Physics 8.321, Fall 2002
Homework #1

Due **Monday, September 16** by 4:30 PM in the 8.321 homework box in 4-339B.

1. A skew-Hermitian operator A is an operator satisfying $A^\dagger = -A$.

(a) Prove that A can have at most one real eigenvalue (which may be degenerate).

(b) Prove that the commutator of two Hermitian operators is skew-Hermitian.

2. Show that if H and K are both Hermitian operators with positive eigenvalues, then

$$\mathrm{Tr}\, HK \geq 0,$$

and that equality implies that $HK = 0$.

3. Consider a Hermitian operator H whose eigenvectors form a complete orthonormal set, and whose eigenvalues are all positive.

(a) Prove that for any two vectors $|\alpha\rangle, |\beta\rangle$

$$|\langle\alpha|H|\beta\rangle|^2 \leq \langle\alpha|H|\alpha\rangle\langle\beta|H|\beta\rangle$$

(b) Prove that $\mathrm{Tr}\,(H) > 0$.

4. Prove that the equation $AB - BA = \mathbb{1}$ cannot be satisfied by any finite-dimensional matrices A, B.

5. Let U be a unitary operator. Consider the eigenvalue equation

$$U|\lambda\rangle = \lambda|\lambda\rangle\,.$$

(a) Prove that λ is of the form $e^{i\theta}$ with θ real.

(b) Show that if $\lambda \neq \mu$ then $\langle\mu|\lambda\rangle = 0$.

6. (a) Consider two operators A, B that do not necessarily commute. Show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\}$$

where

$$A^0\{B\} = B, \quad A^1\{B\} = [A, B], \quad A^2\{B\} = [A, [A, B]], \text{ etc.}$$

(b) Let $A(x)$ be an operator that depends on a continuous parameter x. Derive the following identity

$$e^{-iA(x)} \frac{d}{dx} e^{iA(x)} = i \sum_{n=0}^{\infty} \frac{(-i)^n}{(n+1)!} A^n\{\frac{dA}{dx}\}\,.$$

7. (a) Show that the set of N x N complex matrices form a vector space of dimension N^2.

(b) Show that Tr $(A^\dagger B)$ defines an inner product on this vector space.

(c) Show that the set of 2 x 2 matrices is spanned by the basis

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

How can these matrices be used to form an orthonormal basis?

(d) Find the spectrum and eigenvectors for each of the matrices in (c)

(e) Prove that

$$\exp(i\theta\, \sigma \cdot \mathbf{n}) = \cos\theta + i\, \sigma \cdot \mathbf{n}\, \sin\theta$$

where $\mathbf{n}$ is a unit 3-vector.

(f) Prove that if $\mathbf{A}, \mathbf{B}$ are two vector operators that commute with σ, it follows that

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})\mathbb{1} + i\sigma \cdot \mathbf{A}\mathrm{x}\mathbf{B}$$