cap \Lambda^\lambda$ is a basis of Λ^λ. In particular, for any $\alpha \in R$, $\mathbf{B} \cap \Lambda^\alpha$ is a single element; we denote it by b^α.

Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ and let $\Lambda_{\mathcal{A}}$ be the $\mathcal{A}$-submodule of Λ generated by $\mathbf{B}$. It is known that $L_{\mathcal{A}}$ is stable under $E_i^{(s)}, F_i^{(s)}$ for $i \in I, s \in N$.

By [L4, 19.3.4], there is a unique $\mathbf{Q}$-linear isomorphism $^-: \Lambda \to \Lambda$ such that $\overline{u\eta} = \bar{u}\eta$ for all $u \in \mathbf{U}$. By [L4, 19.1.2], there is a unique bilinear form $(,) : \Lambda \times \Lambda \to \mathbf{Q}(v)$ such that $(\eta, \eta) = 1$ and $(E_i x, x') = (x, v_i K_i^{i \cdot i/2} F_i x')$, $(F_i x, x') = (x, v_i K_i^{-i \cdot i/2} E_i x')$, $(K_y x, x') = (x, K_y x')$ for all $i \in I, y \in Y$ and x, x' in Λ.

1.2. By [L4, 19.3.5],

(a) *an element $b \in \Lambda$ satisfies $\pm b \in \mathbf{B}$ if and only if $b \in \Lambda_{\mathcal{A}}$, $\bar{b} = b$ and $(b, b) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$.*

1.3. By [Ka] (see also [L4, 16.1.4]), for any $i \in I$ there is a unique $\mathbf{Q}(v)$-linear map $\tilde{F}_i : \Lambda \to \Lambda$ such that the following holds: if $x \in \Lambda^\lambda$, $E_i x = 0$ and $s \in \mathbf{N}$, then $\tilde{F}_i(F_i^{(s)} x) = F_i^{(s+1)} x$. Moreover, there is a unique $\mathbf{Q}(v)$-linear map $\tilde{E}_i : \Lambda \to \Lambda$ such that the following holds: if $x \in \Lambda^\lambda$, $F_i x = 0$ and $s \in \mathbf{N}$, then $\tilde{E}_i(E_i^{(s)} x) = E_i^{(s+1)} x$. Let $\mathbf{A} = \mathbf{Q}(v) \cap \mathbf{Q}[[v^{-1}]]$. Let $\Lambda_{\mathbf{A}}$ be the $\mathbf{A}$-submodule of Λ generated by $\mathbf{B}$. For any $x \in \Lambda_{\mathbf{A}}$ let $\underline{x}$ be the image of x in $\underline{\Lambda} := \Lambda_{\mathbf{A}}/v^{-1}\Lambda_{\mathbf{A}}$. Note that $\{\underline{b}; b \in \mathbf{B}\}$ is a $\mathbf{Q}$-basis of $\underline{\Lambda}$. By [Ka] (see also [L4, 20.1.4]), for any $i \in I$, $\tilde{F}_i, \tilde{E}_i$ preserve $\Lambda_{\mathbf{A}}, v^{-1}\Lambda_{\mathbf{A}}$ hence they induce $\mathbf{Q}$-linear maps $\underline{\Lambda} \to \underline{\Lambda}$ (denoted again by $\tilde{F}_i, \tilde{E}_i$). From [Ka] (see also [L4, 20.1.4]) we see also that

(a) *$\tilde{F}_i : \underline{\Lambda} \to \underline{\Lambda}, \tilde{E}_i : \underline{\Lambda} \to \underline{\Lambda}$ act in the basis $\{\underline{b}; b \in \mathbf{B}\}$ by matrices with all entries in $\{0, 1\}$.*

In the case where $e = 1$, the results in this subsection are not needed; in this case, instead of (a), we could use the positivity of the matrix entries of $E_i : \Lambda \to \Lambda$, $F_i : \Lambda \to \Lambda$ proved in [L4, 22.1.7].

1.4. Let $\alpha \in R$, $i \in I$ be such that $q_{i,\alpha} = 0, p = p_{i,\alpha} \geq 1$. Then we have $\langle i, \alpha \rangle = -p$. Let $Z^0 = b^\alpha \in \Lambda^\alpha$. We have $F_i Z^0 \in \Lambda^{\alpha - i'}$ hence $F_i Z^0 = 0$. We define $Z^k \in \Lambda^{\alpha + ki'}$ for $k = 1, \dots, p$ by the inductive formula

(a) $Z^k = [k]_i^{-1} E_i Z^{k-1} = \tilde{E}_i^k Z^0$.

Using $F_i Z^0 = 0$ together with (a) and the commutation formula between E_i, F_i we see by induction on k that for $k = 1, \dots, p$ we have

(b) $F_i Z^k = [p - k + 1]_i Z^{k-1}$.

1.5. We preserve the setup of 1.4. We show that for $k \in [0, p-1]$ we have

$$(a) \qquad (Z^{k+1}, Z^{k+1}) = \frac{1 - v_i^{-2p+2k}}{1 - v_i^{-2k-2}} (Z^k, Z^k).$$

We have $E_iZ^k=[k+1]_iZ^{k+1}$ hence using 1.4(b):

$$\begin{aligned}
&[k+1]_i^2(Z^{k+1},Z^{k+1})=(E_iZ^k,E_iZ^k)=(Z^k,v_iK_i^{i\cdot i/2}F_iE_iZ^k)\\
&=(Z^k,v_iK_i^{i\cdot i/2}E_iF_iZ^k)-(Z^k,v_iK_i^{i\cdot i/2}\frac{K_i^{i\cdot i/2}-K_i^{-i\cdot i/2}}{v_i-v_i^{-1}}Z^k)\\
&=(v_i^{\langle i,\alpha+ki'\rangle+1}[k]_i[p-k+1]_i-\frac{v_i^{2\langle i,\alpha+ki'\rangle+1}-v_i}{v_i-v_i^{-1}})(Z^k,Z^k)\\
&=(v_i^{-p+2k+1}[k]_i[p-k+1]_i-\frac{v_i^{-2p+4k+1}-v_i}{v_i-v_i^{-1}})(Z^k,Z^k).
\end{aligned}$$

We have

$$\begin{aligned}
&(v_i-v_i^{-1})^2(v_i^{-p+2k+1}[k]_i[p-k+1]_i-\frac{v_i^{-2p+4k+1}-v_i}{v_i-v_i^{-1}})\\
&=v_i^{-p+2k+1}(v_i^k-v_i^{-k})(v_i^{p-k+1}-v_i^{-p+k-1})-(v_i^{-2p+4k+1}-v_i)(v_i-v_i^{-1})\\
&=v_i^{2k+2}-v_i^2-v_i^{-2p+4k}+v_i^{-2p+2k}-v_i^{-2p+4k+2}+v_i^2+v_i^{-2p+4k}-1\\
&=v_i^{2k+2}+v_i^{-2p+2k}-v_i^{-2p+4k+2}-1=(v_i^{-2p+2k}-1)(1-v_i^{2k+2}).
\end{aligned}$$

Thus

$$(Z^{k+1},Z^{k+1})=\frac{(v_i^{-2p+2k}-1)(1-v_i^{2k+2})}{(v_i^{k+1}-v_i^{-k-1})^2}(Z^k,Z^k)$$

and (a) follows.

1.6. We preserve the setup of 1.4. We must have $p\in\{1,2,3\}$.

Assume first that $p=1$. From 1.5(a) we have $(Z^1,Z^1)=(Z^0,Z^0)$.

Assume now that $p=2$. Then from 0.4(b) we have $v_i=v$ and from 1.5(a) we have

$(Z^1,Z^1)=\frac{1-v^{-4}}{1-v^{-2}}(Z^0,Z^0)$,

$(Z^2,Z^2)=\frac{1-v^{-2}}{1-v^{-4}}(Z^1,Z^1)=(Z^0,Z^0)$.

Assume next that $p=3$. Then from 0.4(b) we have $v_i=v$ and from 1.5(a) we have

$(Z^1,Z^1)=\frac{1-v^{-6}}{1-v^{-2}}(Z^0,Z^0)$,

$(Z^2,Z^2)=(Z^1,Z^1)$.

$(Z^3,Z^3)=\frac{1-v^{-2}}{1-v^{-6}}(Z^2,Z^2)=(Z^0,Z^0)$.

1.7. We preserve the setup of 1.6. We show:

(a) We have $Z^k=b^{\alpha+ki'}$ for $k=0,1,\dots,p$.

Since $Z^0\in\mathbf{B}$, we have $Z^0\in\Lambda_{\mathcal{A}}$, $\bar{Z}^0=Z^0$, $(Z^0,Z^0)\in 1+v^{-1}\mathbf{Z}(v^{-1})$. From the formulas in 1.6 we see that $(Z^k,Z^k)\in 1+v^{-1}\mathbf{Z}(v^{-1})$ for $k=0,1,\dots,p$. For $k=1,\dots,p$ we have $E_iZ^{k-1}=[k]_iZ^k$ hence for $k=0,1,\dots,p$ we have $Z^k=$

$E_i^{(k)}Z^0 \in \Lambda_{\mathcal{A}}$. From $Z^k = E_i^{(k)}Z^0$ we see also that $\bar{Z}^k = \overline{E_i^{(k)}}\overline{Z^0} = E_i^{(k)}Z^0 = Z^k$. Using 1.2(a) we see that $\epsilon Z^k \in \mathbf{B}$ for some $\epsilon \in \{1, -1\}$. By 1.4(a), we have $\underline{Z^k} = \tilde{E}_i^k \underline{Z^0}$. Using this together with and 1.3(a), we see that $\epsilon = 1$ so that $Z^k \in \mathbf{B}$. Since $Z^k \in \Lambda^{\alpha+ki'}$, we see that $Z^k = b^{\alpha+ki'}$.

1.8. Let $i \in I, \tilde{\alpha} \in R$ be such that $p_{i,\tilde{\alpha}} > 0$ (or equivalently such that $\tilde{\alpha} + i' \in R$). We show:

(a) $$E_i b^{\tilde{\alpha}} = [q_{i,\tilde{\alpha}} + 1]_i b^{\tilde{\alpha}+i'}$$

Let $\alpha = \tilde{\alpha} - q_{i,\tilde{\alpha}} i' \in R$. We have $q_{i,\alpha} = 0$, $p_{i,\alpha} = p_{i,\tilde{\alpha}} + q_{i,\tilde{\alpha}} > 0$. We set $Z^0 = b^{\alpha}$. We then define Z^k with $k \in [1, p_{i,\alpha}]$ in terms of α, Z^0 as in 1.4. Note that $E_i Z^{k-1} = [k]_i Z^k$ for any $k \in [1, p_{i,\alpha}]$. Taking $k = q_{i,\tilde{\alpha}}+1$ (so that $k \in [1, p_{i,\alpha}]$) we deduce

$$E_i Z^{q_{i,\tilde{\alpha}}} = [q_{i,\tilde{\alpha}} + 1]_i Z^{q_{i,\tilde{\alpha}}+1}.$$

By 1.7(a) we have $Z^{q_{i,\tilde{\alpha}}} = b^{\tilde{\alpha}}$, $Z^{q_{i,\tilde{\alpha}}+1} = b^{\tilde{\alpha}+i'}$. This proves (a).

Here is a special case of (a); we assume that $i \neq j$ in I:

(b) If $\langle j, i' \rangle < 0$ then $E_j b^{i'} = b^{i'+j'}$; if $\langle j, i' \rangle = 0$ then $E_j b^{i'} = 0$.

It is enough to use that $p_{j,i'} = -\langle j, i' \rangle$ (we have $q_{j,i'} = 0$ since $i' - j' \notin R$).

1.9. Let $i \in I, \tilde{\alpha} \in R$ be such that $q_{i,\tilde{\alpha}} > 0$ (or equivalently such that $\tilde{\alpha} - i' \in R$). We show:

(a) $$F_i b^{\tilde{\alpha}} = [p_{i,\tilde{\alpha}} + 1]_i b^{\tilde{\alpha}-i'}.$$

Let $\alpha = \tilde{\alpha} - q_{i,\tilde{\alpha}} i' \in R$. We have $q_{i,\alpha} = 0$, $p_{i,\alpha} = p_{i,\tilde{\alpha}} + q_{i,\tilde{\alpha}} > 0$. We set $Z^0 = b^{\alpha}$. We then define Z^k with $k \in [1, p_{i,\alpha}]$ in terms of α, Z^0 as in 1.4. Note that $F_i Z^k = [p_{i,\alpha} - k + 1]_i Z^{k-1}$ for $k \in [1, p_{i,\alpha}]$. Taking $k = q_{i,\tilde{\alpha}}$ (so that $k \in [1, p_{i,\alpha}]$) we deduce

$$F_i Z^{q_{i,\tilde{\alpha}}} = [p_{i,\tilde{\alpha}} + 1]_i Z^{q_{i,\tilde{\alpha}}-1}.$$

By 1.7(a) we have $Z^{q_{i,\tilde{\alpha}}} = b^{\tilde{\alpha}}$, $Z^{q_{i,\tilde{\alpha}}-1} = b^{\tilde{\alpha}-i'}$. This proves (a).

Here is a special case of (a); we assume that $i \neq j$ in I:

(b) If $\langle j, i' \rangle < 0$ then $F_j b^{-i'} = b^{-i'-j'}$; if $\langle j, i' \rangle = 0$, then $F_j b^{-i'} = 0$.

It is enough to use that $q_{j,-i'} = \langle j, -i' \rangle$ (we have $p_{j,-i'} = 0$ since $-i' + j' \notin R$).

1.10. Let $i \in I$; we set $t_i = E_i b^{-i'} \in \Lambda^0$. We show

(a) $$F_i t_i = (v_i + v_i^{-1}) b^{-i'}.$$

Indeed,

$$F_i t_i = F_i E_i b^{-i'} = E_i F_i b^{-i'} - \frac{K_i^{i \cdot i/2} - K_i^{-i \cdot i/2}}{v_i - v_i^{-1}} b^{-i'}$$
$$= \frac{v_i^2 - v_i^{-2}}{v_i - v_i^{-1}} b^{i'} = (v_i + v_i^{-1}) b^{-i'}.$$

We show:

(b) $$(t_i, t_i) = (1 + v_i^{-2})(b^{-i'}, b^{-i'}).$$

Indeed, using (a) we have

$$(t_i, t_i) = (E_i b^{-i'}, t_i) = (b^{-i'}, v_i K_i^{i\cdot i/2} F_i t_i) = (b^{-i'}, v_i K_i^{i\cdot i/2}(v_i + v_i^{-1}) b^{-i'}) =$$
$$(v_i + v_i^{-1}) v_i^{-\langle i, i'\rangle + 1}(b^{-i'}, b^{-i'}) = (1 + v_i^{-2})(b^{-i'}, b^{-i'}).$$

From (b) we see that $(t_i, t_i) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$; from the definitions we have also $t_i \in \Lambda_{\mathcal{A}}$ and $\bar{t}_i = t_i$; it follows that $\epsilon t_i \in \mathbf{B}$ for some $\epsilon \in \{1, -1\}$. Now from $t_i = E_i b^{-i'}$ and $F_i b^{-i'} = 0$ we see that $t_i = \tilde{E}_i b^{-i'}$ hence $\underline{t_i} = \tilde{E}_i \underline{b^{-i'}}$. Using this together with 1.3(a) and we see that $\epsilon = 1$ hence

(c) $$t_i \in \mathbf{B}.$$

We show:

(d) $$\text{If } i \neq j, \text{ then } F_i t_j = [-\langle j, i'\rangle]_j b^{-i'}.$$

We have $F_i t_j = F_i E_j b^{-j'} = E_j F_i b^{-j'}$. This is 0 if $\langle i, j'\rangle = 0$ since by 1.9(b) we have $F_i b^{-j'} = 0$ (so in this case (a) holds). Now assume that $\langle i, j'\rangle < 0$. Then using 1.9(b) and 1.8(a) we have

$$E_j F_i b^{-j'} = E_j b^{-i'-j'} = [q_{j,-i'-j'} + 1]_j b^{-i'}.$$

Note that $p_{j,-i'-j'} = 1$ since $-i' - j' + j' \in R$, $-i' - j' + 2j' \notin R$. Hence $q_{j,-i'-j'} - 1 = \langle j, -i' - j'\rangle = -2 - \langle j, i'\rangle$ that is, $q_{i,-i'-j'} + 1 = -\langle j, i'\rangle$. This completes the proof of (d).

We show:

(e) $$(E_i t_i, E_i t_i) = [2]_i^2 (b^{-i'}, b^{-i'}).$$

Indeed, using (b) we have

$$(E_i t_i, E_i t_i) = (t_i, v_i K_i^{i\cdot i/2} F_i E_i t_i) = (t_i, v_i K_i^{i\cdot i/2} E_i F_i t_i)$$
$$- (t_i, v_i K_i^{i\cdot i/2} \frac{K_i^{i\cdot i/2} - K_i^{-i\cdot i/2}}{v_i - v_i^{-1}} t_i) = [2]_i (t_i, v_i K_i^{i\cdot i/2} E_i b^{-i'}) = [2]_i (t_i, v_i K_i^{i\cdot i/2} t_i)$$
$$= [2]_i (t_i, v_i t_i) = [2]_i^2 (b^{-i'}, b^{-i'}),$$

proving (e).

From (e) we get $([2]_i^{-1} E_i t_i, [2]_i^{-1} E_i t_i) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$. We have $[2]_i^{-1} E_i t_i = E_i^{(2)} b^{-i'} \in \Lambda_{\mathcal{A}}$. Moreover, we have clearly $\overline{[2]_i^{-1} E_i t_i} = [2]_i^{-1} E_i t_i$. Using 1.2(a) we deduce that $\epsilon [2]_i^{-1} E_i t_i \in \mathbf{B}$ for some $\epsilon \in \{1, -1\}$. Since $[2]_i^{-1} E_i t_i \in \Lambda^{i'}$, we must have $\epsilon [2]_i^{-1} E_i t_i = b^{i'}$. Thus we have $\epsilon E_i^{(2)} b^{-i'} = b^{i'}$. Since $F_i b^{-i'} = 0$ it follows that $\tilde{E}_i^2 b^{-i'} = \epsilon b^{i'}$ and $\tilde{E}_i^2 \underline{b^{-i'}} = \epsilon \underline{b^{i'}}$. Using 1.3(a), we deduce that $\epsilon = 1$. Thus,

(f) $$E_i t_i = [2]_i b^{i'}.$$

1.11. Let $i \in I$. We set $\tilde{t}_i = F_i b^{i'} \in \Lambda^0$. We show:

(a) $$E_i \tilde{t}_i = [2]_i b^{i'}.$$

Indeed,

$$E_i \tilde{t}_i = E_i F_i b^{i'} = F_i E_i b^{i'} + \frac{K_i^{i\cdot i/2} - K_i^{-i\cdot i/2}}{v_i - v_i^{-1}} b^{i'} = \frac{v_i^2 - v_i^{-2}}{v_i - v_i^{-1}} b^{i'} = [2]_i b^{i'}.$$

We show:

(b) $$(\tilde{t}_i, \tilde{t}_i) = [2]_i v_i^{-1} (b^{i'}, b^{i'}).$$

Indeed, using (a) we have:

$$(\tilde{t}_i, \tilde{t}_i) = (F_i b^{i'}, \tilde{t}_i) = (b^{i'}, v_i K_i^{-i\cdot i/2} E_i \tilde{t}_i) = (b^{i'}, v_i K_i^{-i\cdot i/2} [2]_i b^{i'})$$
$$= [2]_i v_i^{-1} (b^{i'}, b^{i'}).$$

From (b) we see that $(\tilde{t}_i, \tilde{t}_i) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$; from the definitions we have also $\tilde{t}_i \in \Lambda_{\mathcal{A}}$ and $\bar{\tilde{t}}_i = \tilde{t}_i$; using 1.2(a) we see that $\epsilon \tilde{t}_i \in \mathbf{B}$ for some $\epsilon \in \{1, -1\}$. From $\tilde{t}_i = F_i b^{i'}$, $E_i b^{i'} = 0$ we see that $\tilde{t}_i = \tilde{F}_i b^{i'}$ hence $\underline{\tilde{t}_i} = \tilde{F}_i \underline{b^{i'}}$. Using this and 1.3(a) we deduce that $\epsilon = 1$ so that

(c) $$\tilde{t}_i \in \mathbf{B}.$$

We show:

(d) $$(\tilde{t}_i, t_i) = \pm(1 + v_i^{-2})(b^{i'}, b^{i'}).$$

Indeed, using 1.10(f) we have

$$(\tilde{t}_i, t_i) = (F_i b^{i'}, t_i) = (b^{i'}, v_i K_i^{-i\cdot i/2} E_i t_i) = (b^{i'}, v_i K_i^{-i\cdot i/2} [2]_i b^{i'})$$
$$= v_i^{-1} [2]_i (b^{i'}, b^{i'}) = (1 + v_i^{-2})(b^{i'}, b^{i'})$$

hence $(\tilde{t}_i, t_i) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$. If $\tilde{t}_i \neq t_i$ then, since $\tilde{t}_i \in \mathbf{B}$ and $t_i \in \mathbf{B}$, we would have $(\tilde{t}_i, t_i) \in v^{-1}\mathbf{Z}[v^{-1}]$ (see [L4, 19.3.3]), contradicting (d). Thus we hve $\tilde{t}_i = t_i$ and

(e) $$F_i b^{i'} = t_i.$$

We show:

(f) $$\text{If } i \neq j, \text{ then } E_i t_j = [-\langle j, i' \rangle]_j b^{i'}.$$

Using (e) we have $E_i t_j = E_i F_j b^{j'} = F_j E_i b^{j'}$. This is 0 if $\langle i, j' \rangle = 0$ since by 1.8(b) we have $E_i b^{j'} = 0$ (so in this case (f) holds). Now assume that $\langle i, j' \rangle < 0$. Then using 1.8(b) and 1.9(a) we have

$$F_j E_i b^{j'} = F_j b^{i'+j'} = [p_{j,i'+j'} + 1]_j b^{i'}.$$

Note that $q_{j,i'+j'} = 1$ since $i' + j' - j' \in R$, $i' + j' - 2j' \notin R$. Hence $1 - p_{j,i'+j'} = \langle j, i' + j' \rangle = 2 + \langle j, i' \rangle$ that is, $p_{i,i'+j'} + 1 = -\langle j, i' \rangle$. This completes the proof of (f).

1.12. We show:

(a) *If* $\alpha \in R^1$, *then* $(b^\alpha, b^\alpha) = 1 + v^{-2} + \cdots + v^{-2(e-1)} = v^{-e+1}[e]$. *If* $\alpha \in R^e$, *then* $(b^\alpha, b^\alpha) = 1$.

Note that when $e = 1$ we have $R^1 = R^e$ and the two formulas in (a) are compatible with each other.

We first prove (a) for $\alpha \in R^+$ by descending induction on $h(\alpha)$. If $h(\alpha) = h(\alpha_0)$ then $\alpha = \alpha_0$ and we have $b^\alpha = \eta$ so that $(b^\alpha, b^\alpha) = (\eta, \eta) = 1$. Now assume that $\alpha \in R^+$, $h(a) < h(\alpha_0)$. We can find $\alpha' \in R^+$, $i \in I$ such that $q_{i,\alpha'} = 0$, $p = p_{i,\alpha'} \geq 1$ and $\alpha = \alpha' + ki'$ where $k \in \{0, 1, \ldots, p-1\}$. Then $h(\alpha' + pi') > h(\alpha)$ hence $(\alpha' + pi', \alpha' + pi')$ is given by the formula in (a). Assume first that $p = 1$. Then $\alpha = \alpha'$ and by 1.6 and 1.7(a) we have $(b^\alpha, b^\alpha) = (b^{\alpha'+i'}, b^{\alpha'+i'})$. By 0.4(c), either both $\alpha, \alpha + i'$ belong to R^e or both belong to R^1; (a) follows in this case. Next assume that $p > 1$. By 0.4(b) we have $p = e$ and $\alpha' + pi' \in R^e$. Hence $(b^{\alpha'+pi'}, b^{\alpha'+pi'}) = 1$. If $k = 0$ then $\alpha \in R^e$ (see 0.4(b)) and by 1.6 and 1.7(a) we have $(b^\alpha, b^\alpha) = (b^{\alpha'+pi'}, b^{\alpha'+pi'})$; (a) follows in this case. If $k > 0$, $k < p$ then $\alpha \in R^1$ (see 0.4(b)) and by 1.6 and 1.7(a) we have $(b^\alpha, b^\alpha) = (1 + v^{-2} + \cdots + v^{-2(e-1)})(b^{\alpha'+pi'}, b^{\alpha'+pi'})$; (a) follows in this case. This completes the proof of (a) assuming that $\alpha \in R^+$.

We now prove (a) for $\alpha \in R^-$ by induction on $h(-\alpha) \geq 1$. Let $i \in I$. Recall that $\tilde{t}_i, t_i$ satisfy $\tilde{t}_i = t_i$ (see 1.11), $(t_i, t_i) = [2]_i v_i^{-1}(b^{-i'}, b^{-i'})$ (see 1.10(b)) and $(\tilde{t}_i, \tilde{t}_i) = [2]_i v_i^{-1}(b^{i'}, b^{i'})$ (see 1.11(b)). It follows that

$$(b^{-i'}, b^{-i'}) = (b^{i'}, b^{i'}). \tag{b}$$

In particular, (a) holds when $h(-\alpha) = 1$. We now assume that $\alpha \in R^-$ and $h(-\alpha) \geq 2$. We can find $\alpha' \in R^-$, $i \in I$ such that $q_{i,\alpha'} = 0$, $p = p_{i,\alpha'} \geq 1$ and $\alpha = \alpha' + ki'$ where $k \in \{0, 1, \ldots, p-1\}$. Then $h(-(\alpha' + pi')) < h(-\alpha)$ hence $(\alpha' + pi', \alpha' + pi')$ is given by the formula in (a). The rest of the proof is a repetition of the first part of the proof. Assume first that $p = 1$. Then $\alpha = \alpha'$ and by 1.6 and 1.7(a) we have $(b^\alpha, b^\alpha) = (b^{\alpha'+i'}, b^{\alpha'+i'})$. By 0.4(c), either both $\alpha, \alpha + i'$ belong to R^e or both belong to R^1; (a) follows in this case. Next assume that $p > 1$. By 0.4(b) we have $p = e$ and $\alpha' + pi' \in R^e$. Hence $(b^{\alpha'+pi'}, b^{\alpha'+pi'}) = 1$. If $k = 0$ then $\alpha \in R^e$ (see 0.4(b)) and by 1.6 and 1.7(a) we have $(b^\alpha, b^\alpha) = (b^{\alpha'+pi'}, b^{\alpha'+pi'})$; (a) follows in this case. If $k > 0$, $k < p$ then $\alpha \in R^1$ (see 0.4(b)) and by 1.6 and 1.7(a) we have $(b^\alpha, b^\alpha) = (1 + v^{-2} + \cdots + v^{-2(e-1)})(b^{\alpha'+pi'}, b^{\alpha'+pi'})$; (a) follows in this case. This completes the proof of (a) assuming that $\alpha \in R^-$; hence (a) is proved in all cases.

1.13. We show:

(a) *If* $i \in I^1$ *then* $(t_i, t_i) = (1 + v^{-2})(1 + v^{-2} + \cdots + v^{-2(e-1)})$. *If* $i \in I^e$ *then* $(t_i, t_i) = 1 + v_i^{-2} = 1 + v^{-2e}$.

Note that when $e = 1$ we have $I^1 = I^e$ and the two formulas in (a) are compatible with each other.

From 1.10(b) we have $(t_i, t_i) = [2]_i v_i^{-1}(b^{-i'}, b^{-i'})$. Using 1.12(a) we see that (a) holds.

In the remainder of this subsection we fix $i \neq j$ in I. We show:

(b) *If at least one of i, j is in I^1 and $i \cdot j \neq 0$ then $(t_i, t_j) = v^{-e}[e]$. If both i, j are in I^e and $i \cdot j \neq 0$ then $(t_i, t_j) = v^{-e}$. If $i \cdot j = 0$ then $(t_i, t_j) = 0$.*

Using 1.10(d), we have

$$\begin{aligned}(t_i, t_j) &= (E_i b^{-i'}, t_j) = (b^{-i'}, v_i K_i^{i \cdot i/2} F_i t_j) \\ &= [-\langle j, i'\rangle]_j (b^{-i'}, v_i K_i^{i \cdot i/2} b^{-i'}) = v_i^{-1}[-\langle j, i'\rangle]_j (b^{-i'}, b^{-i'}).\end{aligned}$$

We see that if $\langle j, i'\rangle = 0$ then $(t_i, t_j) = 0$.

Now assume that $\langle j, i'\rangle \neq 0$.

If $i \in I^e, j \in I^e$ then $\langle j, i'\rangle = -1$ and $(t_i, t_j) = v^{-e}$.

If $i \in I^e, j \in I^1$ then $\langle j, i'\rangle = -e$ and $(t_i, t_j) = v^{-e}[e]$.

If $i \in I^1, j \in I^e$ then $(t_i, t_j) = (t_j, t_i) = v^{-e}[e]$.

If $i \in I^1, j \in I^1$ then $\langle j, i'\rangle = -1$ and $(t_i, t_j) = v^{-1}(1 + v^{-2} + \cdots + v^{-2(e-1)}) = v^{-e}[e]$.

This completes the proof of (b).

1.14. We show:

(a) *The elements $\{t_i; i \in I\}$ are distinct.*

Let $i \neq j$ in I. If we had $t_i = t_j$, then we would have $(t_i, t_j) \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$, see 1.13(a). But 1.13(b) shows that $(t_i, t_j) \in v^{-1}\mathbf{Z}[v^{-1}]$. This completes the proof of (a).

Let $\mathfrak{E} = \{b^\alpha; \alpha \in R\} \sqcup \{t_i, i \in I\}$. By (a), this is a subset of Λ rather than a multiset. We show:

(b) *We have $\mathbf{B} = \mathfrak{E}$.*

Since $t_i \in \mathbf{B}$ for any $i \in I$, we have $\mathfrak{E} \subset \mathbf{B}$. Clearly we have $\sharp(\mathfrak{E}) = \sharp(R) + \sharp(I)$. Since we have also $\sharp(\mathbf{B}) = \sharp(R) + \sharp(I)$, it follows that $\mathfrak{E} = \mathbf{B}$, proving (b).

1.15. We prove the existence part of 0.6(a). It is enough to prove that the elements $X_\alpha = b^\alpha$ and t_i satisfy the requirements of 0.6(a). Now 0.6(a)(i) holds by definition; 0.6(a)(ii) is immediate; 0.6(a)(iii) has been verified earlier in this section. This proves the existence part of 0.6(a) and at the same time proves 0.6(b) (see 1.14(b)).

2. Applications

2.1. Let $i \in I$, $k \in \mathbf{Z}_{>0}$. From 0.6 we see that the action of $E_i^{(k)}$, $F_i^{(k)}$ in the basis $\mathfrak{E}$ of Λ is given by the following formulas.

$$E_i^{(k)} X_\alpha = \frac{[q_{i,\alpha} + k]_i^!}{[q_{i,\alpha}]_i^![k]_i^!} X_{\alpha + ki'} \text{ if } \alpha \in R, \alpha \neq -i', k \leq p_{i,\alpha},$$

$$E_i^{(k)} X_\alpha = 0 \text{ if } \alpha \in R, \alpha \neq -i', k > p_{i,\alpha},$$

$$E_i X_{-i'} = t_i, E_i^{(2)} X_{-i'} = X_{i'}, E_i^{(k)} X_{-i'} = 0 \text{ if } k \geq 3,$$

$$E_i t_j = [|\langle j, i' \rangle|]_j X_{i'}, E_i^{(k)} t_j = 0 \text{ if } k \geq 2,$$

$$F_i^{(k)} X_\alpha = \frac{[p_{i,\alpha} + k]_i^!}{[p_{i,\alpha}]_i^! [k]_i^!} X_{\alpha - ki'} \text{ if } \alpha \in R, \alpha \neq i', k \leq q_{i,\alpha},$$

$$F_i^{(k)} X_\alpha = 0 \text{ if } \alpha \in R, \alpha \neq i', k > q_{i,\alpha},$$

$$F_i X_{i'} = t_i, F_i^{(2)} X_{i'} = X_{-i'}, F_i^{(k)} X_{i'} = 0 \text{ if } k \geq 3,$$

$$F_i t_j = [|\langle j, i' \rangle|]_j X_{-i'}, F_i^{(k)} t_j = 0 \text{ if } k \geq 2.$$

In particular, we see that $E_i^{(k)}$, $F_i^{(k)}$ act through matrices with all entries in $\mathbf{N}[v, v^{-1}]$. (In the case where $e = 1$ this is already known from [L4, 22.1.7].)

2.2. If v is specialized to 1, the $\mathbf{U}$-module Λ becomes a simple module over the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$ corresponding to the adjoint representation $\Lambda|_{v=1}$ of $\mathfrak{g}$; this module inherits a $\mathbf{Q}$-basis $\{X_\alpha; \alpha \in R\} \sqcup \{t_i; i \in I\}$ in which the elements e_i, f_i of $\mathfrak{g}$ defined by E_i, F_i act by matrices with entries in $\mathbf{N}$. Let $z \in \mathbf{Q}$. Then for $i \in I$, the exponentials $x_i(z) = \exp(ze_i), y_i(z) = \exp(zf_i)$ are well defined endomorphisms of $\Lambda|_{v=1}$. Their action in the basis above can be described using the formulas in 2.1:

$$x_i(z) X_\alpha = \sum_{0 \leq k \leq p_{i,\alpha}} \frac{(q_{i,\alpha} + k)!}{q_{i,\alpha}! k!} z^k X_{\alpha + ki'} \text{ if } \alpha \in R, \alpha \neq -i',$$

$$x_i(z) X_{-i'} = X_{-i'} + z t_i + z^2 X_{i'},$$

$$x_i(z) t_j = t_j + |\langle j, i' \rangle| z X_{i'} \text{ if } j \in I,$$

$$y_i(z) X_\alpha = \sum_{0 \leq k \leq q_{i,\alpha}} \frac{(p_{i,\alpha} + k)!}{p_{i,\alpha}! k!} z^k X_{\alpha - ki'} \text{ if } \alpha \in R, \alpha \neq i',$$

$$y_i(z) X_{i'} = X_{i'} + z t_i + z^2 X_{-i'},$$

$$y_i(z) t_j = t_j + |\langle j, i' \rangle| z X_{-i'} \text{ if } j \in I.$$

2.3. Now let k be any field and let V be the k-vector space with basis $\{X_\alpha; \alpha \in R\} \sqcup \{t_i; i \in I\}$. For any $i \in I$ and $z \in k$ we define $x_i(z) \in GL(V)$, $y_i(z) \in GL(V)$ by the formulas in 2.2 (which involve only integer coefficients). The subgroup of $GL(V)$ generated by the elements $x_i(z), y_i(z)$ for various $i \in I, z \in k$ is the Chevalley group [Ch] over k associated to $\mathfrak{g}$.

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