Dirac operators and the very strange formula for Lie superalgebras

DIRAC OPERATORS AND THE VERY STRANGE FORMULA FOR LIE SUPERALGEBRAS

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ABSTRACT. Using a super-affine version of Kostant's cubic Dirac operator, we prove a very strange formula for quadratic finite-dimensional Lie superalgebras with a reductive even subalgebra.

1. INTRODUCTION

The goal of this paper is to provide an approach to the strange and very strange formulas for a wide class of finite dimensional Lie superalgebras. Let us recall what these formulas are in the even case. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let Δ^+ be a set of positive roots for the set Δ of $\mathfrak{h}$-roots in $\mathfrak{g}$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ be the corresponding Weyl vector. Freudenthal and de Vries discovered in [6] the following remarkable relation between the square length of ρ in the Killing form κ and the dimension of $\mathfrak{g}$:

$$\kappa(\rho,\rho) = \frac{\dim \mathfrak{g}}{24}.$$

They called this the *strange formula*. It can be proved in several very different ways (see e.g. [5], [2]), and it plays an important role in the proof of the Macdonald identities. Indeed, the very strange formula enters as a transition factor between the Euler product $\varphi(x) = \prod_{i=1}^{\infty}(1-x^i)$ and Dedekind's η-function $\eta(x) = x^{\frac{1}{24}}\varphi(x)$.

In [8] Kac gave a representation theoretic interpretation of the Macdonald identities as denominator identities for an affine Lie algebra. Moreover, using the modular properties of characters of the latter algebras, he provided in [10] a multivariable generalization of them and a corresponding transition identity that he named the *very strange formula*. Here the representation theoretic interpretation of the formulas involves an affine Lie algebra, which is built up from a simple Lie algebra endowed with a finite order automorphism. To get the very strange formula from a "master formula" it was also required that the characteristic polynomial of the automorphism has rational coefficients. A more general form, with no rationality hypothesis, is proved in [14], where it is also used to estimate the asymptotic behavior at cusps of the modular forms involved in the character of an highest weight module. Let us state this version of the very strange formula, for simplicity of exposition just in the case of inner automorphisms. Let σ be an automorphism of order m of type $(s_0, s_1, \ldots, s_n; 1)$ (see [11, Chapter 8]). Let $\mathfrak{g} = \oplus_{\bar{j}\in\mathbb{Z}/m\mathbb{Z}}\,\mathfrak{g}^{\bar{j}}$ be the eigenspace decomposition with respect to σ. Define $\lambda_s \in \mathfrak{h}^*$ by $\kappa(\lambda_s, \alpha_i) = \frac{s_i}{2m}$, $1 \le i \le n$; here

$\{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots of $\mathfrak{g}$. Then

$$\kappa(\rho - \lambda_s, \rho - \lambda_s) = \frac{\dim \mathfrak{g}}{24} - \frac{1}{4m^2} \sum_{j=1}^{m-1} j(m-j) \dim \mathfrak{g}^{\bar{j}}. \tag{1.1}$$

Much more recently, we provided a vertex algebra approach to this formula (in a slight generalized version where an elliptic automorphism is considered, cf. [12]) as a byproduct of our attempt to reproduce Kostant's theory of the cubic Dirac operator in affine setting. Our proof is based on two main ingredients:

(a). An explicit vertex algebra isomorphism $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}}) \cong V^{k+g,1}(\mathfrak{g})$, where $V^k(\mathfrak{g})$ is the affine vertex algebra of noncritical level k and $F(\overline{\mathfrak{g}})$ is the fermionic vertex algebra of $\mathfrak{g}$ viewed as a purely odd space.

(b). A nice formula for the λ-bracket of the Kac-Todorov Dirac field $G \in V^{k+g,1}(\mathfrak{g})$ with itself.

Indeed, using (a), we can let the zero mode G_0 of G act on the tensor product of representations of $V^k(\mathfrak{g}), F(\overline{\mathfrak{g}})$. Since we are able to compute $G_0 \cdot (v \otimes 1)$, where v is an highest weight vector of an highest weight module for the affinization of $\mathfrak{g}$, the expression for $[G_\lambda G]$ obtained in step (b) yields a formula which can be recast in the form (1.1) (cf. [12, Section 6]).

Now we discuss our work in the super case. A finite dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called quadratic if it carries a supersymmetric bilinear form (i.e. symmetric on $\mathfrak{g}_0$, skewsymmetric on $\mathfrak{g}_1$, and $\mathfrak{g}_0$ is orthogonal to $\mathfrak{g}_1$), which is non-degenerate and invariant. We say that a complex quadratic Lie superalgebra $\mathfrak{g}$ is of *basic type* if $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$. In Theorem 6.2 we prove a *very strange formula* (cf. (6.5)) for basic type Lie superalgebras endowed with an indecomposable elliptic automorphism (see Definition 2.1) which preserves the invariant form. When the automorphism is the identity, this formula specializes to the strange formula (6.4), which has been proved for Lie superalgebras of defect zero in [10] and for general basic classical Lie superalgebras in [13], using case by case combinatorial calculations. The proof using the Weyl character formula as in [6] or the proof using modular forms as in [14] are not applicable in this setting.

Although the proof proceeds along the lines of what we did in [12] for Lie algebras, we have to face several technical difficulties. We single out two of them. First, we have to build up a twisted Clifford-Weil module for $F(\overline{\mathfrak{g}})$; this requires a careful choice of a maximal isotropic subspace in $\mathfrak{g}^{\bar{0}}$. In Section 2 we prove that the class of Lie superalgebras of basic type is closed under taking fixed points of automorphisms and in Section 3 we show that Lie superalgebras of basic type admit a triangular decomposition. This implies the existence a "good" maximal isotropic subspace.

Secondly, the isomorphism in (a) is given by formulas which are different w.r.t. the even case, and this makes subtler the computation of the square of the Dirac field under λ-product. We have also obtained several simplifications with respect to the exposition given in [12].

Some of our results have been (very sketchily) announced in [16].

2. Setup

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra of basic type, i.e.,

(1) $\mathfrak{g}_0$ is reductive subalgebra of $\mathfrak{g}$, i.e., the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}$ is completely reducible;

(2) $\mathfrak{g}$ is *quadratic*, i.e., $\mathfrak{g}$ admits a nondegenerate invariant supersymmetric bilinear form $(\cdot,\cdot)$.

Note that condition (1) implies that $\mathfrak{g}_0$ is a reductive Lie algebra and that $\mathfrak{g}_1$ is completely reducible as a $\mathfrak{g}_0$-module. Examples are given by the simple basic classical Lie superalgebras and the contragredient finite dimensional Lie superalgebras with a symmetrizable Cartan matrix (in particular, $gl(m,n)$). There are of course examples of different kind, like a symplectic vector space regarded as a purely odd abelian Lie superalgebra. An inductive classification is provided in [1].

We say that $\mathfrak{g}$ is $(\cdot,\cdot)$-irreducible if the form restricted to a proper ideal is degenerate.

Definition 2.1. An automorphism σ of $\mathfrak{g}$ is said indecomposable if $\mathfrak{g}$ cannot be decomposed as an orthogonal direct sum of two nonzero σ-stable ideals.

We say that σ is elliptic if it is diagonalizable with modulus 1 eigenvalues.

Let σ be an indecomposable elliptic automorphism of $\mathfrak{g}$ which is parity preserving and leaves the form invariant. If $j \in \mathbb{R}$, set $\bar{j} = j + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. Set $\mathfrak{g}^{\bar{j}} = \{x \in \mathfrak{g} \mid \sigma(x) = e^{2\pi\sqrt{-1}j}x\}$. Let $\mathfrak{h}^0$ be a Cartan subalgebra of $\mathfrak{g}^{\bar{0}}$.

Proposition 2.1. *If $\mathfrak{g}$ is of basic type, then $\mathfrak{g}^{\bar{0}}$ is of basic type.*

Proof. Since σ is parity preserving, it induces an automorphism of $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is reductive, we have that $\mathfrak{g}_0^{\bar{0}}$ is also reductive. Since σ preserves the invariant bilinear form, we have that $(\mathfrak{g}^{\bar{i}}, \mathfrak{g}^{\bar{j}}) \neq 0$ if and only if $\bar{i} = -\bar{j}$. Thus $(\cdot,\cdot)_{|\mathfrak{g}^{\bar{0}}\times\mathfrak{g}^{\bar{0}}}$ is nondegenerate. Since $\mathfrak{g}_0^{\bar{0}}$ is reductive, $\mathfrak{h}^0$ is abelian, thus it is contained in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Since $\mathfrak{g}_1$ is completely reducible as a $\mathfrak{g}_0$-module, $\mathfrak{h}$ acts semisimply on $\mathfrak{g}_1$, hence $\mathfrak{h}^0$ acts semisimply on $\mathfrak{g}_1^{\bar{0}}$. Thus $\mathfrak{g}_1^{\bar{0}}$ is a semisimple $\mathfrak{g}_0^{\bar{0}}$-module. □

It is well-known that we can choose as Cartan subalgebra for $\mathfrak{g}$ the centralizer $\mathfrak{h}$ of $\mathfrak{h}^0$ in $\mathfrak{g}_0$. In particular we have that $\sigma(\mathfrak{h}) = \mathfrak{h}$. If $\mathfrak{a}$ is any Lie superalgebra, we let $\mathfrak{z}(\mathfrak{a})$ denote its center.

3. The structure of basic type Lie superalgebras

The goal of this section is to prove that a basic type Lie superalgebra admits a triangular decomposition. We will apply this result in the next sections to $\mathfrak{g}^{\bar{0}}$, which, by Proposition 2.1, is of basic type.

Since $\mathfrak{g}_0$ is reductive, we can fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ and a set of positive roots for $\mathfrak{g}_0$. If $\lambda \in \mathfrak{h}^*$, let h_λ be, as usual, the unique element of $\mathfrak{h}$ such that $(h_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$. Let $V(\lambda)$ denote the irreducible representation of $\mathfrak{g}_0$ with highest weight $\lambda \in (\mathfrak{h})^*$.

Then we can write

$$\mathfrak{g} = \mathfrak{h} + [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \sum_{\lambda \in \mathfrak{h}^*} V(\lambda). \tag{3.1}$$

Decompose now the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ as

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}'', \qquad \mathfrak{h}' = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}].$$

Let M_{triv} be the isotypic component of the trivial $[\mathfrak{g}_0,\mathfrak{g}_0]$-module in $\mathfrak{g}_1$. Decompose it into isotypic components for $\mathfrak{g}_0$ as

$$M_{triv}=\oplus_{\lambda\in\Lambda}M(\lambda)=M(0)\oplus M'_{triv}, \tag{3.2}$$

where $M'_{triv}=\oplus_{0\neq\lambda\in\Lambda}M(\lambda)$. Then

$$\begin{aligned}\mathfrak{g}^{(1)}&:=[\mathfrak{g},\mathfrak{g}]\\ &=\mathfrak{h}'+[\mathfrak{g}_0,\mathfrak{g}_0]\oplus\sum_{0\neq\lambda\in\mathfrak{h}^*}V(\lambda)\\ &=\mathfrak{h}'+[\mathfrak{g}_0,\mathfrak{g}_0]\oplus M'_{triv}\oplus\sum_{\lambda\in\mathfrak{h}^*,\dim V(\lambda)>1}V(\lambda).\end{aligned} \tag{3.3}$$

Lemma 3.1. *If in decomposition* (3.1) *we have that* $\dim V(\lambda)=1$*, then* $\lambda(\mathfrak{h}')=0$.

Proof. Let $h\in\mathfrak{h}'$. If $h\in[\mathfrak{g}_0,\mathfrak{g}_0]$ the claim is obvious; if $h\in[\mathfrak{g}_1,\mathfrak{g}_1]$ then we may assume that $h=[x_\mu,x_{-\mu}]=h_\mu$, μ being a $\mathfrak{h}$-weight of $\mathfrak{g}_1$. Assume first $\mu\pm\lambda\neq 0$. Then, for $v_\lambda\in V(\lambda)$, we have

$$0=[v_{-\lambda},[v_\lambda,x_\mu]]=\mu(h_\lambda)x_\mu-[v_\lambda,[v_{-\lambda},x_\mu]]=\mu(h_\lambda)x_\mu \tag{3.4}$$

so that $\mu(h_\lambda)=0$ or $\lambda(h_\mu)=0$. It remains to deal with the case $\mu=\pm\lambda$. We have

$$[[v_\lambda,v_{-\lambda}],v_{\pm\lambda}]=\pm||\lambda||^2v_{\pm\lambda}.$$

This finishes the proof, since $||\lambda||^2=0$. Indeed, $[v_\lambda,v_\lambda]$ is a weight vector of weight 2λ. This implies either $\lambda=0$, and we are done, or $[v_\lambda,v_\lambda]=0$. In the latter case $||\lambda||^2v_\lambda=[[v_\lambda,v_{-\lambda}],v_\lambda]=0$ by the Jacobi identity. □

In turn, by Lemma 3.1,

$$\begin{aligned}\mathfrak{g}^{(2)}&:=[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}]\\ &=\mathfrak{h}'+[\mathfrak{g}_0,\mathfrak{g}_0]\oplus(\sum_{\lambda\in\mathfrak{h}^*,\,\dim V(\lambda)>1}V(\lambda)).\end{aligned}$$

Finally define

$$\underline{\mathfrak{g}}=\mathfrak{g}^{(2)}/(\mathfrak{z}(\mathfrak{g})\cap\mathfrak{g}^{(2)}).$$

Lemma 3.2.

(1) *The radical of the restriction of the invariant form to* $\mathfrak{g}^{(2)}$ *equals* $\mathfrak{z}(\mathfrak{g})\cap\mathfrak{g}^{(2)}$.
(2) $\underline{\mathfrak{g}}$ *is an orthogonal direct sum of quadratic simple Lie superalgebras.*

Proof. It suffices to show that if $x\in\mathfrak{g}^{(2)}$ belongs to the radical of the (restricted) form, then it belongs to the center of $\mathfrak{g}$. We know that $(x,[y,z])=0$ for all $y,z\in\mathfrak{g}^{(1)}$; invariance of the form implies that $[x,y]$ belongs to the radical of the form restricted to $\mathfrak{g}^{(1)}$. This in turn means that $([x,y],[w,t])=0$ for all $w,t\in\mathfrak{g}$. Therefore, $0=([x,y],[w,t])=([[x,y],w],t)\,\forall\,t\in\mathfrak{g}$. Since the form on $\mathfrak{g}$ is nondegenerate, we have that $[x,y]\in\mathfrak{z}(\mathfrak{g})$ for any $y\in\mathfrak{g}^{(1)}$. If $x\in\mathfrak{g}_1$ and $y\in\mathfrak{g}_0^{(1)}$, then $[x,y]\in\mathfrak{z}(\mathfrak{g})\cap(\sum_{\lambda\in\mathfrak{h}^*,\,\lambda_{|\mathfrak{h}'}\neq0}V(\lambda))=\{0\}$. This imples that x commutes with $\mathfrak{g}_0^{(1)}$. Since $x\in\sum_{\lambda\in\mathfrak{h}^*,\,\lambda_{|\mathfrak{h}'}\neq0}V(\lambda)$, we have $x=0$. If $x\in\mathfrak{g}_0$, then, if $y\in\sum_{\lambda\in\mathfrak{h}^*,\,\lambda_{|\mathfrak{h}'}\neq0}V(\lambda)$, we have $[x,y]\in\mathfrak{z}(\mathfrak{g})\cap(\sum_{\lambda\in\mathfrak{h}^*,\,\lambda_{|\mathfrak{h}'}\neq0}V(\lambda))=\{0\}$. If $y\in\mathfrak{g}_0^{(2)}$, then $[x,y]\in[\mathfrak{g}_0,\mathfrak{g}_0]\cap\mathfrak{z}(\mathfrak{g})=\{0\}$. So $x\in\mathfrak{z}(\mathfrak{g}^{(2)})$. This implies that $x\in\mathfrak{h}'$, so it commutes also with M_{triv} and $\mathfrak{h}$, hence $x\in\mathfrak{z}(\mathfrak{g})$, as required.

To prove the second statement, it suffices to show that there does not exist an isotropic ideal in $\underline{\mathfrak{g}}$. Indeed, if this is the case and $\mathfrak{i}$ is a minimal ideal, then by minimality either $\mathfrak{i} \subseteq \mathfrak{i}^\perp$ or $\mathfrak{i} \cap \mathfrak{i}^\perp = \{0\}$. Since we have excluded the former case, we have $\underline{\mathfrak{g}} = \mathfrak{i} \oplus \mathfrak{i}^\perp$ with $\mathfrak{i}$ a simple Lie superalgebra endowed with a non degenerate form and we can conclude by induction.

Suppose that $\mathfrak{i}$ is an isotropic ideal. If $x \in \mathfrak{g}^{(2)}$, we let $\pi(x)$ be its image in $\underline{\mathfrak{g}}$. If $\mathfrak{i}_1 \neq \{0\}$, we have that there is $\pi(V(\lambda)) \subset \mathfrak{i}_1$. We can choose an highest weight vector v_λ in $V(\lambda)$ and a vector $v_{-\lambda} \in \mathfrak{g}_1$ of weight $-\lambda$ such that $(v_\lambda, v_{-\lambda}) = 1$. Then $\pi(h_\lambda) = [\pi(v_\lambda), \pi(v_{-\lambda})] \in \mathfrak{i}$. Note that $h_\lambda \notin \mathfrak{z}(\mathfrak{g})$. In fact, since $\dim V(\lambda) > 1$, $\lambda_{|\mathfrak{h}\cap[\mathfrak{g}_0,\mathfrak{g}_0]} \neq 0$. On the other hand, if $[h_\lambda, \mathfrak{g}^{(2)})] \neq 0$, then there is $0 \neq v_\mu \in (\mathfrak{g}^{(2)})_\mu$ such that $[h_\lambda, v_\mu] = \lambda(h_\mu) v_\mu \neq 0$. In particular $\pi(v_\mu) \in \mathfrak{i}$. Choose $v_{-\mu} \in (\mathfrak{g}^{(2)})_{-\mu}$ such $(v_\mu, v_{-\mu}) \neq 0$. Then $\pi(v_{-\mu}) = -\frac{1}{\lambda(h_\mu)}[\pi(h_\lambda), \pi(v_{-\mu})] \in \mathfrak{i}$. But then $\mathfrak{i}$ is not isotropic. It follows that $h_\lambda \in \mathfrak{z}(\mathfrak{g}^{(2)}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}$, which is absurd. Then $\mathfrak{i} = \mathfrak{i}_0$, hence $\mathfrak{i} \subset \mathfrak{z}(\underline{\mathfrak{g}}_0)$. Since $[\mathfrak{i}, \underline{\mathfrak{g}}_1] = 0$, we have that $\pi^{-1}(\mathfrak{i}) \subset \mathfrak{z}(\mathfrak{g}^{(2)})$. Since $\mathfrak{z}(\mathfrak{g}^{(2)}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}$, we have $\mathfrak{i} = \{0\}$. □

At this point we have the following decomposition:

$$\mathfrak{g} = (\mathfrak{h} + [\mathfrak{g}_0, \mathfrak{g}_0]) \oplus \sum_{\lambda \in \mathfrak{h}^*, \dim V(\lambda) > 1} V(\lambda) \oplus M_{triv}.$$

Let, as in the proof of Lemma 3.2, $\pi : \mathfrak{g}^{(2)} \to \underline{\mathfrak{g}}$ be the projection. Since $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}_0, \mathfrak{g}_0] = 0$, we see that $[\mathfrak{g}_0, \mathfrak{g}_0] = \pi([\mathfrak{g}_0, \mathfrak{g}_0])$, hence we can see the set of positive roots for $\mathfrak{g}_0$ as a set of positive roots for $\underline{\mathfrak{g}}_0$. By Lemma 3.2, we have

$$\underline{\mathfrak{g}} = \bigoplus_{i=1}^{k} \underline{\mathfrak{g}}(i), \tag{3.5}$$

with $\underline{\mathfrak{g}}(i)$ simple ideals. It is clear that $\mathfrak{g}$ acts on $\underline{\mathfrak{g}}$ and that the projection π intertwines the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1^{(2)}$ with that on $\underline{\mathfrak{g}}_1$. Since $[\mathfrak{g}_0, \mathfrak{g}_0] = \pi([\mathfrak{g}_0, \mathfrak{g}_0])$, we see that $\underline{\mathfrak{g}}(i)_1$ is a $[\mathfrak{g}_0, \mathfrak{g}_0]$-module. Since the decomposition (3.5) is orthogonal, we see that the $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules $\underline{\mathfrak{g}}(i)_1$ are inequivalent. It follows that that $\mathfrak{z}(\mathfrak{g}_0)$ stabilizes $\underline{\mathfrak{g}}(i)_1$, thus $\underline{\mathfrak{g}}(i)_1$ is a $\mathfrak{g}_0$-module.

We now discuss the $\mathfrak{g}_0$-module structure of $\underline{\mathfrak{g}}(i)_1$. By the classification of simple Lie superalgebras, either $\underline{\mathfrak{g}}(i)_1 = V(i)$ with $V(i)$ self dual irreducible $\underline{\mathfrak{g}}_0$-module or there is a polarization (with respect to $(\cdot,\cdot)$) $\underline{\mathfrak{g}}(i)_1 = V \oplus V^*$ with V an irreducible $\underline{\mathfrak{g}}_0$-module. In the first case $\underline{\mathfrak{g}}(i)_1$ is an irreducible $\mathfrak{g}_0$-module. In the second case, since the action of $\mathfrak{h}$ is semisimple, $\underline{\mathfrak{g}}(i)$ decomposes as $V_1(i) \oplus V_2(i)$, with $V_j(i)$ $(j = 1, 2)$ irreducible $\mathfrak{g}_0$-modules. If $V_1(i)$ is not self dual, then the decomposition $\underline{\mathfrak{g}}(i) = V_1(i) \oplus V_2(i)$ is a polarization. If $V_1(i) = V_1(i)^*$ and $V_2(i) = V_2(i)^*$ then the center of $\mathfrak{g}_0$ acts trivially on $\underline{\mathfrak{g}}(i)_1$, thus V and V^* are actually $\mathfrak{g}_0$-modules. We can therefore choose $V_1(i) = V$ and $V_2(i) = V^*$.

The simple ideals $\underline{\mathfrak{g}}(i)$ are basic classical Lie superalgebras. By the classification of such algebras (see [9]) there is a contragredient Lie superalgebra $\tilde{\mathfrak{g}}(i)$ such that $\underline{\mathfrak{g}}(i) = [\tilde{\mathfrak{g}}(i), \tilde{\mathfrak{g}}(i)]/\mathfrak{z}([\tilde{\mathfrak{g}}(i), \tilde{\mathfrak{g}}(i)])$. Choose Chevalley generators $\{\tilde{e}_j, \tilde{f}_j\}_{j \in J(i)}$ for $\tilde{\mathfrak{g}}(i)$. Let $\underline{e}_j, \underline{f}_j$ be their image in $\underline{\mathfrak{g}}(i)$.

We claim that $\underline{e}_j, \underline{f}_j$ are $\mathfrak{h}$-weight vectors. The vectors $\underline{e}_j, \underline{f}_j$ are root vectors for $\underline{\mathfrak{g}}(i)$. If the roots of $\underline{e}_j, \underline{f}_j$ have multiplicity one, then, $\underline{e}_j, \underline{f}_j$ must be $\mathfrak{h}$-stable, hence they are $\mathfrak{h}$-weight vectors. If there are roots of higher multiplicity then $\underline{\mathfrak{g}}(i)$ is of type $A(1,1)$. Let $\overline{d}$ be the derivation on $\underline{\mathfrak{g}}(i)$ defined by setting $\overline{d}(\underline{\mathfrak{g}}_0) = 0$, $\overline{d}(v) = v$ for $v \in V_1(i)$, and $\overline{d}(v) = -v$ for $v \in V_2(i)$. Then it is not hard to check that $\tilde{\mathfrak{g}}(i) = \mathbb{C}d \oplus \mathbb{C}c \oplus \underline{\mathfrak{g}}(i)$ with bracket defined as in Exercise 2.10 of [11]. If the roots of $\underline{e}_j, \underline{f}_j$ have multiplicity two, then $\tilde{e}_j, \tilde{f}_j$ are odd root vectors of $\tilde{\mathfrak{g}}(i)$. In particular $\tilde{e}_j$ is in $V_1(i)$ and $\tilde{f}_j$ is in $V_2(i)$. This implies that $\underline{e}_j$ is in $V_1(i)$ and $\underline{f}_j$ is in $V_2(i)$. Since $\mathfrak{z}(\mathfrak{g}_0)$ acts as multiple of the identity on $V_1(i)$ and $V_2(i)$, we see that $\underline{e}_j, \underline{f}_j$ are $\mathfrak{h}$-weight vectors also in this case.

Since π restricted to $[\mathfrak{g}_0, \mathfrak{g}_0] + \mathfrak{g}_1^{(2)}$ is an isomorphism, we can define e_j, f_j to be the unique elements of $[\mathfrak{g}_0, \mathfrak{g}_0] + \mathfrak{g}_1^{(2)}$ such that $\pi(e_j) = \underline{e}_j$ and $\pi(f_j) = \underline{f}_j$. Since $\underline{e}_j, \underline{f}_j$ are $\mathfrak{h}$-weight vectors, we have that e_j, f_j are root vectors for $\mathfrak{g}$.

Set $J = \cup_i J(i)$. We can always assume that the positive root vectors of $\underline{\mathfrak{g}}_0$ are in the algebra spanned by $\{\underline{e}_j \mid j \in J\}$.

Let $\alpha_j \in \mathfrak{h}^*$ be the weight of e_j. We note that the weight of f_j is $-\alpha_j$ for any $j \in J$. One way to check this is the following: if $j \in J(i)$, there is an invariant form $< \cdot, \cdot >$ on $\tilde{\mathfrak{g}}(i)$ such that $< \tilde{e}_j, \tilde{f}_j > \neq 0$. Since $\underline{\mathfrak{g}}(i)$ is simple, the form $(\cdot, \cdot)$ is a (nonzero) multiple of the form induced by $< \cdot, \cdot >$. In particular $(\underline{e}_j, \underline{f}_j) \neq 0$. Since $(e_j, f_j) = (\underline{e}_j, \underline{f}_j)$, we see that the root of f_j is $-\alpha_j$.

Subdivide Λ in (3.2) as $\Lambda = \{0\} \cup \Lambda^+ \cup \Lambda^-$ with $\Lambda^+ \cap \Lambda^- = \emptyset$ and $\Lambda^- = -\Lambda^+$ (which is possible since the form $(\cdot, \cdot)$ is nondegenerate on M'_{triv}). Choose a basis $\{e_i^\lambda \mid i = 1, \dots, \dim M(\lambda)\}$ in $M(\lambda)$ for $\lambda \in \Lambda^+$ and let $\{f_i^\lambda\} \subset M(-\lambda)$ be the dual basis. Also $(\cdot, \cdot)_{|M(0)\times M(0)}$ is nondegenerate, hence we can find a polarization $M(0) = M^+ \oplus M^-$. Let $\{e_i^0\}$ and $\{f_i^0\}$ be a basis of M^+ and its dual basis in M^-, respectively.

We now check that relations

$$[e_i, f_j] = \delta_{ij} h_i, \ i, j \in J \quad [e_i, f_j^\lambda] = [e_i^\lambda, f_j] = 0, \ j \in J \quad [e_i^\lambda, f_j^\mu] = \delta_{\lambda,\mu}\delta_{i,j} h_i^\lambda,$$

hold for $\{e_j, f_j\}_{j\in J} \cup \{e_i^\lambda, f_i^\lambda\}_{\lambda \in \Lambda^+ \cup \{0\}}$. Assume now $i \neq j$, $i, j \in J$. Then, since $[\underline{e}_i, \underline{f}_j] = 0$, $[e_i, f_j] \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)} \subset \mathfrak{h}'$. This implies that $\alpha_i = \alpha_j$ so $[e_i, f_j] \in \mathbb{C}h_{\alpha_i}$. If e_i is even then α_i is a root of $\mathfrak{g}_0$ so $\pi(h_{\alpha_i}) \neq 0$, hence $[e_i, f_j] = 0$. If e_i is odd and $\pi(h_{\alpha_i}) = 0$, since $[\underline{e}_i, \underline{f}_i] = (\underline{e}_i, \underline{f}_i)\pi(h_{\alpha_i}) = 0$, we have that $[\underline{e}_i, \underline{f}_j] = 0$ for any $j \in J$. In particular, $\underline{e}_i$ is a lowest weight vector for $\underline{\mathfrak{g}}_0$. On the other hand, since $h_{\alpha_i} \in \mathfrak{z}(\mathfrak{g})$, we have in particular that $\alpha(h_{\alpha_i}) = 0$ for any root of $\mathfrak{g}_0$. This implies that $\mathbb{C}\underline{e}_i$ is stable under the adjoint action of $\underline{\mathfrak{g}}_0$. This is absurd since $\underline{\mathfrak{g}}_1$ does not have one-dimensional $\underline{\mathfrak{g}}_0$-submodules.

If $\lambda, \mu \in \Lambda^+ \cup \{0\}$, then $[e_i^\lambda, f_j^\mu]$ is in the center of $\mathfrak{g}_0$, hence $[e_i^\lambda, f_j^\mu] = \delta_{i,j}\delta_{\lambda,\mu} h_\lambda$. Moreover it is obvious that $[e_h^\lambda, f_j] = [e_j, f_h^\lambda] = 0$ if e_j, f_j are even. It remains to check that $[e_h^\lambda, f_j] = [e_j, f_h^\lambda] = 0$ when e_j, f_j are odd.

This follows from the more general

Lemma 3.3.

$$[M_{triv}, \mathfrak{g}_1^{(2)}] = 0.$$

Proof. Choose $x \in M(\lambda)$ and $y \in V(\mu)$ with $\dim V(\mu) > 1$. It is enough to show that $([x, y], z) = 0$ for any $z \in \mathfrak{g}_0$. Observe that, since $\mathbb{C}x$ and $V(\mu)^*$ are inequivalent as $\mathfrak{g}_0$-modules, we have that $(x, V(\mu)) = 0$. Since $([x, y], z) = (x, [y, z])$ and $[y, z]$ is in $V(\mu)$, we have the claim. □

The outcome of the above construction is that we have a triangular decomposition

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}_-, \tag{3.6}$$

where $\mathfrak{n}$ (resp. $\mathfrak{n}_-$) is the algebra generated by $\{e_j, \mid j \in J\} \cup \{e_i^\lambda \mid \lambda \in \Lambda^+ \cup \{0\}\}$ (resp. $\{f_j, \mid j \in J\} \cup \{f_i^\lambda \mid \lambda \in \Lambda^+ \cup \{0\}\}$). By Lemma 3.3, we see that $[e_h^\lambda, e_j] = 0$. It follows that

$$\mathfrak{n} = \mathfrak{n}_{triv} \oplus \mathfrak{e},$$

where $\mathfrak{n}_{triv}$ is the algebra generated by $\{e_i^\lambda \mid \lambda \in \Lambda^+ \cup \{0\}\}$ and $\mathfrak{e}$ is the algebra generated by $\{e_j, \mid j \in J\}$. Notice that

$$\mathfrak{n}_{triv} = M^+ \oplus \sum_{\lambda \in \Lambda^+} M(\lambda).$$

This follows from the fact that the right hand side is an abelian subalgebra. Since $\mathfrak{e} \subset \mathfrak{g}^{(2)}$, we have the orthogonal decomposition

$$\mathfrak{n} = M^+ \oplus (\sum_{\lambda \in \Lambda^+} M(\lambda)) \oplus (\mathfrak{n} \cap \mathfrak{g}^{(2)}). \tag{3.7}$$

Choose any maximal isotropic subspace $\mathfrak{h}^+$ in $\mathfrak{h}$. The previous constructions imply the following fact.

Lemma 3.4. *$\mathfrak{h}^+ + \mathfrak{n}$ is a maximal isotropic subspace in $\mathfrak{g}$.*

Proof. We first prove that $\mathfrak{n}$ is isotropic. By (3.7), it is enough to check that $M^+ \oplus (\sum_{\lambda \in \Lambda^+} M(\lambda))$ and $(\mathfrak{n} \cap \mathfrak{g}^{(2)})$ are isotropic. By construction M^+ is isotropic. Moreover, if $\lambda \neq -\mu$, then $M(\lambda)^*$ and $M(\mu)$ are inequivalent, thus $(M(\lambda), M(\mu)) = 0$. This implies that $M(\lambda)$ is isotropic if $\lambda \neq 0$ and $(M(\lambda), M(\mu)) = 0$ if $\lambda \neq \mu$, $\lambda, \mu \in \Lambda^+ \cup \{0\}$.

If $x, y \in \mathfrak{n} \cap \mathfrak{g}^{(2)}$ and $\pi(x) \in \underline{\mathfrak{g}}(i)$, $\pi(y) \in \underline{\mathfrak{g}}(j)$ with $i \neq j$, then $(x, y) = (\pi(x), \pi(y)) = 0$. If $i = j$, let $p : [\tilde{\mathfrak{g}}(i), \tilde{\mathfrak{g}}(i)] \to \underline{\mathfrak{g}}(i)$ be the projection. Let $\tilde{\mathfrak{n}}(i)$ be the algebra spanned by the $\{\tilde{e}_j\}_{j \in J(i)}$. Then $\pi(x), \pi(y) \in p(\tilde{\mathfrak{n}}(i))$. Recall that the weights of $\tilde{\mathfrak{n}}(i)$ are a set of positive roots for $\tilde{\mathfrak{g}}(i)$, and, since $\tilde{\mathfrak{g}}(i)$ is contragredient α and $-\alpha$ cannot be both positive roots for $\tilde{\mathfrak{g}}(i)$. This implies that $\tilde{\mathfrak{n}}(i)$ is an isotropic subspace of $\tilde{\mathfrak{g}}(i)$ for any invariant form of $\tilde{\mathfrak{g}}(i)$. Since $\underline{\mathfrak{g}}(i)$ is simple, $(\cdot, \cdot)$ is induced by an invariant form on $\tilde{\mathfrak{g}}(i)$ so $p(\tilde{n}(i))$ is isotropic.

Clearly, $(\mathfrak{h}, \mathfrak{n}) = (\mathfrak{h}, \mathfrak{n} \cap \mathfrak{g}_0) = 0$, since $\mathfrak{n} \cap \mathfrak{g}_0$ is the nilradical of a Borel subalgebra. Note that $(\mathfrak{n}, \mathfrak{g}) = (\mathfrak{n}, \mathfrak{n}_-)$ so $\mathfrak{n}$ and $\mathfrak{n}_-$ are non degenerately paired. Thus $\mathfrak{n}$ is a maximal isotropic subspace of $\mathfrak{n} + \mathfrak{n}_-$. Since $\mathfrak{h}$ and $\mathfrak{n} + \mathfrak{n}_-$ are orthogonal, the result follows. □

Proposition 3.5.

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-. \tag{3.8}$$

Proof. Having (3.6) at hand, it remains to prove that the sum is direct. This follows from Lemma 3.4: indeed, if $x \in \mathfrak{n} \cap \mathfrak{n}_-$, then x would be in the radical of the form. □

4. The super affine vertex algebra

Set $\bar{\mathfrak{g}} = P\mathfrak{g}$, where P is the parity reversing functor. In the following, we refer the reader to [3] for basic definitions and notation regarding Lie conformal and vertex algebras. In particular, for the reader's convenience we recall Wick's formula, which will be used several times in the following. Let V be a vertex algebra, then

$$[a_\lambda : bc :] =: [a_\lambda b]c : +p(a,b) : b[a_\lambda c] : + \int_0^\lambda [[a_\lambda b]_\mu c] d\mu. \tag{4.1}$$

Consider the conformal algebra $R = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes \bar{\mathfrak{g}}) \oplus \mathbb{C}K \oplus \mathbb{C}\bar{K}$ with λ-products

$$[a_\lambda b] = [a,b] + \lambda(a,b)K, \tag{4.2}$$

$$[a_\lambda \bar{b}] = \overline{[a,b]}, \quad [\bar{a}_\lambda b] = p(b)\overline{[a,b]}, \tag{4.3}$$

$$[\bar{a}_\lambda \bar{b}] = (b,a)\bar{K}, \tag{4.4}$$

K, $\bar{K}$ being even central elements. Let $V(R)$ be the corresponding universal vertex algebra, and denote by $V^{k,1}(\mathfrak{g})$ its quotient by the ideal generated by $K - k|0\rangle$ and $\overline{K} - |0\rangle$. The vertex algebra $V^{k,1}(\mathfrak{g})$ is called the super affine vertex algebra of level k. The relations are the same used in [12] for even variables. We remark that the order of a, b in the r.h.s. of (4.4) is relevant.

Recall that one defines the current Lie conformal superalgebra $Cur(\mathfrak{g})$ as

$$Cur(\mathfrak{g}) = (\mathbb{C}[T] \otimes \mathfrak{g}) + \mathbb{C}\mathcal{K}$$

with $T(\mathcal{K}) = 0$ and the λ-bracket defined for $a, b \in 1 \otimes \mathfrak{g}$ by

$$[a_\lambda b] = [a,b] + \lambda(a,b)\mathcal{K}, \quad [a_\lambda \mathcal{K}] = [\mathcal{K}_\lambda \mathcal{K}] = 0.$$

Let $V(\mathfrak{g})$ be its universal enveloping vertex algebra. The quotient $V^k(\mathfrak{g})$ of $V(\mathfrak{g})$ by the ideal generated by $\mathcal{K} - k|0\rangle$ is called the level k affine vertex algebra.

If A is a superspace equipped with a skewsupersymmetric bilinear form $< \cdot, \cdot >$ one also has the Clifford Lie conformal superalgebra

$$C(A) = (\mathbb{C}[T] \otimes A) + \mathbb{C}\overline{\mathcal{K}}$$

with $T(\overline{\mathcal{K}}) = 0$ and the λ-bracket defined for $a, b \in 1 \otimes A$ by

$$[a_\lambda b] = < a, b > \overline{\mathcal{K}}, \quad [a_\lambda \overline{\mathcal{K}}] = [\overline{\mathcal{K}}_\lambda \overline{\mathcal{K}}] = 0.$$

Let V be the universal enveloping vertex algebra of $C(A)$. The quotient of V by the ideal generated by $\overline{\mathcal{K}} - |0\rangle$ is denoted by $F(A)$. Applying this construction to $\bar{\mathfrak{g}}$ with the form $< \cdot, \cdot >$ defined by $< a, b >= (b, a)$ one obtains the vertex algebra $F(\bar{\mathfrak{g}})$.

We define the Casimir operator of $\mathfrak{g}$ as $\Omega_\mathfrak{g} = \sum_i x^i x_i$ if $\{x_i\}$ is a basis of $\mathfrak{g}$ and $\{x^i\}$ its dual basis w.r.t. $(\cdot, \cdot)$ (see [9, pag. 85]). Since $\Omega_\mathfrak{g}$ supercommutes with any element of $U(\mathfrak{g})$, the generalized eigenspaces of its action on $\mathfrak{g}$ are ideals in $\mathfrak{g}$. Observe that $\Omega_\mathfrak{g}$ is a symmetric operator: indeed

$$\begin{aligned}(\Omega_\mathfrak{g}(a), b) &= \sum_i ([x^i, [x_i, a]], b) = \sum_i -p(x^i, [x_i, a])([[x_i, a], x^i], b) \\ &= \sum_i p(x^i)([a, x_i], [x^i, b]) = \sum_i p(x^i)(a, [x_i, [x^i, b]]) = (a, \Omega_\mathfrak{g}(b)).\end{aligned}$$

Since $\Omega_{\mathfrak{g}}$ is symmetric, the generalized eigenspaces provide an orthogonal decomposition of $\mathfrak{g}$. Moreover, since σ preserves the form, we have that $\sigma \circ \Omega_{\mathfrak{g}} = \Omega_{\mathfrak{g}} \circ \sigma$, hence σ stabilizes the generalized eigenspaces. Since σ is assumed to be indecomposable, it follows that that $\Omega_{\mathfrak{g}}$ has a unique eigenvalue. Let $2g$ be such eigenvalue. If the form $(\cdot,\cdot)$ is normalized as in [13, (1.3)], then the number g is called the dual Coxeter number of $\mathfrak{g}$.

Lemma 4.1. *If $\Omega_{\mathfrak{g}} - 2gI \neq 0$ then $g = 0$. Moreover, in such a case, $\Omega_{\mathfrak{g}}(\mathfrak{g})$ is a central ideal.*

Proof. Let $\mathfrak{g} = \sum_{S=1}^{k} \mathfrak{g}(S)$ be a orthogonal decomposition in $(\cdot,\cdot)$-irreducible ideals. Clearly $\Omega_{\mathfrak{g}}(\mathfrak{g}(i)) \subset \mathfrak{g}(i)$, hence we can assume without loss of generality that $\mathfrak{g}$ is $(\cdot,\cdot)$-irreducible.

If x is in the center of $\mathfrak{g}$, then x is orthogonal to $[\mathfrak{g},\mathfrak{g}]$, so, if $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$, then $\mathfrak{g}$ must be centerless. In particular, in this case, $\mathfrak{g} = \underline{\mathfrak{g}}$ is a sum of simple ideals, but, being $(\cdot,\cdot)$-irreducible, it is simple. Since $\Omega_{\mathfrak{g}} - 2gI$ is nilpotent, we have that $(\Omega_{\mathfrak{g}} - 2gI)(\mathfrak{g})$ is a proper ideal of $\mathfrak{g}$, hence $\Omega_{\mathfrak{g}} = 2gI$.

Thus, if $\Omega_{\mathfrak{g}} - 2gI \neq 0$, we must have $\mathfrak{g} \neq [\mathfrak{g},\mathfrak{g}]$. Since $\mathfrak{g} \neq [\mathfrak{g},\mathfrak{g}]$, the form becomes degenerate when restricted to $[\mathfrak{g},\mathfrak{g}]$ and its radical is contained in the center of $\mathfrak{g}$. It follows that the center of $\mathfrak{g}$ is nonzero. Clearly $\Omega_{\mathfrak{g}}$ acts trivially on the center, hence $g = 0$.

Since $\mathfrak{g}^{(2)}$ is an ideal of $\mathfrak{g}$, clearly $\Omega_{\mathfrak{g}}$ acts on it. Since $\Omega_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{g})) = 0$, this action descends to $\underline{\mathfrak{g}}$. Recall that we have $\underline{\mathfrak{g}} = \bigoplus_{i=1}^{k} \underline{\mathfrak{g}}(i)$, with $\underline{\mathfrak{g}}(i)$ simple ideals. We already observed that these ideals are inequivalent as $\mathfrak{g}_0$-modules, thus $\Omega_{\mathfrak{g}}(\underline{\mathfrak{g}}(i)) \subset \underline{\mathfrak{g}}(i)$. Since $\Omega_{\mathfrak{g}}(\underline{\mathfrak{g}}(i))$ is a proper ideal, we see that $\Omega_{\mathfrak{g}}(\underline{\mathfrak{g}}) = 0$. Thus $\Omega_{\mathfrak{g}}(\mathfrak{g}^{(2)}) \subset \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}$. We now check that $\Omega_{\mathfrak{g}}(M_{triv}) = 0$. Let $x \in M(\lambda)$. If $x_i \in \mathfrak{g}_1^{(2)}$, by Lemma 3.3, $[x_i, x] = 0$. If $x_i \in M(\mu)$ with $\mu \neq -\lambda$, then $[x_i, x] = 0$. If $x_i \in M(-\lambda)$, then $[x^i, [x_i, x]] = (x_i, x)[x^i, h_\lambda] = (x_i, x)\|\lambda\|^2 x_i = 0$. It follows that $\Omega_{\mathfrak{g}}(x) = \Omega_{\mathfrak{g}_0}(x) = \|\lambda\|^2 x = 0$. The final outcome is that $\Omega_{\mathfrak{g}}(\mathfrak{g}_1) \subset \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)} \subset \mathfrak{h}'$. Thus, since $\Omega_{\mathfrak{g}}$ preserves parity,

$$\Omega_{\mathfrak{g}}(\mathfrak{g}_1) = \{0\}. \tag{4.5}$$

It follows that $\Omega_{\mathfrak{g}}(\mathfrak{g}_0) = \Omega_{\mathfrak{g}}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_0$. Since $\Omega_{\mathfrak{g}}$ is nilpotent, $\Omega_{\mathfrak{g}}(\mathfrak{g})$ is a nilpotent ideal, hence it intersects trivially $[\mathfrak{g}_0, \mathfrak{g}_0]$. It follows that $[\Omega_{\mathfrak{g}}(\mathfrak{g}), \mathfrak{g}_0] = 0$. Since $\Omega_{\mathfrak{g}}(\mathfrak{g})$ is an ideal contained in $\mathfrak{g}_0$, $[\Omega_{\mathfrak{g}}(\mathfrak{g}), \mathfrak{g}_1] \subset \mathfrak{g}_1 \cap \mathfrak{g}_0 = \{0\}$ as well. The result follows. □

Remark 4.1. Note that we have proved the following fact: if $\mathfrak{g}$ is centerless and $(\cdot,\cdot)$-irreducible then it is simple (cf. [1, Theorem 2.1]).

Set $C_{\mathfrak{g}} = \Omega_{\mathfrak{g}} - 2gI_{\mathfrak{g}}$.

Proposition 4.2. *Assume $k + g \neq 0$. Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and let $\{x^i\}$ be its dual basis w.r.t. $(\cdot,\cdot)$. For $x \in \mathfrak{g}$ set*

$$\widetilde{x} = x - \frac{1}{2}\sum_i :\overline{[x, x_i]}\,\overline{x}^i: + \frac{1}{4(k+g)} C_{\mathfrak{g}}(x). \tag{4.6}$$

The map $x \mapsto \widetilde{x}$, $\overline{y} \mapsto \overline{y}$, induces an isomorphism of vertex algebras $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}}) \cong V^{k+g,1}(\mathfrak{g})$.

Proof. Set $\alpha=\frac{1}{4(k+g)}$. Fix $a,b\in\mathfrak{g}$. Since, by Lemma 4.1, $C_{\mathfrak{g}}(b)$ is central, we get, from Wick formula (4.1), that

$$\begin{aligned}
[a_\lambda\widetilde{b}] &= [a_\lambda b]-\tfrac12\sum_i[a_\lambda :\overline{[b,x_i]}\overline{x}^i:]+\alpha\lambda(a,C_{\mathfrak{g}}(b))K\\
&=[a_\lambda b]-\tfrac12\sum_i(:[a_\lambda\overline{[b,x_i]}]\overline{x}^i:+p(a,\overline{[b,x_i]}):\overline{[b,x_i]}[a_\lambda\overline{x}^i]:)\\
&-\frac12\sum_i\int_0^\lambda[[a_\lambda\overline{[b,x_i]}]_\mu\overline{x}^i]d\mu+\alpha\lambda(a,C_{\mathfrak{g}}(b))K\\
&=[a_\lambda b]-\tfrac12\sum_i(:\overline{[a,[b,x_i]]}\overline{x}^i:+p(a,\overline{[b,x_i]}):\overline{[b,x_i]}\,\overline{[a,x^i]}:)\\
&-\frac12\sum_i\lambda(x^i,[a,[b,x_i]])\overline{K}+\alpha\lambda(a,C_{\mathfrak{g}}(b))K.
\end{aligned}$$

Using the invariance of the form and Jacobi identity, we have

$$\begin{aligned}
[a_\lambda\widetilde{b}] &= [a_\lambda b]-\tfrac12\sum_i(:\overline{[a,[b,x_i]]}\overline{x}^i:-p(a,b):\overline{[b,[a,x_i]]}\overline{x}^i:)\\
&-\frac12\sum_i\lambda(x^i,[a,[b,x_i]])\overline{K}+\alpha\lambda(a,C_{\mathfrak{g}}(b))K\\
&=[a_\lambda b]-\tfrac12\sum_i:\overline{[[a,b],x_i]}\overline{x}^i:\\
&-\frac12\sum_i\lambda(a,[x^i,[x_i,b]])\overline{K}+\alpha\lambda(a,C_{\mathfrak{g}}(b))K\\
&=[a_\lambda b]-\tfrac12\sum_i:\overline{[[a,b],x_i]}\overline{x}^i:-\frac12\lambda(a,\Omega_{\mathfrak{g}}(b))\overline{K}+\alpha\lambda(a,C_{\mathfrak{g}}(b))K \qquad (4.7)
\end{aligned}$$

Next we prove that

$$[\overline{a}_\lambda\widetilde{b}]=0. \qquad (4.8)$$

By Lemma 4.1 and (4.1),

$$\begin{aligned}
[\overline{a}_\lambda\widetilde{b}] &= [\overline{a}_\lambda b]-\tfrac12\sum_i[\overline{a}_\lambda:\overline{[b,x_i]}\overline{x}^i:]\\
&=p(b)\overline{[a,b]}-\tfrac12\sum_i(:[\overline{a}_\lambda\overline{[b,x_i]}]\overline{x}^i:+p(\overline{a},\overline{[b,x_i]}):\overline{[b,x_i]}[\overline{a}_\lambda\overline{x}^i]:)\\
&=p(b)\overline{[a,b]}-\tfrac12\sum_i(([b,x_i],a)\overline{x}^i+p(\overline{a},\overline{[b,x_i]})(x^i,a)\overline{[b,x_i]})\\
&=p(b)\overline{[a,b]}-\tfrac12(p(b)\overline{[a,b]}+p(b)\overline{[a,b]})=0.
\end{aligned}$$

We now compute $[\widetilde{a}_\lambda \widetilde{b}]$. Using (4.8), we find $[\widetilde{a}_\lambda \widetilde{b}] = [a_\lambda \widetilde{b}] + \alpha[C_\mathfrak{g}(a)_\lambda \widetilde{b}]$, hence, by (4.7)

$$\begin{aligned}[\widetilde{a}_\lambda \widetilde{b}] &= [a_\lambda b] - \tfrac{1}{2}\sum_i : \overline{[[a,b],x_i]}\overline{x}^i : -\tfrac{1}{2}\lambda(a,\Omega_\mathfrak{g}(b))\overline{K} + \alpha\lambda(a,C_\mathfrak{g}(b))K + \\ &+ \alpha[C_\mathfrak{g}(a)_\lambda \widetilde{b}] \\ &= [a_\lambda b] - \tfrac{1}{2}\sum_i : \overline{[[a,b],x_i]}\overline{x}^i : -\tfrac{1}{2}\lambda(a,\Omega_\mathfrak{g}(b))\overline{K} + \alpha\lambda(a,C_\mathfrak{g}(b))K + \\ &+ \alpha([C_\mathfrak{g}(a)_\lambda b] - \tfrac{1}{2}\sum_i : \overline{[[C_\mathfrak{g}(a),b],x_i]}\overline{x}^i :) \\ &- \alpha(\frac{1}{2}\lambda(C_\mathfrak{g}(a),\Omega_\mathfrak{g}(b))\overline{K} + \alpha\lambda(C_\mathfrak{g}(a),C_\mathfrak{g}(b))K).\end{aligned}$$

By Lemma (4.1), $[C_\mathfrak{g}(a), b] = 0$. Since $\Omega_\mathfrak{g}(b) \in [\mathfrak{g},\mathfrak{g}]$ and $C_\mathfrak{g}(a)$ is central, we have $(C_\mathfrak{g}(a), \Omega_\mathfrak{g}(b)) = 0$.

The term $(C_\mathfrak{g}(a), C_\mathfrak{g}(b))$ is zero as well: if $g \neq 0$, then, by Lemma 4.1, $C_\mathfrak{g}(b) = 0$, and, if $g = 0$, as above, $(C_\mathfrak{g}(a), C_\mathfrak{g}(b)) = (C_\mathfrak{g}(a), \Omega_\mathfrak{g}(b)) = 0$. Thus, we can write

$$\begin{aligned}[\widetilde{a}_\lambda \widetilde{b}] &= [a_\lambda b] - \tfrac{1}{2}\sum_i : \overline{[[a,b],x_i]}\overline{x}^i : -\tfrac{1}{2}\lambda(a,\Omega_\mathfrak{g}(b))\overline{K} + \alpha\lambda(a,C_\mathfrak{g}(b))K + \\ &+ \alpha([C_\mathfrak{g}(a),b] + \lambda(C_\mathfrak{g}(a),b)K) \\ &= [a_\lambda b] - \tfrac{1}{2}\sum_i : \overline{[[a,b],x_i]}\overline{x}^i : -\tfrac{1}{2}\lambda(a,\Omega_\mathfrak{g}(b))\overline{K} + \alpha\lambda(a,C_\mathfrak{g}(b))K + \\ &+ \lambda(C_\mathfrak{g}(a),b)K).\end{aligned}$$

In the last equality we used the fact that $C_\mathfrak{g}(a)$ is central.

Since $\Omega_\mathfrak{g}$ (hence $C_\mathfrak{g}$) is symmetric, we have

$$[\widetilde{a}_\lambda \widetilde{b}] = [a_\lambda b] - \tfrac{1}{2}\sum_i : \overline{[[a,b],x_i]}\overline{x}^i : -\tfrac{1}{2}\lambda(a,\Omega_\mathfrak{g}(b))\overline{K} + 2\alpha\lambda(a,C_\mathfrak{g}(b))K.$$

Note that $C_\mathfrak{g}([a,b]) = 0$. In fact, for any $z \in \mathfrak{g}$,

$$(C_\mathfrak{g}([a,b]), z) = ([a,b], C_\mathfrak{g}(z)) = (a, [b, C_\mathfrak{g}(z)]) = 0.$$

It follows that $[a_\lambda b] - \frac{1}{2}\sum_i : \overline{[[a,b],x_i]}\overline{x}^i := [a,b] + \lambda(a,b)K - \frac{1}{2}\sum_i : \overline{[[a,b],x_i]}\overline{x}^i := \widetilde{[a,b]} + \lambda(a,b)K$. Hence

$$\begin{aligned}[\widetilde{a}_\lambda \widetilde{b}] &= \widetilde{[a,b]} + \lambda(a,b)K - \tfrac{1}{2}\lambda(a,\Omega_\mathfrak{g}(b))\overline{K} + 2\alpha\lambda(a,C_\mathfrak{g}(b))K \\ &= \widetilde{[a,b]} + \lambda(a,b)K - \lambda g(a,b)\overline{K} - \tfrac{1}{2}\lambda(a,C_\mathfrak{g}(b)\overline{K} \\ &+ 2\alpha\lambda(a,C_\mathfrak{g}(b)K).\end{aligned}$$

Thus, in $V^{k+g,1}(\mathfrak{g})$, we have

$$\begin{aligned}[\widetilde{a}_\lambda \widetilde{b}] &= \widetilde{[a,b]} + \lambda(a,b)(k+g) - \lambda g(a,b) - \tfrac{1}{2}\lambda(a,C_\mathfrak{g}(b))|0\rangle \\ &+ 2\alpha\lambda(a,C_\mathfrak{g}(b))(k+g)|0\rangle\end{aligned}$$

so, recalling that $\alpha = \frac{1}{4(k+g)}$, we get

$$[\widetilde{a}_\lambda \widetilde{b}] = \widetilde{[a,b]} + \lambda k(a,b).$$

We can now finish the proof as in Proposition 2.1 of [12]. □

Remark 4.2. If $g \neq 0$, as in Proposition 2.1 of [12], the map $x \mapsto \widetilde{x}$, $\overline{y} \mapsto \overline{y}$, $\mathcal{K} \mapsto K - g\overline{K}$, $\overline{\mathcal{K}} \mapsto \overline{K}$ defines a homomorphism of Lie conformal algebras $Cur(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}}) \to V(R)$. In particular, if $g \neq 0$, Propositon 4.2 holds true for any $k \in \mathbb{C}$.

Let A be a vector superspace with a non-degenerate bilinear skewsupersymmetric form $(\cdot,\cdot)$ and σ an elliptic operator preserving the parity and leaving the form invariant. If $r \in \mathbb{R}$ let $\bar{r} = r + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. Let $A^{\bar{r}}$ be the $e^{2\pi i r}$ eigenspace of A. We set $L(\sigma, A) = \oplus_{\mu \in \frac{1}{2}+\bar{r}}(t^\mu \otimes A^{\bar{r}})$ and define the bilinear form $<\cdot,\cdot>$ on $L(\sigma, A)$ by setting $< t^\mu \otimes a, t^\nu \otimes b >= \delta_{\mu+\nu,-1}(a,b)$.

If B is any superspace endowed with a non-degenerate bilinear skewsupersymmetric form $<\cdot,\cdot>$, we denote by $\mathcal{W}(B)$ be the quotient of the tensor algebra of B modulo the ideal generated by

$$a \otimes b - p(a,b) b \otimes a - < a, b >, \quad a, b \in B.$$

We now apply this construction to $B = L(\sigma, A)$ to obtain $\mathcal{W}(L(\sigma, A))$. We choose a maximal isotropic subspace L^+ of $L(\sigma, A)$ as follows: fix a maximal isotropic subspace A^+ of $A^{\bar{0}}$, and let

$$L^+ = \bigoplus_{\mu > -\frac{1}{2}} (t^\mu \otimes A^{\bar{\mu}}) \oplus (t^{-\frac{1}{2}} \otimes A^+).$$

We obtain a $\mathcal{W}(L(\sigma, A))$-module $CW(A) = \mathcal{W}(L(\sigma, A))/\mathcal{W}(L(\sigma, A))L^+$ (here CW stands for "Clifford-Weil").

Note that $-I_A$ induces an involutive automorphism of $C(A)$ that we denote by ω. Set $\tau = \omega \circ \sigma$. Then we can define fields

$$Y(a, z) = \sum_{n \in \frac{1}{2}+\bar{r}} (t^n \otimes a) z^{-n-1}, \quad a \in A^{\bar{r}},$$

where we let $t^n \otimes a$ act on $CW(A)$ by left multiplication. Setting furthermore $Y(\overline{\mathcal{K}}, z) = I_A$, we get a τ-twisted representation of $C(A)$ on $CW(A)$ (that descends to a representation of $F(A)$).

Take now $A = \mathfrak{g}$, and let σ be an automorphism of $\mathfrak{g}$ as in Section 2. Let $\mathfrak{g}^{\overline{0}} = \mathfrak{n}^0 \oplus \mathfrak{h}^0 \oplus \mathfrak{n}^0_-$ be the triangular decomposition provided by Proposition 3.5 applied to $\mathfrak{g}^{\overline{0}}$. Choosing an isotropic subspace $\mathfrak{h}^+$ of $\mathfrak{h}^0$ we can choose the maximal isotropic subspace of $\mathfrak{g}^{\overline{0}}$ provided by Lemma 3.4 and construct the corresponding Clifford-Weil module $CW(\overline{\mathfrak{g}})$, which we regard as a τ-twisted representation of $F(\overline{\mathfrak{g}})$.

In light of Proposition 4.2, given a σ-twisted representation M of $V^k(\mathfrak{g})$, we can form $\sigma \otimes \tau$-twisted representation $X(M) = M \otimes CW(\overline{\mathfrak{g}})$ of $V^{k+g,1}(\mathfrak{g})$.

In the above setting, we choose M in a particular class of representations arising from the theory of twisted affine algebras. We recall briefly their construction and refer the reader to [12] for more details.

Let $L'(\mathfrak{g}, \sigma) = \sum_{j \in \mathbb{R}} (t^j \otimes \mathfrak{g}^{\bar{j}}) \oplus \mathbb{C}K$. This is a Lie superalgebra with bracket defined by

$$[t^m \otimes a, t^n \otimes b] = t^{m+n} \otimes [a, b] + \delta_{m,-n} m(a, b) K, \quad m, n \in \mathbb{R},$$

K being a central element.

Let $(\mathfrak{h}^0)' = \mathfrak{h}^0 + \mathbb{C}K$. If $\mu \in ((\mathfrak{h}^0)')^*$, we set $\overline{\mu} = \mu_{|\mathfrak{h}^0}$. Set $\mathfrak{n}' = \mathfrak{n}^0 + \sum_{j>0} t^j \otimes \mathfrak{g}^{\bar{j}}$. Fix $\Lambda \in ((\mathfrak{h}^0)')^*$. A $L'(\mathfrak{g}, \sigma)$-module M is called a highest weight module with highest weight Λ if there is a nonzero vector $v_\Lambda \in M$ such that

$$\mathfrak{n}'(v_\Lambda) = 0,\ \ hv_\Lambda = \Lambda(h)v_\Lambda \text{ for } h \in (\mathfrak{h}^0)',\ \ U(L'(\mathfrak{g},\sigma))v_\Lambda = M. \tag{4.9}$$

Let $\Delta^{\bar{j}}$ be the set of $\mathfrak{h}^0$-weights of $\mathfrak{g}^{\bar{j}}$. If $\mu \in (\mathfrak{h}^0)^*$ and $\mathfrak{m}$ is any $\mathfrak{h}^0$-stable subspace of $\mathfrak{g}$, then we let $\mathfrak{m}_\mu$ be the corresponding weight space. Denote by Δ^0 the set of roots (i. e. the nonzero $\mathfrak{h}^0$-weights) of $\mathfrak{g}^{\bar{0}}$. Set $\Delta^0_+ = \{\alpha \in \Delta^0 \mid \mathfrak{n}^0_\alpha \neq \{0\}\}$.

Since $\mathfrak{n}^0$ and $\mathfrak{n}^0_-$ are non degenerately paired, we have that $-\Delta^0_+ = \{\alpha \in \Delta^0 \mid (\mathfrak{n}_-)_\alpha \neq \{0\}\}$. By the decomposition (3.8) we have $\Delta^0 = \Delta^0_+ \cup -\Delta^0_+$.

Set

$$\rho^{\bar{0}} = \frac{1}{2}\sum_{\alpha\in\Delta^0_+}(\operatorname{sdim}\mathfrak{n}^0_\alpha)\alpha,\quad \rho^{\bar{j}} = \frac{1}{2}\sum_{\alpha\in\Delta^{\bar{j}}}(\operatorname{sdim}\mathfrak{g}^{\bar{j}}_\alpha)\alpha \quad \text{if } \bar{j}\neq\bar{0}, \tag{4.10}$$

$$\rho_\sigma = \sum_{0\le j\le\frac{1}{2}}(1-2j)\rho^{\bar{j}}. \tag{4.11}$$

Finally set

$$z(\mathfrak{g},\sigma) = \frac{1}{2}\sum_{0\le j<1}\frac{j(1-j)}{2}\operatorname{sdim}\mathfrak{g}^{\bar{j}}. \tag{4.12}$$

Here and in the following we denote by sdim$V$ the superdimension $\dim V_0 - \dim V_1$ of a superspace $V = V_0 \oplus V_1$.

If X is a twisted representation of a vertex algebra V (see [12, § 3]) and $a \in V^{\bar{j}}$, we let

$$Y^X(a,z) = \sum_{n\in\bar{j}} a^X_{(n)} z^{-n-1}$$

be the corresponding field. As explained in [12], a highest weight module M for $L'(\mathfrak{g},\sigma)$ of highest weight Λ becomes automatically a σ-twisted representation of $V^k(\mathfrak{g})$ where $k = \Lambda(K)$.

Set

$$L^{\mathfrak{g}} = \frac{1}{2}\sum_i :x^i x_i: \in V^k(\mathfrak{g}), \tag{4.13}$$

$$L^{\bar{\mathfrak{g}}} = \frac{1}{2}\sum_i :T(\bar{x}_i)\bar{x}^i: \in F(\bar{\mathfrak{g}}). \tag{4.14}$$

We can now prove (cf. [12, Lemma 3.2]):

Lemma 4.3. *If M is a highest weight module for $L'(\mathfrak{g},\sigma)$ with highest weight Λ and $\Lambda(K) = k$ then*

$$\sum_i :C_{\mathfrak{g}}(x^i)x_i:^M_{(1)}(v_\Lambda) = \overline{\Lambda}(C_{\mathfrak{g}}(h_{\overline{\Lambda}}))v_\Lambda \tag{4.15}$$

$$(L^{\mathfrak{g}})^M_{(1)}(v_\Lambda) = \frac{1}{2}(\overline{\Lambda}+2\rho_\sigma,\overline{\Lambda})v_\Lambda + kz(\mathfrak{g},\sigma)v_\Lambda. \tag{4.16}$$

Proof. We can and do choose $\{x_i\}$ so that $x_i \in \mathfrak{g}^{\bar{s}_i}$, for some $\bar{s}_i \in \mathbb{R}/\mathbb{Z}$. Let A be any parity preserving operator on $\mathfrak{g}$ which commutes with σ. In particular A preserves

$\mathfrak{g}^{\bar{j}}$ for any $j \in \mathbb{R}$. By (3.4) of [12], we have

$$\begin{aligned}\sum_i :A(x^i)x_i:^M_{(1)} &= \sum_i \left(\sum_{n<-s_i} A(x^i)^M_{(n)}(x_i)^M_{(-n)} + \sum_{n\geq -s_i} p(x_i)(x_i)^M_{(-n)}A(x^i)^M_{(n)} \right) \\ &- \sum_{r\in\mathbb{Z}_+} \binom{-s_i}{r+1} (A(x^i)_{(r)}(x_i))^M_{(-r)}.\end{aligned} \tag{4.17}$$

We choose $s_i \in [0, 1)$, thus

$$\sum_i :A(x^i)x_i:^M_{(1)}(v_\Lambda) =$$
$$\sum_i (p(x_i)(x_i)^M_{(s_i)}A(x^i)^M_{(-s_i)} + s_i[A(x^i), x_i]^M_{(0)} - k\binom{-s_i}{2}(A(x^i), x_i))(v_\Lambda),$$

which we can rewrite as

$$\sum_i (A(x^i)^M_{(-s_i)}(x_i)^M_{(s_i)} + (s_i-1)[A(x^i), x_i]^M_{(0)} - k(\binom{-s_i}{2} - s_i)(A(x^i), x_i))(v_\Lambda) =$$
$$\sum_{i:s_i=0} A(x^i)^M_{(0)}(x_i)^M_{(0)}(v_\Lambda) + \sum_i ((s_i-1)[A(x^i), x_i]^M_{(0)} - k\binom{s_i}{2}(A(x^i), x_i))(v_\Lambda).$$

Assume now that $A = C_\mathfrak{g}$. Then, since $C_\mathfrak{g}(x^i)$ is central, $[C_\mathfrak{g}(x^i), x_i] = 0$. Note also that $\sum_{i:s_i=j}(C_\mathfrak{g}(x^i), x_i)$ is the supertrace of $(C_\mathfrak{g})_{|\mathfrak{g}^{\bar{j}}}$. Since $C_\mathfrak{g}$ is nilpotent, we obtain that $\sum_{i:s_i=j}(C_\mathfrak{g}(x^i), x_i) = 0$. Thus

$$\sum_i :C_\mathfrak{g}(x^i)x_i:^M_{(1)}(v_\Lambda) = \sum_{i:s_i=0} C_\mathfrak{g}(x^i)^M_{(0)}(x_i)^M_{(0)}(v_\Lambda).$$

We choose the basis $\{x_i\}$ by choosing, for each $\alpha \in \Delta^0 \cup \{0\}$, a basis $\{(x_\alpha)_i\}$ of $\mathfrak{g}^{\bar{0}}_\alpha$. Set $\{x^i_\alpha\}$ to be its dual basis in $(\mathfrak{g})_{-\alpha}$. If $x \in \mathfrak{g}_\alpha$, then $0 = C_\mathfrak{g}([h, x_\alpha]) = \alpha(h)C_\mathfrak{g}(x_\alpha)$. If $\alpha \neq 0$ then this implies $C_\mathfrak{g}(x_\alpha) = 0$. If $\alpha = 0$ and $(x_\alpha)_i \in \mathfrak{g}_1$, then, by (4.5), we have that $C_\mathfrak{g}((x_\alpha)_i) = 0$ as well. This implies that, if $\{h_i\}$ is an orthonormal basis of $\mathfrak{h}^0$,

$$\begin{aligned}\sum_i :C_\mathfrak{g}(x^i)x_i:^M_{(1)}(v_\Lambda) &= \sum_i C_\mathfrak{g}(h_i)^M_{(0)}(h_i)^M_{(0)}(v_\Lambda) \\ &= \sum_i \Lambda(C_\mathfrak{g}(h_i))\Lambda(h_i)v_\Lambda = \overline{\Lambda}(C_\mathfrak{g}(h_{\overline{\Lambda}}))v_\Lambda.\end{aligned}$$

Let now $A = Id$. Clearly we can assume that the basis $\{(x_\alpha)_i\}$ of $\mathfrak{g}^{\bar{0}}_\alpha$ is the union of a basis of $\mathfrak{n}^0_\alpha$ and a basis of $(\mathfrak{n}^0_-)_\alpha$ if $\alpha \in \Delta^0_+$, while, if $\alpha = 0$, we can choose the basis $\{(x_\alpha)_i\}$ to be the union of a basis of $\mathfrak{n}^0_\alpha$, a basis of $(\mathfrak{n}^0_-)_\alpha$ and an orthonormal basis $\{h_i\}$ of $\mathfrak{h}^0$. We can therefore write

$$\sum_{i:s_i=0} x^i_{(0)}(x_i)_{(0)} = 2(h_{\rho_0})_{(0)} + \sum_i (h_i)^2_{(0)} + 2\sum_{(x_\alpha)_i\in\mathfrak{n}^0_\alpha} (x^i_\alpha)_{(0)}((x_\alpha)_i)_{(0)}.$$

We find that

$$\sum_i :x^i x_i:^M_{(1)}(v_\Lambda) = (\overline{\Lambda}+2\rho_0,\overline{\Lambda})v_\Lambda + k(\sum_{0<j<1}\frac{j(1-j)}{2}\mathrm{sdim}\,\mathfrak{g}^{\overline{j}})v_\Lambda$$
$$+\sum_{i:s_i>0} s_i[x^i,x_i]^M_{(0)}(v_\Lambda).$$

In order to evaluate $\sum_{i:s_i>0} s_i[x^i,x_i]^M_{(0)}(v_\Lambda)$, we observe that

$$\sum_{i:s_i=s}[x^i,x_i] = \sum_{i:s_i=1-s} p(x_i)[x_i,x^i] = -\sum_{i:s_i=1-s}[x^i,x_i].$$

This relation is easily derived by exchanging the roles of x_i and x^i. Hence

$$\begin{aligned}\sum_{i:s_i>0} s_i[x^i,x_i] &= \sum_{i:\frac{1}{2}>s_i>0} s_i[x^i,x_i] + \sum_{i:1>s_i>\frac{1}{2}} s_i[x^i,x_i]\\ &= \sum_{i:\frac{1}{2}>s_i>0} s_i[x^i,x_i] + \sum_{i:0<s_i<\frac{1}{2}} (s_i-1)[x^i,x_i]\\ &= -\sum_{i:\frac{1}{2}>s_i>0}(1-2s_i)[x^i,x_i].\end{aligned}$$

We can choose x_i in $\mathfrak{g}^{\overline{s}_i}_\alpha$ so that $[x^i,x_i] = -p(x_i)h_\alpha$, hence

$$\sum_i s_i[x^i,x_i] = \sum_{i:0<s_i<\frac{1}{2}} 2(1-2s_i)h_{\rho^{\overline{s}_i}}, \tag{4.18}$$

hence

$$\sum_i s_i[x^i,x_i]^M_{(0)}(v_\Lambda) = \sum_{i:0<s_i<\frac{1}{2}}(1-2s_i)(2\rho^{\overline{s}_i},\overline{\Lambda})v_\Lambda.$$

This completes the proof of (4.15). □

5. Dirac operators

As in the previous section $\{x_i\}$ is a homogeneous basis of $\mathfrak{g}$ and $\{x^i\}$ is its dual basis. The following element of $V^{k,1}(\mathfrak{g})$:

$$G_{\mathfrak{g}} = \sum_i :x^i\overline{x}_i: -\frac{1}{3}\sum_{i,j}:\overline{[x^i,x_j]}\overline{x}^j\overline{x}_i: \tag{5.1}$$

is called the affine Dirac operator. It has the following properties:

$$[a_\lambda G_{\mathfrak{g}}] = \lambda k\overline{a}, \quad [\overline{a}_\lambda G_{\mathfrak{g}}] = a. \tag{5.2}$$

By the sesquilinearity of the λ-bracket, we also have

$$[(G_{\mathfrak{g}})_\lambda a] = p(a)k(\lambda\overline{a}+T(\overline{a})), \quad [(G_{\mathfrak{g}})_\lambda\overline{a}] = p(a)a. \tag{5.3}$$

If $\mathfrak{g}$ is purely even, $G_{\mathfrak{g}}$ is the Kac-Todorov-Dirac field considered in [12] and in [3] as an analogue of Kostant's cubic Dirac operator. It can be proved, by using a suitable Zhu functor $\pi : V^{k,1}(\mathfrak{g}) \to U(\mathfrak{g})\otimes\mathcal{W}(\overline{\mathfrak{g}})$, that $\pi(G_{\mathfrak{g}})$ is the Dirac operator considered by Huang and Pandzic in [7].

Write for shortness G instead of $G_{\mathfrak{g}}$. We want to calculate $[G_\lambda G]$. We proceed in steps. Set

$$\theta(x)=\frac{1}{2}\sum_i :\overline{[x,x_i]}\overline{x}^i:, \tag{5.4}$$

and note that

$$G=\sum_i :x^i\overline{x}_i: -\tfrac{2}{3}\sum_i :\theta(x^i)\overline{x}_i: . \tag{5.5}$$

We start by collecting some formulas.

Lemma 5.1.

$$[a_\lambda\theta(b)]=\theta([a,b])+\frac{1}{2}\lambda(a,\Omega_{\mathfrak{g}}(b)), \tag{5.6}$$

$$[\overline{a}_\lambda\theta(b)]=p(b)\overline{[a,b]}, \tag{5.7}$$

$$[\theta(a)_\lambda b]=\theta([a,b])+\frac{1}{2}\lambda(a,\Omega_{\mathfrak{g}}(b)), \tag{5.8}$$

$$[\theta(a)_\lambda\overline{b}]=\overline{[a,b]}, \tag{5.9}$$

$$[\theta(a)_\lambda\theta(b))]=\theta([a,b])+\frac{1}{2}\lambda(a,\Omega_{\mathfrak{g}}(b)), \tag{5.10}$$

$$[\theta(a)_\lambda\sum_i :x^i\overline{x}_i:]=-\sum_i :(x^i-\theta(x^i))\overline{[x_i,a]}: +\frac{3}{2}\lambda\overline{\Omega_{\mathfrak{g}}(a)}, \tag{5.11}$$

$$[\theta(a)_\lambda\sum_i :\theta(x^i)\overline{x}_i:]=\frac{3}{2}\lambda\overline{\Omega_{\mathfrak{g}}(a)}, \tag{5.12}$$

$$\sum_i :x^i\theta(x_i):=\sum_i :\theta(x^i)x_i:, \tag{5.13}$$

$$\sum_i :\theta(x^i)\theta(x_i):=0. \tag{5.14}$$

Proof. Formulas (5.6) and (5.7) have been proven in the proof of Proposition 4.2. Formulas (5.8) and (5.9) are obtained by applying sesquilinearity of the λ-bracket to (5.6) and (5.7). From (5.9) and (4.3) one derives that

$$[\overline{a}_\lambda(b-\theta(b))]=[(b-\theta(b))_\lambda\overline{a}]=0, \tag{5.15}$$

hence $[\theta(a)_\lambda(b-\theta(b))]=0$. This implies (5.10). Using Wick's formula (4.1), (5.8), and (5.9) we get

$$[\theta(a)_\lambda\sum_i :x^i\overline{x}_i:]=\sum_i(:\theta([a,x^i])\overline{x}_i: +p(a,x_i):x^i\overline{[a,x_i]}:)+\frac{3}{2}\lambda\overline{\Omega_{\mathfrak{g}}(a)}.$$

Note that

$$\sum_i :\theta(x^i)\overline{[x_i,a]}:=\sum_i :\theta([a,x^i])\overline{x}_i: \tag{5.16}$$

so (5.11) follows. Likewise

$$[\theta(a)_\lambda \sum_i :\theta(x^i)\overline{x}_i:] = \sum_i (:\theta([a,x^i])\overline{x}_i: + p(a,x_i):\theta(x^i)\overline{[a,x_i]}:) + \frac{3}{2}\lambda\overline{\Omega_{\mathfrak{g}}(a)}.$$

so, by (5.16), (5.12) follows as well. For (5.13), it is enough to apply formula (1.39) of [3] and (5.6). Finally,

$$\begin{aligned}\sum_i :\theta(x^i)\theta(x_i): &= \tfrac{1}{2}\sum_{i,r} :\theta(x^i):\overline{[x_i,x_r]}\overline{x}^r:: = \tfrac{1}{2}\sum_{i,r,s} :\theta(x^i)([x_i,x_r],x^s):\overline{x}_s\overline{x}^r:: \\ &= \tfrac{1}{2}\sum_{i,r,s} :\theta(x^i)(x_i,[x_r,x^s]):\overline{x}_s\overline{x}^r:: = \tfrac{1}{2}\sum_{r,s} :\theta([x_r,x^s]):\overline{x}_s\overline{x}^r::\end{aligned}$$

and

$$\begin{aligned}\sum_{r,s} :\theta([x_r,x^s]):\overline{x}_s\overline{x}^r:: &= \sum_{r,s} -p(x_r,x^s)p(\overline{x}_s,\overline{x}_r):\theta([x^s,x_r]):\overline{x}^r\overline{x}_s:: \\ &= -\sum_{r,s} p(x^s)p(x_r):\theta([x^s,x_r]):\overline{x}^r\overline{x}_s:: \\ &= -\sum_{r,s} :\theta([x_s,x^r]):\overline{x}_r\overline{x}^s:: .\end{aligned}$$

so (5.14) holds. □

We start our computation of $[G_\lambda G]$. First observe that, by (5.3),

$$\sum_i [G_\lambda :x^i\overline{x}_i:] = \sum_i :x^i x_i: + k\sum_i :T(\overline{x}_i)\overline{x}^i: + k\frac{\lambda^2}{2}\mathrm{sdim}\mathfrak{g}. \tag{5.17}$$

Next we compute $[G_\lambda \sum_i :\theta(x^i)\overline{x}_i:]$. By (5.5), (5.11), and (5.12),

$$[\theta(a)_\lambda G] = -\sum_i :(x^i - \theta(x^i))\overline{[x_i,a]}: + \frac{1}{2}\lambda\overline{\Omega_{\mathfrak{g}}(a)}$$

and, by sesquilinearity,

$$[G_\lambda \theta(a)] = p(a)\sum_i :(x^i - \theta(x^i))\overline{[x_i,a]}: + p(a)\frac{1}{2}(\lambda+T)(\overline{\Omega_{\mathfrak{g}}(a)}) \tag{5.18}$$

By Wick's formula, (5.18), and (5.3),

$$\begin{aligned}[G_\lambda \sum_i :\theta(x^i)\overline{x}_i:] &= \sum_{i,j} ::(x^j - \theta(x^j))\overline{[x_j,x_i]}:\overline{x}^i:) \\ &+ \frac{1}{2}\sum_i p(x^i)(\lambda+T):\overline{\Omega_{\mathfrak{g}}(x^i)}\overline{x}_i: + \sum_i :\theta(x^i)x_i: \\ &+ \sum_{i,j}\int_0^\lambda [:(x^j - \theta(x^j))\overline{[x_j,x_i]}:_\mu \overline{x}^i]d\mu \\ &+ \frac{1}{2}\sum_i p(x^i)\int_0^\lambda [(\lambda+T)(\overline{\Omega_{\mathfrak{g}}(x^i)})_\mu \overline{x}_i]d\mu).\end{aligned}$$

Let us compute the terms of the above sum one by one. By formula (1.40) of [3], and (5.15) above, we have

$$\sum_{i,j} :: (x^j-\theta(x^j))\overline{[x_j,x_i]} : \overline{x}^i := \sum_{i,j} : (x^j-\theta(x^j)) : \overline{[x_j,x_i]}\overline{x}^i ::$$
$$+\sum_{i,j}\int_0^T d\lambda : (x^j-\theta(x^j))[\overline{[x_j,x_i]}_\lambda \overline{x}^i] :$$
$$=2\sum_j : (x^j-\theta(x^j))\theta(x_j) : -\sum_{i,j}\int_0^T d\lambda : (x^j-\theta(x^j))([x^i,x_i],x_j) :$$
$$=2\sum_j : (x^j-\theta(x^j))\theta(x_j) :$$

Note that $\sum_i p(x^i) : \overline{\Omega_{\mathfrak g}(x^i)}\overline{x}_i := 0$. Indeed

$$\sum_i p(x_i) : \overline{\Omega_{\mathfrak g}(x^i)}\overline{x}_i : = \sum_i : \overline{\Omega_{\mathfrak g}(x_i)}\overline{x}^i := \sum_{i,r}(\Omega_{\mathfrak g}(x_i),x^r) : \overline{x}_r\overline{x}^i :$$
$$=\sum_{i,r}(x_i,\Omega_{\mathfrak g}(x^r)) : \overline{x}_r\overline{x}^i := \sum_r : \overline{x}_r\overline{\Omega_{\mathfrak g}(x^r)} :$$
$$=\sum_r p(\overline{x}_r) : \overline{\Omega_{\mathfrak g}(x^r)}\overline{x}_r := -\sum_r p(x_r) : \overline{\Omega_{\mathfrak g}(x^r)}\overline{x}_r$$

Using formula (1.38) of [3] and (5.15) above we see that

$$\sum_{i,j}\int_0^\lambda [: (x^j-\theta(x^j))\overline{[x_j,x_i]} :_\mu \overline{x}^i]d\mu = \sum_{i,j}\int_0^\lambda (x^j-\theta(x^j))(x_j,[x^i,x_i])d\mu = 0$$

Finally, since $\Omega_{\mathfrak g}-2gId$ is nilpotent, it has zero supertrace, hence

$$\sum_i p(x^i)\int_0^\lambda [(\lambda+T)(\overline{\Omega_{\mathfrak g}(x^i)})_\mu \overline{x}_i]d\mu = \sum_i p(x^i)\int_0^\lambda \lambda(x_i,\Omega_{\mathfrak g}(x^i))d\mu = g\lambda^2\mathrm{sdim}\mathfrak g.$$

It follows that

$$[G_\lambda \sum_i : \theta(x^i)\overline{x}_i :] = 2\sum_j : (x^j-\theta(x^j))\theta(x_j) : +\sum_i : \theta(x^i)x_i : +\frac{g}{2}\lambda^2\mathrm{sdim}\mathfrak g$$

Using (5.13) and (5.14), we can conclude that

$$[G_\lambda \sum_i : \theta(x^i)\overline{x}_i :] = 3\sum_i : x^i\theta(x_i) : -\frac{3}{2}\sum_i : \theta(x^i)\theta(x_i) : +\frac{g}{2}\lambda^2\mathrm{sdim}\mathfrak g,$$

Combining this with (5.17), the final outcome is that

$$[G_\lambda G] = \sum_i : (x^i-\theta(x^i))(x_i-\theta(x_i)) : +k\sum_i : T(\overline{x}_i)\overline{x}^i : +\frac{\lambda^2}{2}(k-\frac{2g}{3})\mathrm{sdim}\mathfrak g. \tag{5.19}$$

Since $x-\theta(x)=\tilde{x}-\frac{1}{4k}C_{\mathfrak g}(x)=\tilde{x}-\frac{1}{4k}\widetilde{C_{\mathfrak g}(x)}$ we have that

$$\sum_i : (x^i-\theta(x^i))(x_i-\theta(x_i)) : = \sum_i : (\tilde{x}^i-\tfrac{1}{4k}C_{\mathfrak g}(x^i))(\tilde{x}_i-\tfrac{1}{4k}C_{\mathfrak g}(x_i)) :$$
$$=\sum_i : \tilde{x}^i\tilde{x}_i : -\tfrac{1}{2k}\sum_i : \widetilde{C_{\mathfrak g}(x^i)}\tilde{x}_i : .$$

We used the fact, that, since $C_{\mathfrak{g}}$ is symmetric, $\sum_i :\widetilde{C_{\mathfrak{g}}(x^i)}\tilde{x}_i: = \sum_i :\tilde{x}^i\widetilde{C_{\mathfrak{g}}(x_i)}:$ and, since $C_{\mathfrak{g}}^2=0$, $\sum_i :\widetilde{C_{\mathfrak{g}}(x^i)}\widetilde{C_{\mathfrak{g}}(x_i)}: = 0$. Thus (5.19) can be rewritten as

$$(5.20)\quad [G_{\mathfrak{g}\lambda}G_{\mathfrak{g}}] = \sum_i (:\tilde{x}^i\tilde{x}_i: - \tfrac{1}{2k}:\widetilde{C_{\mathfrak{g}}(x^i)}\tilde{x}_i: + k:T(\overline{x}_i)\overline{x}^i:) + \frac{\lambda^2}{2}(k-\frac{2g}{3})\mathrm{sdim}\mathfrak{g}.$$

Identifying $V^{k,1}(\mathfrak{g})$ with $V^{k-g}(\mathfrak{g})\otimes F(\overline{\mathfrak{g}})$ we have that (5.20) can be rewritten as

$$(5.21)\quad [G_{\mathfrak{g}\lambda}G_{\mathfrak{g}}] = 2L^{\mathfrak{g}}\otimes|0\rangle - \tfrac{1}{2k}\sum_i :C_{\mathfrak{g}}(x^i)x_i:\otimes|0\rangle + 2k|0\rangle\otimes L^{\overline{\mathfrak{g}}} + \tfrac{\lambda^2}{2}(k-\tfrac{2}{3}g)\mathrm{sdim}\mathfrak{g}.$$

Recall that, given a highest weight representation M of $L'(\mathfrak{g},\sigma)$, we constructed a $\sigma\otimes\tau$-twisted representation $X=X(M)$ of $V^{k,1}(\mathfrak{g})$. Setting $(G_{\mathfrak{g}})^X_n = (G_{\mathfrak{g}})^X_{(n+1/2)}$, we can write the field $Y^X(G_{\mathfrak{g}},z)$ as

$$Y^X(G_{\mathfrak{g}},z) = \sum_{n\in\mathbb{Z}} G^X_n z^{-n-\frac{3}{2}}.$$

Using the fact that $(G^X_0)^2 = \frac{1}{2}[G^X_0,G^X_0]$ and (5.21), we have

(5.22)

$$(G^X_0)^2 = (L^{\mathfrak{g}} - \tfrac{1}{4k}\sum_i :C_{\mathfrak{g}}(x^i)x_i:)^M_{(1)}\otimes I_{CW(\overline{\mathfrak{g}})} - kI_M\otimes(L^{\overline{\mathfrak{g}}})^{CW(\overline{\mathfrak{g}})}_{(1)} - \tfrac{1}{16}(k-\tfrac{2}{3}g)(\mathrm{sdim}\mathfrak{g})\,I_X.$$

From now on we will write $a_{(n)}$ instead of $a^V_{(n)}$ when there is no risk of confusion for the twisted representation V.

Lemma 5.2. *In $CW(\overline{\mathfrak{g}})$, if $x\in\mathfrak{g}^{\overline{s}}$ and $n>0$, we have $\theta(x)_{(n)}\cdot 1 = 0$.*

Proof. Choose the basis $\{x_i\}$ of $\mathfrak{g}$, so that $x_i\in\mathfrak{g}^{\overline{s}_i}$. We can clearly assume $s,s_i\in[0,1)$. We apply formula (3.4) of [12] to get

$$\theta(x)_{(n)} = \sum_{i,m<s+s_i-\frac{1}{2}} \overline{[x,x_i]}_{(m)}\overline{x}^i_{(n-m-1)} - p(x,x_i)p(x)p(x_i)\sum_{i,m\ge s+s_i-\frac{1}{2}} \overline{x}^i_{(n-m-1)}\overline{[x,x_i]}_{(m)}$$
$$-\sum_i \binom{s+s_i-\frac{1}{2}}{1}[\overline{[x,x_i]}_{(0)}\overline{x}^i]_{(n-1)}.$$

If $m<s+s_i-\frac{1}{2}$, then $n-m-1>n-s-s_i-\frac{1}{2}$. Since $n\in\overline{s}$, $n>0$ and $s\in[0,1)$, we have $n-s\ge 0$. Thus $n-m-1>-s_i-\frac{1}{2}$. Since $s_i\in[0,1)$ and $n-m-1\in-\overline{s}_i+\frac{1}{2}$, we see that $n-m-1\ge -s_i+\frac{1}{2}>-\frac{1}{2}$. It follows that $\overline{x}^i_{(n-m-1)}\cdot 1=0$. If $m>s+s_i-\frac{1}{2}$, then $m>-\frac{1}{2}$ so $\overline{[x,x_i]}_{(m)}\cdot 1=0$. Since $[\overline{[x,x_i]}_{(0)}\overline{x}^i] = (x^i,[x,x_i])|0\rangle$, we see that, since $n>0$, $[\overline{[x,x_i]}_{(0)}\overline{x}^i]_{(n-1)}=0$. We therefore obtain that

$$\theta(x)_{(n)} = -p(x,x_i)p(x)p(x_i)\sum_i \overline{x}^i_{(n-s-s_i-\frac{1}{2})}\overline{[x,x_i]}_{(s+s_i-\frac{1}{2})}.$$

If $s>0$ or $s_i>0$ then $\overline{[x,x_i]}_{(s+s_i-\frac{1}{2})}\cdot 1=0$, so we can assume $s=0$ and get that

$$\theta(x)_{(n)} = -p(x,x_i)p(x)p(x_i)\sum_{i:s_i=0}\overline{x}^i_{(n-\frac{1}{2})}\overline{[x,x_i]}_{(-\frac{1}{2})} = -\sum_{i:s_i=0}\overline{[x,x_i]}_{(-\frac{1}{2})}\overline{x}^i_{(n-\frac{1}{2})}.$$

Observing that, since $n>0$, $\overline{x}^i_{(n-\frac{1}{2})}\cdot 1=0$ we get the claim. ∎

Lemma 5.3. *In $CW(\bar{\mathfrak{g}})$ we have that*

$$\sum_{i,j:s_i=s_j=0} p(\overline{[x^i,x_j]},\overline{x}^j)(\overline{x}^j)_{(-\frac{1}{2})}(\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1 = 6(\overline{h}_{\rho_0})_{(-\frac{1}{2})}\cdot 1.$$

Proof. Clearly we can choose the basis $\{x_i\}$ of $\mathfrak{g}^{\overline{0}}$ to be homogeneous with respect to the triangular decomposition $\mathfrak{g}^{\overline{0}} = \mathfrak{n}^0 \oplus \mathfrak{h}^0 \oplus \mathfrak{n}^0_-$. We can also assume that the x_i are $\mathfrak{h}^0$-weight vectors. Let μ_i be the weight of x_i. Set $\mathfrak{b}^0 = \mathfrak{h}^0 \oplus \mathfrak{n}^0$ and $\mathfrak{b}^0_- = \mathfrak{h}^0 \oplus \mathfrak{n}^0_-$. Then

$$\begin{aligned}
&\sum_{i,j:s_i=s_j=0} p(\overline{[x^i,x_j]},\overline{x}^j)(\overline{x}^j)_{(-\frac{1}{2})}(\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1\\
&\quad= \sum_{i,j:x_i\in\mathfrak{b}^0_-,s_j=0} p(\overline{[x^i,x_j]},\overline{x}^j)(\overline{x}^j)_{(-\frac{1}{2})}(\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1\\
&\quad= \sum_{i,j:x_i\in\mathfrak{b}^0_-,s_j=0} (\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}^j)_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1\\
&\quad+ \sum_{i,j:x_i\in\mathfrak{b}^0_-,s_j=0} p(\overline{[x^i,x_j]},\overline{x}^j)([x^i,x_j],x^j)(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1
\end{aligned}$$

Since $[x_j,x^j]\in\mathfrak{h}^0$, we have that

$$\begin{aligned}
&\sum_{i,j:x_i\in\mathfrak{b}^0_-,s_j=0} p(\overline{[x^i,x_j]},\overline{x}^j)([x^i,x_j],x^j)(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1 =\\
&- \sum_{i,j:s_i=s_j=0} (x^i,[x^j,x_j])(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1 = 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{i,j:s_i=s_j=0} p(\overline{[x^i,x_j]},\overline{x}^j)(\overline{x}^j)_{(-\frac{1}{2})}(\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1\\
&\quad= \sum_{i,j:x_i\in\mathfrak{b}^0_-,s_j=0} (\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}^j)_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1\\
&\quad= \sum_{i,j:x_i,\in\mathfrak{b}^0_-,s_j=0} p(\overline{x}_i,\overline{x}_j)(\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}^j)_{(-\frac{1}{2})}\cdot 1\\
&\quad+ \sum_{i:x_i\in\mathfrak{b}^0_-} (\overline{[x^i,x_i]})_{(-\frac{1}{2})}\cdot 1
\end{aligned}$$

Since $\sum_{i:x_i\in\mathfrak{b}^0_-}(\overline{[x^i,x_i]})_{(-\frac{1}{2})}\cdot 1 = \sum_{i:x_i\in\mathfrak{n}_-} p(x_i)(\overline{h}_{-\mu_i})_{(-\frac{1}{2})}\cdot 1 = 2(\overline{h}_{\rho_0})_{(-\frac{1}{2})}\cdot 1$, we need only to check that

$$\sum_{i,j:x_i,\in\mathfrak{b}^0_-,s_j=0} p(\overline{x}_i,\overline{x}_j)(\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}^j)_{(-\frac{1}{2})}\cdot 1 = 4(\overline{h}_{\rho_0})_{(-\frac{1}{2})}\cdot 1. \tag{5.23}$$

Now

$$\sum_{i,j:x_i,\in\mathfrak{b}^0_-,s_j=0} p(\overline{x}_i,\overline{x}_j)(\overline{[x^i,x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}^j)_{(-\frac{1}{2})}\cdot 1 =$$

$$\sum_{i,j:x_i,\in\mathfrak{b}^0_-,s_j=0} p(x_i)(\overline{[x^j,x^i]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}_j)_{(-\frac{1}{2})}\cdot 1 =$$

$$\sum_{i,j:x_i,x_j\in\mathfrak{b}^0_-} p(x_i)(\overline{[x^j,x^i]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}_j)_{(-\frac{1}{2})}\cdot 1 =$$

$$\sum_{i,j:x_i,x_j\in\mathfrak{b}^0_-} p(x_i)(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}_j)_{(-\frac{1}{2})}(\overline{[x^j,x^i]})_{(-\frac{1}{2})}\cdot 1$$

$$+\sum_{i,j:x_i,x_j\in\mathfrak{b}^0_-} p(x_i)(x_i,[x^j,x^i])(\overline{x}_j)_{(-\frac{1}{2})}\cdot 1$$

$$+\sum_{i,j:x_i,x_j\in\mathfrak{b}^0_-} p(x_i)p(\overline{[x^j,x^i]},\overline{x}_i)(\overline{x}_i)_{(-\frac{1}{2})}(x_j,[x^j,x^i])\cdot 1.$$

Since $[x^j,x^i]\in\mathfrak{b}^0_-$ only when $x^j,x^i\in\mathfrak{h}^0$, we see that

$$\sum_{i,j:x_i,x_j\in\mathfrak{b}^0_-} p(x_i)(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}_j)_{(-\frac{1}{2})}(\overline{[x^j,x^i]})_{(-\frac{1}{2})}\cdot 1 = 0.$$

Moreover both

$$\sum_{i,j:x_i,x_j\in\mathfrak{b}^0_-} p(x_i)(x_i,[x^j,x^i])(\overline{x}_j)_{(-\frac{1}{2})}\cdot 1$$

and

$$\sum_{i,j:x_i,x_j\in\mathfrak{b}^0_-} p(x_i)p(\overline{[x^j,x^i]},\overline{x}_i)(\overline{x}_i)_{(-\frac{1}{2})}(x_j,[x^j,x^i])\cdot 1$$

are equal to $\sum_{x_i\in\mathfrak{b}^0_-}(\overline{[x^i,x_i]})_{(-\frac{1}{2})}\cdot 1 = 2(\overline{h}_{\rho_0})_{(-\frac{1}{2})}\cdot 1$. This proves (5.23), hence the statement.

□

6. The very strange formula

We are interested in calculating $G_0^X(v_\Lambda\otimes 1)$, v_Λ being a highest weight vector of a $L'(\mathfrak{g},\sigma)$-module M with highest weight Λ such that $\Lambda(K)=k-g$.

Since σ preserves the form $(\cdot,\cdot)$, we have that $\sigma\Omega_\mathfrak{g}=\Omega_\mathfrak{g}\sigma$. It follows that $\Omega_\mathfrak{g}$ stabilizes $\mathfrak{g}^{\overline{j}}$ for any j. Recall, furthermore, that $\Omega_\mathfrak{g}(\mathfrak{g})$ is contained in the radical of the form restricted to $[\mathfrak{g},\mathfrak{g}]$. In particular $\Omega_\mathfrak{g}(\mathfrak{g})\subset\mathfrak{h}$. We can therefore choose the maximal isotropic subspace $\mathfrak{h}^+$ of $\mathfrak{h}^0$ so that $\Omega_\mathfrak{g}(\mathfrak{g}^{\overline{0}})\subset\mathfrak{h}^+$. With this choice we are now ready to prove the following result.

Proposition 6.1.

$$G_0^X(v_\Lambda\otimes 1)=v_\Lambda\otimes(\overline{h}_{\overline{\Lambda}+\rho_\sigma})_{(-\frac{1}{2})}\cdot 1. \tag{6.1}$$

Proof. Since $C_\mathfrak{g}$ is symmetric, we can rewrite G_0^X as

$$G_0^X=\sum_i :\tilde{x}^i\overline{x}_i:_{(\frac{1}{2})}+\tfrac{1}{3}\sum_i :\theta(x^i)\overline{x}_i:_{(\frac{1}{2})}-\tfrac{1}{4k}\sum_i :\tilde{x}^i\overline{C_\mathfrak{g}(x_i)}:_{(\frac{1}{2})}.$$

With easy calculations one proves that

$$\sum_i :\tilde{x}^i\overline{x}_i:_{(\frac{1}{2})}(v_\Lambda\otimes 1)=v_\Lambda\otimes(\overline{h_{\overline{\Lambda}}})_{(-\frac{1}{2})}\cdot 1. \tag{6.2}$$

Next we observe that, since $C_\mathfrak{g}(x)\in\mathfrak{h}^+$ when $x\in\mathfrak{g}^{\overline{0}}$, we have that

$$\sum_i :\tilde{x}^i\overline{C_\mathfrak{g}(x_i)}:_{(\frac{1}{2})}(v_\Lambda\otimes 1)=0. \tag{6.3}$$

It remains to check the action of $\sum_i :\theta(x^i)\overline{x}_i:_{(\frac{1}{2})}$ on 1. Choose the basis $\{x_i\}$ of $\mathfrak{g}$, so that $x_i\in\mathfrak{g}^{\overline{s}_i}$. We can clearly assume $s_i\in[0,1)$. We apply formula (3.4) of [12] to get

$$\begin{aligned}\sum_i :\theta(x^i)\overline{x}_i:_{(\frac{1}{2})}=&\sum_{i,m<-s_i}\theta(x^i)_{(m)}(\overline{x}_i)_{(-m-\frac{1}{2})}+\sum_{i,m\geq -s_i}(\overline{x}_i)_{(-m-\frac{1}{2})}\theta(x^i)_{(m)}\\&-\sum_i\binom{-s_i}{1}\overline{[x^i,x_i]}_{(-\frac{1}{2})}.\end{aligned}$$

If $m<-s_i$ then $-m>0$ so $(\overline{x}_i)_{(-m-\frac{1}{2})}\cdot 1=0$. Since $s_i\in[0,1)$, if $m>-s_i$ then $m>0$. By Lemma 5.2, $\theta(x^i)_{(m)}\cdot 1=0$. Thus

$$\begin{aligned}\sum_i :\theta(x^i)\overline{x}_i:_{(\frac{1}{2})}&=\sum_i(\overline{x}_i)_{(s_i-\frac{1}{2})}\theta(x^i)_{(-s_i)}+\sum_i s_i\overline{[x^i,x_i]}_{(-\frac{1}{2})}\\&=\sum_i\theta(x^i)_{(-s_i)}(\overline{x}_i)_{(s_i-\frac{1}{2})}+\sum_i p(x_i)\overline{[x_i,x^i]}_{(-\frac{1}{2})}+\sum_i s_i\overline{[x^i,x_i]}_{(-\frac{1}{2})}.\end{aligned}$$

If $s_i>0$ then $(\overline{x}_i)_{(s_i-\frac{1}{2})}\cdot 1=0$. Observe also that $\sum_i p(x_i)\overline{[x_i,x^i]}_{(-\frac{1}{2})}=0$. Thus

$$\sum_i :\theta(x^i)\overline{x}_i:_{(\frac{1}{2})}=\sum_{i:s_i=0}\theta(x^i)_{(0)}(\overline{x}_i)_{(-\frac{1}{2})}+\sum_i s_i\overline{[x^i,x_i]}_{(-\frac{1}{2})}.$$

Now, applying formula (3.4) of [12], we get

$$\begin{aligned}&\theta(x^i)_{(0)}=\\&\tfrac{1}{2}\sum_{j,m<s_j-\frac{1}{2}}(\overline{[x^i,x_j]})_{(m)}(\overline{x}^j)_{(-m-1)}+\tfrac{1}{2}p(\overline{[x^i,x_j]},\overline{x}^j)\sum_{j,m\geq s_j-\frac{1}{2}}(\overline{x}^j)_{(-m-1)}(\overline{[x^i,x_j]})_{(m)}\\&-\tfrac{1}{2}\sum_j\binom{s_j-\frac{1}{2}}{1}(x^j,[x^i,x_j])I_{CW(\mathfrak{g})}.\end{aligned}$$

hence

$$\begin{aligned}\theta(x^i)_{(0)}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1=&\tfrac{1}{2}\sum_{j,m<s_j-\frac{1}{2}}(\overline{[x^i,x_j]})_{(m)}(\overline{x}^j)_{(-m-1)}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1\\&+\tfrac{1}{2}p(\overline{[x^i,x_j]},\overline{x}^j)\sum_{j,m\geq s_j-\frac{1}{2}}(\overline{x}^j)_{(-m-1)}(\overline{[x^i,x_j]})_{(m)}(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1\\&-\tfrac{1}{2}\sum_j\binom{s_j-\frac{1}{2}}{1}(x^j,[x^i,x_j])(\overline{x}_i)_{(-\frac{1}{2})}\cdot 1.\end{aligned}$$

If $m < s_j - \frac{1}{2}$, then $-m-1 \geq -s_j + \frac{1}{2} > -\frac{1}{2}$. It follows that $(\overline{x}^j)_{(-m-1)}(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1 = p(\overline{x}_j, \overline{x}_i)(\overline{x}_i)_{(-\frac{1}{2})}(\overline{x}^j)_{(-m-1)} \cdot 1 = 0$. If $s_j > 0$ and $m \geq s_j - \frac{1}{2}$ or $s_j = 0$ and $m > s_j - \frac{1}{2}$, then $(\overline{[x^i, x_j]})_{(m)}(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1 = p(\overline{[x^i, x_j]}, \overline{x}_i)(\overline{x}_i)_{(-\frac{1}{2})}(\overline{[x^i, x_j]})_{(m)} \cdot 1 = 0$. Thus

$$\sum_{i:s_i=0} \theta(x^i)_{(0)}(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1 = \tfrac{1}{2} \sum_{i,j:s_i=s_j=0} p(\overline{[x^i, x_j]}, \overline{x}^j)(\overline{x}^j)_{(-\frac{1}{2})}(\overline{[x^i, x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1$$
$$- \tfrac{1}{2} \sum_{i:s_i=0,j} \binom{s_j - \frac{1}{2}}{1}(x^j, [x^i, x_j])(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1.$$

Next we compute

$$\begin{aligned}
&\sum_{i:s_i=0,j} \binom{s_j - \frac{1}{2}}{1}(x^j, [x^i, x_j])(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1 = \\
&- \sum_{i:s_i=0,j} \binom{s_j - \frac{1}{2}}{1} p(x^j, x_i)([x^j, x_j], x^i)(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1 = \\
&- \sum_{i,j} \binom{s_j - \frac{1}{2}}{1}([x^j, x_j], x^i)(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1 = \\
&- \sum_{j} \binom{s_j - \frac{1}{2}}{1} \overline{[x^j, x_j]}_{(-\frac{1}{2})} \cdot 1 \\
&= - \sum_j s_j \overline{[x^j, x_j]}_{(-\frac{1}{2})}.
\end{aligned}$$

The final outcome is that

$$\sum_i : \theta(x^i)\overline{x}_i :_{(\frac{1}{2})} \cdot 1 = \tfrac{1}{2} \sum_{i,j:s_i=s_j=0} p(\overline{[x^i, x_j]}, \overline{x}^j)(\overline{x}^j)_{(-\frac{1}{2})}(\overline{[x^i, x_j]})_{(-\frac{1}{2})}(\overline{x}_i)_{(-\frac{1}{2})} \cdot 1$$
$$+ \tfrac{3}{2} \sum_i s_i \overline{[x^i, x_i]}_{(-\frac{1}{2})} \cdot 1.$$

By (4.18), we see that

$$\sum_{i:0 \leq s_i < 1} s_i [x^i, x_i] = 2 \sum_{0 < j \leq \frac{1}{2}} h_{\rho^{\overline{j}}}.$$

Combining this observation with Lemma 5.3, we see that

$$\tfrac{1}{3} \sum_i : \theta(x^i)\overline{x}_i :_{(\frac{1}{2})} \cdot 1 = (\overline{h}_{\rho_\sigma})_{(-\frac{1}{2})} \cdot 1.$$

This, together with (6.2) and (6.3), gives the statement. □

Theorem 6.2. *Let $\mathfrak{g}$ be a basic type Lie superalgebra and let σ be an indecomposable elliptic automorphism preserving the bilinear form. Let $2g$ be the eigenvalue of the Casimir operator in the adjoint representation. Let ρ_σ be defined by (4.11) and $z(\mathfrak{g}, \sigma)$ by (4.12). Set $\rho = \rho_{Id}$. Then we have:*

(Strange formula).

$$||\rho||^2 = \frac{g}{12} \mathrm{sdim}\mathfrak{g}. \tag{6.4}$$

(Very strange formula).

$$||\rho_\sigma||^2 = g(\frac{\mathrm{sdim}\mathfrak{g}}{12} - 2z(\mathfrak{g},\sigma)) \tag{6.5}$$

Remark 6.1. If $\mathfrak{z}(\mathfrak{g})$ is non-zero, then it contains an eigenvector of the Casimir operator with zero eigenvalue, hence $g = 0$, and the very strange formula amounts to saying that ρ_σ is isotropic.

Proof. Let $\{v_i\}_{i\in\mathbb{Z}_+}$ be a basis of $CW(\bar{\mathfrak{g}})$ with $v_0 = 1$. Write $(L^{\bar{\mathfrak{g}}})^{CW(\bar{\mathfrak{g}})}_{(1)} \cdot 1 = \sum_i c_i v_i$. If M_0 is a highest weight module with highest weight $\Lambda = -\rho_\sigma + k\Lambda_0$ then $L^{\bar{\mathfrak{g}}}_{(1)}(v_\Lambda \otimes 1) = \sum c_i(v_\Lambda \otimes v_i)$ with the coefficents c_i that do not depend on k. By Proposition 6.1, $G_0(v_\Lambda \otimes 1) = 0$. Applying (5.22) and Lemma 4.3, we find that

$$\begin{aligned}0 = (-\frac{1}{2}\|\rho_\sigma\|^2 + (k-g)z(\mathfrak{g},\sigma) - \tfrac{1}{4k}\rho_\sigma(C_{\mathfrak{g}}(h_{\rho_\sigma})) - \tfrac{1}{16}(k - \frac{2g}{3})\mathrm{sdim}\mathfrak{g})(v_\Lambda \otimes 1)\\ - \sum_i c_i k(v_\Lambda \otimes v_i).\end{aligned}$$

Since this equality holds for any k, we see that $c_i = 0$ if $i > 0$. Moreover the coefficient of $v_\Lambda \otimes 1$ must vanish. This coefficient is

$$\tfrac{1}{k}\left(-\tfrac{1}{4}\rho_\sigma(C_{\mathfrak{g}}(h_{\rho_\sigma})) + k(-\frac{1}{2}\|\rho_\sigma\|^2 - gz(\mathfrak{g},\sigma) + \tfrac{g}{24}\mathrm{sdim}\mathfrak{g}) + k^2(z(\mathfrak{g},\sigma) - \tfrac{1}{16}\mathrm{sdim}\mathfrak{g} - c_0)\right),$$

so, again by the genericity of k, we obtain

$$\rho_\sigma(C_{\mathfrak{g}}(h_{\rho_\sigma})) = 0, \tag{6.6}$$

$$-\frac{1}{2}\|\rho_\sigma\|^2 - gz(\mathfrak{g},\sigma) + \tfrac{g}{24}\mathrm{sdim}\mathfrak{g} = 0, \tag{6.7}$$

$$z(\mathfrak{g},\sigma) - \tfrac{1}{16}\mathrm{sdim}\mathfrak{g} = c_0. \tag{6.8}$$

Formula (6.7) is (6.5) which specializes clearly to (6.4) when $\sigma = I_{\mathfrak{g}}$. □

As byproduct of the proof of Theorem 6.2 we also obtain

Proposition 6.3.

1. $(L^{\bar{\mathfrak{g}}})^{CW(\bar{\mathfrak{g}})}_0 \cdot 1 = z(\mathfrak{g},\sigma) - \tfrac{1}{16}\mathrm{sdim}\mathfrak{g}$.
2. *If M is a highest weight $L(\mathfrak{g})'$-module with highest weight Λ, then*

$$\begin{aligned}&(G^X_0)^2(v_\Lambda \otimes 1) =\\ &\tfrac{1}{2}\left((\overline{\Lambda} + 2\rho_\sigma, \overline{\Lambda}) - \tfrac{1}{2k}\overline{\Lambda}(C_{\mathfrak{g}}(h_{\overline{\Lambda}})) + \frac{g}{12}\mathrm{sdim}\mathfrak{g} - 2gz(\mathfrak{g},\sigma)\right)(v_\Lambda \otimes 1).\end{aligned} \tag{6.9}$$

Proof. We saw in the proof of Proposition 6.2 that $(L^{\bar{\mathfrak{g}}})^{CW(\bar{\mathfrak{g}})}_0 \cdot 1 = c_0$ and (6.8) gives our formula for c_0.

Again by (5.22) and Lemma 4.3,

$$\begin{aligned}G_0^2(v_\Lambda \otimes 1) = (\frac{1}{2}(\overline{\Lambda} + 2\rho_\sigma, \overline{\Lambda}) - \tfrac{1}{4k}\overline{\Lambda}(C_{\mathfrak{g}}(\overline{\Lambda})) + (k-g)z(\mathfrak{g},\sigma))v_\Lambda \otimes 1\\ - \tfrac{1}{16}(k - \frac{2g}{3})\mathrm{sdim}\mathfrak{g}(v_\Lambda \otimes 1)v_\Lambda \otimes 1 - kv_\Lambda \otimes (L^{\bar{\mathfrak{g}}})^{CW(\bar{\mathfrak{g}})}_0 \cdot 1).\end{aligned}$$

Using the first equality we get the second claim. □

References

[1] S. Benayadi, *Quadratic Lie superalgebras with the completely reducible action of the even part on the odd part*, J. Algebra 223 (2000) 344–366.

[2] J. Burns, *An elementary proof of the "strange formula" of Freudenthal and de Vries.* Q. J. Math. 51 (2000), no. 3, 295–297.

[3] A. De Sole and V. G. Kac, *Finite vs affine $W-$algebras.* Jpn. J. Math. **1** (2006), 137–261.

[4] M. Chuah, *Finite order automorphisms on contragredient Lie superalgebras*, J. Algebra, **351**, (2012), 138–159.

[5] H. D. Fegan, B. Steer, *On the "strange formula"' of Freudenthal and de Vries.* Math. Proc. Cambridge Philos. Soc. 105 (1989), no. 2, 249–252.

[6] H. Freudenthal, H. de Vries, *Linear Lie Groups*, Academic Press, New York, 1969.

[7] J.-S. Huang and P. Pandžić, *Dirac cohomology for Lie superalgebras.* Transform. Groups **10** (2005), 201–209.

[8] V. G. Kac, *Infinite-dimensional Lie algebras, and the Dedekind η-function.* (Russian) Funkcional. Anal. i Prilozen. 8 (1974), no. 1, 77–78.

[9] V. G. Kac, *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96.

[10] V. G. Kac, *Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function and the very strange formula.* Adv. in Math. 30 (1978), no. 2, 85–136.

[11] V. G. Kac, *Infinite dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.

[12] V. G. Kac, P. Möseneder Frajria and P. Papi, *Multiplets of representations, twisted Dirac operators and Vogan's conjecture in affine setting*, Advances in Math **217** (2008), no. 6, 2485–2562.

[13] V. G. Kac, M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory.* Lie theory and geometry, 415–456, Progr. Math., 123, Birkhäuser Boston

[14] V. G. Kac, D. Peterson *Infinite-dimensional Lie algebras, theta functions and modular forms.* Adv. in Math. 53 (1984), no. 2, 125–264.

[15] V. G. Kac and I. T. Todorov, *Superconformal current algebras and their unitary representations*, Comm. Math. Phys. **102** (1985), no. 2, 337–347.

[16] P. Papi, *Dirac operators in the affine setting*, Oberwolfach Reports, Vol. 6 n.1, 2009, 835–837

[17] V. Serganova *Automorphisms of simple Lie superalgebras.* Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984), no. 3, 585–598.

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