Irreducible modules over finite simple Lie pseudoalgebras II. Primitive pseudoalgebras of type K

IRREDUCIBLE MODULES OVER FINITE SIMPLE LIE PSEUDOALGEBRAS II. PRIMITIVE PSEUDOALGEBRAS OF TYPE K

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Abstract. One of the algebraic structures that has emerged recently in the study of the operator product expansions of chiral fields in conformal field theory is that of a Lie conformal algebra. A Lie pseudoalgebra is a generalization of the notion of a Lie conformal algebra for which $\mathbb{C}[\partial]$ is replaced by the universal enveloping algebra H of a finite-dimensional Lie algebra. The finite (i.e., finitely generated over H) simple Lie pseudoalgebras were classified in our previous work [BDK]. The present paper is the second in our series on representation theory of simple Lie pseudoalgebras. In the first paper we showed that any finite irreducible module over a simple Lie pseudoalgebra of type W or S is either an irreducible tensor module or the kernel of the differential in a member of the pseudo de Rham complex. In the present paper we establish a similar result for Lie pseudoalgebras of type K, with the pseudo de Rham complex replaced by a certain reduction called the contact pseudo de Rham complex. This reduction in the context of contact geometry was discovered by Rumin.

Contents

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1. Introduction

The present paper is the second in our series of papers on representation theory of simple Lie pseudoalgebras, the first of which is [BDK1].

Recall that a *Lie pseudoalgebra* is a (left) module L over a cocommutative Hopf algebra H, endowed with a pseudo-bracket

$$L\otimes L\to (H\otimes H)\otimes_H L\,,\quad a\otimes b\mapsto [a*b]\,,$$

which is an H-bilinear map of H-modules, satisfying some analogs of the skewsymmetry and Jacobi identity of a Lie algebra bracket (see [BD], [BDK]).

In the case when H is the base field $\mathbf{k}$, this notion coincides with that of a Lie algebra. Any Lie algebra $\mathfrak{g}$ gives rise to a Lie pseudoalgebra $\operatorname{Cur}\mathfrak{g}=H\otimes\mathfrak{g}$ over H with pseudobracket

$$[(1\otimes a)*(1\otimes b)]=(1\otimes 1)\otimes_H [a,b]\,,$$

extended to the whole $\operatorname{Cur}\mathfrak{g}$ by H-bilinearity.

In the case when $H=\mathbf{k}[\partial]$, the algebra of polynomials in an indeterminate ∂ with the comultiplication $\Delta(\partial)=\partial\otimes 1+1\otimes\partial$, the notion of a Lie pseudoalgebra coincides with that of a *Lie conformal algebra* [K]. The main result of [DK] states that in this case any finite (i.e., finitely generated over $H=\mathbf{k}[\partial]$) simple Lie pseudoalgebra is isomorphic either to $\operatorname{Cur}\mathfrak{g}$ with simple finite-dimensional $\mathfrak{g}$, or to the Virasoro pseudoalgebra $\operatorname{Vir}=\mathbf{k}[\partial]\ell$, where

$$[\ell*\ell]=(1\otimes\partial-\partial\otimes 1)\otimes_{\mathbf{k}[\partial]}\ell\,,$$

provided that $\mathbf{k}$ is algebraically closed of characteristic 0.

In [BDK] we generalized this result to the case when $H = U(\mathfrak{d})$, where $\mathfrak{d}$ is any finite-dimensional Lie algebra. The generalization of the Virasoro pseudoalgebra is $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ with the pseudobracket

$$[(1 \otimes a) * (1 \otimes b)] = (1 \otimes 1) \otimes_H (1 \otimes [a,b]) + (b \otimes 1) \otimes_H (1 \otimes a) - (1 \otimes a) \otimes_H (1 \otimes b)\,.$$

The main result of [BDK] is that all nonzero subalgebras of the Lie pseudoalgebra $W(\mathfrak{d})$ are simple and non-isomorphic, and, along with $\operatorname{Cur}\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, they provide a complete list of finitely generated over H simple Lie pseudoalgebras, provided that $\mathbf{k}$ is algebraically closed of characteristic 0. Furthermore, in [BDK] we gave a description of all subalgebras of $W(\mathfrak{d})$. Namely, a complete list consists of the "primitive" series $S(\mathfrak{d},\chi)$, $H(\mathfrak{d},\chi,\omega)$ and $K(\mathfrak{d},\theta)$, and their "current" generalizations.

In [BDK1] we constructed all *finite* (i.e., finitely generated over $H = U(\mathfrak{d})$) irreducible modules over the Lie pseudoalgebras $W(\mathfrak{d})$ and $S(\mathfrak{d},\chi)$. The simplest nonzero module over $W(\mathfrak{d})$ is $\Omega^0(\mathfrak{d}) = H$, given by

$$(f \otimes a) * g = -(f \otimes ga) \otimes_H 1\,, \quad f,g \in H\,,\ a \in \mathfrak{d}\,. \tag{1.1}$$

A generalization of this construction, called a tensor $W(\mathfrak{d})$-module, is as follows [BDK1]. First, given a Lie algebra $\mathfrak{g}$, define the semidirect sum $W(\mathfrak{d}) \ltimes \operatorname{Cur}\mathfrak{g}$ as a direct sum as H-modules, for which $W(\mathfrak{d})$ is a subalgebra and $\operatorname{Cur}\mathfrak{g}$ is an ideal, with the following pseudobracket between them:

$$[(f \otimes a) * (g \otimes b)] = -(f \otimes ga) \otimes_H (1 \otimes b)\,,$$

where $f, g \in H$, $a \in \mathfrak{d}$, $b \in \mathfrak{g}$. Given a finite-dimensional $\mathfrak{g}$-module V_0, we construct a representation of $W(\mathfrak{d}) \ltimes \operatorname{Cur}\mathfrak{g}$ in $V = H \otimes V_0$ by (cf. (1.1)):

$$\big((f \otimes a) \oplus (g \otimes b)\big) * (h \otimes v) = -(f \otimes ha) \otimes_H (1 \otimes v) + (g \otimes h) \otimes_H (1 \otimes bv)\,, \tag{1.2}$$

where $f, g, h \in H$, $a \in \mathfrak{d}$, $b \in \mathfrak{g}$, $v \in V_0$.

Next, we define an embedding of $W(\mathfrak{d})$ in $W(\mathfrak{d}) \ltimes \operatorname{Cur}(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$ by

$$1 \otimes \partial_i \mapsto (1 \otimes \partial_i) \oplus \big((1 \otimes \partial_i) \oplus (1 \otimes \operatorname{ad}\partial_i + \sum_j \partial_j \otimes e_i^j)\big)\,, \tag{1.3}$$

where $\{\partial_i\}$ is a basis of $\mathfrak{d}$ and $\{e_i^j\}$ a basis of $\mathfrak{gl}\,\mathfrak{d}$, defined by $e_i^j(\partial_k) = \delta_k^j \partial_i$. Composing this embedding with the action (1.2) of $W(\mathfrak{d}) \ltimes \operatorname{Cur}\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d}$, we obtain a $W(\mathfrak{d})$-module $V = H \otimes V_0$ for each $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module V_0. This module is called a *tensor* $W(\mathfrak{d})$*-module* and is denoted $\mathcal{T}(V_0)$.

The main result of [BDK1] states that any finite irreducible $W(\mathfrak{d})$-module is a unique quotient of a tensor module $\mathcal{T}(V_0)$ for some finite-dimensional irreducible $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module V_0, describes all cases when $\mathcal{T}(V_0)$ are not irreducible, and provides an explicit construction of their irreducible quotients, called the *degenerate* $W(\mathfrak{d})$*-modules*. Namely, we prove in [BDK1] that all degenerate $W(\mathfrak{d})$-modules occur as images of the differential d in the Π-twisted pseudo de Rham complex of $W(\mathfrak{d})$-modules

$$0 \to \Omega^0_\Pi(\mathfrak{d}) \xrightarrow{\mathrm{d}} \Omega^1_\Pi(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{\dim\mathfrak{d}}_\Pi(\mathfrak{d})\,. \tag{1.4}$$

Here Π is a finite-dimensional irreducible $\mathfrak{d}$-module and $\Omega^n_\Pi(\mathfrak{d}) = \mathcal{T}(\Pi \otimes \bigwedge^n \mathfrak{d}^*)$ is the space of pseudo n-forms.

In the present paper we construct all finite irreducible modules over the contact Lie pseudoalgebra $K(\mathfrak{d},\theta)$, where $\mathfrak{d}$ is a Lie algebra of odd dimension $2N+1$ and θ is a contact linear function on $\mathfrak{d}$. To any $\theta \in \mathfrak{d}^*$ one can associate a skewsymmetric bilinear form ω on $\mathfrak{d}$, defined by $\omega(a \wedge b) = -\theta([a,b])$. The linear function θ is called *contact* if $\mathfrak{d}$ is a direct sum of subspaces $\bar{\mathfrak{d}} = \ker\theta$ and $\ker\omega$. In this case $\dim\ker\omega = 1$ and there exists a unique element $\partial_0 \in \ker\omega$ such that $\theta(\partial_0) = -1$. Furthermore, the restriction of ω to $\bar{\mathfrak{d}}$ is non-degenerate; hence we can choose dual bases $\{\partial_i\}$ and $\{\partial^i\}$ of $\bar{\mathfrak{d}}$, i.e., $\omega(\partial^i \wedge \partial_j) = \delta^i_j$ for $i, j = 1, \dots, 2N$. Then the element

$$r = \sum_{i=1}^{2N} \partial_i \otimes \partial^i \in H \otimes H$$

is skewsymmetric and independent of the choice of dual bases.

The Lie pseudoalgebra $K(\mathfrak{d}, \theta)$ is defined as a free H-module He of rank 1 with the following pseudobracket:

$$[e * e] = (r + \partial_0 \otimes 1 - 1 \otimes \partial_0) \otimes_H e\,.$$

There is a unique pseudoalgebra embedding of $K(\mathfrak{d}, \theta)$ in $W(\mathfrak{d})$, which is given by

$$e \mapsto -r + 1 \otimes \partial_0\,.$$

We will denote again by e its image in $W(\mathfrak{d})$. Let $\mathfrak{sp}\,\mathfrak{d}$ (respectively $\mathfrak{sp}\,\bar{\mathfrak{d}}$) be the subalgebra of the Lie algebra $\mathfrak{gl}\,\mathfrak{d}$ (resp. $\mathfrak{gl}\,\bar{\mathfrak{d}}$), consisting of $A \in \mathfrak{gl}\,\mathfrak{d}$ (resp. $A \in \mathfrak{gl}\,\bar{\mathfrak{d}}$), such that $\omega(Au \wedge v) = -\omega(u \wedge Av)$ for all $u, v \in \mathfrak{d}$ (resp. $\bar{\mathfrak{d}}$). Let $\mathfrak{csp}\,\mathfrak{d} = \mathfrak{sp}\,\mathfrak{d} \oplus \mathbf{k}I'$, where $I'(\partial_0) = 2\partial_0$, $I'|_{\bar{\mathfrak{d}}} = I_{\bar{\mathfrak{d}}}$, and $\mathfrak{csp}\,\bar{\mathfrak{d}} = \mathfrak{sp}\,\bar{\mathfrak{d}} \oplus \mathbf{k}I_{\bar{\mathfrak{d}}}$. We have an obvious surjective Lie algebra homomorphism of the Lie algebra $\mathfrak{sp}\,\mathfrak{d}$ onto the (simple) Lie algebra $\mathfrak{sp}\,\bar{\mathfrak{d}}$, and of $\mathfrak{csp}\,\mathfrak{d}$ onto $\mathfrak{csp}\,\bar{\mathfrak{d}}$. We show that the image of $e \in W(\mathfrak{d})$ under the map (1.3) lies in $W(\mathfrak{d}) \ltimes \mathrm{Cur}(\mathfrak{d} \oplus \mathfrak{csp}\,\mathfrak{d})$. Hence each $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-module V_0, being a $(\mathfrak{d} \oplus \mathfrak{csp}\,\mathfrak{d})$-module, gives rise to a $K(\mathfrak{d}, \theta)$-module $\mathcal{T}(V_0) = H \otimes V_0$, with the action given by (1.2). These are the *tensor modules* $\mathcal{T}(V_0)$ over $K(\mathfrak{d}, \theta)$.

In the present paper we show that any finite irreducible $K(\mathfrak{d}, \theta)$-module is a unique quotient of a tensor module $\mathcal{T}(V_0)$ for some finite-dimensional irreducible $(\mathfrak{d} \oplus \mathfrak{csp}\bar{\mathfrak{d}})$-module V_0. We describe all cases when the $K(\mathfrak{d}, \theta)$-modules $\mathcal{T}(V_0)$ are not irreducible and give an explicit construction of their irreducible quotients called degenerate $K(\mathfrak{d}, \theta)$-modules. It turns out that all degenerate $K(\mathfrak{d}, \theta)$-modules again appear as images of the differential in a certain complex of $K(\mathfrak{d}, \theta)$-modules, which we call the Π-twisted contact pseudo de Rham complex, obtained by a certain reduction of the Π-twisted pseudo de Rham complex (1.4). The idea of this reduction is borrowed from Rumin's reduction of the de Rham complex on a contact manifold [Ru].

As a corollary of our results we obtain the classification of all degenerate modules over the contact Lie–Cartan algebra K_{2N+1}, along with a description of the corresponding singular vectors given (without proofs) in [Ko]. Moreover, we obtain an explicit construction of these modules.

We will work over an algebraically closed field $\mathbf{k}$ of characteristic 0. Unless otherwise specified, all vector spaces, linear maps and tensor products will be considered over $\mathbf{k}$. Throughout the paper, $\mathfrak{d}$ will be a Lie algebra of odd dimension $2N+1 < \infty$.

2. Preliminaries

In this section we review some facts and notation that will be used throughout the paper.

2.1. **Forms with constant coefficients.** Consider the cohomology complex of the Lie algebra $\mathfrak{d}$ with trivial coefficients:

$$0 \to \Omega^0 \xrightarrow{\mathrm{d}_0} \Omega^1 \xrightarrow{\mathrm{d}_0} \cdots \xrightarrow{\mathrm{d}_0} \Omega^{2N+1}, \qquad \dim \mathfrak{d} = 2N+1\,, \tag{2.1}$$

where $\Omega^n = \bigwedge^n \mathfrak{d}^*$. Set $\Omega = \bigwedge^\bullet \mathfrak{d}^* = \bigoplus_{n=0}^{2N+1} \Omega^n$ and $\Omega^n = \{0\}$ if $n < 0$ or $n > 2N+1$. We will think of the elements of Ω^n as skew-symmetric *n-forms*, i.e., linear maps from $\bigwedge^n \mathfrak{d}$ to $\mathbf{k}$. Then the *differential* d_0 is given by the formula ($\alpha \in \Omega^n$, $a_i \in \mathfrak{d}$):

$$\begin{aligned}(\mathrm{d}_0\alpha)&(a_1 \wedge \cdots \wedge a_{n+1}) \\ &= \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_{n+1})\end{aligned} \tag{2.2}$$

if $n \geq 1$, and $\mathrm{d}_0\alpha = 0$ if $\alpha \in \Omega^0 = \mathbf{k}$. Here, as usual, a hat over a term means that it is omitted in the wedge product.

Recall also that the *wedge product* of two forms $\alpha \in \Omega^n$ and $\beta \in \Omega^p$ is defined by:

$$\begin{aligned}(\alpha \wedge \beta)&(a_1 \wedge \cdots \wedge a_{n+p}) \\ &= \frac{1}{n!p!} \sum_{\pi \in S_{n+p}} (\operatorname{sgn} \pi)\, \alpha(a_{\pi(1)} \wedge \cdots \wedge a_{\pi(n)})\, \beta(a_{\pi(n+1)} \wedge \cdots \wedge a_{\pi(n+p)})\,,\end{aligned} \tag{2.3}$$

where S_{n+p} denotes the symmetric group on $n+p$ letters and $\operatorname{sgn} \pi$ is the sign of the permutation π.

The wedge product, defined by (2.3), makes Ω an associative graded-commutative algebra: for $\alpha \in \Omega^n$, $\beta \in \Omega^p$, $\gamma \in \Omega$, we have

$$\alpha \wedge \beta = (-1)^{np} \beta \wedge \alpha \in \Omega^{n+p}, \qquad (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)\,. \tag{2.4}$$

The differential d_0 is an odd derivation of Ω:

$$\mathrm{d}_0(\alpha \wedge \beta) = \mathrm{d}_0\alpha \wedge \beta + (-1)^n \alpha \wedge \mathrm{d}_0\beta\,. \tag{2.5}$$

For $a \in \mathfrak{d}$, define operators $\iota_a \colon \Omega^n \to \Omega^{n-1}$ by

$$(\iota_a\alpha)(a_1 \wedge \cdots \wedge a_{n-1}) = \alpha(a \wedge a_1 \wedge \cdots \wedge a_{n-1})\,, \qquad a_i \in \mathfrak{d}\,. \tag{2.6}$$

Then each ι_a is also an odd derivation of Ω. For $A \in \mathfrak{gl}\,\mathfrak{d}$, denote by $A\cdot$ its action on Ω; explicitly,

$$(A \cdot \alpha)(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} (-1)^i \alpha(Aa_i \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_n)\,. \tag{2.7}$$

Each $A\cdot$ is an even derivation of Ω:

$$A \cdot (\alpha \wedge \beta) = (A \cdot \alpha) \wedge \beta + \alpha \wedge (A \cdot \beta)\,, \tag{2.8}$$

and we have the following Cartan formula for the coadjoint action of $\mathfrak{d}$:

$$(\operatorname{ad} a)\cdot = \mathrm{d}_0\iota_a + \iota_a\mathrm{d}_0\,. \tag{2.9}$$

The latter implies that $(\operatorname{ad} a)\cdot$ commutes with d_0.

2.2. **Contact forms on $\mathfrak{d}$.** From now on we will assume that the Lie algebra $\mathfrak{d}$ admits a *contact form* $\theta \in \Omega^1 = \mathfrak{d}^*$, i.e., a 1-form such that

$$\theta \wedge \underbrace{\omega \wedge \dots \wedge \omega}_{N} \neq 0\,, \qquad \text{where} \quad \omega = \mathrm{d}_0\theta\,. \tag{2.10}$$

Consider the kernel of ω, i.e., the space of all elements $a \in \mathfrak{d}$ such that $\iota_a\omega = 0$. Equation (2.10) implies that $\ker\omega$ is 1-dimensional and θ does not vanish on it. We let $s \in \ker\omega$ be the unique element for which $\theta(s) = -1$, and let $\bar{\mathfrak{d}} \subset \mathfrak{d}$ be the kernel of θ. Then it is easy to deduce the following lemma (cf. [BDK]).

Lemma 2.1. *With the above notation, we have a direct sum of vector subspaces* $\mathfrak{d} = \bar{\mathfrak{d}} \oplus \mathbf{k}s$ *such that*

$$[a,b] = \omega(a \wedge b)s \mod \bar{\mathfrak{d}}\,, \qquad a,b \in \mathfrak{d}\,. \tag{2.11}$$

The restriction of ω *to* $\bar{\mathfrak{d}} \wedge \bar{\mathfrak{d}}$ *is nondegenerate,* $\iota_s\omega = 0$*, and* $[s,\bar{\mathfrak{d}}] \subset \bar{\mathfrak{d}}$*.*

Note that not every Lie algebra of odd dimension admits a contact form. In particular, it is clear from the above lemma that $\mathfrak{d}$ cannot be abelian. Also, the Lie algebra $\mathfrak{d}$ cannot be simple other than $\mathfrak{sl}_2$ (see [BDK, Example 8.6]). Here are two examples of pairs $(\mathfrak{d},\theta)$ taken from [BDK, Section 8.7].

Example 2.1. Let $\mathfrak{d} = \mathfrak{sl}_2$ with the standard basis $\{e,f,h\}$, and let $\theta(h) = 1$, $\theta(e) = \theta(f) = 0$. Then $s = -h$, $\bar{\mathfrak{d}} = \operatorname{span}\{e,f\}$, and $\omega(e \wedge f) = -1$.

Example 2.2. Let $\mathfrak{d}$ be the Heisenberg Lie algebra with a basis $\{a_i,b_i,c\}$ and the only nonzero brackets $[a_i,b_i] = c$ for $1 \le i \le N$, and let $\theta(c) = 1$, $\theta(a_i) = \theta(b_i) = 0$. Then $s = -c$, $\bar{\mathfrak{d}} = \operatorname{span}\{a_i,b_i\}$, and $\omega(a_i \wedge b_i) = -1$.

Let $\bar\omega$ be the restriction of ω to $\bar{\mathfrak{d}} \wedge \bar{\mathfrak{d}}$. Since $\bar\omega$ is nondegenerate, it defines a linear isomorphism $\phi\colon \bar{\mathfrak{d}} \to \bar{\mathfrak{d}}^*$, given by $\phi(a) = \iota_a\bar\omega$. The inverse map $\phi^{-1}\colon \bar{\mathfrak{d}}^* \to \bar{\mathfrak{d}}$ gives rise to a skew-symmetric element $r \in \bar{\mathfrak{d}} \otimes \bar{\mathfrak{d}}$ such that $\phi^{-1}(\alpha) = (\alpha \otimes \mathrm{id})(r)$ for $\alpha \in \bar{\mathfrak{d}}^*$. Explicitly, let us choose a basis $\{\partial_0,\partial_1,\dots,\partial_{2N}\}$ of $\mathfrak{d}$ such that $\partial_0 = s$ and $\{\partial_1,\dots,\partial_{2N}\}$ is a basis of $\bar{\mathfrak{d}}$, and let $\{x^0,\dots,x^{2N}\}$ be the dual basis of $\mathfrak{d}^*$ so that $\langle x^j,\partial_k\rangle = \delta^j_k$.

We set $\omega_{ij} = \omega(\partial_i \wedge \partial_j)$, and we denote by $(r^{ij})_{i,j=1,\dots,2N}$ the inverse matrix to $(\omega_{ij})_{i,j=1,\dots,2N}$, so that

$$\sum_{j=1}^{2N} r^{ij}\omega_{jk} = \delta^i_k\,, \qquad i,k = 1,\dots,2N\,. \tag{2.12}$$

Then

$$r = \sum_{i,j=1}^{2N} r^{ij}\partial_i \otimes \partial_j = \sum_{i=1}^{2N} \partial_i \otimes \partial^i = -\sum_{i=1}^{2N} \partial^i \otimes \partial_i\,, \tag{2.13}$$

where

$$\partial^i = \sum_{j=1}^{2N} r^{ij}\partial_j\,, \qquad \omega(\partial^i \wedge \partial_k) = \delta^i_k \qquad \text{for} \quad i,k = 1,\dots,2N\,. \tag{2.14}$$

We also have

$$\omega(\partial^i \wedge \partial^j) = \langle x^i,\partial^j\rangle = -r^{ij} = r^{ji}\,. \tag{2.15}$$

Recall that a basis $\{\partial_1, \dots, \partial_{2N}\}$ of $\bar{\mathfrak{d}}$ is called *symplectic* iff it satisfies

$$\omega(\partial_i \wedge \partial_{i+N}) = 1 = -\omega(\partial_{i+N} \wedge \partial_i)\,, \quad \omega(\partial_i \wedge \partial_j) = 0 \quad \text{for} \quad |i-j| \neq N\,. \tag{2.16}$$

In this case we have

$$\partial^i = -\partial_{i+N}\,, \quad \partial^{i+N} = \partial_i\,, \qquad i = 1, \dots, N\,, \tag{2.17}$$

which implies that

$$r = \sum_{i=1}^{N} (\partial_{i+N} \otimes \partial_i - \partial_i \otimes \partial_{i+N})\,. \tag{2.18}$$

Note that, by (2.3),

$$\theta = -x^0\,, \qquad \omega = \frac{1}{2} \sum_{i,j=1}^{2N} \omega_{ij} x^i \wedge x^j\,, \tag{2.19}$$

and when the basis $\{\partial_1, \dots, \partial_{2N}\}$ of $\bar{\mathfrak{d}}$ is symplectic, we have

$$\omega = \sum_{i=1}^{N} x^i \wedge x^{i+N}\,. \tag{2.20}$$

2.3. **The Lie algebras $\mathfrak{sp}\,\bar{\mathfrak{d}}$ and $\mathfrak{csp}\,\bar{\mathfrak{d}}$.** In this subsection we continue to use the notation from the previous one. In particular, recall that $\{\partial_0, \dots, \partial_{2N}\}$ is a basis of $\mathfrak{d}$ and $\{x^0, \dots, x^{2N}\}$ is the dual basis of $\mathfrak{d}^*$, while restriction to nonzero indices gives dual bases of $\bar{\mathfrak{d}}$ and $\bar{\mathfrak{d}}^*$.

We will identify $\operatorname{End}\bar{\mathfrak{d}}$ with $\bar{\mathfrak{d}} \otimes \bar{\mathfrak{d}}^*$ as a vector space. In more detail, the elementary matrix $e_i^j \in \operatorname{End}\bar{\mathfrak{d}}$ is identified with the element $\partial_i \otimes x^j \in \bar{\mathfrak{d}} \otimes \bar{\mathfrak{d}}^*$, where $e_i^j(\partial_k) = \delta_k^j \partial_i$. Notice that $(\partial \otimes x)(\partial') = \langle x, \partial' \rangle \partial$, so that the composition $(\partial \otimes x) \circ (\partial' \otimes x')$ equals $\langle x, \partial' \rangle \partial \otimes x'$. We will adopt a raising index notation for elements of $\operatorname{End}\bar{\mathfrak{d}}$ as well, so that

$$e^{ij} = \partial^i \otimes x^j = \sum_{k=1}^{2N} r^{ik} e_k^j\,, \qquad i \neq 0\,. \tag{2.21}$$

Definition 2.1. We denote by $\mathfrak{sp}\,\bar{\mathfrak{d}} = \mathfrak{sp}(\bar{\mathfrak{d}}, \bar{\omega})$ the Lie algebra of all $A \in \mathfrak{gl}\,\bar{\mathfrak{d}}$ such that $A \cdot \bar{\omega} = 0$.

Since the 2-form $\bar{\omega}$ is nondegenerate, the Lie algebra $\mathfrak{sp}\,\bar{\mathfrak{d}}$ is isomorphic to the *symplectic* Lie algebra $\mathfrak{sp}_{2N}$, and in particular it is simple. It will be sometimes convenient to embed $\mathfrak{sp}\,\bar{\mathfrak{d}}$ in $\mathfrak{gl}\,\mathfrak{d}$ by identifying $\mathfrak{gl}\,\bar{\mathfrak{d}}$ with a subalgebra of $\mathfrak{gl}\,\mathfrak{d}$. We will also consider the Lie subalgebra $\mathfrak{csp}\,\bar{\mathfrak{d}} = \mathfrak{sp}\,\bar{\mathfrak{d}} \oplus \mathbf{k}I'$ of $\mathfrak{gl}\,\mathfrak{d}$, where

$$I' = 2e_0^0 + \sum_{i=1}^{2N} e_i^i \in \mathfrak{gl}\,\mathfrak{d}\,. \tag{2.22}$$

Note that $\mathfrak{csp}\,\bar{\mathfrak{d}}$ is a trivial extension of $\mathfrak{sp}\,\bar{\mathfrak{d}}$ by the central ideal $\mathbf{k}I'$.

Lemma 2.2. *We have*

$$e_k^j \cdot \theta = \delta_{k0}\, x^j\,, \qquad e_0^j \cdot \omega = 0\,, \qquad e^{ij} \cdot \omega = x^i \wedge x^j\,, \quad i \neq 0\,. \tag{2.23}$$

In particular, $A \cdot \theta = A \cdot \omega = 0$ for all $A \in \mathfrak{sp}\,\bar{\mathfrak{d}}$ and

$$I' \cdot \theta = -2\theta\,, \qquad I' \cdot \omega = -2\omega\,, \qquad I' \cdot x^i = -x^i\,, \quad i \neq 0\,. \tag{2.24}$$

Proof. One can deduce from (2.7) that $e^j_k \cdot x^i = -\delta^i_k x^j$. Then the first two equations in (2.23) are immediate from (2.19) and (2.8). To check the third one, we observe that

$$e^j_k \cdot \omega = \sum_{i=1}^{2N} \omega_{ki} x^i \wedge x^j \,, \qquad k \neq 0$$

and then apply (2.21). Finally, (2.24) can be deduced from (2.22) and the above formulas. □

Corollary 2.1. *The elements*

$$f^{ij} = -\frac{1}{2}(e^{ij} + e^{ji}) = f^{ji} \,, \qquad 1 \leq i \leq j \leq 2N \tag{2.25}$$

form a basis of $\mathfrak{sp}\,\bar{\mathfrak{d}}$.

Recalling that $\langle x^i, \partial^j \rangle = -r^{ij} = r^{ji}$, we find

$$e^{ij} \circ e^{kl} = r^{kj} e^{il}, \tag{2.26}$$

so that

$$[e^{ij}, e^{kl}] = r^{kj} e^{il} - r^{il} e^{kj} \tag{2.27}$$

and

$$[f^{ij}, f^{kl}] = \frac{1}{2} \left(r^{ik} f^{jl} + r^{il} f^{jk} + r^{jk} f^{il} + r^{jl} f^{ik} \right). \tag{2.28}$$

Let us also introduce the notation

$$f^j_i = \sum_{a=1}^{2N} \omega_{ia} f^{aj}, \qquad f_{ij} = \sum_{a,b=1}^{2N} \omega_{ia} \omega_{jb} f^{ab} \,. \tag{2.29}$$

Lemma 2.3. (i) *For every* $i = 1, \dots, 2N$ *the elements*

$$h_i = -2 f^i_i, \qquad e_i = f_{ii}, \qquad f_i = -f^{ii} \tag{2.30}$$

constitute a standard $\mathfrak{sl}_2$*-triple.*

(ii) *The element*

$$-\sum_{i,j=1}^{2N} f_{ij} f^{ij} \in U(\mathfrak{sp}\,\bar{\mathfrak{d}}) \tag{2.31}$$

equals the Casimir element corresponding to the invariant bilinear form normalized by the condition that the square length of long roots is 2.

Proof. (i) We have:

$$[f^i_i, f^{ii}] = \sum_{a=1}^{2N} [\omega_{ia} f^{ai}, f^{ii}] = \sum_{a=1}^{2N} \omega_{ia} r^{ai} f^{ii} = f^{ii},$$

and similarly

$$
\begin{aligned}
[f^i_i, f_{ii}] &= \sum_{a,b,c=1}^{2N} [\omega_{ia} f^{ai}, \omega_{ib}\omega_{ic} f^{bc}] \\
&= \frac{1}{2} \sum_{a,b,c=1}^{2N} \omega_{ia}\omega_{ib}\omega_{ic}(r^{ab} f^{ic} + r^{ac} f^{ib} + r^{ib} f^{ac} + r^{ic} f^{ab}) \\
&= \frac{1}{2} \sum_{a,b,c=1}^{2N} (\delta^b_i \omega_{ib}\omega_{ic} f^{ic} + \delta^c_i \omega_{ic}\omega_{ib} f^{ib} - \omega_{ib} r^{bi} \omega_{ia}\omega_{ic} f^{ac} - \omega_{ic} r^{ci} \omega_{ia}\omega_{ib} f^{ab}) = -f_{ii}\,.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
[f^{ii}, f_{ii}] &= [f^{ii}, \sum_{a,b=1}^{2N} \omega_{ia}\omega_{ib} f^{ab}] \\
&= \sum_{a,b=1}^{2N} \omega_{ia}\omega_{ib}(r^{ia} f^{ib} + r^{ib} f^{ia}) \\
&= -\sum_{a,b=1}^{2N} \omega_{ia} r^{ai} \omega_{ib} f^{ib} + \omega_{ib} r^{bi} \omega_{ia} f^{ia} = -2 f^i_i\,,
\end{aligned}
$$

proving part (i).

(ii) Using (2.25) and (2.26), we compute:

$$
\begin{aligned}
f_{ij} f^{kl} &= \sum_{a,b=1}^{2N} \omega_{ia}\omega_{jb} f^{ab} f^{kl} \\
&= \frac{1}{4} \sum_{a,b=1}^{2N} \omega_{ia}\omega_{jb}(r^{kb} e^{al} + r^{lb} e^{ak} + r^{ka} e^{bl} + r^{la} e^{bk})\,.
\end{aligned}
$$

Since by (2.21), $\operatorname{tr} e^{ij} = r^{ij} = -r^{ji}$, we obtain

$$
\operatorname{tr} f_{ij} f^{kl} = \frac{1}{2} \sum_{a,b=1}^{2N} \omega_{ia}\omega_{jb}(r^{kb} r^{al} + r^{lb} r^{ak}) = -\frac{1}{2}(\delta^l_i \delta^k_j + \delta^k_i \delta^l_j)\,.
$$

The trace form is bilinear, symmetric, invariant under the adjoint action, and gives square length 2 for long roots of $\mathfrak{sp}\,\bar{\mathfrak{d}}$ (see, e.g., [FH, Lecture 16]). This proves part (ii). □

The above lemma turns out to be particularly useful when the basis $\{\partial_i\}$ of $\bar{\mathfrak{d}}$ is symplectic (see (2.16)). In this case one has

$$
h_i = e^i_i - e^{N+i}_{N+i}\,, \qquad i = 1, \dots, N\,; \tag{2.32}
$$

hence $\{h_i\}_{i=1,\dots,N}$ is a basis for the diagonal Cartan subalgebra of $\mathfrak{sp}\,\bar{\mathfrak{d}}$ (cf. [FH, Lecture 16]).

Following the notation of [OV], we denote by $R(\lambda)$ the irreducible $\mathfrak{sp}\,\bar{\mathfrak{d}}$-module with highest weight λ. Recall that the highest weight of the vector representation $\bar{\mathfrak{d}}$ is the fundamental weight π_1, and that

$$
\textstyle\bigwedge^n \bar{\mathfrak{d}} \simeq R(\pi_n) \oplus R(\pi_{n-2}) \oplus R(\pi_{n-4}) \oplus \cdots\,, \qquad 0 \le n \le N\,, \tag{2.33}
$$

where π_n are the fundamental weights and we set $R(\pi_0) = \mathbf{k}$, $R(\pi_n) = \{0\}$ if $n < 0$ or $n > N$. The following facts are standard (see, e.g., [OV], Reference Chapter, Table 5).

Lemma 2.4. *With the above notation, we have*:

$$R(\pi_n) \otimes R(\pi_1) \simeq R(\pi_n + \pi_1) \oplus R(\pi_{n-1}) \oplus R(\pi_{n+1}) \,,$$
$$\dim R(\pi_n + \pi_1) > \dim R(\pi_n) \,, \qquad 1 \le n \le N \,.$$

Furthermore, the Casimir element (2.31) *acts on* $R(\pi_n)$ *as scalar multiplication by* $n(2N + 2 - n)/2$.

2.4. **Bases and filtrations of $U(\mathfrak{d})$ and $U(\mathfrak{d})^*$.** Let $\mathfrak{d}$ be a Lie algebra of dimension $2N + 1$ with a basis $\{\partial_0, \partial_1, \dots, \partial_{2N}\}$, as in Section 2.2. Then its universal enveloping algebra $H = U(\mathfrak{d})$ has a basis

$$\partial^{(I)} = \partial_0^{i_0} \cdots \partial_{2N}^{i_{2N}} / i_0! \cdots i_{2N}! \,, \qquad I = (i_0, \dots, i_{2N}) \in \mathbb{Z}_+^{2N+1} \,. \tag{2.34}$$

Recall that the coproduct $\Delta \colon H \to H \otimes H$ is a homomorphism of associative algebras defined by $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$ for $\partial \in \mathfrak{d}$. Then it is easy to see that

$$\Delta(\partial^{(I)}) = \sum_{J+K=I} \partial^{(J)} \otimes \partial^{(K)}. \tag{2.35}$$

The canonical increasing filtration of $U(\mathfrak{d})$ is given by

$$\mathrm{F}^p \, U(\mathfrak{d}) = \mathrm{span}_{\mathbf{k}} \{ \partial^{(I)} \mid |I| \le p \} \,, \qquad \text{where} \quad |I| = i_0 + \dots + i_{2N} \,, \tag{2.36}$$

and it does not depend on the choice of basis of $\mathfrak{d}$. This filtration is compatible with the structure of a Hopf algebra (see, e.g., [BDK, Section 2.2] for more details). We have: $\mathrm{F}^{-1} H = \{0\}$, $\mathrm{F}^0 H = \mathbf{k}$, $\mathrm{F}^1 H = \mathbf{k} \oplus \mathfrak{d}$.

It is also convenient to define a different filtration of $U(\mathfrak{d})$, called the *contact filtration*:

$$\mathrm{F}'^p \, U(\mathfrak{d}) = \mathrm{span}_{\mathbf{k}} \{ \partial^{(I)} \mid |I|' \le p \} \,, \qquad \text{where} \quad |I|' = 2i_0 + i_1 + \dots + i_{N-1} \,. \tag{2.37}$$

This filtration is also compatible with the Hopf algebra structure on $U(\mathfrak{d})$, and we have $\mathrm{F}'^0 H = \mathbf{k}$, $\mathrm{F}'^1 H = \mathbf{k} \oplus \bar{\mathfrak{d}}$, $\mathrm{F}'^2 H \supset \mathbf{k} \oplus \mathfrak{d} = \mathrm{F}^1 H$. It is easy to see that the two filtrations of H are equivalent.

The dual $X = H^* := \mathrm{Hom}_{\mathbf{k}}(H, \mathbf{k})$ is a commutative associative algebra. Define elements $x_I \in X$ by $\langle x_I, \partial^{(J)} \rangle = \delta_I^J$, where, as usual, $\delta_I^J = 1$ if $I = J$ and $\delta_I^J = 0$ if $I \ne J$. Then, by (2.35), we have $x_J x_K = x_{J+K}$ and

$$x_I = (x^0)^{i_0} \cdots (x^{2N})^{i_{2N}} \,, \qquad I = (i_0, \dots, i_{2N}) \in \mathbb{Z}_+^{2N+1} \,, \tag{2.38}$$

where

$$x^i = x_{\varepsilon_i} \,, \qquad \varepsilon_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \,, \quad i = 0, \dots, 2N \,. \tag{2.39}$$

Therefore, X can be identified with the algebra $\mathcal{O}_{2N+1} = \mathbf{k}[[t^0, t^1, \dots, t^{2N}]]$ of formal power series in $2N + 1$ indeterminates.

There are left and right actions of $\mathfrak{d}$ on X by derivations given by

$$\langle \partial x, f \rangle = - \langle x, \partial f \rangle \,, \tag{2.40}$$

$$\langle x \partial, f \rangle = - \langle x, f \partial \rangle \,, \qquad \partial \in \mathfrak{d}, \ x \in X, \ f \in H \,, \tag{2.41}$$

where ∂f and $f \partial$ are products in H. These two actions coincide only when the Lie algebra $\mathfrak{d}$ is abelian. The difference $\partial x - x \partial$ gives the coadjoint action of $\partial \in \mathfrak{d}$ on $x \in X$.

Let $\mathrm{F}_p X = (\mathrm{F}^p H)^\perp$ be the set of elements from $X = H^*$ that vanish on $\mathrm{F}^p H$. Then $\{\mathrm{F}_p X\}$ is a decreasing filtration of X called the *canonical filtration*. It has the properties:

$$\mathrm{F}_{-1} X = X\,, \quad X/\mathrm{F}_0 X \simeq \mathbf{k}\,, \quad \mathrm{F}_0 X/\mathrm{F}_1 X \simeq \mathfrak{d}^*\,, \tag{2.42}$$

$$(\mathrm{F}_n X)(\mathrm{F}_p X) \subset \mathrm{F}_{n+p+1} X\,, \quad \mathfrak{d}(\mathrm{F}_p X) \subset \mathrm{F}_{p-1} X\,, \quad (\mathrm{F}_p X)\mathfrak{d} \subset \mathrm{F}_{p-1} X\,. \tag{2.43}$$

Note that $\mathrm{F}_0 X$ is the unique maximal ideal of X, and $\mathrm{F}_p X = (\mathrm{F}_0 X)^{p+1}$. We define a topology of X by considering $\{\mathrm{F}_p X\}$ as a fundamental system of neighborhoods of 0. We will always consider X with this topology, while H and $\mathfrak{d}$ with the discrete topology. Then X is a linearly compact algebra (see [BDK, Chapter 6]), and the left and right actions of $\mathfrak{d}$ on it are continuous (see (2.43)).

Similar statements hold for the filtration $\mathrm{F}'_p X = (\mathrm{F}'^p H)^\perp$, namely:

$$(\mathrm{F}'_n X)(\mathrm{F}'_p X) \subset \mathrm{F}'_{n+p+1} X\,, \quad \bar{\mathfrak{d}}(\mathrm{F}'_p X) \subset \mathrm{F}'_{p-1} X\,, \quad (\mathrm{F}'_p X)\bar{\mathfrak{d}} \subset \mathrm{F}'_{p-1} X\,, \tag{2.44}$$

$$\partial_0(\mathrm{F}'_p X) \subset \mathrm{F}'_{p-2} X\,, \quad (\mathrm{F}'_p X)\partial_0 \subset \mathrm{F}'_{p-2} X\,. \tag{2.45}$$

We will call $\{\mathrm{F}'_p X\}$ the *contact filtration*. It is equivalent to the canonical filtration $\{\mathrm{F}_p X\}$.

We can consider x^i as elements of $\mathfrak{d}^*$; then $\{x^i\}$ is a basis of $\mathfrak{d}^*$ dual to the basis $\{\partial_i\}$ of $\mathfrak{d}$, i.e., $\langle x^i, \partial_j\rangle = \delta^i_j$. Let c^k_{ij} be the structure constants of $\mathfrak{d}$ in the basis $\{\partial_i\}$, so that $[\partial_i, \partial_j] = \sum c^k_{ij}\partial_k$. Then we have the following formulas for the left and right actions of $\mathfrak{d}$ on X (see, e.g., [BDK1, Lemma 2.2]):

$$\partial_i x^j = -\delta^j_i - \sum_{k<i} c^j_{ik} x^k \mod \mathrm{F}_1 X\,, \tag{2.46}$$

$$x^j \partial_i = -\delta^j_i + \sum_{k>i} c^j_{ik} x^k \mod \mathrm{F}_1 X\,. \tag{2.47}$$

3. Lie Pseudoalgebras and Their Representations

In this section we review the definitions and results about Lie pseudoalgebras from [BDK, BDK1], which will be needed in the paper.

3.1. **Hopf algebra notations.** Let H be a cocommutative Hopf algebra with a coproduct Δ, a counit ε, and an antipode S. We will use the following notation (cf. [Sw]):

$$\Delta(h) = h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}\,, \tag{3.1}$$

$$(\Delta \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \Delta)\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}\,, \tag{3.2}$$

$$(S \otimes \mathrm{id})\Delta(h) = h_{(-1)} \otimes h_{(2)}\,, \qquad h \in H\,. \tag{3.3}$$

Then the axioms of antipode and counit can be written as follows:

$$h_{(-1)}h_{(2)} = h_{(1)}h_{(-2)} = \varepsilon(h), \tag{3.4}$$

$$\varepsilon(h_{(1)})h_{(2)} = h_{(1)}\varepsilon(h_{(2)}) = h, \tag{3.5}$$

while the fact that Δ is a homomorphism of algebras translates as:

$$(fg)_{(1)} \otimes (fg)_{(2)} = f_{(1)}g_{(1)} \otimes f_{(2)}g_{(2)}, \qquad f, g \in H. \tag{3.6}$$

Eqs. (3.4), (3.5) imply the following useful relations:

$$h_{(-1)}h_{(2)} \otimes h_{(3)} = 1 \otimes h = h_{(1)}h_{(-2)} \otimes h_{(3)}. \tag{3.7}$$

The following lemma, which follows from [BDK, Lemma 2.3], plays an important role in the paper.

Lemma 3.1. *For any H-module V, the linear maps*

$$H \otimes V \to (H \otimes H) \otimes_H V\,, \qquad h \otimes v \mapsto (h \otimes 1) \otimes_H v$$

and

$$H \otimes V \to (H \otimes H) \otimes_H V\,, \qquad h \otimes v \mapsto (1 \otimes h) \otimes_H v$$

are isomorphisms of vector spaces.

The dual $X = H^* := \mathrm{Hom}_{\mathbf{k}}(H, \mathbf{k})$ becomes a commutative associative algebra under the product defined by

$$\langle xy, h\rangle = \langle x, h_{(1)}\rangle \langle y h_{(2)}\rangle\,, \qquad h \in H,\ x, y \in X\,. \tag{3.8}$$

X admits left and right actions of H, given by (cf. (2.40), (2.41)):

$$\langle hx, f\rangle = \langle x, S(h)f\rangle\,, \tag{3.9}$$

$$\langle xh, f\rangle = \langle x, fS(h)\rangle\,, \qquad h, f \in H,\ x, y \in X\,. \tag{3.10}$$

They have the following properties:

$$h(xy) = (h_{(1)}x)(h_{(2)}y)\,, \tag{3.11}$$

$$(xy)h = (xh_{(1)})(yh_{(2)})\,, \tag{3.12}$$

$$h(xg) = (hx)g, \qquad h, g \in H,\ x, y \in X. \tag{3.13}$$

3.2. **Lie pseudoalgebras and their representations.** Let us recall the definition of a Lie pseudoalgebra from [BDK, Chapter 3]. A *pseudobracket* on a left H-module L is an H-bilinear map

$$L \otimes L \to (H \otimes H) \otimes_H L\,, \quad a \otimes b \mapsto [a * b]\,, \tag{3.14}$$

where we use the comultiplication $\Delta\colon H \to H \otimes H$ to define $(H \otimes H) \otimes_H L$. We extend the pseudobracket (3.14) to maps $(H^{\otimes 2} \otimes_H L) \otimes L \to H^{\otimes 3} \otimes_H L$ and $L \otimes (H^{\otimes 2} \otimes_H L) \to H^{\otimes 3} \otimes_H L$ by letting:

$$[(h \otimes_H a) * b] = \sum (h \otimes 1)\,(\Delta \otimes \mathrm{id})(g_i) \otimes_H c_i\,, \tag{3.15}$$

$$[a * (h \otimes_H b)] = \sum (1 \otimes h)\,(\mathrm{id} \otimes \Delta)(g_i) \otimes_H c_i\,, \tag{3.16}$$

where $h \in H^{\otimes 2}$, $a, b \in L$, and

$$[a * b] = \sum g_i \otimes_H c_i \qquad \text{with} \quad g_i \in H^{\otimes 2},\ c_i \in L. \tag{3.17}$$

A *Lie pseudoalgebra* is a left H-module equipped with a pseudobracket satisfying the following skewsymmetry and Jacobi identity axioms:

$$[b * a] = -(\sigma \otimes_H \mathrm{id})\,[a * b]\,, \tag{3.18}$$

$$[[a * b] * c] = [a * [b * c]] - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,[b * [a * c]]\,. \tag{3.19}$$

Here, $\sigma\colon H \otimes H \to H \otimes H$ is the permutation of factors, and the compositions $[[a * b] * c]$, $[a * [b * c]]$ are defined using (3.15), (3.16).

The definition of a module over a Lie pseudoalgebras is an obvious modification of the above. A *module* over a Lie pseudoalgebra L is a left H-module V together with an H-bilinear map

$$L \otimes V \to (H \otimes H) \otimes_H V\,, \quad a \otimes v \mapsto a * v \tag{3.20}$$

that satisfies ($a, b \in L$, $v \in V$)

$$[a * b] * v = a * (b * v) - ((\sigma \otimes \mathrm{id}) \otimes_H \mathrm{id})\,(b * (a * v))\,. \tag{3.21}$$

An L-module V will be called *finite* if it is finitely generated as an H-module.

Remark 3.1. If V is a torsion module over H, then the action of L on V is trivial, i.e., $L * V = \{0\}$ (see [BDK, Corollary 10.1]). Notice that this holds whenever V is finite dimensional and $H = U(\mathfrak{d})$ with $\dim \mathfrak{d} > 0$.

Some of the most important Lie pseudoalgebras are described in the following examples (see [BDK]).

Example 3.1. For a Lie algebra $\mathfrak{g}$, the current Lie pseudoalgebra $\mathrm{Cur}\,\mathfrak{g} = H \otimes \mathfrak{g}$ has an action of H by left multiplication on the first tensor factor and a pseudobracket

$$[(f \otimes a) * (g \otimes b)] = (f \otimes g) \otimes_H (1 \otimes [a, b])\,. \tag{3.22}$$

Example 3.2. Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{d}$. Then $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ has the structure of a Lie pseudoalgebra with the pseudobracket

$$\begin{aligned}[(f \otimes a) * (g \otimes b)] = (f \otimes g) &\otimes_H (1 \otimes [a, b]) \\ &- (f \otimes ga) \otimes_H (1 \otimes b) + (fb \otimes g) \otimes_H (1 \otimes a)\,.\end{aligned} \tag{3.23}$$

The formula

$$(f \otimes a) * g = -(f \otimes ga) \otimes_H 1 \tag{3.24}$$

defines the structure of a $W(\mathfrak{d})$-module on H.

Example 3.3. The semidirect sum $W(\mathfrak{d}) \ltimes \mathrm{Cur}\,\mathfrak{g}$ contains $W(\mathfrak{d})$ and $\mathrm{Cur}\,\mathfrak{g}$ as subalgebras and has the pseudobracket

$$[(f \otimes a) * (g \otimes b)] = -(f \otimes ga) \otimes_H (1 \otimes b) \tag{3.25}$$

for $f, g \in H = U(\mathfrak{d})$, $a \in \mathfrak{d}$, $b \in \mathfrak{g}$ (cf. (3.24)).

Let U and V be two L-modules. A map $\beta \colon U \to V$ is a *homomorphism* of L-modules if β is H-linear and satisfies

$$\big((\mathrm{id} \otimes \mathrm{id}) \otimes_H \beta\big)(a * u) = a * \beta(u)\,, \qquad a \in L\,,\ u \in U\,. \tag{3.26}$$

A subspace $W \subset V$ is an *L-submodule* if it is an H-submodule and $L * W \subset (H \otimes H) \otimes_H W$, where $L * W$ is the linear span of all elements $a * w$ with $a \in L$ and $w \in W$. A submodule $W \subset V$ is called *proper* if $W \neq V$. An L-module V is *irreducible* (or *simple*) if it does not contain any nonzero proper L-submodules and $L * V \neq \{0\}$.

Remark 3.2. (i) Let V be a module over a Lie pseudoalgebra L and let W be an H-submodule of V. By Lemma 3.1, for each $a \in L$, $v \in V$, we can write

$$a * v = \sum_{I \in \mathbb{Z}_+^{2N+1}} (\partial^{(I)} \otimes 1) \otimes_H v'_I\,, \qquad v'_I \in V\,, \tag{3.27}$$

where the elements v'_I are uniquely determined by a and v. Then $W \subset V$ is an L-submodule iff it has the property that all $v'_I \in W$ whenever $v \in W$. This follows again from Lemma 3.1.

(ii) Similarly, for each $a \in L$, $v \in V$, we can write

$$a * v = \sum_{I \in \mathbb{Z}_+^{2N+1}} (1 \otimes \partial^{(I)}) \otimes_H v''_I\,, \qquad v''_I \in V\,, \tag{3.28}$$

and W is an L-submodule iff $v''_I \in W$ whenever $v \in W$.

3.3. **Twistings of representations.** Let L be a Lie pseudoalgebra over $H = U(\mathfrak{d})$, and let Π be any finite-dimensional $\mathfrak{d}$-module. In [BDK1, Section 4.2], we introduced a covariant functor T_Π from the category of finite L-modules to itself. In the present paper we will use it only in the special case when all the modules are free as H-modules. For a finite L-module $V = H \otimes V_0$, which is free over H, we choose a $\mathbf{k}$-basis $\{v_i\}$ of V_0, and write the action of L on V in the form

$$a * (1 \otimes v_i) = \sum_j (f_{ij} \otimes g_{ij}) \otimes_H (1 \otimes v_j) \tag{3.29}$$

where $a \in L$, $f_{ij}, g_{ij} \in H$.

Definition 3.1. The *twisting* of V by Π is the L-module $T_\Pi(V) = H \otimes \Pi \otimes V_0$, where H acts by a left multiplication on the first factor and

$$a * (1 \otimes u \otimes v_i) = \sum_j \bigl(f_{ij} \otimes g_{ij\,(1)}\bigr) \otimes_H \bigl(1 \otimes g_{ij\,(-2)} u \otimes v_j\bigr) \tag{3.30}$$

for $a \in L$, $u \in \Pi$.

The facts that $T_\Pi(V)$ is an L-module and that the action of L on it is independent of the choice of basis of V_0 follow from [BDK1, Proposition 4.2]. Let us now recall how T_Π is defined on homomorphisms of L-modules. Consider two finite L-modules, $V = H \otimes V_0$ and $V' = H \otimes V'_0$. Choose $\mathbf{k}$-bases $\{v_i\}$ and $\{v'_i\}$ of V_0 and V'_0, respectively. For a homomorphism of L-modules $\beta\colon V \to V'$, write

$$\beta(1 \otimes v_i) = \sum_j h_{ij} \otimes v'_j\,, \qquad h_{ij} \in H\,. \tag{3.31}$$

Then $T_\Pi(\beta)\colon T_\Pi(V) \to T_\Pi(V')$ is given by

$$T_\Pi(\beta)(1 \otimes u \otimes v_i) = \sum_j h_{ij\,(1)} \otimes h_{ij\,(-2)} u \otimes v'_j\,. \tag{3.32}$$

Thanks to [BDK1, Proposition 4.3], the map $T_\Pi(\beta)$ is a homomorphism of L-modules, independent of the choice of bases.

Note that T_Π can be defined on the category of (free) H-modules. The next result concerns only the H-module structure.

Proposition 3.1. (i) *The functor T_Π is exact on free H-modules, i.e., if $V \xrightarrow{\beta} V' \xrightarrow{\beta'} V''$ is a short exact sequence of finite free H-modules, then the sequence $T_\Pi(V) \xrightarrow{T_\Pi(\beta)} T_\Pi(V') \xrightarrow{T_\Pi(\beta')} T_\Pi(V'')$ is exact.*

(ii) *Let $\beta\colon V \to V'$ be a homomorphism between two free H-modules. If the image of β has a finite codimension over $\mathbf{k}$, then the image of $T_\Pi(\beta)$ has a finite codimension in $T_\Pi(V')$.*

Proof. Consider the linear map

$$F\colon H \otimes \Pi \to H \otimes \Pi\,, \qquad h \otimes u \mapsto h_{(1)} \otimes h_{(-2)} u\,,$$

which was introduced in the proof of [BDK1, Lemma 5.2]. From (3.7) it is easy to see that F is a linear isomorphism and

$$F^{-1}(h \otimes u) = h_{(1)} \otimes h_{(2)} u\,, \qquad h \in H,\ u \in \Pi\,.$$

Since F is a linear isomorphism, both statements of the proposition are true if and only if they are true for $(F^{-1}\otimes \mathrm{id})T_\Pi(\beta)$ instead of $T_\Pi(\beta)$. In this case, they follow easily from the identity

$$(F^{-1}\otimes \mathrm{id})T_\Pi(\beta)(1\otimes u\otimes v_i)=\sum_j h_{ij}\otimes u\otimes v'_j=\sigma_{12}\big(u\otimes \beta(1\otimes v_i)\big)\,,$$

where σ_{12} is the transposition of the first and second factors. □

3.4. **Annihilation algebras of Lie pseudoalgebras.** For a Lie pseudoalgebra L, we set $\mathcal{A}(L)=X\otimes_H L$, where as before $X=H^*$, and we define a Lie bracket on $\mathcal{L}=\mathcal{A}(L)$ by the formula (cf. [BDK, Eq. (7.2)]):

$$[x\otimes_H a, y\otimes_H b]=\sum (xf_i)(yg_i)\otimes_H c_i\,,\quad \text{if}\quad [a*b]=\sum (f_i\otimes g_i)\otimes_H c_i\,. \tag{3.33}$$

Then $\mathcal{L}$ is a Lie algebra, called the *annihilation algebra* of L (see [BDK, Section 7.1]). We define a left action of H on $\mathcal{L}$ in the obvious way:

$$h(x\otimes_H a)=hx\otimes_H a. \tag{3.34}$$

In the case $H=U(\mathfrak{d})$, the Lie algebra $\mathfrak{d}$ acts on $\mathcal{L}$ by derivations. The semidirect sum $\widetilde{\mathcal{L}}=\mathfrak{d}\ltimes\mathcal{L}$ is called the *extended annihilation algebra*.

Similarly, if V is a module over a Lie pseudoalgebra L, we let $\mathcal{A}(V)=X\otimes_H V$, and we define an action of $\mathcal{L}=\mathcal{A}(L)$ on $\mathcal{A}(V)$ by:

$$(x\otimes_H a)(y\otimes_H v)=\sum (xf_i)(yg_i)\otimes_H v_i\,,\quad \text{if}\quad a*v=\sum (f_i\otimes g_i)\otimes_H v_i\,. \tag{3.35}$$

We also define an H-action on $\mathcal{A}(V)$ similarly to (3.34). Then $\mathcal{A}(V)$ is an $\widetilde{\mathcal{L}}$-module [BDK, Proposition 7.1].

When L is a finite H-module, we can define a filtration on $\mathcal{L}$ as follows (see [BDK, Section 7.4] for more details). We fix a finite-dimensional vector subspace L_0 of L such that $L=HL_0$, and set

$$\mathrm{F}_p\,\mathcal{L}=\mathrm{span}_{\mathbf{k}}\{x\otimes_H a\in\mathcal{L}\mid x\in \mathrm{F}_p\,X\,,\ a\in L_0\}\,,\qquad p\geq -1\,. \tag{3.36}$$

The subspaces $\mathrm{F}_p\,\mathcal{L}$ constitute a decreasing filtration of $\mathcal{L}$, satisfying

$$[\mathrm{F}_n\,\mathcal{L},\mathrm{F}_p\,\mathcal{L}]\subset \mathrm{F}_{n+p-\ell}\,\mathcal{L}\,,\quad \mathfrak{d}(\mathrm{F}_p\,\mathcal{L})\subset \mathrm{F}_{p-1}\,\mathcal{L}\,, \tag{3.37}$$

where ℓ is an integer depending only on the choice of L_0. Notice that the filtration just defined depends on the choice of L_0, but the topology that it induces does not [BDK, Lemma 7.2]. We set $\mathcal{L}_p=\mathrm{F}_{p+\ell}\,\mathcal{L}$, so that $[\mathcal{L}_n,\mathcal{L}_p]\subset\mathcal{L}_{n+p}$. In particular, $\mathcal{L}_0$ is a subalgebra of $\mathcal{L}$.

We also define a filtration of $\widetilde{\mathcal{L}}$ by letting $\mathrm{F}_{-1}\,\widetilde{\mathcal{L}}=\widetilde{\mathcal{L}}$, $\mathrm{F}_p\,\widetilde{\mathcal{L}}=\mathrm{F}_p\,\mathcal{L}$ for $p\geq 0$, and we set $\widetilde{\mathcal{L}}_p=\mathrm{F}_{p+\ell}\,\widetilde{\mathcal{L}}$. An $\widetilde{\mathcal{L}}$-module V is called *conformal* if every $v\in V$ is killed by some $\mathcal{L}_p$; in other words, if V is a continuous $\widetilde{\mathcal{L}}$-module when endowed with the discrete topology.

The next two results from [BDK] play a crucial role in our study of representations (see [BDK], Propositions 9.1 and 14.2, and Lemma 14.4).

Proposition 3.2. *Any module V over the Lie pseudoalgebra L has a natural structure of a conformal $\widetilde{\mathcal{L}}$-module, given by the action of $\mathfrak{d}$ on V and by*

$$(x\otimes_H a)\cdot v=\sum \langle xf_i,g_{i(1)}\rangle\, g_{i(2)}v_i\,,\quad \textit{if}\quad a*v=\sum (f_i\otimes g_i)\otimes_H v_i \tag{3.38}$$

for $a\in L$, $x\in X$, $v\in V$.

Conversely, any conformal $\widetilde{\mathcal{L}}$-module V has a natural structure of an L-module, given by

$$a * v = \sum_{I \in \mathbb{Z}_+^{2N+1}} \bigl(S(\partial^{(I)}) \otimes 1\bigr) \otimes_H \bigl((x_I \otimes_H a) \cdot v\bigr) . \tag{3.39}$$

Moreover, V is irreducible as an L-module iff it is irreducible as an $\widetilde{\mathcal{L}}$-module.

Lemma 3.2. *Let L be a finite Lie pseudoalgebra and V be a finite L-module. For $p \geq -1-\ell$, let*

$$\ker_p V = \{v \in V \mid \mathcal{L}_p\, v = 0\},$$

so that, for example, $\ker_{-1-\ell} V = \ker V$ and $V = \bigcup \ker_p V$. Then all vector spaces $\ker_p V / \ker V$ are finite dimensional. In particular, if $\ker V = \{0\}$, then every vector $v \in V$ is contained in a finite-dimensional subspace invariant under $\mathcal{L}_0$.

4. Primitive Lie Pseudoalgebras of Type K

Here we introduce the main objects of our study: the Lie pseudoalgebra $K(\mathfrak{d},\theta)$ and its annihilation algebra $\mathcal{K}$ (see [BDK, Chapter 8]). We will review the (unique) embedding of $K(\mathfrak{d},\theta)$ into $W(\mathfrak{d})$ and the induced embedding of annihilation algebras. Throughout this section, $\mathfrak{d}$ will be a Lie algebra of odd dimension $2N+1$, and $\theta \in \mathfrak{d}^*$ will be a contact form, as in Section 2.2. As before, let $H = U(\mathfrak{d})$.

4.1. **Definition of $K(\mathfrak{d},\theta)$.** Recall the elements $r \in \mathfrak{d}\otimes\mathfrak{d}$ and $s \in \mathfrak{d}$ introduced in Section 2.2 and notice that r is skew-symmetric. It was shown in [BDK, Lemma 8.7] that r and s satisfy the following equations:

$$[r, \Delta(s)] = 0 \,, \tag{4.1}$$

$$([r_{12}, r_{13}] + r_{12}s_3) + \text{cyclic} = 0 \,, \tag{4.2}$$

where we use the standard notation $r_{12} = r \otimes 1$, $s_3 = 1 \otimes 1 \otimes s$, etc., and "cyclic" denotes terms obtained by applying the two nontrivial cyclic permutations.

Definition 4.1. The Lie pseudoalgebra $K(\mathfrak{d},\theta)$ is defined as a free H-module of rank one, He, with the following pseudobracket

$$[e * e] = (r + s \otimes 1 - 1 \otimes s) \otimes_H e \,. \tag{4.3}$$

The fact that $K(\mathfrak{d},\theta)$ is a Lie pseudoalgebra follows from (4.1), (4.2); see [BDK, Section 4.3]. By [BDK, Lemma 8.3], there is an injective homomorphism of Lie pseudoalgebras

$$\iota\colon K(\mathfrak{d},\theta) \to W(\mathfrak{d}) \,, \qquad e \mapsto -r + 1 \otimes s \,, \tag{4.4}$$

where $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ is from Example 3.2. Moreover, this is the unique nontrivial homomorphism from $K(\mathfrak{d},\theta)$ to $W(\mathfrak{d})$ [BDK, Theorem 13.7]. From now on, we will often identify $K(\mathfrak{d},\theta)$ with its image in $W(\mathfrak{d})$ and will write simply e instead of $\iota(e)$. In the notation of Section 2.2, we have the formula

$$e = 1 \otimes \partial_0 - \sum_{i=1}^{2N} \partial_i \otimes \partial^i \,. \tag{4.5}$$

4.2. **Annihilation algebra of $W(\mathfrak{d})$.** Let $\mathcal{W} = \mathcal{A}(W(\mathfrak{d}))$ be the annihilation algebra of the Lie pseudoalgebra $W(\mathfrak{d})$ (see Section 3.4). Since $W(\mathfrak{d}) = H \otimes \mathfrak{d}$, we have $\mathcal{W} = X \otimes_H (H \otimes \mathfrak{d}) \simeq X \otimes \mathfrak{d}$, so we can identify $\mathcal{W}$ with $X \otimes \mathfrak{d}$. Then the Lie bracket in $\mathcal{W}$ becomes ($x, y \in X,\ a, b \in \mathfrak{d}$):

$$[x \otimes a, y \otimes b] = xy \otimes [a,b] - x(ya) \otimes b + (xb)y \otimes a\,, \tag{4.6}$$

while the left action of H on $\mathcal{W}$ is given by: $h(x \otimes a) = hx \otimes a$. The Lie algebra $\mathfrak{d}$ acts on $\mathcal{W}$ by derivations. We denote by $\widetilde{\mathcal{W}}$ the extended annihilation algebra $\mathfrak{d} \ltimes \mathcal{W}$, where

$$[\partial, x \otimes a] = \partial x \otimes a\,, \qquad \partial, a \in \mathfrak{d},\ x \in X\,. \tag{4.7}$$

We choose $L_0 = \mathbf{k} \otimes \mathfrak{d}$ as a subspace of $W(\mathfrak{d})$ such that $W(\mathfrak{d}) = HL_0$, and we obtain the following filtration of $\mathcal{W}$:

$$\mathcal{W}_p = \mathrm{F}_p\, \mathcal{W} = \mathrm{F}_p\, X \otimes_H L_0 \equiv \mathrm{F}_p\, X \otimes \mathfrak{d}\,, \qquad p \geq -1\,. \tag{4.8}$$

This is a decreasing filtration of $\mathcal{W}$, satisfying $\mathcal{W}_{-1} = \mathcal{W}$ and $[\mathcal{W}_i, \mathcal{W}_j] \subset \mathcal{W}_{i+j}$. Note that $\mathcal{W}/\mathcal{W}_0 \simeq \mathbf{k} \otimes \mathfrak{d} \simeq \mathfrak{d}$ and $\mathcal{W}_0/\mathcal{W}_1 \simeq \mathfrak{d}^* \otimes \mathfrak{d}$.

Lemma 4.1 ([BDK1])**.** *For $x \in \mathrm{F}_0\, X$, $a \in \mathfrak{d}$, the map*

$$(x \otimes a) \mod \mathcal{W}_1 \mapsto -a \otimes (x \bmod \mathrm{F}_1\, X) \tag{4.9}$$

is a Lie algebra isomorphism from $\mathcal{W}_0/\mathcal{W}_1$ to $\mathfrak{d} \otimes \mathfrak{d}^* \simeq \mathfrak{gl}\,\mathfrak{d}$. Under this isomorphism, the adjoint action of $\mathcal{W}_0/\mathcal{W}_1$ on $\mathcal{W}/\mathcal{W}_0$ coincides with the standard action of $\mathfrak{gl}\,\mathfrak{d}$ on $\mathfrak{d}$.

The action (3.24) of $W(\mathfrak{d})$ on H induces a corresponding action of the annihilation algebra $\mathcal{W}$ on $\mathcal{A}(H) \equiv X$:

$$(x \otimes a)y = -x(ya), \qquad x, y \in X,\ a \in \mathfrak{d}. \tag{4.10}$$

Since $\mathfrak{d}$ acts on X by continuous derivations, the Lie algebra $\mathcal{W}$ acts on X by continuous derivations. The isomorphism $X \simeq \mathcal{O}_{2N+1}$ from Section 2.4 induces a Lie algebra homomorphism $\mathcal{W} \to W_{2N+1} = \operatorname{Der} \mathcal{O}_{2N+1}$. In fact, this is an isomorphism compatible with the filtrations [BDK1, Proposition 3.1]. Recall that the canonical filtration of the Lie–Cartan algebra W_{2N+1} is given by

$$\mathrm{F}_p\, W_{2N+1} = \Bigl\{ \sum_{i=0}^{2N} f_i \frac{\partial}{\partial t^i} \Bigm| f_i \in \mathrm{F}_p\, \mathcal{O}_{2N+1} \Bigr\}\,, \tag{4.11}$$

where $\mathrm{F}_p\, \mathcal{O}_{2N+1}$ is the $(p+1)$-st power of the maximal ideal $(t^0, \dots, t^{2N})$ of $\mathcal{O}_{2N+1}$.

The Euler vector field

$$E := \sum_{i=0}^{2N} t^i \frac{\partial}{\partial t^i} \in \mathrm{F}_0\, W_{2N+1} \tag{4.12}$$

gives rise to a grading of $\mathcal{O}_{2N+1}$ and a grading $W_{2N+1;j}$ $(j \geq -1)$ of W_{2N+1} such that

$$\mathrm{F}_p\, W_{2N+1} = \prod_{j \geq p} W_{2N+1;j}\,, \quad \mathrm{F}_p\, W_{2N+1}/\, \mathrm{F}_{p+1}\, W_{2N+1} \simeq W_{2N+1;p}\,. \tag{4.13}$$

We define the *contact filtration* of $\mathcal{W}$ by (see (2.37)):

$$\mathcal{W}'_p = \mathrm{F}'_p\, \mathcal{W} = (\mathrm{F}'_p\, X \otimes \bar{\mathfrak{d}}) \oplus (\mathrm{F}'_{p+1}\, X \otimes \mathbf{k}s)\,. \tag{4.14}$$

Introduce the *contact Euler vector field*

$$E' := 2t^0 \frac{\partial}{\partial t^0} + \sum_{i=1}^{2N} t^i \frac{\partial}{\partial t^i} \in \mathrm{F}_0\, W_{2N+1} \cap \mathrm{F}'_0\, W_{2N+1}\,. \tag{4.15}$$

Then the adjoint action of E' decomposes W_{2N+1} as a direct product of eigenspaces $W'_{2N+1;j}$ $(j \geq -1)$, on which $\operatorname{ad} E'$ acts as multiplication by j. One defines

$$\mathrm{F}'_p\, W_{2N+1} = \prod_{j \geq p} W'_{2N+1;j} \tag{4.16}$$

so that

$$\mathrm{F}'_p\, W_{2N+1} / \mathrm{F}'_{p+1}\, W_{2N+1} \simeq W'_{2N+1;p}\,. \tag{4.17}$$

The filtration $\{\mathrm{F}'_p\, W_{2N+1}\}$ induces on W_{2N+1} the same topology as the filtration $\{\mathrm{F}_p\, W_{2N+1}\}$.

4.3. **Annihilation algebra of $K(\mathfrak{d}, \theta)$.** We define a filtration on the annihilation algebra $\mathcal{K} = \mathcal{A}(K(\mathfrak{d}, \theta))$ by

$$\mathcal{K}'_p = \mathrm{F}'_p\, \mathcal{K} = \mathrm{F}'_{p+1}\, X \otimes_H e\,, \qquad p \geq -2\,. \tag{4.18}$$

This filtration is equivalent to the one defined in Section 3.4 by choosing $L_0 = \mathbf{k}e$, because the filtrations $\{\mathrm{F}'_p\, X\}$ and $\{\mathrm{F}_p\, X\}$ are equivalent.

Recall that the canonical injection ι of the subalgebra $K(\mathfrak{d}, \theta)$ in $W(\mathfrak{d})$ induces an injective Lie algebra homomorphism $\mathcal{A}(\iota)\colon \mathcal{K} \to \mathcal{W}$ that allows us to view $\mathcal{K}$ as a subalgebra of $\mathcal{W}$. In more detail, by (4.5) we have

$$\mathcal{A}(\iota)(x \otimes_H e) = x \otimes \partial_0 - \sum_{i=1}^{2N} x\partial_i \otimes \partial^i\,, \qquad x \in X\,. \tag{4.19}$$

Lemma 4.2. *The contact filtrations of $\mathcal{K}$ and $\mathcal{W}$ are compatible, i.e., one has $\mathcal{K}'_p = \mathcal{K} \cap \mathcal{W}'_p$. In particular, $[\mathcal{K}'_m, \mathcal{K}'_n] \subset \mathcal{K}'_{m+n}$.*

Proof. Any element of $\mathcal{K}'_p$ has the form $x \otimes_H e$ with $x \in \mathrm{F}'_{p+1}\, X$. Then, by (4.19), (4.14) and (2.44), its image in $\mathcal{W}$ lies in $\mathcal{W}'_p$. Therefore, $\mathcal{K}'_p \subset \mathcal{K} \cap \mathcal{W}'_p$. The opposite inclusion is proved similarly. □

Composing the isomorphism $\mathcal{W} \to W_{2N+1}$ with the injection $\mathcal{K} \to \mathcal{W}$, one obtains a map $\phi\colon \mathcal{K} \to W_{2N+1}$, whose image however does not coincide with $K_{2N+1} \subset W_{2N+1}$. Recall that K_{2N+1} is the Lie subalgebra of W_{2N+1} consisting of vector fields preserving the standard contact form $\mathrm{d}t^0 + \sum_{i=1}^N t^i \mathrm{d}t^{N+i}$ up to multiplication by a function, i.e., by an element of $\mathcal{O}_{2N+1}$ (see [BDK, Chapter 6] and the references therein).

Proposition 4.1. *There exists a ring automorphism ψ of $\mathcal{O}_{2N+1}$ such that the induced Lie algebra automorphism ψ of W_{2N+1} satisfies $\phi(\mathcal{K}) = \psi(K_{2N+1})$.*

Proof. The proof is similar to that of [BDK1, Proposition 3.6]. The image $\phi(\mathcal{K})$ is the Lie algebra of all vector fields preserving a certain contact form up to multiplication by an element of $\mathcal{O}_{2N+1}$ [BDK1, Proposition 8.3]. We can find a change of variables conjugating this contact form to the standard contact form $\mathrm{d}t^0 + \sum_{i=1}^N t^i \mathrm{d}t^{N+i}$. Hence, there exists an automorphism ψ of $\mathcal{O}_{2N+1}$ such that $\phi(\mathcal{K}) = \psi(K_{2N+1})$. □

We will denote by $\mathcal{E}'$ the lifting to $\mathcal{K}$ of the contact Euler vector field $E' \in K_{2N+1}$, that is $\mathcal{E}' = \phi^{-1}\psi(E')$.

Remark 4.1. The adjoint action of $\mathcal{E}'$ on $\mathcal{K}$ is semisimple, as it translates the semisimple action of E' on K_{2N+1}. As the automorphism ψ can be chosen so that the induced homomorphism on the associated graded Lie algebra equals the identity, one can easily show that the adjoint action of $\mathcal{E}'$ on $\mathcal{K}$ preserves each $\mathcal{K}'_n$ and that it equals multiplication by n on $\mathcal{K}'_n/\mathcal{K}'_{n+1}$.

4.4. **The normalizer $\mathcal{N}_{\mathcal{K}}$.** It is well known that all derivations of the Lie–Cartan algebras of type W are inner. This fact was used in [BDK1, Section 3.3] to prove that the centralizer of $\mathcal{W}$ in $\widetilde{\mathcal{W}}$ consists of elements $\widetilde{\partial}$ ($\partial \in \mathfrak{d}$) so that the map $\partial \mapsto \widetilde{\partial}$ is an isomorphism of Lie algebras. We have

$$\widetilde{\partial} = \partial + 1 \otimes \partial - \operatorname{ad}\partial \mod \mathcal{W}_1, \qquad \partial \in \mathfrak{d}\,, \tag{4.20}$$

where $\operatorname{ad}\partial$ is understood as an element of $\mathfrak{gl}\,\mathfrak{d} \simeq \mathcal{W}_0/\mathcal{W}_1$.

Proposition 4.2. *Elements $\widetilde{\partial}$ span a Lie subalgebra $\widetilde{\mathfrak{d}} \subset \widetilde{\mathcal{K}}$ isomorphic to $\mathfrak{d}$. The normalizer $\mathcal{N}_{\mathcal{K}}$ of $\mathcal{K}'_p$ in $\widetilde{\mathcal{K}}$ coincides with $\widetilde{\mathfrak{d}} \oplus \mathcal{K}'_0$ and is independent of $p \geq 0$. There is a decomposition as a direct sum of subspaces $\widetilde{\mathcal{K}} = \mathfrak{d} \oplus \mathcal{N}_{\mathcal{K}}$.*

Proof. Since all derivations of $\mathcal{K} \simeq K_{2N+1}$ are inner, there exist elements $\widehat{\partial} \in \widetilde{\mathcal{K}}$ centralizing $\mathcal{K}$ and such that $\widehat{\partial} = \partial \mod \mathcal{K}$ for $\partial \in \mathfrak{d}$. Then $\widehat{\partial} - \widetilde{\partial} \in \mathcal{W}$ centralizes $\mathcal{K}$, which implies $\widehat{\partial} = \widetilde{\partial}$, because the centralizer of $\mathcal{K}$ in $\mathcal{W}$ is zero. Therefore, the centralizer of $\mathcal{K}$ in $\widetilde{\mathcal{K}}$ coincides with the centralizer $\widetilde{\mathfrak{d}}$ of $\mathcal{W}$ in $\widetilde{\mathcal{W}}$. The other statements follow as in [BDK1, Proposition 3.3]. □

The above proposition implies that for every $\partial \in \mathfrak{d}$ the element $\widetilde{\partial} - \partial \in \mathcal{W}$ lies in the subalgebra $\mathcal{K}$, and hence it can be expressed as a Fourier coefficient $x \otimes_H e$ for suitable $x \in X$. In order to do so, let us compute the images of the first few Fourier coefficients of e under the identification of $\mathcal{K}$ as a subalgebra of $\mathcal{W}$.

Lemma 4.3. *The embedding $\mathcal{A}(\iota)\colon \mathcal{K} \to \mathcal{W}$ identifies the following elements*:

(i) $$1 \otimes_H e \mapsto 1 \otimes \partial_0\,;$$

(ii) $$x^j \otimes_H e \mapsto 1 \otimes \partial^j + x^j \otimes \partial_0 - \sum_{0<i<k} c^j_{ik} x^k \otimes \partial^i \mod \mathcal{W}'_1 \cap \mathcal{W}_1\,;$$

(iii) $$x^0 \otimes_H e \mapsto x^0 \otimes \partial_0 - \sum_{0<i<k} \omega_{ik} x^k \otimes \partial^i \mod \mathcal{W}'_1 \cap \mathcal{W}_1\,;$$

(iv) $$x^i x^j \otimes_H e \mapsto 2 f^{ij} \mod \mathcal{W}_1\,, \qquad i,j \neq 0\,;$$

(v) $$x^0 x^j \otimes_H e \mapsto x^0 \otimes \partial^j \mod \mathcal{W}'_1 \cap \mathcal{W}_1\,, \qquad j \neq 0\,;$$

(vi) $$x^i x^j x^k \otimes_H e \mapsto 0 \mod \mathcal{W}'_1 \cap \mathcal{W}_1\,, \qquad i,j,k \neq 0\,.$$

Proof. The proof is straightforward, using (4.19), (2.47), and (2.11). Note that elements $f^{ij} \in \mathfrak{gl}\,\mathfrak{d}$, defined in (2.25), need to be understood by means of the identification $\mathfrak{gl}\,\mathfrak{d} = \mathcal{W}_0/\mathcal{W}_1$ given in Lemma 4.1. □

Notice that $\mathcal{K}$ (respectively $\mathcal{K}'_0$, $\mathcal{K}'_1$) is spanned over $\mathbf{k}$ by elements (i)-(vi) (resp. (iii)-(vi), (v)-(vi)) modulo $\mathcal{K}'_2$. Also, $\mathcal{K}'_2 \subset \mathcal{W}'_2 \subset \mathcal{W}_1$, by Lemma 4.2 and $\mathrm{F}'_2\, X \subset \mathrm{F}_1\, X$, which follows from $\mathrm{F}^1\, H \subset \mathrm{F}'^2\, H$.

In the proof of next proposition, we will use the following abelian Lie subalgebra of $\mathfrak{gl}\,\mathfrak{d}$:

$$\mathfrak{c}_0 = x^0 \otimes \bar{\mathfrak{d}} = \operatorname{span}\{e^{i0}\}_{1\le i\le 2N} = \operatorname{span}\{e_i^0\}_{1\le i\le 2N} \subset \mathfrak{gl}\,\mathfrak{d}\,. \tag{4.21}$$

Note that the semidirect sum $\mathfrak{c}_0 \rtimes \mathfrak{csp}\,\bar{\mathfrak{d}} \subset \mathfrak{gl}\,\mathfrak{d}$ is a Lie algebra containing $\mathfrak{c}_0$ as an abelian ideal.

Proposition 4.3. *We have* $\mathcal{K}_0'/\mathcal{K}_1' \simeq \mathfrak{sp}\,\bar{\mathfrak{d}} \oplus \mathbf{k}I' = \mathfrak{csp}\,\bar{\mathfrak{d}}$.

Proof. Since elements (iii)-(vi) in the previous lemma all lie in $\mathcal{W}_0$, and $\mathcal{K}_2' \subset \mathcal{W}_1$, we have $\mathcal{K}_0' \subset \mathcal{W}_0$. Moreover $\mathcal{W}_1 \subset \mathcal{W}_0$ is an ideal, so the inclusion $\mathcal{K} \to \mathcal{W}$ induces a well-defined Lie algebra homomorphism $\pi\colon \mathcal{K}_0' \to \mathcal{W}_0/\mathcal{W}_1 \simeq \mathfrak{gl}\,\mathfrak{d}$. Observe now that in $\mathcal{W}_0/\mathcal{W}_1$ one has

$$\begin{aligned} -I' &= 2x^0 \otimes \partial_0 + \sum_{i=1}^{2N} x^i \otimes \partial_i \\ &= 2x^0 \otimes \partial_0 + \sum_{i,j=1}^{2N} \omega_{ij} x^i \otimes \partial^j \\ &= 2x^0 \otimes_H e + 2 \sum_{0<i<j} \omega_{ij} f^{ij} \mod \mathcal{W}_1\,. \end{aligned} \tag{4.22}$$

As a consequence, $I' \in \mathfrak{gl}\,\mathfrak{d}$ lies in the image of π. By Lemma 4.3, π is injective on the linear span of elements (iii)-(v). The image of π equals $\mathfrak{c}_0 \rtimes \mathfrak{csp}\,\bar{\mathfrak{d}}$, and π maps the ideal $\mathcal{K}_1' \subset \mathcal{K}_0'$ onto the ideal $\mathfrak{c}_0 \subset \mathfrak{c}_0 \rtimes \mathfrak{csp}\,\bar{\mathfrak{d}}$, so that π induces an isomorphism between $\mathcal{K}_0'/\mathcal{K}_1'$ and $\mathfrak{csp}\,\bar{\mathfrak{d}}$. ☐

Corollary 4.1. *Elements* $\widetilde{\partial} \in \widetilde{\mathcal{K}}$ *satisfy the following* ($j \ne 0$):

$$\widetilde{\partial}_0 - \partial_0 = 1 \otimes_H e - \operatorname{ad}\partial_0 \mod \mathcal{K}_1'\,, \tag{4.23}$$

$$\widetilde{\partial}^j - \partial^j = x^j \otimes_H e - \Bigl(\operatorname{ad}\partial^j + x^j \otimes \partial_0 - \sum_{0<i<k} c_{ik}^j x^k \otimes \partial^i\Bigr) \mod \mathcal{K}_1'\,. \tag{4.24}$$

Proof. Follows from (4.20), Lemma 4.3 and Propositions 4.2 and 4.3. ☐

The above two statements imply:

Corollary 4.2. *Elements*

$$\operatorname{ad}\partial_0\,, \quad \operatorname{ad}\partial^j - e_0^j + \sum_{0<i<k} c_{ik}^j e^{ik}\,, \quad j \ne 0 \tag{4.25}$$

lie in $\mathfrak{sp}\,\bar{\mathfrak{d}}$.

Proof. Indeed, they must lie in $\mathfrak{csp}\,\bar{\mathfrak{d}}$ but the matrix coefficient multiplying e_0^0 is zero in both cases. ☐

Similarly to [BDK, BDK1], we will say that an $\mathcal{N}_{\mathcal{K}}$-module V is *conformal* if $\mathcal{K}_p'$ acts trivially on it for some $p \ge 1$.

Proposition 4.4. *The subalgebra* $\mathcal{K}_1' \subset \mathcal{N}_{\mathcal{K}}$ *acts trivially on any irreducible finite-dimensional conformal* $\mathcal{N}_{\mathcal{K}}$*-module. Irreducible finite-dimensional conformal* $\mathcal{N}_{\mathcal{K}}$*-modules are in one-to-one correspondence with irreducible finite-dimensional modules over the Lie algebra* $\mathcal{N}_{\mathcal{K}}/\mathcal{K}_1' \simeq \mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$.

Proof. The proof is the same as in [BDK1, Proposition 3.4]. Let V be a finite-dimensional irreducible conformal $\mathcal{N}_{\mathcal{K}}$-module; then it is an irreducible module over the finite-dimensional Lie algebra $\mathfrak{g} = \mathcal{N}_{\mathcal{K}}/\mathcal{K}'_p = \widetilde{\mathfrak{d}} \oplus (\mathcal{K}'_0/\mathcal{K}'_p)$ for some $p \geq 1$. We apply [BDK1, Lemma 3.4] for $I = \mathcal{K}'_1/\mathcal{K}'_p$ and $\mathfrak{g}_0 = (\mathbf{k}\mathcal{E}' + \mathcal{K}'_p)/\mathcal{K}'_p$. Note that by Lemma 4.2, one has $I \subset \operatorname{Rad}\mathfrak{g}$, and $[\mathcal{E}', \mathcal{K}'_p] \subset \mathcal{K}'_p$. Moreover, the adjoint action of $\mathcal{E}'$ on $\mathcal{K}'_1$ is invertible. Thus, the adjoint action of $\mathcal{E}'$ is injective on I, and I acts trivially on V. We can then take $p = 1$, in which case $\mathfrak{g} = \widetilde{\mathfrak{d}} \oplus (\mathcal{K}'_0/\mathcal{K}'_1) \simeq \mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$. □

5. Singular Vectors and Tensor Modules

We start this section by recalling an important class of modules over the Lie pseudoalgebra $W(\mathfrak{d})$ called tensor modules. Restricting such modules to $K(\mathfrak{d}, \theta)$ leads us to the definition of a tensor module over $K(\mathfrak{d}, \theta)$. By investigating singular vectors, we show that every irreducible module is a homomorphic image of a tensor module. We continue to use the notation of Section 2.

5.1. **Tensor modules for $W(\mathfrak{d})$.** Consider a Lie algebra $\mathfrak{g}$ with a finite-dimensional representation V_0. Then the semidirect sum Lie pseudoalgebra $W(\mathfrak{d}) \ltimes \operatorname{Cur}\mathfrak{g}$ from Example 3.3 acts on the free H-module $V = H \otimes V_0$ as follows (see [BDK1, Remark 4.3]):

$$\big((f \otimes a) \oplus (g \otimes b)\big) * (h \otimes u) = -(f \otimes ha) \otimes_H (1 \otimes u) + (g \otimes h) \otimes_H (1 \otimes bu)\,, \tag{5.1}$$

where $f, g, h \in H = U(\mathfrak{d})$, $a \in \mathfrak{d}$, $b \in \mathfrak{g}$, $u \in V_0$. This combines the usual action of $\operatorname{Cur}\mathfrak{g}$ on V with the $W(\mathfrak{d})$-action on H given by (3.24).

By [BDK1, Remark 4.6], there is an embedding of Lie pseudoalgebras $W(\mathfrak{d}) \hookrightarrow W(\mathfrak{d}) \ltimes \operatorname{Cur}(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$ given by

$$1 \otimes \partial_i \mapsto (1 \otimes \partial_i) \oplus \Big((1 \otimes \partial_i) \oplus (1 \otimes \operatorname{ad}\partial_i + \sum_j \partial_j \otimes e_i^j)\Big)\,. \tag{5.2}$$

Composing this embedding with the above action (5.1) for $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d}$, we obtain a $W(\mathfrak{d})$-module $V = H \otimes V_0$ for every $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module V_0. This module V is called a *tensor module* and denoted $\mathcal{T}(V_0)$. The action of $W(\mathfrak{d})$ on $\mathcal{T}(V_0)$ is given explicitly by [BDK1, Eq. (4.30)], which we reproduce here for convenience:

$$\begin{aligned}(1 \otimes \partial_i) * (1 \otimes u) &= (1 \otimes 1) \otimes_H (1 \otimes (\operatorname{ad}\partial_i)u) + \sum_j (\partial_j \otimes 1) \otimes_H (1 \otimes e_i^j u) \\ &\quad - (1 \otimes \partial_i) \otimes_H (1 \otimes u) + (1 \otimes 1) \otimes_H (1 \otimes \partial_i u)\,.\end{aligned} \tag{5.3}$$

If Π is a finite-dimensional $\mathfrak{d}$-module and V_0 is a finite-dimensional $\mathfrak{gl}\,\mathfrak{d}$-module, then their exterior tensor product $\Pi \boxtimes V_0$ is defined as the $(\mathfrak{d} \oplus \mathfrak{gl}\,\mathfrak{d})$-module $\Pi \otimes V_0$, where $\mathfrak{d}$ acts on the first factor and $\mathfrak{gl}\,\mathfrak{d}$ acts on the second one. Following [BDK1], in this case the tensor module $\mathcal{T}(\Pi \boxtimes V_0)$ will also be denoted as $\mathcal{T}(\Pi, V_0)$. Then

$$\mathcal{T}(\Pi, V_0) = T_\Pi(\mathcal{T}(\mathbf{k}, V_0))\,, \tag{5.4}$$

where T_Π is the twisting functor from Definition 3.1.

5.2. **Tensor modules for** $K(\mathfrak{d},\theta)$**.** We will identify $K(\mathfrak{d},\theta)$ with a subalgebra of $W(\mathfrak{d})$ via embedding (4.4). Then $K(\mathfrak{d},\theta) = He$ where $e \in W(\mathfrak{d})$ is given by (4.5). Introduce the H-linear map $\tau\colon W(\mathfrak{d}) \to \operatorname{Cur} \mathfrak{gl}\,\mathfrak{d}$ given by (cf. (5.2))

$$\tau(h \otimes \partial_i) = h \otimes \operatorname{ad} \partial_i + \sum_{j=0}^{2N} h\partial_j \otimes e_i^j\,, \qquad h \in H\,. \tag{5.5}$$

Then the image of e under the map (5.2) has the form $e \oplus (e \oplus \tau(e))$.

Definition 5.1. We define a linear map $\operatorname{ad}^{\mathfrak{sp}}\colon \mathfrak{d} \to \mathfrak{sp}\,\bar{\mathfrak{d}}$ by $\operatorname{ad}^{\mathfrak{sp}} \partial_0 = \operatorname{ad} \partial_0$ and

$$\operatorname{ad}^{\mathfrak{sp}} \partial^k = \operatorname{ad} \partial^k - e_0^k + \frac{1}{2} \sum_{i,j=1}^{2N} c_{ij}^k e^{ij}\,, \qquad k \neq 0\,. \tag{5.6}$$

Remark 5.1. The fact that the image of $\operatorname{ad}^{\mathfrak{sp}}$ is inside $\mathfrak{sp}\,\bar{\mathfrak{d}}$ follows from Corollary 4.2 (cf. (2.25), (4.25)). One can show that $\operatorname{ad}^{\mathfrak{sp}} \partial^k$ is obtained from $\operatorname{ad} \partial^k$ by first restricting it to $\bar{\mathfrak{d}} \subset \mathfrak{d}$ and then projecting onto $\mathfrak{sp}\,\bar{\mathfrak{d}}$. This implies that the map $\operatorname{ad}^{\mathfrak{sp}}$ does not depend on the choice of basis.

Lemma 5.1. *With the above notation, we have*

$$\tau(e) = (\mathrm{id} \otimes \operatorname{ad}^{\mathfrak{sp}})(e) + \frac{1}{2}\partial_0 \otimes I' - \sum_{i=1}^{2N} \partial_i\partial_0 \otimes e^{i0} + \sum_{i,j=1}^{2N} \partial_i\partial_j \otimes f^{ij}\,. \tag{5.7}$$

Proof. Using (2.25) and the H-linearity of τ, we find for $i \neq 0$

$$\begin{aligned}\tau(\partial_i \otimes \partial^i) &= \partial_i \otimes \operatorname{ad} \partial^i + \sum_{j=0}^{2N} \partial_i\partial_j \otimes e^{ij} \\ &= \partial_i \otimes \operatorname{ad} \partial^i + \partial_i\partial_0 \otimes e^{i0} - \sum_{j=1}^{2N} \partial_i\partial_j \otimes f^{ij} + \frac{1}{2} \sum_{j=1}^{2N} [\partial_i, \partial_j] \otimes e^{ij}\,.\end{aligned}$$

By (2.11) and (2.22), we have

$$[\partial_i, \partial_j] = \omega_{ij}\partial_0 + \sum_{k=1}^{2N} c_{ij}^k \partial_k\,, \qquad i,j \neq 0 \tag{5.8}$$

and

$$\sum_{i,j=1}^{2N} \omega_{ij}\partial_0 \otimes e^{ij} = -\sum_{j=1}^{2N} \partial_0 \otimes e_j^j = \partial_0 \otimes 2(e_0^0 - I')\,.$$

The rest of the proof is straightforward. □

Recall the definition of the abelian subalgebra $\mathfrak{c}_0 \subset \mathfrak{gl}\,\mathfrak{d}$ given in (4.21).

Corollary 5.1. *With the above notation, we have*: $\tau(e) \in \operatorname{Cur}(\mathfrak{c}_0 \rtimes \mathfrak{csp}\,\bar{\mathfrak{d}})$.

Therefore, the image of e under map (5.2) lies in $W(\mathfrak{d}) \ltimes \operatorname{Cur} \mathfrak{g}$ where $\mathfrak{g} := \mathfrak{d} \oplus (\mathfrak{c}_0 \rtimes \mathfrak{csp}\,\bar{\mathfrak{d}})$. Hence, every finite-dimensional $\mathfrak{g}$-module V_0 gives rise to a $K(\mathfrak{d},\theta)$-module $H \otimes V_0$ with an action given by (5.1). An important special case is when $\mathfrak{c}_0$ acts trivially on V_0. Since $\mathfrak{c}_0$ is an ideal in $\mathfrak{g}$, having such a representation is equivalent to having a representation of the Lie algebra $\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}} \simeq \mathfrak{g}/\mathfrak{c}_0$.

Definition 5.2. (i) Let V_0 be a finite-dimensional representation of $\mathfrak{d}\oplus\mathfrak{csp}\,\bar{\mathfrak{d}}$. Then the above $K(\mathfrak{d},\theta)$-module $H\otimes V_0$ is called a *tensor module* and will be denoted as $\mathcal{T}(V_0)$.

(ii) Let $V_0=\Pi\boxtimes U$, where Π is a finite-dimensional $\mathfrak{d}$-module and U is a finite-dimensional $\mathfrak{csp}\,\bar{\mathfrak{d}}$-module. Then the module $\mathcal{T}(V_0)$ will also be denoted as $\mathcal{T}(\Pi,U)$.

(iii) Let V_0 be as in part (ii), and assume $I'\in\mathfrak{csp}\,\bar{\mathfrak{d}}$ acts on U as multiplication by a scalar $c\in\mathbf{k}$. Then the module $\mathcal{T}(V_0)$ will also be denoted as $\mathcal{T}(\Pi,U,c)$, and similarly the $\mathfrak{csp}\,\bar{\mathfrak{d}}$-module structure on U will be denoted (U,c).

The action of $e\in K(\mathfrak{d},\theta)$ on a tensor module $\mathcal{T}(V_0)=H\otimes V_0$ is given explicitly by (cf. (4.5), (5.1), (5.7)):

$$
\begin{aligned}
e*(1\otimes u) &= -e\otimes_H(1\otimes u)+(1\otimes 1)\otimes_H\big(1\otimes(\partial_0+\operatorname{ad}\partial_0)u\big)\\
&\quad-\sum_{k=1}^{2N}(\partial_k\otimes 1)\otimes_H\big(1\otimes(\partial^k+\operatorname{ad}^{\mathfrak{sp}}\partial^k)u\big)+\frac{1}{2}(\partial_0\otimes 1)\otimes_H(1\otimes I'u)\\
&\quad+\sum_{i,j=1}^{2N}(\partial_i\partial_j\otimes 1)\otimes_H(1\otimes f^{ij}u)\,,\qquad u\in V_0\,.
\end{aligned}
\tag{5.9}
$$

Remark 5.2. More generally, if $\mathfrak{c}_0$ does not act trivially on V_0, the above action (5.9) is modified by adding the term

$$-\sum_{i=1}^{2N}(\partial_i\partial_0\otimes 1)\otimes_H(1\otimes e^{i0}u)$$

to the right-hand side (cf. Lemma 5.1).

As in [BDK1], in the sequel it will be convenient to modify the above definition of tensor module. Let R be a finite-dimensional $(\mathfrak{d}\oplus\mathfrak{csp}\,\bar{\mathfrak{d}})$-module, with an action denoted as ρ_R. We equip R with the following modified action of $\mathfrak{d}\oplus\mathfrak{csp}\,\bar{\mathfrak{d}}$ (cf. [BDK1, Eqs. (6.7), (6.8)]):

$$
\begin{aligned}
\partial u &= (\rho_R(\partial)+\operatorname{tr}(\operatorname{ad}\partial))u\,, &&\partial\in\mathfrak{d}\,,\ u\in R\,,\\
Au &= (\rho_R(A)-\operatorname{tr}A)u\,, &&A\in\mathfrak{csp}\,\bar{\mathfrak{d}}\,,\ u\in R\,.
\end{aligned}
\tag{5.10}
$$

Note that, in fact, $\operatorname{tr}A=0$ for $A\in\mathfrak{sp}\,\bar{\mathfrak{d}}$ and $\operatorname{tr}I'=2N+2$.

Definition 5.3. Let R be a finite-dimensional $(\mathfrak{d}\oplus\mathfrak{csp}\,\bar{\mathfrak{d}})$-module with an action ρ_R. Then the tensor module $\mathcal{T}(R)$, where R is considered with the modified action (5.10), will be denoted as $\mathcal{V}(R)$. As in Definition 5.2, we will also use the notation $\mathcal{V}(\Pi,U)$ and $\mathcal{V}(\Pi,U,c)$ when $R=\Pi\boxtimes U$ and I' acts on U as multiplication by a scalar c.

The above definition can be made more explicit as follows:

$$
\begin{aligned}
\mathcal{V}(\Pi,U,c) &= \mathcal{T}(\Pi\otimes\mathbf{k}_{\operatorname{tr}\operatorname{ad}},U,c-2N-2)\,,\\
\mathcal{T}(\Pi,U,c) &= \mathcal{V}(\Pi\otimes\mathbf{k}_{-\operatorname{tr}\operatorname{ad}},U,c+2N+2)\,,
\end{aligned}
\tag{5.11}
$$

where for a trace form χ on $\mathfrak{d}$ we denote by $\mathbf{k}_\chi$ the corresponding 1-dimensional $\mathfrak{d}$-module.

Remark 5.3. (cf. [BDK1, Remark 6.2]). Let R be a finite-dimensional representation of $\mathfrak{d}\oplus\mathfrak{csp}\,\bar{\mathfrak{d}}$, or more generally, of $\mathfrak{d}\oplus(\mathfrak{c}_0\rtimes\mathfrak{csp}\,\bar{\mathfrak{d}})$. Using the map π from the proof of Proposition 4.3, whose image is $\mathfrak{c}_0\rtimes\mathfrak{csp}\,\bar{\mathfrak{d}}$, we endow R with an action of $\mathcal{N}_{\mathcal{K}}=\widetilde{\mathfrak{d}}\oplus\mathcal{K}_0'$.

Moreover, $\mathfrak{c}_0$ acts trivially on R if and only if $\mathcal{K}'_1$ does. Then Propositions 3.2, 4.2 and 4.3 imply that, as a $\widetilde{\mathcal{K}}$-module, the tensor module $\mathcal{V}(R)$ is isomorphic to the induced module $\operatorname{Ind}_{\mathcal{N}_\mathcal{K}}^{\widetilde{\mathcal{K}}} R$.

The action of $K(\mathfrak{d},\theta)$ on $\mathcal{V}(R)$ can be derived from (5.9) and (5.10). We will need the following explicit form of this action.

Proposition 5.1. *The action of $K(\mathfrak{d},\theta)$ on a tensor module $\mathcal{V}(R)$ is given by:*

$$\begin{aligned} e*(1\otimes u) &= (1\otimes 1)\otimes_H \bigl(1\otimes\rho_R(\partial_0+\operatorname{ad}\partial_0)u-\partial_0\otimes u\bigr)\\ &\quad-\sum_{k=1}^{2N}(\partial_k\otimes 1)\otimes_H\bigl(1\otimes\rho_R(\partial^k+\operatorname{ad}^{\mathfrak{sp}}\partial^k)u-\partial^k\otimes u\bigr)\\ &\quad+\frac12(\partial_0\otimes 1)\otimes_H\bigl(1\otimes\rho_R(I')u\bigr)+\sum_{i,j=1}^{2N}(\partial_i\partial_j\otimes 1)\otimes_H\bigl(1\otimes\rho_R(f^{ij})u\bigr)\,. \end{aligned} \tag{5.12}$$

Proof. Let us compare (5.12) to (5.9), using (5.10) and the fact that

$$\begin{aligned} -(1\otimes 1)\otimes_H(\partial_0\otimes u)&+\sum_{i=1}^{2N}(\partial_i\otimes 1)\otimes_H(\partial^i\otimes u)\\ &=-e\otimes_H(1\otimes u)-(\partial_0\otimes 1)\otimes_H(1\otimes u)+\sum_{i=1}^{2N}(\partial_i\partial^i\otimes 1)\otimes_H(1\otimes u)\,. \end{aligned}$$

Noting that $\operatorname{ad}\partial_0\in\mathfrak{sp}\,\bar{\mathfrak{d}}$ and $\operatorname{tr}\operatorname{ad}\partial_0=0$, we see that (5.12) reduces to the following identity

$$\sum_{i=1}^{2N}\partial_i\partial^i=-N\partial_0-\sum_{k=1}^{2N}(\operatorname{tr}\operatorname{ad}\partial^k)\partial_k\,.$$

By (2.11)–(2.14), we have:

$$\begin{aligned} 2\sum_{i=1}^{2N}\partial_i\partial^i&=\sum_{i=1}^{2N}[\partial_i,\partial^i]=\sum_{i,j=1}^{2N}r^{ij}[\partial_i,\partial_j]\\ &=\sum_{i,j=1}^{2N}r^{ij}\omega_{ij}\partial_0+\sum_{i,j,k=1}^{2N}r^{ij}c_{ij}^k\partial_k\,, \end{aligned} \tag{5.13}$$

and the coefficient of ∂_0 in the right-hand side is indeed $-2N$. On the other hand, for $k\neq 0$ the fact that $\operatorname{ad}^{\mathfrak{sp}}\partial^k\in\mathfrak{sp}\,\bar{\mathfrak{d}}$ implies

$$0=\operatorname{tr}\operatorname{ad}^{\mathfrak{sp}}\partial^k=\operatorname{tr}\operatorname{ad}\partial^k+\frac12\sum_{i,j=1}^{2N}r^{ij}c_{ij}^k\,,$$

using that $\operatorname{tr}e^{ij}=r^{ij}$. This completes the proof. □

Remark 5.4. Computing directly

$$\operatorname{tr}\operatorname{ad}\partial^k=\sum_{i=1}^{2N}r^{ki}\operatorname{tr}\operatorname{ad}\partial_i=\sum_{i,j=1}^{2N}r^{ki}c_{ij}^j\,,$$

we obtain the identities

$$\sum_{i,j=1}^{2N} r^{ki} c_{ij}^{j} + \frac{1}{2} \sum_{i,j=1}^{2N} r^{ij} c_{ij}^{k} = 0\,, \qquad k \neq 0\,.$$

5.3. **Singular vectors.** The annihilation algebra $\mathcal{K}$ of $K(\mathfrak{d},\theta)$ has a decreasing filtration $\{\mathcal{K}'_p\}_{p\geq -2}$ (see (4.18)). For a $\mathcal{K}$-module V, we denote by $\ker_p V$ the set of all $v \in V$ that are killed by $\mathcal{K}'_p$. A $\mathcal{K}$-module V is called *conformal* iff $V = \bigcup \ker_p V$. For any $p \geq 0$ the normalizer of $\mathcal{K}'_p$ in $\widetilde{\mathcal{K}}$ is equal to $\mathcal{N}_{\mathcal{K}}$ due to Proposition 4.2. Therefore, each $\ker_p V$ is an $\mathcal{N}_{\mathcal{K}}$-module, and in fact, $\ker_p V$ is a representation of the finite-dimensional Lie algebra $\mathcal{N}_{\mathcal{K}}/\mathcal{K}'_p = \widetilde{\mathfrak{d}} \oplus (\mathcal{K}'_0/\mathcal{K}'_p)$. In particular, by Proposition 4.3, $\mathcal{N}_{\mathcal{K}}/\mathcal{K}'_1$ is isomorphic to the direct sum of Lie algebras $\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$.

Equivalence of the filtrations $\{\mathcal{K}_p\}$ and $\{\mathcal{K}'_p\}$, along with Proposition 3.2, implies that any $K(\mathfrak{d},\theta)$-module has a natural structure of a conformal $\widetilde{\mathcal{K}}$-module and vice versa.

Definition 5.4. For any $K(\mathfrak{d},\theta)$-module V, a *singular vector* is an element $v \in V$ such that $\mathcal{K}'_1 \cdot v = 0$. The space of singular vectors in V will be denoted by $\operatorname{sing} V$. We will denote by $\rho_{\mathrm{sing}} \colon \mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}} \to \mathfrak{gl}(\operatorname{sing} V)$ the representation obtained from the $\mathcal{N}_{\mathcal{K}}$-action on $\operatorname{sing} V \equiv \ker_1 V$ via the isomorphism $\mathcal{N}_{\mathcal{K}}/\mathcal{K}'_1 \simeq \mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$.

It follows that a vector $v \in V$ is singular if and only if

$$e * v \in (\mathrm{F}'^2 H \otimes \mathbf{k}) \otimes_H V\,, \tag{5.14}$$

or equivalently

$$e * v \in (\mathbf{k} \otimes \mathrm{F}'^2 H) \otimes_H V\,. \tag{5.15}$$

Proposition 5.2. *For any nonzero finite $K(\mathfrak{d},\theta)$-module V, the vector space* $\operatorname{sing} V$ *is nonzero and the space* $\operatorname{sing} V/\ker V$ *is finite dimensional.*

Proof. Finite dimensionality of $\ker_p V/\ker V$ for all p follows from Lemma 3.2. To prove that $\operatorname{sing} V \neq \{0\}$, we may assume without loss of generality that $\ker V = \{0\}$. Since the $\mathcal{K}$-module V is conformal, $\ker_p V$ is nonzero for some $p \geq 0$. Note that $\ker_p V$ is preserved by the normalizer $\mathcal{N}_{\mathcal{K}}$. Choose an irreducible $\mathcal{N}_{\mathcal{K}}$-submodule $U \subset \ker_p V$. As U is finite dimensional, Proposition 4.4 shows that the action of $\mathcal{K}'_1$ on U is trivial, hence $U \subset \operatorname{sing} V$. □

Note that, by definition,

$$\rho_{\mathrm{sing}}(\partial) v = \widetilde{\partial} \cdot v\,, \qquad \partial \in \mathfrak{d}\,,\ v \in \operatorname{sing} V\,, \tag{5.16}$$

and, due to Lemma 4.3(iv),

$$\rho_{\mathrm{sing}}(f^{ij}) v = \frac{1}{2} (x^i x^j \otimes_H e) \cdot v\,, \qquad v \in \operatorname{sing} V\,. \tag{5.17}$$

The next result describes the action of $K(\mathfrak{d},\theta)$ on a singular vector. It can be derived from Remark 5.3, but for completeness we give a direct proof.

Proposition 5.3. *Let V be a $K(\mathfrak{d},\theta)$-module and $v \in V$ be a singular vector. Then the action of $K(\mathfrak{d},\theta)$ on v is given by*

$$\begin{aligned} e * v = & \sum_{i,j=1}^{2N} (\partial_i \partial_j \otimes 1) \otimes_H \rho_{\rm sing}(f^{ij}) v + \frac{1}{2} (\partial_0 \otimes 1) \otimes_H \rho_{\rm sing}(I') v \\ & - \sum_{k=1}^{2N} (\partial_k \otimes 1) \otimes_H \big(\rho_{\rm sing}(\partial^k + \operatorname{ad}^{\mathfrak{sp}} \partial^k) v - \partial^k v \big) \\ & + (1 \otimes 1) \otimes_H \big(\rho_{\rm sing}(\partial_0 + \operatorname{ad} \partial_0) v - \partial_0 v \big) . \end{aligned} \tag{5.18}$$

Proof. As $\mathcal{K}'_1$ acts trivially on a singular vector v, Proposition 3.2 implies that

$$\begin{aligned} e * v = & \sum_{0<i<j} (S(\partial_i \partial_j) \otimes 1) \otimes_H (x^i x^j \otimes_H e) \cdot v \\ & + \frac{1}{2} \sum_{i=1}^{2N} (S(\partial_i^2) \otimes 1) \otimes_H ((x^i)^2 \otimes_H e) \cdot v \\ & + \sum_{k=0}^{2N} (S(\partial_k) \otimes 1) \otimes_H (x^k \otimes_H e) \cdot v \\ & + (1 \otimes 1) \otimes_H (1 \otimes_H e) \cdot v . \end{aligned} \tag{5.19}$$

On the other hand, by Corollary 4.2 and Lemma 4.3(iv), we have for $k \neq 0$:

$$(1 \otimes_H e) \cdot v = \widetilde{\partial}_0 \cdot v - \partial_0 v + \rho_{\rm sing}(\operatorname{ad} \partial_0) v , \tag{5.20}$$

$$(x^k \otimes_H e) \cdot v = \widetilde{\partial}^k \cdot v - \partial^k v + \rho_{\rm sing}\Big(\operatorname{ad} \partial^k - e^k_0 + \sum_{0<i<j} c^k_{ij} e^{ij} \Big) v . \tag{5.21}$$

Now we rewrite the first summand on the right-hand side of (5.19) using that

$$S(\partial_i \partial_j) = \partial_j \partial_i = \frac{1}{2} (\partial_i \partial_j + \partial_j \partial_i) - \frac{1}{2} [\partial_i, \partial_j] .$$

Then, thanks to (5.17), the first two summands become

$$\sum_{i,j=1}^{2N} (\partial_i \partial_j \otimes 1) \otimes_H \rho_{\rm sing}(f^{ij}) v - \sum_{0<i<j} ([\partial_i, \partial_j] \otimes 1) \otimes_H \rho_{\rm sing}(f^{ij}) v .$$

This shows that the first summand in (5.18) matches with (5.19). By (5.16), (5.20), the last summands in (5.18) and (5.19) are also equal.

It remains to rewrite

$$\sum_{0<i<j} ([\partial_i, \partial_j] \otimes 1) \otimes_H \rho_{\rm sing}(f^{ij}) v + \sum_{k=0}^{2N} (\partial_k \otimes 1) \otimes_H (x^k \otimes_H e) \cdot v$$

so that it matches the negative of the second and third terms in the right-hand side of (5.18). Recalling the commutation relations (5.8), we obtain

$$\begin{aligned} & (\partial_0 \otimes 1) \otimes_H \Big((x^0 \otimes_H e) \cdot v + \rho_{\rm sing}\big(\sum_{0<i<j} \omega_{ij} f^{ij} \big) v \Big) \\ & + \sum_{k=1}^{2N} (\partial_k \otimes 1) \otimes_H \Big((x^k \otimes_H e) \cdot v + \rho_{\rm sing}\big(\sum_{0<i<j} c^k_{ij} f^{ij} \big) v \Big) . \end{aligned}$$

By (4.22), the first summand is equal to $-\frac{1}{2}(\partial_0 \otimes 1) \otimes_H \rho_{\mathrm{sing}}(I')v$.

Finally, by (5.21), (2.25) and (5.6), we have

$$\begin{aligned}
&(x^k \otimes_H e) \cdot v + \rho_{\mathrm{sing}}\big(\sum_{0<i<j} c_{ij}^k f^{ij} \big) v \\
&\quad = \widetilde{\partial}^k \cdot v - \partial^k v + \rho_{\mathrm{sing}} \Big(\mathrm{ad}\, \partial^k - e_0^k + \sum_{0<i<j} c_{ij}^k e^{ij} + \sum_{0<i<j} c_{ij}^k f^{ij} \Big) v \\
&\quad = \widetilde{\partial}^k \cdot v - \partial^k v + \rho_{\mathrm{sing}}(\mathrm{ad}^{\mathfrak{sp}}\, \partial^k) v \,.
\end{aligned}$$

This completes the proof. □

Corollary 5.2. *Let V be a $K(\mathfrak{d},\theta)$-module and let R be a nonzero $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-submodule of $\mathrm{sing}\, V$. Denote by HR the H-submodule of V generated by R. Then HR is a $K(\mathfrak{d},\theta)$-submodule of V. In particular, if V is irreducible, then $V = HR$.*

Proof. By (5.18), $K(\mathfrak{d},\theta) * R \subset (H \otimes H) \otimes_H HR$, and by H-bilinearity, $K(\mathfrak{d},\theta) * HR \subset (H \otimes H) \otimes_H HR$. □

Corollary 5.3. *Let R be a finite-dimensional $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-module with an action ρ_R. Then for the tensor $K(\mathfrak{d},\theta)$-module $\mathcal{V}(R) = H \otimes R$, we have $\mathbf{k} \otimes R \subset \mathrm{sing}\, \mathcal{V}(R)$ and*

$$\rho_{\mathrm{sing}}(A)(1 \otimes u) = 1 \otimes \rho_R(A) u \,, \qquad A \in \mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}, \;\; u \in R \,. \tag{5.22}$$

We will call elements of $\mathbf{k} \otimes R \subset \mathcal{V}(R)$ *constant* vectors. Combining the above results, we obtain the following theorem.

Theorem 5.1. *Let V be an irreducible finite $K(\mathfrak{d},\theta)$-module, and let R be an irreducible $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-submodule of $\mathrm{sing}\, V$. Then V is a homomorphic image of $\mathcal{V}(R)$. In particular, every irreducible finite $K(\mathfrak{d},\theta)$-module is a quotient of a tensor module.*

Proof. Comparing (5.18) and (5.12), we see that the canonical projection $\mathcal{V}(R) = H \otimes R \to HR$ is a homomorphism of $K(\mathfrak{d},\theta)$-modules. However, $HR = V$ by Corollary 5.2. □

We will now show that reducibility of a tensor module depends on the existence of nonconstant singular vectors.

Definition 5.5. An element v of a $K(\mathfrak{d},\theta)$-module V is called *homogeneous* if it is an eigenvector for the action of $\mathcal{E}' \in \mathcal{K}$.

Remark 5.5. Note that the homogeneous components of a singular vector are still singular, so that a classification of singular vectors will follow from a description of homogeneous ones.

Lemma 5.2. *Let R be an irreducible representation of $\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$. Then any nonzero proper $K(\mathfrak{d},\theta)$-submodule M of $\mathcal{V}(R)$ does not contain nonzero constant vectors, i.e., $M \cap (\mathbf{k} \otimes R) = \{0\}$.*

Proof. Both M and $\mathbf{k} \otimes R \subset \mathrm{sing}\, \mathcal{V}(R)$ are $\mathcal{N}_{\mathcal{K}}$-stable, and the same is true of their intersection M_0. Since $\mathcal{K}'_1$ acts trivially on M_0, it is a representation of $\mathcal{N}_{\mathcal{K}} / \mathcal{K}'_1 \simeq \mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$. The claim now follows from the irreducibility of $\mathbf{k} \otimes R \simeq R$. □

Corollary 5.4. *If $\mathrm{sing}\, \mathcal{V}(R) = \mathbf{k} \otimes R$, then the $K(\mathfrak{d},\theta)$-module $\mathcal{V}(R)$ is irreducible.*

Proof. Assume there is a nonzero proper submodule M. Then M must contain some nonzero singular vector. However, $M \cap \operatorname{sing} \mathcal{V}(R) = \{0\}$ by Lemma 5.2. $\square$

Proposition 5.4. *Every nonconstant homogeneous singular vector in $\mathcal{V}(R)$ is contained in a nonzero proper submodule. In particular, $\mathcal{V}(R)$ is irreducible if and only if* $\operatorname{sing} \mathcal{V}(R) = \mathbf{k} \otimes R$.

Proof. Recall that, by Remark 5.3, we have $\mathcal{V}(R) = \operatorname{Ind}_{\mathcal{N}_\mathcal{K}}^{\widetilde{\mathcal{K}}} R$. The Lie algebra $\mathcal{K}$ is graded by the eigenspace decomposition of $\operatorname{ad} \mathcal{E}'$. If $\mathfrak{k}_n$ denotes the graded summand of eigenvalue n, then one has the direct sum decomposition of Lie algebras

$$\mathcal{N}_\mathcal{K} = \widetilde{\mathfrak{d}} \oplus \mathcal{K}'_0 = \widetilde{\mathfrak{d}} \oplus \prod_{j \geq 0} \mathfrak{k}_j$$

and the decomposition of vector spaces

$$\widetilde{\mathcal{K}} = (\mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1}) \oplus \mathcal{N}_\mathcal{K}.$$

Since $\mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1}$ is a graded Lie algebra, its universal enveloping algebra is also graded. Then $\mathcal{V}(R) = \operatorname{Ind}_{\mathcal{N}_\mathcal{K}}^{\widetilde{\mathcal{K}}} R$ is isomorphic to $U(\mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1}) \otimes R$, which can be endowed with a $\mathbb{Z}$-grading by setting elements from R to have degree zero, and elements from $\mathfrak{k}_{-i}$ to have degree $-i$. Thus submodules of $\mathcal{V}(R)$ contain all $\mathcal{E}'$-homogeneous components of their elements, i.e., they are graded submodules.

It is now easy to show that every homogeneous singular vector v, say of degree $d < 0$, is contained in some nonzero proper $\widetilde{\mathcal{K}}$-submodule of $\mathcal{V}(R)$. Indeed $U(\widetilde{\mathcal{K}})v = U(\mathfrak{k}_{-2} \oplus \mathfrak{k}_{-1})v$ is a nonzero submodule of $\mathcal{V}(R)$ lying in degrees $\leq d$, and it intersects R trivially, since R lies in degree zero. $\square$

5.4. **Filtration of tensor modules.** After filtering the Lie algebra $\mathcal{K}$ using the contact filtration F' of X, it is convenient to filter tensor $K(\mathfrak{d}, \theta)$-modules using the contact filtration of H. We therefore define

$$\mathrm{F}'^p\, \mathcal{V}(R) = \mathrm{F}'^p\, H \otimes R\,, \qquad p = -1, 0, \dots\,. \tag{5.23}$$

As usual, $\mathrm{F}'^{-1}\, \mathcal{V}(R) = \{0\}$ and $\mathrm{F}'^0\, \mathcal{V}(R) = \mathbf{k} \otimes R$. It will also be convenient to agree that $\mathrm{F}'^{-2}\, \mathcal{V}(R) = \{0\}$. The associated graded space is defined accordingly, and we have isomorphisms of vector spaces

$$\mathrm{gr}'^p\, \mathcal{V}(R) \simeq \mathrm{gr}'^p\, H \otimes R. \tag{5.24}$$

Note that, since $\mathfrak{d} = \bar{\mathfrak{d}} \oplus \mathbf{k}\partial_0$ and the degree of ∂_0 equals two, $\mathrm{gr}'^p\, H$ is isomorphic to the direct sum $\bigoplus_{i=0}^{\lfloor p/2 \rfloor} S^{p-2i} \bar{\mathfrak{d}}$. Here $\lfloor p/2 \rfloor$ denotes the largest integer not greater than $p/2$, which is $p/2$ for p even and $(p-1)/2$ for p odd.

Lemma 5.3. *For every $p \geq 0$, we have*:

(i) $$\bar{\mathfrak{d}} \cdot \mathrm{F}'^p\, \mathcal{V}(R) \subset \mathrm{F}'^{p+1}\, \mathcal{V}(R)\,,$$

(ii) $$\partial_0 \cdot \mathrm{F}'^p\, \mathcal{V}(R) \subset \mathrm{F}'^{p+2}\, \mathcal{V}(R)\,,$$

(iii) $$\mathcal{N}_\mathcal{K} \cdot \mathrm{F}'^p\, \mathcal{V}(R) \subset \mathrm{F}'^p\, \mathcal{V}(R)\,,$$

(iv) $$\widetilde{\mathcal{K}} \cdot \mathrm{F}'^p\, \mathcal{V}(R) \subset \mathrm{F}'^{p+2}\, \mathcal{V}(R)\,,$$

(v) $$\mathcal{K}'_1 \cdot \mathrm{F}'^p\, \mathcal{V}(R) \subset \mathrm{F}'^{p-1}\, \mathcal{V}(R)\,.$$

Proof. The proof of (i) and (ii) is clear, as the action of elements in $\mathfrak{d}$ is by left multiplication on the left factor of $\mathcal{V}(R) = H \otimes R$. In particular, this implies $\mathfrak{d} \cdot \mathrm{F}'^p\, \mathcal{V}(R) \subset \mathrm{F}'^{p+2}\, \mathcal{V}(R)$. Before proceeding with proving (iii)–(v), observe that (4.23), (4.24) imply

$$\partial^i \in \widetilde{\mathfrak{d}} + \mathcal{K}'_{-1}, \qquad \partial_0 \in \widetilde{\mathfrak{d}} + \mathcal{K}'_{-2},$$

so that $\bar{\mathfrak{d}} \subset \widetilde{\mathfrak{d}} + \mathcal{K}'_{-1}$. Also notice that $\widetilde{\mathcal{K}} = \widetilde{\mathfrak{d}} + \mathcal{K}'_{-2}$, which implies $[\widetilde{\mathcal{K}}, \mathcal{K}'_p] \subset \mathcal{K}'_{p-2}$. Moreover $\mathcal{K}'_{-1} + \mathcal{N}_{\mathcal{K}} = \bar{\mathfrak{d}} + \mathcal{N}_{\mathcal{K}}$, as $\widetilde{\mathfrak{d}} \subset \mathcal{N}_{\mathcal{K}}$. Then we have:

$$\begin{aligned}
&[\bar{\mathfrak{d}}, \mathcal{K}'_1] \subset [\widetilde{\mathfrak{d}} + \mathcal{K}'_{-1}, \mathcal{K}'_1] \subset [\mathcal{K}'_{-1}, \mathcal{K}'_1] \subset \mathcal{K}'_0 \subset \mathcal{N}_{\mathcal{K}}\,,\\
&[\partial_0, \mathcal{K}'_1] \subset [\widetilde{\mathcal{K}}, \mathcal{K}'_1] \subset \mathcal{K}'_{-1} \subset \bar{\mathfrak{d}} + \mathcal{N}_{\mathcal{K}}\,,\\
&[\bar{\mathfrak{d}}, \mathcal{N}_{\mathcal{K}}] \subset [\widetilde{\mathfrak{d}} + \mathcal{K}'_{-1}, \widetilde{\mathfrak{d}} + \mathcal{K}'_0] \subset \widetilde{\mathfrak{d}} + \mathcal{K}'_{-1} \subset \mathcal{K}'_{-1} + \mathcal{N}_{\mathcal{K}} \subset \bar{\mathfrak{d}} + \mathcal{N}_{\mathcal{K}}\,,\\
&[\partial_0, \mathcal{N}_{\mathcal{K}}] \subset \widetilde{\mathcal{K}} = \mathfrak{d} + \mathcal{N}_{\mathcal{K}}\,.
\end{aligned}$$

Now (iii) can be proved by induction as in the case of $W(\mathfrak{d})$ (see [BDK1, Lemma 6.3]), the basis of induction $p = 0$ following from $\mathrm{F}'^0\, \mathcal{V}(R) \subset \operatorname{sing} \mathcal{V}(R)$. As for $p > 0$, notice that

$$\mathrm{F}'^p\, \mathcal{V}(R) = \mathrm{F}'^0\, \mathcal{V}(R) + \bar{\mathfrak{d}}\, \mathrm{F}'^{p-1}\, \mathcal{V}(R) + \partial_0\, \mathrm{F}'^{p-2}\, \mathcal{V}(R).$$

Then:

$$\begin{aligned}
\mathcal{N}_{\mathcal{K}}(\bar{\mathfrak{d}}\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)) &\subset \bar{\mathfrak{d}}(\mathcal{N}_{\mathcal{K}}\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)) + [\bar{\mathfrak{d}}, \mathcal{N}_{\mathcal{K}}]\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)\\
&\subset \bar{\mathfrak{d}}\, \mathrm{F}'^{p-1}\, \mathcal{V}(R) + (\bar{\mathfrak{d}} + \mathcal{N}_{\mathcal{K}})\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)\\
&\subset \bar{\mathfrak{d}}\, \mathrm{F}'^{p-1}\, \mathcal{V}(R) + \mathrm{F}'^{p-1}\, \mathcal{V}(R) \subset \mathrm{F}'^p\, \mathcal{V}(R),
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathcal{N}_{\mathcal{K}}(\partial_0\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)) &\subset \partial_0(\mathcal{N}_{\mathcal{K}}\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)) + [\partial_0, \mathcal{N}_{\mathcal{K}}]\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)\\
&\subset \mathfrak{d}\, \mathrm{F}'^{p-2}\, \mathcal{V}(R) + (\mathfrak{d} + \mathcal{N}_{\mathcal{K}})\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)\\
&\subset \mathfrak{d}\, \mathrm{F}'^{p-2}\, \mathcal{V}(R) + \mathcal{N}_{\mathcal{K}}\, \mathrm{F}'^{p-2}\, \mathcal{V}(R) \subset \mathrm{F}'^p\, \mathcal{V}(R).
\end{aligned}$$

It is now immediate to prove (iv) from $\widetilde{\mathcal{K}} = \mathfrak{d} + \mathcal{N}_{\mathcal{K}}$.

Finally, (v) can analogously be showed by induction on p: when $p = 0$, we have $\mathrm{F}'^0\, \mathcal{V}(R) \subset \operatorname{sing} \mathcal{V}(R)$, hence $\mathcal{K}'_1\, \mathrm{F}'^0\, \mathcal{V}(R) = \{0\}$ by definition of $\operatorname{sing} \mathcal{V}(R)$. When $p > 0$, we observe that

$$\begin{aligned}
\mathcal{K}'_1(\bar{\mathfrak{d}}\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)) &\subset \bar{\mathfrak{d}}(\mathcal{K}'_1\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)) + [\bar{\mathfrak{d}}, \mathcal{K}'_1]\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)\\
&\subset \bar{\mathfrak{d}}(\mathrm{F}'^{p-2}\, \mathcal{V}(R)) + \mathcal{N}_{\mathcal{K}}\, \mathrm{F}'^{p-1}\, \mathcal{V}(R)\\
&\subset \mathrm{F}'^{p-1}\, \mathcal{V}(R),
\end{aligned}$$

and that

$$\begin{aligned}
\mathcal{K}'_1(\partial_0\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)) &\subset \partial_0(\mathcal{K}'_1\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)) + [\partial_0, \mathcal{K}'_1]\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)\\
&\subset \partial_0(\mathrm{F}'^{p-3}\, \mathcal{V}(R)) + (\bar{\mathfrak{d}} + \mathcal{N}_{\mathcal{K}})\, \mathrm{F}'^{p-2}\, \mathcal{V}(R)\\
&\subset \mathrm{F}'^{p-1}\, \mathcal{V}(R).
\end{aligned}$$

This completes the proof. □

The above lemma implies that both $\mathcal{N}_{\mathcal{K}}$ and its quotient $\mathcal{N}_{\mathcal{K}}/\mathcal{K}'_1 = \widetilde{\mathfrak{d}} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$ act on each space $\operatorname{gr}'^p\, \mathcal{V}(R)$. The next result describes the action of $\mathcal{N}_{\mathcal{K}}/\mathcal{K}'_1$ more explicitly.

Lemma 5.4. *The action of* $\widetilde{\mathfrak{d}} \simeq \mathfrak{d}$ *and* $\mathcal{K}'_0/\mathcal{K}'_1 \simeq \mathfrak{csp}\,\bar{\mathfrak{d}} = \mathfrak{sp}\,\bar{\mathfrak{d}} \oplus \mathbf{k}I'$ *on the space* $\mathrm{gr}'^p\,\mathcal{V}(R) \simeq \mathrm{gr}'^p\,H \otimes R$ *is given by:*

$$\widetilde{\partial} \cdot (f \otimes u) = f \otimes \rho_R(\partial)u, \tag{5.25}$$

$$A \cdot (\bar{f}\partial_0^i \otimes u) = (A\bar{f})\partial_0^i \otimes u + \bar{f}\partial_0^i \otimes \rho_R(A)u, \tag{5.26}$$

$$I' \cdot (f \otimes u) = p\, f \otimes u + f \otimes \rho_R(I')u, \tag{5.27}$$

where $A \in \mathfrak{sp}\,\bar{\mathfrak{d}}, f \in \mathrm{gr}'^p\,H, \bar{f} \in S^{p-2i}\bar{\mathfrak{d}}, u \in R$*, and* $A\bar{f}$ *denotes the standard action of* $\mathfrak{sp}\,\bar{\mathfrak{d}} \subset \mathfrak{gl}\,\mathfrak{d}$ *on* $\bar{\mathfrak{d}}$*.*

Proof. The proof is similar to that of Lemmas 6.4 and 6.5 from [BDK1]. □

Corollary 5.5. *We have an isomorphism of* $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$*-modules*

$$\mathrm{gr}'^p\,\mathcal{V}(\Pi, U, c) \simeq \bigoplus_{i=0}^{\lfloor p/2 \rfloor} \Pi \boxtimes (S^{p-2i}\bar{\mathfrak{d}} \otimes U, c+p).$$

Proof. Follows immediately from Lemma 5.4. □

6. Tensor Modules of de Rham Type

The main goal of this section is to define an important complex of $K(\mathfrak{d},\theta)$-modules, called the contact pseudo de Rham complex. We continue to use the notation of Sections 2.1 and 2.2.

6.1. **The Rumin complex.** As before, let $\theta \in \mathfrak{d}^*$ be a contact form, and let $\bar{\mathfrak{d}} \subset \mathfrak{d}$ be the kernel of θ. Consider the wedge powers $\Omega^n = \bigwedge^n \mathfrak{d}^*$ and $\bar{\Omega}^n = \bigwedge^n \bar{\mathfrak{d}}^*$. Then we have a short exact sequence

$$0 \to \Theta\Omega^{n-1} \to \Omega^n \to \bar{\Omega}^n \to 0\,, \tag{6.1}$$

where Θ is the operator of left wedge multiplication with θ, i.e., $\Theta(\alpha) = \theta \wedge \alpha$. For $\alpha \in \Omega^n$, we will denote by $\bar{\alpha} \in \bar{\Omega}^n$ its projection via (6.1).

The direct sum decomposition $\mathfrak{d} = \bar{\mathfrak{d}} \oplus \mathbf{k}s$ gives a splitting of the sequence (6.1). In more detail, elements $\bar{\alpha} \in \bar{\Omega}^n$ are identified with n-forms $\alpha \in \Omega^n$ such that $\iota_s\alpha = 0$. Thus we have a direct sum $\Omega^n = \Theta\Omega^{n-1} \oplus \bar{\Omega}^n$. Then $\Theta^2 = 0$ implies that $\ker \Theta|_{\Omega^n} = \Theta\Omega^{n-1}$, and we get a natural isomorphism

$$\Theta\Omega^n \xrightarrow{\sim} \bar{\Omega}^n, \quad \theta \wedge \alpha \mapsto \bar{\alpha}\,. \tag{6.2}$$

The 2-form $\omega = \mathrm{d}_0\theta$ can be identified with $\bar{\omega}$, because $\iota_s\omega = 0$. Denote by Ψ (respectively, $\bar{\Psi}$) the operator of left wedge multiplication with ω (respectively, $\bar{\omega}$). Consider the images and kernels of $\bar{\Psi}$:

$$\bar{I}^n = \bar{\Psi}\bar{\Omega}^{n-2} \subset \bar{\Omega}^n\,, \qquad \bar{K}^n = \ker \bar{\Psi}|_{\bar{\Omega}^n} \subset \bar{\Omega}^n\,. \tag{6.3}$$

Since $\bar{\omega}$ is nondegenerate, we have $\bar{I}^n = \bar{\Omega}^n$ for $n \geq N+1$ and $\bar{K}^n = 0$ for $n \leq N-1$. In particular, $\bar{\Psi}\colon \bar{\Omega}^{N-1} \to \bar{\Omega}^{N+1}$ is an isomorphism. More generally, for all $m = 0, \dots, N$, the maps $\bar{\Psi}^m\colon \bar{\Omega}^{N-m} \to \bar{\Omega}^{N+m}$ are isomorphisms.

Lemma 6.1. *The composition of natural maps* $\bar{K}^N \hookrightarrow \bar{\Omega}^N \twoheadrightarrow \bar{\Omega}^N/\bar{I}^N$ *is an isomorphism. More generally, the composition*

$$\bar{K}^{N+m} \hookrightarrow \bar{\Omega}^{N+m} \xrightarrow{(\bar{\Psi}^m)^{-1}} \bar{\Omega}^{N-m} \twoheadrightarrow \bar{\Omega}^{N-m}/\bar{I}^{N-m}$$

is an isomorphism for all $m = 0, \dots, N$*.*

Proof. To show surjectivity, take any $\alpha \in \bar{\Omega}^{N-m}$. We want to find $\beta \in \bar{K}^{N+m}$ such that $\alpha - (\bar{\Psi}^m)^{-1}\beta \in \bar{I}^{N-m}$. Since $\bar{\Psi}^{m+2}\colon \bar{\Omega}^{N-m-2} \to \bar{\Omega}^{N+m+2}$ is an isomorphism, there is $\gamma \in \bar{\Omega}^{N-m-2}$ such that $\bar{\Psi}^{m+2}\gamma = \bar{\Psi}^{m+1}\alpha$. Then $\beta = \bar{\Psi}^m(\alpha - \bar{\Psi}\gamma)$ satisfies the above conditions.

To prove injectivity, we need to show that $\bar{\Psi}^m \bar{I}^{N-m} \cap \bar{K}^{N+m} = \{0\}$. If $\alpha \in \bar{\Psi}^m \bar{I}^{N-m} \cap \bar{K}^{N+m}$, then $\alpha = \bar{\Psi}^{m+1}\rho$ for some $\rho \in \bar{\Omega}^{N-m-2}$. But then $\bar{\Psi}^{m+2}\rho = \bar{\Psi}\alpha = 0$, which implies $\rho = 0$ and $\alpha = 0$. □

Since $A\cdot\bar{\omega} = 0$ for $A \in \mathfrak{sp}\,\bar{\mathfrak{d}}$ and the action of A is an even derivation of the wedge product (see Lemma 2.2 and (2.8)), it follows that $\bar{I}^n$ and $\bar{K}^n$ are $\mathfrak{sp}\,\bar{\mathfrak{d}}$-submodules of $\bar{\Omega}^n$. Furthermore, the map $\bar{\Psi}$ is an $\mathfrak{sp}\,\bar{\mathfrak{d}}$-homomorphism. In particular, the isomorphism from Lemma 6.1 commutes with the action of $\mathfrak{sp}\,\bar{\mathfrak{d}}$. Recall that $R(\pi_n)$ denotes the n-th fundamental representation of $\mathfrak{sp}\,\bar{\mathfrak{d}}$, and $R(\pi_0) = \mathbf{k}$.

Lemma 6.2. *We have isomorphisms of* $\mathfrak{sp}\,\bar{\mathfrak{d}}$*-modules*

$$\bar{\Omega}^n/\bar{I}^n \simeq \bar{K}^{2N-n} \simeq R(\pi_n)\,, \qquad 0 \le n \le N\,.$$

Proof. This is well known; see, e.g., [FH, Lecture 17]. □

Following [Ru], we consider the spaces

$$I^n = \Psi\Omega^{n-2} + \Theta\Omega^{n-1} \subset \Omega^n, \qquad K^n = \ker\Psi|_{\Omega^n} \cap \ker\Theta|_{\Omega^n} \subset \Omega^n. \tag{6.4}$$

Using $\Theta\Psi = \Psi\Theta$ and (6.1), we obtain a short exact sequence

$$0 \to \Theta\Omega^{n-1} \to I^n \to \bar{I}^n \to 0\,, \tag{6.5}$$

while (6.2) gives a natural isomorphism

$$K^n \xrightarrow{\sim} \bar{K}^{n-1}, \quad \theta\wedge\alpha \mapsto \bar{\alpha}\,. \tag{6.6}$$

The above equations imply that $I^n = \Omega^n$ for $n \ge N+1$ and $K^n = 0$ for $n \le N$. It is also clear that $\Omega^n/I^n \simeq \bar{\Omega}^n/\bar{I}^n$ for all n.

The "constant-coefficient" *Rumin complex* [Ru] is the following complex of $\mathfrak{csp}\,\bar{\mathfrak{d}}$-modules

$$0 \to \Omega^0/I^0 \xrightarrow{\mathrm{d}_0} \cdots \xrightarrow{\mathrm{d}_0} \Omega^N/I^N \xrightarrow{\mathrm{d}_0^{\mathrm{R}}} K^{N+1} \xrightarrow{\mathrm{d}_0} \cdots \xrightarrow{\mathrm{d}_0} K^{2N+1}\,, \tag{6.7}$$

where the map $\mathrm{d}_0^{\mathrm{R}}$ is defined as in [Ru]. We will need the "pseudo" version of this complex defined in Section 6.3 below. The latter is a contact counterpart of the pseudo de Rham complex from [BDK, BDK1], which we review in the next subsection.

6.2. **Pseudo de Rham complex.** Following [BDK], we define the spaces of *pseudoforms* $\Omega^n(\mathfrak{d}) = H\otimes\Omega^n$ and $\Omega(\mathfrak{d}) = H\otimes\Omega = \bigoplus_{n=0}^{2N+1}\Omega^n(\mathfrak{d})$. They are considered as H-modules, where H acts on the first factor by left multiplication. We can identify $\Omega^n(\mathfrak{d})$ with the space of linear maps from $\bigwedge^n\mathfrak{d}$ to H, and $H^{\otimes 2}\otimes_H \Omega^n(\mathfrak{d})$ with $\mathrm{Hom}(\bigwedge^n\mathfrak{d}, H^{\otimes 2})$. For $g \in H$, $\alpha \in \Omega$, we will write the element $g\otimes\alpha \in \Omega(\mathfrak{d})$ as $g\alpha$; in particular, we will identify Ω with $\mathbf{k}\otimes\Omega \subset \Omega(\mathfrak{d})$.

Let us consider $H = U(\mathfrak{d})$ as a left $\mathfrak{d}$-module with respect to the action $a\cdot h = -ha$, where ha is the product of $a \in \mathfrak{d} \subset H$ and $h \in H$ in H. Then consider the cohomology complex of $\mathfrak{d}$ with coefficients in H:

$$0 \to \Omega^0(\mathfrak{d}) \xrightarrow{\mathrm{d}} \Omega^1(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{2N+1}(\mathfrak{d})\,. \tag{6.8}$$

Explicitly, the *differential* d is given by the formula ($\alpha \in \Omega^n(\mathfrak{d})$, $a_i \in \mathfrak{d}$):

$$
\begin{aligned}
&(\mathrm{d}\alpha)(a_1 \wedge \cdots \wedge a_{n+1}) \\
&\qquad = \sum_{i<j} (-1)^{i+j} \alpha([a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_{n+1}) \\
&\qquad + \sum_i (-1)^i \alpha(a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_{n+1})\, a_i \qquad \text{if } n \geq 1, \\
&(\mathrm{d}\alpha)(a_1) = -\alpha a_1 \qquad \text{if } \alpha \in \Omega^0(\mathfrak{d}) = H.
\end{aligned} \tag{6.9}
$$

Notice that d is H-linear. The sequence (6.8) is called the *pseudo de Rham complex*. It was shown in [BDK, Remark 8.1] that the n-th cohomology of the complex $(\Omega(\mathfrak{d}), \mathrm{d})$ is trivial for $n \neq 2N+1 = \dim \mathfrak{d}$, and it is 1-dimensional for $n = 2N+1$. In particular, the sequence (6.8) is exact.

Example 6.1. For $\alpha = 1 \in H = \Omega^0(\mathfrak{d})$, Eq. (6.9) gives

$$
-\,\mathrm{d}1 = \epsilon := \sum_{i=0}^{2N} \partial_i \otimes x^i \in H \otimes \mathfrak{d}^* = \Omega^1(\mathfrak{d})\,. \tag{6.10}
$$

Next, we introduce H-bilinear maps

$$
*\colon W(\mathfrak{d}) \otimes \Omega^n(\mathfrak{d}) \to H^{\otimes 2} \otimes_H \Omega^n(\mathfrak{d}) \tag{6.11}
$$

by the formula [BDK]:

$$
\begin{aligned}
(w * \gamma)(a_1 \wedge \cdots \wedge a_n) &= -(f \otimes ga)\, \alpha(a_1 \wedge \cdots \wedge a_n) \\
&+ \sum_{i=1}^{n} (-1)^i (f a_i \otimes g)\, \alpha(a \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_n) \\
&+ \sum_{i=1}^{n} (-1)^i (f \otimes g)\, \alpha([a, a_i] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge a_n) \in H^{\otimes 2},
\end{aligned} \tag{6.12}
$$

where $n \geq 1$, $w = f \otimes a \in W(\mathfrak{d})$ and $\gamma = g\alpha \in \Omega^n(\mathfrak{d})$. When $\gamma = g \in \Omega^0(\mathfrak{d}) = H$, we let $w * \gamma = -f \otimes ga$. Note that the latter coincides with the action (3.24) of $W(\mathfrak{d})$ on H.

It was shown in [BDK,BDK1] that maps (6.11) provide each $\Omega^n(\mathfrak{d})$ with a structure of a $W(\mathfrak{d})$-module. These modules are instances of *tensor modules* as introduced in [BDK1], namely $\Omega^n(\mathfrak{d}) = \mathcal{T}(\mathbf{k}, \Omega^n)$ (see Section 5.1). The action of $W(\mathfrak{d})$ commutes with d, i.e.,

$$
w * (\mathrm{d}\gamma) = ((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})(w * \gamma) \tag{6.13}
$$

for $w \in W(\mathfrak{d})$, $\gamma \in \Omega^n(\mathfrak{d})$.

Let us extend the wedge product in Ω to a product in $\Omega(\mathfrak{d})$ by setting

$$
(f\alpha) \wedge (g\beta) = (fg)(\alpha \wedge \beta)\,, \qquad \alpha, \beta \in \Omega\,, \quad f, g \in H\,.
$$

In a similar way, we also extend it to products

$$
\begin{aligned}
(h \otimes_H (f \otimes \alpha)) \wedge \beta &= h \otimes_H (f \otimes (\alpha \wedge \beta))\,, \\
\alpha \wedge (h \otimes_H (g \otimes \beta)) &= h \otimes_H (g \otimes (\alpha \wedge \beta))\,, \qquad h \in H^{\otimes 2}\,.
\end{aligned}
$$

Lemma 6.3. *For any $w \in W(\mathfrak{d})$, $\alpha \in \Omega^n$ and $\beta \in \Omega$, we have*:

$$
\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}_0\alpha \wedge \beta + (-1)^n \alpha \wedge \mathrm{d}\beta\,, \tag{6.14}
$$

$$
w * (\alpha \wedge \beta) = (w * \alpha) \wedge \beta + \alpha \wedge (w * \beta) + w \otimes_H (\alpha \wedge \beta)\,. \tag{6.15}
$$

Proof. Since d_0 is an odd derivation of the wedge product, by subtracting (2.5) from (6.14), we obtain that (6.14) is equivalent to:

$$(\mathrm{d}-\mathrm{d}_0)(\alpha\wedge\beta)=(-1)^n\alpha\wedge(\mathrm{d}-\mathrm{d}_0)\beta\,.$$

On the other hand, comparing (6.14) with (2.5) and (2.3), we see that

$$(\mathrm{d}-\mathrm{d}_0)\alpha=-\epsilon\wedge\alpha\,, \tag{6.16}$$

where ϵ is defined by (6.10). Then (6.14) follows from the associativity and graded-commutativity of the wedge product (see (2.4)).

By H-linearity, it is enough to prove (6.15) in the case $w=1\otimes\partial_i$. Then by (5.3) we have

$$(1\otimes\partial_i)*\alpha-(1\otimes\partial_i)\otimes_H\alpha=(1\otimes1)\otimes_H(\operatorname{ad}\partial_i)\cdot\alpha+\sum_{j=0}^{2N}(\partial_j\otimes1)\otimes_H e_i^j\cdot\alpha\,.$$

Using that $(\operatorname{ad}\partial_i)$ and e_i^j are even derivations of the wedge product (see (2.8)) completes the proof. ☐

6.3. **Contact pseudo de Rham complex.** As before, let $\Omega^n(\mathfrak{d})=H\otimes\Omega^n$, $\Omega(\mathfrak{d})=\bigoplus_{n=0}^{2N+1}\Omega^n(\mathfrak{d})$ be the spaces of pseudoforms. We extend the operators Θ and Ψ defined in Section 6.1 to $\Omega(\mathfrak{d})$ by H-linearity. We also set $I^n(\mathfrak{d})=H\otimes I^n$ and $K^n(\mathfrak{d})=H\otimes K^n$. From (6.14) and $\omega=\mathrm{d}_0\theta$, we deduce:

$$\mathrm{d}\Psi=\Psi\mathrm{d},\quad \mathrm{d}\Theta=\Psi-\Theta\mathrm{d}\,, \tag{6.17}$$

where d is given by (6.9). This implies that $\mathrm{d}I^n(\mathfrak{d})\subset I^{n+1}(\mathfrak{d})$ and $\mathrm{d}K^n(\mathfrak{d})\subset K^{n+1}(\mathfrak{d})$. Therefore, we have the induced complexes

$$0\to\Omega^0(\mathfrak{d})/I^0(\mathfrak{d})\xrightarrow{\mathrm{d}}\Omega^1(\mathfrak{d})/I^1(\mathfrak{d})\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}\Omega^N(\mathfrak{d})/I^N(\mathfrak{d}) \tag{6.18}$$

and

$$K^{N+1}(\mathfrak{d})\xrightarrow{\mathrm{d}}K^{N+2}(\mathfrak{d})\xrightarrow{\mathrm{d}}\cdots\xrightarrow{\mathrm{d}}K^{2N+1}(\mathfrak{d}). \tag{6.19}$$

Lemma 6.4 (cf. [Ru]). *The sequences* (6.18) *and* (6.19) *are exact.*

Proof. First, to show exactness at the term $\Omega^n(\mathfrak{d})/I^n(\mathfrak{d})$ in (6.18) for $n\le N-1$, take $\alpha\in\Omega^n(\mathfrak{d})$ such that $\mathrm{d}\alpha\in I^{n+1}(\mathfrak{d})$. This means $\mathrm{d}\alpha=\Theta\beta+\Psi\gamma$ for some $\beta\in\Omega^n(\mathfrak{d})$, $\gamma\in\Omega^{n-1}(\mathfrak{d})$. Then $\mathrm{d}(\alpha-\Theta\gamma)=\mathrm{d}\alpha-\Psi\gamma+\Theta\mathrm{d}\gamma=\Theta(\beta+\mathrm{d}\gamma)$; hence, by changing the representative α mod $I^n(\mathfrak{d})$, we can assume that $\gamma=0$. Now we have $0=\mathrm{d}^2\alpha=\mathrm{d}\Theta\beta=\Psi\beta-\Theta\mathrm{d}\beta$. Then $\Psi(\Theta\beta)=0$, i.e., $\Theta\beta\in K^{n+1}(\mathfrak{d})$. But $K^{n+1}(\mathfrak{d})=0$ for $n\le N-1$; thus $\Theta\beta=0$ and $\mathrm{d}\alpha=0$. It follows that $\alpha=\mathrm{d}\rho$ for some $\rho\in\Omega^{n-1}(\mathfrak{d})$.

To prove exactness at the term $K^n(\mathfrak{d})$ in (6.19) for $n\ge N+2$, take $\alpha\in K^n(\mathfrak{d})$ such that $\mathrm{d}\alpha=0$. Then $\alpha=\mathrm{d}\beta$ for some $\beta\in\Omega^{n-1}(\mathfrak{d})$. Since $I^{n-1}(\mathfrak{d})=\Omega^{n-1}(\mathfrak{d})$ for $n\ge N+2$, we can write $\beta=\Theta\gamma+\Psi\rho$ for some $\gamma\in\Omega^{n-2}(\mathfrak{d})$, $\rho\in\Omega^{n-3}(\mathfrak{d})$. But since $\mathrm{d}(\Psi\rho)=\mathrm{d}(\Theta\mathrm{d}\rho)$, by replacing γ with $\gamma+\mathrm{d}\rho$, we can assume that $\rho=0$. Then $\mathrm{d}\beta=-\Theta\mathrm{d}\gamma+\Psi\gamma$ and $\Theta\alpha=0$ implies $\Theta\Psi\gamma=0$. Therefore, $\beta=\Theta\gamma\in K^{n-1}(\mathfrak{d})$, which completes the proof. ☐

Now, following [Ru], we will construct a map $\mathrm{d}^{\mathrm{R}}\colon\Omega^N(\mathfrak{d})/I^N(\mathfrak{d})\to K^{N+1}(\mathfrak{d})$ that connects the complexes (6.18) and (6.19), which we will call the *Rumin map.* Since $I^{N+1}(\mathfrak{d})=\Omega^{N+1}(\mathfrak{d})$, for every $\alpha\in\Omega^N(\mathfrak{d})$ we can write $\mathrm{d}\alpha=\Theta\beta+\Psi\gamma$ for some $\beta \in \Omega^N(\mathfrak{d})$, $\gamma \in \Omega^{N-1}(\mathfrak{d})$. Then, as in the proof of Lemma 6.4, we have $\mathrm{d}\widetilde{\alpha} = \Theta(\beta + \mathrm{d}\gamma) \in K^{N+1}(\mathfrak{d})$ for $\widetilde{\alpha} = \alpha - \Theta\gamma$. We let $\mathrm{d}^{\mathrm{R}}\alpha = \mathrm{d}\widetilde{\alpha}$.

We have to prove that $\mathrm{d}^{\mathrm{R}}\alpha$ is independent of the choice of $\widetilde{\alpha}$ and depends only on the class of α mod $I^N(\mathfrak{d})$. First, if $\mathrm{d}\alpha = \Theta\beta+\Psi\gamma = \Theta\beta'+\Psi\gamma'$, then $\Theta\Psi(\gamma-\gamma') = 0$, which implies $\Theta(\gamma - \gamma') \in K^N(\mathfrak{d})$. But $K^N(\mathfrak{d}) = 0$; hence, $\widetilde{\alpha} = \alpha - \Theta\gamma = \alpha - \Theta\gamma'$. Next, consider the case when $\alpha \in I^N(\mathfrak{d})$. Write $\alpha = \Theta\mu + \Psi\rho$; then $\mathrm{d}\alpha = \Theta(-\mathrm{d}\mu) + \Psi(\mu + \mathrm{d}\rho)$ and $\mathrm{d}^{\mathrm{R}}\alpha = \Theta((-\mathrm{d}\mu) + \mathrm{d}(\mu + \mathrm{d}\rho)) = 0$, as desired.

Using the Rumin map d^{R}, we can combine the two complexes (6.18) and (6.19).

Proposition 6.1 (cf. [Ru])**.** *The sequence*

$$0 \to \Omega^0(\mathfrak{d})/I^0(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^N(\mathfrak{d})/I^N(\mathfrak{d}) \xrightarrow{\mathrm{d}^{\mathrm{R}}} K^{N+1}(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} K^{2N+1}(\mathfrak{d})$$

is an exact complex.

Proof. In the preceding discussion we have shown that d^{R} is well defined. Next, it is clear by construction that $\mathrm{d}^{\mathrm{R}}\mathrm{d} = 0$ and $\mathrm{d}\mathrm{d}^{\mathrm{R}} = 0$. Due to Lemma 6.4, it remains only to check exactness at the terms $\Omega^N(\mathfrak{d})/I^N(\mathfrak{d})$ and $K^{N+1}(\mathfrak{d})$.

First, let $\alpha \in \Omega^N(\mathfrak{d})$ be such that $\mathrm{d}^{\mathrm{R}}\alpha = 0$. Then $\mathrm{d}\widetilde{\alpha} = \mathrm{d}^{\mathrm{R}}\alpha = 0$; hence $\widetilde{\alpha} = \mathrm{d}\beta$ for some $\beta \in \Omega^{N-1}(\mathfrak{d})$. Then $\alpha + I^N(\mathfrak{d}) = \widetilde{\alpha} + I^N(\mathfrak{d}) = \mathrm{d}(\beta + I^{N-1}(\mathfrak{d}))$.

Now let $\alpha \in K^{N+1}(\mathfrak{d})$ be such that $\mathrm{d}\alpha = 0$. Then $\alpha = \mathrm{d}\beta$ for some $\beta \in \Omega^N(\mathfrak{d})$. Since $\mathrm{d}\beta \in K^{N+1}(\mathfrak{d})$, we can take $\widetilde{\beta} = \beta$, and $\mathrm{d}^{\mathrm{R}}\beta = \mathrm{d}\widetilde{\beta} = \alpha$. □

We will call the complex from Proposition 6.1 the *contact pseudo de Rham complex.*

6.4. **$K(\mathfrak{d},\theta)$-action on the contact pseudo de Rham complex.** Here we prove that the contact pseudo de Rham complex is a complex of $K(\mathfrak{d},\theta)$-modules, and we realize its members as tensor modules.

First, we show that the members of the Rumin complex (6.7) are $\mathfrak{csp}\,\bar{\mathfrak{d}}$-modules. Recall that the Lie algebra $\mathfrak{gl}\,\mathfrak{d}$ acts on the space Ω^n of constant coefficient n-forms via (2.7), and this action is by even derivations (see (2.8)).

Lemma 6.5. *For every n, we have: $\mathfrak{csp}\,\bar{\mathfrak{d}} \cdot I^n \subset I^n$ and $\mathfrak{csp}\,\bar{\mathfrak{d}} \cdot K^n \subset K^n$. In addition, $\mathfrak{c}_0 \cdot \Omega^n \subset I^n$ and $\mathfrak{c}_0 \cdot K^n = \{0\}$. Hence the $\mathfrak{gl}\,\mathfrak{d}$-action on Ω^n induces actions of $\mathfrak{csp}\,\bar{\mathfrak{d}}$ on Ω^n/I^n and K^n, and the trivial action of $\mathfrak{c}_0$ on them.*

Proof. By Lemma 2.2, $A \cdot \alpha = c\alpha$ for $A \in \mathfrak{csp}\,\bar{\mathfrak{d}}$, $\alpha \in \{\theta, \omega\}$ and some $c \in \mathbb{C}$. Then by (2.8), $A \cdot (\alpha \wedge \beta) = \alpha \wedge (c\beta + A \cdot \beta)$ for all $\beta \in \Omega$. This implies $A \cdot I^n \subset I^n$ and $A \cdot K^n \subset K^n$.

Next, recall that $\mathfrak{c}_0 = \operatorname{span}\{e^0_k\}_{k\neq0}$ and $e^0_k \cdot x^i = -\delta^i_k x^0 = \delta^i_k\theta$. Then

$$e^0_k \cdot (x^{i_1} \wedge \cdots \wedge x^{i_n}) = \theta \wedge x^{i_2} \wedge \cdots \wedge x^{i_n} \,, \qquad \text{if} \quad k = i_1 \,,$$

and is zero if $k \neq i_s$ for all s. Therefore, $\mathfrak{c}_0 \cdot \Omega^n \subset \Theta\Omega^{n-1} \subset I^n$.

Now, if $\alpha \in K^n$, by (6.6) we can write $\alpha = \theta \wedge \beta$ for some $\beta \in \Omega^{n-1}$. Then for $k \neq 0$ we have $e^0_k \cdot \beta = \theta \wedge \gamma$ for some $\gamma \in \Omega^{n-2}$, and we find

$$e^0_k \cdot \alpha = e^0_k \cdot (\theta \wedge \beta) = \theta \wedge (e^0_k \cdot \beta) = \theta \wedge (\theta \wedge \gamma) = 0 \,,$$

using that $e^0_k \cdot \theta = 0$. □

Lemma 6.6. *We have isomorphisms of $\mathfrak{csp}\,\bar{\mathfrak{d}}$-modules*

$$\Omega^n/I^n \simeq (R(\pi_n), -n) \,, \quad K^{2N+1-n} \simeq (R(\pi_n), -2N-2+n) \,, \qquad 0 \leq n \leq N \,.$$

Proof. Recall that we have isomorphisms of $\mathfrak{sp}\,\bar{\mathfrak{d}}$-modules $\Omega^n/I^n \simeq \bar{\Omega}^n/\bar{I}^n$ and $K^n \simeq \bar{K}^{n-1}$ (see (6.6)). The $\mathfrak{sp}\,\bar{\mathfrak{d}}$-action on these modules is described in Lemma 6.2. Finally, to determine the action of I', we use (2.8), (2.24) and (6.6). We obtain that I' acts as $-n$ on $\bar{\Omega}^n \subset \Omega^n$ and as $-n-1$ on K^n. □

Here is the main result of this section.

Theorem 6.1. *The contact pseudo de Rham complex*

$$0 \to \Omega^0(\mathfrak{d})/I^0(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^N(\mathfrak{d})/I^N(\mathfrak{d}) \xrightarrow{\mathrm{d^R}} K^{N+1}(\mathfrak{d}) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} K^{2N+1}(\mathfrak{d})$$

is an exact complex of $K(\mathfrak{d},\theta)$*-modules. Its members are tensor modules, namely*

$$\Omega^n(\mathfrak{d})/I^n(\mathfrak{d}) = \mathcal{T}(\mathbf{k}, \Omega^n/I^n) = \mathcal{T}(\mathbf{k}, R(\pi_n), -n)$$

and

$$K^n(\mathfrak{d}) = \mathcal{T}(\mathbf{k}, K^n) = \mathcal{T}(\mathbf{k}, R(\pi_{2N+1-n}), -n-1).$$

Proof. Recall that all $\Omega^n(\mathfrak{d}) = \mathcal{T}(\mathbf{k}, \Omega^n)$ are tensor modules for $W(\mathfrak{d})$; see Section 5.1 and [BDK1]. In particular, $e * (1 \otimes \alpha)$ is given by Remark 5.2 for $\alpha \in \Omega^n$. By Lemma 6.5, $\mathfrak{c}_0$ acts trivially on K^n and K^n is a $\mathfrak{csp}\,\bar{\mathfrak{d}}$-module. Therefore, for $\alpha \in K^n$, the action $e * (1 \otimes \alpha)$ is given by (5.9). By definition, this means that $K^n(\mathfrak{d}) = H \otimes K^n = \mathcal{T}(\mathbf{k}, K^n)$ has the structure of a tensor $K(\mathfrak{d},\theta)$-module. The same argument applies to the quotient $\Omega^n(\mathfrak{d})/I^n(\mathfrak{d}) = \mathcal{T}(\mathbf{k}, \Omega^n/I^n)$.

The exactness of the complex was established in Proposition 6.1. It remains to prove that the maps of the complex are homomorphisms of $K(\mathfrak{d},\theta)$-modules. For d, this follows by construction from the fact that $\mathrm{d}\colon \Omega^n(\mathfrak{d}) \to \Omega^{n+1}(\mathfrak{d})$ is a homomorphism of $W(\mathfrak{d})$-modules. In order to prove it for $\mathrm{d^R}$, we need the next lemma, which can be deduced from Remark 5.2 and Lemma 2.2.

Lemma 6.7. *Identifying* $\alpha \in \Omega^n$ *with* $1 \otimes \alpha \in \Omega^n(\mathfrak{d}) = H \otimes \Omega^n$*, we have*:

$$e * \theta = -(e + \partial_0 \otimes 1) \otimes_H \theta\,, \tag{6.20}$$

$$e * \omega = -(e + \partial_0 \otimes 1) \otimes_H \omega - \sum_{i=1}^{2N} (\partial_i \partial_0 \otimes 1) \otimes_H (\theta \wedge x^i)\,. \tag{6.21}$$

Now take an $\alpha \in \Omega^N(\mathfrak{d})$ and write $\mathrm{d}\alpha = \Theta\beta + \Psi\gamma = \theta \wedge \beta + \omega \wedge \gamma$. Then, by definition, $\mathrm{d^R}\alpha = \mathrm{d}(\alpha - \theta \wedge \gamma)$. Using that d is a homomorphism (see (6.13)), we obtain

$$e * (\mathrm{d^R}\alpha) = \bigl((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d}\bigr)\bigl(e * \alpha - e * (\theta \wedge \gamma)\bigr)\,.$$

Then we find from (6.15) and (6.20) that

$$e * (\theta \wedge \gamma) = \theta \wedge \gamma'\,, \qquad \gamma' = e * \gamma - (\partial_0 \otimes 1) \otimes_H \gamma\,.$$

On the other hand, using again (6.15), (6.20) and (6.21), we compute

$$((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d})(e * \alpha) = e * (\mathrm{d}\alpha) = e * (\theta \wedge \beta) + e * (\omega \wedge \gamma) = \theta \wedge \beta' + \omega \wedge \gamma'$$

for some β', where γ' is as above. Then

$$((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d^R})(e * \alpha) = \bigl((\mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{d}\bigr)\bigl(e * \alpha - \theta \wedge \gamma'\bigr)\,,$$

which coincides with $e * (\mathrm{d^R}\alpha)$. This completes the proof of the theorem. □

6.5. **Twisted contact pseudo de Rham complex.** For any choice of a finite-dimensional $\mathfrak{d}$-module Π, one may apply the twisting functor T_Π from Section 3.3 to Theorem 6.1 and obtain a corresponding exact complex of $K(\mathfrak{d},\theta)$-modules

$$0 \to \mathcal{T}(\Pi,\mathbf{k},0) \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi,R(\pi_1),-1) \xrightarrow{\mathrm{d}_\Pi} \cdots \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi,R(\pi_N),-N) \xrightarrow{\mathrm{d}^{\mathrm{R}}_\Pi}$$
$$\mathcal{T}(\Pi,R(\pi_N),-N-2) \xrightarrow{\mathrm{d}_\Pi} \cdots \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi,R(\pi_1),-2N-1) \xrightarrow{\mathrm{d}_\Pi} \mathcal{T}(\Pi,\mathbf{k},-2N-2),$$

where we used the notation $\mathrm{d}_\Pi = T_\Pi(\mathrm{d})$ and $\mathrm{d}^{\mathrm{R}}_\Pi = T_\Pi(\mathrm{d}^{\mathrm{R}})$. In the rest of the paper, we will suppress the reference to Π, and write d instead of d_Π and d^{R} instead of $\mathrm{d}^{\mathrm{R}}_\Pi$ whenever there is no possibility of confusion. If we set

$$\begin{aligned} \mathcal{V}^\Pi_p &= \mathcal{V}(\Pi,R(\pi_p),p) = \mathcal{T}(\Pi\otimes\mathbf{k}_{\mathrm{tr\,ad}},R(\pi_p),p-2N-2) \\ \mathcal{V}^\Pi_{2N+2-p} &= \mathcal{V}(\Pi,R(\pi_p),2N+2-p) = \mathcal{T}(\Pi\otimes\mathbf{k}_{\mathrm{tr\,ad}},R(\pi_p),-p), \end{aligned} \tag{6.22}$$

for $0\le p\le N$, where $R(\pi_0)=\mathbf{k}$ denotes the trivial representation of $\mathfrak{sp}\,\bar{\mathfrak{d}}$, then we obtain an exact sequence of $K(\mathfrak{d},\theta)$-modules

$$0 \to \mathcal{V}^\Pi_{2N+2} \xrightarrow{\mathrm{d}} \mathcal{V}^\Pi_{2N+1} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{V}^\Pi_{N+2} \xrightarrow{\mathrm{d}^{\mathrm{R}}} \mathcal{V}^\Pi_N \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{V}^\Pi_1 \xrightarrow{\mathrm{d}} \mathcal{V}^\Pi_0. \tag{6.23}$$

The above exact complex will be useful in the study of reducible tensor modules and in the computation of their singular vectors. We will be using notation (6.22) throughout the rest of the paper. Notice that $\mathcal{V}^\Pi_{N+1}$ is not defined.

7. Irreducibility of Tensor Modules

We will investigate submodules of tensor modules, and prove a criterion for irreducibility of tensor modules. Throughout the section, R will be an irreducible $(\mathfrak{d}\oplus\mathfrak{csp}\,\bar{\mathfrak{d}})$-module with an action denoted ρ_R, and $\mathcal{V}(R)$ the corresponding tensor module.

7.1. **Coefficients of elements and submodules.** Note that every element $v\in\mathcal{V}(R)=H\otimes R$ can be written uniquely in the form

$$v = \sum_{I\in\mathbb{Z}_+^{2N+1}} \partial^{(I)}\otimes v_I\,, \qquad v_I\in R\,. \tag{7.1}$$

Definition 7.1. The nonzero elements v_I in (7.1) are called *coefficients* of $v\in\mathcal{V}(R)$. For a submodule $M\subset\mathcal{V}(R)$, we denote by $\operatorname{coeff} M$ the subspace of R linearly spanned by all coefficients of elements from M.

It will be convenient to introduce the notation

$$\psi(u) = \sum_{i,j=1}^{2N} \partial_i\partial_j\otimes\rho_R(f^{ij})u\,, \qquad u\in R\,. \tag{7.2}$$

Lemma 7.1. *If $v\in\mathcal{V}(R)$ is given by (7.1), then*

$$\begin{aligned} e*v &= \sum_I (1\otimes\partial^{(I)})\otimes_H\psi(v_I) \\ &\quad + \text{ terms in } (\mathbf{k}\otimes\partial^{(I)}H)\otimes_H(\mathrm{F}^1 H\otimes(\mathbf{k}+\rho_R(\mathfrak{sp}\,\bar{\mathfrak{d}}+\mathfrak{d}))\cdot v_I)\,. \end{aligned} \tag{7.3}$$

In particular, the coefficient multiplying $1\otimes\partial^{(I)}$ equals $\psi(v_I)$ modulo $\mathrm{F}^1\,\mathcal{V}(R)$.

Proof. We rewrite (5.12) using the fact that

$$(\partial_i \otimes 1) \otimes_H v = (1 \otimes 1) \otimes_H \partial_i v - (1 \otimes \partial_i) \otimes_H v \tag{7.4}$$

for any $v \in \mathcal{V}(R)$. We obtain:

$$\begin{aligned} e * (1 \otimes u) &= (1 \otimes 1) \otimes_H \Big(\psi(u) - \sum_{k=1}^{2N} \partial_k \otimes \rho_R(\partial^k) u\Big) \\ &\quad + \text{ terms in } (\mathbf{k} \otimes H) \otimes_H \Big(\mathfrak{d} \otimes \big(\mathbf{k} + \rho_R(\mathfrak{sp}\,\bar{\mathfrak{d}})\big) \cdot u + \mathbf{k} \otimes \big(\mathbf{k} + \rho_R(\mathfrak{sp}\,\bar{\mathfrak{d}} + \mathfrak{d})\big) \cdot u\Big). \end{aligned} \tag{7.5}$$

Then plugging in (7.1) and applying H-bilinearity completes the proof. □

Remark 7.1. For $v \in \operatorname{sing} \mathcal{V}(R)$, we have by (5.18)

$$e * v = \sum_{i,j=1}^{2N} (1 \otimes \partial_i \partial_j) \otimes_H \rho_{\mathrm{sing}}(f^{ij}) v + \text{ terms in } (\mathbf{k} \otimes \mathrm{F}^1 H) \otimes_H \mathcal{V}(R). \tag{7.6}$$

Lemma 7.2. *For any nonzero proper $K(\mathfrak{d}, \theta)$-submodule $M \subset \mathcal{V}(R)$, we have* $\operatorname{coeff} M = R$.

Proof. Pick a nonzero element $v = \sum_I \partial^{(I)} \otimes v_I$ contained in M. Then Lemma 7.1 shows that M contains an element congruent to $\psi(v_I)$ modulo $\mathrm{F}^1 \mathcal{V}(R)$, thus coefficients of $\psi(v_I)$ lie in $\operatorname{coeff} M$ for all I. This proves that $\mathfrak{sp}\,\bar{\mathfrak{d}}(\operatorname{coeff} M) \subset \operatorname{coeff} M$. Similarly, one can write

$$\begin{aligned} e * v &= \sum_I (1 \otimes \partial^{(I)}) \otimes_H \Big(\psi(v_I) - \sum_{k=1}^{2N} \partial_k \otimes \rho_R(\partial^k) v_I\Big) \\ &\quad + \text{ terms in } (\mathbf{k} \otimes \partial^{(I)} H) \otimes_H \Big(\mathfrak{d} \otimes \rho_R(\mathfrak{sp}\,\bar{\mathfrak{d}} + \mathbf{k}) v_I + \mathbf{k} \otimes \big(\mathbf{k} + \rho_R(\mathfrak{sp}\,\bar{\mathfrak{d}} + \mathfrak{d})\big) v_I\Big), \end{aligned}$$

showing that $\rho_R(\partial^k) v_I \in \operatorname{coeff} M$ for all I and all $k = 1, \dots, 2N$. Thus, $\bar{\mathfrak{d}}(\operatorname{coeff} M) \subset \operatorname{coeff} M$. However, $\bar{\mathfrak{d}}$ generates $\mathfrak{d}$ as a Lie algebra, hence $\mathfrak{d}$ stabilizes $\operatorname{coeff} M$ as well. Then $\operatorname{coeff} M$ is a nonzero $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-submodule of R. Irreducibility of R now gives that $\operatorname{coeff} M = R$. □

Corollary 7.1. *Let M be a nonzero proper $K(\mathfrak{d}, \theta)$-submodule of $\mathcal{V}(R)$. Then for every $u \in R$ there is an element in M that coincides with $\psi(u)$ modulo $\mathrm{F}^1 \mathcal{V}(R)$.*

Proof. As $\operatorname{coeff} M = R$, it is enough to prove the statement for coefficients of elements $v \in M$. Since M is a $K(\mathfrak{d}, \theta)$-submodule of $\mathcal{V}(R)$, the coefficient multiplying $1 \otimes \partial^{(I)}$ in (7.3) still lies in M and it equals $\psi(v_I)$ modulo $\mathrm{F}^1 \mathcal{V}(R)$. □

7.2. **An irreducibility criterion.** The results of the previous subsection make it possible to prove a sufficient condition for irreducibility of $\mathcal{V}(R)$ when the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-action on R is nontrivial. We first need the following lemma.

Lemma 7.3. *Assume the $K(\mathfrak{d}, \theta)$-tensor module $\mathcal{V}(R)$ contains a nonzero proper submodule. Then the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-action on R satisfies*

$$\sum f^{ab} f^{cd}(u) = 0\,, \qquad u \in R\,, \tag{7.7}$$

for all $1 \le a, b, c, d \le 2N$, where the sum is over all permutations of a, b, c, d.

Proof. Let M be a nonzero proper $K(\mathfrak{d},\theta)$-submodule of $\mathcal{V}(R)$ and $v \in M$ be an element equal to $\psi(u)$ modulo $\mathrm{F}^1\,\mathcal{V}(R)$ (see Corollary 7.1). Let us express $e * v$ in the form $\sum_I (1 \otimes \partial^{(I)}) \otimes_H u_I$ using (5.12) and (7.4). If $|I| > 4$ then $u_I = 0$; moreover if $|I| = 4$ then u_I lies in $\mathbf{k} \otimes R$. By Lemma 5.2 these coefficients must cancel with each other, and they give exactly (7.7). □

Now we can prove the main result of this section.

Theorem 7.1. *If the $K(\mathfrak{d},\theta)$-tensor module $\mathcal{V}(\Pi,U,c)$ is not irreducible, then U is either the trivial representation of $\mathfrak{sp}\,\bar{\mathfrak{d}}$ or is isomorphic to $R(\pi_i)$ for some $i = 1,\dots,N$.*

Proof. Lowering indices in (7.7) gives the following equivalent identity:

$$(f_a^b f_c^d + f_{ac} f^{bd} + f_a^d f_c^b + f_c^b f_a^d + f^{bd} f_{ac} + f_c^d f_a^b) \cdot u = 0,$$

for all $1 \le a,b,c,d \le 2N$ and $u \in R$. Specializing to $a = b = c = d = i$ we obtain:

$$4 f_i^i f_i^i + f_{ii} f^{ii} + f^{ii} f_{ii} = 0\,.$$

Recalling that elements (2.30) form a standard $\mathfrak{sl}_2$-triple, this can be rewritten as

$$h_i^2 - e_i f_i - f_i e_i = 0\,.$$

As h_i^2 is a linear combination of $h_i^2 - (e_i f_i + f_i e_i) = 0$ and the Casimir element of $\langle e_i, h_i, f_i \rangle \simeq \mathfrak{sl}_2$, it acts on any irreducible $\mathfrak{sl}_2$-submodule $W \subset U$ as a scalar, which forces h_i^2 to be equal either 0 or 1. Then the eigenvalue of the action of $h_i = -2 f_i^i$ on a highest weight vector in U is also either 0 or 1.

When the basis $\partial_1,\dots,\partial_{2N}$ of $\bar{\mathfrak{d}}$ is symplectic with respect to ω, elements $h_i = e_i^i - e_{N+i}^{N+i}$ form a basis of the diagonal Cartan subalgebra of $\mathfrak{sp}\,\bar{\mathfrak{d}}$ (see (2.32)). Let $U = R(\lambda)$ be an irreducible representation of $\mathfrak{sp}\,\bar{\mathfrak{d}}$ with highest weight $\lambda = \sum_i \lambda_i \pi_i$. For the standard choice of simple roots, the eigenvalue of $h_1 = e_1^1 - e_{N+1}^{N+1}$ on the highest weight vector λ is $\sum_i \lambda_i \le 1$ (cf. [FH, Lecture 16]). Since λ_i are non-negative integers, λ must be 0 or one of the fundamental weights π_i. □

Remark 7.2. The module $\mathcal{V}(\Pi,U,c)$ is always irreducible if the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-module U is infinite-dimensional irreducible. In order to show this, it suffices to prove that the factor of $U(\mathfrak{sp}\,\bar{\mathfrak{d}})$ by the ideal generated by relations (7.7) is finite-dimensional. It is enough to prove this for the associated graded algebra $S(\mathfrak{sp}\,\bar{\mathfrak{d}})$: letting $a = b = c = d$ in (7.7), we get $(f^{aa})^2 = 0$; then letting in (7.7) $a = b$, $c = d$, we get $(f^{ab})^2 = -4 f^{aa} f^{bb}$, hence $(f^{ab})^4 = 0$ for all a, b, proving the claim (we are grateful to C. De Concini for this argument). Similarly, the tensor modules for Lie pseudoalgebras of W and S types in [BDK1] are irreducible if the corresponding modules U are irreducible infinite-dimensional.

We are left with investigating irreducibility of all tensor modules for which U is isomorphic to some $R(\pi_i)$ or to the trivial representation $R(\pi_0) = \mathbf{k}$. We will do so by explicitly constructing all singular vectors contained in nonzero proper submodules of $\mathcal{V}(\Pi,U,c)$, and thus determining conditions on the scalar value c of the action of I'. A central tool for the classification of singular vectors is the following proposition, which enables us to bound the degree of singular vectors.

Proposition 7.1. *Let $v \in \mathcal{V}(R)$ be a singular vector contained in a nonzero proper $K(\mathfrak{d},\theta)$-submodule M, and assume that the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-action on R is nontrivial. Then v*

is of degree at most two in the contact filtration, i.e., it is of the form

$$v = \sum_{i,j=1}^{2N} \partial_i \partial_j \otimes v_{ij} + \sum_{k=0}^{2N} \partial_k \otimes v_k + 1 \otimes \widetilde{v}\,. \tag{7.8}$$

Proof. Write $v = \sum_I \partial^{(I)} \otimes v_I$. Then Lemma 7.1, together with (7.6), shows that $\psi(v_I) = 0$ whenever $|I|' \geq 2$. As the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-action on R is nontrivial, $\psi(v_I) = 0$ implies $v_I = 0$. □

Our next goal is to characterize singular vectors of degree at most two in all modules that do not satisfy the irreducibility criterion given in Theorem 7.1, and thus to obtain a classification of reducible tensor modules.

8. Computation of Singular Vectors

In this section, we will be concerned with tensor modules of the form $\mathcal{V}(\Pi, U, c)$, where Π is an irreducible finite-dimensional representation of $\mathfrak{d}$, and U is either the trivial $\mathfrak{sp}\,\bar{\mathfrak{d}}$-module or one the fundamental representations. Our final result states that such a tensor module contains singular vectors if and only if it shows up in a twist of the contact pseudo de Rham complex, and that in such cases singular vectors may be described in terms of the differentials.

8.1. **Singular vectors in $\mathcal{V}(\Pi, \mathbf{k}, c)$.** Here we treat separately the case $U \simeq \mathbf{k}$. Since the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-action is trivial, now (5.12) can be rewritten as

$$\begin{aligned} e * (1 \otimes u) = &\sum_{k=1}^{2N} (\partial_k \otimes 1) \otimes_H \big(\partial^k \otimes u - 1 \otimes \rho_R(\partial^k) u\big) \\ &+ (\partial_0 \otimes 1) \otimes_H (1 \otimes cu/2) - (1 \otimes 1) \otimes_H \big(\partial_0 \otimes u - 1 \otimes \rho_R(\partial_0) u\big)\,. \end{aligned} \tag{8.1}$$

Using (5.13), we can also write

$$\begin{aligned} e * (1 \otimes u) = &-\sum_{k=1}^{2N} (1 \otimes \partial_k) \otimes_H \big(\partial^k \otimes u - 1 \otimes \rho_R(\partial^k) u\big) \\ &- (1 \otimes \partial_0) \otimes_H (1 \otimes cu/2) + \text{ terms in } (\mathbf{k} \otimes \mathbf{k}) \otimes_H \mathrm{F}^1\, \mathcal{V}(R)\,. \end{aligned} \tag{8.2}$$

Proposition 8.1. *We have*:

(i) $\operatorname{sing} \mathcal{V}(\Pi, \mathbf{k}, c) = \mathrm{F}^0\, \mathcal{V}(\Pi, \mathbf{k}, c)$ *for* $c \neq 0$;

(ii) $\operatorname{sing} \mathcal{V}(\Pi, \mathbf{k}, 0) = \mathrm{F}'^1\, \mathcal{V}(\Pi, \mathbf{k}, 0)$;

(iii) $\mathcal{V}(R) = \mathcal{V}(\Pi, \mathbf{k}, c)$ *is irreducible if and only if* $c \neq 0$.

Proof. (i) Let $v = \sum_I \partial^{(I)} \otimes v_I \in \mathcal{V}(R)$ be a singular vector, and assume that $v_I \neq 0$ for some I with $|I| > 0$. If n is the maximal value of $|I|$ for such I, choose among all $I = (i_0, i_1, \dots, i_{2N})$ with $|I| = n$ one with largest possible i_0. If we use (8.2) to compute $e * v$ and express the result in the form

$$\sum_J (1 \otimes \partial^{(J)}) \otimes_H u_J\,, \qquad u_J \in \mathcal{V}(R)\,, \tag{8.3}$$

then the coefficient multiplying $1 \otimes \partial^{(I+\varepsilon_0)}$ equals $-(i_0+1) c v_I/2$. Since v is singular, this must vanish if $|I| > 0$, and $c \neq 0$ gives a contradiction with $v_I \neq 0$.

(ii) In the same way as in part (i), we show that $\operatorname{sing} \mathcal{V}(\Pi, \mathbf{k}, 0) \subset \mathrm{F}^1\, \mathcal{V}(\Pi, \mathbf{k}, 0)$. Indeed, computing the coefficient multiplying $1 \otimes \partial^{(I+\varepsilon_k)}$, we see that $|I| > 1$ implies

$v_I = 0$. Now, constant vectors are clearly singular, and using $u = \partial_i \otimes v_i$ $(i \neq 0)$ in (8.1) easily shows u to be singular for all choices of $v_i \in R$. We are left with showing that $\partial_0 \otimes v_0$ $(v_0 \neq 0)$ is never a singular vector. Once again, substituting this in (8.1) and expressing the result as in (8.3) gives nonzero terms multiplying $\partial_0\partial_k \otimes 1$ for all $k \neq 0$.

(iii) If $c \neq 0$, then $\mathcal{V}(\Pi, \mathbf{k}, c)$ has no nonconstant singular vectors, hence it is irreducible by Corollary 5.4. As far as $\mathcal{V}(\Pi, \mathbf{k}, 0)$ is concerned, direct inspection of (8.1) shows that elements $\partial \otimes u - 1 \otimes \rho_R(\partial)u$ $(\partial \in \mathfrak{d}, u \in R)$ generate over H a proper $K(\mathfrak{d}, \theta)$-submodule of $\mathcal{V}(\Pi, \mathbf{k}, 0)$. □

Corollary 8.1. *We have* $\operatorname{sing} \mathcal{V}_0^\Pi = \mathrm{F}^0\, \mathcal{V}_0^\Pi + \mathrm{d}\, \mathrm{F}^0\, \mathcal{V}_1^\Pi$.

Proof. The $(\mathfrak{d} \oplus \mathfrak{csp}\, \bar{\mathfrak{d}})$-submodule $\mathrm{d}\, \mathrm{F}^0\, \mathcal{V}_1^\Pi \subset \operatorname{sing} \mathcal{V}_0^\Pi$ contains nonconstant elements, so it has a nonzero projection to $\mathrm{gr}'^1\, \mathcal{V}_1^\Pi$. However, Corollary 5.5 shows that this is isomorphic to $\Pi \boxtimes \bar{\mathfrak{d}}$, whence it is irreducible. □

We will now separately classify singular vectors of degree one and two in all other cases.

8.2. **Classification of singular vectors of degree one.** Our setting is the following: $V = \mathcal{V}(R) = H \otimes R$ is a reducible $K(\mathfrak{d}, \theta)$ tensor module, R is isomorphic to $\Pi \boxtimes U$ as a $(\mathfrak{d} \oplus \mathfrak{csp}\, \bar{\mathfrak{d}})$-module, where both Π and U are irreducible, and $U = R(\pi_n)$ as an $\mathfrak{sp}\, \bar{\mathfrak{d}}$-module for some $1 \leq n \leq N$. We are also given a nonzero proper submodule $M \subset V$. Note that by assumption U is not the trivial $\mathfrak{sp}\, \bar{\mathfrak{d}}$-representation. We look for singular vectors of degree one, i.e., of the form

$$v = \sum_{i=0}^{2N} \partial_i \otimes v_i + 1 \otimes \widetilde{v}\,, \tag{8.4}$$

which are contained in M. Note that every such singular vector is uniquely determined by its degree one part. Indeed, if v and v' are two such vectors agreeing in degree one, then $v - v'$ is a singular vector contained in $M \cap (\mathbf{k} \otimes R) = \{0\}$ (see Lemma 5.2).

Lemma 8.1. *If $v \in M$ is a singular vector written as in* (8.4)*, then $v_0 = 0$.*

Proof. Compute $e * v$ using (5.12). Then if we write $e * v = \sum_I (\partial^{(I)} \otimes 1) \otimes_H v_I$, the coefficient multiplying $\partial_0\partial_i\partial_j \otimes 1$ for $i \leq j$ is $\rho_R(f^{ij})v_0 \in M \cap (\mathbf{k} \otimes R)$, up to a nonzero multiplicative constant. Hence $\rho_R(f^{ij})v_0 = 0$ for all i, j, which implies $v_0 = 0$ as the $\mathfrak{sp}\, \bar{\mathfrak{d}}$-action is nontrivial. □

Proposition 8.2. *Let v, v' be nonzero singular vectors of degree one contained in nonzero proper submodules M, M' of a tensor module $\mathcal{V}(R) = \mathcal{V}(\Pi, U, c)$, as above. If $v = v' \mod \mathrm{F}^0\, \mathcal{V}(R)$, then $v = v'$.*

Proof. By Lemma 8.1 and Corollary 5.5, I' acts on $(\operatorname{sing} \mathcal{V}(R) \cap \mathrm{F}^1\, \mathcal{V}(R)) / \mathrm{F}^0\, \mathcal{V}(R)$ via multiplication by $c + 1$, and it obviously acts on $\mathrm{F}^0\, \mathcal{V}(R)$ via multiplication by c. Then $\operatorname{sing} \mathcal{V}(R) \cap \mathrm{F}^1\, \mathcal{V}(R)$ is isomorphic to the direct sum of the $c + 1$ and c eigenspaces with respect to I'.

Any $\widetilde{\mathcal{K}}$-submodule of V is in particular stable under the action of I', so it contains the I'-eigenspace components of all of its singular vectors. However, a nonzero proper submodule M cannot contain constant singular vectors. Thus, singular vectors must lie in the $c{+}1$-eigenspace, and their constant coefficient part is determined by their degree one part, independently on the choice of the submodule M. □

So far, we have showed that singular vectors of degree one also have degree one in the contact filtration, and that those contained in a nonzero submodule must be homogeneous (i.e. eigenvectors) with respect to the action of I'. Notice that since all constant vectors are singular, a singular vector of degree one stays singular if we alter or suppress its constant part.

Lemma 8.2. *A nonzero element* $v = \sum_{k=1}^{2N} \partial_k \otimes v_k \in H \otimes R$ *is a singular vector in* $\mathcal{V}(R) = \mathcal{V}(\Pi, U, c)$ *for at most one value of* c.

Proof. Compute $e * v = \sum_I (\partial^{(I)} \otimes 1) \otimes_H v_I$ using (5.12). For $k \neq 0$, the coefficient multiplying $\partial_0\partial_k \otimes 1$ equals $-1/2 \otimes cv_k$ plus a linear combination of terms of the form $1 \otimes \rho_R(f^{ij})v_k$ that arise from reordering terms multiplying $\partial_i\partial_j\partial_k \otimes 1$; such terms are however independent of the choice of c. All such coefficients must vanish when v is singular. If this happens for two distinct values of c, we obtain $v_k = 0$ for all k, a contradiction with $v \neq 0$. □

Theorem 8.1. *Assume that the action of* $\mathfrak{sp}\,\bar{\mathfrak{d}}$ *on* U *is nontrivial. If* $V = \mathcal{V}(\Pi, U, c)$ *contains singular vectors of degree one, then* $V = \mathcal{V}_p^\Pi$ *for some* $1 \leq p \leq 2N+1, p \neq N+1$*. More precisely,* $\operatorname{sing} \mathcal{V}_p^\Pi \cap \mathrm{F}^1\, \mathcal{V}_p^\Pi = \mathrm{F}^0\, \mathcal{V}_p^\Pi + \mathrm{d}\,\mathrm{F}^0\, \mathcal{V}_{p+1}^\Pi$.

Proof. By Theorem 7.1, $\mathcal{V}(R) = \mathcal{V}(\Pi, U, c)$ is irreducible unless $U = R(\pi_p)$ for some $1 \leq p \leq N$. Lemma 8.1 and Corollary 5.5 show that singular vectors of degree one in a nonzero proper $K(\mathfrak{d}, \theta)$-submodule M project faithfully to a $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-submodule of $\mathrm{gr}'^1\, \mathcal{V}(R)$ isomorphic to $\Pi \boxtimes (\bar{\mathfrak{d}} \otimes U, c+1)$. We can explicitly decompose $\bar{\mathfrak{d}} \otimes U$ as a direct sum of irreducibles using Lemma 2.4. One has:

$$\bar{\mathfrak{d}} \otimes R(\pi_1) \simeq R(2\pi_1) \oplus \mathbf{k} \oplus R(\pi_2)\,,$$
$$\bar{\mathfrak{d}} \otimes R(\pi_p) \simeq R(\pi_p + \pi_1) \oplus R(\pi_{p-1}) \oplus R(\pi_{p+1}), \qquad \text{if } 1 < p < N\,,$$
$$\bar{\mathfrak{d}} \otimes R(\pi_N) \simeq R(\pi_N + \pi_1) \oplus R(\pi_{N-1})\,.$$

For all values of $1 \leq p \leq N$, the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-module $R(\pi_p + \pi_1)$ satisfies the irreducibility criterion stated in Theorem 7.1, and its dimension is larger than $\dim R(\pi_p)$. We can therefore proceed as in [BDK1, Lemma 7.8] to conclude that no singular vectors will have a nonzero projection to this summand.

However, by the construction of the contact pseudo de Rham complex, the tensor module $\mathcal{V}(\Pi, R(\pi_p), c)$ contains singular vectors projecting to the summand $R(\pi_{p+1})$ when $c = p$ and to the summand $R(\pi_{p-1})$ if $c = 2N+2-p$. Lemma 8.2 shows now that these are the only values of c for which there are singular vectors projecting to such components, whereas Proposition 8.2 implies that those are the only homogeneous singular vectors. □

8.3. **Classification of singular vectors of degree two.** In all of this section, $\mathcal{V}(R) = \mathcal{V}(\Pi, U, c)$ will be a tensor module containing a singular vector v of degree two. Due to Proposition 7.1, we may assume that

$$v = \sum_{i,j=1}^{2N} \partial_i\partial_j \otimes v_{ij} + \sum_{i=0}^{2N} \partial_i \otimes v_i + 1 \otimes \widetilde{v} \tag{8.5}$$

where $v_{ij} = v_{ji}$ for all i, j. We already know by Proposition 5.4 that $\mathcal{V}(R)$ is reducible, hence $U = R(\pi_p)$ for some p by Theorem 7.1 and Proposition 8.1. Our goal is to describe all possible v's, and show that the only tensor modules possessing them is $\mathcal{V}(\Pi, R(\pi_N), N)$. Recall the definition of $\psi(u)$ given by (7.2).

Lemma 8.3. *We have* $f^{\alpha\beta}(v) = \psi(v_{\alpha\beta}) \mod \mathrm{F}^1\, \mathcal{V}(R)$*.*

Proof. Use (7.5) to compute $e * v$ and compare it with (7.6). □

This shows that for some $u \in R$ there exists a singular vector coinciding with $\psi(u)$ modulo $\mathrm{F}^1\, \mathcal{V}(R)$, since if v is a singular vector of degree two, then $v_{\alpha\beta} \neq 0$ for some choice of α, β.

Lemma 8.4. *Let* v, v' *be singular vectors of degree two in* $\mathcal{V}(R)$*, and assume that* $v = v' \mod \mathrm{F}^1\, \mathcal{V}(R)$*. Then* $v_0 = v'_0$*.*

Proof. Apply Lemma 8.1 to the singular vector of degree one $v - v'$. □

Note that since I' acts on singular vectors, the projection operator p_2 of $\mathcal{V}(R) = \mathcal{V}(\Pi, U, c)$ to the $c+2$ eigenspace with respect to I' maps $\operatorname{sing} \mathcal{V}(R) \cap \mathrm{F}^2\, \mathcal{V}(R)$ to itself. If a nonzero proper submodule M of $\mathcal{V}(R)$ contains a singular vector v of degree two, then it also contains $p_2 v$. We will say that $p_2 v$ is a *homogeneous* singular vector of degree two.

Lemma 8.5. *For every* $u \in R$ *there exists a unique homogeneous singular vector*

$$\phi(u) = \psi(u) \mod \mathrm{F}^1\, \mathcal{V}(R)\,. \tag{8.6}$$

Elements $\phi(u)$ *depend linearly on* u *and satisfy*:

$$f^{\alpha\beta}(\phi(u)) = \phi(f^{\alpha\beta}(u))\,, \tag{8.7}$$

$$\widetilde{\partial} \cdot \phi(u) = \phi(\widetilde{\partial} \cdot u)\,. \tag{8.8}$$

Moreover, if v *is a homogeneous singular vector of degree two as in* (8.5)*, then*

$$f^{\alpha\beta}(v) = \phi(v_{\alpha\beta})\,. \tag{8.9}$$

Proof. We know that for some $0 \neq u \in R$ we can find a singular vector v equal to $\psi(u)$ modulo $\mathrm{F}^1\, \mathcal{V}(R)$. Then its projection $p_2 v$ to the $c+2$-eigenspace of I' is still singular and coincides with v up to lower degree terms. If we are able to show that (8.7) and (8.8) hold whenever both sides make sense, then the set of all $u \in R$ for which $\phi(u)$ is defined is a nonzero $(\widetilde{\mathfrak{d}} \oplus \mathfrak{csp}\, \bar{\mathfrak{d}})$-submodule of R, hence all of it by irreducibility.

So, say $\phi(u)$ is an element as above. By Lemma 8.3, we know that $f^{\alpha\beta}(\phi(u))$ coincides with $\psi(f^{\alpha\beta}(u))$ up to lower degree terms. Moreover, as I' commutes with $\mathfrak{sp}\, \bar{\mathfrak{d}}$, the vector $f^{\alpha\beta}(\phi(u))$ is still homogeneous, thus showing (8.7). The proof of (8.9) is completely analogous. Similarly, Lemma 5.4 implies (8.8), as the action of I' commutes with that of $\widetilde{\mathfrak{d}}$. □

Corollary 8.2. *The map* $\phi\colon R \to \operatorname{sing} \mathcal{V}(R)$ *is a well-defined injective* $(\mathfrak{d} \oplus \mathfrak{sp}\, \bar{\mathfrak{d}})$*-homomorphism, and the action of* $\mathfrak{sp}\, \bar{\mathfrak{d}}$ *maps* $p_2 \operatorname{sing} \mathcal{V}(R)$ *to the image of* ϕ*.*

Proof. Since we are assuming that the action of $\mathfrak{sp}\, \bar{\mathfrak{d}}$ on R is nontrivial, the map $\psi\colon R \to \mathcal{V}(R)$ is injective. Then, by (8.6), ϕ is also injective. □

Corollary 8.3. *The space* $p_2 \operatorname{sing} \mathcal{V}(R)$ *does not contain trivial* $\mathfrak{sp}\, \bar{\mathfrak{d}}$*-summands.*

Proof. If $v \in p_2 \operatorname{sing} \mathcal{V}(R)$ lies in a trivial summand, then $0 = f^{\alpha\beta}(v) = \phi(v_{\alpha\beta})$ for all α, β. But ϕ is injective, hence $v_{\alpha\beta} = 0$ for all α, β, a contradiction with v being of degree two. Therefore $v = 0$. □

The above results can be summarized as follows.

Theorem 8.2. *The map $\phi\colon R \to p_2 \operatorname{sing} \mathcal{V}(R)$ is an isomorphism of $(\mathfrak{d} \oplus \mathfrak{sp}\,\bar{\mathfrak{d}})$-modules. The action of I' on $p_2 \operatorname{sing} \mathcal{V}(R)$ is via scalar multiplication by $c+2$. All homogeneous singular vectors of degree two in $\mathcal{V}(R)$ are of the form $\phi(u)$ for $u \in R$.*

A classification of singular vectors of degree two will now follow by computing the action of $e \in K(\mathfrak{d}, \theta)$ on vectors of the form $\phi(u)$. In the computations we will need some identities, which hold in any associative algebra. We will denote by $[x, y] = xy - yx$ the usual commutator and by

$$\{x_1, \dots, x_n\} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)}$$

the complete symmetrization of the product.

Lemma 8.6. *For any elements a, b, c, d in an associative algebra, we have*:

$$abc = \{a, b, c\} + \frac{1}{2}\Big(\{a, [b, c]\} + \{b, [a, c]\} + \{c, [a, b]\}\Big) + \frac{1}{6}\Big([a, [b, c]] + [[a, b], c]\Big),$$

$$\begin{aligned} abcd = \{a, b, c, d\} \\ &+ \frac{1}{2}\Big(\{a, b, [c, d]\} + \{a, c, [b, d]\} + \{a, d, [b, c]\} \\ &\qquad + \{b, c, [a, d]\} + \{b, d, [a, c]\} + \{c, d, [a, b]\}\Big) \\ &+ \frac{1}{4}\Big(\{[a, b], [c, d]\} + \{[a, c], [b, d]\} + \{[a, d], [b, c]\}\Big) \\ &+ \frac{1}{6}\Big(\{a, [b, [c, d]]\} + \{a, [[b, c], d]\} + \{b, [a, [c, d]]\} + \{b, [[a, c], d]\} \\ &\qquad + \{c, [a, [b, d]]\} + \{c, [[a, b], d]\} + \{d, [a, [b, c]]\} + \{d, [[a, b], c]\}\Big) \\ &+ \frac{1}{6}\Big([[[c, d], b], a] - [[[b, d], c], a]\Big) \\ &+ \frac{1}{12}\Big([[a, b], [c, d]] + [[a, c], [b, d]] + [[a, d], [b, c]]\Big)\,. \end{aligned}$$

Proof. It is a lengthy but standard computation. The authors have checked it using Maple. ☐

Now let us write

$$\phi(u) = \psi(u) + \sum_{k=0}^{2N} \partial_k \otimes v_k + 1 \otimes \widetilde{v}\,, \qquad u \in R\,, \tag{8.10}$$

for some $v_k, \widetilde{v} \in R$, which may depend on u.

Lemma 8.7. *If the above vector $\phi(u)$ is singular, then $v_0 = (c/2 - N - 1)u$.*

Proof. We use (5.12) to compute $e * \phi(u) = \sum_I (\partial^{(I)} \otimes 1) \otimes_H v_I$. If $0 < a < b$, then the coefficient multiplying $\partial_0 \partial_a \partial_b \otimes 1$ equals $I' \cdot f^{ab}(u) - 2 f^{ab}(v_0)+$ commutators that are obtained from reordering terms of the form $\partial_i \partial_j \partial_k \partial_l$ in the associative algebra $H = U(\mathfrak{d})$. These can be computed using Lemma 8.6, leading to

$$-2 f^{ab}(v_0) + I' \cdot f^{ab}(u) + 2 \sum_{i,j=1}^{2N} \omega_{ij} [f^{ia}, f^{jb}](u) = f^{ab}\big(I' \cdot u - 2v_0 - (2N+2)u\big)\,,$$

where $\sum_{ij} \omega_{ij}[f^{ai}, f^{bj}] = -(N+1)f^{ab}$ due to (2.28).

Since $\phi(u)$ is a singular vector, this coefficient must vanish for all $a < b$, and a similar computation can be done when $a = b$. Since the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-action on R is nontrivial, we obtain that $I' \cdot u - 2v_0 - (2N+2)u = 0$. To finish the proof, observe that $I' \cdot u = cu$ for all $u \in R$. □

Lemma 8.8. *If $\phi(u)$ is singular, then*

$$cv_0 = \sum_{a,b=1}^{2N} f_{ab} f^{ab}(u)\,. \tag{8.11}$$

Proof. Compute the coefficient multiplying $\partial_0^2 \otimes 1$ as in Lemma 8.7, using Lemma 8.6 in order to explicitly compute commutators arising from terms $\partial_0\partial_i\partial_j$, which cancel, and $\partial_i\partial_j\partial_k\partial_l$. The final result is

$$-\frac{1}{2} I' \cdot v_0 + \frac{1}{2} \sum_{i,j,k,l=1}^{2N} \omega_{ik}\omega_{jl} f^{ij} f^{kl}(u) = -\frac{1}{2} I' \cdot v_0 + \frac{1}{2} \sum_{k,l} f_{kl} f^{kl}(u)\,,$$

which is a constant element, and must therefore vanish. □

Corollary 8.4. *If $\phi(u)$ is singular for $0 \neq u \in R = \Pi \boxtimes (R(\pi_p), c)$, then c equals either $2N+2-p$ or p. In other words, the only tensor modules that may possess singular vectors of degree 2 are of the form $\mathcal{V}_p^{\Pi}$ or $\mathcal{V}_{2N+2-p}^{\Pi}$.*

Proof. Substitute Lemma 8.7 into Lemma 8.8, to obtain

$$\frac{1}{2} c^2 u - (N+1)cu - \sum_{a,b=1}^{2N} f_{ab} f^{ab}(u) = 0\,.$$

Recall by Lemmas 2.3 and 2.4 that $-\sum_{a,b=1}^{2N} f_{ab} f^{ab}$ equals the Casimir element of $\mathfrak{sp}\,\bar{\mathfrak{d}}$ and acts on $R(\pi_p)$ via multiplication by $p(2N+2-p)/2$. Hence we obtain

$$c^2 - (2N+2)c + p(2N+2-p) = 0,$$

whose only solutions are $c = p$ and $c = 2N+2-p$. □

Corollary 8.5. *Let U be a nontrivial irreducible $\mathfrak{sp}\,\bar{\mathfrak{d}}$-module. Then a tensor module $V = \mathcal{V}(\Pi, U, c)$ is reducible if and only if it is of the form $\mathcal{V}_p^{\Pi}$ for some $1 \leq p \leq 2N+1$, $p \neq N+1$.*

Proof. The image of differentials constitute proper submodules of each tensor module showing up in the contact pseudo de Rham complex (6.23). Conversely, Theorem 8.1 and Corollary 8.4 show that there are no other tensor modules possessing nonconstant singular vectors. □

Theorem 8.3. *The only tensor modules over $K(\mathfrak{d}, \theta)$ possessing singular vectors of degree two are those of the form $\mathcal{V}_N^{\Pi}$.*

Proof. If $\mathcal{V}(R) = \mathcal{V}(\Pi, R(\pi_p), c)$ has singular vectors of degree two, then we have a nonzero homomorphism

$$\mathcal{V}(\Pi, R(\pi_p), c+2) \to \mathcal{V}(\Pi, R(\pi_p), c)\,.$$

However, if $\mathcal{V}(\Pi, R(\pi_p), c+2)$ is irreducible, then this map is injective, and its image has the same rank as $\mathcal{V}(R)$. Hence, it is a proper cotorsion submodule $M \simeq \mathcal{V}(\Pi, R(\pi_p), c+2)$ in $\mathcal{V}(R)$, and the action of $K(\mathfrak{d}, \theta)$ on $\mathcal{V}(R)/M$ is trivial by

Remark 3.1. This means that $e * \mathcal{V}(R) \subset (H \otimes H) \otimes_H M$. But a direct inspection of (5.12) shows that $e * \mathcal{V}(R) = (H \otimes H) \otimes_H \mathcal{V}(R)$, which is a contradiction.

We conclude that $\mathcal{V}(\Pi, R(\pi_p), c+2)$ and $\mathcal{V}(\Pi, R(\pi_p), c)$ are both reducible. By Corollary 8.5 and (6.22), c and $c+2$ must add up to $2N+2$. Hence, $c = p = N$ and $\mathcal{V}(R) = \mathcal{V}(\Pi, R(\pi_N), N) = \mathcal{V}_N^\Pi$. □

9. Classification of Irreducible Finite $K(\mathfrak{d}, \theta)$-Modules

We already know that all $K(\mathfrak{d}, \theta)$-modules belonging to the exact complex (6.23) are reducible, as the image of each differential provides a nonzero submodule. Further, Corollary 8.5 shows that these are the only reducible tensor modules $\mathcal{V}(R)$, when R is an irreducible finite-dimensional representation of $\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}}$. However, by Proposition 5.2 and Theorem 5.1, every finite irreducible $K(\mathfrak{d}, \theta)$-module is a quotient of some $\mathcal{V}(R)$, where R is an irreducible finite-dimensional $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-module. Thus, classifying irreducible quotients of all (reducible) tensor modules $\mathcal{V}(R)$ will yield a classification of all irreducible finite $K(\mathfrak{d}, \theta)$-modules.

Remark 9.1. By Theorems 8.1 and 8.3, each of the reducible tensor modules from (6.23) contains a unique irreducible $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-summand of nonconstant singular vectors.

Lemma 9.1. *Let V and W be $K(\mathfrak{d}, \theta)$-modules, and assume V is generated over H by its singular vectors. If $f\colon V \to W$ is a $K(\mathfrak{d}, \theta)$-homomorphism, then $f(V)$ is also H-linearly generated by its singular vectors.*

Proof. This follows immediately from $f(\operatorname{sing} V) \subset \operatorname{sing} W$. □

Theorem 9.1. *The image modules $\mathrm{d}^{\mathrm{R}}\mathcal{V}_{N+2}^\Pi$ and $\mathrm{d}\mathcal{V}_{p+1}^\Pi$, where $0 \le p \le 2N+1$, $p \ne N, N+1$ are the unique nonzero proper $K(\mathfrak{d}, \theta)$-submodules of $\mathcal{V}_N^\Pi$ and $\mathcal{V}_p^\Pi$, respectively.*

Proof. We first claim that these submodules are irreducible, hence minimal. By Proposition 5.2 and Remark 9.1, it is enough to show that they are H-linearly generated by their singular vectors. This follows from Lemma 9.1.

To prove that there are no other nonzero proper submodules, it is enough to show that these minimal submodules are also maximal. Equivalently, the quotients $\mathcal{V}_N^\Pi / \mathrm{d}^{\mathrm{R}}\mathcal{V}_{N+2}^\Pi$ and $\mathcal{V}_p^\Pi / \mathrm{d}\mathcal{V}_{p+1}^\Pi$ are irreducible, which follows from exactness of the complex (6.23). □

The above results lead to the main theorem of the paper.

Theorem 9.2. *A complete list of non-isomorphic finite irreducible $K(\mathfrak{d}, \theta)$-modules is as follows*:

(i) *Tensor modules $\mathcal{V}(\Pi, U)$ where Π is an irreducible finite-dimensional representation of $\mathfrak{d}$ and U is a nontrivial irreducible finite-dimensional $\mathfrak{csp}\,\bar{\mathfrak{d}}$-module not isomorphic to $(R(\pi_p), p)$ or $(R(\pi_p), 2N+2-p)$ with $1 \le p \le N$,*

(ii) *Images of differentials in the twisted contact pseudo de Rham complex (6.23), namely $\mathrm{d}^{\mathrm{R}}\mathcal{V}_{N+2}^\Pi$ and $\mathrm{d}\mathcal{V}_n^\Pi$ where $1 \le n \le 2N+1$, $n \ne N+1, N+2$. Here Π is again an irreducible finite-dimensional $\mathfrak{d}$-module.*

Remark 9.2. The image $\mathrm{d}\mathcal{V}_{2N+2}^\Pi$ of the first member of the complex (6.23) is isomorphic to $\mathcal{V}_{2N+2}^\Pi = \mathcal{V}(\Pi, R(\pi_0), 2N+2)$ and it is included in part (i) of the above theorem.

Recall that representations of the Lie pseudoalgebra $K(\mathfrak{d},\theta)$ are in one-to-one correspondence with conformal representations of the extended annihilation algebra $\widetilde{\mathcal{K}}$ (see [BDK] and Proposition 3.2). The latter is a direct sum of Lie algebras $\widetilde{\mathcal{K}} = \widetilde{\mathfrak{d}} \oplus \mathcal{K}$ where $\widetilde{\mathfrak{d}} \simeq \mathfrak{d}$ and $\mathcal{K}$ is isomorphic to the Lie–Cartan algebra K_{2N+1} (see Propositions 4.1 and 4.2). Thus, from our classification of finite irreducible $K(\mathfrak{d},\theta)$-modules we can deduce a classification of irreducible conformal K_{2N+1}-modules. In this way we recover the results of I.A. Kostrikin, which were stated in [Ko] without proof.

In order to state the results, we first need to set up some notation. Let R be a finite-dimensional representation of $\mathfrak{csp}\,\bar{\mathfrak{d}}$. Using that $\mathfrak{csp}\,\bar{\mathfrak{d}} \simeq \mathcal{K}'_0/\mathcal{K}'_1$, we endow R with an action of $\mathcal{K}'_0$ such that $\mathcal{K}'_1$ acts trivially (see Proposition 4.3). We also view R as a $(\mathfrak{d} \oplus \mathfrak{csp}\,\bar{\mathfrak{d}})$-module with a trivial action of $\mathfrak{d}$, and as before we write $R = (U, c)$. Then, by Remark 5.3 and Proposition 4.2, the induced $\mathcal{K}$-module $\operatorname{Ind}_{\mathcal{K}'_0}^{\mathcal{K}} R$ is isomorphic to the tensor $K(\mathfrak{d},\theta)$-module $\mathcal{V}(R) = \mathcal{V}(\mathbf{k}, U, c)$. Finally, let us recall the $\mathbb{Z}$-grading of V introduced in the proof of Proposition 5.4. In this setting, Theorems 9.1 and 9.2 along with Remark 7.2 imply the following.

Corollary 9.1. (i) [Ko]. *Every nonconstant homogeneous singular vector in $V = \mathcal{V}(\mathbf{k}, U, c)$ has degree 1 or 2. The space S of such singular vectors is an $\mathfrak{sp}\,\bar{\mathfrak{d}}$-module, and the quotient of V by the $\mathcal{K}$-submodule generated by S is an irreducible $\mathcal{K}$-module. All singular vectors of degree 1 in V are listed in cases* (a), (b) *below, while all singular vectors of degree 2 are listed in* (c):

(a) $\quad U = R(\pi_p)\,, \quad c = p\,, \quad S = R(\pi_{p+1})\,, \qquad 0 \le p \le N-1\,.$

(b) $\quad U = R(\pi_p)\,, \quad c = 2N+2-p\,, \quad S = R(\pi_{p-1})\,, \qquad 1 \le p \le N\,.$

(c) $\quad U = R(\pi_N)\,, \quad c = N\,, \quad S = R(\pi_N)\,.$

(ii) [Ko]. *If the $\mathfrak{sp}\,\bar{\mathfrak{d}}$-module U is infinite-dimensional irreducible, then $\mathcal{V}(\mathbf{k}, U, c)$ does not contain nonconstant singular vectors.*

(iii) *If a $\mathcal{K}$-module $\mathcal{V}(\mathbf{k}, U, c)$ is not irreducible, then its* (*unique*) *irreducible quotient is isomorphic to the topological dual of the kernel of the differential of a member of the Rumin complex over formal power series.*

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