AN INVESTIGATION OF THE FLUTTER OF LOW DENSITY WINGS

by

ROSE MARIE PRATT

S.B., University of Oklahoma

1950

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

at

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1952

Signature of Author: ___________________________ Rose Marie Pratt

Certified by: _________________________________ Holt Ashley, Supervisor

Shatswell Ober, Chairman
Departmental Committee on Graduate Students
Aero.
Thesis
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ABSTRACT OF THESIS

AN INVESTIGATION OF THE FLUTTER OF LOW DENSITY WINGS

by Rose Marie Pratt

Submitted to the Department of Aeronautical Engineering on January 12, 1952, in partial fulfillment of the requirements for the degree of Master of Science

The conventional type of flutter analysis is known to yield erroneous results in certain cases (Ref. 1). One of these is for a bending-torsion flutter at low values of the density parameter $\mu$ ($\mu$ = wing density $\div$ air density). There the theory is unconservative; that is, it predicts either no flutter or flutter at a speed higher than that given by experiment.

For this type of flutter various analyses are carried out on two particular families of wings using $\mu$ as one of the variables. One section considers the energy expended by the aerodynamic forces acting on the wing at flutter. The forces necessary to keep the wing oscillating in simple harmonic motion at various frequencies and free stream velocities near the flutter values are also determined. The important information to be learned from these analyses is that generally the forces applied and energy expended decrease steadily with decreasing $\mu$; however, the energy due to the lift force is found to increase suddenly at a low value of $\mu$.

Feeling that perhaps a small change in the magnitude of the aerodynamic forces or their phase angles might produce a large change in flutter speed, a series of flutter calculations is carried out making variations in the aerodynamic coefficients. It was found that, by certain arbitrary changes in one of the coefficients, $L_\alpha$, the "theoretical" flutter speed could be made to agree with the experimental values. However, the changes necessary are of such a large magnitude as to leave doubt as to the desirability of attempting to correct the theory in this manner.

The design and construction of a low $\mu$ wing model with an aileron to be tested in the wind tunnel is discussed. Experimental flutter conditions for various aileron frequencies at one value of $\mu$ are given, and a comparison is made with theory. However, the small amount of experimental and theoretical data gathered is not sufficient to enable any conclusions to be drawn other than that the theory appears to give unreliable results.

Thesis Supervisor: Holt Ashley
Assistant Professor of Aeronautical Engineering
Professor Joseph S. Newell
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

Dear Professor Newell:

In accordance with the regulations of the Faculty, I hereby submit a thesis entitled, "An Investigation of the Flutter of Low Density Wings," in partial fulfillment of the requirements for the degree of Master of Science.

Very truly yours,

Signature redacted
Rose Marie Pratt
ACKNOWLEDGEMENTS

For encouragement and assistance throughout this study the author is indebted to many people—

To Professor Holt Ashley for his untiring guidance and constructive criticism in all matters pertaining to this thesis.

To Mr. Larry Beckley of the Aeroelastic Laboratory for his invaluable aid in the design and construction of the model.

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And to Miss Marjorie Bruckner who typed the final copy.

To all go the author's sincere thanks.
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I. INTRODUCTION

Experiments carried out by the N.A.C.A. (Ref. 1) have shown that the conventional type of bending-torsion flutter analysis is quite unconservative at low values of the density parameter $\mu$. $\mu$ is defined as $m/\pi b^2$, the mass of the wing-unit length divided by the mass of a cylinder of air of unit length whose diameter is equal to the wing chord. Typical experimental values of $V_f$, the flutter speed, vs. $\mu$ and the corresponding theoretical values are shown in Fig. 7 taken from Ref. 1. The theory generally begins to give unconservative results for values of $\mu$ in the range 4 to 7 although for some configurations the discrepancy appears for values of $\mu$ as high as 16.

Since some of the light airplanes in production today have wings with a value of $\mu$ between 3 and 4 while other planes have $\mu$'s below 10, it is desirable that a more accurate method be developed to predict the flutter of these wings. An attempt to discover the important parameters causing the trouble is discussed in Sections I to III.

However, it is known that in the majority of flutter analyses the flutter mode occurring at the lowest speed is a combination of an aileron vibration mode with the torsion and/or bending mode. Therefore it is of practical as well as of theoretical importance to determine if the same unconservatism occurs in this type of flutter. An attempt to begin investigation along this line is discussed in Sections IV and V.
The logical method to begin such investigations is to examine the conventional method of flutter analysis. Essentially the method, as developed by Theodorsen (Ref. 2), Theodorsen and Garrick (Ref. 3), and others, is to set up the equations of motion of the system with as many degrees of freedom as is deemed necessary for the specific problem and solve them under the conditions imposed at flutter. These equations may be set up using the method of Lagrange Equations (Ref. 4) or by the more elementary principles of mechanics.

The definition of flutter used for these solutions is that it is a condition in which the system is oscillating in simple harmonic motion with all degrees of freedom having the same frequency (although not necessarily the same phase angle).

There are five major approximations usually made in setting up the final equations of motion; these are:

1. The system which actually has an infinite number of degrees of freedom is approximated by one with only a few degrees of freedom.

2. The mode shapes corresponding to each degree of freedom are generally approximated by the mode shapes of some easily computed system. e.g., the first bending mode of a wing might be approximated by the first natural bending mode of a uniform beam.

3. By an application of thin airfoil theory, the aerodynamic forces and moments acting on an airfoil are approximated by those acting on a flat plate.

4. The analyses of Refs. 5 and 6 assume that the actual
air forces acting at each spanwise station of the wing may be replaced by the two dimensional or sectional values at that station. There are methods of taking into account the effect of aspect ratio (Ref. 7) but for a conventional wing planform they are not generally used.

5. Structural damping, if considered, is assumed to be of the hysteresis type.

Although in examining these one might doubt the advisability of basing a theory on such approximations, they seem to be necessary due to the complexity of the flutter problem. Fortunately, over a wide range of flutter variables the theory gives reasonable, slightly conservative answers. It is only in the range of small $\mu$ that trouble is experienced.

Following the line of reasoning that at low values of $\mu$ the effect of the aerodynamic forces would be increased in relation to the inertia forces, this thesis deals primarily with the study of the forces acting on a wing at and near flutter.
II. BENDING-TORSION FLUTTER

a.) Derivation of the Flutter Equations

Although the bending-torsion-aileron flutter is of more practical importance, the complexity of analysis makes it advisable to begin any study of the phenomenon of flutter by examining the two-degree-of-freedom bending-torsion case. To further simplify the problem all investigations will be carried out on a rectangular plan-form wing which has constant mass, inertia and stiffness properties in the spanwise direction. The wing, which actually has an infinite number of degrees of freedom will be approximated by one with only two—a bending of the wing elastic axis and a twisting of the wing about the elastic axis. The elastic axis (e.a.) is defined as the locus of the elastic centers of the various wing sections, or as the locus of points which will not be deflected when a pure couple is applied to the wing. For both the two and three degree of freedom cases the notation of Fig. 1 will be used with all quantities positive as shown.

Figure 1
The equations of motion of the system may be set up by use of the Lagrange equations (Ref. 4) with $h$, the vertical displacement of the e.a., and $\alpha$, the angle of attack of the wing, as the generalized coordinates. $h$ and $\alpha$ will be assumed to be of the form

$$h = h_r(t) f_h(y) = \bar{h} e^{i\omega t} f_h(y) = h e^{i\omega t} f_h(y)$$

$$\alpha = \alpha_r(t) f_\alpha(y) = \bar{\alpha} e^{i\omega t} f_\alpha(y) = \alpha e^{i(\omega t + \phi)} f_\alpha(y)$$

where $\bar{h}$ and $\bar{\alpha}$ are complex constants denoting both the amplitude of the resulting motions and their phase angles. Requiring the motions to be sinusoidal with frequency $\omega$ at each spanwise station is consistent with the definition of flutter; i.e., the condition in which oscillatory motions of the system in all its degrees of freedom neither increase nor decrease in amplitude. The flutter mode considered is the symmetrical one; and the mode shape functions $f_h(y)$ and $f_\alpha(y)$ will be assumed to be proportional to the first natural mode in bending of a uniform beam and the first natural mode in torsion of a uniform rod. As was pointed out in the introduction, although an assumption of this type is generally made in flutter analyses, it should be kept in mind as a possible source of error. These modes are

$$f_h(y) = 1.3622 \left( \cos 0.597 \frac{\pi y}{L} - \cos h 0.597 \frac{\pi y}{L} \right) \sin 0.597 \frac{\pi y}{L} - \sin h 0.597 \frac{\pi y}{L}$$

$$f_\alpha(y) = \sin \frac{\pi y}{2L}$$

The equations of motion are
\[
\ddot{h}_t \int_0^L \rho f_h^2(y) \, dy + \ddot{\alpha} \int_0^L f_h(y) \, dy + m \omega_h^2 \int_0^L f_h^2(y) \, dy
\]
\[
= \int_0^L P f_h(y) \, dy
\]
\[
\ddot{\alpha} \int_0^L I_\alpha f_\alpha^2(y) \, dy + \ddot{h}_t \int_0^L f_h(y) f_\alpha(y) \, dy + I_\alpha \omega_\alpha^2 \int_0^L f_\alpha^2(y) \, dy
\]
\[
= \int_0^L M f_\alpha(y) \, dy
\]

\(S_\alpha\) is the static unbalance and \(I_\alpha\) the moment of inertia of the wing while \(\omega_h\) and \(\omega_\alpha\) are the first natural bending and torsion frequencies of the wing, respectively. These equations are derived in the Appendix. \(P\) is the vertical force applied at the e.a. and \(M\) the moment applied about the e.a. by the aerodynamic forces acting on the wing. Dots over a quantity indicate differentiation with respect to time.

To save a great deal of computational labor, \(P\) and \(M\) may be expressed in terms of tabulated dimensionless aerodynamic coefficients by using equations 1:19 and 1:20 in Ref. 5. These expressions are

\[
P = \pi \rho b^3 \omega^2 e^{1/\omega t} \left\{ \frac{\bar{f}_h}{b} L_h + \bar{\alpha} f_\alpha \left[ L_\alpha - L_h \left( \frac{1}{2} + a \right) \right] \right\}
\]
\[
M = \pi \rho b^4 \omega^2 e^{1/\omega t} \left\{ \frac{\bar{F}_h}{b} \left[ M_h - L_h \left( \frac{1}{2} + a \right) \right] + \bar{\alpha} f_\alpha \left[ M_\alpha - L_\alpha \left( \frac{1}{2} + a \right) \right]
\right\}
\]
\[
- M_h \left( \frac{1}{2} + a \right) + L_h \left( \frac{1}{2} + a \right)^2 \right\}
\]

where \(M_h = \frac{1}{2}\) and \(L_h\), \(L_\alpha\) and \(M_\alpha\) are tabulated vs. \(1/k\) in Ref. 5. Substituting Eqs. (4) into Eqs. (3) will give the flutter equations; however, it is more convenient to use them
in a non-dimensional form. This may be accomplished by introducing certain dimensionless parameters \( \mu, \omega, \chi, \tau \) -- the mass, frequency, static unbalance and radius of gyration terms. The equations may be written in their final form as

\[
\frac{1}{L} \int_0^L f_h^2 \, dy \left\{ \mu \left[ 1 - \left( \frac{\omega h}{\omega_h} \right)^2 \right] + L_h^2 \frac{\hbar}{b} \right. \\
+ \frac{1}{L} \int_0^L f_h f_\chi \, dy \left\{ \mu \chi + \left[ L_\omega - L_h \left( \frac{3}{2} + a \right) \right] \right\} \frac{\hbar}{b} = 0
\]

(5)

\[
\frac{1}{L} \int_0^L f_h f_\chi \, dy \left\{ \mu \chi + \left[ \frac{3}{2} - L_h \left( \frac{3}{2} + a \right) \right] \right\} \frac{\hbar}{b} \\
+ \frac{1}{L} \int_0^L f_\chi^2 \, dy \left\{ \mu \chi^2 \left( 1 - \omega \right) + M_\chi - L_\omega \left( \frac{3}{2} + a \right) - \frac{3}{2} \left( \frac{3}{2} + a \right) + L_h \left( \frac{3}{2} + a \right)^2 \right\} \frac{\hbar}{b} = 0
\]

These are of the form

\[
\bar{A} \frac{\hbar}{b} + \bar{B} \bar{\chi} = 0
\]

\[
\bar{D} \frac{\hbar}{b} + \bar{E} \bar{\chi} = 0
\]

(6)

b.) **Basic Solution of the Flutter Equations**

In order for there to exist a non-trivial solution to the set of homogeneous equations (6), the determinant of the coefficients must vanish. This leads to the complex flutter determinant

\[
\begin{vmatrix} \bar{A} & \bar{B} \\ \bar{D} & \bar{E} \end{vmatrix} = 0
\]

(7)

Upon substitution of the properties of a particular model, the terms of this determinant become functions of \( 1/k \) and \( \Omega \).
theoretically, expanding the complex determinant then leads to two real equations to be solved for $1/k$ and $\omega$. However, due to the complexity with which $1/k$ appears in the aerodynamic coefficients $L_h$, $L_\alpha$ and $M_\alpha$ the standard procedure is to introduce some new variable and solve the determinant for several values of $1/k$. Generally the auxiliary variable is a damping coefficient, $g$; a value of $1/k$ must then be found to give the proper value of $g$ (usually assumed zero or some small positive quantity).

Since in the present investigation $\mu$ is the important parameter it was chosen as the auxiliary variable to be determined along with $\omega = \left(\frac{\omega}{\omega_f}\right)^2$ for a given value of $1/k$. Expanding the complex determinant and setting its real and imaginary parts equal to zero leads to two equations in $\omega$ and $\mu$. However, it is found that they may be written in simpler form by setting $\mu = \frac{1}{k}$ and multiplying the real equation by $k^2$ and the imaginary one by $k$. This leads to two equations of the form

$$C_1 + C_2 \, k + C_3 \, \omega + C_4 \, k^2 + C_5 \, k \, \omega + C_6 \, \omega^2 = 0$$

$$C_7 + C_8 \, k + C_9 \, \omega = 0$$

where the coefficients $C_1$ to $C_9$ are functions only of $1/k$ for a given wing configuration. This is similar to a method worked out in Ref. 8. Equations (8) are now in a convenient form to be solved for $K$ and $\omega$.

Knowing $K$ and $\omega$, for several values of $1/k$, the flutter speed and frequency may be found at various values of $\mu$. 

-8-
Generally for each value of $1/k$, the theory will lead to two values of $\mu$ at which flutter occurs although this is not found to be true experimentally. (See, for example, Figs. 2 and 7.) Other variables of interest are the phase angle $\phi$ between the $\alpha$ motion and $h$ motion and the dimensionless ratio $\frac{\alpha}{h/b}$ of the amplitude of the $\alpha$ motion to $h$ motion. Expressions for these may be derived by setting $\bar{h} = h$ and $$\bar{x} = e^{i\phi}$$ in Eqs. (6). The result is

$$A \frac{h}{b} + B e^{i\phi} \alpha = 0$$

and leads

$$\tan \phi = \frac{A_I B_R - A_R B_I}{A_R B_R + A_I B_I}$$

and

$$\frac{\alpha}{h/b} = \frac{A_R}{B_I \sin \phi - B_R \cos \phi}$$

where $A = A_R + iA_I$, etc.

For the following numerical work two different sets of wing parameters have been used. Those of model A were chosen to be representative of present light airplanes while model B was taken from Ref. 1 where it is called model 17-32-4. Model properties are listed in the Appendix.

Figure 2 shows the results of the calculations for model A. Each value of $1/k$ down to 1.82 led to two real values of $\mu$ at which flutter will occur; lower values gave imaginary
FIG. 2 - VARIATION OF FLUTTER PARAMETERS WITH MODEL-A
answers to the solution of the quadratic equation in \( \mu \) which results from Eqs. (8). At low values of \( \mu \) the curves become double-valued in \( \mu \) indicating a minimum value of \( \mu \) at which flutter may exist; further discussion of this point is given in Part c of this Section. The flutter frequency \( \omega_f \) is seen to remain fairly constant over a wide range of \( \mu \) somewhat above the theoretical minimum. These results are substantiated in Refs. 1 and 3; however, in some cases \( \omega_f \) increases rapidly at \( \mu_{\text{min}} \) rather than decreasing as in Fig. 2. For this particular model, the phase angle \( \phi_\infty \) decreases uniformly as decreases and the flutter becomes more predominately h motion for low \( \mu \)'s.

c.) Limiting Case of Zero Mass

It has been noted in Fig. 2 that there appears to be a minimum value of \( \mu \) for which bending-torsion flutter occurs. In order to determine that this is indeed true theoretically, the equations can be examined in the limiting case of a massless wing. It will be assumed that the ratio \( dm/m \) remains constant as \( m \to 0 \) so that

\[
\frac{r_\infty^2}{I_{\infty}} = \frac{I_{\infty}}{mb^2} = \frac{1}{b^2} \int_{-b}^{b} (x-ba)dm/m \]

remains constant; similarly \( x_\infty = \frac{S_\infty}{mb} \) retains its original value in the limiting case. Now \( \omega_{h} \sim 1/\sqrt{m} \) and \( \omega_{\infty} \sim 1/\sqrt{I_{\infty}} \) will approach infinity as \( m \to 0 \); but \( \frac{\omega_{h}}{\omega_{\infty}} \sim r_\infty \) will remain constant.

Since the massless wing does possess stiffness and has aerodynamic forces acting upon it, it will be assumed to possess some finite flutter frequency \( \omega_f \) when placed in a moving stream of air of density \( \rho \) at velocity \( V_f \). Consequently,
\( \frac{\omega_i}{\omega_i} \rightarrow \infty \) as \( m \rightarrow 0 \). However, the parameter \( \mu \eta \)

\[ m/n b \left( \frac{\omega_i}{\omega_i} \right)^2 \sim \frac{m}{\rho} \omega_i L_a \sim 1/\rho \omega_i^2 r_a^2 \]

will remain constant.

Equations (5) then take the form

\[
\frac{1}{L} \int_0^L f_h^2 dy \left[ -\left( \frac{\omega_i}{\omega_i} \right)^2 \psi + L_h \right] \frac{h}{b} + \frac{1}{L} \int_0^L f_h f_x dy \left[ L_h (\frac{h}{b} + a) \right] = 0 \quad (12)
\]

\[
\frac{1}{L} \int_0^L f_h f_x^2 dy \left[ \frac{1}{2} - L_h (\frac{h}{b} + a) \right] \frac{h}{b} + \frac{1}{L} \int_0^L f_x^2 dy \left[ -\frac{h}{b}^2 \psi + \omega_i \right]
\]

\[ - L_h (\frac{h}{b} + a) - \frac{1}{2}(\frac{h}{b} + a) + L_h (\frac{h}{b} + a)^2 \] \[ \omega_i \]

where \( \psi = \frac{GJ}{4} b \frac{2}{4} \omega_i^2 L_a^2 r_a^2 \sim 1/\omega_i^2 \), and \( GJ \) is the torsional stiffness of the wing. The determinant resulting from Eqs. (12) can now be solved for the required values of \( 1/k \) and \( \psi \). For a given value of \( 1/k \), the real and imaginary parts of the expanded determinant lead to a linear and a quadratic equation in \( \psi \).

Using the data of model B, no value of \( 1/k \) from 0 to 3.75 would lead to a real value of \( \psi \) satisfying both the linear and the quadratic equations derived from the determinant. Since the curves of \( \psi \) vs. \( 1/k \) determined from each equation for \( \psi \) were found to diverge with increasing \( 1/k \), it seems likely that no flutter condition does exist for the zero mass case.

Since there exists a minimum value of \( 1/k \) below which the flutter determinant, Eq. (7), will not lead to a real value of \( \mu \) and since large values of \( 1/k \) lead to a doubling back of the curves of \( 1/k \) vs. \( \mu \), it is seen that there is indeed a \( \mu_{\text{min}} \) below which flutter will not occur theoretically.
This information does not agree with that obtained experimentally in Ref. 1 as shown for model B in Fig. 7, and it is this discrepancy which motivates the investigation discussed in this thesis.

d.) **Forcing the Wing in Simple Harmonic Motion**

To begin the study of the nature of the aerodynamic forces acting on the wing it was decided to determine the force which must be applied at the wing elastic axis to keep it "fluttering" at various values of \( V \) and \( \omega \); that is, for a wing in a given airstream \( V \), to determine the sinusoidal force \( \frac{h}{b}(F_R + iF_I) e^{i\omega t} \) which must be applied to cause each wing section to undergo the motions \( -e^{i\omega t} \) and \( e^{i\omega t} \). This is easily done by modifying Eqs. (6) to read

\[
A \frac{h}{b} e^{i\omega t} + B = e^{i(\omega t + \phi_\alpha)} = \frac{h}{b} (F_R + iF_I) e^{i\omega t}
\]

\[
D \frac{h}{b} e^{i\omega t} + E = e^{i(\omega t + \phi_\alpha)} = 0
\]

Equations (13) may be solved to give

\[
F_R + iF_I = \frac{A}{E} \frac{E - B D}{\overline{E}}
\]

For a comparison, three values of \( \mu \) were chosen: 5.114, 3.456 and 2.576. For each of these \( F_R + iF_I \) was determined at various combinations of \( V \) and \( \omega \) near the flutter values. Curves of \( F_I \) vs. \( F_R \) are plotted for each value of \( V \) in Figs. 3 to 5; lines of constant \( \omega \) are also shown. It should be noted that each figure is plotted to a different scale. The requirement that both \( F_R \) and \( F_I \) be zero at flutter leads to
FIG. 3 - FORCE APPLIED AT ELASTIC AXIS TO PRODUCE SINUSOIDAL OSCILLATIONS

Model A
\[ \mu = 5.114 \]
\[ V_f = 486.75 \text{ ft./sec.} \]
\[ \omega_f = 103.02 \text{ rad./sec.} \]
Model A

\[ \mu = 3.456 \]

\[ V_f = 468.27 \text{ ft./sec.} \]

\[ \omega_f = 102.07 \text{ rad./sec.} \]

FIG. 4—FORCE APPLIED AT ELASTIC AXIS TO PRODUCE SINUSOIDAL OSCILLATIONS
Model A

\(\mu = 2.576\)

\(V_f = 504.75 \text{ ft./sec.}\)

\(\omega_f = 100.75 \text{ rad./sec.}\)

FIG. 5 - FORCE APPLIED AT ELASTIC AXIS TO PRODUCE SINUSOIDAL OSCILLATIONS
an alternate method of solving the flutter equations; this
method is examined in some detail in Ref. 9.

The immediate conclusion that can be drawn is that as the
wing mass, $\pi \rho b^2 L / \mu$ decreases, so do the aerodynamic forces
acting on the wing near flutter. It is also noted that for
this model the force becomes more in phase with the h motion;
this might be expected from an examination of Fig. 2 which
shows that the magnitude of the h motion at flutter increases
with respect to $\alpha$ at low values of $\mu$.

The fact that the magnitude of the forcing function de-
creases with decreasing $\mu$ suggests that the forces acting on
the wing should be examined more closely. The hypothesis that
a slight error in the theoretical aerodynamic forces causes
the discrepancy between theory and experiment can be corroborated
to some extent by Figs. 3 to 5. For a small variation in aero-
dynamic forces would cause a small change in the value of the
forcing function $F_R + iF_I$ at a $\mu$ of 5.114 as can be seen in
Fig. 3; however, at a $\mu$ of 2.576, the small change would have
a much more pronounced effect on the flutter solution. This
indicates that use of the theoretical aerodynamic force terms
might possibly lead to results well within the realm of
engineering accuracy at moderate and high values of $\mu$ but be
entirely unsatisfactory at low $\mu$'s. With this thought in mind,
the work done by the aerodynamic forces at flutter will be
examined in Section II, Part e.
e.) Energy Balance at Flutter

Continuing with the hypothesis that it is the aerodynamic coefficients that are in error, the following examination can be made of the work done by these coefficients at flutter.

Equations (4) for the lift and moment can be written symbolically as

\[
P = \pi \rho b^2 \omega^2 (P_h h/b + P_\alpha \alpha) \tag{15}
\]

\[
M = \pi \rho b^2 \omega^2 (M_h h/b + M_\alpha \alpha)
\]

where \( P_h \) represents the non-dimensional lift due to vertical translation and \( M_h \), the moment due to change in \( \alpha \), etc. Actually, however, the lift is only the real part of \( P \) and the vertical translation is the real part of \( h \) or \( hf_h(y)\cos \omega t \). Consequently, the total work done by the aerodynamic forces during one cycle of flutter motion can be written as

\[
\text{work} = \int_0^L \left[ \int_0^b \text{R}\left\{ P \right\} \text{R}\left\{ \frac{d}{dt} \left( \frac{h}{b} \right) \right\} \, dt + \int_0^b \text{R}\left\{ M \right\} \text{R}\left\{ \frac{d\alpha}{dt} \right\} \, dt \right] \, dy
\]

\[
= \pi \rho b^2 L \omega^2 h^2 \left[ W_{hh} + W_{h\alpha} + W_{\alpha h} + W_{\alpha\alpha} \right] \tag{16}
\]

where \( \text{R}\left\{ P \right\} \) indicates the real part of the force \( P \), and the non-dimensional work terms are as follows:

\[
W_{hh} = \left( \frac{\omega_f}{\omega_d} \right)^2 \frac{1}{L} \int_0^L f_h^2 \, dy P_{hI} \sin \phi \tag{17}
\]

\[
W_{h\alpha} = \left( \frac{\omega_f}{\omega_d} \right)^2 \left( \frac{\alpha}{h/b} \right) \frac{1}{L} \int_0^L f_h f_\alpha \, dy \left[ P_{\alpha R} \sin \phi + P_{hI} \cos \phi \right]
\]
\[ W_{\alpha h} = \left( \frac{w}{w_\alpha} \right)^2 \left( \frac{a}{h/b} \right)^2 \int_0^L f_h f_\alpha \, dy \left[ m_{hI} \cos \phi_\alpha - m_{hR} \sin \phi_\alpha \right] \]
\[ W_{\alpha \alpha} = \left( \frac{w}{w_\alpha} \right)^2 \left( \frac{a}{h/b} \right)^2 \int_0^L f_\alpha^2 \, dy \quad m_{I} \]

\( \Phi_l \) indicates the imaginary part of \( P_h \) and \( P_{hR} \) the real part. \( W_{hh} \) represents the work done by the lift in vertical translation, \( W_{h\alpha} \) the work done by the lift as \( \alpha \) changes, etc. The details of Eqs. (16) and (17) are given in the Appendix.

The significance of these work terms is that at flutter, since we are not including the effects of damping, their sum is zero, or
\[ W_{hh} + W_{h\alpha} + W_{\alpha h} + W_{\alpha \alpha} = 0 \quad (18) \]

This can be derived by examining the basic equations. If Eqs. (1) and (4) are substituted in Eqs. (3) and the first divided by \( e^{i\omega t} \) and the second by \( e^{i(\omega t + \phi_\alpha)} \), the imaginary parts of the resulting equations can be written as
\[ -\omega^2 S\sin \phi_\alpha \int_0^L f_h f_\alpha \, dy = \pi \frac{b^3}{\omega^2} \int_0^L f_h^2 \, dy (P_{hI}) \]
\[ + \alpha \int_0^L f_h f_\alpha \, dy (P_{hR} \sin \phi_\alpha + P_{hI} \cos \phi_\alpha) \]
\[ \omega^2 h S\sin \phi_\alpha \int_0^L f_h f_\alpha \, dy = \pi \frac{b^4}{\omega^2} \int_0^L f_h f_\alpha \, dy (m_{hI} \cos \phi_\alpha - ???) \]
\[ - m_{hR} \sin \phi_\alpha) + \alpha \int_0^L f_\alpha^2 \, dy (m_{I}) \quad (19) \]

Multiplying the first of Eqs. (19) by \( \frac{h}{b} \), the second by \( \alpha \), and adding leads immediately to \( 1/P \cdot \text{Work} = 0 \) and so to Eq. (18). This derivation is similar to one carried out in Ref. 10 for
a typical section flutter analysis. This leads to another criteria for flutter, that of zero work done by the aerodynamic forces; and Ref. 10 develops this into a method for determining flutter solutions.

However, the interest here lies in the individual work terms, not their sum, so they are plotted as a function of $\mu$ in Fig. 6 for Model A. It is to be noted immediately that $W_{hh}$ and $W_{\alpha\alpha}$ are negative. This could have been predicted beforehand for it is known that generally one degree of freedom flutter cases do not exist; consequently, energy must always be taken out of any such system. The gradual decrease of these work terms with decreasing $\mu$ fits in nicely with the results of Section II, Part d, indicating that the forces acting on the wing decrease at lower $\mu$'s. The interesting result, however, is that $W_{h\alpha}$ and $W_{hh}$ increase suddenly near $\mu_{min}$. Continuing the hypothesis that it is the aerodynamic coefficients that cause the flutter discrepancies, Fig. 6 suggests that it is the lift terms, or using the notation of Ref. 5, it is $L_{\alpha}$ and $L_{h}$ that should be examined more closely. This is done in Section III.
FIG. 6 - WORK / CYCLE DONE BY AERODYNAMIC FORCES AT FLUTTER - MODEL A
III. VARIATION OF AERODYNAMIC COEFFICIENTS--
BENDING-TORSION FLUTTER

This section deals with an investigation of the effect
of variation of certain of the aerodynamic coefficients
defined in Eqs. (4) and Ref. 5. The work began with an attempt
to use in the flutter analysis experimental values of these
coefficients rather than the theoretically determined values.
In Ref. 10 the aerodynamic forces acting on an oscillating
airfoil have been determined experimentally and plotted vs.
$1/k$ in terms of certain functions $L_T$, the lift in translation,
$L_P$, $M_T$, and $M_T$. The coefficients of Eqs. (4) may be determined
from these through the following equations:

\[
\begin{align*}
L_h &= \frac{2}{k^2} L_T \\
L_\alpha &= \frac{2}{k^2} L_h (\frac{1}{2} + \alpha) = \frac{2}{k^2} L_P \\
M_h &= \frac{2}{k^2} L_h (\frac{1}{2} + \alpha) = \frac{2}{k^2} M_T \\
M_\alpha &= \frac{2}{k^2} L_h (\frac{1}{2} + \alpha) = \frac{2}{k^2} M_P
\end{align*}
\]

The analyses described in this section are carried out on
model B; it was chosen for examination from among those
examined in Ref. 1 because experimental data was obtained for
it at a lower value of $\mu$ than for other models. The N.A.C.A.
theoretical analysis was carried out considering the wing to
have three degrees of freedom. The mode shapes were considered
as being proportional to the first two bending modes of a
uniform cantilever beam and the first torsion mode of a uniform
rod. In Fig. 7 the N.A.C.A. theoretical and experimental
results are plotted vs. $\mu$ along with the theoretical result
obtained considering two degrees of freedom as in Section II.
FIG. 7 - FLUTTER PARAMETERS vs $\mu$ - MODEL B

- Experiment
- A - Theory, 2° freedom
- B - Theory, 3° freedom
It should be noted that improvement of the mode shape approximation through an additional degree of freedom of the system led to a delay of the turning up of the $1/k$ vs. $\mu$ curves in this case; nevertheless both theoretical results show the same tendency to diverge from the experimental results at low values of $\mu$.

At a $1/k$ of 3.33 the experimental values of the aerodynamic coefficients were substituted for the theoretical and the flutter analysis carried out; unfortunately, it led to imaginary values of $\mu$. An examination of the experimental data for the magnitude of $L_T$ and for its phase in Ref. 10 seems to indicate an inconsistency in the results when compared to theory; that is, over part of the range of $1/k$ the experimental values are smaller than theory but suddenly become larger. Such behavior does not occur in the plots of $L_p$, $M_T$, and $M_p$. This seeming inconsistency is mentioned by the author of Ref. 10 and, with the results above, leads to doubt as to the desirability of using such data. Assuming that there is an error in the determination of $R_L\{L_T\}$, Eqs. (20) show that this introduces an error, not only in $L_h$ but also in the real parts of the other desired coefficients; this is due to the fact that the four equations must be solved simultaneously. The comparison between theory and experiment for a $1/k$ of 3.33 is given on the following page in Table 1.
Table 1

<table>
<thead>
<tr>
<th>Theory</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_h$</td>
<td>-0.1950 - 4.4333 i</td>
</tr>
<tr>
<td>$L_\infty$</td>
<td>-15.4731 - 3.7822 i</td>
</tr>
<tr>
<td>$M_h$</td>
<td>0.5</td>
</tr>
<tr>
<td>$M_\infty$</td>
<td>0.375 - 3.3333 i</td>
</tr>
</tbody>
</table>

Since the experimental aerodynamic coefficients could not be used in the flutter analysis, it was decided to make small arbitrary changes in the theoretical values and examine their effect upon the flutter solution. A $1/k$ of 3.33 was again chosen for these first variations; and, noting from Table 1 that the major discrepancy is in the real parts, they were examined first. The results are shown in Fig. 8a. A 10% change in $R\{L_h\}$ and in $R\{M_\infty\}$ had only a small effect on the flutter condition at this value of $1/k$ compared to the effect of $L_\infty$. A 10% decrease in $|R\{L_\infty\}|$ gave imaginary values of $\mu$. This indicates why using experimental values of these coefficients gave an imaginary value of $\mu$ since Table 1 shows that the experimental value of $|R\{L_\infty\}|$ is about 30% less than the theoretical value. Since the energy term $W_h\infty$, which was mentioned in Section II, Part e, as a possible source of error, is a function of $R\{L_\infty\}$ it was decided to examine it more closely.

First, however, the effect of the imaginary part of $L_\infty$...
FIG. 8 - EFFECT OF VARIATION OF AERODYNAMIC COEFFICIENTS ON BENDING-TORSION FLUTTER THEORY MODEL B

\[ \frac{1}{k} \]

- 10% \( R\{L_\alpha\} \)
- 10% \( R\{L_h\} \)
- 10% \( R\{M_\alpha\} \)

- 5% in phase of \( L_\alpha \)
- 5% in phase of \( L_\alpha \)
was examined; Fig. 8b shows the results. At this value of $1/k$

$$L_\alpha = -15.4730 - 3.7822 \; i = 15.9285 \; e^{193.736^\circ}.$$ Changing the magnitude of $L_\alpha$ had about the same effect as changing the real part, while changing the imaginary part had little effect.

These changes all had the same effect of either spreading the curve out or squeezing it together. However, changing the phase tends to shift the curve to the left or right at this value of $1/k$. Increasing the phase 5% shifted the flutter points to the right giving one value of $\mu$ of 21.511 which was not plotted in Fig. 8b.

Returning to $R \{L_\alpha \}$, Fig. 9 shows the results for a 10% increase in the magnitude of this coefficient at various values of $1/k$. This change can be seen to have a much greater effect upon the flutter speed at low values of $\mu$ than at high $\mu$'s; in fact it even decreases $\mu_{\text{min}}$ giving a finite flutter speed at a value of $\mu$ for which the theoretical values of $L_\alpha$ predict no flutter. It is seen that at high $\mu$'s changing $R \{L_\alpha \}$ has negligible effect on the flutter frequency $\omega_f$. These results lend support to the hypothesis that there might exist small errors in the aerodynamic coefficients capable of giving erroneous results at small $\mu$'s while giving satisfactory results for engineering purposes at high $\mu$'s.

Noting with interest the trend of the flutter solutions due to changes in $R \{L_\alpha \}$, the study was continued to determine if it would be possible to reach the low values of $1/k$ determined experimentally. Once the original flutter analysis has been set up in tabular form and carried out for each value of
FIG. 9 - EFFECT ON BENDING-TORSION FLUTTER THEORY OF INCREASING $|R\{L_\alpha\}|$ 10% - MODEL B

- Experiment
- A - Theory
- B - Theory with $|R\{L_\alpha\}|$ increased 10%
$1/k$, the auxiliary analyses with variations in $R \{L_\alpha\}$ can be carried out fairly rapidly; Fig. 10 shows the results for model B. For each value of $1/k$ the resulting $\mu$ at flutter is plotted vs. $\%$ change in $R \{L_\alpha\}$. The vertices of the parabola-like curves were found by interpolation between changes in $R \{L_\alpha\}$ giving imaginary values of $\mu$ and those giving real values.

Also plotted in Fig. 10 as a dashed line is the locus of points giving the experimental values of $1/k$ at various values of $\mu$. This shows the quite interesting result that it is possible to duplicate experimental values of $1/k$ at each $\mu$ by certain changes in only one of the aerodynamic coefficients. A plot of $\%$ change in $R \{L_\alpha\}$ vs. $1/k$ is given in Fig. 11; also shown is the theoretical value of $R \{L_\alpha\}$ and the "corrected" values necessary to give the experimental values of $1/k$.

As can be seen from Figs. 10 and 11 the variation in $R \{L_\alpha\}$ must become much larger as $\mu$ decreases to duplicate the experimental curve. As Fig. 12 shows, these values of $L_\alpha$ duplicate, not only $1/k$, but also $V_f$ and $\omega_f$ quite well for this particular model.

There is considerable doubt, however, that the theory could be in error enough to require a 42% change in one of the terms as is shown to be necessary here. This would indicate that the error is due to more than just $R \{L_\alpha\}$, and that a more systematic study of the theory is necessary. The same conclusion is also reached by examining the data of Ref. 1. There it is seen that the experimental and theoretical curves
FIG. 10 - $\mu$ AT FLUTTER vs % CHANGE IN $|R(L_\infty)|$ AT VARIOUS VALUES OF $1/k$ - MODEL B
Fig. 11 - $R\{L_\alpha\}$ vs $\frac{1}{k}$ and variations in $R\{L_\alpha\}$

Model B
Fig. 12 - Flutter parameters vs μ for model B
Experimental results and theory using varied R{L_α}
of \( V_f/b_0 \) vs. \( \mu \) cross at points corresponding to widely different values of \( 1/k \) for various models. However, a correction to only one of the aerodynamic coefficients will cause theory and "corrected" theory curves to cross at the same value of \( 1/k \).

Unfortunately, due to the large amount of computations involved, there has not been time to examine any other possibilities. Section II, Part e, leads to a suggestion for future work however; as is shown there, it is \( W_{h\alpha} \) and \( W_{hh} \) that are suspected of being in error. Written out more fully they are

\[
W_{hh} = \left( \frac{\omega}{\omega_0} \right)^2 \frac{1}{L} \int_0^L f_h^2 \, dy \cdot I \{ L_h \} \tag{17a}
\]

\[
W_{h\alpha} = \left( \frac{\omega}{\omega_0} \right)^2 \left( \frac{\alpha}{h/b} \right) \frac{1}{L} \int_0^L f_h f_{\alpha} \, dy \left[ P_{\alpha R} \sin \phi_{\alpha} + I \{ L_h \} \cos \phi_{\alpha} \right] \tag{17b}
\]

where

\[
P_{\alpha R} = R \left\{ L_{\alpha} - \left( \frac{1}{2} + a \right) L_h \right\}
\]

The imaginary part of \( L_h \) is seen to enter in both of these energy terms. Since it was \( L_T = (k^2/2) L_h \) in Ref. 10 that was shown to exhibit strange behavior when compared with theory, perhaps it is this coefficient that should be examined more closely. Admittedly, juggling these aerodynamic coefficients is not the desired method of approach to problems of this nature; however, the complexity of the basic flutter equations and the difficulty of tracing the effects of important variables in any other manner seems of necessity to lead to such an experimental method of attack.
IV. DESIGN AND CONSTRUCTION OF A LOW DENSITY WING

The low flutter problem has been discussed thus far only in terms of the two degree of freedom bending-torsion case. If the same discrepancy in flutter speeds exists for the wing-aileron combination the problem takes on added importance from a practical standpoint. Consequently the design and construction of a low density wing has been attempted.

a.) The Design

The testing done by the N.A.C.A. at Langley Field was facilitated by the fact that a variable density tunnel was at their disposal; therefore, increasing the fluid density by changing from air to a heavier gas had the same effect as decreasing the wing density. But, with no such tunnel available, the problem becomes one of building a wing whose mass can be varied while keeping other parameters such as $x$, $r$, and the elastic axis location constant. The desire to reach a $\mu$ of two posed a critical design problem for previous wings built at M.I.T. generally had $\mu$'s of at least twice that value. To aid in construction and in the theoretical analysis the model was designed as a rectangular semispan wing with constant properties spanwise.

The wing properties were scaled down from values typical of light airplanes today to give a semispan of five feet. The velocity ratio was taken to give a flutter speed of around 40 mph. However, due to an error in the preliminary calculations, the wing stiffnesses in bending and torsion were made three
times as large as they should have been. This resulted in flutter speeds nearly twice as high as anticipated and led to the loss of the model during the wind tunnel tests.

The sectional type of construction was used for the wing in the belief that it would offer the lightest model as well as make it relatively easy to measure the model properties. This type of construction was first suggested and is discussed in some detail in Ref. 11; Ref. 12 also discusses this construction. The wing was to be divided into 8 rigid sections whose functions were to provide the proper airfoil shape and the desired mass and inertia properties. All the stiffness in bending and torsion was to be carried by a spar-torque tube assembly to which the rigid sections were attached.

After examination of several types of spar design, the lightest configuration tested was found to be two I-beam spars of equal stiffnesses connected by 8 torque tubes; with this arrangement the elastic axis of the wing should then be half way between the spars. To conserve on weight the spars were made of spruce, for use of magnesium would have nearly doubled the weight. The torque tubes, which contributed most of the torsional stiffness to the model, were machined from 1\(\frac{1}{4}\) inch diameter magnesium tubing. Details of the torque tubes and of the rest of the model are shown in Fig. 13; Figs. 14 to 19 are photographs showing the model in various stages of completion. Sample calculations of the spar and torque tube sizes are given in the Appendix. The torque tubes were attached to each spar by two aluminum machine screws and for added
FIG. 14 — Spar and torque tube assembly in tunnel mount.

FIG. 15 — Completed wing sections, tip and aileron ready for assembly.
FIG. 16 - Torque tube, aileron and two wing sections.

FIG. 17 - Completed tip section with torque tube and sample section of spar installed.
FIG. 18 - Assembled wing before covering.

FIG. 19 - Wing mounted in flutter tunnel.
strength the joints were also glued. The joints were further strengthened by small squares of magnesium sheet under the head of each machine screw. The holes in the torque tube flanges were to add glue surface as well as to lighten the tubes.

The sections were built of balsa, covered with tissue paper and given one coat of dope. The four ribs in each section were of 1/16 inch sheet balsa and were joined as one structure by 1/16 inch square balsa stringers, and by a solid balsa trailing edge; also the leading edge was covered with 1/32 inch sheet balsa to give strength to the sections and provide the proper leading edge contour. The continual battle to keep the weight to a minimum led to the use of paper for a covering. Although there was some doubt as to the advisability of using such a fragile covering, it was found to stand up quite well even at wind tunnel speeds above 80 mph. Adding only seven grams per section to the wing mass, and with most of that due to the dope, it provided a quite adequate covering. The sections were attached to the basic structure at only one point—the center of the torque tubes. A balsa block was glued to the center flange of the torque tube and to the two center ribs of the section as shown in Fig. 17. Holes somewhat larger than the spars were cut in all the ribs so that there would be no binding when the wing was deflected; this prevented the sections from adding any stiffness to the model. A 1/16 inch-thick wingtip was hollowed out of a balsa block and attached to the last section; this added 1 1/2 inch to the original 60 inch wingspan.
The aileron, although it covered three of the wing sections, was made in one piece; this was deemed satisfactory since it was to be hinged only at the ends and not attached to the middle section. Consequently, no stiffness was given to the structure by the aileron. Thin sheets of magnesium were glued to the ribs at the hinge points and the aileron was provided with 1/16 inch diameter aluminum tubing for hinges. Figures 16 and 17 show the hinge system to some extent. This simple hinge system was necessitated by the desire to conserve weight, and an equally simple type of aileron restraint was devised. It consisted of a small spring attached to each side of the aileron and to the wing rib at the inboard aileron tip; Fig. 13 shows the exact arrangement. With this, the aileron stiffness could easily be changed by merely changing the size of the springs.

The support mechanism for the model in the tunnel was one originally built for another wing and was capable of allowing freedom in both vertical translation and in roll. However, for the tests contemplated for this model, the support was to be locked allowing no motion of the wing root. A section of the wind tunnel wall is removable and the wing extended through this into the tunnel while the support was attached to heavy steel channels in the tunnel wall. A streamlined "fuselage" covered the part of the mount extending into the tunnel as may be seen in Fig. 19. There was also a motor-driven mechanism allowing the angle of attack of the wing to be varied at will by the operator during tests. This enabled the wing to be
adjusted to a position of near zero lift at all speeds. Complete details of the mounting system are given in Ref. 13.

b.) **Balancing the Wing**

After the torque tubes were glued to the wing sections, each unit was checked to determine its center of gravity and moment of inertia. The effects of the spars, covering and aileron were added to determine the basic values of $x_\alpha$, $r_\alpha$ and the mass. Although values representative of the two non-dimensional quantities in light planes were 0.285 and 0.5, it was found impossible to obtain them without increasing the weight more than was desired. In fact, it was decided not to attempt to achieve the same value of $r_\alpha$ in each section at the lightest configuration in order to obtain as low a $\mu$ as possible. A certain amount of mass in the form of lead weights was added to each section to make both the mass and $x_\alpha$ values the same in each section. These were deemed the most important parameters for the study at hand; with the weight added $r_\alpha$ could not have been made constant without adding the weights outside the wing. The weights were fastened to small wooden platforms attached to the two middle ribs of each section in the proper chordwise location. In the third section, part of the weight was taken up by a small accelerometer weighing 10.6 grams placed near the trailing edge to record the frequencies of vibration of the wing. A table of the basic wing properties and the added masses and their locations is given in the Appendix.

It was planned to test the model at higher values of $\mu$
simply by removing a piece of the paper covering of each section and adding weights in the proper positions; the only requirement would be that $x$ and $r$ remain the same. This would cause a decrease in the natural frequencies of the wing since the stiffness would remain constant while $m$ increased. However, for data plotted in terms of $V_{x}/b\omega_{x}$, the results should be consistent with the theoretical work computed with a constant $\omega_{x}$ as in Section II.

c.) Measurement of Wing Properties

Before the model was covered with tissue paper it was tested for elastic axis position and for bending and torsional stiffness. To determine the e.a. a pure couple was applied to the wing with the root support rigidly clamped and the deflections of the wing measured at two points. From these readings, the point of no deflection, or the e.a., was easily found. This test also led to the torsional stiffness $GJ$ of the wing given by the equation

$$GJ = \frac{\text{moment}}{d\alpha/dy}$$  \hspace{1cm} (21)

where $d\alpha/dy$ is the angular deflection per unit length.

The bending stiffness $EI$ was determined by measuring the deflection of the elastic axis for a given load applied to the wing. The $EI$ was then determined through the cantilever beam formula

$$EI = \frac{\text{weight} \cdot d^{3}}{3 \cdot \text{deflection}}$$  \hspace{1cm} (22)

where $d$ is the distance from the root to the point where the
load was applied and the readings taken. Table 2 shows the results of the tests together with the computed values.

Table 2

<table>
<thead>
<tr>
<th></th>
<th>e. a. location from leading edge-inches</th>
<th>EI (pound ft²)</th>
<th>GJ (pound ft²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>computed</td>
<td>6.0</td>
<td>1216</td>
<td>318</td>
</tr>
<tr>
<td>experimental</td>
<td>6.4</td>
<td>909</td>
<td>262</td>
</tr>
</tbody>
</table>

It is difficult to conceive why there should be such a discrepancy in the elastic axis location, for the spars were both cut from the same piece of spruce and each torque tube was tested separately before assembly to insure that they all had the same stiffness. A great deal of difficulty was experienced at the beginning in getting the wing support sufficiently anchored to prevent movement of the spars at the root, and the original results were discarded. Although the tests were later repeated with added root supports and better results were obtained, a possible source of error in both the e.a. position and in the stiffness values might be in the wing root fitting.

The desirability of using the wooden spars was questioned as the tests proceeded. In the static tests it was difficult to judge results for the model did not tend to return to its original position upon removal of the weights, and the tests were not always repeatable to the expected accuracy. The later loss of the model due to failure of the spars at the root also helped lead to the recommendation that future models
incorporate magnesium spars in spite of the increased weight.

Although the natural frequencies in bending and torsion could be approximately determined from the EI and GJ values, it is more accurate to measure these frequencies in the tunnel. A small jet of air was directed onto the wing at the tip from both above and below. The air from the jets was varied in such a manner to apply a sinusoidal force to the wing tip; the frequency of the force could be varied until the natural frequency of the model was reached. Frequency readings were taken on a brush oscillograph connected to the small accelerometer in the model; these were checked by noting the frequency of the applied force. Details of the air-jet shaker may be found in Ref. 9.

These tests gave an $\omega_h$ of 6.3 cps and an $\omega_\alpha$ of 17.2 cps. The EI and GJ computed from these values are 1005 and 338 pound feet$^2$ respectively. This value of GJ throws even more doubt into the validity of the results of the static tests as given in Table 2. However, the value of the e.a. location of 6.4 inches back of the leading edge was used in the theoretical calculations described in Section V.

The aileron stiffness per unit span, $C_\beta$, for each spring configuration was determined by measuring the deflection of the aileron under a statically applied moment. From a knowledge of $C_\beta$ the aileron frequencies were determined by the formula

$$\omega_\beta = \sqrt{C_\beta / I_c}$$

where $I_c$ is the moment of inertia of the aileron per unit length about its hinge point. The four springs used all gave
values of $\omega_3/\omega_\infty$ below 0.6. As the stiffness was increased, it tended to increase the aileron hinge friction due to the type of hinge and restraint system used. Perhaps, for future models, a different type of restraint should be devised to achieve larger values of $C_\beta$ more easily and eliminate the effect of varying aileron friction.
V. TEST RESULTS AND THEORETICAL CALCULATIONS FOR MODEL C

a.) Tests

The wing described in Section IV (model C) was tested in the M.I.T. 5 x 7\(\frac{1}{2}\) foot wind tunnel at only one value of \(\mu_0\). \(\mu_0\) based on standard air density was 2.17; however, the value of \(\mu\) during the tests was around 2.2. The wing was mounted rigidly in the tunnel wall but in such a manner as to allow the angle of attack to be changed by the operator. Cords attached to the two spars led through holes in the tunnel floor to give the operator some control over the model. They could be used to excite oscillations of the wing and also to stop the aileron flutter modes when desired. The small accelerometer in the model fed data into a brush-type oscillograph which recorded the flutter frequencies; this data was deemed accurate to within two per cent while the flutter speeds were within one per cent of the actual values.

The wing was tested with six different aileron restraints which gave values of \(\omega_p/\omega_\alpha\) of 0, 0.05, 0.075, 0.216, 0.541 and \(\infty\). An \(\omega_p\) of 0 corresponds to an aileron with no restraints while a value of \(\infty\) indicates the aileron locked, or the two-degree-of-freedom case. The results of the tests are given in Table 3 and are plotted in Fig. 21.
Table 3
Test Data for Model 0

<table>
<thead>
<tr>
<th>$\omega_p/\omega_n$</th>
<th>$V_f$ (mph)</th>
<th>$\omega_f$ (ops)</th>
<th>$V_f/b\omega_n$</th>
<th>$\omega_f/\omega_n$</th>
<th>$1/k$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>78.0</td>
<td>12.53</td>
<td>1.27</td>
<td>0.73</td>
<td>1.74</td>
<td>2.22</td>
</tr>
<tr>
<td>0.05</td>
<td>75.6</td>
<td>12.83</td>
<td>1.24</td>
<td>0.75</td>
<td>1.66</td>
<td>2.18</td>
</tr>
<tr>
<td>0.075</td>
<td>79.2</td>
<td>12.91</td>
<td>1.29</td>
<td>0.75</td>
<td>1.72</td>
<td>2.22</td>
</tr>
<tr>
<td>0.216</td>
<td>77.0</td>
<td>12.69</td>
<td>1.26</td>
<td>0.74</td>
<td>1.70</td>
<td>2.18</td>
</tr>
<tr>
<td>0.541</td>
<td>74.8</td>
<td>12.42</td>
<td>1.22</td>
<td>0.72</td>
<td>1.69</td>
<td>2.18</td>
</tr>
<tr>
<td>$\infty$</td>
<td>93.6</td>
<td>---</td>
<td>1.53</td>
<td>--</td>
<td>--</td>
<td>2.18</td>
</tr>
</tbody>
</table>

For the five three-degree-of-freedom cases the flutter speeds and frequencies were nearly constant; during these tests the model seemed to be fluttering in a torsion-aileron mode, for very little bending motion was observed. As was to be expected, the bending-torsion flutter with aileron locked occurred at a higher speed. It began with such violence that the spars failed at the root as the wing deflected downward; breaking loose from the mount, the model was destroyed. This was indeed unfortunate for the data obtained covers such a limited range of aileron frequencies, and only one value of $\mu$ so it is difficult to gain much information from it.

b.) Two-Degree-of-Freedom Calculations

Theoretical flutter calculations for the bending-torsion case ($\omega_p/\omega_n = \infty$) were carried out as in Section II, Part b, and plotted in Fig. 20. The one experimental point obtained at a $\mu$ of 2.18 is seen to be in the range where the theory
predicts no flutter and is in about the position to be expected from examining the plots of Ref. 1.

At two values of $1/k$ the theoretical calculations were reworked using for the real part of $L_\alpha$ the "corrected" values from Fig. 11 which gave the proper $1/k$ values at flutter for model B. These points are also plotted in Fig. 20. Here they tend to over-correct the theory, and definitely show that it is not $R\{L_\alpha\}$ alone that must be in error. However, some encouragement is gained from this examination for the points so obtained are in the desired range of values and indicate that $R\{L_\alpha\}$ is probably one of the most important parameters affecting the flutter solutions.

c.) Three-Degree-of-Freedom Calculations

The three-degree-of-freedom flutter equations are derived in a manner similar to Eqs. (3) for two degrees of freedom. In Lagrange’s Equations the kinetic and potential energies must be corrected to include the effect of the aileron and the virtual work equation must be modified to include the work done by the aerodynamic moment acting about the aileron hinge axis. The expression for this moment is given in Ref. 5; it should be noted that the expressions for the lift and moment given in Eqs. (4) must also be modified by some additional aileron terms. The aileron motion was assumed to be of the form

$$\beta = \beta_r(t) = \bar{\beta} e^{i \omega t} = \bar{\beta} e^{i(\omega t + \phi_\beta)}$$

(24)

Since the aileron was built as one unit, its motion is not a function of $y$, the spanwise coordinate.
FIG. 20 - FLUTTER PARAMETERS vs $\mu$ - MODEL C - $\frac{\omega_b}{\omega_\alpha} = \infty$

EXPERIMENTAL $\frac{V_f}{b \omega_\alpha}$

USING THEORY WITH $R\{L_\alpha\}$ INCREASED 10%
With the addition of the new variable $\beta$, three equations of motion are necessary to determine the motion of the system.

When $\alpha$, $h$ and $h$ are assumed of the form given in Eqs. (1) and (24), the final equations of motion may be written in the form

$$\begin{align*}
\bar{A} & \bar{h}/b + \bar{F} \bar{\alpha} + \bar{G} \bar{\beta} = 0 \\
\bar{B} & \bar{h}/b + \bar{F} \bar{\alpha} + \bar{F} \bar{\beta} = 0 \\
\bar{C} & \bar{h}/b + \bar{F} \bar{\alpha} + \bar{F} \bar{\beta} = 0
\end{align*}$$

(25)


where the complex coefficients are

$$\begin{align*}
\bar{A} &= \frac{1}{L} \int_0^L f_h^2 dy \left\{ \mu \left[ 1 - \frac{(\omega h)^2}{\omega^2} \right] + L_h \right\} \\
\bar{B} &= \frac{1}{L} \int_0^L f_h f_\alpha dy \left\{ \mu x_\alpha + L_\alpha - L_h (\frac{1}{2} + a) \right\} \\
\bar{C} &= \frac{1}{L} \int_0^L f_h dy \left\{ \mu x_\beta + L_\beta - L_z (c-e) \right\} \\
\bar{D} &= \frac{1}{L} \int_0^L f_h f_\alpha dy \left\{ \mu x_\alpha + \frac{1}{2} - L_h (\frac{1}{2} + a) \right\} \\
\bar{E} &= \frac{1}{L} \int_0^L f_\alpha^2 dy \left\{ \mu \alpha^2 (1 - \Omega) + M_\alpha - \frac{1}{2} (\frac{1}{2} + a) - L_\alpha (\frac{1}{2} + a) + L_h (\frac{1}{2} + a)^2 \right\} \\
\bar{F} &= \frac{1}{L} \int_0^L f_\alpha dy \left\{ \mu \left[ \mathbf{r}_\alpha^2 + (c-a) x_\beta \right] + M_\beta - L_\beta (\frac{1}{2} + a) - M_z (c-e) \right\} \\
\bar{G} &= \frac{1}{L} \int_0^L f_h dy \left\{ \mu x_\beta + T_h - P_h (c-e) \right\}
\end{align*}$$

(26)
\[ \bar{H} = \frac{1}{L} \int_{d_2L}^{d_1L} r_\alpha \, dy \left\{ \mu \left[ \frac{r_\beta^2}{\omega_\alpha^2} + (c-a) x_\beta \right] + T_\alpha - T_h (\frac{1}{2} + a) - P_\alpha (c-e) + P_h (c-e)(\frac{1}{2} + a)^2 \right\} \]

\[ \bar{J} = (d_2 - d_1) \left\{ \mu \frac{r_\beta^2}{\omega_\alpha^2} \left[ 1 - \left( \frac{\omega_\beta}{\omega_\alpha} \right)^2 \Omega \right] + T_\beta + P_z (c-e)^2 - (P_\beta + T_z) (c-e)^2 \right\} \]

The terms \( T_\alpha, P_h, M_z, \) etc. are non-dimensional aerodynamic coefficients introduced because of the effect of the aileron. They are defined in Ref. 5 and given in tables for various values of \( 1/k \) and \( e \). The solutions of the homogeneous set of equations (25) reduces to the determinant

\[
\begin{vmatrix}
A & B & C \\
\bar{D} & \bar{E} & \bar{F} \\
\bar{G} & \bar{H} & \bar{J}
\end{vmatrix} = 0 \tag{27}
\]

Since \( \mu \) appears in each of the nine elements of the determinant it does not seem feasible to leave it as a variable; and, of course, \( 1/k \) values must be picked. Consequently, another unknown must be chosen along with \( \Omega \); the logical one to choose is \( \frac{\omega_\beta}{\omega_\alpha} \) since the flutter solution is desired for several values of this parameter. \( \Omega \) appears only in the diagonal elements \( \bar{A}, \bar{E} \) and \( \bar{J} \) while \( \left( \frac{\omega_\beta}{\omega_\alpha} \right)^2 \) appears only in \( \bar{J} \). When expanded, the complex determinant leads to two equations in \( \left( \frac{\omega_\beta}{\omega_\alpha} \right)^2 \) and \( \Omega \). Since both are linear in \( \left( \frac{\omega_\beta}{\omega_\alpha} \right)^2 \), it may be eliminated easily leaving a single equation of fourth degree to be solved for \( \Omega \). These quartics were solved by Graeffe's root-squaring method (Ref. 4).
Four values of $1/k$ were chosen in the range of values given by experiment, and the flutter solutions carried out. Values of $1/k$ used were 1.46, 1.67, 1.835 and 2.00. Solution of the quartic gave two real roots and a pair of complex roots. The values are given in Table 4 and plotted in Fig. 21 along with the experimental data.

Table 4
Theoretical Flutter Data for Model C

<table>
<thead>
<tr>
<th>1/k</th>
<th>$\omega_b/\omega_d$</th>
<th>$\omega_f/\omega_d$</th>
<th>$V_f/b\omega_d$</th>
<th>$\omega_b/\omega_d$</th>
<th>$\omega_f/\omega_d$</th>
<th>$V_f/b\omega_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.46</td>
<td>0.498</td>
<td>0.574</td>
<td>0.839</td>
<td>1.183</td>
<td>1.107</td>
<td>1.616</td>
</tr>
<tr>
<td>1.67</td>
<td>0.408</td>
<td>0.477</td>
<td>0.786</td>
<td>1.151</td>
<td>1.000</td>
<td>1.671</td>
</tr>
<tr>
<td>1.835</td>
<td>0.324</td>
<td>0.385</td>
<td>0.706</td>
<td>1.157</td>
<td>0.914</td>
<td>1.678</td>
</tr>
<tr>
<td>2.00</td>
<td>0.279</td>
<td>0.344</td>
<td>0.687</td>
<td>1.217</td>
<td>0.875</td>
<td>1.751</td>
</tr>
</tbody>
</table>

The interesting thing to note is that the theory gives conservative flutter speeds at low values of $\frac{\omega_b}{\omega_d}$. However, the values of $1/k$ and $\frac{\omega_f}{\omega_d}$ given are seen to be varying quite rapidly; also the results given for the second root of the quartic, at values of $\frac{\omega_b}{\omega_d} > 1$, seem to behave in a rather strange manner. Due to the lack of both experimental and theoretical points it is difficult to draw any conclusions from Fig. 21. It must be remembered that the wing under discussion is a very light one with larger-than-normal amounts of stiffness, and that it is in the $\mu$ range where the bending-torsion
FIG. 21 - FLUTTER PARAMETERS vs $\frac{\omega_f}{\omega_\infty}$ FOR MODEL C
THEORY AND EXPERIMENT
flutter theory gives erroneous results. The only conclusion that can be drawn at this time is that the theory appears to be unreliable and it would be desirable to continue the investigation further.

The best method to study the problem would be to plot curves of flutter parameters vs. $\mu$ similar to those in Fig. 20 for various values of $\frac{\omega_3}{\omega_x}$; looking at just one value of $\mu$ as in Fig. 21 is not at all satisfactory. But due to lack of time and the complexity of the three-degree flutter analysis, it was not possible to determine theoretical values of the flutter parameters at other values of $\mu$. 
VI. SUMMARY AND RECOMMENDATIONS

Several interesting facts in regard to the nature of the aerodynamic forces acting at low $\mu$'s were discovered for the bending-torsion flutter case. It was shown that the forces which must be applied to the wing to keep it oscillating decrease markedly at low values of $\mu$. Also, the work done by the aerodynamic forces and moments in bending and torsion motions decrease in magnitude until some low $\mu$ is reached; then $W_{hh}$ and $W_{h\alpha}$ increase rapidly, the first negatively and the second positively. $W_{hh}$ and $W_{h\alpha}$ are functions of $L_h$ and $L_\alpha$ and are suggested as the terms most likely to be causing trouble. Changing $R \{L_\alpha\}$ is shown to have a marked effect on the flutter speed while $L_h$ is known from Ref. 10 to behave differently in experiments than the theory predicts. It would seem that $L_h$ would be the aerodynamic coefficient that should be examined more closely in future studies.

The three-degree-of-freedom results are inconclusive due to the small amount of data obtained. It has been demonstrated that a low $\mu$ wing can be built, and if rebuilt with lower stiffnesses, it is thought that the whole series of tests at various values of $\mu$ could be carried out without danger to the wing. Stiffnesses of $1/3$ of the values given to model C would decrease the flutter speed to around 40 or 50 mph. However, even at this lower speed, it is suggested that magnesium spars be used rather than wood. It is thought that there would be enough improvement in the model characteristics
to warrant the added weight. Also, it should be noted that decreasing the total stiffnesses will decrease the weight to some extent; however, preliminary calculations show that the decrease would be less than the weight added by the magnesium spars.

As one final suggestion—if future studies show that some types of bending-torsion-aileron flutter are indeed conservative at all values of $\mu$—perhaps it can be shown that only flutter involving large amounts of bending are in error. This would be in agreement with the results reported here. For the experimental flutter mode of model 0 seemed to be mainly a torsion-aileron mode, while the theoretical study carried out on model A indicated that the magnitude of the $h$ motion increased as $\mu$ decreased. The predominance of $h$ motion would make more pronounced the effect of $L_h$ and of $W_{hh}$, the negative work term shown to increase suddenly in magnitude at low $\mu$; this by itself would tend to increase the stability of the system and so increase the flutter speed since it indicates that energy is taken out of the system. This points out even more clearly the need to make a thorough examination of the effects of variations in $L_h$ keeping in mind the experimental results of Ref. 10.
VII. APPENDIX

a.) Model Properties

Three models were used for the theoretical analyses. Model A was chosen to represent a typical light airplane while model B was picked because of the large amount of experimental data (Ref. 1) available on it at low $\mu$'s; both of these models did not have the aileron degree of freedom. Model C, discussed in Sections IV and V, was used to study the effect of the aileron. The model properties are given in Table 4.

Table 5
Model Properties

<table>
<thead>
<tr>
<th>Model</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>-0.4</td>
<td>-0.628</td>
<td>-0.36</td>
</tr>
<tr>
<td>$x_{\alpha}$</td>
<td>0.285</td>
<td>0.27</td>
<td>0.185</td>
</tr>
<tr>
<td>$r_{\alpha}^2$</td>
<td>0.624</td>
<td>0.336</td>
<td>0.3127</td>
</tr>
<tr>
<td>$(\omega_h/\omega_\alpha)^2$</td>
<td>0.3333</td>
<td>0.5566</td>
<td>0.1333</td>
</tr>
<tr>
<td>$b$ (feet)</td>
<td>2.5</td>
<td>0.500</td>
<td>0.833</td>
</tr>
<tr>
<td>$\omega_\alpha$ (OPS)</td>
<td>18.59</td>
<td>21.65</td>
<td>17.15</td>
</tr>
<tr>
<td>$c$</td>
<td>--</td>
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<td>0.65</td>
</tr>
<tr>
<td>$e$</td>
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<td>0.6</td>
</tr>
<tr>
<td>$x_{\phi}$</td>
<td>--</td>
<td>--</td>
<td>0.001</td>
</tr>
<tr>
<td>$r_{\phi}^2$</td>
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<td>0.0764</td>
</tr>
<tr>
<td>$d_1$</td>
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<td>--</td>
<td>0.612</td>
</tr>
<tr>
<td>$d_2$</td>
<td>--</td>
<td>--</td>
<td>0.980</td>
</tr>
</tbody>
</table>
b.) Mass and Inertia Properties of Model C

Table 5 gives the basic properties of the model as measured, and the balance weights added for the tests. The column labeled "section and torque tube" concerns the basic uncovered balsa section with torque tube glued to the ribs as shown in Fig. 17. The torque tubes weighed approximately 20 grams apiece. The aileron mass and center of gravity location values include the effect of the covering and the balance weights added to the aileron leading edge. The mass and c.g. location of the covering was measured on a sample section and its moment of inertia was approximately computed from that data. The root section is labeled Section 1; Section 8 includes the effect of the added balsa tip. Quantities labeled I are moments of inertia about the expected elastic axis location of 6 inches from the leading edge while the final value, Iea, is about the measured e.a. 6.4 inches back. Lead weights were added to each section to give the model a mass of 13.73 gms/inch. In the last three sections the weights were divided into two lumps— one placed at the leading edge and the other just in front of the aileron; this was done to increase the moment of inertia of these sections as much as possible. The average Iea was 2.887 gm.ft^2/in. or 21.65 gm.ft^2/section. If the wing could have been tested at a higher $\mu$, it was planned to add the new balance weights in such a manner as to make the Iea as well as the mass and c.g. location, the same in each section.
<table>
<thead>
<tr>
<th>Quantity</th>
<th>Section</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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</thead>
<tbody>
<tr>
<td>section and</td>
<td>mass</td>
<td>55.6</td>
<td>56.4</td>
<td>56.4</td>
<td>52.6</td>
<td>56.3</td>
<td>55.2</td>
<td>48.8</td>
<td>65.5</td>
</tr>
<tr>
<td>torque tube</td>
<td>c.g. (in)</td>
<td>7.10</td>
<td>6.91</td>
<td>6.86</td>
<td>6.97</td>
<td>7.29</td>
<td>7.10</td>
<td>6.58</td>
<td>7.30</td>
</tr>
<tr>
<td></td>
<td>I (gm ft²)</td>
<td>7.774</td>
<td>7.350</td>
<td>7.628</td>
<td>7.310</td>
<td>8.750</td>
<td>7.305</td>
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<td></td>
<td>c.g.</td>
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<td></td>
<td>I</td>
<td>3.021</td>
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<td></td>
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<td>--</td>
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<td>1.95</td>
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<td>6.77</td>
<td>6.84</td>
<td>7.04</td>
<td>7.90</td>
<td>7.67</td>
<td>7.95</td>
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<td>15.6</td>
<td>19.4</td>
<td>15.7</td>
<td>1.52</td>
<td>3.48</td>
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<td>16.37</td>
<td>16.54</td>
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<td>14.98</td>
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<td>10.90</td>
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<tr>
<td></td>
<td>c.g.</td>
<td>--</td>
<td>--</td>
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<td>8.25</td>
<td>8.25</td>
<td>8.25</td>
<td>8.25</td>
<td>8.25</td>
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</table>
c.) Derivation of Equations (2)

The equation of motion of the model can most easily be derived by application of the Lagrange equations. These are

$$\frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \quad i = 1, 2$$

(A1)

where $q_i$ are the variables $\alpha$ and $h$; $T$ is the total kinetic energy of the system; $U$ is the potential energy and $Q_i$ is defined by

$$\delta W = \sum_{i=1}^{2} Q_i \delta q_i$$

(A2)

where $\delta W$ is the work done by the external forces during a small "virtual displacement" $\delta q_i$.

Assuming the wing to be clamped at the midspan $T$, $U$ and $Q_i$ may be calculated for the semispan. For $T$,

$$T = \int_{0}^{L} T_s (y) \, dy$$

(A3)

where $T_s (y)$ is the kinetic energy per unit span.

$$T_s = \frac{1}{2} \int \frac{V^2}{\text{chord}} \, dm = \frac{1}{2} \int_{-b}^{b} \left[ \dot{h} + (x - ba) \alpha \right]^2 \, m \, dy$$

(A4)

where the dots indicate differentiation with respect to time.

Substituting Eq. (A4) into (A3) and noting that $S = \int_{-b}^{b} (x - ba) \, m \, dy$ and $I_{\alpha} = \int_{-b}^{b} (x - ba)^2 \, m \, dy$ leads to
\[ T = \frac{1}{2} \int_0^L (m \dot{h}^2 + I \dot{\alpha}^2 + 2S \dot{\alpha} \ddot{\alpha}) \, dy \]

and, from Eqs. (1)

\[ T = \frac{1}{2} \int_0^L f_h^2 \, dy + \frac{1}{2} I \int_0^L \dot{\alpha}^2 \, dy + \int_0^L \dot{\alpha} \ddot{\alpha} \, dy \]

The potential energy will be (from simple beam and rod theory)

\[ U = \frac{1}{2} \int_0^L EI \left( \frac{d^2h}{dy^2} \right)^2 \, dy + \frac{1}{2} \int_0^L GJ \left( \frac{d\alpha}{dy} \right)^2 \, dy \]

\[ = \frac{1}{2} \int_0^L f_h^2 \, dy + \frac{1}{2} I \int_0^L \omega^2 \alpha^2 \, dy \]

The equalities in Eqs. (A6) occur because in free vibrations at the bending or torsion natural frequencies, the maximum kinetic and potential energies must be equal.

The work done by the external (aerodynamic) forces during an arbitrary displacement is

\[ \delta W = \int_0^L P \delta h + M \delta \alpha \]

\[ = \delta h_r \int_0^L P f_h \, dy + \delta \alpha_r \int_0^L M f_\alpha \, dy \]

Substituting Eqs. (A5) to (A7) in (A1) leads immediately to Eqs. (3) of Section II. A similar analysis including the effects of the aileron would lead to Eqs. (25) of Section V.

The integrals of the mode shape functions given in Eqs. (3) may be easily computed. For example,

\[ \int_0^L f_h(y) f_\alpha(y) \, dy = k \int_0^L \sin \delta y (\cos \lambda y - \cos h \lambda y) \, dy \]

\[ + \int_0^L \sin \delta y (\sin \lambda y - \sin h \lambda y) \, dy \]
\[
\begin{align*}
&= \frac{-A}{2} \left[ \frac{\cos(\delta - \lambda) y}{\delta - \lambda} + \frac{\cos(\delta + \lambda) y}{\delta + \lambda} \right]_0^L + \frac{1}{2} \left[ \frac{\sin(\delta - \lambda) y}{\delta - \lambda} - \frac{\sin(\delta + \lambda) y}{\delta + \lambda} \right]_0^L \\
&= \frac{-A}{\lambda^2 + \delta^2} \left[ \sin h \lambda y \sin \delta y - \cos h \lambda y \cos \delta y \right]_0^L \\
&= \frac{-1}{\lambda^2 + \delta^2} \left[ \cos h \lambda y \sin \delta y - \sin h \lambda y \cos \delta y \right]_0^L \\
&= -1.94456 L
\end{align*}
\]

where \( K = \frac{\sin \lambda L + \sin h \lambda L}{\cos \lambda L + \cos h \lambda L} \) and \( \lambda L = 0.597 \pi \) and \( \delta L = \pi/2 \)

The other integrals have the following values:

\[
\begin{align*}
1/L \int_0^L \left[ f_h(y) \right]^2 dy &= 9.08725 \\
1/L \int_0^L \left[ f_\alpha(y) \right]^2 dy &= 0.5
\end{align*}
\]

Other integrals necessary in the three-degree-of-freedom flutter analysis are

\[
\begin{align*}
1/L \int \frac{d^2 L}{d_1 L} f_h(y) dy &= -1.58701 \\
1/L \int \frac{d^2 L}{d_1 L} f_\alpha(y) dy &= 0.34380
\end{align*}
\]

c.) Derivation of Equations (16) and (17) in Section II

The work done by the aerodynamic forces during one cycle of flutter motion can be written \([\text{Eqs. (16)}]\) as

\[
\text{work} = \int_0^L \left[ b \int \text{cycle} \left\{ R \left\{ P \right\} R \left\{ \frac{\text{d}^2}{\text{dt}^2} \right\} \right\} \text{dy} + \int \text{cycle} \left\{ R \left\{ M \right\} R \left\{ \frac{\text{d}^2}{\text{dt}^2} \right\} \right\} \text{dy} \right]
\]

-63-
With P and M represented as in Eqs. (15) and h and \( \alpha \) given in Eqs. (1), their real parts may be expressed as

\[
R\{P\} = \pi \rho b^3 \omega^2 \left[ P_{hR} \cos \omega t - P_{hI} \sin \omega t + P_{\alpha R} \cos (\omega t + \phi) - P_{\alpha I} \sin (\omega t + \phi) \right]
\]

(A8)

\[
R\{M\} = \pi \rho b^4 \omega^2 \left[ M_{hR} \cos \omega t - M_{hI} \sin \omega t + M_{\alpha R} \cos (\omega t + \phi) - M_{\alpha I} \sin (\omega t + \phi) \right]
\]

Also we have

\[
R \left\{ \frac{d}{dt} \left( \frac{h}{b} \right) \right\} = R \left\{ i \omega \frac{h}{b} \right\} = -f_h(y) \omega h \sin \omega t
\]

(A9)

Substituting Eqs. (A8) and (A9) in Eq. (16) will lead to the work terms given in Eqs. (17). For example, \( W_{h_\alpha} \) may be determined as follows:

\[
(\text{work})_{h_\alpha} = b \int_0^L \int_0^{2\pi} \left[ P_{\alpha} \right] \cdot R \left\{ \frac{d}{dt} \left( \frac{h}{b} \right) \right\} dt dy
\]

\[
= \pi \rho b^4 \omega^2 \left( \frac{h}{b} \right) <x> \int_0^L \int_0^{2\pi} \left[ P_{\alpha I} \sin (\omega t + \phi) - P_{\alpha R} \cos (\omega t + \phi) \right] \sin \omega t d(\omega t) dy
\]

\[
= \pi \rho b^4 \omega^2 \left( \frac{h}{b} \right) <x> \int_0^L \int_0^{2\pi} \left[ (P_{\alpha I} \sin \phi - P_{\alpha R} \cos \phi) \cos \omega t + (P_{\alpha R} \sin \phi + P_{\alpha I} \cos \phi) \sin \omega t \right] \sin \omega t d(\omega t) dy
\]

\[
= \pi^2 \rho b^2 L \omega^2 \left( \frac{h}{b} \right) W_{h_\alpha}
\]

The other terms are derived in a similar manner.

e.) Calculation of Spar and Torque Tube Dimensions

The dimensions of the spruce spar and magnesium torque
tube were computed to give the model an EI of 1216.2 pound ft. and a GJ of 318 pound ft. The properties of the materials are given in Table 7.

Table 7

Properties of Model Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>E (psi)</th>
<th>G (psi)</th>
<th>weight (gm/in.³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spruce</td>
<td>$1.3 \times 10^6$</td>
<td>$7.6 \times 10^4$</td>
<td>6.56</td>
</tr>
<tr>
<td>Magnesium</td>
<td>$6.5 \times 10^6$</td>
<td>$2.23 \times 10^6$</td>
<td>29.48</td>
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</table>

All of the bending stiffness must be given by the two I beam spars. To simplify the calculations and construction both spars were made the same size. Dividing the desired EI by the value of E for spruce gives a value of I, the moment of inertia about the neutral axis of the two spar cross-sections, of $0.1346 \text{ in.}^4$. The values of web thickness of 0.16 inch, web height of 1 inch, and flange width of 0.24 inch left only the total height of the spar cross-section to be determined. The moment of inertia of a rectangular cross section about its neutral axis is given by $\frac{bh^3}{12}$ where b is the width and h the height. The desired height may thus be determined from

$$I = 0.1346 = \frac{2}{12} \left[ (.24)h^3 - (.08)(1)^3 \right]$$  \hspace{1cm} (A10)

$$h = 1.5 \text{ inches}$$

The spars contribute a small amount to the torsional stiffness through the differential binding effect and also through their own torsional stiffness. These may be determined approximately as follows:

$$GJ_{\text{spars}} = 2G \phi b^3 h = 3.04 \text{ #ft.}^2$$  \hspace{1cm} (All)
where \( b \) was taken to be the web thickness and \( h \) the spar height. \( \beta \) is an experimental factor given in Ref. 14 as a function of \( h/b \). It was taken as \( \beta = 0.3 \) for this calculation. The \( \text{GJ} \) due to differential bending is

\[
\text{GJ}_{\text{d.b.}} = \frac{3 \beta^2 \text{EI}}{2 \text{L}^2} = \frac{3(0.3)^2 1216.2}{2 \left[0.7(61.5) \right]^2} = 33.1 \text{ ft.}^2
\]  

where \( d \) is the distance between spars and \( L \) is the distance from the root to the point where the stiffness is desired; the value taken for \( L \) was at 0.7 of the wing semispan. Since the effect here is not constant along the span, this is an approximation, and one that should be made only when the differential bending stiffness is small compared to the total stiffness.

The stiffness of the torque tubes must be the desired stiffness minus the spar effects, or

\[
\text{GJ}_{\text{torque tubes}} = \text{GJ}_{\text{total}} - \text{GJ}_{\text{spars}} - \text{GJ}_{\text{d.b.}} = 281.9 \text{ ft.}^2
\]  

\( \text{GJ} \) of the torque tubes is given by

\[
\text{GJ}_{\text{tt}} = \text{GJ} \frac{d}{\Delta y}
\]  

where \( d \) is the length of the torque tube and \( \Delta y \) is the spacing between torque tubes; \( \Delta y \) was equal to 7.5 inches for the model with 8 torque tubes. \( J \) is the polar moment of inertia of a torque tube cross-section and \( G \) is the modulus of elasticity in shear for the material. Application of Eq. (A14) leads to a value of \( J \) of 0.0174 in. Since \( J \) for a solid circular cross-section is \( \frac{\pi R^4}{4} \), the value of \( J \)
for the hollow torque tube with an outer diameter of 0.95 inch can be written as

\[ J = 0.0174 = \pi/2 \left[ \left( \frac{2}{5} \right)^4 - R^4 \right] \]

where \( R = 0.448 \) inch

These calculations are the results of a trial and error process, for several values of each of the picked quantities were chosen and the calculations carried out in order to determine the combinations weighing the least. The final dimensions gave a spar weight of 3.1 gms/inch and a torque tube weight of 2.5 gms/inch.
The subscript $h$ refers to the bending degree of freedom of the wing, $\alpha$ to the torsional and $\beta$ to the aileron degree of freedom.

- $a$: wing elastic axis location measured from the wing mid-chord in fractions of the wing semichord, positive aft
- $b$: semichord of the wing
- $c$: aileron hinge axis location measured from the wing mid-chord in fractions of the wing semichord
- $d_1, d_2$: aileron location measured from the wing root in fractions of the wing semichord
- $e$: aileron leading edge location measured from the wing midchord in fractions of the wing semichord
- $f(y)$: spanwise mode shape function
- $h$: wing displacement in a direction perpendicular to the undisturbed airflow, positive downward
- $k$: reduced frequency of the wing; $k = \frac{b\omega}{V}$
- $m$: mass of wing and aileron per unit span
- $x_\alpha$: wing center of gravity location measured from the elastic axis position in fractions of the wing semichord, positive aft
- $x_\beta$: aileron center of gravity location measured from the aileron hinge axis in fractions of the wing semichord
- $\bar{A}, \bar{E}, \text{etc.}$: flutter determinant elements
- $A, B, C$: model designation letters
- $C_\xi$: torsional stiffness of aileron about its hinge axis per unit span
- $EI$: bending stiffness of the wing
- $GJ$: torsional stiffness of the wing
- $I \{\cdots\}$: indicates imaginary part of a complex quantity
- $I_\alpha$: moment of inertia per unit span of wing about the elastic axis
NOMENCLATURE (Cont'd.)

$I_{\phi}$ moment of inertia per unit span of aileron about its hinge axis

$L$ semispan of the wing

$L_p, L_\alpha, L_h$, etc. - non-dimensional aerodynamic lift coefficients

$M$ aerodynamic moment acting on the wing about the elastic axis per unit span, positive for increasing

$M_h, M_\alpha$ part of total moment (non-dimensional) due to $h$ motion, and $\alpha$ motion, respectively

$M_p, M_\alpha, M_h$, etc. - non-dimensional aerodynamic moment coefficients

$P$ aerodynamic lift acting on the wing per unit span, positive downward

$R \{ \cdot \cdot \}$ indicates real part of a complex quantity

$S_\alpha$ static moment of the wing per unit span about the elastic axis

$S_\beta$ static moment of the aileron per unit span about its hinge axis

$V$ forward speed of the wing

$V_f$ critical flutter speed of the wing

$\alpha$ angular displacement of the wing about the elastic axis, positive for increasing angle of attack

$\beta$ rotation of the aileron about its hinge axis, positive for increasing angle of attack

$\mu = 1/\kappa$ non-dimensional wing mass parameter; $\mu = m/\pi \rho b^2$

$\rho$ air density

$\phi_\alpha, \phi_\beta$ time phase angles that wing torsion and aileron deflection angles lead the bending deflection, respectively

$\omega$ vibration frequency of the wing

$\omega_f$ critical flutter frequency of the wing

$\omega_h, \omega_\alpha, \omega_\beta$ natural uncoupled frequencies of the wing in bending, in torsion and of the aileron oscillating about its hinge axis, respectively
REFERENCES


REFERENCES (Cont'd.)


