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KALMAN ESTIMATION FOR A CLASS OF RATIONAL ISOTROPIC RANDOM FIELDS*

Ahmed H. Tewfik, Bernard C. Levy and Alan S. Willsky

Laboratory for Information and Decision Systems
and the Department of Electrical Engineering and
Computer Science, M.I.T., Cambridge, MA. 02139

Abstract

This paper considers the application of Kalman estimation theory to the problem of estimating two-dimensional isotropic random fields, whose equations are expressed in terms of the Laplacian, given some noisy observations on a finite disk. It is shown that this problem is equivalent to that of solving a countably infinite number of one-dimensional estimation problems. Markovian models for the one-dimensional processes are developed and the associated Kalman filters are shown to be asymptotically stable. The desired field estimate is then obtained by combining the smoothed estimates resulting from each of the one-dimensional problems weighted in an appropriate fashion.

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I. Introduction

Isotropic random fields are random fields whose autocorrelation function is invariant under both rotations and translations. Apart from the fact that the isotropic property is the natural extention of the notion of stationarity in one dimension, isotropic fields deserve a special attention because they arise in a number of physical problems of interest among which we can mention the study of sound propagation in the ocean ([1] Chapter 10, [2]), the investigation of the temperature and pressure distributions in the atmosphere at a constant altitude ([3], [4]), and the analysis of turbulence in fluid mechanics [5]. In this paper we shall study isotropic fields defined by the integral

$$z(\underline{r}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} C\, K_0(A|\underline{r}-\underline{r}'|)\, w(d\underline{r}'), \underline{r} \in \mathbb{R}^2 \tag{1.1}$$

where C is a matrix of size pxn, A a positive definite symmetric matrix of size nxn, and $w(d\underline{r})$ a random zero-mean two-dimensional Gaussian orthogonal measure with

$$E[|w(d\underline{r})|^2] = Q\, d\underline{r}\ , \tag{1.2}$$

where Q is a non-negative definite matrix which commutes with A. $K_0(Ar)$ denotes here the matrix modified Bessel function of second kind and of order zero given by

$$K_0(Ar) = -\,(\ell n(Ar) + \gamma I) \sum_{k=0}^{\infty} \frac{(1/2\ Ar)^{2k}}{k!\,\Gamma(k+1)} + \sum_{k=1}^{\infty} \frac{(1/2\ Ar)^{2k}}{k!} \left(\sum_{i=1}^{k} i^{-1}\right)\ , \tag{1.3}$$

where γ is Euler's constant. Heuristically $z(\underline{r})$ can be described in differential form by the 2-D state-space model

$$(I_n\nabla^2 - A^2)\; x(\underline{r}) = u(\underline{r}) \tag{1.4a}$$

$$z(\underline{r}) = Cx(\underline{r}) \quad , \tag{1.4b}$$

where $u(\underline{r}) = \frac{\partial^2 w(\underline{r})}{\partial x \partial y}$ is a two-dimensional zero-mean white Gaussian noise process of intensity Q. Note that the partial differential equation (1.4a) does not specify uniquely the state process $x(\underline{r})$. An asymptotic condition must also be imposed which has for effect to specify the Green's function

$$G(\underline{r},\underline{r}') = K_0\,(A\,|\underline{r}-\underline{r}'|) \tag{1.5}$$

appearing in solution (1.1). This condition is that the state process $x(\underline{r})$ must have <u>finite variance</u> as $|\underline{r}|$ tends to infinity. It is also worth observing that since the partial differential equation (1.4a) involves the Laplacian, it can be used to model a large class of physical phenomena, such as potential problems with uniformly distributed random sources in a lossy medium, where the loss is described here by A^2. We now show that $x(\underline{r})$ and $z(\underline{r})$ are indeed isotropic processes.

<u>Theorem 1.1</u>: The state process $x(\underline{r})$ defined by equation (1.4a), or equivalently by setting $C = I$ in equation (1.1), is an isotropic process with autocorrelation function

$$R_x(\underline{r},\underline{s}) = E[x(\underline{r})x^T(\underline{s})] = \frac{|\underline{r}-\underline{s}|}{4\pi}\, QA^{-1}K_1(A|\underline{r}-\underline{s}|) \quad , \tag{1.6}$$

where $K_1(Ar)$ is the matrix modified Bessel function defined by

$$K_1(Ar) = -\,(\frac{d}{dr}K_0(Ar))A^{-1} \quad . \tag{1.7}$$

This implies that the output process $z(\underline{r})$ is isotropic with autocorrelation function

$$R_z(\underline{r},\underline{s}) = CR_x(\underline{r},\underline{s})C^T \quad . \tag{1.8}$$

Proof: We will first show that $R_x(\underline{r},\underline{s})$ is invariant under translation. By definition we have

$$R_x(\underline{r},\underline{s}) = E[x(\underline{r})x(\underline{s})]$$

$$= \frac{Q}{4\pi^2}\int d\underline{r}'\, K_0(A|\underline{r}-\underline{r}'|)\, K_0(A|\underline{s}-\underline{r}'|)$$

$$= \frac{Q}{4\pi^2}\int_{-\infty}^{\infty} du' \int_{-\infty}^{\infty} dv'\, K_0(A[(u_1-u')^2 + (v_1-v')^2]^{1/2})\,\cdot$$

$$K_0(A[(u_2-u')^2 + (v_2-v')^2]^{1/2})$$

where $\underline{r} = (u_1,v_1)$ and $s = (u_2,v_2)$. Now, perform the transformation

$$u'' = h_1 + u'$$

$$v'' = h_2 + v'$$

to obtain

$$R_x(\underline{r},\underline{s}) = \frac{Q}{4\pi^2}\int_{-\infty}^{\infty} du''\int_{-\infty}^{\infty} dv''\, K_0(A[(u_1+h_1-u'')^2 + (v_1+h_2-v'')^2]^{1/2})$$

$$\cdot\; K_0(A[(u_2+h_1-u'')^2 + (v_2+h_2-v'')^2]^{1/2})$$

$$= R_x(\underline{r}+\underline{h},\ \underline{s}+\underline{h}) \quad ,$$

where $\underline{h} = h_1\hat{\underline{i}} + h_2\hat{j}$. This shows that $R_x(\underline{r},\underline{s})$ is invariant under translation. Using this fact, we can write

$$R_x(\underline{r}',\underline{s}) = R_x(\underline{r},0) \quad ,$$

with $\underline{r} = \underline{r}'-\underline{s}$. Hence

$$R_x(\underline{r}',\underline{s}) = \frac{Q}{4\pi^2}\int d\underline{r}''\; K_0(A|\underline{r}-\underline{r}''|)\; K_0(A|\underline{r}''|)$$

$$= \frac{Q}{4\pi^2} K_0(A|\underline{r}|) * K_0(A|\underline{r}|)$$

where * denotes the convolution operation. Thus

$$R_x(\underline{r}',\underline{s}) = \frac{Q}{4\pi^2}\int_\theta^\infty \int_0^{2\pi} K_0(Ar'')\; K_0(A(r^2+r''^2-2rr''\cos\theta)^{1/2})r''dr''d\theta$$

$$= R_x(|\underline{r}|) ,$$

where $|\underline{r}| = r$, which shows that $R_x(\underline{r},\underline{s})$ is also invariant under rotation.

Finally, by using the properties of Hankel transforms and the fact that

$$R_x(\underline{r},\underline{s}) = \frac{Q}{4\pi^2} K_0(A|r-s|) * K_0(A|\underline{r}-\underline{s}|) , \tag{1.9}$$

we have

$$R_x(\underline{r},\underline{s}) = \frac{|\underline{r}-\underline{s}|}{4\pi}\; QA^{-1}K_1(A|\underline{r}-\underline{s}|) .$$

■

The motivation for considering isotropic fields which admit the integral representation (1.1), or equivalently, which are described by the partial differential equation (1.4a), is that the spectral density

$$S_z(\lambda) = C(\lambda^2 I+A^2)^{-1}\; Q(\lambda^2 I+A^2)^{-1}C^T$$

$$= C(\lambda^2 I+A^2)^{-2}QC^T \tag{1.10}$$

which is obtained from (1.9) is <u>rational</u>. Furthermore, since A is symmetric and positive definite, the poles of the spectrum $S_z(p)$ obtained by setting $p = j\lambda$ in (1.10) are all real and occur in sets of four: p=a (twice) and $p = -a$ (twice), where $a \geq 0$ is an eigenvalue of A. To see how random fields of this type appear physically, consider a random field $z(\cdot)$ which is obtained by passing 2-D white noise through a linear, stable, rational filter $F(\lambda^2)$ which is invariant under translations and rotations. Then, if $p = j\lambda$, the poles of $F(\cdot)$ appear in pairs $p=\pm a$, where a is real (this is a consequence of the circular symmetry of $F(\cdot)$), and the spectrum of $z(\cdot)$ is given by

$$S_z(\lambda) = F(\lambda^2)F(\lambda^2) \quad . \tag{1.11}$$

This implies that the field $z(\cdot)$ has a realization of the form (1.1) or (1.4). We see therefore that the class of random fields with a spectral density of the form (1.10) is quite large. It is, in fact, the analog of the class of stationary processes which are obtained by passing white noise through a finite dimensional, linear time-invariant filter in one dimension.

In the remainder of this paper we shall develop a theory of Kalman filtering and smoothing for the process $z(\underline{r})$. Specifically, we shall consider the problem of finding the best estimate $\hat{z}(\underline{r}|R)$ of $z(\underline{r})$ given some noisy observations $y(\underline{r})$ of $z(\underline{r})$ where

$$y(\underline{r}) = z(\underline{r}) + v(\underline{r}) \qquad r \leq R \; , \tag{1.12}$$

and where $v(\underline{r})$ is a two-dimensional zero-mean white Gaussian noise process such that

$$E[v(\underline{r})v(\underline{s})] = V\delta(\underline{r}-\underline{s}) \tag{1.13}$$

$$E[v(\underline{r})u(\underline{s})] = 0.$$

Here V is a positive definite matrix and $\delta(\cdot)$ denotes a two dimensional impulse function. In Section 2 we shall show that the above problem is equivalent to a countably infinite number of orthogonal one-dimensional estimation problems. We will then develop a state-space Markov realization for each of the one dimensional estimation problems in Section 3. Kalman estimation theory can then be applied to each problem separately and the resulting estimates can be combined in a proper fashion to obtain $\hat{z}(\underline{r}|R)$. The stability of the Kalman filters associated with each one-dimensional problem is studied in Section 4.

II. Fourier Series Expansions for Isotropic Fields

Following [6] we expand each isotropic field appearing in equation (1.12) in Fourier series

$$y(\underline{r}) = \sum_{n=-\infty}^{\infty} y_n(r) e^{jn\theta} \tag{2.1a}$$

$$z(\underline{r}) = \sum_{n=-\infty}^{\infty} z_n(r) e^{jn\theta} \tag{2.1b}$$

$$v(\underline{r}) = \sum_{n=-\infty}^{\infty} v_n(r) \; e^{jn\theta} \quad , \tag{2.1c}$$

where the Fourier coefficients

$$y_n(r) \frac{1}{2\pi} \int_0^{2\pi} y(r,\theta) e^{-jn\theta} \; d\theta \tag{2.2a}$$

$$z_n(r) = \frac{1}{2\pi} \int_0^{2\pi} z(r,\theta) e^{-jn\theta} d\theta \tag{2.2b}$$

$$v_n(r) = \frac{1}{2\pi} \int_0^{2\pi} v(r,\theta) e^{-jn\theta} d\theta \tag{2.2c}$$

define some one-dimensional estimation problems

$$y_n(r) = z_n(r) + v_n(r) \qquad 0 \leq r \leq R \; . \tag{2.3}$$

The main feature of this expansion is that the Fourier coefficients of different orders are uncorrelated, i.e.

$$E[z_n(r) z_m^*(s)] = E[v_n(r) v_m^*(s)]$$

$$= E[z_n(r) v_m^*(s)] = 0 \tag{2.4}$$

for $n \neq m$ ([7]). Hence $\hat{z}(\underline{r}|R) = E[z(\underline{r})|Y_R]$, where Y_R denotes the Hilbert space spanned by the observation process $y(\underline{r})$ for $r \leq R$, can be written as

$$\hat{z}(\underline{r}|R) = \sum_{n=-\infty}^{\infty} E[z_n(r)|Y_R^n] \quad e^{jn\theta} , \tag{2.5}$$

where $Y_R^n = H(y_n(r), 0 \leq r \leq R)$ is the Hilbert space spanned by the nth Fourier coefficient of the observations. The two-dimensional estimation problem has thus been reduced to a countably infinite number of one-dimensional estimation problems. In practice one would consider only a finite set of N of these one dimensional estimation problems, where N increases in a non-linear fashion with increasing distance from the origin of the location where $z(\underline{r})$ is to be estimated, and decreases with increasing allowable error covariances. Thus, if one is interested in the value of $\hat{z}(\underline{0}|R)$ at the origin, one would use only the zeroth order Fourier processes.

To obtain the Fourier expansion of the fields specified by equation (1.1) and equation (1.12) we shall make use of the following identity

$$K_0(A|\underline{r}-\underline{r}'|) = I_0(Ar_<) \; K_0(Ar_>)$$

$$+ \sum_{n=-\infty}^{-1} I_n(Ar_<) \; K_n(Ar_>) \; \cos n(\theta-\theta')$$

$$+ \sum_{n=1}^{\infty} I_n(Ar_<) \; K_n(Ar_>) \; \cos n(\theta-\theta') , \tag{2.6}$$

where

$$\underline{r} = (r,\theta), \; \underline{r}' = (r',\theta') ,$$

$$r_< = \min (r,r') \quad r_> = \max (r,r') , \tag{2.7}$$

and $I_n(Ar)$ and $K_n(Ar)$ are modified Bessel functions of the first and second kind respectively and are defined by

$$I_n(Ar) = (\frac{1}{2} Ar)^n \sum_{k=0}^{\infty} \frac{(\frac{1}{2} Ar)^{2k}}{k!\ \Gamma(n+k+1)} \tag{2.8}$$

$$K_n(Ar) = \frac{1}{2} (\frac{1}{2} Ar)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (-1/4\ A^2 r^2)^k + (-1)^{n+1} \ell n(\frac{Ar}{2})\ I_n(Ar)$$

$$+ (-1)^n \frac{1}{2} (Ar)^n \sum_{k=0}^{\infty} (-2\gamma + \sum_{i=1}^{k} \frac{1}{i} + \sum_{i=1}^{k+n} \frac{1}{i}) (\frac{Ar}{2})^{2k} / k!(n+k)!. \tag{2.9}$$

Upon multiplying both sides of (1.12) by $\frac{e^{-jn\theta}}{2\pi}$ and integrating from 0 to 2π we obtain

$$y_n(r) = z_n(r) + v_n(r), \tag{2.10}$$

where

$$z_n(r) = C(\int_0^r dr'\ r'\ I_n(Ar') K_n(Ar) u_n(r')$$

$$+ \int_r^{\infty} dr'\ r'\ I_n(Ar)\ K_n(Ar') u_n(r')) \tag{2.11}$$

$$u_n(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta)\ e^{-jn\theta} d\theta\ . \tag{2.12}$$

The two noise processes $u_n(r)$ and $v_n(r)$ are zero-mean white Gaussian noise processes with covariance

$$E\left[\begin{bmatrix} u_n(r) \\ v_n(r) \end{bmatrix} [u_n^T(s)\ \ v_n^T(s)]\right] = \begin{bmatrix} \frac{Q}{2\pi r} & 0 \\ 0 & \frac{V}{2\pi r} \end{bmatrix} \delta(r-s)\ .$$

Note that the noise intensities vary as $\frac{1}{r}$. This can be explained as follows. The noise processes $u_n(r)$ and $v_n(r)$ are obtained by averaging the white noises $u(\underline{r})$ and $v(\underline{r})$ weighted by $e^{-jn\theta}$ on annular strips centered at the origin.

Since the total noise energy associated with $u(\underline{r})$ and $v(\underline{r})$ is constant independent of position, then the total energy associated with $u_n(r)$ and $v_n(r)$ on any strip must be constant, and since the area of an annular strip varies as r, the noise intensities of $u_n(r)$ and $v_n(r)$ must vary as $\frac{1}{r}$. In the next section we develop a state-space realization for equations (2.10) and (2.11).

III. State-Space Realizations of the Fourier Processes

Let us define two new processes $\xi_n(r)$ and $\eta_n(r)$ as

$$\xi_n(r) = \int_0^r dr' \; r' \; I_n(Ar')u_n(r') \tag{3.1}$$

$$\eta_n(r) = \int_r^\infty dr' \; r' \; K_n(Ar')u_n(r') \quad . \tag{3.2}$$

Clearly

$$z_n(r) = CK_n(Ar)\xi_n(r) + CI_n(Ar)\eta_n(r) \quad . \tag{3.4}$$

By differentiating (3.1) and (3.2) we obtain

$$\dot{\xi}_n(r) = rI_n(Ar)u_n(r) \tag{3.5}$$

$$\dot{\eta}_n(r) = -r \; K_n(Ar)u_n(r) \quad , \tag{3.6}$$

with the limiting conditions $\xi_n(0) = 0$ and $\eta_n(\infty) = 0$. Note that $\xi_n(\cdot)$ is propagating in a radially outwards direction, while $\eta_n(\cdot)$ is propagating in a radially inwards direction. The set of equations (3.4) - (3.6) constitutes a state-space description of process $z_n(\cdot)$, but it is <u>non-causal</u>. In order to apply Kalman filtering techniques to this state-space model, we must transform it into an ordinary forwards propagating model. This can be done by applying the method of Verghese and Kailath [8], for constructing backwards Markovian models, to the problem of reversing the direction of propagation of $\eta_n(\cdot)$. Let $X_r = \sigma\{\eta_n(r'): 0 \le r' \le r\}$ denote the sigma field generated by the process $\eta_n(r')$ for $r' \le r$. Then

$$\begin{aligned} E[u_n(r) | X_r] &= E[u_n(r)\eta_n^T(r)] \; (E[\eta_n(r)\eta_n^T(r)])^{-1}\eta_n(r) \\ &= \frac{2}{r} K_n(Ar) \; D_n^{-1}(Ar)\eta_n(r) \quad , \end{aligned} \tag{3.7}$$

where

$$D_n(Ar) = r(K_{n-1}(Ar)K_{n+1}(Ar) - K_n^2(Ar)) \,, \tag{3.8}$$

and where we have used the fact that Q and A commute. Equations (3.5) and (3.6) can now be rewritten as

$$\dot{\xi}_n(r) = 2K_n(Ar)I_n(Ar)D_n^{-1}(Ar)\eta_n(r) + rI_n(Ar)\tilde{u}_n(r) \tag{3.9a}$$

$$\dot{\eta}_n(r) = -2K_n^2(Ar)D_n^{-1}(Ar)\eta_n(r) - rK_n(Ar)\tilde{u}_n(r) \,, \tag{3.9b}$$

where $\tilde{u}_n(r)$ is an X_r-martingale having the same intensity $Q/2\pi r$ as $u_n(r)$. By applying the state transformation

$$T_n(r) = \begin{bmatrix} K_n(Ar) & 0 \\ 0 & I_n(Ar) \end{bmatrix} \tag{3.10}$$

to equations (3.9), and using equations (2.10) and (3.4), we obtain the following state-space Markovian model for the nth order Fourier coefficients

$$\dot{x}_n(r)=A_n(r)x_n(r) + B_n(r)\tilde{u}_n(r) \tag{3.11a}$$

$$y_n(r) = C_n(r)x_n(r) + v_n(r) \quad , \tag{3.11b}$$

with

$$A_n(r) = \begin{bmatrix} A\dot{K}_n(Ar)K_n^{-1}(Ar) & 2K_n^2(Ar)D_n^{-1}(Ar) \\ 0 & A\dot{I}_n(Ar)I_n^{-1}(Ar) - 2K_n^2(Ar)D_n^{-1}(Ar) \end{bmatrix} \tag{3.12}$$

$$B_n(r) = \begin{bmatrix} rI_n(Ar)\,K_n(Ar) \\ \\ -rI_n(Ar)\,K_n(Ar) \end{bmatrix} \tag{3.13}$$

$$C_n(r) = [C \quad C] \quad , \tag{3.14}$$

and where $\tilde{u}_n(r)$ and $v_n(r)$ are two-independent zero-mean white Gaussian noise processes with

$$E\left[\begin{bmatrix} \tilde{u}_n(r) \\ \\ v_n(r) \end{bmatrix} \begin{bmatrix} \tilde{u}_n^T(s) & v_n^T(s) \end{bmatrix}\right] = \begin{bmatrix} \frac{Q}{2\pi r} & 0 \\ 0 & \frac{V}{2\pi r} \end{bmatrix} \delta(r-s) \quad . \tag{3.15}$$

The initial conditions associated with system (3.11) are

$$E[x_n(0)] = 0, \tag{3.16}$$

and

$$E[x_n(0)x_n^T(0)] = \begin{bmatrix} 0 & 0 \\ \\ 0 & \frac{Q}{4\pi} A^{-2} \delta_{n0} \end{bmatrix} \tag{3.17}$$

where δ_{n0} denotes the Kronecker delta function. Equation (3.17) is a direct result of the fact that as r tends to zero the covariance of the process $x_n(r)$ behaves as

$$E[x_n(\varepsilon)x_n^T(\varepsilon)] \sim \begin{bmatrix} \frac{\varepsilon^2}{16\pi n^2(n+1)}\,Q & 0 \\ \\ 0 & \frac{\varepsilon^2}{16\pi n^2(n+1)}\,Q \end{bmatrix} \tag{3.18}$$

for $n \neq 0$ or 1, as

$$E[x_1(\varepsilon)x_1^T(\varepsilon)] \sim \begin{bmatrix} \frac{\varepsilon^2}{32\pi} Q & 0 \\ 0 & -\frac{\varepsilon^2 Q}{16\pi}(I+2\ell n(A\varepsilon)) \end{bmatrix} \tag{3.19}$$

for $n=1$, and as

$$E[x_0(\varepsilon)x_0^T(\varepsilon)] \sim \begin{bmatrix} \frac{\varepsilon^2 Q}{4\pi}\ell n^2(A\varepsilon)(I-\frac{(A\varepsilon)^2}{4}) & 0 \\ 0 & \frac{QA^{-2}}{4\pi}(I-((A\varepsilon)\ell n(A\varepsilon))^2) \end{bmatrix}. \tag{3.20}$$

It is interesting to examine the asymptotic form of the system (3.11) as r tends to zero and to infinity. Note that as r tends to zero the only process of interest is the zeroth order Fourier process. As r tends to zero the matrices $A_0(r)$, $B_0(r)$ and $C_0(r)$ tend to the following matrices

$$A_0(r) \to \bar{A}_0(r) = \begin{bmatrix} \frac{1}{r}\ell n^{-1}(Ar) & 2A^2 r\ell n^2(Ar) \\ 0 & \frac{A^2 r}{2} - 2A^2 r\ell n^2(Ar) \end{bmatrix} \tag{3.21}$$

$$B_0(r) \to \bar{B}_0(r) = \begin{bmatrix} -r\ell n(Ar) \\ r\ell n(Ar) \end{bmatrix} \tag{3.22}$$

$$C_0(r) = [C \quad C] \quad . \tag{3.23}$$

Clearly the product $A_0(r)x(r)$ is ill-behaved as r tends to zero. To circumvent this difficulty we make the following state-space transformation

$$\tilde{x}_0(r) = P_0(r)x_0(r) \tag{3.24a}$$

$$P_0(r) = \begin{bmatrix} -\ell n^{-1}(Ar) & 0 \\ 0 & I \end{bmatrix} , \tag{3.24b}$$

to obtain the following model for the zeroth order Fourier process

$$\dot{\tilde{x}}_0(r) = \tilde{A}_0(r)\tilde{x}_0(r) + \tilde{B}_0(r)\tilde{u}_0(r) \tag{3.25a}$$

$$y_0(r) = \tilde{C}_0(r)\tilde{x}_0(r) + v_0(r) \quad , \tag{3.25b}$$

with

$$\tilde{A}_0(r) = \begin{bmatrix} -AK_1(Ar)K_0^{-1}(Ar) - \frac{1}{r}\ell n^{-1}(Ar) & -2AK_0^{2}(Ar)\ell n^{-1}(Ar) \cdot D_0^{-1}(Ar) \\ 0 & AI_1(Ar)I_0^{-1}(Ar) - 2AK_0^{2}(Ar) \cdot D_0^{-1}(Ar) \end{bmatrix} \tag{3.26}$$

$$\tilde{B}_0(r) = \begin{bmatrix} -rK_0(Ar)\ell n^{-1}(Ar) \\ -rK_0(Ar)\ell n^{-1}(Ar) \end{bmatrix} \tag{3.27}$$

$$\tilde{C}_0(r) = [-\ell n(Ar) \quad I] \quad , \tag{3.28}$$

and with initial conditions

$$E[\tilde{x}_0(0)] = 0 \tag{3.29a}$$

$$E[\tilde{x}_0(0)\tilde{x}_0^T(0)] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{QA^{-2}}{4\pi} \end{bmatrix} . \tag{3.29b}$$

The model (3.21) is well behaved near the origin since

$$\lim_{r\to 0} \tilde{A}_0(r) = \lim_{r\to 0} \tilde{B}_0(r) = 0 \tag{3.30a}$$

$$\lim_{r\to 0} \tilde{A}_0(r)\tilde{x}_0(r) = 0 \tag{3.30b}$$

$$\lim_{r\to 0} \tilde{C}_0(r)\tilde{x}_0(r) = z_0(0) \quad , \tag{3.30c}$$

and should be used for the zeroth order Fourier processes instead of model (3.11) near the origin.

On the other hand as r tends to infinity the system (3.11) tends to

$$\dot{x}_n(r) = \bar{\bar{A}}x_n(r) + \bar{\bar{B}}\tilde{u}_n(r) \tag{3.31a}$$

$$y_n(r) = \bar{\bar{C}}x_n(r) + v_n(r) \quad , \tag{3.31b}$$

with

$$\bar{\bar{A}} = \begin{bmatrix} -A & 2A \\ 0 & -A \end{bmatrix} \tag{3.32}$$

$$\bar{\bar{B}} = \begin{bmatrix} \frac{1}{2} A^{-1} \\ -\frac{1}{2} A^{-1} \end{bmatrix} \tag{3.33}$$

$$\bar{\bar{C}} = [C \quad C] \quad . \tag{3.34}$$

It is interesting to note that $\bar{\bar{A}}$, $\bar{\bar{B}}$ and $\bar{\bar{C}}$ do not depend on the order of the Fourier coefficients. This reflects the fact that as r tends to infinity all Fourier processes have an equal importance in the sense that to obtain meaningful results one would have to retain a very large number of terms in the expansions (2.1a) to (2.1c). Furthermore, $\bar{\bar{A}}$, $\bar{\bar{B}}$ and $\bar{\bar{C}}$ are constant matrices and $\bar{\bar{A}}$ is stable, an observation that will be used in the sequel to show that the Kalman filter associated with the model (3.11) is stable.

Models (3.31) and (3.11) define a singular estimation problem as r tends to infinity. This is due to the fact that the intensity of the noise processes $\tilde{u}_n(r)$ and $v_n(r)$ varies as r^{-1}. To overcome this difficulty we introduce the following <u>normalized processes</u>

$$\bar{x}_n(r) = r^{1/2} x_n(r) \tag{3.35a}$$

$$\bar{u}_n(r) = r^{1/2} \tilde{u}_n(r) \tag{3.35b}$$

$$\bar{v}_n(r) = r^{1/2} v_n(r) \tag{3.35c}$$

$$\bar{y}_n(r) = r^{1/2} y_n(r) \tag{3.35d}$$

$$\bar{z}_n(r) = r^{1/2} z_n(r) \quad . \tag{3.35e}$$

The normalized processes $\bar{y}_n(r)$ and $\bar{x}_n(r)$ have a Markovian state-space model of the form

$$\dot{\bar{x}}_n(r) = (A_n(r) + \frac{1}{2r} I)\, \bar{x}_n(r) + B_n(r)\bar{u}_n(r) \tag{3.36a}$$

$$\bar{y}_n(r) = C_n(r)\bar{x}_n(r) + \bar{v}_n(r), \tag{3.36b}$$

where $A_n(r)$, $B_n(r)$ and $C_n(r)$ are defined by equations (3.12) - (3.14), and where $\bar{u}_n(r)$ and $\bar{v}_n(r)$ are two uncorrelated zero-mean Gaussian noise processes with intensities

$$E\left[\begin{bmatrix}\bar{u}_n(r)\\ \\ \bar{v}_n(r)\end{bmatrix}[\bar{u}_n^T(s) \quad \bar{v}_n^T(s)]\right] = \begin{bmatrix}\frac{Q}{2\pi} & 0\\ 0 & \frac{V}{2\pi}\end{bmatrix}\delta(r-s) . \tag{3.37}$$

The initial condition associated with system (3.36) is $\bar{x}_n(0) = 0$ with probability one. Note that as r tends to infinity the estimation problem is non-singular. Further, as r tends to infinity $(A_n(r) + \frac{1}{2r} I)$, $B_n(r)$ and $C_n(r)$ tend to $\bar{\bar{A}}$, $\bar{\bar{B}}$ and $\bar{\bar{C}}$ respectively, where $\bar{\bar{A}}$, $\bar{\bar{B}}$ and $\bar{\bar{C}}$ are defined by equations (3.32) - (3.34). A nice way to interpret the asymptotic model is to note that

$$\bar{\bar{C}}(j\lambda-\bar{\bar{A}})^{-1}\bar{\bar{B}} = C(j\lambda+A)^{-2} , \tag{3.38}$$

where $C(j\lambda+A)^{-2}Q^{1/2}$ is a stable spectral factor of $S_z(\lambda)$. Models (3.11), (3.31) and (3.36) can now be used to develop Kalman filters for the processes $z_n(r)$. Filtering corresponds to processing the data in a radially outward direction while smoothing involves processing the data in both the outwards and inwards radial directions, and combining the results of the outwards and inwards processing operations via standard smoothing formulas, such as the two-filter smoothing formula, or the Rauch-Tung-Striebel, or innovations formulas. By combining the smoothed estimates $\hat{z}_n(r|R)$ one can then obtain $\hat{z}(\underline{r}|R)$ via equation (2.5).

IV. Stability and Asymptotic Properties of the Kalman Filters

We begin this section by showing that the system (3.10) is exponentially stable. To do this we will need the following lemma which is an adaptation of a result of Coddington and Levinson ([9], p. 314).

Lemma 4.1: Let

$$\dot{x} = Ax + f(t,x) \tag{4.1}$$

where A is a real constant matrix with eigenvalues all having negative real parts. Let f be real, continuous for small $|x|$ and $t \geq 0$, and such that

$$f(t,x) = 0(|x|) \quad \text{as} \quad |x| \to 0$$

uniformly in t, $t \geq 0$. Then, the system (4.1) is exponentially stable.

Proof: Let $\phi(t)$ be a solution of (4.1). So long as $\phi(t)$ exists, it follows from (4.1) that

$$\phi(t) = e^{At}\phi(0) + \int_0^t e^{A(t-s)} f(s, \phi(s))ds \quad . \tag{4.2}$$

Because the real parts of the characteristic roots of A are negative, there exist positive constants K and σ such that

$$|e^{At}| \leq Ke^{-\sigma t} \qquad t \geq 0 \quad . \tag{4.3}$$

Hence from (4.3) and (4.2) we have

$$|\sigma(t)| \leq K \; |\phi(0)|e^{-\sigma t} + K \int_0^t e^{-\sigma(t-s)} |f(s,\phi(s))|ds \; .$$

Given $\varepsilon > 0$, there exists a δ such that $|f(t,x)| \leq \frac{\varepsilon|x|}{k}$ for $|x| \leq \delta$, by assumption. Thus, as long as $|\phi(t)| \leq \delta$, it follows that

$$e^{\sigma t}|\phi(t)| \leq K|\phi(0)| + \varepsilon \int_0^t e^{\sigma s}|\phi(s)|ds \quad .$$

This inequality yields

$$e^{\sigma t}|\phi(t)| \leq K\phi(0)e^{\varepsilon t} \quad ,$$

or

$$|\phi(t)| \leq K|\phi(0)|e^{-(\sigma-\varepsilon)t} \quad t \geq 0 \quad . \qquad \blacksquare \quad (4.4)$$

Lemma 4.1 can now be used to prove the following theorem

Theorem 4.1: The systems defined by equations (3.11) and (3.36) are exponentially stable.

Proof: The proof follows by writing

$$A_n(r) = \bar{\bar{A}} + A'_n(r)$$

where $\bar{\bar{A}}$ is defined by equation (3.28). By taking f(r,x) in Lemma 4.1 as

$$f(r, x) = A_n'(r)x(r) \quad ,$$

and noting that

$$\lim_{r\to\infty} A_n'(r) = 0 \quad ,$$

we obtain the desired result for system (3.11) by invoking the above mentioned lemma. A similar proof can be constructed for system (3.36) by absorbing $\frac{1}{2r} I$ into $A_n'(r)$. ■

We can now state and prove the main result of this section.

Theorem 4.2: The Kalman filters associated with the models (3.11) and (3.36) are asymptotically stable. Furthermore, the error covariances associated with the normalized processes converge to a non-negative definite matrix $\bar{P}$ as r tends to infinity, where $\bar{P}$ is the solution of the algebraic Riccati equation

$$0 = \bar{\bar{A}}\bar{P} + \bar{P}\bar{\bar{A}}^T + \bar{\bar{B}}Q\bar{\bar{B}}^T - \bar{P}\bar{\bar{C}}^T V^{-1}\bar{\bar{C}}\bar{P} \quad . \qquad (4.5)$$

Proof. The result follows by direct application of Theorem 4.10 of [10].

V. Conclusion

In this paper we have studied the class of rational isotropic processes $z(\underline{r})$ $\underline{r} \in \mathbb{R}$ defined by equation (1.1). We showed that the problem of estimating $z(\underline{r})$ given some noisy measurements on a disk of radius R is equivalent to a countably infinite set of one-dimensional estimation problems. Markovian models for the one-dimensional problems were developed and Kalman estimation theory was used to obtain a smoothed estimate of $z(\underline{r})$ given the noisy measurements. Finally, the Kalman filters associated with the one dimensional problems were shown to be asymptotically stable.

In view of the fact that in one dimension, Kalman filtering theory applies to nonstationary processes as well as to stationary processes, it ought to be possible to generalize our work to the case where the matrices A and C appearing in equations (1.4) are functions of $r = |\underline{r}|$ and where the intensity of the noise processes $u(\underline{r})$ and $v(\underline{r})$ are also functions of r.

Finally, further studies based on the ideas introduced in [5], [11]-[13] will be required to understand and characterize rational isotropic fields which obey equations containing not only the Laplacian, but also gradients, curls or divergences. This will require the study of isotropic 2-D vector fields, in addition to the scalar fields that we examined here.

REFERENCES

[1] N.S. Burdic, Underwater Acoustic System Analysis, Prentice Hall, Inc., Englewood Cliffs, N.J., 1984.

[2] H. Cox, "Spatial correlation in arbitrary noise fields with application to ambient sea noise," Jour. Acoustical Society of America, Vol. 54, No. 5, pp. 1289-1301, 1973.

[3] K.H. Bergman, "Multivariate Analysis of Temperature and Wind using Optimum Interpolation", Mon. Wea. Rev., Vol. 107, pp. 1423-1444, 1979.

[4] P.R. Julian and A. Cline, "The Direct Estimation of Spatial Wavenumber Spectra of Atmospheric Variables," Jour. Atmospheric Sci., Vol. 31, pp. 1526-1539, 1974.

[5] A.S. Monin and A.M. Yaglom, Statistical Fluid Mechanics: Mechanics of Turbulence, Volume 2, M.I.T. Press, Cambridge, MA., 1975.

[6] B.C. Levy and J.N. Tsitsiklis, "A Fast Algorithm for Linear Estimation of Two-Dimensional Isotropic Random Fields," to appear in IEEE Trans. Information Theory.

[7] E. Wong, "Two-Dimensional Random Fields and Representation of Images," SIAM J. Applied Math., Vol. 16, No. 4, pp. 756-770, July 1968.

[8] G. Verghese and T. Kailath, "A Further Note on Backwards Markovian Models", IEEE Trans. Info. Theory, Vol. IT-25, No. 1, pp. 121-124, January 1979.

[9] E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York, N.Y. 1955.

[10] H. Kwakernaak and R. Sivan, Linear Optimal Control Systems, J. Wiley and Sons, Inc., New York, N.Y. 1972.

[11] H.P. Robertson, "The Invariant Theory of Isotropic Turbulence," Proc. Cambridge Phil. Soc., Vol. 36, pp. 209-223, 1940.

[12] K. Ito, "Isotropic Random Current", Proc. 3rd Berkeley Symposium on Math. Stat. and Prob., pp. 125-132, 1956.

[13] E. Wong, "Markovian Random Fields," Proc. 23rd IEEE Conf. on Decision and Control, Las Vegas, NV, Dec. 1984, pp. 1447-1450.