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Gromov-Witten/Pairs correspondence for the quintic 3-fold

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GROMOV-WITTEN/PAIRS CORRESPONDENCE
FOR THE QUINTIC 3-FOLD

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CONTENTS

0. INTRODUCTION

0.1. **Overview.** The main result of the paper is a proof of the Gromov-Witten/ Pairs (GW/P) correspondence for several compact Calabi-Yau 3-folds (including all Calabi-Yau complete intersections in products of projective spaces). The GW/P correspondence was first stated in terms of the Donaldson-Thomas theory of ideal sheaves in [17, 18] and is often referred to as the *Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) conjecture.*

(i) Via the Gromov-Witten theory of the moduli of stable maps to a 3-fold X, the generating series of curve counts is defined with a genus parameter u.

(ii) Via the Donaldson-Thomas theory of the moduli of ideal sheaves on X, the generating series of sheaf counts is defined with an Euler characteristic parameter q.

The MNOP conjecture equates the generating series (i) and (ii) after the nontrivial change of variables

$$-q = e^{iu}.$$

For Calabi-Yau 3-folds, the formulation in terms of stable pairs [34] was proven to be equivalent in [4, 40].

Our proof here uses much of the development of the Donaldson-Thomas theory of 3-folds in the past decade. The first input is a series of papers culminating in [21] which establish the MNOP conjecture for nonsingular, quasi-projective, *toric* 3-folds. The results essentially concern the 3-fold toric vertex (first studied in the Calabi-Yau case in [1]). The argument of [21] uses

- the proof of the MNOP conjecture for local curves established in the papers [5, 28, 29],
- the proof of the MNOP conjecture for 3-folds $A_n \times \mathbf{P}^1$, where A_n is the holomorphic symplectic resolution of the standard A_n-surface singularity, in the papers [16, 19, 20].

A basic idea introduced in [21] is the notion of the *capped* 3-fold vertex—a reorganization of the standard localization formula which respects the MNOP conjecture.

The second input is the foundation of the theory of stable pairs developed in [34–36]. Stable pairs are much better behaved than the Donaldson-Thomas theory of ideal sheaves since there are no floating points. Real differences between stable pairs and ideal sheaves appear in the study of *descendent* invariants involving the integration of the slant products of the Chern characters of the tautological sheaves over the moduli space. The generating functions of descendent invariants for stable pairs are conjectured to be rational functions (while the parallel generating functions for ideal sheaves are known to be irrational).

The third input is the study of descendent invariants for the stable pairs theory of 3-folds:

- the proof of the rationality of the generating series for *toric* 3-folds in the papers [30, 31],
- the formulation (and proof in the toric case) of a GW/P correspondence for descendents in [33].

For the toric arguments in [33], a *capped* 3-fold descendent vertex is introduced.

Given a 3-fold X and a nonsingular divisor $D \subset X$, there are *relative* stable pairs and Gromov-Witten theories [13, 15, 18, 30]. The interaction of the descendent theory with the relative theory plays a crucial role. We approach compact Calabi-Yau 3-fold via degeneration to toric geometries. In order to prove the GW/P correspondence, we prove appropriate GW/P correspondence for *all* the simpler descendent and relative geometries which arise in the degeneration process.

A crucial case concerns the geometry of a $\mathbf{P}^1$-bundle,

$$\pi : \mathbf{P}_S \to S,$$

over a surface S relative to a section of π. We prove GW/P correspondences in case S is a toric surface, a $K3$ surface, or a projective bundle over a higher genus curve C. The proofs systematically use the descendent theory of 3-folds. Once the GW/P correspondences for these special relative geometries are established, then the degeneration scheme of [22] can be used to prove the GW/P correspondence for any compact Calabi-Yau 3-fold which admits a good degeneration.

0.2. Descendents in Gromov-Witten theory.

Let X be a nonsingular projective 3-fold. Gromov-Witten theory is defined via integration over the moduli space of stable maps. Let $\overline{M}_{g,r}(X, \beta)$ denote the moduli space of r-pointed stable maps

from connected genus g curves to X representing the class $\beta \in H_2(X, \mathbb{Z})$. Let

$$\mathrm{ev}_i : \overline{M}_{g,r}(X, \beta) \to X,$$

$$\mathbb{L}_i \to \overline{M}_{g,r}(X, \beta)$$

denote the evaluation maps and the cotangent line bundles associated to the marked points. Let $\gamma_1, \ldots, \gamma_r \in H^*(X, \mathbb{Q})$, and let

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}_{g,n}(X, \beta), \mathbb{Q}).$$

The descendent fields, denoted by $\tau_k(\gamma)$, correspond to the classes $\psi_i^k \mathrm{ev}_i^*(\gamma)$ on the moduli space of maps. Let

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \right\rangle_{g,\beta} = \int_{[\overline{M}_{g,r}(X,\beta)]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i)$$

denote the descendent Gromov-Witten invariants. Foundational aspects of the theory are treated, for example, in [2, 3, 14].

Let C be a possibly disconnected curve with at worst nodal singularities. The genus of C is defined by $1 - \chi(\mathcal{O}_C)$. Let $\overline{M}'_{g,r}(X, \beta)$ denote the moduli space of maps with possibly disconnected domain curves C of genus g with *no* collapsed connected components. The latter condition requires each connected component of C to represent a nonzero class in $H_2(X, \mathbb{Z})$. In particular, C must represent a nonzero class β.

We define the descendent invariants in the disconnected case by

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \right\rangle'_{g,\beta} = \int_{[\overline{M}'_{g,r}(X,\beta)]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i).$$

The associated partition function is defined by[1]

$$(1) \qquad \mathsf{Z}'_{\mathsf{GW}}\Big(X; u \ \Big| \ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta = \sum_{g \in \mathbb{Z}} \Big\langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \Big\rangle'_{g,\beta} u^{2g-2}.$$

Since the domain components must map nontrivially, an elementary argument shows the genus g in the sum (1) is bounded from below. The descendent insertions in (1) should match the (genus independent) virtual dimension,

$$(2) \qquad \dim [\overline{M}'_{g,r}(X, \beta)]^{vir} = \int_\beta c_1(T_X) + r.$$

If X is a nonsingular projective toric 3-fold, then the descendent invariants can be lifted to equivariant cohomology. Let

$$\mathbf{T} = (\mathbb{C}^*)^3$$

be the 3-dimensional algebraic torus acting on X. Let s_1, s_2, s_3 be the equivariant first Chern classes of the standard representations of the three factors of $\mathbf{T}$. The equivariant cohomology of the point is well-known to be

$$H^*_{\mathbf{T}}(\bullet) = \mathbb{Q}[s_1, s_2, s_3] \ .$$

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[1]Our notation follows [18, 21] and emphasizes the role of the moduli space $\overline{M}'_{g,r}(X, \beta)$. The degree 0 collapsed contributions will not appear anywhere in our paper.

For equivariant classes $\gamma_i \in H^*_{\mathbf{T}}(X, \mathbb{Q})$, the descendent invariants

$$\left\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \right\rangle'_{g,\beta} = \int_{[\overline{M}'_{g,r}(X,\beta)]^{vir}} \prod_{i=1}^{r} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \ \in H^*_{\mathbf{T}}(\bullet)$$

are well-defined. In the equivariant setting, the descendent insertions may exceed the virtual dimension (2). The equivariant partition function

$$\mathsf{Z}'_{\mathsf{GW}}\Big(X; u \ \Big| \ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \Big)^{\mathbf{T}}_{\beta} \in \mathbb{Q}[s_1, s_2, s_3]((u))$$

is a Laurent series in u with coefficients in $H^*_{\mathbf{T}}(\bullet)$.

If X is a nonsingular quasi-projective toric 3-fold, the equivariant Gromov-Witten invariants of X are still well-defined[2] by localization residues [5]. In the quasi-projective case,

$$\mathsf{Z}'_{\mathsf{GW}}\Big(X; u \ \Big| \ \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \Big)^{\mathbf{T}}_{\beta} \in \mathbb{Q}(s_1, s_2, s_3)((u)) \ .$$

For the study of the Gromov-Witten theory of toric 3-folds, the open geometries play an important role.

0.3. **Descendents in the theory of stable pairs.** Let X be a nonsingular projective 3-fold, and let $\beta \in H_2(X, \mathbb{Z})$ be a nonzero class. We consider next the moduli space of stable pairs[3]

$$[\mathcal{O}_X \xrightarrow{s} F] \in P_n(X, \beta),$$

where F is a pure sheaf supported on a Cohen-Macaulay subcurve of X, s is a morphism with 0-dimensional cokernel, and

$$\chi(F) = n, \quad [F] = \beta.$$

The space $P_n(X, \beta)$ carries a virtual fundamental class obtained from the deformation theory of complexes in the derived category [34].

Since $P_n(X, \beta)$ is a fine moduli space, there exists a universal sheaf

$$\mathbb{F} \to X \times P_n(X, \beta);$$

see Section 2.3 of [34]. For a stable pair $[\mathcal{O}_X \to F] \in P_n(X, \beta)$, the restriction of $\mathbb{F}$ to the fiber

$$X \times [\mathcal{O}_X \to F] \ \subset \ X \times P_n(X, \beta)$$

is canonically isomorphic to F. Let

$$\pi_X \colon X \times P_n(X, \beta) \to X,$$

$$\pi_P \colon X \times P_n(X, \beta) \to P_n(X, \beta)$$

be the projections onto the first and second factors. Since X is nonsingular and $\mathbb{F}$ is π_P-flat, $\mathbb{F}$ has a finite resolution by locally free sheaves.[4] Hence, the Chern

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[2] A quasi-projective toric variety X has a finite skeleton of 1-dimensional projective torus orbits. For a stable map to X to be torus fixed, the image must lie in the 1-dimensional skeleton. Hence, the torus fixed locus of the moduli space of stable maps is compact.

[3] See [34] for a foundational development. An introduction to the subject of stable pairs can be found in [38].

[4] Both X and $P_n(X, \beta)$ carry ample line bundles.

character of the universal sheaf $\mathbb{F}$ on $X \times P_n(X, \beta)$ is well-defined. By definition, the operation

$$\pi_{P*}\big(\pi_X^*(\gamma) \cdot \mathrm{ch}_{2+i}(\mathbb{F}) \cap \pi_P^*(\,\cdot\,)\big) \colon H_*(P_n(X, \beta)) \to H_*(P_n(X, \beta))$$

is the action of the descendent $\tau_i(\gamma)$, where $\gamma \in H^*(X, \mathbb{Z})$.

For nonzero $\beta \in H_2(X, \mathbb{Z})$ and arbitrary $\gamma_i \in H^*(X, \mathbb{Q})$, define the stable pairs invariant with descendent insertions by

$$\Big\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \Big\rangle_{n, \beta} \;=\; \int_{[P_n(X, \beta)]^{vir}} \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \;.$$

The partition function is

$$\mathsf{Z}_{\mathsf{P}}\Big(X; q \,\Big|\, \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta = \sum_n \Big\langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \Big\rangle_{n, \beta} q^n.$$

Since $P_n(X, \beta)$ is empty for sufficiently negative n, the partition function is a Laurent series in q. The following conjecture was made in [35].

Conjecture 1. *The partition function* $\mathsf{Z}_{\mathsf{P}}\big(X; q \mid \prod_{i=1}^r \tau_{k_i}(\gamma_i)\big)_\beta$ *is the Laurent expansion of a rational function in* q.

Let X be a nonsingular quasi-projective toric 3-fold. The stable pairs descendent invariants can be lifted to equivariant cohomology (and defined by residues in the open case). For equivariant classes $\gamma_i \in H^*_{\mathbf{T}}(X, \mathbb{Q})$, we see

$$\mathsf{Z}_{\mathsf{P}}\Big(X; q \,\Big|\, \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta^{\mathbf{T}} \in \mathbb{Q}(s_1, s_2, s_3)((q))$$

is a Laurent series in q with coefficients in $H^*_{\mathbf{T}}(\bullet)$. A central result of [30,31] is the following rationality property.

Toric rationality. *Let X be a nonsingular quasi-projective toric 3-fold. The partition function*

$$\mathsf{Z}_P\Big(X; q \,\Big|\, \prod_{i=1}^{r} \tau_{k_i}(\gamma_i)\Big)_\beta^{\mathbf{T}}$$

is the Laurent expansion in q *of a rational function in the field* $\mathbb{Q}(q, s_1, s_2, s_3)$.

The above rationality result implies Conjecture 1 when X is a nonsingular projective toric 3-fold. The corresponding statement for the equivariant Gromov-Witten descendent partition function is expected (from calculational evidence) to be false.

0.4. **Descendent correspondence.** Let X be a nonsingular projective 3-fold. Let $\widehat{\alpha} = (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{\widehat{\ell}})$ be a partition of length $\widehat{\ell}$. Let

$$\iota_\Delta : \Delta \to X^{\widehat{\ell}}$$

be the inclusion of the small diagonal in the product $X^{\widehat{\ell}}$. For $\gamma \in H^*(X, \mathbb{Q})$, we write

$$\gamma \cdot \Delta = \iota_{\Delta *}(\gamma) \in H^*(X^{\widehat{\ell}}, \mathbb{Q}) \,.$$

By Künneth decomposition, we have

$$\gamma \cdot \Delta = \sum_{j_1, \ldots, j_{\widehat{\ell}}} c^\gamma_{j_1, \ldots, j_{\widehat{\ell}}}\, \theta_{j_1} \otimes \cdots \otimes \theta_{j_{\widehat{\ell}}},$$

where $\{\theta_j\}$ is a basis of $H^*(X, \mathbb{Q})$. We define the descendent insertion $\tau_{\widehat{\alpha}}(\gamma)$ by

$$(3) \qquad \tau_{\widehat{\alpha}}(\gamma) = \sum_{j_1,\ldots,j_{\widehat{\ell}}} c^\gamma_{j_1,\ldots,j_{\widehat{\ell}}} \, \tau_{\widehat{\alpha}_1-1}(\theta_{j_1}) \cdots \tau_{\widehat{\alpha}_\ell-1}(\theta_{j_{\widehat{\ell}}}) \, .$$

Three basic examples are the following:

- If $\widehat{\alpha} = (\widehat{a}_1)$, then
$$\tau_{(\widehat{a}_1)}(\gamma) = \tau_{\widehat{a}_1-1}(\gamma) \, .$$
  The convention of shifting the descendent by 1 allows us to index descendent insertions by standard partitions $\widehat{\alpha}$. The shift by 1 is natural from the point of view of relative/descendent correspondences and follows the notation of [33].

- If $\widehat{\alpha} = (\widehat{a}_1, \widehat{a}_2)$ and $\gamma = 1$ is the identity class, then
$$\tau_{(\widehat{a}_1, \widehat{a}_2)}(1) = \sum_{j_1,j_2} c^1_{j_1,j_2} \tau_{\widehat{a}_1-1}(\theta_{j_1}) \, \tau_{\widehat{a}_2-1}(\theta_{j_2}) \, ,$$
  where $\Delta = \sum_{j_1,j_2} c^1_{j_1,j_2} \, \theta_{j_1} \otimes \theta_{j_2}$ is the standard Künneth decomposition of the diagonal in X^2.

- If γ is the class of a point, then
$$\tau_{\widehat{\alpha}}(\mathsf{p}) = \tau_{\widehat{\alpha}_1-1}(\mathsf{p}) \cdots \tau_{\widehat{\alpha}_\ell-1}(\mathsf{p}).$$

By the multilinearity of descendent insertions, formula (3) does not depend upon the basis choice $\{\theta_j\}$.

A central result of [33] is the construction of a universal correspondence matrix $\widetilde{\mathsf{K}}$ indexed by partitions α and $\widehat{\alpha}$ of positive size with[5]

$$\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} \in \mathbb{Q}[i, c_1, c_2, c_3]((u))$$

and $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}} = 0$ unless $|\alpha| \geq |\widehat{\alpha}|$. Via the substitution

$$(4) \qquad\qquad\qquad c_i = c_i(T_X),$$

the elements of $\widetilde{\mathsf{K}}$ act by cup product on the cohomology of X with $\mathbb{Q}[i]((u))$-coefficients. The coefficients $\widetilde{\mathsf{K}}_{\alpha,\widehat{\alpha}}$ are constructed from the capped descendent vertex [33].

The matrix $\widetilde{\mathsf{K}}$ is used to define a correspondence rule

$$(5) \qquad \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \;\; \mapsto \;\; \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \, .$$

The formula for the right side of (5) requires a sum over all set partitions P of $\{1,\ldots,\ell\}$. For such a set partition P, each element $S \in P$ is a subset of $\{1,\ldots,\ell\}$. Let α_S be the associated subpartition of α, and let

$$\gamma_S = \prod_{i \in S} \gamma_i.$$

In case all cohomology classes γ_j are even, we define the right side of (5) by

$$(6) \qquad \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{P \text{ set partition of } \{1,\ldots,\ell\}} \;\; \prod_{S \in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \cdot \gamma_S) \, .$$

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[5]Here, $i^2 = -1$.

The last sum is over all partitions $\widehat{\alpha}$ of positive size, but by the vanishing

$$\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} = 0 \quad \text{unless} \quad |\alpha_S| \geq |\widehat{\alpha}| \,,$$

the summation index may be restricted to partitions $\widehat{\alpha}$ of positive size bounded by $|\alpha_S|$.

The leading term of the descendent correspondence is calculated in [33],

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = (iu)^{\ell(\alpha)-|\alpha|}\,\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) + \cdots.$$

The leading term occurs in the contribution of the maximal set partition

$$\{1\}\cup\{2\}\cup\cdots\cup\{\ell\} = \{1,2,\ldots,\ell\}$$

in ℓ parts; see [33, Section 7]. In case $\alpha = 1^\ell$ has all parts equal to 1, the leading term is the entire formula,

$$\overline{\tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)} = \tau_0(\gamma_1)\cdots\tau_0(\gamma_\ell)\,.$$

In the presence of odd cohomology, a natural sign must be included in (6). We may write set partitions P of $\{1,\ldots,\ell\}$ indexing the sum on the right side of (6) as

$$S_1\cup\cdots\cup S_{|P|} = \{1,\ldots,\ell\}.$$

The parts S_i of P are unordered, but we choose an ordering for each P. We then obtain a permutation of $\{1,\ldots,\ell\}$ by moving the elements to the ordered parts S_i (and respecting the original order in each group). The permutation, in turn, determines a sign $\sigma(P)$ determined by the anti-commutation of the associated odd classes. We then write

$$\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} = \sum_{P \text{ set partition of } \{1,\ldots,\ell\}} (-1)^{\sigma(P)} \prod_{S_i\in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_{S_i},\widehat{\alpha}}\cdot\gamma_{S_i})\,.$$

The descendent $\overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}$ is easily seen to have the same commutation rules with respect to odd cohomology as $\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)$.

To state the descendent correspondence proposed in [33] for all nonsingular projective 3-folds X, the basic degree

$$d_\beta = \int_\beta c_1(X) \ \in \mathbb{Z}$$

associated to the class $\beta \in H_2(X,\mathbb{Z})$ will be required.

Conjecture 2. *For $\gamma_i \in H^*(X,\mathbb{Q})$, we have*

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(X; q \,\Big|\, \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)\Big)_\beta$$
$$= (-iu)^{d_\beta}\mathsf{Z}'_{\mathsf{GW}}\Big(X; u \,\Big|\, \overline{\tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)}\,\Big)_\beta$$

under the variable change $-q = e^{iu}$.

By Conjecture 1, the stable pairs descendent series on the left is expected to be a rational function in q, so the change of variables is well-defined.

If X is a nonsingular quasi-projective toric 3-fold, all terms of the descendent correspondence have **T**-equivariant interpretations. We take the equivariant Künneth decomposition in (3), and the equivariant Chern classes $c_i(T_X)$ with respect to the canonical **T**-action on T_X in (4). The toric case is proven in [33].

Toric correspondence. *For* $\gamma_i \in H^*_{\mathbf{T}}(X, \mathbb{Q})$, *we have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_\mathsf{P}\left(X; q \left| \tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)\right)\right._\beta^{\mathbf{T}}$$
$$= (-iu)^{d_\beta} \mathsf{Z}'_\mathsf{GW}\left(X; u \left| \overline{\tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)}\right.\right)_\beta^{\mathbf{T}}$$

under the variable change $-q = e^{iu}$ *for all nonsingular quasi-projective toric 3-folds* X.

0.5. **Complete intersections.** Let X be a Fano or Calabi-Yau complete intersection in a product of projective spaces,

$$X \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m} \ .$$

The main result of the paper is the proof of the descendent correspondence for even classes.

Theorem 1. *Let* X *be a Fano or Calabi-Yau complete intersection 3-fold in a product of projective spaces, and let* $\gamma_i \in H^{2*}(X, \mathbb{Q})$ *be even classes. Then,*

$$\mathsf{Z}_\mathsf{P}\left(X; q \left| \tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)\right)\right._\beta \ \in \mathbb{Q}(q) \ ,$$

and we have the correspondence

$$(-q)^{-d_\beta/2} \mathsf{Z}_\mathsf{P}\left(X; q \left| \tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)\right)\right._\beta$$
$$= (-iu)^{d_\beta} \mathsf{Z}'_\mathsf{GW}\left(X; u \left| \overline{\tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)}\right.\right)_\beta$$

under the variable change $-q = e^{iu}$.

If we specialize Theorem 1 to the case where all descendents are primary or stationary, we obtain the explicit correspondence conjectured first in [18] for the Donaldson-Thomas theory of ideal sheaves.

Corollary 1. *Let* X *be a Fano or Calabi-Yau complete intersection 3-fold in a product of projective spaces, and let* $\gamma_i \in H^{2*}(X, \mathbb{Q})$ *be even classes of positive degree. Then,*

$$\mathsf{Z}_\mathsf{P}\left(X; q \left| \prod_{i=1}^r \tau_0(\gamma_i) \prod_{j=1}^s \tau_{k_j}(\mathsf{p})\right.\right)_\beta \ \in \mathbb{Q}(q) \ ,$$

and we have the correspondence

$$(-q)^{-d_\beta/2} \mathsf{Z}_\mathsf{P}\left(X; q \left| \prod_{i=1}^r \tau_0(\gamma_i) \prod_{j=1}^s \tau_{k_j}(\mathsf{p})\right.\right)_\beta$$
$$= (-iu)^{d_\beta}(iu)^{-\sum k_j} \mathsf{Z}'_\mathsf{GW}\left(X; u \left| \prod_{i=1}^r \tau_0(\gamma_i) \prod_{j=1}^s \tau_{k_j}(\mathsf{p})\right.\right)_\beta$$

under the variable change $-q = e^{iu}$.

If we specialize Theorem 1 further to the Calabi-Yau case (with no descendent insertions), we obtain the following result.

Corollary 2. *Let X be a Calabi-Yau complete intersection 3-fold in a product of projective spaces. Then,*

$$\mathsf{Z}_\mathsf{P}\Big(X;q\Big)_\beta \in \mathbb{Q}(q) \ ,$$

and we have the correspondence

$$\mathsf{Z}_\mathsf{P}\Big(X;q\Big)_\beta = \mathsf{Z}'_\mathsf{GW}\Big(X;u\Big)_\beta$$

under the variable change $-q = e^{iu}$.

Corollary 2 together with the Donaldson-Thomas/Pandharipande-Thomas (DT/PT) correspondence proven by Toda [40] and Bridgeland [4] implies the original GW/DT correspondence [17] in case X is a Calabi-Yau complete intersection in a product of projective spaces.

0.6. **Bogomol'nyi-Prasad-Sommerfield counts.** For complete intersection Calabi-Yau 3-folds, Theorem 1 is closely related to the Bogomol'nyi-Prasad-Sommerfield (BPS) structure conjectured by Gopakumar and Vafa [6] in 1998.

The method of [6] was to consider limits of type IIA string theory which may be conjecturally analyzed in M-theory. A remarkable proposal was made in [6] for the form of the Gromov-Witten potential F^X of a Calabi-Yau 3-fold X. Let

$$\mathsf{F}^X(u,v) = \sum_{g\geq 0} u^{2g-2}\mathsf{F}^X_g(v), \qquad \mathsf{F}^X_g(v) = \sum_{0\neq\beta\in H_2(X,\mathbb{Z})} N^X_{g,\beta}\, v^\beta,$$

where $N^X_{g,\beta}$ is the (connected) genus g Gromov-Witten invariant of X in curve class β. For each curve class $\beta \in H_2(X,\mathbb{Z})$ and genus g, there is conjecturally an *integer* $n^X_{g,\beta}$ counting BPS states in the associated M-theory. For fixed β, the count $n^X_{g,\beta}$ is conjectured to be nonzero for only finitely many g. The formula predicted in [6] is

$$(7) \qquad \mathsf{F}^X(u,v) = \sum_{g\geq 0}\sum_{\beta\neq 0} n^X_{g,\beta}u^{2g-2}\sum_{d>0}\frac{1}{d}\Big(\frac{\sin(du/2)}{u/2}\Big)^{2g-2}v^{d\beta}.$$

The BPS form (7) places integrality constraints on the Gromov-Witten invariants.

We can uniquely *define* invariants $n^X_{g,\beta} \in \mathbb{Q}$ by (7). Neither the integrality nor the vanishing of $n^X_{g,\beta}$ for sufficiently high g is then clear. As a corollary of Theorem 1, we obtain the following result.

Corollary 3. *Let X be a Calabi-Yau complete intersection 3-fold in a product of projective spaces, and let $\beta \in H_2(X,\mathbb{Z})$:*

   (i) *After the variable change* $-q = e^{iu}$,

$$\mathsf{F}^X_\beta(q) = \mathrm{Coeff}_{v^\beta}\big[\mathsf{F}^X\big] \in \mathbb{Q}(q)$$

     is a rational function invariant under $q \leftrightarrow q^{-1}$.
  (ii) *If, for all divisors* $\tilde\beta|\beta$, $n^X_{g,\tilde\beta}$ *vanishes for all sufficiently large g, then*

$$n^X_{g,\beta} \in \mathbb{Z}, \quad \forall g \geq 0 \ .$$

Corollary 3 follows easily from Theorem 1 and the results of Section 3 of [34]. The rationality of part (i) is slightly weaker than the full Gopakumar-Vafa predicted BPS form, but becomes equivalent with the vanishing assumed in (ii). A proof of the integrality of $n^X_{g,\beta}$ has recently been claimed in [10]. The method is analytic but eventually reduces the integrality to the local curves calculation of [5]. The vanishing (ii) is open.

0.7. **Plan of the paper.** We will prove Theorem 1 via the degeneration scheme established in [22]. To control the Gromov-Witten and stable pairs theories of Fano and Calabi-Yau complete intersections in products of projective spaces, we must prove GW/P correspondences for relative and descendent insertions in several simpler geometries.

Let $D \subset X$ be a nonsingular divisor in a nonsingular 3-fold X. The first step in the proof of Theorem 1 is to formulate a GW/P descendent correspondence for the relative geometry X/D. The interaction of the descendents with the relative divisor is explained in Section 1 with a full correspondence proposed in Conjecture 4 of Section 1.3.

The degeneration scheme of [22] requires the study of $\mathbf{P}^1$-bundles

$$\pi : \mathbf{P}_S \to S$$

over surfaces S relative to a section of π where S is either

  (i) a toric surface,
  (ii) a $K3$ surface,
  (iii) or a $\mathbf{P}^1$-bundle over a higher genus curve C.

Sections 2–6 are devoted to the proofs of descendent correspondences for the relative surface geometries (i)–(iii).

The toric case (i) is studied in Section 2. For the $K3$ surface, the results of Section 8 of [33] establish special cases. The required descendent correspondence for $\mathbf{P}_{K3}$ is proven in Section 3 after the fully equivariant relative descendent correspondence for the 3-fold cap is established.

The technically most difficult results concern the surface geometries (iii). We study higher genus curves by degeneration to genus 0. The method requires establishing correspondences for special surface geometries in Section 4 and the introduction of bi-relative residue theories in Section 5. The odd cohomology of the higher genus curves, discussed in Section 6, is controlled by the strategy first employed in [27].

The degeneration scheme and the proof of Theorem 1 is presented in Section 7. In fact, our methods are valid in any context in which the Fano or Calabi-Yau 3-folds can be efficiently degenerated. As an example, the GW/P correspondence for the Enriques Calabi-Yau is discussed in Section 7.6.

The application of relative and descendent methods to the GW/P correspondences for nontoric Calabi-Yau geometries has been one of the major motivations for our work in [30–33]. The recent proof [37] of the full Katz-Klemm-Vafa conjecture for the Gromov-Witten theory of $K3$ surfaces uses the GW/P correspondences for nontoric hypersurface Calabi-Yau 3-folds established here.

1. Relative theories

1.1. **Definitions.** Let X be a nonsingular 3-fold with a nonsingular divisor $D \subset X$. Relative Gromov-Witten and relative stable pairs theories enumerate curves with specified tangency to the divisor D. See [13, 15, 18, 30] for a technical discussion of relative theories.

In Gromov-Witten theory, relative conditions are represented by a partition μ of the integer $\int_\beta [D]$, each part μ_i of which is marked by a cohomology class $\delta_i \in H^*(D, \mathbb{Z})$,

(8) $$\mu = ((\mu_1, \delta_1), \ldots, (\mu_\ell, \delta_\ell)) .$$

The numbers μ_i record the multiplicities of intersection with D while the cohomology labels δ_i record where the tangency occurs. More precisely, let $\overline{M}'_{g,r}(X/D, \beta)_\mu$ be the moduli space of stable relative maps with tangency conditions μ along D. To impose the full boundary condition, we pull back the classes δ_i via the evaluation maps

$$(9) \qquad\qquad \overline{M}'_{g,r}(X/D, \beta)_\mu \to D$$

at the points of tangency. Also, the tangency points are considered to be unordered.[6]

In the stable pairs theory, the relative moduli space admits a natural morphism to the Hilbert scheme of d points in D,

$$P_n(X/D, \beta) \to \mathrm{Hilb}(D, \int_\beta [D]) \ .$$

Cohomology classes on $\mathrm{Hilb}(D, \int_\beta [D])$ may thus be pulled back to the relative moduli space. We will work in the *Nakajima basis* of $H^*(\mathrm{Hilb}(D, \int_\beta [D]), \mathbb{Q})$ indexed by a partition μ of $\int_\beta [D]$ labeled by cohomology classes of D as in (8). For example, the class

$$\big| \mu \big\rangle \in H^*(\mathrm{Hilb}(D, \int_\beta [D]), \mathbb{Q}) \ ,$$

with all cohomology labels equal to the identity, is $\prod \mu_i^{-1}$ times the Poincaré dual of the closure of the subvariety formed by unions of schemes of length

$$\mu_1, \ldots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of D.

The conjectural relative GW/P correspondence for primary fields [18] equates the partition functions of the theories.

Conjecture 3. *For $\gamma_i \in H^*(X, \mathbb{Q})$, we have*

$$(-q)^{-d_\beta/2} \, \mathsf{Z}_\mathsf{P}\Big(X/D; q \ \Big| \ \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \,\Big| \mu\Big)_\beta$$
$$= (-iu)^{d_\beta + \ell(\mu) - |\mu|} \, \mathsf{Z}'_\mathsf{GW}\Big(X/D; u \ \Big| \ \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \,\Big| \mu\Big)_\beta,$$

after the change of variables $e^{iu} = -q$.

As before, $\mathsf{Z}_\mathsf{P}\big(X/D; q \mid \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \,\big| \mu\big)_\beta$ is conjectured to be a rational function of q. Conjecture 3 is made for every boundary condition (8).

1.2. Diagonal classes.
To state our results for the Gromov-Witten/Pairs descendent correspondence in the relative case, a discussion of diagonal classes is required.

For the absolute geometry X, the product X^s naturally parameterizes s ordered (possibly coincident) points on X. For the relative geometry X/D, the moduli space of s ordered (possibly coincident) points

$$(p_1, \ldots, p_s) \in X/D$$

---

[6]The evaluation maps are well-defined only after ordering the points. We define the theory first with ordered tangency points. The unordered theory is then defined by dividing by the automorphisms of the cohomology weighted partition μ.

is a more subtle space. The points are not allowed to lie on the relative divisor D. When the points approach D, the target X degenerates. The resulting moduli space $(X/D)^s$ is a nonsingular variety. Let

$$\Delta_{\mathsf{rel}} \subset (X/D)^s$$

consisting of the small diagonal where all the points p_i are coincident. As a variety, Δ_{rel} is isomorphic to X.

The space $(X/D)^s$ is a special case of well-known constructions in relative geometry. For example, $(X/D)^2$ consists of 6 strata:
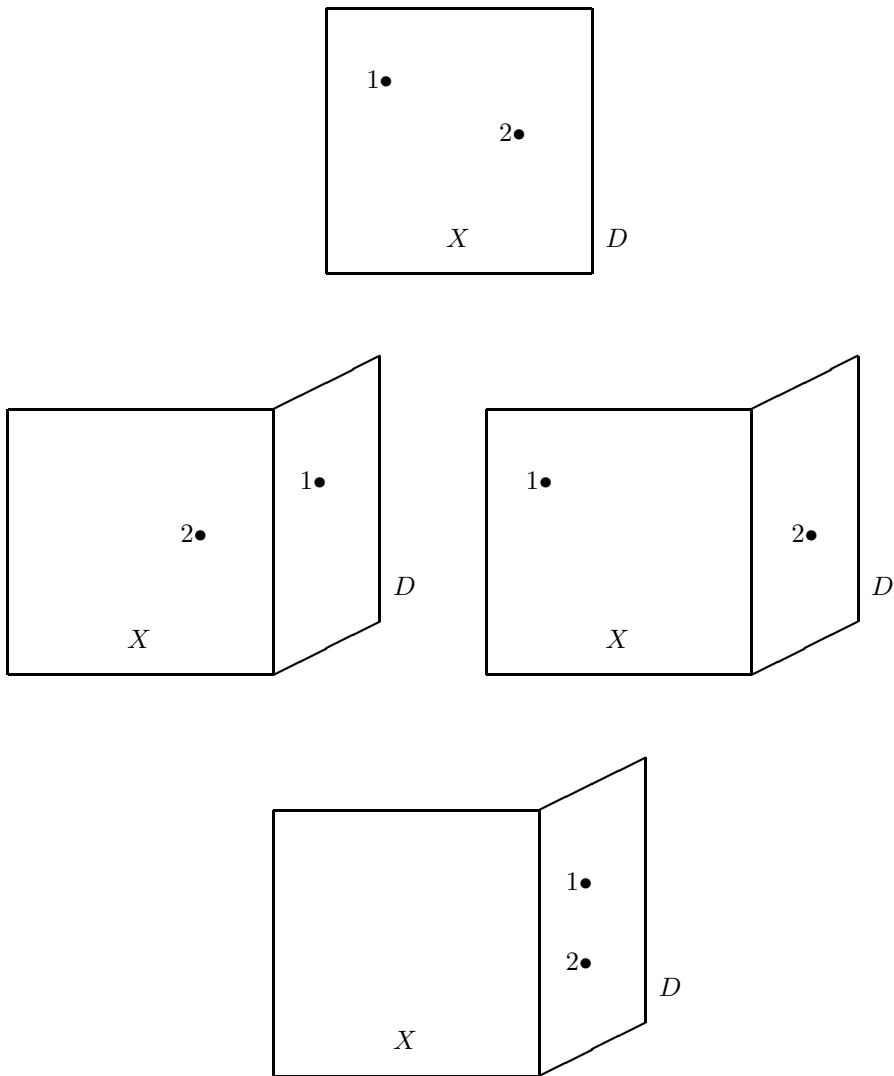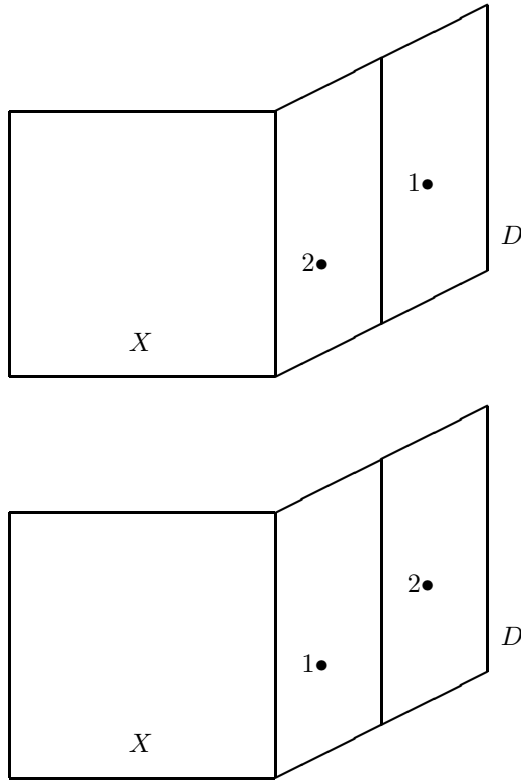
As a variety, $(X/D)^2$ is the blowup of X^2 along D^2. And, $\Delta_{\mathsf{rel}} \subset (X/D)^2$ is the strict transform of the standard diagonal.

Select a subset S of cardinality s from the r markings of the moduli space of maps. Just as $\overline{M}'_{g,r}(X, \beta)$ admits a canonical evaluation to X^s via the selected markings, the moduli space $\overline{M}'_{g,r}(X/D, \beta)_\mu$ admits a canonical evaluation

$$\mathrm{ev}_S : \overline{M}'_{g,r}(X/D, \beta)_\mu \to (X/D)^s,$$

well-defined by the definition of a relative stable map (the markings never map to the relative divisor). The class

$$\mathrm{ev}_S^*(\Delta_{\mathsf{rel}}) \in H^*(\overline{M}'_{g,r}(X/D, \beta)_\mu)$$

plays a crucial role in the relative descendent correspondence.

By forgetting the relative structure, we obtain a projection

$$\pi : (X/D)^s \to X^s \ .$$

The product contains the standard diagonal $\Delta \subset X^s$. However,

$$\pi^*(\Delta) \neq \Delta_{\mathsf{rel}} \ .$$

The former has more components in the relative boundary if $D \neq \emptyset$.

1.3. **Relative descendent correspondence.** Let $\widehat{\alpha}$ be a partition of length $\widehat{\ell}$. Let Δ_{rel} be the cohomology class of the small diagonal in $(X/D)^{\widehat{\ell}}$. For a cohomology class γ of X, let

$$\gamma \cdot \Delta_{\mathsf{rel}} \in H^*((X/D)^{\widehat{\ell}}, \mathbb{Q}).$$

We define the relative descendent insertion $\tau_\alpha(\gamma)$ by

$$(10) \qquad \tau_{\widehat{\alpha}}(\gamma) = \psi_1^{\widehat{\alpha}_1 - 1} \cdots \psi_{\hat{\ell}}^{\widehat{\alpha}_\ell - 1} \cdot \mathrm{ev}^*_{1,\ldots,\hat{\ell}}(\gamma \cdot \Delta_{\mathsf{rel}}) \ .$$

In case, $D = \emptyset$, definition (10) specializes to (3).

Let $\Omega_X[D]$ denote the locally free sheaf of differentials with logarithmic poles along D. Let

$$T_X[-D] = \Omega_X[D]^{\,\vee}$$

denote the dual sheaf of tangent fields with logarithmic zeros.

For the relative geometry X/D, we let the coefficients of $\widetilde{\mathsf{K}}$ act on the cohomology of X via the substitution

$$c_i = c_i(T_X[-D])$$

instead of the substitution $c_i = T_X$ used in the absolute case. Then, we define

$$(11) \qquad \overline{\tau_{\alpha_1 - 1}(\gamma_1) \cdots \tau_{\alpha_\ell - 1}(\gamma_\ell)} = \sum_{P \text{ set partition of } \{1,\ldots,l\}} \prod_{S \in P} \sum_{\widehat{\alpha}} \tau_{\widehat{\alpha}}(\widetilde{\mathsf{K}}_{\alpha_S,\widehat{\alpha}} \cdot \gamma_S)$$

as before via (10) instead of (3).

Definition (11) is for even classes γ_i. In the presence of odd γ_i, a sign has to be included exactly as in the absolute case.

Conjecture 4. *For $\gamma_i \in H^*(X, \mathbb{Q})$, we have*

$$(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}}\Big(X/D; q \ \Big| \tau_{\alpha_1 - 1}(\gamma_1) \cdots \tau_{\alpha_\ell - 1}(\gamma_\ell) \ \Big| \ \mu \Big)_\beta$$

$$= (-iu)^{d_\beta + \ell(\mu) - |\mu|} \mathsf{Z}'_{\mathsf{GW}}\Big(X/D; u \ \Big| \ \overline{\tau_{a_1 - 1}(\gamma_1) \cdots \tau_{\alpha_\ell - 1}(\gamma_\ell)} \ \Big| \ \mu \Big)_\beta$$

under the variable change $-q = e^{iu}$.

In addition, the stable pairs descendent series on the left is conjectured to be a rational function in q, so the change of variables is well-defined. Conjecture 4 is also well-defined in the equivariant case with respect to a group action on X preserving the relative divisor D. Definition (10) lifts canonically to the equivariant cohomology. The coefficients of $\widetilde{\mathsf{K}}$ act on the equivariant cohomology of X via the equivariant Chern classes $c_i(T_X[-D])$.

1.4. **Degeneration.** There is no difficulty in proving the compatibility of Conjectures 2 and 4 with respect to the degeneration formula. In fact, both definition (10) and the replacement of T_X by $T_X[-D]$ are required for compatibility with the degeneration formula. Definition (10) canonically lifts the diagonal splittings which occur in the correspondence for the absolute case.

The log tangent bundle arises for the following reason. Let

$$\pi : \mathfrak{X} \to B$$

be a nonsingular 4-fold fibered over an irreducible nonsingular base curve B. Let X be a nonsingular fiber, and let

$$X_1 \cup_D X_2$$

be a reducible special fiber consisting of two nonsingular 3-folds intersecting transversally along a nonsingular surface D. Let $T_{\mathfrak{X}}[-X_1 - X_2]$ be the tangent bundle of the total space $\mathfrak{X}$ with logarithmic zeros along $X_1 \cup_D X_2$. The basic restriction property

$$c(T_{\mathfrak{X}}[-X_1 - X_2])|_{X_i} = c(T_{X_i}[-D])$$

holds on the special fiber. The Chern classes of the tangent bundle of a general fiber of π therefore are extended by the Chern classes of the log tangent bundle of the special fiber.

Since the compatibility with degeneration will play an important role in the paper, we state the result (a formal consequence of the usual degeneration formula in Gromov-Witten theory [12, 13]).

Compatibility with degeneration. *Let $\gamma_1, \ldots, \gamma_\ell$ be cohomology classes on the total space $\mathfrak{X}$. We have*

$$\mathsf{Z}'_{\mathsf{GW}}\Big(X \,\Big|\, \overline{\tau_{a_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)} \,\Big)_\beta$$

$$= \sum \mathsf{Z}'_{\mathsf{GW}}\Big(X_1/D \,\Big|\, \overline{\prod_{i \in I_1} \tau_{a_i-1}(\gamma_i)} \,\Big|\, \mu \Big)_{\beta_1} \, \mathfrak{z}(\mu) u^{2\ell(\mu)}$$

$$\cdot \, \mathsf{Z}'_{\mathsf{GW}}\Big(X_2/D \,\Big|\, \overline{\prod_{i \in I_2} \tau_{a_i-1}(\gamma_i)} \,\Big|\, \mu^\vee \Big)_{\beta_2} .$$

The sum is over all marking distributions and curve class splittings

$$I_1 \cup I_2 = \{1, \ldots, \ell\}, \quad \beta = \beta_1 + \beta_2,$$

and all boundary conditions μ along D.

The boundary conditions μ are partitions weighted by elements of a fixed basis of $H^*(D, \mathbb{Q})$. The boundary condition $\mu^\vee$ has the same parts as μ but with weights given by dual elements of the dual[7] basis of $H^*(D, \mathbb{Q})$. The gluing factor is defined by

$$(12) \qquad\qquad \mathfrak{z}(\mu) = \prod_{i=1}^{\ell(\mu)} \mu_i \cdot |\mathrm{Aut}(\mu)|.$$

The first factor in (12) is simply the product of the parts of μ. The second term is the order of the symmetry group of μ *as a weighted partition*.

1.5. Relative results.

The first results about the descendent correspondence in the relative case concern projective bundles over a nonsingular surface S. Let

$$L_0, L_\infty \to S$$

be two line bundles. The projective bundle[8]

$$\mathbf{P}_S = \mathbf{P}(L_0 \oplus L_\infty) \to S$$

admits sections

$$S_i = \mathbf{P}(L_i) \subset \mathbf{P}_S .$$

We will establish the relative descendent correspondence of Conjecture 4 for $\mathbf{P}_S/S_\infty$ and $\mathbf{P}_S/S_0 \cup S_\infty$ when S is a toric surface.

There is a canonical $\mathbb{C}^*$-action on $\mathbf{P}_S$ by scaling the coordinates on the $\mathbf{P}^1$-fibers,

$$(13) \qquad\qquad \xi \cdot [l_0, l_\infty] = [\xi l_0, l_\infty], \qquad \xi \in \mathbb{C}^* .$$

We denote by t the generator of the equivariant cohomology of $\mathbb{C}^*$. We will prove Conjecture 4 for $\mathbf{P}_S$ equivariantly with respect to the fiberwise $\mathbb{C}^*$-action (13).

---

[7]With respect to the intersection pairing.

[8]We always follow the convention of projectivization by 1-dimension subspaces.

Theorem 2. *Let S be a nonsingular projective toric surface. For classes $\gamma_i \in H^*_{\mathbb{C}^*}(\mathbf{P}_S, \mathbb{Q})$, we have*

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathbf{P}_S/S_\infty; q \; \Big| \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \; \Big| \; \mu\Big)^{\mathbb{C}^*}_\beta \in \mathbb{Q}(q, t)$$

and the correspondence

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(\mathbf{P}_S/S_\infty; q \; \Big| \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \; \Big| \; \mu\Big)^{\mathbb{C}^*}_\beta$$

$$= (-iu)^{d_\beta+\ell(\mu)-|\mu|}\mathsf{Z}'_{\mathsf{GW}}\Big(\mathbf{P}_S/S_\infty; u \; \Big| \; \overline{\tau_{a_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \; \Big| \; \mu\Big)^{\mathbb{C}^*}_\beta$$

under the variable change $-q = e^{iu}$.

The parallel result holds when the projective bundle geometry is taken relative to both sections.

Theorem 3. *Let S be a nonsingular projective toric surface. Consider the relative geometry $\mathbf{P}_S/S_0 \cup S_\infty$. For $\gamma_i \in H^*_{\mathbb{C}^*}(\mathbf{P}_S, \mathbb{Q})$, we have*

$$\mathsf{Z}_{\mathsf{P}}\Big(\nu \; \Big| \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \; \Big| \; \mu\Big)^{\mathbb{C}^*}_\beta \in \mathbb{Q}(q, t)$$

and the correspondence

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(\nu \; \Big| \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \; \Big| \; \mu\Big)^{\mathbb{C}^*}_\beta$$

$$= (-iu)^{d_\beta+\ell(\nu)-|\nu|+\ell(\mu)-|\mu|}\mathsf{Z}'_{\mathsf{GW}}\Big(\nu \; \Big| \; \overline{\tau_{a_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \; \Big| \; \mu\Big)^{\mathbb{C}^*}_\beta$$

under the variable change $-q = e^{iu}$.

Theorems 2 and 3 will be proven in Section 2. We will use the absolute toric correspondence and the relative projective bundle geometries to prove Theorem 1 in Section 7.

2. Proofs of Theorems 2 and 3

2.1. Conventions. Localization with respect to the fiberwise $\mathbb{C}^*$-action will play a central role in the proofs of the descendent correspondence for the relative projective bundle geometries. We will use the localization formula for $\mathbf{P}_S/S_\infty$ in a capped form following [21, 33]. We review the constructions here.

Since the fiberwise $\mathbb{C}^*$ acts trivially on S, we have the simple characterization

$$H^*_{\mathbb{C}^*}(S, \mathbb{Q}) = H^*(S, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[t] \ .$$

Via the $\mathbb{C}^*$-invariant projection

$$\pi : \mathbf{P}_S \to S \ ,$$

there is a canonical pullback

$$\pi^* : H^*_{\mathbb{C}^*}(S, \mathbb{Q}) \to H^*_{\mathbb{C}^*}(\mathbf{P}_S, \mathbb{Q}) \ .$$

The localized $\mathbb{C}^*$-equivariant cohomology of $\mathbf{P}_S$ is a free module of rank 2 over the localized $\mathbb{C}^*$-equivariant cohomology of S,

$$(14) \qquad H^*_{\mathbb{C}^*}(\mathbf{P}_S, \mathbb{Q})_{\frac{1}{t}} \stackrel{\cong}{=} H^*_{\mathbb{C}^*}(S, \mathbb{Q})_{\frac{1}{t}} \cdot [S_0] \; \oplus H^*_{\mathbb{C}^*}(S, \mathbb{Q})_{\frac{1}{t}} \cdot [S_\infty].$$

The normal bundles of S_0 and S_∞ in $\mathbf{P}_S$ are

$$\mathcal{N}^* = L_\infty \otimes L_0^* \quad \text{and} \quad \mathcal{N} = L_0 \otimes L_\infty^*,$$

respectively. Under the isomorphism (14), we have

$$(15) \qquad \pi^*(\gamma) \;=\; \frac{\gamma}{-t-N}[S_0] + \frac{\gamma}{t+N}[S_\infty], \qquad \gamma \in H^*_{\mathbb{C}^*}(S, \mathbb{Q}),$$

where $N = c_1(\mathcal{N}) \in H^*(S, \mathbb{Q})$. Equation (15) is the Atiyah-Bott localization formula for the fiberwise $\mathbb{C}^*$-action on $\mathbf{P}_S$.

Let $L \in H_2(S, \mathbb{Z})$ be a fixed ample polarization of S. We will measure the S-degree of curve classes on $\mathbf{P}_S$ via π pushforward followed by intersection with L,

$$L_\beta = \int_S L \cdot \pi_*(\beta) \ .$$

Let $[\mathbf{P}] \in H_2(\mathbf{P}_S, \mathbb{Z})$ be the class of a fiber of π. We have an exact sequence

$$(16) \qquad 0 \longrightarrow \mathbb{Z}[\mathbf{P}] \longrightarrow H_2(\mathbf{P}_S, \mathbb{Z}) \xrightarrow{\pi_*} H_2(S, \mathbb{Z}) \longrightarrow 0 \ .$$

The only effective curve classes with $L_\beta = 0$ are multiples of $[\mathbf{P}]$.

The inclusions of S via S_0 and S_∞ determine two sections of the surjection in (16). Let

$$\mathrm{Eff}(S_0), \ \mathrm{Eff}(S_\infty) \subset H_2(\mathbf{P}_S, \mathbb{Z})$$

denote the effective curve classes supported on S_0 and S_∞, respectively.

2.2. Log tangent bundle.
The definition of the descendent correspondence

$$\tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell) \quad \mapsto \quad \overline{\tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)}$$

for the relative geometry $\mathbf{P}_S/S_\infty$ requires the Chern classes of the log tangent bundle $T_{\mathbf{P}_S}[-S_\infty]$.

Similarly, for the relative geometry $\mathbf{P}_S/S_0 \cup S_\infty$, the Chern classes of $T_{\mathbf{P}_S}[-S_0 - S_\infty]$ are required.

Lemma 1. *The total Chern classes are*

$$\begin{aligned}
c(T_{\mathbf{P}_S}[-S_\infty]) &= c(\pi^* T_S) \cdot (1 + [S_0]) \ , \\
c(T_{\mathbf{P}_S}[-S_0 - S_\infty]) &= c(\pi^* T_S)
\end{aligned}$$

in the $\mathbb{C}^*$-equivariant cohomology of $\mathbf{P}_S$ for the fiberwise action.

In both cases, the restriction of the Chern classes to S_∞ involves only classes pulled back from S via π. We leave the elementary derivation of Lemma 1 to the reader.

2.3. Capped localization.

2.3.1. *Capping over S_0.* Let $P_n(\mathbf{P}_S/S_\infty, \beta)_\mu$ be the moduli space of stable pairs[9] with the boundary condition given by μ. Let α be a partition of positive size, and let

$$\Gamma = (\gamma_1, \ldots, \gamma_\ell), \quad \gamma_i \in H^*(S, \mathbb{Q})$$

---

[9]The moduli space $P_n(\mathbf{P}_S/S_\infty, \beta)_\mu \subset P_n(\mathbf{P}_S/S_\infty, \beta)$ is defined as the inverse image, via the boundary evaluation, of a cycle $V \xhookrightarrow{\iota} \mathrm{Hilb}(S_\infty)$ representing the class μ. The virtual class of $P_n(\mathbf{P}_S/S_\infty, \beta)_\mu$ is defined by refined intersection

$$[P_n(\mathbf{P}_S/S_\infty, \beta)_\mu]^{vir} = \iota^! [P_n(\mathbf{P}_S/S_\infty, \beta)]^{vir} \ ,$$

using the nonsingularity of $\mathrm{Hilb}(S_\infty)$. For a representative cycle, Nakajima's construction may be used.

be a vector of cohomology classes. Let

$$\tau_\alpha(\Gamma_0) = \tau_{\alpha_1-1}(\gamma_1[S_0]) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell[S_0])$$

be the associated descendent insertion over S_0. We can study the partition functions

(17) $$\sum_n q^n \int_{[P_n(\mathbf{P}_S/S_\infty, \beta)_\mu]^{vir}} \tau_\alpha(\Gamma_0),$$

$$\sum_g u^{2g-2} \int_{[\overline{M}'_{g,\star}(\mathbf{P}_S/S_\infty, \beta)_\mu]^{vir}} \overline{\tau_\alpha(\Gamma_0)}$$

via localization with respect to the fiberwise $\mathbb{C}^*$-action. Recall, $\overline{\tau_\alpha(\Gamma_0)}$ is defined by (11) and is a sum of terms. For the stable maps moduli space, the number of markings depends upon the summand of $\overline{\tau_\alpha(\Gamma_0)}$ and is denoted by $\star$.

The stable pairs capped descendent over S_0 is a sum of particular localization contributions to (17). Let

$$U_{n,\beta,\mu} \subset P_n(\mathbf{P}_S/S_\infty, \beta)_\mu$$

be the open locus corresponding to stable pairs which *do not carry components of positive S-degree in the rubber over* S_∞. The open set $U_{n,\beta,\mu}$ is $\mathbb{C}^*$-invariant and has compact $\mathbb{C}^*$-fixed locus. Indeed, the fixed locus

$$U_{n,\beta,\mu}^{\mathbb{C}^*} \subset U_{n,\beta,\mu}$$

consists precisely of the $\mathbb{C}^*$-fixed loci of $P_n(\mathbf{P}_S/S_\infty, \beta)_\mu$ with no components of positive S-degree in the rubber over S_∞. Unless the curve class β is of the form

(18) $$\beta = \beta_0 + |\mu|[\mathbf{P}], \quad \beta_0 \in \mathrm{Eff}(S_0),$$

the open set $U_{n,\beta,\mu}$ is empty. The stable pairs capped descendent over S_0 is

(19) $$\mathsf{C}_0^\mathsf{P}(\tau_\alpha(\Gamma_0), \beta)_\mu = \sum_n q^n \int_{[U_{n,\beta,\mu}]^{vir}} \tau_\alpha(\Gamma_0) \quad \in \mathbb{Q}[t, \tfrac{1}{t}]((q)),$$

well-defined by $\mathbb{C}^*$-residues.[10]   If condition (18) is not satisfied, $\mathsf{C}_0^\mathsf{P}(\tau_\alpha(\Gamma_0), \beta)_\mu$ vanishes.

For the Gromov-Witten theory, we consider the parallel open set

$$\widetilde{U}_{g,\beta,\mu} \subset \overline{M}'_{g,\star}(\mathbf{P}_S/S_\infty, \beta)_\mu$$

corresponding to stable maps which *do not carry curves of positive S-degree in the rubber over* S_∞. The open set $\widetilde{U}_{g,\beta,\mu}$ is $\mathbb{C}^*$-invariant and has a compact $\mathbb{C}^*$-fixed locus. We again define the Gromov-Witten capped descendent over S_0 via

---

[10]We have presented the definition of the stable pairs capped descendent $\mathsf{C}_0^\mathsf{P}(\tau_\alpha(\Gamma_0), \beta)_\mu$ to parallel as closely as possible the definition of the Gromov-Witten capped descendent $\mathsf{C}_0^\mathsf{GW}(\tau_\alpha(\Gamma_0), \beta)_\mu$. Instead of considering $P_n(\mathbf{P}_S/S_\infty, \beta)_\mu$ as a space (after a fixed representative of the Nakajima basis element $|\mu\rangle$ is chosen), we could alternatively arrive at the *same* definition of $\mathsf{C}_0^\mathsf{P}(\tau_\alpha(\Gamma_0), \beta)_\mu$ via the $\mathbb{C}^*$-residue

$$\mathsf{C}_0^\mathsf{P}(\tau_\alpha(\Gamma_0), \beta)_\mu = \sum_n q^n \int_{[U_{n,\beta}]^{vir}} \tau_\alpha(\Gamma_0) \cup \mathrm{ev}^*(\mu),$$

where $U_{n,\beta} \subset P_n(\mathbf{P}_S/S_\infty, \beta)$ is the open locus corresponding to stable pairs which *do not carry components of positive S-degree in the rubber over* S_∞ and ev is the boundary map to the Hilbert scheme of points of S_∞.

$\mathbb{C}^*$-residues,

$$(20) \qquad \mathsf{C}_0^{\mathsf{GW}}\left(\overline{\tau_\alpha(\Gamma_0)}, \beta\right)_\mu = \sum_g u^{2g-2} \int_{[\widetilde{U}_{g,\beta,\mu}]^{vir}} \overline{\tau_\alpha(\Gamma_0)} \quad \in \mathbb{Q}[t, \frac{1}{t}]((u)).$$

The capped descendent (20) vanishes unless condition (18) is satisfied.

2.3.2. *Capping over S_∞.* We can similarly define the capped contribution over S_∞. Let

$$\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty) = \tau_{\widehat{\alpha}_1 - 1}(\widehat{\gamma}_1[S_\infty]) \cdots \tau_{\widehat{\alpha}_\ell - 1}(\widehat{\gamma}_\ell[S_\infty]) \ .$$

Consider the integrals

$$(21) \qquad \sum_n q^n \int_{[P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)_{\nu,\mu}]^{vir}} \tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty),$$

$$\sum_g u^{2g-2} \int_{[\overline{M}'_{g,\star}(\mathbf{P}_S/S_0 \cup S_\infty, \beta)_{\nu,\mu}]^{vir}} \overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}$$

via localization with respect to the fiberwise $\mathbb{C}^*$-action.

The stable pairs capped descendent over S_∞ is again a sum of particular localization contributions to (21). Let

$$W_{n,\beta,\nu,\mu} \subset P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)_{\nu,\mu}$$

be the open locus corresponding to stable maps which *do not carry components of positive S-degree in the rubber over S_0*. The open set $W_{n,\beta,\nu,\mu}$ is $\mathbb{C}^*$-invariant and has a compact $\mathbb{C}^*$-fixed locus. The fixed locus

$$W_{n,\beta,\nu,\mu}^{\mathbb{C}^*} \subset W_{n,\beta,\nu,\mu}$$

consists precisely of the $\mathbb{C}^*$-fixed loci of $P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)_{\nu,\mu}$ with no components of positive S-degree in S_0. Unless the curve class β satisfies

$$(22) \qquad \beta = |\nu|[\mathbf{P}] + \beta_\infty, \quad \beta_\infty \in \mathrm{Eff}(S_\infty),$$

the open set $W_{n,\beta,\nu,\mu}$ is empty. The stable pairs capped descendent over S_∞ is

$$(23) \qquad \mathsf{C}_\infty^{\mathsf{P}}(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), \beta)_{\nu,\mu} = \sum_n q^n \int_{[W_{n,\beta,\nu,\mu}]^{vir}} \tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty) \quad \in \mathbb{Q}[t, \frac{1}{t}]((q))$$

well-defined by $\mathbb{C}^*$-residues. The capped descendent (23) vanishes unless condition (22) is satisfied.

For Gromov-Witten theory, we consider the parallel open set

$$\widetilde{W}_{g,\beta,\nu,\mu} \subset \overline{M}'_{g,\star}(\mathbf{P}_S/S_0 \cup S_\infty, \beta)_{\nu,\mu}$$

corresponding to stable maps which *do not carry curves of positive S-degree in the rubber over S_0*. The open set $\widetilde{W}_{g,\beta,\nu,\mu}$ is $\mathbb{C}^*$-invariant and has a compact $\mathbb{C}^*$-fixed locus. We define the Gromov-Witten capped descendent over S_∞ via $\mathbb{C}^*$-residues,

$$(24) \qquad \mathsf{C}_\infty^{\mathsf{GW}}\left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}, \beta\right)_{\nu,\mu} = \sum_g u^{2g-2} \int_{[\widetilde{W}_{g,\beta,\nu,\mu}]^{vir}} \overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)} \quad \in \mathbb{Q}[t, \frac{1}{t}]((u)).$$

The capped descendent (24) vanishes unless condition (22) is satisfied.

2.3.3. *Capped localization formula.* Let $\Phi = (\phi_1, \ldots, \phi_f)$ be a graded basis of $H^*(S, \mathbb{Q})$, and let $\phi_1^\vee, \ldots, \phi_f^\vee$ be the dual basis satisfying

$$\int_S \phi_i \cdot \phi_j^\vee = \delta_{ij} \ .$$

We take the cohomological weights of the relative boundary condition μ to lie in the basis Φ. Let $\mu^\vee$ then denote the boundary condition obtained by replacing each ϕ_i by the Poincaré dual class $\phi_i^\vee$.

Let $\beta \in H_2(\mathbf{P}_S, \mathbb{Z})$ be a curve class. A splitting of β of type $d \geq 0$ is a pair of curve classes β_0, β_∞ of $\mathbf{P}_S$ satisfying

$$\beta_0 \in \mathrm{Eff}(S_0), \quad \beta_\infty \in \mathrm{Eff}(S_\infty), \quad \text{and} \quad \beta_0 + d[\mathbf{P}] + \beta_\infty = \beta.$$

We will often denote the type of a splitting by

$$\beta = \beta_0 + d[\mathbf{P}] + \beta_\infty \ .$$

A given $\beta \in H_2(\mathbf{P}_S, \mathbb{Z})$ admits only finitely many such splittings.

The capped localization formula for $\mathbf{P}_S/S_\infty$ is easy to state in terms of the capped descendents over S_0 and S_∞. First consider the stable pairs partition function[11]

$$\mathsf{Z}_{\beta,\mu}^{\mathsf{P}} \left(\tau_\alpha(\Gamma_0) \cdot \tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty) \right)^{\mathbb{C}^*}$$
$$= \sum_n q^n \int_{[P_n(\mathbf{P}_S/S_\infty, \beta)_\mu]^{vir}} \prod_i \tau_{\alpha_i - 1}(\gamma_i[S_0]) \cdot \prod_j \tau_{\widehat{\alpha}_j - 1}(\widehat{\gamma}_j[S_\infty]).$$

The capped localization formula is

$$\mathsf{Z}_{\beta,\mu}^{\mathsf{P}} \left(\tau_\alpha(\Gamma_0) \cdot \tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty) \right)^{\mathbb{C}^*}$$
$$= \sum \mathsf{C}_0^{\mathsf{P}}(\tau_\alpha(\Gamma_0), \beta_0 + d[\mathbf{P}])_\nu \frac{(-1)^{|\nu| - \ell(\nu)} \mathfrak{z}(\nu)}{q^{|\nu|}} \ \mathsf{C}_\infty^{\mathsf{P}}(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), d[\mathsf{P}] + \beta_\infty)_{\nu^\vee, \mu}).$$

The sum on the right side is the triple sum

$$\sum_{d \geq 0} \sum_{\beta_0 + d[\mathsf{P}] + \beta_\infty = \beta} \sum_{|\nu| = d} \ .$$

The gluing factor $\mathfrak{z}(\nu)$ is defined by (12).

The parallel partition function[12] in the Gromov-Witten theory is

$$\mathsf{Z}_{\beta,\mu}'^{\mathsf{GW}} \left(\overline{\tau_\alpha(\Gamma_0)} \cdot \overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)} \right)^{\mathbb{C}^*}$$
$$= \sum_g u^{2g-2} \int_{[\overline{M}_{g,\star}'(\mathbf{P}_S/S_\infty, \beta)_\mu]^{vir}} \overline{\prod_i \tau_{\alpha_i - 1}(\gamma_i[S_0])} \cdot \overline{\prod_j \tau_{\widehat{\alpha}_j - 1}(\widehat{\gamma}_j[S_\infty])},$$

---

[11] We depart slightly from the notation of the Introduction for more efficient presentation of the data.

[12] Since S_0 and S_∞ are disjoint, we have

$$\overline{\tau_\alpha(\Gamma_0) \cdot \tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)} = \overline{\tau_\alpha(\Gamma_0)} \cdot \overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}$$

by definition (11).

and the capped localization formula is

$$
Z'^{\mathsf{GW}}_{\beta,\mu}\left(\overline{\tau_\alpha(\Gamma_0)}\cdot\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}\right)^{\mathbb{C}^*}
$$
$$
= \mathsf{C}^{\mathsf{GW}}_0\left(\overline{\tau_\alpha(\Gamma_0)},\beta_0+d[\mathbf{P}]\right)_\nu \; \mathfrak{z}(\nu)\, u^{2\ell(\nu)}\; \mathsf{C}^{\mathsf{GW}}_\infty\left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)},d[\mathbf{P}]+\beta_\infty\right)_{\nu^\vee,\mu},
$$

where again the sum on the right is the triple sum

$$
\sum_{d\geq 0}\;\sum_{\beta_0+d[\mathsf{P}]+\beta_\infty=\beta}\;\sum_{|\nu|=d}\;.
$$

The idea of capping localization contributions has been used extensively in [21, 33]. The main properties are the following:

- By definition, the capped contributions differ from the bare residue contributions by just edge contributions and 1-legged S-degree 0 contributions on the far vertex.
- The capped localization formula is obtained from the standard localization formula by redistributing edge and 1-legged S-degree 0 contributions (no new geometric derivation is required).
- The capped contributions, unlike the bare contributions, are conjectured to have well-behaved rationality and GW/P correspondence properties.

2.3.4. *Capped edge.* In the capped localization formulas of [21, 33], capped edge terms appear as follows:

$$
Z^{\mathsf{P}}_{d,\nu,\mu} \;=\; \sum_n q^n \int_{[P_n(\mathbf{P}_S/S_0\cup S_\infty,d[\mathbf{P}])_\mu]^{vir}} 1\;,
$$
$$
Z'^{\mathsf{GW}}_{d,\nu,\mu} \;=\; \sum_g u^{2g-2}\int_{[\overline{M}_g(\mathbf{P}_S/S_0\cup S_\infty,d[\mathbf{P}])_\mu]^{vir}} 1,
$$

where $d=|\nu|=|\mu|$. By the following result, the capped edges here are trivial, and hence need not be included in the capped localization formulas in our geometry.

Lemma 2. *We have the evaluations*

$$
Z^{\mathsf{P}}_{d,\nu,\mu^\vee} \;=\; \delta_{\nu,\mu}\,\frac{(-1)^{|\nu|-\ell(\nu)}}{\mathfrak{z}(\nu)}\,q^d,
$$
$$
Z'^{\mathsf{GW}}_{d,\nu,\mu^\vee} \;=\; \delta_{\nu,\mu}\,\frac{1}{\mathfrak{z}(\nu)}\,u^{-2\ell(\nu)}\;.
$$

Proof. We use the standard degeneration of $\mathbf{P}_S/S_0\cup S_\infty$ to

$$
\mathbf{P}_S/S_0\cup S_\infty \;\cup\; \mathbf{P}_S/S_0\cup S_\infty\;.
$$

For the stable pairs, the degeneration formula for the capped edges is

$$
Z^{\mathsf{P}}_{d,\nu,\mu^\vee} = \sum_\lambda Z^{\mathsf{P}}_{d,\nu,\lambda^\vee}\,(-1)^{|\lambda|-\ell(\lambda)}\,\mathfrak{z}(\lambda)\,q^{-|\lambda|}\,Z^{\mathsf{P}}_{d,\lambda,\mu^\vee}\;.
$$

The capped edge evaluation follows immediately. A parallel argument is valid in the Gromov-Witten theory. $\qquad\square$

2.4. **Proof of Theorem 2.**

2.4.1. *Correspondence over S_0.* We will use the capped localization formulas together with $\mathbb{C}^*$-equivariant descendent correspondences for the capped contributions over S_0 and S_∞ to prove Theorem 2.

To study the contributions over S_0, we require the full torus action. Since S is a toric surface, a 2-dimension torus T acts on S. We lift T to the line bundles L_0 and L_1. Let $\mathbf{T}$ be the full 3-dimensional torus acting on the relative geometry $\mathbf{P}_S/S_\infty$,

$$\mathbf{T} = T \times \mathbb{C}^*,$$

where the second factor is the fiberwise $\mathbb{C}^*$.

The capped contribution over S_0 is not difficult to understand. The open sets

$$(25) \qquad U_{n,\beta,\mu} \subset P_n(\mathbf{P}_S/S_\infty, \beta)_\mu, \quad \widetilde{U}_{g,\beta,\mu} \subset \overline{M}'_{g,\star}(\mathbf{P}_S/S_\infty, \beta)_\mu$$

after localization with respect to the $\mathbf{T}$-action yield only the standard capped descendent vertices at the T-fixed points of S_0.

We consider capped contributions over S_0 in curve class

$$\beta = \beta_0 + d[\mathbf{P}], \quad \beta_0 \in \mathrm{Eff}(S_0).$$

Let μ be a boundary condition along S_∞ with $|\mu| = d$.

Proposition 3. *The $\mathbb{C}^*$-equivariant descendent correspondence for the capped contributions over S_0 holds. We have* $\mathsf{C}_0^\mathsf{P}\left(\tau_\alpha(\Gamma_0), \beta\right)_\mu \in \mathbb{Q}(q,t)$ *and*

$$(-q)^{-d_\beta/2} \, \mathsf{C}_0^\mathsf{P}\left(\tau_\alpha(\Gamma_0), \beta\right)_\mu = (-iu)^{d_\beta + \ell(\mu) - |\mu|} \, \mathsf{C}_0^\mathsf{GW}\left(\overline{\tau_\alpha(\Gamma_0)}, \beta\right)_\mu$$

under the variable change $-q = e^{iu}$.

Proof. We apply $\mathbf{T}$-equivariant localization to the open sets (25) to express capped contributions in terms of descendent vertices [33]. We then apply the GW/P correspondence established in Theorem 8 of [33]. Since the $\mathbb{C}^*$-fixed locus is compact (for the fiberwise $\mathbb{C}^*$-action), we may set the equivariant parameters of T to 0. $\qquad\square$

2.4.2. *Correspondence over S_∞.* The next step is to prove a descendent correspondence for the capped contributions over S_∞. Consider capped contributions over S_∞ in curve class

$$\beta = d[\mathbf{P}] + \beta_\infty, \quad \beta_\infty \in \mathrm{Eff}(S_\infty).$$

Let ν, μ be boundary conditions along S_0 and S_∞ with $|\nu| = d$.

Proposition 4. *The $\mathbb{C}^*$-equivariant descendent correspondence for the capped contributions over S_∞ holds. We have* $\mathsf{C}_\infty^\mathsf{P}\left(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), \beta\right)_{\nu,\mu} \in \mathbb{Q}(q,s)$ *and*

$$(-q)^{-d_\beta/2} \, \mathsf{C}_\infty^\mathsf{P}\left(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), \beta\right)_{\nu,\mu} = (-iu)^{d_\beta + \ell(\nu) - |\nu| + \ell(\mu) - |\mu|} \, \mathsf{C}_\infty^\mathsf{GW}\left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}, \beta\right)_{\nu,\mu}$$

under the variable change $-q = e^{iu}$.

Theorem 2 is an immediate consequence of Propositions 3 and 4 and the capped localization formulas for $\mathbf{P}_S/S_\infty$. Proposition 4 is harder to prove than Proposition 3 because of the possibility of curves of positive S-degree in the rubber over S_∞. The GW/P correspondence for the descendent vertex [33] does not directly apply. The proof of Proposition 4 is given in Sections 2.4.3–2.4.6.

2.4.3. *Induction strategy.* If β is not an effective curve class, both capped descendent contributions over S_∞ vanish and Proposition 4 is trivial.

We will prove Proposition 4 for effective curve classes by induction on L_β and the length $\ell(\widehat{\alpha})$. The base case is

$$L_\beta = 0 \quad \text{and} \quad \ell(\widehat{\alpha}) = 0 \ .$$

If $L_\beta = 0$, then $\beta_\infty = 0$. If $\ell(\widehat{\alpha})$ is also 0, the capped contribution over S_∞ is equal to the capped edge term determined by Lemma 2. Proposition 4 for $\beta_\infty = 0$ and $\ell(\widehat{\alpha}) = 0$ is then easily seen to hold.

Consider capped contributions over S_∞ in curve class

$$\beta = d[\mathbf{P}] + \beta_\infty, \quad \beta_\infty \in \mathrm{Eff}(S_\infty).$$

Let ν and μ be relative conditions along S_0 and S_∞ with $|\nu| = d$. We take the cohomology weights of ν and μ to lie in the basis

$$\Phi = (\phi_1, \ldots, \phi_f)$$

of $H^*(S, \mathbb{Q})$. Let $\deg(\nu)$ and $\deg(\mu)$ be the sum of the (complex) degrees[13] of the cohomology weights of ν and μ, respectively. The codimensions of the relative conditions ν and μ are

$$\theta(\nu) = |\nu| - \ell(\nu) + \deg(\nu) \quad \text{and} \quad \theta(\mu) = |\mu| - \ell(\mu) + \deg(\mu) \ .$$

For the vector $\widehat{\Gamma} = (\widehat{\gamma}_1, \ldots, \widehat{\gamma}_\ell)$ associated to the descendent insertion, define[14]

$$(26) \qquad \deg(\widehat{\Gamma}) = \frac{1}{2} \sum_{i=1}^{\ell} \deg(\widehat{\gamma}_i), \quad \widehat{\gamma}_i \in H^{\deg(\widehat{\gamma}_i)}(S, \mathbb{Q}) \ .$$

The maximum value of $\deg(\widehat{\Gamma})$ is 2ℓ.

We will prove Proposition 4 for the capped contributions

$$(27) \qquad \mathsf{C}^{\mathsf{P}}_\infty \big(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), \beta \big)_{\nu, \mu}, \qquad \mathsf{C}^{\mathsf{GW}}_\infty \left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}, \beta \right)_{\nu, \mu} \ .$$

By induction, we assume Proposition 4 has been established for all capped contributions

$$\mathsf{C}^{\mathsf{P}}_\infty \big(\tau_{\alpha'}(\Gamma'_\infty), \beta' \big)_{\nu', \mu'}, \qquad \mathsf{C}^{\mathsf{GW}}_\infty \left(\overline{\tau_{\alpha'}(\Gamma'_\infty)}, \beta' \right)_{\nu', \mu'}$$

satisfying at least one of the following four conditions:

- $L_{\beta'} < L_\beta$,
- $L_{\beta'} = L_\beta$ and $\ell(\alpha') < \ell(\widehat{\alpha})$,
- $L_{\beta'} = L_\beta$, $\ell(\alpha') = \ell(\widehat{\alpha})$, and $\deg(\Gamma') > \deg(\widehat{\Gamma})$,
- $L_{\beta'} = L_\beta$, $\ell(\alpha') < \ell(\widehat{\alpha})$, $\deg(\Gamma') = \deg(\widehat{\Gamma})$, and $\theta(\nu') < \theta(\nu)$.

Via the third condition, we include a reverse induction over $\deg(\widehat{\Gamma})$. Since $\deg(\widehat{\Gamma}) \leq 2\ell$, the reverse induction is possible.

The proof of the induction step requires the $\mathbb{C}^*$-localization formula for the capped descendent contributions over S_∞ in terms of rubber moduli spaces. A review of the basic facts is presented in Sections 2.4.4 and 2.4.5.

---

[13]We will always use the complex grading (which is $\frac{1}{2}$ of the real grading).

[14]Again, we use the complex grading.

2.4.4. *Rubber geometry.* The capped contributions (27) over S_∞ are defined via $\mathbb{C}^*$-residues. The $\mathbb{C}^*$-localization formula for the capped contributions has three parts:

   (i) rubber integrals over S_0,
   (ii) edge terms,
   (iii) rubber integrals over S_∞.

The edge terms for stable pairs and stable maps are determined by Lemma 2. We discuss the rubber integrals here.

Consider first rubber[15] geometry for the moduli of stable pairs. Let

$$P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)^\circ_{\epsilon,\delta} \subset P_n(\mathbf{P}/S_0 \cup S_\infty, \beta)_{\epsilon,\delta}$$

denote the open set with finite stabilizers for the fiberwise $\mathbb{C}^*$-action and *no* destabilization over S_∞. The rubber moduli space,

$$P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)^{\sim}_{\epsilon,\delta} = P_n(\mathbf{P}/S_0 \cup S_\infty, \beta)^\circ_{\epsilon,\delta} \,/\, \mathbb{C}^*,$$

denoted by a superscripted tilde, is determined by the (stack) quotient. The rubber moduli space carries a virtual fundamental class,

$$[P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)^{\sim}_{\epsilon,\delta}]^{vir}.$$

The fiberwise $\mathbb{C}^*$-action is lost after the quotient, the fiberwise $\mathbb{C}^*$ acts trivially on the rubber moduli space.

The rubber moduli space $P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)^{\sim}_{\epsilon,\delta}$ carries cotangent lines associated to S_0 and S_∞. A construction can be found in Section 1.5.2 of [22]. Let

$$\Psi_0, \Psi_\infty \in H^2(P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)^{\sim}_{\epsilon,\delta}, \mathbb{Q})$$

denote the associated cotangent line classes.

The $\mathbb{C}^*$-localization formula for the capped descendent contribution over S_∞ for stable pairs is

$$(28) \quad \mathsf{C}^{\mathsf{P}}_\infty(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), \beta)_{\nu,\mu}$$

$$= \sum_{|\lambda|=d} \mathsf{R}^{\mathsf{P}}_{d[\mathbf{P}]}\left(\frac{1}{-\Psi_\infty - t}\right)_{\nu,\lambda} \frac{(-1)^{|\lambda|-\ell(\lambda)}\mathfrak{z}(\lambda)}{q^d}$$

$$\cdot \mathsf{R}^{\mathsf{P}}_\beta\left(\frac{1}{-\Psi_0 + t} \cdot \prod_{i=1}^{\ell} \tau_{\widehat{\alpha}_i-1}\big((t+N)\widehat{\gamma}_i\big)\right)_{\lambda^\vee,\mu}.$$

Here, $\mathsf{R}^{\mathsf{P}}_{d[\mathbf{P}]}$ denotes the generating series for rubber integrals over $S_0 \subset \mathbf{P}_S$ of curve class $d[\mathbf{P}]$ with inverse normal factor[16] $\left(\frac{1}{-\Psi_\infty - t}\right)$. Similarly, $\mathsf{R}^{\mathsf{P}}_\beta$ denotes the generating series of rubber integrals over $S_\infty \subset \mathbf{P}_S$ of curve class β with inverse normal factor $\left(\frac{1}{-\Psi_0 + t}\right)$ and descendent insertions. There are only two virtual normal directions in the relative geometry here.

---

[15]We follow the terminology and conventions of the rubber discussion in [30] for stable pairs and [22] for Gromov-Witten theory.

[16]The normal factor is the tensor of the tangent line $-\Psi_\infty$ of the rubber moduli with the tangent line $-t$ on the target fiber (which is pure weight).

For Gromov-Witten theory, a parallel discussion yields the $\mathbb{C}^*$-localization formula

$$(29) \quad \mathsf{C}_\infty^{\mathsf{GW}}\left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}, \beta\right)_{\nu,\mu}$$

$$= \sum_{|\lambda|=d} \mathsf{R}_{d[\mathbf{P}]}^{\prime\,\mathsf{GW}}\left(\frac{1}{-\Psi_\infty - t}\right)_{\nu,\lambda} \mathfrak{z}(\lambda) u^{2\ell(\lambda)}$$

$$\cdot \mathsf{R}_\beta^{\prime\,\mathsf{GW}}\left(\frac{1}{-\Psi_0 + t} \cdot \overline{\prod_{i=1}^{\ell} \tau_{\widehat{\alpha}_i - 1}\big((t+N)\widehat{\gamma}_i\big)}\right)_{\lambda^\vee,\mu} .$$

2.4.5. *Virtual dimensions.* The virtual dimensions of the stable pairs and stable map spaces are

$$\dim\,[P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)_{\nu,\mu}]^{vir} \;\; = \;\; d_\beta - \theta(\nu) - \theta(\mu) \;,$$
$$\dim\,[\overline{M}_{g,\ell}'(\mathbf{P}_S/S_\infty, \beta)_{\nu,\mu}]^{vir} \;\; = \;\; d_\beta + \ell - \theta(\nu) - \theta(\mu) \;.$$

The virtual dimensions of the rubber moduli space are 1 less,

$$\dim\,[P_n(\mathbf{P}_S/S_0 \cup S_\infty, \beta)_{\nu,\mu}^{\sim}]^{vir} \;\; = \;\; d_\beta - \theta(\nu) - \theta(\mu) - 1 \;,$$
$$\dim\,[\overline{M}_{g,\ell}'(\mathbf{P}_S/S_\infty, \beta)_{\nu,\mu}^{\sim}]^{vir} \;\; = \;\; d_\beta + \ell - \theta(\nu) - \theta(\mu) - 1 \;.$$

2.4.6. *Proof of the induction step.* We return to the proof of Proposition 4 via the induction strategy of Section 2.4.3. We must prove the descendent correspondence for the capped contributions (27) assuming the induction hypothesis. The analysis divides into two cases.

Case I. $|\widehat{\alpha}| - 2\ell(\widehat{\alpha}) + \deg(\widehat{\Gamma}) \geq \mathrm{d}_\beta - \theta(\nu) - \theta(\mu)$.

Under the hypothesis of Case I, we will prove the vanishing of both sides of the descendent correspondence of Proposition 4 for capped contributions over S_∞ by a straightforward dimension analysis.

First, consider the moduli space of stable pairs. Formula (28) expresses

$$\mathsf{C}_\infty^{\mathsf{P}}\,(\tau_\alpha(\Gamma_\infty), \beta)_{\nu,\mu}, \quad \beta = d[\mathbf{P}] + \beta_\infty$$

in terms of integrals over rubber moduli spaces. The rubber over S_0 carries curve classes with S-degree 0. In formula (28), if

$$\theta(\nu) + \theta(\lambda) > 2d,$$

then the rubber integrals over S_0 vanish (since the virtual dimension[17] of the rubber moduli spaces over S_0 is $2d - 1$). Therefore,

$$\theta(\lambda^\vee) \geq \theta(\nu) \;.$$

As a consequence, the virtual dimensions of the rubber moduli spaces over S_∞ in (28) never exceed

$$d_\beta - \theta(\nu) - \theta(\mu) - 1 \;.$$

The dimension of the integrand on the rubber of ∞ is at least the dimension of

$$\dim\Big(\prod_{i=1}^{\ell} \tau_{\widehat{\alpha}_i - 1}\big(\widehat{\gamma}_i\big)\Big) = |\widehat{\alpha}| - 2\ell(\widehat{\alpha}) + \deg(\widehat{\Gamma}) > d_\beta - \theta(\nu) - \theta(\mu) - 1 \;,$$

---

[17]The leading q term of $\mathsf{R}_{d[\mathbf{P}]}^{\mathsf{P}}\left(\frac{1}{-\Psi_\infty + t}\right)_{\nu,\lambda}$, given by the intersection pairing between ν and λ, is degenerate.

where the inequality is by the hypothesis of Case I. We conclude *every* rubber integral[18] over S_∞ in (28) vanishes and hence

$$\mathsf{C}^\mathsf{P}_\infty\left(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty),\beta\right)_{\nu,\mu} = 0\ .$$

The argument for the vanishing of $\mathsf{C}^\mathsf{GW}_\infty\left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)},\beta\right)_{\nu,\mu}$ is identical. We use the compatibility of the correspondence with grading established in Proposition 24 of [33] and the identification of the log tangent bundle of Lemma 1. Degree can be interchanged between the cotangent lines and Chern class of $T_{\mathbf{P}_S}(-S_0-S_\infty)$. However, since

$$c(T_{\mathbf{P}_S}[-S_0-S_\infty]) = c(\pi^* T_S),$$

the dimension calculus for the vanishing remains unchanged. We conclude

$$\mathsf{C}^\mathsf{GW}_\infty\left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)},\beta\right)_{\nu,\mu} = 0\ .$$

Proposition 4 is established in Case I.

Case II. $|\widehat{\alpha}| - 2\ell(\widehat{\alpha}) + \deg(\widehat{\Gamma}) < \mathrm{d}_\beta - \theta(\nu) - \theta(\mu)$.

The capped contributions need not vanish under the hypothesis of Case II. However, we will find parallel inductive relations to establish the descendent correspondence of Proposition 4.

To each partition λ weighted by cohomology classes of S in the basis Φ,

$$(30) \qquad ((\lambda_1,\delta_1),\ldots,(\lambda_{\ell(\lambda)},\delta_{\ell(\lambda)}))\ ,\quad |\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i,\quad \delta_i \in \Phi\ ,$$

we associate a descendent insertion over S_0,

$$\tau[\lambda] = \tau_{\lambda_1-1}(\delta_1[S_0])\cdots\tau_{\lambda_{\ell(\lambda)}-1}(\delta_{\ell(\lambda)}[S_0])\ .$$

The dimension of the descendent insertion $\tau[\lambda]$ equals $\theta(\lambda)$.

Let Λ_ν be the set of cohomology weighted partitions (30) defined by

$$\Lambda_\nu = \left\{\lambda\ \Big|\ |\lambda| = |\nu|,\ \ \theta(\lambda) = \theta(\nu)\ \right\}\ .$$

Since there are only finitely many partitions (30) satisfying $|\lambda| = |\nu|$, the set Λ_ν is finite.

For each cohomology weighted partition $\lambda \in \Lambda_\nu$, consider the stable pairs and Gromov-Witten generating series

$$(31)\quad \mathsf{Z}^\mathsf{P}_{\beta,\mu}\left(\tau[\lambda]\cdot\tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\mathrm{Id}})\right)^{\mathbb{C}^*}$$

$$= \sum_n q^n \int_{[P_n(\mathbf{P}_S/S_\infty,\beta)_\mu]^{vir}} \prod_i \tau_{\lambda_i-1}(\delta_i[S_0])\cdot\prod_j \tau_{\widehat{\alpha}_j-1}(\widehat{\gamma}_j).$$

$$\mathsf{Z}'^\mathsf{GW}_{\beta,\mu}\left(\overline{\tau[\lambda]\cdot\tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\mathrm{Id}})}\right)^{\mathbb{C}^*}$$

$$= \sum_g u^{2g-2} \int_{[\overline{M}'_{g,\star}(\mathbf{P}_S/S_\infty,\beta)_\mu]^{vir}} \overline{\prod_i \tau_{\lambda_i-1}(\delta_i[S_0])\cdot\prod_j \tau_{\widehat{\alpha}_j-1}(\widehat{\gamma}_j)}.$$

---

[18]All the rubber integrals are nonequivariant (there is no $\mathbb{C}^*$-action). For a nonvanishing result, the integrand cannot exceed the virtual dimension.

The dimension of the integrand for the stable pair series (31) is

$$\theta(\nu) + |\widehat{\alpha}| - 2\ell(\widehat{\alpha}) + \deg(\widehat{\Gamma}),$$

and the dimension of the moduli space of stable pairs is $d_\beta - \theta(\mu)$. By the hypothesis of Case II, the integrand dimension is *strictly less* than the dimension of the moduli space. By compactness of the geometry, the series (31) vanishes identically,

$$(32) \qquad \mathsf{Z}^{\mathsf{P}}_{\beta,\mu}\left(\tau[\lambda] \cdot \tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\mathrm{Id}})\right)^{\mathbb{C}^*} = 0 \ .$$

An identical dimension count shows

$$(33) \qquad \mathsf{Z}'^{\mathsf{GW}}_{\beta,\mu}\left(\overline{\tau[\lambda] \cdot \tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\mathrm{Id}})}\right)^{\mathbb{C}^*} = 0 \ .$$

The relations (32) and (33) will be used to uniquely determine the capped contributions

$$(34) \qquad \mathsf{C}^{\mathsf{P}}_{\infty}\left(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\infty}), \beta\right)_{\nu,\mu} \ , \quad \mathsf{C}^{\mathsf{GW}}_{\infty}\left(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\infty})}, \beta\right)_{\nu,\mu} \ , \quad \beta = d[\mathbf{P}] + \beta_{\infty} \ .$$

Moreover, the determinations will be sufficiently compatible to prove the correspondence of Proposition 4 for (34).

We will expand relations (32) and (33) using the capped localization formula. First, we write

$$(35) \qquad \tau_{\widehat{\alpha}_j - 1}(\widehat{\gamma}_j) = \tau_{\widehat{\alpha}_j - 1}\left(\frac{\widehat{\gamma}_j}{-t - N}[S_0]\right) + \tau_{\widehat{\alpha}_j - 1}\left(\frac{\widehat{\gamma}_j}{t + N}[S_{\infty}]\right)$$

using the basic identity (15). We have already proven the descendent correspondence for almost all the terms of the parallel capped localization formulas for (32) and (33). The correspondence is proven for all the capped contributions over S_0 by Proposition 3. Also, the correspondence is proven for all the capped contributions over S_{∞} which are covered by the induction hypothesis of Section 2.4.3. We can write

$$0 = \sum_{|\rho|=d} \mathsf{C}^{\mathsf{P}}_0(\tau[\lambda], d[\mathbf{P}])_\rho \frac{(-1)^{|\rho|-\ell(\rho)}\mathfrak{z}(\rho)}{q^{|\rho|}} \left(\frac{1}{t}\right)^{\ell(\widehat{\alpha})} \mathsf{C}^{\mathsf{P}}_{\infty}(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\infty}), \beta)_{\rho^\vee,\mu} + \cdots,$$

$$0 = \sum_{|\rho|=d} \mathsf{C}^{\mathsf{GW}}_0(\overline{\tau[\lambda]}, d[\mathbf{P}])_\rho \ \mathfrak{z}(\rho) u^{2\ell(\rho)} \left(\frac{1}{t}\right)^{\ell(\widehat{\alpha})} \mathsf{C}^{\mathsf{GW}}_{\infty}(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_{\infty})}, \beta)_{\rho^\vee,\mu} + \cdots,$$

where the sums are over all cohomology weighted partitions ρ of d. The dots stand for terms covered by the first three inductive conditions:

- lower S-degree over S_{∞},
- fewer descendent insertions over S_{∞},
- higher descendent degree over S_{∞}.

The induction condition over the descendent degree is used to replace

$$\frac{\widehat{\gamma}_j}{t + N} = \frac{\widehat{\gamma}_j}{t} - \frac{\widehat{\gamma}_j N}{t^2} + \frac{\widehat{\gamma}_j N^2}{t^3}$$

by the leading term (note $N^3 = 0$ in $H^*(S, \mathbb{Q})$).

Using the fourth induction condition, the relations can be simplified further. The capped contributions over S_0,

$$\mathsf{C}^{\mathsf{P}}_0(\tau[\lambda], d[\mathbf{P}])_\rho, \quad \mathsf{C}^{\mathsf{GW}}_0(\overline{\tau[\lambda]}, d[\mathbf{P}])_\rho,$$

have a curve class of S-degree 0. Hence, the capped contributions equal the full stable pairs and Gromov-Witten partition functions

$$\mathsf{C}_0^{\mathsf{P}}(\tau[\lambda], d[\mathbf{P}])_\rho \;\; = \;\; \mathsf{Z}_{d[\mathbf{P}],\rho}^{\mathsf{P}}(\tau[\lambda])^{\mathbb{C}^*} \;,$$
$$\mathsf{C}_0^{\mathsf{GW}}(\overline{\tau[\lambda]}, d[\mathbf{P}])_\rho \;\; = \;\; \mathsf{Z}'_{d[\mathbf{P}],\rho}{}^{\mathsf{GW}}(\overline{\tau[\lambda]})^{\mathbb{C}^*} \;.$$

Since the moduli space $P_n(\mathbf{P}_S, d[\mathbf{P}])_\rho$ has virtual dimension $2d - \theta(\rho)$, we see only terms with

$$\theta(\lambda) + \theta(\rho) \geq 2d$$

occur in the stable pairs relation. A parallel dimension count yields the same conclusion on the Gromov-Witten side. When the inequality is strict, we have

$$\theta(\lambda) + \theta(\rho) > 2d \quad \implies \quad \theta(\rho^\vee) < \theta(\lambda) = \theta(\nu),$$

so the terms are covered by the fourth induction condition.

   The final forms we find for the principal terms on the right side of the relations (32) and (33) are the following:

$$\sum_{|\rho|=d,\ \theta(\rho)=\theta(\nu^\vee)} \mathsf{C}_0^{\mathsf{P}}(\tau[\lambda], d[\mathbf{P}])_\rho \frac{(-1)^{|\rho|-\ell(\rho)}\mathfrak{z}(\rho)}{q^{|\rho|}} \left(\frac{1}{t}\right)^{\ell(\widehat{\alpha})} \mathsf{C}_\infty^{\mathsf{P}}(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), \beta)_{\rho^\vee,\mu} + \cdots,$$

$$\sum_{|\rho|=d,\ \theta(\rho)=\theta(\nu^\vee)} \mathsf{C}_0^{\mathsf{GW}}(\overline{\tau[\lambda]}, d[\mathbf{P}])_\rho\, \mathfrak{z}(\rho) u^{2\ell(\rho)} \left(\frac{1}{t}\right)^{\ell(\widehat{\alpha})} \mathsf{C}_\infty^{\mathsf{GW}}(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}, \beta)_{\rho^\vee,\mu} + \cdots.$$

The capped contributions

$$\mathsf{C}_\infty^{\mathsf{P}}(\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty), \beta)_{\rho^\vee,\mu}, \quad \mathsf{C}_\infty^{\mathsf{GW}}(\overline{\tau_{\widehat{\alpha}}(\widehat{\Gamma}_\infty)}, \beta)_{\rho^\vee,\mu}$$

as ρ varies yield exactly $|\Lambda_\nu|$ unknowns. As λ varies, we obtain exactly $|\Lambda_\nu|$ equations. The coefficients of the system are nonsingular by Proposition 6 of [31] on the stable pairs side (and therefore also on the Gromov-Witten side by Proposition 3). Hence, the relations uniquely determine all the unknowns including (34). Since the descendent correspondences have already been proven for all of the terms besides the unknowns, we conclude Proposition 4 holds for (34). The induction step has been established. □

2.5. **Proof of Theorem 3.** The capped localization formulas for stable pairs and stable maps for the relative geometry $\mathbf{P}_S/S_0 \cup S_\infty$ have contributions over S_0 and S_∞. Both take the form of the capped contributions over S_∞ for the relative geometry $\mathbf{P}_S/S_\infty$. Hence, both are covered by the descendent correspondence of Proposition 4. Theorem 3 follows immediately. □

2.6. **Nontoric surfaces.** Let S be a nonsingular projective surface (not necessarily toric) with line bundles

$$L_0, L_\infty \to S \;.$$

As a consequence of Conjecture 2, Theorems 2 and 3 should hold for nontoric S exactly as stated.

   In fact, our proofs of Theorems 2 and 3 are valid for *any* nonsingular projective surface S for which Proposition 3, concerning the correspondence for capped descendent contributions of $\mathbf{P}_S/S_\infty$ over S_0, holds. The toric hypothesis for S was only used to establish Proposition 3 via the descendent correspondence of [33] for capped vertices in toric geometry.

In order to prove Theorem 1, we will require Theorem 2 for particular nontoric surfaces. Let

$$\epsilon : S \to C$$

be a surface S expressed as a $\mathbf{P}^1$-bundle over a curve of genus g. Let

$$L_0^C, L_\infty^C \to C$$

be line bundles. We will prove Proposition 3 for S and the line bundles

(36) $$\epsilon^* L_0^C, \ \epsilon^* L_\infty^C \to S$$

in Section 6. As a consequence, Theorem 2 will also hold for the geometry determined by the data (36).

In the proof of Theorem 1, $K3$ surfaces will also appear. A special case of Theorem 2 for $K3$ surfaces S (in the nonequivariant limit) has been established in Proposition 26 of [33]. In Proposition 10 of Section 3.8, we will prove the results we require for $K3$ surfaces.

3. Descendent correspondence for the cap

3.1. **Overview.** The 1-leg *cap* is the total space of the trivial bundle,

(37) $$N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \to \mathbf{P}^1 \ ,$$

relative to the fiber

$$N_\infty \subset N$$

over $\infty \in \mathbf{P}^1$. The total space N naturally carries an action of a 3-dimensional torus

$$\mathbf{T} = T \times \mathbb{C}^* \ .$$

Here, T acts by scaling the factors of N and preserving the relative divisor N_∞. The $\mathbb{C}^*$-action on the base $\mathbf{P}^1$ which fixes the points $0, \infty \in \mathbf{P}^1$ lifts to an additional $\mathbb{C}^*$-action on N fixing N_∞. Let the tangent weights at $0, \infty \in \mathbf{P}^1$ with respect to the last $\mathbb{C}^*$-factor be $-s_3$ and s_3, respectively.[19]

The equivariant cohomology ring $H^*_{\mathbf{T}}(\bullet)$ is generated by the Chern classes s_1, s_2, and s_3 of the standard representation of the three $\mathbb{C}^*$-factors. Following [30], we define

(38) $$\mathsf{Z}^\mathsf{P}_{d,\eta}\left(\prod_{j=1}^k \tau_{i_j}([0]) \ \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty]) \right)^{\mathsf{cap},\mathbf{T}}$$

$$= \sum_{n \in \mathbb{Z}} q^n \int_{[P_n(N/N_\infty,d)_\eta]^{vir}} \prod_{j=1}^k \tau_{i_j}([0]) \ \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty]) \ ,$$

---

[19]The tangent weight conventions here match [30].

by $\mathbf{T}$-equivariant residues. By Theorem 3 of [30], the partition function (38) is a Laurent series in q of a rational function[20] in $\mathbb{Q}(q, s_1, s_2, s_3)$. Let

$$(39) \quad \mathsf{Z}'^{\mathsf{GW}}_{d,\eta}\left(\overline{\prod_{j=1}^{k} \tau_{i_j}([0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty])} \right)^{\mathsf{cap},\mathbf{T}}$$

$$= \sum_{g \in \mathbb{Z}} u^{2g-2} \int_{[\overline{M}'_{g,\star}(N/N_\infty,d)_\eta]^{vir}} \overline{\prod_{j=1}^{k} \tau_{i_j}([0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty])} \ ,$$

be the parallel Gromov-Witten partition function.

Our goal here is to prove the relative descendent correspondence of Conjecture 4 for the fully $\mathbf{T}$-equivariant partition functions (38) and (39).

Theorem 4. *For the cap geometry N/N_∞, we have*

$$(-q)^{-d}\mathsf{Z}^{\mathsf{P}}_{d,\eta}\Big(\prod_{j=1}^{k} \tau_{i_j}([0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty])\Big)^{\mathsf{cap},\mathbf{T}}$$

$$= (-iu)^{|\eta|+\ell(\eta)} \mathsf{Z}'^{\mathsf{GW}}_{d,\eta}\Big(\overline{\prod_{j=1}^{k} \tau_{i_j}([0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty])} \ \Big)^{\mathsf{cap},\mathbf{T}}$$

under the variable change $-q = e^{iu}$.

The proof of Theorem 4, given in Sections 3.2–3.7, follows the strategy of the proof of Theorem 3 of [33]. The main idea is to intertwine an induction on the depth of the descendent theories with the localization formula.

3.2. *T-depth.* For N defined by (37), let $S \subset N$ be the relative divisor associated to the points $p_1, \ldots, p_r \in \mathbf{P}^1$. We consider the T-equivariant stable pairs theory of N/S with respect to the scaling action.

The *T-depth m* theory of N/S consists of all T-equivariant series

$$(40) \qquad\qquad \mathsf{Z}^{\mathsf{P}}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j'=1}^{k'} \tau_{i'_{j'}}(1) \prod_{j=1}^{k} \tau_{i_j}(\mathsf{p}) \right)^{N/S,T} ,$$

where $k' \leq m$. Here, $\mathsf{p} \in H^2(\mathbf{P}^1, \mathbb{Z})$ is the class of a point, and the η^i are partitions determining the relative conditions along $\pi^{-1}(p_i)$. The T-depth m theory has at most m descendents of 1 and arbitrarily many descendents of p in the integrand. The *T-depth m* theory of N/S is *correspondent* if Conjecture 4 holds for all T-depth m series (40),

$$(-q)^{-d}\mathsf{Z}^{\mathsf{P}}_{d,\eta^1,\ldots,\eta^r}\left(\prod_{j'=1}^{k'} \tau_{i'_{j'}}(1) \prod_{j=1}^{k} \tau_{i_j}(\mathsf{p}) \right)^{N/S,T}$$

$$= (-iu)^{2d+\sum_l \left(\ell(\eta^l)-|\eta^l| \right)} \mathsf{Z}'^{\mathsf{GW}}_{d,\eta^1,\ldots,\eta^r}\left(\overline{\prod_{j'=1}^{k'} \tau_{i'_{j'}}(1) \prod_{j=1}^{k} \tau_{i_j}(\mathsf{p})} \right)^{N/S,T} .$$

---

[20]By Theorem 5 of [30], the poles in q of the partition function occur only at roots of unity.

The T-depth 0 theory concerns only descendents of p. By taking the specialization $s_3 = 0$, we have

$$\mathsf{Z}^{\mathsf{P}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}(\mathsf{p})\right)^{\mathsf{cap},T} = \mathsf{Z}^{\mathsf{P}}_{d,\eta}\left(\prod_{j=1}^{k}\tau_{i_j}([0])\right)^{\mathsf{cap},\mathbf{T}}\Bigg|_{s_3=0}.$$

The parallel relation holds for Gromov-Witten theory. By the descendent correspondence for the 1-leg capped vertex [33], we see the T-depth 0 theory of the cap is correspondent.

Lemma 5. *The T-depth 0 theory of N/S is correspondent.*

Proof. By the degeneration formula, all the descendents $\tau_{i_j}(\mathsf{p})$ can be degenerated onto a cap. The T-depth 0 theory of the cap is correspondent. The theories of local curves without any insertions are correspondent by [24, 29]. Hence, the result follows by the compatibility of Conjecture 4 with the degeneration formula. $\square$

3.3. **Induction I.** To establish the descendent correspondence for the T-depth m theory of N/S, the following result is required.

Lemma 6. *The descendent correspondence for the T-depth m theory of the cap implies the descendent correspondence of the T-depth m theory of N/S.*

Proof. We must prove the descendent correspondence for the T-depth m theories of N relative to $p_1, \ldots, p_r \in \mathbf{P}^1$. If $r = 1$, the geometry is the cap and the correspondence of the T-depth m theories is given. Assume the correspondence holds for r. We will show the correspondence holds for $r + 1$.

Let $p(d)$ be the number of partitions of size $d > 0$. Consider the $\infty \times p(d)$ matrix M_d, indexed by monomials

$$L = \prod_{i \geq 0} \tau_i(\mathsf{p})^{n_i}$$

in the descendents of p and partitions μ of d, with coefficient $\mathsf{Z}^{\mathsf{P}}_{d,\mu}(L)^{\mathsf{cap},T}$ in position (L, μ). The lowest Euler characteristic for a degree d stable pair on the cap is d. The leading q^d coefficients of M_d are well-known to be of maximal rank.[21] Hence, the full matrix M_d is also of maximal rank.

Consider N relative to $r + 1$ points in T-depth m,

(41) $$\mathsf{Z}^{\mathsf{P}}_{d,\eta^1,\ldots,\eta^r,\mu}\left(\prod_{j'=1}^{k'}\tau_{i_{j'}'}(1)\prod_{j=1}^{k}\tau_{i_j}(\mathsf{p})\right)^{N/S,T}.$$

We will determine the series (41) from the T-depth m series relative to r points,

(42) $$\mathsf{Z}^{\mathsf{P}}_{d,\eta^1,\ldots,\eta^r}\left(L\prod_{j'=1}^{k'}\tau_{i_{j'}'}(1)\prod_{j=1}^{k}\tau_{i_j}(\mathsf{p})\right)^{N/S,T}$$

defined by all monomials L in the descendents of p.

---

[21]The leading q^d coefficients are obtained from the Chern characters of the tautological rank d bundle on $\mathrm{Hilb}(N_\infty, d)$. The Chern characters generate the ring $H^*_T(\mathrm{Hilb}(N_\infty, d), \mathbb{Q})$ after localization as can easily be seen in the T-fixed point basis. A more refined result is discussed in Proposition 9 of [30].

Consider the T-equivariant degeneration of N by bubbling off a single cap at a point not equal to $p_1, \ldots, p_r$. All the descendents of p remain on the original N in the degeneration except for those in L which distribute to the cap. By induction on m, we need only analyze the terms of the degeneration formula in which the descendents of 1 distribute away from the cap. Then, since M_d has full rank, the invariants (41) are determined by the invariants (42).

The parallel inductive construction for Gromov-Witten theory determines

$$(43) \qquad \mathsf{Z}'^{\mathsf{GW}}_{d,\eta^1,\ldots,\eta^r,\mu} \left(\overline{\prod_{j'=1}^{k'} \tau_{i'_{j'}}(1) \prod_{j=1}^{k} \tau_{i_j}(\mathsf{p})} \right)^{N/S,T}$$

in terms of the T-depth m series relative to r points,

$$(44) \qquad \mathsf{Z}'^{\mathsf{GW}}_{d,\eta^1,\ldots,\eta^r} \left(L \overline{\prod_{j'=1}^{k'} \tau_{i'_{j'}}(1) \prod_{j=1}^{k} \tau_{i_j}(\mathsf{p})} \right)^{N/S,T} ,$$

the T-depth m theory of the cap, and theories of lower T-depth. By the compatibility of the descendent correspondence with the degeneration formula, the determinations of the T-depth m theories of N relative $r+1$ points in $\mathbf{P}^1$ respect the descendent correspondence. $\qquad \square$

The 1-leg *tube* is the total space of the trivial bundle,

$$N = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \to \mathbf{P}^1 ,$$

relative to the fibers

$$N_0, N_\infty \subset N$$

over both $0, \infty \in \mathbf{P}^1$. The tube carries a fiberwise T-action as well as a full $\mathbf{T}$-action. Lemma 6 implies the following result which will be half of our induction argument relating the descendent theory of the cap and the tube.

Lemma 7. *The descendent correspondence for the T-depth m theory of the cap implies the descendent correspondence for the T-depth m theory of the tube.*

3.4. **T-depth.** The $\mathbf{T}$-*depth* m theories of the cap consists of all the $\mathbf{T}$-equivariant series

$$(45) \qquad \mathsf{Z}^{\mathsf{P}}_{d,\eta} \left(\prod_{j=1}^{k} \tau_{i_j}([0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty]) \right)^{\mathsf{cap},\mathbf{T}} ,$$

$$\mathsf{Z}'^{\mathsf{GW}}_{d,\eta} \left(\overline{\prod_{j=1}^{k} \tau_{i_j}([0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty])} \right)^{\mathsf{cap},\mathbf{T}} ,$$

where $k' \leq m$. Here, $0 \in \mathbf{P}^1$ is the nonrelative $\mathbf{T}$-fixed point and $\infty \in \mathbf{P}^1$ is the relative point. The $\mathbf{T}$-depth m theory of the cap is *correspondent* if Conjecture 4 holds for all depth m stable pairs and Gromov-Witten partition functions (45).

Lemma 8. *The descendent correspondence for the $\mathbf{T}$-depth m theory of the cap implies the descendent correspondence for the T-depth m theory of the cap.*

Proof. The identity class $1 \in H_T^*(\mathbf{P}^1, \mathbb{Z})$ may be expressed in terms of the $\mathbf{T}$-fixed point classes,

$$1 = -\frac{[0]}{s_3} + \frac{[\infty]}{s_3} \; .$$

We can calculate at most m descendents of 1 in the T-equivariant theory via at most m descendents of $[\infty]$ in the $\mathbf{T}$-equivariant theory (followed by the specialization $s_3 = 0$). $\square$

3.5. Induction II.

The first half of our induction argument was established in Lemma 7. The second half relates the tube back to the cap with an increase in depth.

Lemma 9. *The descendent correspondence for the T-depth m theory of the tube implies the descendent correspondence for the $\mathbf{T}$-depth $m+1$ theory of the cap.*

Proof. The result follows from the $\mathbf{T}$-equivariant localization formula for the cap in terms of the T-equivariant theory of the tube (already used in [30]). We first review the formula.

For the theory of stable pairs, consider the partition function

$$(46) \qquad \mathsf{Z}_{d,\eta}^{\mathsf{P}}\left(\prod_{j=1}^{k} \tau_{i_j}([0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}([\infty])\right)^{\mathsf{cap},\mathbf{T}} .$$

We will write the $\mathbf{T}$-equivariant localization formula for (46), as a sum over set partitions

$$\mathsf{R} = (R_1, R_2, \ldots, R_{r(\mathsf{R})}), \quad R_i \subset \{1, \ldots, k'\},$$

satisfying the following conditions:

- R_i are nonempty and disjoint,
- $R_1 \cup R_2 \cup \cdots \cup R_{r(\mathsf{R})} = \{1, \ldots, k'\}$,
- $\min\{j' \in R_i\} > \min\{j' \in R_{i+1}\}$.

Let m_i be the minimal index in R_i. As a consequence of the third condition, $m_{r(\mathsf{R})} = 1 \in R_{r(\mathsf{R})}$. The formula for the partition function (46) is

$$\sum_{\mathsf{R}} s_3^{k'-r(\mathsf{R})} \mathsf{Z}_{d,\eta^1}^{\mathsf{P}}\left(\prod_{j=1}^{k} \tau_{i_j}([0])\right)^{\mathsf{cap},\mathbf{T}} \frac{g^{\eta^1 \tilde{\eta}^1}}{q^d} \mathsf{Z}_{d,\tilde{\eta}^1,\eta^2}^{\mathsf{P}}\left(\tau_{i'_{m_1}}(\mathsf{p}) \prod_{j' \in R_1^*} \tau_{i'_{j'}}(1)\right)^{\mathsf{tube},T}$$

$$\cdot \frac{g^{\eta^2 \tilde{\eta}^2}}{q^d} \mathsf{Z}_{d,\tilde{\eta}^2,\eta^3}^{\mathsf{P}}\left(\tau_{i'_{m_2}}(\mathsf{p}) \prod_{j' \in R_2^*} \tau_{i'_{j'}}(1)\right)^{\mathsf{tube},T}$$

$$\cdots$$

$$\cdot \frac{g^{\eta^r \tilde{\eta}^r}}{q^d} \mathsf{Z}_{d,\tilde{\eta}^r,\eta}^{\mathsf{P}}\left(\tau_{i'_{m_r}}(\mathsf{p}) \prod_{j' \in R_r^*} \tau_{i'_{j'}}(1)\right)^{\mathsf{tube},T} ,$$

where the metric term is

$$g^{\eta \tilde{\eta}} = (s_1 s_2)^{\ell(\eta)}(-1)^{|\eta|-\ell(\eta)}\mathfrak{z}(\eta) \cdot \delta_{\eta,\tilde{\eta}} \; .$$

The above $\mathbf{T}$-equivariant formula is proven via localization and the rubber calculus; see Section 7.2 of [30].

For a partition function (46) of $\mathbf{T}$-depth $m+1$, the right side of the $\mathbf{T}$-equivariant localization formula is in terms of the $\mathbf{T}$-depth 0 theory of the cap and the T-depth m theory of the tube. Consider next the Gromov-Witten partition function,

$$(47) \qquad \mathsf{Z}'^{\,\mathsf{GW}}_{d,\eta}\left(\overline{\prod_{j=1}^{k}\tau_{i_j}([0])\prod_{j'=1}^{k'}\tau_{i'_{j'}}([\infty])}\right)^{\mathsf{cap},\mathbf{T}}.$$

The $\mathbf{T}$-equivariant localization formula for (47) is

$$\sum_{\mathsf{R}} s_3^{k'-r(\mathsf{R})}\,\mathsf{Z}'^{\,\mathsf{GW}}_{d,\eta^1}\left(\overline{\prod_{j=1}^{k}\tau_{i_j}([0])}\right)^{\mathsf{cap},\mathbf{T}}\frac{h^{\eta^1\tilde{\eta}^1}}{u^{-2\ell(\eta^1)}}\mathsf{Z}'^{\,\mathsf{GW}}_{d,\tilde{\eta}^1,\eta^2}\left(\overline{\tau_{i'_{m_1}}(\mathsf{p})\prod_{j'\in R_1^*}\tau_{i'_{j'}}(1)}\right)^{\mathsf{tube},T}$$

$$\cdot\frac{h^{\eta^2\tilde{\eta}^2}}{u^{-2\ell(\eta^2)}}\mathsf{Z}'^{\,\mathsf{GW}}_{d,\tilde{\eta}^2,\eta^3}\left(\overline{\tau_{i'_{m_2}}(\mathsf{p})\prod_{j'\in R_2^*}\tau_{i'_{j'}}(1)}\right)^{\mathsf{tube},T}$$

$$\cdots$$

$$\cdot\frac{h^{\eta^r\tilde{\eta}^r}}{u^{-2\ell(\eta^r)}}\mathsf{Z}'^{\,\mathsf{GW}}_{d,\tilde{\eta}^r,\eta}\left(\overline{\tau_{i'_{m_r}}(\mathsf{p})\prod_{j'\in R_r^*}\tau_{i'_{j'}}(1)}\right)^{\mathsf{tube},T},$$

where the metric is now

$$h^{\eta\tilde{\eta}} = (s_1 s_2)^{\ell(\eta)}\mathfrak{z}(\eta)\cdot\delta_{\eta,\tilde{\eta}}.$$

The proof is again via standard localization and rubber calculus.

The descendent correspondence of Conjecture 4 is formally compatible with the above $\mathbf{T}$-equivariant localization formulas. Since the right sides concern only the $\mathbf{T}$-depth 0 theory of the cap and the T-depth m theory of the tube, Lemma 9 is an immediate consequence. $\qquad\square$

3.6. **Gromov-Witten side.** The stable pairs localization formula for (46) in Section 3.5 was explained in [30]. While the Gromov-Witten side is parallel, we present the first cases here to help the reader.

To start, we write the localization formula for $\mathbf{T}$-depth 1 series for the cap as

$$\mathsf{Z}'^{\,\mathsf{GW}}_{d,\eta}\left(\overline{\prod_{j=1}^{k}\tau_{i_j}([0])\cdot\tau_{i'_1}([\infty])}\right)^{\mathsf{cap},\mathbf{T}}$$

$$= \mathsf{Z}'^{\,\mathsf{GW}}_{d,\eta}\left(\overline{\prod_{j=1}^{k}\tau_{i_j}([0])\cdot\overline{\tau_{i'_1}([\infty])}}\right)^{\mathsf{cap},\mathbf{T}}$$

$$= \sum_{|\mu|=d}\mathsf{W}^{\mathsf{Vert}}_{\mu}\left(\overline{\prod_{j=1}^{k}\tau_{i_j}([0])}\right)\cdot\mathsf{W}^{(0,0)}_{\mu}\cdot\mathsf{S}^{\mu}_{\eta}(\overline{\tau_{i'_1}}),$$

where the rubber term on the right is

$$\mathsf{S}^{\mu}_{\eta}(\overline{\tau_{i'_1}}) \;\;=\;\; \sum_{g} u^{2g-2}\left\langle \mu \;\left|\; \frac{s_3\overline{\tau_{i'_1}}}{s_3-\psi_0} \;\right|\; \eta \right\rangle^{\sim}_{g,d}.$$

Here, $\mathsf{W}^{\mathsf{Vert}}_{\mu}$ and $\mathsf{W}^{0,0}_{\mu}$ denote the Gromov-Witten vertex and edge terms.

The rubber term simplifies via the topological recursion relation for ψ_0 after writing

$$(48) \qquad \frac{s_3}{s_3 - \psi_0} = 1 + \frac{\psi_0}{s_3 - \psi_0} \ .$$

We find the relation

$$\mathsf{S}_\eta^\mu(\overline{\tau_{i'_1}}) \quad = \quad \sum_{|\tilde\eta|=d} \mathsf{S}_{\tilde\eta}^\mu \cdot \frac{h^{\tilde\eta\tilde\eta}}{u^{-2\ell(\tilde\eta)}} \cdot \mathsf{Z}_{d,\tilde\eta,\eta}'^{\,\mathsf{GW}}\left(\overline{\tau_{i'_1}([\infty])}\right)^{\mathsf{tube},T},$$

where the rubber term on the right is

$$\mathsf{S}_\eta^\mu \quad = \quad \sum_g u^{2g-2} \left\langle \mu \ \left| \ \frac{1}{s_3 - \psi_0} \ \right| \ \eta \right\rangle_{g,d}^{\sim}.$$

The leading 1 on the right side of (48) corresponds to the degenerate leading term of $\mathsf{S}_{\tilde\eta}^\mu$. The topological recursion applied to the ψ_0 prefactor of the second term produces the rest of $\mathsf{S}_{\tilde\eta}^\mu$. We have also used here the identification of the log tangent bundle on the destabilized cap.

After reassembling the localization formula, we find

$$\mathsf{Z}_{d,\eta}'^{\,\mathsf{GW}}\left(\overline{\prod_{j=1}^k \tau_{i_j}([0]) \cdot \tau_{i'_1}([\infty])}\right)^{\mathsf{cap},\mathbf{T}}$$

$$= \sum_{|\tilde\eta|=d} \mathsf{Z}_{d,\tilde\eta}'^{\,\mathsf{GW}}\left(\overline{\prod_{j=1}^k \tau_{i_j}([0])}\right)^{\mathsf{cap},\mathbf{T}} \cdot \frac{h^{\tilde\eta\tilde\eta}}{u^{-2\ell(\tilde\eta)}} \cdot \mathsf{Z}_{d,\tilde\eta,\eta}'^{\,\mathsf{GW}}\left(\overline{\tau_{i'_1}([\infty])}\right)^{\mathsf{tube},T},$$

which is equivalent to the first case of the Gromov-Witten formula of Section 3.5.

The higher cases of the Gromov-Witten localization formula of Section 3.5 are proven by expanding definition (11) of the descendent correspondence and following the rubber calculus. Consider

$$\mathsf{Z}_{d,\eta}'^{\,\mathsf{GW}}\left(\overline{\prod_{j=1}^k \tau_{i_j}([0]) \cdot \tau_{i'_1}([\infty])\tau_{i'_2}([\infty])}\right)^{\mathsf{cap},\mathbf{T}}$$

$$= \mathsf{Z}_{d,\eta}'^{\,\mathsf{GW}}\left(\overline{\prod_{j=1}^k \tau_{i_j}([0]) \cdot \overline{\tau_{i'_1}([\infty])\tau_{i'_2}([\infty])}}\right)^{\mathsf{cap},\mathbf{T}},$$

where we have

$$(49) \quad \overline{\tau_{i'_1}([\infty])\tau_{i'_2}([\infty])} = s_3 \sum_\alpha \tau_\alpha(\widetilde{K}_{(i'_1+1,i'_2+1),\alpha} \cdot [\infty])$$

$$+ \sum_\delta \tau_\delta(\widetilde{K}_{(i'_1+1),\delta} \cdot [\infty]) \cdot \sum_\epsilon \tau_\epsilon(\widetilde{K}_{(i'_2+1),\epsilon} \cdot [\infty])$$

by definition. The first summand on the right of (49) is obtained from the set partition $\{1,2\}$ and the second term from the set partition $\{1\}\cup\{2\}$. After applying

localization and the rubber calculus to the $\{1,2\}$ term, we obtain the $\{1,2\}$ term of

$$
(50) \qquad \sum_{|\tilde{\eta}|=d} s_3 \mathsf{Z}'^{\mathsf{GW}}_{d,\tilde{\eta}} \left(\overline{\prod_{j=1}^{k} \tau_{i_j}([0])} \right)^{\mathsf{cap},\mathbf{T}} \cdot \frac{h^{\tilde{\eta}\tilde{\eta}}}{u^{-2\ell(\tilde{\eta})}} \cdot \mathsf{Z}'^{\mathsf{GW}}_{d,\tilde{\eta},\eta} \left(\overline{\tau_{i'_1}(\mathsf{p})\tau_{i'_2}(1)} \right)^{\mathsf{tube},T} .
$$

After applying localization and the rubber calculus to the $\{1\} \cup \{2\}$ term of (49), we obtain the $\{1\} \cup \{2\}$ term of (50) plus the full series

$$
(51) \qquad \sum_{|\tilde{\mu}|,|\tilde{\eta}|=d} \mathsf{Z}'^{\mathsf{GW}}_{d,\tilde{\mu}} \left(\overline{\prod_{j=1}^{k} \tau_{i_j}([0])} \right)^{\mathsf{cap},\mathbf{T}} \cdot \frac{h^{\tilde{\mu}\tilde{\mu}}}{u^{-2\ell(\tilde{\mu})}} \cdot \mathsf{Z}'^{\mathsf{GW}}_{d,\tilde{\mu},\tilde{\eta}} \left(\overline{\tau_{i'_2}(\mathsf{p})} \right)^{\mathsf{tube},T}
$$
$$
\cdot \frac{h^{\tilde{\eta}\tilde{\eta}}}{u^{-2\ell(\tilde{\eta})}} \cdot \mathsf{Z}'^{\mathsf{GW}}_{d,\tilde{\eta},\eta} \left(\overline{\tau_{i'_1}(\mathsf{p})} \right)^{\mathsf{tube},T} .
$$

Combining (50) and (51) exactly yields the Gromov-Witten formula of Section 3.5 for two insertions over ∞.

3.7. **Proof of Theorem 4.** Lemmas 7–9 together provide an induction which establishes the descendent correspondence for the $\mathbf{T}$-depth m theory of the cap for all m. □

Since the classes of the $\mathbf{T}$-fixed points $0,\infty \in \mathbf{P}^1$ generate $H^*_{\mathbf{T}}(\mathbf{P}^1,\mathbb{Z})$ after localization, Theorem 4 is a $\mathbf{T}$-equivariant correspondence for the full descendent theory of the cap.

3.8. $K3$ **surfaces.** For a surface S, following the notation of Section 1.5, let

$$
\mathbf{P}_S = \mathbf{P}(L_0 \oplus L_\infty) \to S, \quad S_i = \mathbf{P}(L_i) \subset \mathbf{P}_S .
$$

Proposition 10. *Let S be a nonsingular projective $K3$ surface. For classes $\gamma_i \in H^*(S,\mathbb{Q})$, we have*

$$
\mathsf{Z}_{\mathsf{P}} \left(\mathbf{P}_S/S_\infty; q \; \middle| \tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell) \; \middle| \; \mu \right)_\beta \in \mathbb{Q}(q)
$$

and the correspondence

$$
(-q)^{-d_\beta/2} \mathsf{Z}_{\mathsf{P}} \left(\mathbf{P}_S/S_\infty; q \; \middle| \tau_{\alpha_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell) \; \middle| \; \mu \right)_\beta
$$
$$
= (-iu)^{d_\beta+\ell(\mu)-|\mu|} \mathsf{Z}'_{\mathsf{GW}} \left(\mathbf{P}_S/S_\infty; u \; \middle| \; \overline{\tau_{a_1-1}(\gamma_1) \cdots \tau_{\alpha_\ell-1}(\gamma_\ell)} \; \middle| \; \mu \right)_\beta
$$

under the variable change $-q = e^{iu}$.

Proof. If the cohomology insertions γ_i are supported on S_0, then the above correspondence is proven in Proposition 26 of [33]. The support hypotheses for γ_i were needed there since, for the $\mathbf{T}$-equivariant cap, only the correspondence for descendents of the nonrelative point had been proven in [33]. Theorem 4 now removes the need for the support hypothesis. The proof of Proposition 26 together with Theorem 4 yields the result. □

4. The geometry $\mathbf{P}^1 \times \mathbb{C} \times \mathbf{P}^1 \,/\, \mathbf{P}^1 \times \mathbb{C}$

4.1. **Overview.** Let Y denote the the quasi-projective variety $\mathbf{P}^1 \times \mathbb{C} \times \mathbf{P}^1$. For clarity, we will denote the first factor by $\mathbf{P}^1_1$ and the third factor by $\mathbf{P}^1_3$. Let

$$\pi_1 : Y \to \mathbf{P}^1_1, \quad \pi_3 : Y \to \mathbf{P}^1_3$$

denote the projections onto the first and last factors.

The variety Y admits an action of the 3-torus

$$\mathbf{T} = \mathbb{C}^*_1 \times \mathbb{C}^*_2 \times \mathbb{C}^*_3 \ .$$

The first factor $\mathbb{C}^*_1$ of $\mathbf{T}$ acts on $\mathbf{P}^1_1$ with fixed points $0, \infty \in \mathbf{P}^1_1$ with tangent weights $-s_1, s_1$, respectively. The factor $\mathbb{C}^*_2$ acts on $\mathbb{C}$ with fixed point $0 \in \mathbb{C}$ with tangent weight $-s_2$. Finally, $\mathbb{C}^*_3$ acts on $\mathbf{P}^1_3$ with fixed points $0, \infty \in \mathbf{P}^1_3$ with tangent weights $-s_3, s_3$, respectively.

Define the divisors $Y_0, Y_\infty \subset Y$ to be the fibers of π_3 over $0, \infty \in \mathbf{P}^1_3$,

$$Y_0 = \mathbf{P}^1_1 \times \mathbb{C} \times \{0\}, \quad Y_\infty = \mathbf{P}^1_1 \times \mathbb{C} \times \{\infty\}.$$

Both Y_0 and Y_∞ are preserved by the $\mathbf{T}$-action. Let

$$[0], [\infty] \in H^2_{\mathbf{T}}(Y, \mathbb{Q})$$

denote the classes of Y_0 and Y_∞, respectively.

The projection π_1 is equivariant with respect to the projection of $\mathbf{T}$ onto $\mathbb{C}^*_1$. We will view

$$\theta_j, \theta'_{j'} \in H^*_{\mathbb{C}^*_1}(\mathbf{P}^1_1, \mathbb{Q})$$

as classes in $H^*_{\mathbf{T}}(Y, \mathbb{Q})$ via pullback by π_1.

Since Y_∞ is preserved by the $\mathbf{T}$-action, we can define

$$(52) \quad \mathsf{Z}^{\mathsf{P}}_{\beta, \eta} \left(\prod_{j=1}^k \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty]) \right)^{Y/Y_\infty, \mathbf{T}}$$

$$= \sum_{n \in \mathbb{Z}} q^n \int_{[P_n(Y/Y_\infty, \beta)_\eta]^{vir}} \prod_{j=1}^k \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty]) \ ,$$

by $\mathbf{T}$-equivariant residues. Here, $\beta \in H_2(Y, \mathbb{Z})$ is a curve class (specified by degrees along the two $\mathbf{P}^1$-factors), and η is a boundary condition along Y_∞. The parallel Gromov-Witten partition function is

$$(53) \quad \mathsf{Z}'^{\mathsf{GW}}_{\beta, \eta} \left(\overline{\prod_{j=1}^k \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty])} \right)^{Y/Y_\infty, \mathbf{T}}$$

$$= \sum_{g \in \mathbb{Z}} u^{2g-2} \int_{[\overline{M}'_{g,\star}(Y/Y_\infty, \beta)_\eta]^{vir}} \overline{\prod_{j=1}^k \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty])} \ .$$

Our goal here is to prove the relative descendent correspondence of Conjecture 4 for the fully $\mathbf{T}$-equivariant partition functions (52) and (53).

Theorem 5. *For the relative geometry Y/Y_∞, we have*

$$\mathsf{Z}^{\mathsf{P}}_{\beta, \eta} \Big(\prod_{j=1}^k \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty]) \Big)^{Y/Y_\infty, \mathbf{T}} \in \mathbb{Q}(q, s_1, s_2, s_3)$$

and the correspondence

$$(-q)^{-d_\beta/2} \mathsf{Z}^{\mathsf{P}}_{\beta,\eta} \Big(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty]) \Big)^{Y/Y_\infty, \mathbf{T}}$$

$$= (-iu)^{d_\beta + \ell(\eta) - |\eta|} \mathsf{Z}'^{\mathsf{GW}}_{\beta,\eta} \Big(\overline{\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty])} \Big)^{Y/Y_\infty, \mathbf{T}}$$

under the variable change $-q = e^{iu}$.

The proof of Theorem 5, given in Sections 4.2–4.4, again proceeds by induction on the depth of the descendent theories. A study of capped rubber for the geometry $Y/Y_0 \cup Y_\infty$ is required for the base case of the induction.

4.2. **Depth induction.** The proof of Theorem 4 can be exactly followed to establish Theorem 5. To start, we define the two notions of depth for the geometry Y.

Let $S \subset Y$ be the relative divisor $\cup_i \pi_3^{-1}(p_i)$ associated to the points $p_1, \ldots, p_r \in \mathbf{P}_3^1$. Let

$$T = \mathbb{C}_1^* \times \mathbb{C}_2^* \subset \mathbf{T}$$

be the first two factors of the 3-torus. We consider the T-equivariant stable pairs theory of Y/S. The T-*depth m* theory of Y/S consists of all T-equivariant series

$$(54) \qquad \mathsf{Z}^{\mathsf{P}}_{\beta,\eta^1,\ldots,\eta^r} \left(\prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'} \cdot 1) \prod_{j=1}^{k} \tau_{i_j}(\theta_j \cdot \mathsf{p}) \right)^{Y/S,T},$$

where $k' \leq m$. Here, $\mathsf{p} \in H^2(Y, \mathbb{Z})$ denotes the class of a fiber of π_3, and the η^i are partitions determining the relative conditions along $\pi^{-1}(p_i)$. Following exactly the proof of Lemma 7, we obtain the following result.

Lemma 11. *The descendent correspondence for the T-depth m theory of Y/Y_∞ implies the descendent correspondence for the T-depth m theory of the $Y/Y_0 \cup Y_\infty$.* $\qquad \square$

The stable $\mathbf{T}$-*depth m* theory of Y/Y_∞ consists of all the $\mathbf{T}$-equivariant series

$$(55) \qquad \mathsf{Z}^{\mathsf{P}}_{\beta,\eta} \left(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[\infty]) \right)^{Y/Y_\infty, \mathbf{T}},$$

where $k' \leq m$.

The proofs of Lemmas 8 and 9 are formal and remain valid for the geometry Y/Y_∞. As a result, we obtain the following two results.

Lemma 12. *The descendent correspondence for the $\mathbf{T}$-depth m theory of Y/Y_∞ implies the descendent correspondence for the T-depth m theory of the Y/Y_∞.* $\quad \square$

Lemma 13. *The descendent correspondence for the T-depth m theory of the tube implies the descendent correspondence for the $\mathbf{T}$-depth $m+1$ theory of the cap.* $\quad \square$

Lemmas 11–13 together establish a recursion which reduces Theorem 5 to the base case of the $\mathbf{T}$-depth 0 theory of Y/Y_∞.

4.3. **T-depth 0.** The last step in the proof of Theorem 5 is to establish the descendent correspondence in the base case of **T**-depth 0.

Proposition 14. *For the relative geometry Y/Y_∞, we have*

$$\mathsf{Z}^{\mathsf{P}}_{\beta,\eta}\Big(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0]) \Big)^{Y/Y_\infty,\mathbf{T}} \ \in \mathbb{Q}(q, s_1, s_2, s_3)$$

and the correspondence

$$(-q)^{-d_\beta/2}\mathsf{Z}^{\mathsf{P}}_{\beta,\eta}\Big(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0]) \Big)^{Y/Y_\infty,\mathbf{T}}$$

$$= (-iu)^{d_\beta+\ell(\eta)-|\eta|}\mathsf{Z}'^{\mathsf{GW}}_{d,\eta}\Big(\overline{\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0])} \Big)^{Y/Y_\infty,\mathbf{T}}$$

under the variable change $-q = e^{iu}$.

We can write the partition function for Y/Y_∞ via capped localization for both stable pairs and Gromov-Witten theory. The capped contributions over Y_0 are 2-leg capped toric descendent vertices and satisfy the descendent correspondence by [33]. The interesting new terms in the capped localization formula occur over Y_∞—the capped rubber contributions. The capped rubber contributions carry *no descendent insertions.*

To prove the correspondence for the capped rubber contributions over Y_∞, we follow the technique developed in [21] where the capped rubber contributions for

$$\mathcal{A}_n \times \mathbf{P}^1 \,/\mathcal{A}_n \times \{\infty\}$$

over ∞ were studied. Via the differential equations constructed in Sections 3.2 of [21], the analysis of Section 3.4 can be applied to our capped rubber contributions. The proof of Lemma 6 of [21] is valid here. As a result the correspondence for the capped rubber contributions of Y/Y_∞ over Y_∞ is equivalent to the following nonrubber correspondence.

We consider the relative geometry $Y \,/\, Y_0 \cup Y_\infty$ with respect to the 2-torus T-action by the first two factors $T \subset \mathbf{T}$. Let $\gamma \in H^*_{\mathbb{C}^*}(\mathbf{P}^1_1, \mathbb{Q})$ be the class of the fixed point $\infty \in \mathbf{P}^1_1$.

Proposition 15. *For the relative geometry $Y/Y_0 \cup Y_\infty$, we have*

$$\mathsf{Z}^{\mathsf{P}}_{\beta,\nu,\mu}\Big(\tau_0(\gamma[0]) \Big)^{Y/Y_0\cup Y_\infty,T} \ \in \mathbb{Q}(q, s_1, s_2)$$

and the correspondence

$$(-q)^{-d_\beta/2}\mathsf{Z}^{\mathsf{P}}_{\beta,\nu,\mu}\Big(\tau_0(\gamma[0]) \Big)^{Y/Y_0\cup Y_\infty,T}$$

$$= (-iu)^{d_\beta+\ell(\nu)-|\nu|+\ell(\mu)-|\mu|}\mathsf{Z}'^{\mathsf{GW}}_{\beta,\nu,\mu}\Big(\overline{\tau_0(\gamma[0])} \Big)^{Y/Y_0\cup Y_\infty,T}$$

under the variable change $-q = e^{iu}$.

By basic properties of the descendent correspondence [33],

$$\overline{\tau_0(\gamma[0])} = \tau_0(\gamma[0]) \ .$$

Proposition 14 is a consequence of Proposition 15 together with the recursion of Lemmas 11–13. Hence the proof of Theorem 5 will be complete once Proposition 15 is established.

4.4. **Proof of Proposition 15.** The curve class $\beta \in H_2(Y, \mathbb{Z})$ is a linear combination of the curves

$$[\mathbf{P}_1^1] = \mathbf{P}_1^1 \times \{0\} \times \{0\}, \qquad [\mathbf{P}_3^1] = \{0\} \times \{0\} \times \mathbf{P}_3^1 .$$

If β is a multiple of $[\mathbf{P}_3^1]$, then Proposition 15 reduces immediately to the T-equivariant descendent correspondence of local curves [33].

Let $\mathbf{Y} = \mathbf{P}_1^1 \times \mathbf{P}_2^1 \times \mathbf{P}_3^1$. We view the projective variety $\mathbf{Y}$ as a $\mathbf{T}$-equivariant compactification of the quasi-projective variety Y,

$$\mathbf{P}_1^1 \times \mathbb{C} \times \mathbf{P}_3^1 \subset \mathbf{P}_1^1 \times \mathbf{P}_2^1 \times \mathbf{P}_3^1.$$

Let $\mathbf{Y}_0$ and $\mathbf{Y}_\infty$ be the π_3-fibers of $\mathbf{Y}$ over $0, \infty \in \mathbf{P}_3^1$. Our proof of Proposition 15 will be obtained from studying the partition functions

$$(56) \qquad \mathsf{Z}^{\mathsf{P}}_{\beta,\nu,\mu}\Big(\tau_0(\gamma[0])\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T}, \qquad \mathsf{Z}'^{\,\mathsf{GW}}_{\beta,\nu,\mu}\Big(\overline{\tau_0(\gamma[0])}\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T}$$

for the compact relative geometry $\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty$. We will consider curve classes

$$\beta = d_1[\mathbf{P}_1^1] + d_3[\mathbf{P}_3^1] \in H_2(\mathbf{Y}, \mathbb{Z})$$

for which $d_1 > 0$ and $d_3 \geq 0$.

If $d_3 > 0$, the relative conditions ν and μ in (56) will be taken to be of a special form. The relative condition ν is a partition of d_3 weighted by $H_T^*(\mathbf{P}_1^1 \times \mathbf{P}_2^1, \mathbb{Q})$. We *require* the weights of all the parts ν_i to be the pullbacks of the classes of the $\mathbb{C}_1^*$-fixed points $0, \infty \in \mathbf{P}_1^1$ *except* for the weight of the part ν_1. For ν_1, we take the weight to be the class of one of the following T-fixed points:

$$(0,0), (\infty, 0) \in \mathbf{P}_1^1 \times \mathbf{P}_2^1 .$$

For μ, we require *all* weights to be the pullbacks of the classes of $0, \infty \in \mathbf{P}_1^1$.

The moduli space of stable pairs $P_n(\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, \beta)_{\nu,\mu}$ has virtual dimension $2d_1 + 2d_3$ minus the constraints imposed by the boundary conditions. The number of constraints imposed by ν is $d_3 + 1$ and by μ is d_3. Hence, the virtual dimension of the stable pairs space is

$$2d_1 + 2d_3 - 2d_3 - 1 .$$

The integrand $\tau_0(\gamma[0])$ imposes another constraint, so the virtual dimension of the integrals in the stable pairs partition function (56) is $2d_1 - 2$. The parallel dimension analysis for the Gromov-Witten partition function (56) also yields $2d_1 - 2$.

Lemma 16. *For $d_3 > 0$ with our special boundary conditions ν and μ, we have*

$$\mathsf{Z}^{\mathsf{P}}_{\beta,\nu,\mu}\Big(\tau_0(\gamma[0])\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T} \in \mathbb{Q}(q, s_1, s_2)$$

and the correspondence

$$(-q)^{-d_\beta/2} \mathsf{Z}^{\mathsf{P}}_{\beta,\nu,\mu}\Big(\tau_0(\gamma[0])\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T}$$

$$= (-iu)^{d_\beta + \ell(\nu) - |\nu| + \ell(\mu) - |\mu|} \mathsf{Z}'^{\,\mathsf{GW}}_{\beta,\nu,\mu}\Big(\overline{\tau_0(\gamma[0])}\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T}$$

under the variable change $-q = e^{iu}$.

Proof. We can assume $d_1 > 0$, and then $2d_1 - 2 \geq 0$. If $2d_1 - 2 > 0$, both the stable pairs and Gromov-Witten partition functions vanish by the compactness of the geometry. If $2d_1 - 2 = 0$, then both partition functions are independent of the equivariant parameters s_1 and s_2. The required matching then follows from Theorem 3.   □

We can apply T-equivariant localization to both sides of the correspondence of Lemma 16. Via the action of the second factor of T, the T-equivariant contributions occur with $\mathbf{P}_2^1$ coordinate either 0 or ∞ (remember the curve class β is degree 0 over $\mathbf{P}_2^1$). The localization contributions where the entire curve β and all the boundary conditions lie over $0 \in \mathbf{P}_2^1$ yield[22] the residue invariants appearing in Proposition 15. All the other terms in the localization formula can be expressed as the residue invariants of Proposition 15 (over 0 or $\infty \in \mathbf{P}_2^1$) with lesser β. Hence the T-equivariant localization relation applied to Lemma 16 reduces Proposition 15 to the case where $d_3 = 0$.

To prove the $d_3 = 0$ case of Proposition 15, we replace Lemma 16 with a different partition function. Let

$$\gamma_0 \in H_T^*(\mathbf{P}_1^1 \times \mathbf{P}_2^1, \mathbb{Q})$$

be the class of the point $(\infty, 0)$. Alternatively, γ_0 is the intersection of γ with the divisor over $0 \in \mathbf{P}_2^1$. Hence, γ_0 restricted to $\mathbf{P}_1^1 \times \{0\} \times \mathbf{P}_3^1$ is $-s_2\gamma$.

Lemma 17. *For $d_3 = 0$, we have*

$$\mathsf{Z}_{\beta,\emptyset,\emptyset}^{\mathsf{P}}\Big(\tau_0(\gamma_0[0])\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T} \in \mathbb{Q}(q, s_1, s_2)$$

and the correspondence

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\beta,\emptyset,\emptyset}^{\mathsf{P}}\Big(\tau_0(\gamma_0[0])\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T}$$

$$= (-iu)^{d_\beta}\mathsf{Z}_{\beta,\emptyset,\emptyset}'^{\mathsf{GW}}\Big(\overline{\tau_0(\gamma_0[0])}\Big)^{\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty, T}$$

under the variable change $-q = e^{iu}$.

Proof. The dimension analysis used in the proof of Lemma 16 is also valid here and yields the result.   □

Finally, we can apply T-equivariant localization to both sides of the correspondence of Lemma 17. The localization contributions where the entire curve β lies over $0 \in \mathbf{P}_2^1$ yield[23] the residue invariants appearing in Proposition 15. All the other terms in the localization formula include unconstrained curves over $\infty \in \mathbf{P}_2^1$ with positive $[\mathbf{P}_1^1]$ components—all such contributions vanish.[24] The T-equivariant localization relation applied to Lemma 17 completes the proof of Proposition 15.   □

Proposition 15 was the last step in the proof of Proposition 14. The proof of Proposition 14 completes the proof of Theorem 5.

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[22]Up to a harmless s_2 factor obtained from the cohomology weight of the part ν_1.

[23]Up to a harmless s_2 factor obtained from γ_0.

[24]The proof can be obtained inductively from the geometry $\mathbf{Y}/\mathbf{Y}_0 \cup \mathbf{Y}_\infty$ by considering the integrand $\tau_0(\gamma_0)$. We leave the details as an exercise for the reader.

5. Bi-relative residue theories

5.1. Overview. In order to prove Theorem 1, we must first extend Theorem 2 to nontoric surfaces S which are projective bundles over higher genus curves (as discussed in Section 2.6). Our strategy is to extend Proposition 3 to such surfaces. The extension of Theorem 2 then follows as a consequence.

In order to extend Proposition 3 to projective bundles S over higher genus curves, we will degenerate S. To carry out the argument, we will require a descendent correspondence for a particular residue theory in a bi-relative 3-fold geometry.

5.2. Bi-relative geometry. Following the notation of Section 4.4, let

$$\mathbf{Y} = \mathbf{P}_1^1 \times \mathbf{P}_2^1 \times \mathbf{P}_3^1 , \quad \mathbf{Y}_\infty = \mathbf{P}_1^1 \times \mathbf{P}_2^1 \times \{\infty\} ,$$

and let $\mathbf{D}_\infty \subset \mathbf{Y}$ be the divisor

$$\mathbf{D}_\infty = \mathbf{P}_1^1 \times \{\infty\} \times \mathbf{P}_3^1 .$$

We will consider the bi-relative 3-fold geometry

$$(57) \qquad\qquad\qquad \mathbf{Y} \ / \ \mathbf{Y}_\infty \cup \mathbf{D}_\infty .$$

Since the divisors $\mathbf{Y}_\infty$ and $\mathbf{D}_\infty$ intersect, the full stable pairs and Gromov-Witten theories of the geometry (57) are not described by standard relative techniques [9, 12].

Fortunately, we are only interested here in a corner of the bi-relative geometry (57) which can be approached by standard relative geometry—the residue theory over $0 \in \mathbf{P}_2^1$. To define the residues over $0 \in \mathbf{P}_2^1$, curves intersecting $\mathbf{Y}_\infty \cap \mathbf{D}_\infty$ do *not* arise, so the standard relative methods are sufficient.

The descendent correspondence for residue theory of (57) over $0 \in \mathbf{P}_2^1$ will play a crucial role in the extension of Proposition 3 and Theorem 2.

5.3. Construction I. Consider the moduli space of stable pairs $P_n(\mathbf{Y}/\mathbf{Y}_\infty, \beta)_\eta$ with curve class

$$\beta = d_1[\mathbf{P}_1^1] + d_2[\mathbf{P}_2^1] + d_3[\mathbf{P}_3^1]$$

and $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariant relative condition η along $\mathbf{Y}_\infty$ with cohomology weights supported on

$$\mathbf{P}_1^1 \times \{0\} \times \{\infty\} \subset \mathbf{Y}_\infty .$$

Define the open set

$$V_{n,\beta,\eta} \subset P_n(\mathbf{Y}/\mathbf{Y}_\infty, \beta)_\eta$$

to be the locus of stable pairs for which $\mathbf{D}_\infty$ is not a zero divisor.

More precisely, a stable pair in the relative geometry $\mathbf{Y}/\mathbf{Y}_\infty$ is supported on a destabilization $\widetilde{\mathbf{Y}}$ of $\mathbf{Y}$ along $\mathbf{Y}_\infty$. The original divisor $\mathbf{D}_\infty$ degenerates to the reducible divisor

$$\widetilde{\mathbf{D}}_\infty = \pi_2^{-1}(\infty) \subset \widetilde{\mathbf{Y}}, \quad \pi_2 : \widetilde{\mathbf{Y}} \to \mathbf{P}_2^1 .$$

We define $V_{n,\beta,\eta}$ to be the open set of stable pairs for which $\widetilde{\mathbf{D}}_\infty$ is not a zero divisor.[25] In other words, the stable pair is *transverse* to $\widetilde{\mathbf{D}}_\infty$: there are *no free*

---

[25]The moduli space $P_n(\mathbf{Y}/\mathbf{Y}_\infty, \beta)_\eta$ is *not* relative to $\widetilde{\mathbf{D}}_\infty$, so transversality along $\widetilde{\mathbf{D}}_\infty$ is a nontrivial condition. There is no bubbling along $\widetilde{\mathbf{D}}_\infty$.

points (the cokernel of the stable pair) on $\widetilde{\mathbf{D}}_\infty$. Via the intersection with $\widetilde{\mathbf{D}}_\infty$, we obtain a canonical map[26]

$$\epsilon : V_{n,\beta,\eta} \to \mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2) \ .$$

Here, $\mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2)$ is the Hilbert scheme[27] of d_2 points of the surface $\mathbf{P}_1^1 \times \mathbf{P}_3^1$ relative to the divisor $\mathbf{P}_1^1 \times \{\infty\}$.

The original 3-torus $\mathbf{T}$ acting on $\mathbf{Y}$ acts on $V_{n,\beta,\eta}$. While $V_{n,\beta,\eta}$ is certainly not compact, the $\mathbb{C}_2^*$-fixed point locus is compact—all features of the stable pair occur on

$$\widetilde{\mathbf{D}}_0 = \pi_2^{-1}(0) \subset \widetilde{\mathbf{Y}} \ .$$

A $\mathbb{C}_2^*$-fixed stable pair in $V_{n,\beta,\eta}$ meets $\widetilde{\mathbf{D}}_\infty$ transversely. On $\widetilde{\mathbf{Y}} \setminus \widetilde{\mathbf{D}}_0$, $\mathbb{C}_2^*$-fixed stable pairs are simply the pullbacks of 0-dimensional subschemes of $\widetilde{\mathbf{D}}_\infty$. All components of positive degree over $\mathbf{P}_1^1 \times \mathbf{P}_3^1$ of $\mathbb{C}_2^*$-fixed curves associated to stable pairs in $V_{n,\beta,\eta}$ lie over $0 \in \mathbf{P}_2^1$.

Let $\theta_j, \theta'_{j'} \in H_{\mathbb{C}_1^*}^*(\mathbf{P}_1^1, \mathbb{Q})$ be as in Section 4.1. Let

$$[0,0], [0,\infty] \in H_{\mathbb{C}_2^* \times \mathbb{C}_3^*}^*(\mathbf{P}_2^1 \times \mathbf{P}_3^1)$$

denote the classes of the points $(0,0)$ and $(0,\infty)$, respectively. Let

$$\phi \in H_{\mathbb{C}_1^* \times \mathbb{C}_3^*}^*(\mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2), \mathbb{Q}) \ .$$

We define the uncapped residue descendent series

$$\mathsf{V}_\beta^{\mathsf{P}} \left(\prod_{j=1}^k \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]) \right)_{\eta,\phi}^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}$$

$$= \sum_{n \in \mathbb{Z}} q^n \int_{[V_{n,\beta,\eta}]^{vir}} \prod_{j=1}^k \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]) \cup \epsilon^*(\phi)$$

by $\mathbf{T}$-equivariant residues.

5.4. Construction II.

Next, we consider the moduli of stable pairs for the relative geometry

$$(58) \qquad\qquad\qquad \mathbf{Y} / \mathbf{Y}_\infty \cup \mathbf{D}_\infty$$

with curve classes $d_2[\mathbf{P}_2^1]$. Since

$$[\mathbf{P}_2^1] \cdot \mathbf{Y}_\infty = 0 \ ,$$

the curves never meet $\mathbf{Y}_\infty$. So the delicate study of geometry relative to the singularities of $\mathbf{Y}_\infty \cup \mathbf{D}_\infty$ can be completely avoided. The moduli space

$$P_n(\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, d_2[\mathbf{P}_2^1])$$

is easily constructed. The projections of the curves to

$$\mathbf{P}_1^1 \times \{0\} \times \mathbf{P}_3^1$$

are never allowed to meet

$$\mathbf{P}_1^1 \times \{0\} \times \{\infty\} \subset \mathbf{Y}_\infty \ .$$

---

[26]The map involves possible stabilization. Stabilization here contracts bubbled components which have continuous automorphisms fixing the boundary sheaf.

[27]The Hilbert scheme of points of a surface relative to a divisor is a special case of the relative ideal sheaf moduli for 3-folds. See [39] for a discussion and study.

Bubbling occurs along $\mathbf{Y}_\infty$ to keep the projections away. The points of the resulting moduli space correspond to stable pairs which meet $\widetilde{\mathbf{D}}_\infty$ away from the intersection with $\mathbf{Y}_\infty$. Hence, the deformation theory and virtual class are standard.

The boundary conditions along $\mathbf{D}_\infty$ are defined via the canonical map

$$\epsilon : P_n(\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, d_2[\mathbf{P}_2^1]) \to \mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2) .$$

While any element of the cohomology of $\mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2)$ imposes a boundary condition, special elements corresponding to the Nakajima basis of the cohomology of the Hilbert scheme of points in the absolute case will play a distinguished role.

Let μ be the partition of d_2 weighted by the cohomology of the surface $\mathbf{P}_1^1 \times \mathbf{P}_3^1$. Explicitly,

$$(59) \qquad \mu = \{(\mu_1, \omega_1), \ldots, (\mu_\ell, \omega_\ell)\}, \quad d_2 = \sum_{i=1}^{\ell} \mu_i, \quad \omega_i \in H^*_{\mathbb{C}_1^* \times \mathbb{C}_3^*}(\mathbf{P}_1^1 \times \mathbf{P}_3^1, \mathbb{Q}).$$

Such a weighted partition determines an element

$$\mathsf{Nak}(\mu) \in H^*_{\mathbb{C}_1^* \times \mathbb{C}_3^*}(\mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2), \mathbb{Q})$$

by the following construction. Recall

$$\left(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}\right)^\ell \to \mathbf{P}_1^1 \times \mathbf{P}_3^1$$

is the space of ordered points in the relative surface geometry; see Section 1.2. The cohomology weights ω_i pull back canonically to the space of points $\left(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}\right)^\ell$. Let

$$\mathsf{C}_\mu \subset \left(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}\right)^\ell \times \mathrm{Hilb}\left(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2\right)$$

be the closure of the locus of distinct points in $(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\})^\ell$ carrying punctual subschemes of lengths $\mu_1, \ldots, \mu_{\ell(\mu)}$. Let

$$\mathsf{Nak}(\mu) = \frac{1}{\mathfrak{z}(\mu)} \mathrm{pr}_{2*}\left(\mathsf{C}_\mu \cdot \mathrm{pr}_1^*(\omega_1 \cup \cdots \cup \omega_\ell)\right)$$

with respect to the projections of

$$(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\})^\ell \times \mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2)$$

onto the first and second factors.

Let $\mathbf{D}_0 \subset \mathbf{Y}$ be the divisor lying over $0 \in \mathbf{P}_2^1$. We can also consider the rubber moduli spaces of stable pairs

$$P_n(\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_0 \cup \mathbf{D}_\infty, d_2[\mathbf{P}_2^1])^\sim$$

which arise in the boundary of $P_n(\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, d_2[\mathbf{P}_2^1])$ over $\mathbf{D}_\infty$. In addition to the boundary map ϵ_∞ associated to $\mathbf{D}_\infty$, there is also a boundary map

$$\epsilon_0 : P_n(\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_0 \cup \mathbf{D}_\infty, d_2[\mathbf{P}_2^1])^\sim \to \mathrm{Hilb}(\mathbf{P}_1^1 \times \mathbf{P}_3^1 / \mathbf{P}_1^1 \times \{\infty\}, d_2)$$

obtained by the intersection with $\widetilde{\mathbf{D}}_0$.

As in Section 2.4.4, we have the cotangent line classes

$$\Psi_0, \Psi_\infty \in H^2_{\mathbb{C}_1^* \times \mathbb{C}_3^*}(P_n(\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_0 \cup \mathbf{D}_\infty, d_2[\mathbf{P}_2^1])^\sim, \mathbb{Q}) .$$

Define the rubber series

$$
\mathsf{R}^{\mathsf{P}}_{d_2[\mathbf{P}^1_2]} \left(\frac{1}{-\Phi_0 + s_2} \right)^{\mathbf{T}}_{\phi,\mu}
$$

$$
= \sum_{n \in \mathbb{Z}} q^n \int_{[P_n(\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_0 \cup \mathbf{D}_\infty, d_2[\mathbf{P}^1_2])^\sim]^{vir}} \frac{1}{-\Phi_0 + s_2} \cdot \epsilon_0^*(\phi) \cup \epsilon_\infty^*(\mathsf{Nak}(\mu)) \ .
$$

Here, $\phi \in H^*_{\mathbb{C}^*_1 \times \mathbb{C}^*_3}(\mathrm{Hilb}(\mathbf{P}^1_1 \times \mathbf{P}^1_3 \ / \ \mathbf{P}^1_1 \times \{\infty\}, d_2), \mathbb{Q})$ is an arbitrary class.

5.5. Definition of the bi-relative residue.
We define the bi-relative capped descendent residue theory

$$
\mathsf{C}^{\mathsf{P}}_0 \left(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]), \beta \right)^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}_{\eta,\mu}
$$

by the formula

$$
\sum_i \mathsf{V}^{\mathsf{P}}_\beta \left(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]) \right)^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}_{\eta,\phi_i}
$$

$$
\cdot \, q^{-d_2} \, \mathsf{R}^{\mathsf{P}}_{d_2[\mathbf{P}^1_2]} \left(\frac{1}{-\Phi_0 + s_2} \right)^{\mathbf{T}}_{\phi_i^\vee,\mu},
$$

where the sum is over the components of a $\mathbb{C}^*_1 \times \mathbb{C}^*_3$-equivariant Künneth decomposition

$$
\sum_{i=1}^{f} \phi_i \otimes \phi_i^\vee = [\Delta] \in H_{\mathbb{C}^*_1 \times \mathbb{C}^*_3}(\mathrm{Hilb} \times \mathrm{Hilb}, \mathbb{Q})
$$

of the diagonal of $\mathrm{Hilb}(\mathbf{P}^1_1 \times \mathbf{P}^1_3 \ / \ \mathbf{P}^1_1 \times \{\infty\}, d_2)$.

5.6. Motivation.
We have given above a rigorous definition of the bi-relative capped descendent residue theory. If we had a complete definition of the stable pairs theory of the bi-relative geometry $\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty$, the definition of

$$
(60) \qquad \mathsf{C}^{\mathsf{P}}_0 \left(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]), \beta \right)^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}_{\eta,\mu}
$$

as a capped residue theory would be immediate. Since we are interested in the residue theory over $0 \in \mathbf{P}^1_2$, the stable pairs do not interact with the singularities of $\mathbf{Y}_\infty \cup \mathbf{D}_\infty$, and we are able to define (60) by hand.

5.7. Gromov-Witten theory.
Following every step of the stable pairs construction, we can also define a bi-relative capped descendent residue theory for stable maps,

$$
(61) \qquad \mathsf{C}^{\mathsf{GW}}_0 \left(\prod_{j=1}^{k} \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]), \beta \right)^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}_{\eta,\mu} \ .
$$

Moreover, the depth induction techniques of Sections 3 and 4 applied to both the descendent insertions and to the parts of μ yield the following correspondence.

Theorem 6. *We have*

$$\mathsf{C}_0^\mathsf{P}\Big(\prod_{j=1}^k \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]), \beta\Big)_{\eta,\mu}^{\mathbf{Y}/\mathbf{Y}_\infty\cup\mathbf{D}_\infty,\mathbf{T}} \in \mathbb{Q}(q,s_1,s_2,s_3)$$

and the correspondence

$$(-q)^{-d_\beta/2}\mathsf{C}_0^\mathsf{P}\Big(\prod_{j=1}^k \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty]), \beta\Big)_{\eta,\mu}^{\mathbf{Y}/\mathbf{Y}_\infty\cup\mathbf{D}_\infty,\mathbf{T}}$$

$$= (-iu)^{d_\beta+\ell(\eta)-|\eta|+\ell(\mu)-|\mu|}\overline{\mathsf{C}_0^\mathsf{GW}\Big(\prod_{j=1}^k \tau_{i_j}(\theta_j[0,0]) \prod_{j'=1}^{k'} \tau_{i'_{j'}}(\theta'_{j'}[0,\infty])\,, \beta\Big)}_{\eta,\mu}^{\mathbf{Y}/\mathbf{Y}_\infty\cup\mathbf{D}_\infty,\mathbf{T}}$$

under the variable change $-q = e^{iu}$.

Proof. We take the relative condition μ of the form (59) to have cohomology weights

$$\omega_i = \gamma_i[0] \quad \text{or} \quad \gamma_i[\infty],$$

where $\gamma_i \in H^*_{\mathbb{C}_1^*}(\mathbf{P}_1^1,\mathbb{Q})$ and $[0],[\infty] \in H^*_{\mathbb{C}_3^*}(\mathbf{P}_3^1,\mathbb{Q})$ are the classes of the $\mathbb{C}_3^*$-fixed points.

To prove Theorem 6, we exactly follow the depth induction used in the proof of Theorems 4 and 5. The depth count has two components:

- the number of descendent insertions of the form $\tau_{i'_{j'}}(\theta'_{j'}[0,\infty])$,
- the number of parts of μ with weights of the form $\gamma[\infty]$.

There is no difficulty in including the parts of μ over $\infty \in \mathbf{P}_3^1$ in the $\mathbf{T}$-equivariant localization formula of Lemma 9. The descendents over $\infty \in \mathbf{P}_3^1$ were used to rigidify the rubber—we can also use the parts of μ to rigidify the rubber. The outcome is a reduction of Theorem 6 to the base case where all the descendent insertions and parts of μ lie over $0 \in \mathbf{P}_3^1$.

Theorem 6 in the base case concerns only 3-leg descendent vertices at the points $(0,0,0),(\infty,0,0) \in \mathbf{Y}$ and the capped rubber contributions over $\infty \in \mathbf{P}_3^1$. The GW/P correspondence for the 3-leg descendent vertex has been established in [33]. The correspondence for the capped rubber of $\infty \in \mathbf{P}_3^1$ has already been treated in Section 4.3 via Proposition 15. □

5.8. Degeneration.
Let S be a nonsingular projective surface equipped with two line bundles L_0 and L_∞. Let

$$\pi : \mathbf{P}_S \to S$$

be the $\mathbf{P}^1$-bundle obtained from the projectivization of the sum $L_0 \oplus L_\infty$. The fiberwise $\mathbb{C}^*$-action on $\mathbf{P}_S$ leaves the divisor

$$S_\infty = \mathbf{P}(L_\infty) \subset \mathbf{P}_S$$

invariant. Let $C \subset S$ be a nonsingular curve, and let

$$\mathbf{P}_C = \pi^{-1}(C) \subset \mathbf{P}_S \,.$$

Via the fiberwise $\mathbb{C}^*$-action, we can define capped bi-relative residue theories for the geometry

$$\mathbf{P}_S / \mathbf{P}_C \cup S_\infty$$

for stable pairs and stable maps. The constructions of Sections 5.3–5.5 apply here: only the fiberwise $\mathbb{C}_2^*$-action was needed there to define the bi-relative residue theories. We therefore have capped bi-relative residue theories

$$(62) \quad \mathsf{C}_0^{\mathsf{P}}\left(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j),\beta\right)_{\eta,\mu}^{\mathbf{P}_S/\mathbf{P}_C\cup S_\infty,\mathbb{C}^*} \quad , \quad \mathsf{C}_0^{\mathsf{GW}}\left(\overline{\prod_{j=1}^{k}\tau_{i_j}(\gamma_j)},\beta\right)_{\eta,\mu}^{\mathbf{P}_S/\mathbf{P}_C\cup S_\infty,\mathbb{C}^*} \quad ,$$

where $\gamma_1,\ldots,\gamma_k \in H^*_{\mathbb{C}^*}(\mathbf{P}_S,\mathbb{Q})$ are classes supported on S_0, η is a $\mathbb{C}^*$-equivariant boundary condition along $\mathbf{P}_C$ with support on $\mathbf{P}_C \cap S_0$, and

$$\mathrm{Nak}(\mu) \in H^*(\mathrm{Hilb}(S\,/\,C,|\mu|),\mathbb{Q})$$

is a Nakajima element.

The capped bi-relative residue theories occur naturally in the degeneration formula. Let

$$\pi : \mathfrak{S} \to \Delta$$

be a nonsingular 3-fold fibered over an irreducible nonsingular base curve Δ. Let S be a nonsingular fiber, and let

$$A \cup_C B$$

be a reducible special fiber consisting of two nonsingular surfaces intersecting transversally along a nonsingular surface C. Let

$$\mathfrak{L}_0, \mathfrak{L}_\infty \to \mathfrak{S}$$

be two line bundles. The degeneration

$$(63) \qquad\qquad \mathbf{P}_{\mathfrak{S}} = \mathbf{P}(\mathfrak{L}_0 \oplus \mathfrak{L}_\infty) \to B$$

is a nonsingular 4-fold with a reducible fiber

$$\mathbf{P}_{S_1} \cup_{\mathbf{P}_C} \mathbf{P}_{S_2}\ .$$

Let $[\mathbf{P}] \in H_2(\mathbf{P}_{\mathfrak{S}},\mathbb{Z})$ be the curve class of the $\mathbf{P}^1$-fiber.

To write the degeneration formula corresponding to the geometry (63), we require the following notation:

- Let $\beta = \beta_{S_0} + d[\mathbf{P}] \in H_2(\mathbf{P}_S,\mathbb{Z})$ where $\beta_{S_0} \in H_2(S_0,\mathbb{Z})$.
- Let $\gamma_1,\ldots,\gamma_k \in H^*_{\mathbb{C}^*}(\mathbf{P}_{\mathfrak{S}},\mathbb{Q})$ be classes supported on $\mathfrak{S}_0$.
- Let μ be a partition of d with cohomology weights lying in $H^*(\mathfrak{S}_\infty,\mathbb{Q})$.

The degeneration formula for stable pairs is

$$\mathsf{C}_0^{\mathsf{P}}\Big(\prod_{j=1}^{k}\tau_{i_j}(\gamma_j),\beta\Big)_{\mu}^{\mathbf{P}_S/S_\infty,\mathbb{C}^*}$$

$$= \sum \mathsf{C}_0^{\mathsf{P}}\Big(\prod_{j\in J_1}\tau_{i_j}(\gamma_j),\beta_1\Big)_{\eta,\mu^1}^{\mathbf{P}_A/\mathbf{P}_C\cup A_\infty,\mathbb{C}^*} (-1)^{|\eta|-\ell(\eta)}\mathfrak{z}(\eta)q^{-|\eta|}$$

$$\cdot\, \mathsf{C}_0^{\mathsf{P}}\Big(\prod_{i\in J_2}\tau_{i_j}(\gamma_j),\beta_2\Big)_{\eta^\vee,\mu^2}^{\mathbf{P}_B/\mathbf{P}_C\cup B_\infty,\mathbb{C}^*}\ .$$

The sum is over all distributions of descendents, distributions of μ, and curve class splittings

$$J_1 \cup J_2 = \{1,\ldots,k\}, \quad \mu = \mu^1 \cup \mu^2, \quad \beta = \beta_1 + \beta_2,$$

where we have

$$\beta_1 = \beta_{A_0} + d_1[\mathbf{P}], \quad \beta_2 = \beta_{B_0} + d_2[\mathbf{P}]$$

with $\beta_{A_0} \in H_2(A_0, \mathbb{Z})$, $\beta_{B_0} \in H_2(B_0, \mathbb{Z})$, and $d_1 + d_2 = d$. The sum is also over a basis η of $H^*_{\mathbb{C}^*}(\mathbf{P}_C \setminus \mathbf{P}_C \cap \mathfrak{S}_\infty, \mathbb{Q})$ supported on $\mathbf{P}_C \cap \mathfrak{S}_0$.

The above degeneration formula is a straightforward consequence of the standard degeneration formula for stable pairs residue theories and the definition of the bi-relative integrals.[28] We leave the details to the reader.

The degeneration formula for Gromov-Witten theory takes a parallel form,

$$
\mathsf{C}_0^{\mathsf{GW}}\Big(\overline{\prod_{j=1}^{k} \tau_{i_j}(\gamma_j)}, \beta\Big)_\mu^{\mathbf{P}_S/S_\infty, \mathbb{C}^*}
$$

$$
= \sum \mathsf{C}_0^{\mathsf{GW}}\Big(\overline{\prod_{j \in J_1} \tau_{i_j}(\gamma_j)}, \beta_1\Big)_{\eta, \mu^1}^{\mathbf{P}_A/\mathbf{P}_C \cup A_\infty, \mathbb{C}^*} \mathfrak{z}(\eta) u^{2\ell(\eta)}
$$

$$
\cdot \mathsf{C}_0^{\mathsf{GW}}\Big(\overline{\prod_{i \in J_2} \tau_{i_j}(\gamma_j)}, \beta_2\Big)_{\eta^\vee, \mu^2}^{\mathbf{P}_B/\mathbf{P}_C \cup B_\infty, \mathbb{C}^*}
$$

with the same summation conventions. The correspondence

$$
\prod_{j=1}^{k} \tau_{i_j}(\gamma_j) \mapsto \overline{\prod_{j=1}^{k} \tau_{i_j}(\gamma_j)}
$$

is defined via the conventions of Sections 1.3 for relative geometries. The relative diagonals and log tangent bundle are used.

The above degeneration formulas are compatible with the natural generalization of Conjecture 4 for capped bi-relative residue theories.

Conjecture 5. *For the theories* (62)*, we have*

$$
\mathsf{C}_0^{\mathsf{P}}\Big(\prod_{j=1}^{k} \tau_{i_j}(\gamma_j), \beta\Big)_{\eta, \mu}^{\mathbf{P}_S/\mathbf{P}_C \cup S_\infty, \mathbb{C}^*} \in \mathbb{Q}(q, t)
$$

and the correspondence

$$
(-q)^{-d_\beta/2} \mathsf{C}_0^{\mathsf{P}}\Big(\prod_{j=1}^{k} \tau_{i_j}(\gamma_j), \beta\Big)_{\eta, \mu}^{\mathbf{P}_S/\mathbf{P}_C \cup S_\infty, \mathbb{C}^*}
$$

$$
= (-iu)^{d_\beta + \ell(\eta) - |\eta| + \ell(\mu) - |\mu|} \mathsf{C}_0^{\mathsf{GW}}\Big(\overline{\prod_{j=1}^{k} \tau_{i_j}(\gamma_j)}, \beta\Big)_{\eta, \mu}^{\mathbf{P}_S/\mathbf{P}_C \cup S_\infty, \mathbb{C}^*}
$$

under the variable change $-q = e^{iu}$*.*

The conditions imposed on β, γ_j, μ, and η in Conjecture 5 are as discussed for the degeneration formula.

5.9. **Review.** Theorems 4–6 are parallel results. The strategy of depth induction is the main idea in the proof of Theorem 4 for descendents on the cap. The base case is the correspondence for the 1-leg capped descendent vertex of [33]. For the relative geometry $\mathbf{P}^1 \times \mathbb{C} \times \mathbf{P}^1 / \mathbf{P}^1 \times \mathbb{C}$, the same depth induction is valid, but the base case, settled in Proposition 14, is new. Finally, for the bi-relative geometry

---

[28]Since the curves of the residue theory with positive S-degree lie in S_0 before degeneration, the limits lie in A_0 and B_0 after degeneration.

of Theorem 6, the relative constraints along $\mathbf{D}_\infty$ are new. Fortunately, the relative insertions fit into the original depth induction.

Theorem 6 and the degeneration formula of Section 5.8 are the main technical results which will be needed to study descendent correspondences for projective bundles over curves.

6. Projective bundles over higher genus curves

6.1. **Overview.** Let C be a nonsingular projective curve of genus g equipped with a rank 2 vector bundle $\Lambda \to C$ and two line bundles

$$L_0^C, L_\infty^C \to C .$$

Let S be the nonsingular projective surface obtained by the projectivization of Λ,

$$S = \mathbf{P}(\Lambda) \xrightarrow{\epsilon} X .$$

The projective bundle

(64) $$\mathbf{P}_S = \mathbf{P}(\epsilon^* L_0^C \oplus \epsilon^* L_\infty^C) \to S$$

admits sections

$$S_i = \mathbf{P}(\epsilon^* L_i^C) \subset \mathbf{P}_S .$$

We will establish here the relative descendent correspondence of Conjecture 4 for $\mathbf{P}_S/S_\infty$.

Relative projective bundle geometries over toric surfaces were studied in Section 2. We follow here the conventions and constructions of Section 2. Let

$$\Gamma = (\gamma_1, \ldots, \gamma_\ell), \quad \gamma_i \in H^*(S, \mathbb{Q}) .$$

Since S has odd cohomology, the classes γ_i may be of odd (real) degree.

We consider capped contributions over S_0 in curve class

$$\beta = \beta_0 + d[\mathbf{P}], \quad \beta_0 \in \mathrm{Eff}(S_0).$$

Let μ be a boundary condition along S_∞ with $|\mu| = d$,

$$\mu = \{(\mu_1, \omega_1), \ldots, (\mu_{\ell(\mu)}, \omega_{\ell(\mu)})\}$$

with $\omega_i \in H^*(S_\infty, \mathbb{Q})$. Again, the classes ω_i may be of odd (real) degree.

Proposition 18. *The $\mathbb{C}^*$-equivariant descendent correspondence for the capped contributions over S_0 holds for the geometry* (64),

$$\mathsf{C}_0^\mathsf{P} (\tau_\alpha(\Gamma_0), \beta)_\mu^{\mathbf{P}_S/S_\infty, \mathbb{C}^*} \in \mathbb{Q}(q, t),$$

and we have

$$(-q)^{-d_\beta/2} \, \mathsf{C}_0^\mathsf{P} (\tau_\alpha(\Gamma_0), \beta)_\mu^{\mathbf{P}_S/S_\infty, \mathbb{C}^*} = (-iu)^{d_\beta + \ell(\mu) - |\mu|} \, \mathsf{C}_0^\mathsf{GW} \left(\overline{\tau_\alpha(\Gamma_0)}, \beta \right)_\mu^{\mathbf{P}_S/S_\infty, \mathbb{C}^*}$$

under the variable change $-q = e^{iu}$.

The proof of Proposition 18 will be given in Sections 6.2–6.6. In the toric case studied in Section 2, Proposition 3 was shown to formally imply Theorem 2. For the geometry (64), Proposition 18 implies the descendent correspondence by the same argument.

Theorem 7. *For the relative geometry* $\mathbf{P}_S/S_\infty$ *associated to* (64) *and classes* $\gamma_i \in H^*_{\mathbb{C}^*}(\mathbf{P}_S, \mathbb{Q})$, *we have*

$$\mathsf{Z}_{\mathsf{P}}\Big(\mathbf{P}_S/S_\infty; q \,\Big|\, \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \,\Big|\, \mu\Big)^{\mathbb{C}^*}_\beta \in \mathbb{Q}(q,t)$$

and the correspondence

$$(-q)^{-d_\beta/2}\mathsf{Z}_{\mathsf{P}}\Big(\mathbf{P}_S/S_\infty; q \,\Big|\, \tau_{\alpha_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell) \,\Big|\, \mu\Big)^{\mathbb{C}^*}_\beta$$

$$= (-iu)^{d_\beta+\ell(\mu)-|\mu|}\mathsf{Z}'_{\mathsf{GW}}\Big(\mathbf{P}_S/S_\infty; u \,\Big|\, \overline{\tau_{a_1-1}(\gamma_1)\cdots\tau_{\alpha_\ell-1}(\gamma_\ell)} \,\Big|\, \mu\Big)^{\mathbb{C}^*}_\beta$$

under the variable change $-q = e^{iu}$. □

The parallel descendent correspondence holds when the projective bundle geometry $\mathbf{P}_S/S_0 \cup S_\infty$ is taken relative to both sections.

6.2. **Torus actions.** If Λ splits as a sum of line bundles on C,

(65) $$\Lambda = \Lambda_0 \oplus \Lambda_\infty \,,$$

then $S = \mathbf{P}(\Lambda)$ admits a fiberwise $\mathbb{C}^*$-action by scaling. The 3-fold

(66) $$\mathbf{P}_S = \mathbf{P}(\epsilon^* L_0^C \oplus \epsilon^* L_\infty^C)$$

then carries a 2-dimensional torus action

$$\mathbb{C}^*_1 \times \mathbb{C}^*_2 \times \mathbf{P}_S \to \mathbf{P}_S,$$

where $\mathbb{C}^*_1$ is the scaling associated to the splitting (65) and $\mathbb{C}^*_2$ is the scaling associated to the splitting (66).

In case Λ splits, we will prove the natural $\mathbb{C}^*_1 \times \mathbb{C}^*_2$-equivariant lift of Proposition 18,

$$\mathsf{C}^{\mathsf{P}}_0\left(\tau_\alpha(\Gamma_0), \beta\right)^{\mathbf{P}_S/S_\infty, \mathbb{C}^*_1 \times \mathbb{C}^*_2}_\mu \in \mathbb{Q}(q,s),$$

and we have

$$(-q)^{-d_\beta/2}\,\mathsf{C}^{\mathsf{P}}_0\left(\tau_\alpha(\Gamma_0), \beta\right)^{\mathbf{P}_S/S_\infty, \mathbb{C}^*_1 \times \mathbb{C}^*_2}_\mu$$

$$= (-iu)^{d_\beta+\ell(\mu)-|\mu|}\,\mathsf{C}^{\mathsf{GW}}_0\left(\overline{\tau_\alpha(\Gamma_0)}, \beta\right)^{\mathbf{P}_S/S_\infty, \mathbb{C}^*_1 \times \mathbb{C}^*_2}_\mu$$

under the variable change $-q = e^{iu}$.

Since every rank 2 bundle Λ is deformation equivalent to a split bundle, we can assume Λ is split in the proof of Proposition 18. We will prove the above $\mathbb{C}^*_1 \times \mathbb{C}^*_2$-equivariant correspondence (which, of course, then implies the $\mathbb{C}^*_2$-equivariant statement of Proposition 18).

6.3. **Invertibility.** Before proving Proposition 18, we will require an auxiliary result for the capped residue theory

(67) $$\mathsf{C}^{\mathsf{P}}_0\Big(\prod_{j=1}^k \tau_{i_j}(\theta_j[0,0]),\ d[\mathbf{P}^1_3]\Big)^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}_{\eta, \emptyset}$$

derived from the analysis of stable pairs descendents in [32].

Let $\mathcal{P}(d,2)$ be the set of pairs of partitions (η^0, η^∞) satisfying

$$|\eta^0| + |\eta^\infty| = d.$$

We define the boundary condition $\eta = (\eta^0, \eta^\infty)$ of (67) by weighting the parts of η^0 with the class of the point $(0, 0, \infty) \in \mathbf{Y}_\infty$ and the parts of η^∞ with the class of the point $(\infty, 0, \infty) \in \mathbf{Y}_\infty$.

Let $\mathbb{Q}^{\mathcal{P}(d,2)}$ denote the linear space of functions from $\mathcal{P}(d, 2)$ to the field $\mathbb{Q}(q, s_1, s_2, s_3)$. Let $\mathsf{p}_0, \mathsf{p}_\infty \in H^*_{\mathbb{C}_1^*}(\mathbf{P}_1^1, \mathbb{Q})$ be the classes of the fixed points $0, \infty \in \mathbf{P}_1^1$. Let

$$(68) \qquad \tilde{\tau}(\mathsf{p}) = \sum_{i=0}^\infty c_i^0 \tau_i(\mathsf{p}_0[0,0]) + \sum_{i=0}^\infty c_i^\infty \tau_i(\mathsf{p}_\infty[0,0])$$

be a *finite* linear combination of descendents. For $\mathsf{w} \geq 0$, define a function on $\mathcal{P}(d, 2)$ by

$$\gamma_\mathsf{w} : \mathcal{P}(d, 2) \to \mathbb{Q}(q, s_1, s_2, s_3), \quad \eta \mapsto \mathsf{C}_0^\mathsf{P}\Big(\tilde{\tau}(\mathsf{p})^\mathsf{w}, \ d[\mathbf{P}_3^1]\Big)^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}_{\eta, \emptyset} \ .$$

Here, C_0^P is defined by a multilinear expansion of the insertion $\tilde{\tau}(\mathsf{p})^\mathsf{w}$.

Lemma 19. *For $d \geq 0$, there exists a linear combination $\tilde{\tau}(\mathsf{p})$ for which the set of functions,*

$$\{\gamma_0, \gamma_1, \gamma_2, \dots\},$$

spans $\mathbb{Q}^{\mathcal{P}(d,2)}$.

Proof. The spanning statement concerns the linear algebra of the field of rational functions $\mathbb{Q}(q, s_1, s_2, s_3)$. We must prove nondegeneracy of the set $\{\gamma_0, \gamma_1, \gamma_2, \dots\}$. We will require only the leading q term of

$$\mathsf{C}_0^\mathsf{P}\Big(\tilde{\tau}(\mathsf{p})^\mathsf{w}, \ d[\mathbf{P}_3^1]\Big)^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty, \mathbf{T}}_{\eta, \emptyset} \ .$$

The matter is then an assertion about the cohomology of the Hilbert scheme of points of $\mathbf{Y}_\infty$.

After changing the basis of η to the $\mathbb{C}_1^* \times \mathbb{C}_2^*$-fixed points of the Hilbert scheme of points of $\mathbf{Y}_\infty$ at $(0, 0, \infty)$ and $(\infty, 0, \infty)$, the action of $\tilde{\tau}(\mathsf{p})$ is determined in Section 1.2 of [32]. The operator $\tau_k(\mathsf{p}_0[0,0])$ is diagonal in the fixed point basis with eigenvalues given by symmetric functions in the weights[29] of the structure sheaf of the fixed point of the Hilbert scheme of $\mathbf{Y}_\infty$ at $(0, 0, \infty)$. Modulo lower order symmetric functions, the eigenvalue of $\tau_k(\mathsf{p}_0[0,0])$ is simply the kth power sum.

Since the power sums determine all symmetric functions, we can find a finite linear combination

$$\tilde{\tau}(\mathsf{p}) = \sum_{i=0}^\infty c_i^0 \tau_i(\mathsf{p}_0[0,0]) + \sum_{i=0}^\infty c_i^\infty \tau_i(\mathsf{p}_\infty[0,0])$$

with distinct eigenvalues on the $\mathbb{C}_1^* \times \mathbb{C}_2^*$-fixed points of the Hilbert scheme of $\mathbf{Y}_\infty$ at $(0, 0, \infty)$ and $(\infty, 0, \infty)$. By the Vandermonde determinant, the lemma is proven. $\square$

---

[29]The weights depend only on s_1 and s_2.

Lemma 19 is similar to Lemma 5.6 of [27]. The parallel result for

$$\mathsf{C}_0^{\mathsf{GW}}\left(\overline{\tilde{\tau}(\mathsf{p})^{\mathsf{w}}},\ d[\mathbf{P}_3^1]\right)_{\eta,\emptyset}^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty,\mathbf{T}}$$

follows from the correspondence of Theorem 6.

We have used the full $\mathbf{T}$-action to prove Lemma 19. However, the weight s_3 of the third factor $\mathbb{C}_3^*$ of $\mathbf{T}$ is not needed in the argument. Hence, Lemma 19 holds for

$$\gamma_{\mathsf{w}} : \mathcal{P}(d,2) \to \mathbb{Q}(q,s_1,s_2), \quad \eta \mapsto \mathsf{C}_0^{\mathsf{P}}\left(\tilde{\tau}(\mathsf{p})^{\mathsf{w}},\ d[\mathbf{P}_3^1]\right)_{\eta,\emptyset}^{\mathbf{Y}/\mathbf{Y}_\infty \cup \mathbf{D}_\infty,\mathbb{C}_1^* \times \mathbb{C}_2^*}.$$

6.4. **Even theory.** We first prove Proposition 18 in case all the cohomology insertions γ_j and all the cohomology weights ω_i are of even (real) degree.

If the underlying curve C is $\mathbf{P}^1$, Proposition 18 specializes to Proposition 3 and is established. Consider a fiber F of

$$(69) \qquad\qquad \epsilon : R = \mathbf{P}(\Lambda_0 \oplus \Lambda_\infty) \to \mathbf{P}^1.$$

We can degenerate R to the normal cone of F,

$$R \ \rightsquigarrow \ R \cup_F \mathbf{P}^1 \times \mathbf{P}^1 \ .$$

We can degenerate the line bundles

$$(70) \qquad\qquad \Lambda_0,\ \Lambda_\infty,\ \epsilon^* L_0^C,\ \epsilon^* L_\infty^C$$

so the restrictions to $\mathbf{P}^1 \times \mathbf{P}^1$ are all trivial. By the restriction of Theorem 6 to the subtorus $\mathbb{C}_1^* \times \mathbb{C}_2^*$, Lemma 19 restricted to $\mathbb{C}_1^* \times \mathbb{C}_2^*$, and the compatibility of the descendent correspondence with the degeneration formula, we conclude Conjecture 5 holds $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariantly for $\mathbf{P}_R/\mathbf{P}_F \cup R_\infty$. Repeating the argument for another fiber F' of ϵ shows Conjecture 5 holds $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariantly for $\mathbf{P}_R/\mathbf{P}_{F \cup F'} \cup R_\infty$.

Next suppose E is a genus 1 carrying line bundles Λ_0, Λ_∞, L_0^E, and L_∞^E with

$$\epsilon : S = \mathbf{P}(\Lambda_0 \oplus \Lambda_\infty) \to E \ .$$

We can degenerate E to a nodal rational curve. The line bundles carried by E can be taken to specialize to line bundles on the nodal curve. Since there is no vanishing even cohomology for the degeneration, we conclude Conjecture 5 holds $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariantly for the genus 1 case $\mathbf{P}_S/S_\infty$ as a consequence of the genus 0 case $\mathbf{P}_R/\mathbf{P}_{F \cup F'} \cup R_\infty$.

Since we know Conjecture 5 holds $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariantly for the genus 1 case $\mathbf{P}_S/S_\infty$, degeneration to the normal cone to fibers of ϵ and Lemma 19 restricted to $\mathbb{C}_1^* \times \mathbb{C}_2^*$ prove Conjecture 5 holds $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariantly for the genus 1 cases

$$(71) \qquad\qquad \mathbf{P}_S/\mathbf{P}_F \cup S_\infty \ , \quad \mathbf{P}_S/\mathbf{P}_{F \cup F'} \cup S_\infty \ .$$

Finally, if C is the curve of arbitrary genus g, we can degenerate C to a chain of g elliptic curves. Since there is no vanishing cohomology, we deduce Conjecture 5 in the even case from the geometries (71). $\qquad\square$

6.5. **Odd theory.**

6.5.1. *Reduction to genus 1.* Suppose C is a genus g curve carrying line bundles Λ_0, Λ_∞, L_0^C, and L_∞^C with

$$\epsilon : S = \mathbf{P}(E_0 \oplus E_\infty) \to C \ .$$

We now consider Conjecture 5 equivariantly with respect to $\mathbb{C}_1^* \times \mathbb{C}_2^*$ for classes γ_j and ω_i in full generality.

Since the degeneration of C to a chain of genus 1 curves has no vanishing cohomology, we may assume C is of genus 1. By the $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariant methods of Section 6.4, Conjecture 5 for (71) follows from Conjecture 5 for $\mathbf{P}_S/S_\infty$.

We may further simplify the geometry by degenerating C to the normal cone to a point $p \in C$,

$$C \ \rightsquigarrow \ C \cup_p \mathbf{P}^1,$$

and requiring the line bundles Λ_0, Λ_∞, $\epsilon^* L_0^C$, $\epsilon^* L_\infty^C$ to be trivial on C in the limit. We then obtain a degeneration

$$S \ \rightsquigarrow \ \mathbf{P}^1 \times C \ \cup_F \ R \ ,$$

where R is of the form (69). Since Conjecture 5 has already been proven $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariantly for $\mathbf{P}_R/\mathbf{P}_F \cup R_\infty$, we need only prove Conjecture 5 holds $\mathbb{C}_1^* \times \mathbb{C}_2^*$-equivariantly for the special case

$$(72) \qquad \epsilon : S = \mathbf{P}(\mathcal{O}_E \oplus \mathcal{O}_E) \to E, \quad L_0^E = \mathcal{O}_E, \quad L_\infty^E = \mathcal{O}_E, \quad g(E) = 1 \ .$$

Explicitly, the geometry is

$$\mathbf{P}_S/S_\infty = \mathbf{P}^1 \times \mathbf{P}^1 \times E \ / \ \mathbf{P}^1 \times \{\infty\} \times E.$$

6.6. **Proof of Proposition 18.** Consider the stable pair and Gromov-Witten theories

$$(73) \qquad \mathsf{C}_0^{\mathsf{P}}\left(\tau_\alpha(\Gamma_0), \beta\right)_\mu^{\mathbf{P}_S/S_\infty, \mathbb{C}_1^* \times \mathbb{C}_2^*} \ , \quad \mathsf{C}_0^{\mathsf{GW}}\left(\overline{\tau_\alpha(\Gamma_0)}, \beta\right)_\mu^{\mathbf{P}_S/S_\infty, \mathbb{C}_1^* \times \mathbb{C}_2^*}$$

for the genus 1 geometry (72). Both are uniquely determined from the corresponding even theories by the following four properties:

  (i) Algebraicity of the virtual class,
  (ii) Degeneration formulas for the relative theory in the presence of odd cohomology,
  (iii) Monodromy invariance of the relative theory,
  (iv) Elliptic vanishing relations.

The properties (i)–(iv) were used in [27] to determine the full relative Gromov-Witten descendents of target curves in terms of the descendents of even classes.

The results of Section 5 of [27] are entirely formal and apply verbatim to the theories (73). Lemma 19 replaces Lemma 5.6 of [27]. Let

$$\mathsf{L}_0, \mathsf{L}_\infty \in H_{\mathbb{C}_1^* \times \mathbb{C}_2^*}^*(\mathbf{P}_1^1 \times \mathbf{P}_2^1 \times E, \mathbb{Q})$$

be the classes of the curves

$$\{0\} \times \{0\} \times E \quad \text{and} \quad \{\infty\} \times \{0\} \times E,$$

respectively. For $\gamma \in H^*(E, \mathbb{Q})$, let[30]

$$\tilde{\tau}(\gamma) = \sum_{i=0}^{\infty} c_i^0 \tau_i(\mathsf{L}_0 \gamma) + \sum_{i=0}^{\infty} c_i^\infty \tau_i(\mathsf{L}_\infty \gamma).$$

We start, as in (5.11) of [27], by studying the descendents

(74) $$\mathsf{C}_0^\mathsf{P} \left(\tilde{\tau}(\alpha)\tilde{\tau}(\beta), d_1[\mathbf{P}_1^1] + d_3[E] \right)_{\eta, \emptyset}^{\mathbf{P}_S / S_\infty, \mathbb{C}_1^* \times \mathbb{C}_2^*},$$

$$\mathsf{C}_0^\mathsf{GW} \left(\overline{\tilde{\tau}(\alpha)\tilde{\tau}(\beta)}, d_1[\mathbf{P}_1^1] + d_3[E] \right)_{\eta, \emptyset}^{\mathbf{P}_S / S_\infty, \mathbb{C}_1^* \times \mathbb{C}_2^*}$$

for the geometry (72) where

$$\alpha, \beta \in H^1(E, \mathbb{Q}), \quad \alpha \cup \beta = 1 .$$

Exactly following [27], the descendents (74) are determined from the even theory. Since the relations (i)–(iv) respect the descendent correspondence, we deduce Proposition 18 from the even case proven in Section 6.4. Also, the rationality of the even theory implies the rationality of the full theory. When the invertibility of Lemma 19 is applied here, an induction on d_1 is necessary.

The method developed in Section 5 of [27] proceeds to handle all descendent insertions. For the case studied in [27], the descendents

$$\tau_n(\gamma), \quad \gamma \in H^*(E, \mathbb{Q})$$

are labeled only by the integer n. Here, the insertions are of the form

$$\tau_n(\mathsf{L}_0 \gamma), \quad \tau_n(\mathsf{L}_\infty \gamma), \quad (n, \mathsf{L}_0 \gamma), \quad (n, \mathsf{L}_\infty \gamma), \quad \gamma \in H^*(E, \mathbb{Q}),$$

where the latter two types[31] are relative conditions of μ. The insertion labeling is the only difference. The reduction to the even descendents exactly follows Section 5 of [27]. $\square$

6.7. Products $S \times C$.
Let S be a nonsingular projective toric surface equipped with the action of a 2-dimensional torus T. Let C be a nonsingular projective curve of genus g. A simpler result than Theorem 7 is the following.

Proposition 20. *For $\gamma_i \in H_T^*(S \times C, \mathbb{Q})$, we have*

$$\mathsf{Z}_\mathsf{P}\left(S \times C; q \ \middle| \ \tau_{\alpha_1 - 1}(\gamma_1) \cdots \tau_{\alpha_\ell - 1}(\gamma_\ell) \right)_\beta^T \in \mathbb{Q}(q, s_1, s_2)$$

and the correspondence

$$(-q)^{-d_\beta / 2} \mathsf{Z}_\mathsf{P}\left(S \times C; q \ \middle| \ \tau_{\alpha_1 - 1}(\gamma_1) \cdots \tau_{\alpha_\ell - 1}(\gamma_\ell) \right)_\beta^T$$

$$= (-iu)^{d_\beta} \mathsf{Z}_\mathsf{GW}'\left(S \times C; u \ \middle| \ \overline{\tau_{a_1 - 1}(\gamma_1) \cdots \tau_{\alpha_\ell - 1}(\gamma_\ell)} \right)_\beta^T$$

under the variable change $-q = e^{iu}$.

---

[30]Here, the coefficients c_i^0 and c_i^∞ are taken so Lemma 19 is valid. When γ is a class of a point in E, we recover the $\mathbb{C}_1^* \times \mathbb{C}_2^*$ specialization of (68).

[31]The relative conditions *must* be treated on the same footing as the descendent insertions as the relative weights may also be odd.

Proof. Let $p_1, \ldots, p_n$ denote the T-fixed points of S. By considering localization for stable pairs and stable maps on $S \times C$ with respect to the torus T, we can reduce the descendent correspondence to a local result for $\mathbb{C}^2 \times C$ with caps in the two $\mathbb{C}^2$ directions. The localization formula is in terms of n such capped $\mathbb{C}^2 \times C$ geometries (connected by simple capped edge geometries).

Consider the capped geometry $\mathbb{C}^2 \times C$. If all the descendent insertions γ_i have an even (real) cohomological degree, we can reduce to the case where $g(C) = 0$ by our standard degeneration and relative arguments. Crucial here is a tri-relative residue theory for

$$(75) \qquad \qquad \mathbb{C}^2 \times \mathbf{P}^1 \ / \ \mathbb{C}^2 \times \{\infty\}$$

defined by completely parallel constructions to the bi-relative case considered in Section 5. The tri-relative geometry has caps in the two $\mathbb{C}^2$ directions on (75). The proof of the GW/P descendent correspondence for the tri-relative cap (75) exactly follows the proof Theorem 4 with the two relative directions corresponding to $\mathbb{C}^2$ handled as in the proof Theorem 6.

To control the odd descendents, we follow the strategy of Section 6.5 (and [27]). We reduce to the case where $g(C) = 1$ and express the full theory in terms of the even theory. Lemma 19 for T still applies.

We leave the straightforward details here to the reader. The capped edges are 1-leg geometries and are easily handled. $\qquad \square$

7. PROOF OF THEOREM 1

7.1. Overview. We have now proven the GW/P descendent correspondences in sufficiently many geometries to deduce Theorem 1. The idea is to degenerate the complete intersection by factoring the equations. We present the proof carefully for the quintic in $\mathbf{P}^4$ following the scheme of [22]. The argument for general Fano and Calabi-Yau complete intersections in products of projective spaces is identical.

7.2. Simple theories. A *complete intersection pair* (V, W) is a nonsingular complete intersection

$$V \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m} \ ,$$

together with a nonsingular divisor $W \subset V$ cut out by a hypersurface in $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$. In particular,

$$W \subset \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$$

is also a complete intersection.

A class $\gamma \in H^*(V, \mathbb{Q})$ is *simple* if γ lies in the image of the restriction map

$$H^*(\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}, \mathbb{Q}) \to H^*(V, \mathbb{Q}).$$

If V is a nonsingular complete intersection of dimension 3, the simple cohomology of V equals the even cohomology by the Lefschetz results.

The simple Gromov-Witten and stable pairs theories of V consist of the integrals of descendents of simple classes. Similarly, the simple Gromov-Witten and stable pairs theories of the relative geometry V/W consists of integrals of descendents of simple classes with *no restrictions on the cohomology classes of W in the relative constraints.*

7.3. Fano and Calabi-Yau hypersurfaces in $\mathbf{P}^4$.

7.3.1. *Notation.* The following notation for curves, surfaces, and 3-folds will be convenient for our study:

(i) let $C_{d_1,d_2} \subset \mathbf{P}^3$ be a nonsingular complete intersection of type (d_1, d_2),
(ii) let $S_d \subset \mathbf{P}^3$ be a nonsingular surface of degree d,
(iii) let $T_d \subset \mathbf{P}^4$ be a nonsingular 3-fold of degree d.

Finally, let $\mathbf{P}^3[d_1, d_2]$ be the blowup of $\mathbf{P}^3$ along C_{d_1,d_2}.

7.3.2. *Degeneration for the quintic.* Let $\mathsf{Z}^{\mathsf{P}}(T_5^\star)$ denote the simple stable pairs descendent theory of the Calabi-Yau quintic 3-fold.[32] Factoring the quintic equation yields a degeneration,

$$(76) \qquad\qquad T_5 \rightsquigarrow T_4 \cup_{S_4} \mathbf{P}^3[4, 5],$$

where $S_4 \subset T_4$ is a linear section and $S_4 \subset \mathbf{P}^3[4, 5]$ is the strict transform of a quartic containing $C_{4,5} \subset \mathbf{P}^3$. See Section 0.5.4 of [22] for a detailed construction of the degeneration (76). The degeneration formula expresses $\mathsf{Z}_{\mathsf{P}}(T_5^\star)$ in terms of the relative theories of the special fibers. We write the relation schematically as

$$\mathsf{Z}_{\mathsf{P}}(T_5^\star) \ \rightsquigarrow \ \mathsf{Z}_{\mathsf{P}}(T_4^\star/S_4) \text{ and } \mathsf{Z}_{\mathsf{P}}(\mathbf{P}^3[4, 5]/S_4) \ .$$

Similarly, in Gromov-Witten theory, we have the determination

$$\mathsf{Z}'_{\mathsf{GW}}(T_5^\star) \ \rightsquigarrow \ \mathsf{Z}'_{\mathsf{GW}}(T_4^\star/S_4), \text{ and } \mathsf{Z}'_{\mathsf{GW}}(\mathbf{P}^3[4, 5]/S_4) \ .$$

By the compatibility of the descendent correspondence with degeneration, the descendent correspondences for $T_4^\star/S_4$ and $\mathbf{P}^3[4, 5]/S_4$ imply Theorem 1 for T_5.

7.3.3. *Descendent correspondence for $\mathbf{P}^3[4, 5]/S_4$.* Let us start with $\mathbf{P}^3[4, 5]/S_4$. Degeneration to the normal cone of

$$S_4 \subset \mathbf{P}^3[4, 5]$$

yields the determination[33]

$$(77) \qquad\qquad \mathsf{Z}(\mathbf{P}^3[4, 5]) \ \rightsquigarrow \ \mathsf{Z}(\mathbf{P}^3[4, 5]/S_4) \text{ and } \mathsf{Z}(\mathbf{P}_{S_4}/S_4) \ .$$

We know the descendent correspondence for all projective bundle geometries $\mathbf{P}_{S_4}/S_4$ by Proposition 10. By the invertibility of Proposition 6 of [31], the determination (77) can be reversed,

$$\mathsf{Z}(\mathbf{P}^3[4, 5]/S_4) \ \rightsquigarrow \ \mathsf{Z}(\mathbf{P}^3[4, 5]) \text{ and } \mathsf{Z}(\mathbf{P}_{S_4}/S_4) \ .$$

Hence, the descendent correspondence for $\mathbf{P}^3[4, 5]$ implies the descendent correspondence for $\mathbf{P}^3[4, 5]/S_4$.

An approach to the blowup $\mathbf{P}^3[4, 5]$ is explained in Section 3.1 of [22]. Let $S_4 \subset \mathbf{P}^3$ contain $C_{4,5}$. Degeneration to the normal cone of $S_4 \subset \mathbf{P}^3$ yields

$$(78) \qquad\qquad \mathbf{P}^3 \rightsquigarrow \mathbf{P}^3 \cup_{S_4} \mathbf{P}_{S_4}$$

for the projective bundle geometry

$$(79) \qquad\qquad \pi : \mathbf{P}_{S_4} = \mathbf{P}(\mathcal{O}_{S_4} \oplus \mathcal{O}_{S_4}(4)) \to S_4, \quad \mathbf{P}_{S_4}/(S_4)_\infty \ .$$

---

[32]Here and below, the superscript $\star$ will indicate simple theories.
[33]When no superscript appears on the partition function, the statement is understood to hold for both stable pairs and Gromov-Witten theory.

The original curve $C_{4,5} \subset \mathbf{P}^3$ has a limit in $(S_4)_0 \subset \mathbf{P}_{S_4}$. After blowing up the degeneration (78) along the moving curve $C_{4,5}$, we obtain

$$\mathbf{P}^3[4,5] \rightsquigarrow \mathbf{P}^3 \cup_{S_4} X, \qquad X = \mathcal{B}_{C_{4,5}}(\mathbf{P}_{S_4}) \, .$$

Here X is the blowup of $\mathbf{P}_{S_4}$ along $C_{4,5} \subset (S_4)_0$. In order to prove the descendent correspondence for $\mathbf{P}^3[4,5]$, we need only prove the descendent correspondence for $\mathbf{P}^3/S_4$ and X/S_4. Using the established descendent correspondences for the toric variety $\mathbf{P}^3$ and projective bundles over S_4 and the invertibility of Proposition 6 of [31], we need only prove the descendent correspondence for X.

To study X, we repeat the blowup construction of the previous paragraph. Let $\mathbf{P}_{S_4}$ be the projective bundle (79), and let

$$\mathbf{P}_{C_{4,5}} \subset \mathbf{P}_{S_4}$$

be the divisor lying over $C_{4,5} \subset S_4$ via π. Degeneration to the normal cone of $\mathbf{P}_{C_{4,5}} \subset \mathbf{P}_{S_4}$ yields

(80) $$\mathbf{P}_{S_4} \rightsquigarrow \mathbf{P}_{S_4} \cup_{\mathbf{P}_{C_{4,5}}} \mathbf{P}_{\mathbf{P}_{C_{4,5}}} \, ,$$

for the projective bundle geometry

(81) $$\mathbf{P}_{\mathbf{P}_{C_{4,5}}} = \mathbf{P}(\mathcal{O}_{C_{4,5}} \oplus \mathcal{O}_{C_{4,5}}(5)) \times_{C_{4,5}} \mathbf{P}(\mathcal{O}_{C_{4,5}} \oplus \mathcal{O}_{C_{4,5}}(4)) \to C_{4,5}$$

relative to

$$\mathbf{P}_{C_{4,5}} = \mathbf{P}(\mathcal{O}_{C_{4,5}}(5)) \times_{C_{4,5}} \mathbf{P}(\mathcal{O}_{C_{4,5}} \oplus \mathcal{O}_{C_{4,5}}(4)) \, .$$

The original curve $C_{4,5} \subset (S_4)_0 \subset \mathbf{P}_{S_4}$ has a limit equal to

$$C_{4,5} = \mathbf{P}(\mathcal{O}_{C_{4,5}}) \times_{C_{4,5}} \mathbf{P}(\mathcal{O}_{C_{4,5}}) \subset \mathbf{P}(\mathcal{O}_{C_{4,5}} \oplus \mathcal{O}_{C_{4,5}}(5)) \times_{C_{4,5}} \mathbf{P}(\mathcal{O}_{C_{4,5}} \oplus \mathcal{O}_{C_{4,5}}(4)) \, .$$

After blowing up the degeneration (78) along the moving curve $C_{4,5}$, we obtain

$$X \rightsquigarrow \mathbf{P}_{S_4} \cup_{\mathbf{P}_{C_{4,5}}} Y \, ,$$

$$Y = \mathcal{B}_{C_{4,5}}(\mathbf{P}(\mathcal{O}_{C_{4,5}} \oplus \mathcal{O}_{C_{4,5}}(5)) \times_{C_{4,5}} \mathbf{P}(\mathcal{O}_{C_{4,5}} \oplus \mathcal{O}_{C_{4,5}}(4))) \, .$$

In order to prove the descendent correspondence for Y, we need only prove the descendent correspondence for $\mathbf{P}_{S_4}/\mathbf{P}_{C_{4,5}}$ and $Y/\mathbf{P}_{C_{4,5}}$.

As we have seen in (81), $\mathbf{P}_{\mathbf{P}_{C_{4,5}}}$ is a projective bundle over $\mathbf{P}_{C_{4,5}}$ of the form required by Theorem 7. Hence, the descendent correspondence holds for $\mathbf{P}_{S_4}/\mathbf{P}_{C_{4,5}}$ by inverting

$$\mathsf{Z}(\mathbf{P}_{S_4}) \rightsquigarrow \mathsf{Z}(\mathbf{P}_{S_4}/\mathbf{P}_{C_{4,5}}) \text{ and } \mathsf{Z}(\mathbf{P}_{\mathbf{P}_{C_{4,5}}}/S_4) \, .$$

The invertibility is possible again by Proposition 6 of [31]. Similarly, the descendent correspondence for Y implies the descendent correspondence for $Y/\mathbf{P}_{C_{4,5}}$.

The last step in proving the descendent correspondence for $\mathbf{P}^3[4,5]/S_4$ is to prove the descendent correspondence for Y. The 3-fold Y is a bundle over $C_{4,5}$ with fiber equal to $\mathcal{B}_p(\mathbf{P}^1 \times \mathbf{P}^1)$, the blowup of $\mathbf{P}^1 \times \mathbf{P}^1$ at a point. By further degeneration arguments,[34] we can reduce to the case

$$\mathcal{B}_p(\mathbf{P}^1 \times \mathbf{P}^1) \times C_{4,5}$$

covered by Proposition 20. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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[34]We can degenerate

$$C_{4,5} \rightsquigarrow C_{4,5} \cup \mathbf{P}^1$$

and require all the twisting of Y to lie over $\mathbf{P}^1$.

7.4. **Proof of Theorem 1.** We turn now to $T_4^\star/S_4$. The normal cone degeneration

$$T_4 \rightsquigarrow T_4 \cup_{S_4} \mathbf{P}_{S_4}$$

and invertibility yields the determination

$$\mathsf{Z}(T_4^\star/S_4) \rightsquigarrow \mathsf{Z}(T_4^\star) \text{ and } \mathsf{Z}(\mathbf{P}_{S_4}/S_4) .$$

Hence, the descendent correspondence for $T_5^\star$ follows from the descendent correspondence for $T_4^\star$.

By factoring the quartic equation defining $T_4 \subset \mathbf{P}^3$, degree reduction can be continued. The full reduction scheme for the quintic is

$$
\begin{aligned}
\mathsf{Z}(T_5^\star) &\rightsquigarrow \mathsf{Z}(T_4^\star/S_4) \text{ and } \mathsf{Z}(\mathbf{P}^3[4,5]/S_4), \\
\mathsf{Z}(T_4^\star/S_4) &\rightsquigarrow \mathsf{Z}(T_4^\star) \text{ and } \mathsf{Z}(\mathbf{P}_{S_4}/S_4), \\
\mathsf{Z}(T_4^\star) &\rightsquigarrow \mathsf{Z}(T_3^\star/S_3) \text{ and } \mathsf{Z}(\mathbf{P}^3[3,4]/S_3), \\
\mathsf{Z}(T_3^\star/S_3) &\rightsquigarrow \mathsf{Z}(T_3^\star) \text{ and } \mathsf{Z}(\mathbf{P}_{S_3}/S_3), \\
\mathsf{Z}(T_3^\star) &\rightsquigarrow \mathsf{Z}(T_2^\star/S_2) \text{ and } \mathsf{Z}(\mathbf{P}^3[2,3]/S_2), \\
\mathsf{Z}(T_2^\star/S_2) &\rightsquigarrow \mathsf{Z}(T_2^\star) \text{ and } \mathsf{Z}(\mathbf{P}_{S_2}/S_2), \\
\mathsf{Z}(T_2^\star) &\rightsquigarrow \mathsf{Z}(T_1^\star/S_1) \text{ and } \mathsf{Z}(\mathbf{P}^3[1,2]/S_1), \\
\mathsf{Z}(T_1^\star/S_1) &\rightsquigarrow \mathsf{Z}(T_1^\star) \text{ and } \mathsf{Z}(\mathbf{P}_{S_1}/S_1) .
\end{aligned}
$$

The end points of the scheme are T_1^* (which is toric), projective bundles over toric and $K3$ surfaces, and blown-up projective spaces. The descendent correspondence has been established for all the end points—the blown-up projective spaces are handled by the method of Section 7.3.3.

We have proven the GW/P descendent correspondence for the even theories of all Fano hypersurfaces in $\mathbf{P}^4$. We can degenerate all Fano and Calabi-Yau 3-fold complete intersections by an identical factoring argument. The outcome is a proof of Theorem 1. $\qquad\square$

7.5. **Proof of Corollary 1.** To prove Corollary 1, we start with the descendent correspondence of Theorem 1. The initial term results of Theorem 2 of [33] then imply the Corollary. $\qquad\square$

7.6. **The Enriques Calabi-Yau.** As a further example, we prove the GW/P correspondence for the Enriques Calabi-Yau 3-fold studied in [11, 23].

Let σ act freely on the product $K3 \times E$ by an Enriques involution σ_{K3} on the $K3$ and by -1 on the elliptic curve. By definition, the quotient

$$Q = (K3 \times E)/\langle\sigma\rangle$$

is an *Enriques Calabi-Yau 3-fold*. Since $K3 \times E$ carries a holomorphic 3-form invariant under σ, the canonical class is trivial

$$\omega_Q = \mathcal{O}_Q.$$

By projection on the right,

(82) $$Q \to E/\langle-1\rangle = \mathbf{P}^1$$

is a $K3$ fibration with four double Enriques fibers.

Let $\mathrm{inv}_{\mathbf{P}^1}$ be an involution of $\mathbf{P}^1$ with two fixed points. Let τ act freely on the product $K3 \times \mathbf{P}^1$ by $(\sigma_{K3}, \mathrm{inv}_{\mathbf{P}^1})$. Let

$$R = (K3 \times \mathbf{P}^1)/\langle\tau\rangle$$

denote the quotient. By projecting left,

$$R \to K3/\langle \sigma_{K3} \rangle = S \tag{83}$$

is a projective bundle over the Enriques surface S. Two sections of the bundle (83) are obtained from the fixed points of $\mathrm{inv}_{\mathbf{P}^1}$. By projecting right,

$$R \to \mathbf{P}^1/\langle \mathrm{inv}_{\mathbf{P}^1} \rangle$$

is a $K3$ fibration with two double Enriques fibers.

By degenerating the $K3$ fibration (82), we find a degeneration of the Enriques Calabi-Yau Q,

$$Q \leadsto R \cup_{K3} R,$$

where the intersection $K3$ is a common fiber; see [23, Section 1.4]. Hence the GW/P correspondence for Q is reduced to the GW/P descendent correspondence for $R/K3$. The latter reduces to the GW/P descendent correspondence for R by degeneration to the normal cone and Proposition 10.

The Enriques surface S degenerates[35] to a union along an elliptic curve of a $\mathbf{P}^1$-bundle over an elliptic curve and the rational elliptic surface; see [23, Section 1.3]. We use the corresponding degeneration of (83) to prove the GW/P descendent correspondence for R. We obtain the following result.

Proposition 21. *Let Q be the Enriques Calabi-Yau, and let $\beta \in H_2(Q, \mathbb{Z})/\mathrm{tor}$ be a curve class. Then,*

$$\mathsf{Z}_{\mathsf{P}}\Big(Q; q\Big)_{\beta} \in \mathbb{Q}(q) ,$$

and we have the correspondence

$$\mathsf{Z}_{\mathsf{P}}\Big(Q; q\Big)_{\beta} = \mathsf{Z}'_{\mathsf{GW}}\Big(Q; u\Big)_{\beta}$$

under the variable change $-q = e^{iu}$.

Acknowledgments

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