MAXIMALITY OF ALGEBRAS OF HOLOMORPHIC FUNCTIONS

by

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ABSTRACT

This thesis deals with Banach algebras of holomorphic functions of several complex variables. An attempt is made to relate properties of the algebra with properties of its space of maximal ideals, a compact subset of \( \mathbb{C}^n \). In particular, problems of maximality with respect to certain conditions are discussed.

Reinhardt compacta are treated in great detail. In particular, it is shown that for the closure \( A(P) \) of the algebra of functions holomorphic on the polycylinder, \( A(P) \), is maximal among all subalgebras of the algebra of continuous functions on the polycylinder having the torus as maximum modulus set. That is, if \( B \) is an algebra of continuous functions on \( P \); if for every \( f \) in \( B \), \( ||f||_P = ||f||_T \), then if \( B \) contains \( A(P) \) we must have \( B = A(P) \). This is not true for the algebra \( A(B) \), the corresponding algebra of the unit ball. The subalgebra \( M \) of the algebra of continuous functions on the unit ball for which this is true is constructed, and all superalgebras of \( M \) are constructed. The Silov boundary for \( A(R) \), \( R \) any Reinhardt compactum, is determined and it is proven that every boundary point not on the Silov boundary must be an interior point of a variety on the topological boundary.

Then we discuss one-dimensional varieties in order to determine what effect singularities have on the maximality theorem of John Wermer for Riemann surfaces. It is proven that the algebra of holomorphic functions on a bounded domain bounded by finitely many analytic curves on a one-dimensional variety is contained in just one maximal algebra and contains an ideal of that algebra, and is maximal only if \( \text{int}K \) has no singular points of the variety.

John Wermer has proven (under additional assumptions) the following theorem:

Let \( f_1, \ldots, f_k \) be functions holomorphic in a neighborhood of the unit circle \( \Gamma \); suppose they separate points on \( \Gamma \). Let \( A = \) the closure on \( \Gamma \) of polynomials in \( f_1, \ldots, f_k \). Suppose \( A \neq C(\Gamma) \). Then there exists a Riemann surface \( S \), a domain \( D \subset S \) bounded by a closed analytic curve \( \gamma \) and a homeomorphism \( \varphi : \overline{D} \rightarrow \gamma \) such that
i) $D \cup \gamma$ is compact,

ii) if $f$ is in $A$, $f \circ \xi$ extends into $D$ as an analytic function,

iii) $[f; f$ in $A]$ contains an ideal of the algebra of functions holomorphic in a neighborhood of $D \cup \gamma$.

In this thesis we extend this conclusion to:

Then there exists a variety $V$ contained in a polycylinder $P$ in $C^m$, $m \geq k$, and a domain $D \subset V$ bounded by an analytic curve $\gamma$ and a homeomorphism $\beta: \gamma \rightarrow \xi$ such that $[f \circ \beta; f$ in $A]$ is the closure on $\gamma$ of the algebra of functions holomorphic in a neighborhood of $D \cup \gamma$. Also a proof of this theorem of Wermer's is given using the theory of sheaves (rather than $L^2$ theory as in Wermer's proof); but only for a special case.

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# TABLE OF CONTENTS

## I. INTRODUCTION  
1. Notation and General Facts: Function Algebras  8  
2. Notation and General Facts: Several Complex Variables  11

## II. REINHARDT DOMAINS  
1. Introduction  17  
2. The Polycylinder  18  
3. The Unit Ball  22  
4. Reinhardt compacta  35  
5. Complete Reinhardt compacta in $\mathbb{C}^n$  56

## III. ONE-DIMENSIONAL VARIETIES  
1. Introduction  63  
2. Riemann Surfaces  67  
3. The Normal Model  74  
4. The Maximality Theorem  91

## IV. ON WERMER'S THEOREM  
1. Introduction  98  
2. Wermer's Theorem for Meromorphic Functions  100  
3. Extension of Wermer's Theorem  104

## V. BIBLIOGRAPHY  
107

## VI. BIOGRAPHICAL NOTE  
110
CHAPTER ONE
INTRODUCTION

The main feature of the algebra of functions analytic in the interior of the unit disc and continuous on the closed disc is that it is maximal, that is, there is no proper subalgebra of the algebra of continuous functions on the circle containing it [21]. For the similar algebra on the polycylinder in n-dimensional complex space this is no longer true; this maximality theorem is true for the algebra of analytic functions on a bounded domain bounded by an analytic curve on a Riemann surface [22], but it is not true in general for a similar domain on a variety on a complex manifold, even if the variety is one-dimensional. But in all of these cases it is seen that certain theorems like that of maximality are possible. In one case side restrictions are imposed, in another case one can discover just what the larger algebras are. The emphasis of this thesis is on questions of maximality; in a broader sense it is a survey of certain Banach algebras of holomorphic functions of several variables.
Chapter II deals first with the polycylinder and the unit ball in complex n-space. Their associated algebras of holomorphic functions are contrasted as one with a small Silov boundary in the case of the polycylinder and one with a large Silov boundary in the case of the ball. In the case of the polycylinder, the algebra of holomorphic functions on the torus with space of maximal ideals containing the topological boundary of the polycylinder; in the case of the ball this is not true, there is a larger subalgebra of continuous functions on its Silov boundary. Because of the fibered of the 3-sphere (the topological boundary of the unit ball) as circles over the 2-sphere it is possible to characterize all algebras containing this larger algebra. This is all possible because in the case of the unit ball the Silov boundary and the topological boundary coincide.

Reinhardt compacta are considered, being the first possible generalization of these two elementary cases. The main object is to determine the nature of the Silov boundary of the algebra of holomorphic functions, and to show that (topological boundary)-(Silov boundary) is a union of analytic varieties.

The main aim of Chapter III is to find in what form Wermer's maximality theorem for domains on a Riemann surface holds for domains on a one-dimensional variety. It is proven that such a theorem holds only if the variety is in fact a complex manifold.
(Riemann surface). In the general case it is proven that the algebra of holomorphic functions on a bounded domain bounded by finitely many analytic curves is contained in just one maximal algebra and contains an ideal of that algebra.

John Wermer has proven (under addition assumptions) the following theorem [24]:

Let \( f_1, \ldots, f_k \) be functions holomorphic in a neighborhood of the unit circle \( \Gamma \); suppose they separate points on \( \Gamma \). Let \( A = \) closure on \( \Gamma \) of polynomials in \( f_1, \ldots, f_k \). Suppose \( A \neq C(\Gamma) \). Then there exists a Riemann surface \( S \), a domain \( D \subset S \) bounded by a closed analytic curve \( \partial \) and a homeomorphism \( \psi: \partial \to \Gamma \) such that

1) \( D \cup \partial \) is compact,

2) if \( f \) in \( A \), \( f \circ \psi \) extends into \( D \) as an analytic function,

3) \( [f \circ \psi; f \) in \( A] \) contains an ideal of the algebra of functions holomorphic in a neighborhood of \( D \cup \partial \).

In Chapter IV this conclusion is extended to:

Then there exists a variety \( V \) contained in a polycylinder \( P \) in \( \mathbb{C}^m \), \( m \geq k \) and a domain \( D \subset V \) bounded by an analytic curve \( \partial \) and a homeomorphism \( \varphi: \partial \to \Gamma \) such that \( [f \circ \varphi; f \) in \( A] \) is the closure on \( \partial \) of the algebra of functions holomorphic in a neighborhood of \( D \cup \partial \).

If, in the theorem above it were known that \( f_1, \ldots, f_k \) are holomorphic in a neighborhood of the unit disc, then the Riemann surface \( S \) is the plane
and \( \mathcal{O} \) is the unit circle, only part iii) really has content. If it were known that \( D \cup \mathcal{O} \) is convex with respect to \( A \), then that theorem would have a short proof using the theory of Sheaves. This convexity is the major problem; we show that this is so if \( f_1, \ldots, f_k \) are polynomials.

1. Notation and General Facts: Function Algebras

Let \( X \) be a compact Hausdorff space.

Df 1. A function algebra on \( X \) is a subalgebra \( A \) of \( C(X) \) which is closed in the uniform norm topology.

\[ S(A) = \text{space of maximal ideals of } A = \text{the set of nonzero multiplicative functionals on } A \text{ with the weak topology.} \]

\[ \Gamma(A) = \text{Silov boundary of } A = \text{the smallest closed subset } Y \text{ of } S(A) \text{ satisfying the condition} \]

\[ \|f\|_X = \|f\|_{S(A)} \text{ for all } f \text{ in } A. \]

\( x \) in \( \Gamma(A) \) is called a strong boundary point if and only if there is an \( f \) in \( A \) such that \( |f(x)| > |f(y)| \) for all \( y \neq x \) in \( S(A) \).

The strong boundary points are dense in \( \Gamma(A) \) if \( \Gamma(A) \) is metrizable [5].

Df 2. A function algebra \( M \) on \( X \) is a maximal subalgebra of \( C(X) \)

1) \( M \neq C(X) \)

2) for any algebra \( B, M \not\subseteq B \subseteq C(X), B \) is dense in \( C(X) \).

A function algebra \( A \) is called \( M \)-primary in
1) $A \subseteq M$

ii) $M$ is maximal in $C(X)$

iii) for any algebra $B$, $A \subset B \subset C(X)$ implies $B \subseteq M$ or $B$ is dense in $C(X)$.

**Definition 3.** Let $A$ be a function algebra on $X$. Let $Y$ be a subset of $S(A)$. $A$ is $(Y,X)$-maximal if and only if for any algebra $B$, $A \subset B \subset C(X)$, $S(B) \cap Y$ implies $B = A$.

$A$ is strictly $(S(A),X)$-maximal if and only if for any algebra $B$, $A \subset B \subset C(X)$ and $S(B) = S(A)$ implies $B = A$.

We note that if $M$ is a maximal subalgebra of $X$, then $X = \Gamma(A)$ [15], and if $A$ is $(X,X)$-maximal, then $A$ is a maximal subalgebra of $C(X)$.

In Chapter IV we will use the following general facts about Banach algebras, due to Hoffman and Singer, [14], but here stated differently so as to be applicable to the situations which arise here.

**Theorem 1.** Let $M$ be a maximal algebra of $C(X)$. Let $A \subseteq M$ be a closed subalgebra of $M$ such that $A \supseteq I$, and ideal of $M$. Let $H = \text{hull of } I$ in $S(M)$, $S = \{f \in C(X) \text{ vanishing on } H \cap X\}$. Then if $B$ is a closed subalgebra of $C(X)$, $B \supseteq A$, then either $B \subseteq M$ or $B \supseteq S$.

**Proof:** Suppose there exists $g$ in $B - M$. Let $\mathcal{O} = \{\text{polynomials in } g, \text{ coefficients in } M\}$, then $\mathcal{O}$ is dense in $C(X)$. Let $h$ be in $C(X)$, $p_n$ be in $\mathcal{O}$, $p_n \to h$. Then for $f$ in $I$, $fp_n \to fh$. Since $[fp_n]$ is in $B$, $fh$
is in B. Thus \( B \supset \{ fh; f \text{ in } I, h \text{ in } C(X) \} \). But then B contains the closed algebra generated by this set, which is a closed ideal of \( C(X) \), obviously, \( \mathcal{S} \), since it has the same hull as \( \mathcal{S} \).

**Corollary 1.** Let \( M \) be a maximal subalgebra of \( C(X) \). Let \( H_1, \ldots, H_n \) be pairwise disjoint hulls in \( S(M) \). Let \( A = \{ f \in M; f|_{H_j} \text{ is constant, } j = 1, \ldots, n \} \). Then \( A \) is a maximal subalgebra of \( C(\text{Silov boundary of } A) \) among all subalgebras having \( S(A) \) as space of maximal ideals, i.e., \( A \) is \( (S(A), \Gamma(A)) \)-maximal.

**Proof:** \( A \) is a subalgebra of \( M \), there exists a map \( \pi: S(M) \to S(A) \). Obviously \( \pi \) is a homeomorphism on \( S(M) - (\bigcup_{i=1}^{n} H_i) \) and \( \pi(H_i) = y_1 \), a single point, and \( \pi(X) \supset \Gamma(A) \). Let \( B \subset C(\Gamma(A)) \), and \( S(B) = S(A) \). \( B \) is a closed subalgebra of \( C(X) \), thus either \( B \subset M \), or \( B \supset \{ f \in C(X); f|_{(\bigcup_{j=1}^{n} H_j)} = 0 \} \), since \( \bigcup_{i=1}^{n} H_i \) is a hull, and we can apply the theorem. If \( B \subset M \), since \( S(B) = S(A) \), any \( f \) in \( B \) is such that \( f|_{H_j} = f(y_j) = \text{constant} \). Thus \( B \subset A \), so \( B = A \). If not, then \( B \), as a subalgebra of \( C(\Gamma(A)) \), contains \( \{ f \in C(\Gamma(A)); f(y_1) = 0 \text{ for } y_1 \text{ in } \Gamma(A) \} \). But then, since \( B \) separates \( \Gamma(A) \), obviously \( B = C(\Gamma(A)) \).

**Corollary 2.** Let \( M \) be a maximal subalgebra of \( C(X) \). Let \( H_1, \ldots, H_n \) be disjoint hulls in \( S(M) \), with \( H_j \cap X \neq \emptyset \), \( j = 1, \ldots, n \). Then \( A = \{ f \in M; f|_{H_j \cap X} \text{ is constant, } j = 1, \ldots, n \} \) is a maximal subalgebra of \( C(\Gamma(A)) \).

**Proof:** We calculate the space of maximal ideals.
as above. Letting \( H' = \text{hull}(\text{kernel}(H \cap X)) \) we have
\[
\pi: S(M) \to S(A) \quad \text{onto},
\]
\( \pi: S(M) - \mathcal{J}_{\lambda} H' \) is a homeomorphism,
\( \pi(H'_i) = y_i, \pi(X) = \Gamma(A) \). Let \( B \) be a subalgebra of \( C(\Gamma(A)) \),
\( B \supset A \). If \( B \subset M \), since also \( B \subset C(\Gamma(A)) \), we must have for
all \( f \) in \( B \), \( f \mid_{H'_j} = \text{constant} \). Then, by definition of \( A \),
\( B \subset A \), so \( B = A \). If \( B \not\subset M \), then as in Corollary 1, \( B = C(\Gamma(A)) \).

**Corollary 3.** Let \( M \) be a maximal subalgebra of \( C(X) \),
\( A \) a subalgebra of \( M \). Let \( B \supset A \) be a subalgebra of \( C(X) \).
If \( A \) contains an ideal \( I \) of \( M \) such that \( (\text{hull } I) \cap X \) is a
finite point set, then \( B \subset M \) or \( B = C(S(B)) \).

**Proof:** Let \( (\text{hull } I) \cap X = [x_1, \ldots, x_n] \). Then if \( B \not\subset M \),
by the theorem, \( B \supset \{ f \in C(X); f(x_i) = 0, i = 1, \ldots, n \} \).
Since the Silov boundary of \( B \) is an identification space
of \( X \), we can say that for points \( a_1, \ldots, a_k \) in \( S(B) \),
\( B \supset \{ f \in C(S(B)); f(a_1) = 0 \} \). Since \( B \mid_{a_1} \) is separating,
it is \( C([a_1, \ldots, a_k]) \). Then if \( f \) is in \( B \), there exists
a \( g \) in \( B \) such that \( g(a_1) = \overline{f(a_1)} \). Then \( \overline{f} - g \) is in \( C(S(B)) \).
and is 0 on \( a_1, \ldots, a_n \), so is in \( B \). Then \( \overline{f} = \overline{f} - g + g \) is in \( B \),
so, by the Stone-Weierstrass theorem, \( B = C(X) \).

2. **Notation and General Facts:** Several Complex Variables

**Def 4.** \( \mathbb{C}^n \) = the \( n \)-dimensional vector space over the
complex field.

\( \mathcal{O}^n \) = the sheaf of germs of holomorphic functions
on \( \mathbb{C}^n \). If \( M \) is a complex analytic manifold, \( \mathcal{O}^M \) = the
sheaf of germs of holomorphic functions on \( M \).
Let $M$ be a complex analytic manifold, and $V$ a variety in $M$, i.e., for all $m$ in $M$, there is a neighborhood $U$ of $m$ and finitely many functions $f_1, \ldots, f_K$ in $H^0(U, \mathcal{O}_U^M) = \{\text{functions holomorphic on } U\}$ such that $U \cap V$ is the set $V(f_1, \ldots, f_K)$ of common zeros of $f_1, \ldots, f_K$.

For $m$ in $M$, let $\mathfrak{g}_m$ be the ideal of $\mathcal{O}_m^M$ of germs of holomorphic functions vanishing on $V$. Let $\mathfrak{g}(V)$ be the sheaf on $M$, whose stalk at $m$ is $\mathfrak{g}_m$. The sheafs $\mathfrak{g}(V)$, $\mathcal{O}_m^M/\mathfrak{g}(V)$ are coherent analytic sheafs on $M$. Define $\mathcal{O}^V = \mathcal{O}_m^M/\mathfrak{g}(V)|_V = \text{sheaf of germs of holomorphic functions on } V$.

Let $M$ be a complex analytic manifold, $K$ a compact set on $S$.

$A^0(K) = C(K) \cap H^0(\text{int} K, \mathcal{O}_U^M)$

$A(K)$ = uniform closure on $K$ of the set of all functions holomorphic in a neighborhood of $K$.

$A(K, U) = \text{uniform closure on } K \text{ of } H^0(U, \mathcal{O}_U^M)$, for $U$ a neighborhood of $K$.

Let $K$ be a compact set on $M$.

$K$ is holoholomorphically convex if and only if $K = S(A(K))$.

$K$ is $H^0(U, \mathcal{O}_U^M)$-convex if and only if for $x$ in $U-K$, there exists $f$ in $H^0(U, \mathcal{O}_U^M)$ such that $\|f\|_K < |f(x)|$.

Let $P$ be a subset of $H^0(U, \mathcal{O}_U^M)$.

$K$ is $P$-convex for $x$ in $U-K$, there is an $f$ in $P$ such that $\|f\|_K < |f(x)|$.

$[x \in U ; |f(x)| \leq \|f\|_K] = P$-convex hull of $K$.

Theorem 2. If $U$ is a Stein manifold and $K$ is
$H^o(U,\sigma^U)$-convex, then $K$ is holoconvex and $H^o(U,\sigma^U)$ is dense in $A(K)$ in the uniform norm on $K$.

This is a known fact, I shall only sketch the proof. First, $H^o(U,\sigma^U)$ is dense in $A(K)$: Because $K$ is $H^o(U,\sigma^U)$-convex and $U$ is a Stein manifold, given any neighborhood $V \subset U$ of $K$, there are finitely many functions $f_1, \ldots, f_t$ in $H^o(U,\sigma^U)$ such that $V \supset W = \{ |f_j| < 1; 1 \leq j \leq t \}$ and the mapping $\varphi: W \to \mathbb{C}^t$:

$$\varphi(x) = (f_1(x), \ldots, f_t(x))$$

is one-one and nonsingular. Then $\varphi(W)$ is a variety in $[|z_1| < 1, \ldots, |z_t| < 1] = \mathbb{P}^t$. Let $g$ be in $H^o(V,\sigma^U)$, then $g|_{\varphi(W)}$ is in $H^o(\varphi(W), \sigma^V)$. Then, by Theorem B, there is a function $G$ in $H^o(\mathbb{P}^t,\sigma^t_{\mathbb{P}^t})$ such that $G|_{\varphi(W)} = g|_{\varphi(W)}$. Now $G$ has a uniformly convergent power series expansion in $\mathbb{P}^t$, so the partial series of this expansion converge uniformly to $G$ on any compact set. $\varphi(K)$ is a compact set, so we can find polynomials $P_n$ such that $P_n|_{\varphi(K)} \to G|_{\varphi(K)} = g|_{\varphi(W)}$ uniformly, i.e.,

$$\|P_n - G|_{\varphi(K)} \to 0.$$ 

then

$$\|P_n \circ \varphi - g|_K \to 0.$$ 

But $P_n \circ \varphi = P_n(f_1(x), \ldots, f_t(x))$ is in $H^o(U,\sigma^U)$.

Thus $H^o(U,\sigma^U)$ is dense in $H^o(K,\sigma^U_K)$, so $H^o(U,\sigma^U)$ is dense in $A(K)$.

Now evaluation of a point of $K$ is a homomorphism of $A(K)$, so $K \subset S(A(K))$. Let now $h$ be a nonzero homomorphism of $A(K)$, and let $f_1, \ldots, f_t$ be as in the above.
Let \( h(f_1) = a_1, |a_1| < 1 \), since \( \|f_1\|_K < 1 \). If \( P(X_1, ..., X_t) \) is a polynomial, then

\[
h(P(f_1, ..., f_t)) = P(a_1, ..., a_t).
\]

If \( a = (a_1, ..., a_t) \) is not in \( \Psi(W) \), there is a function \( G \) in \( H^0(P^t, \mathcal{O}^t) \) such that \( G(a) = 1, G|_{\Psi(W)} = 0 \). Then there is a polynomial \( P \) such that \( \|P-G\|_{\Psi(W)U(a)} < 1/2 \), so \( |P(a)| > \|P\|_{\Psi(K)} \), so

\[
|h(P(f_1, ..., f_t))| = |P(a)| > \|P(f_1, ..., f_t)\|_{\Psi(K)},
\]

which is impossible.

Then \( a \) is in \( \Psi(W) \), let \( x = \Psi^{-1}(a) \). If \( g \) is in \( A(K) \), \( g \) is approximable by functions in \( H^0(U, \mathcal{O}^U) \cap H^0(W, \mathcal{O}^W) \), in which the polynomials in \( f_1, ..., f_t \) are dense. Thus \( g \) is approximable by polynomials in \( f_1, ..., f_t \). But for \( P \) a polynomial, \( h(P) = P(x) \), therefore for all \( g \) in \( A(K) \), \( h(g) = g(x) \).

By assumption if \( x \) is not in \( K \), there exists a \( g \) in \( H^0(U, \mathcal{O}^U) \) such that

\[
\|f\|_K < |f(x)|.
\]

Thus we must have \( x \) in \( K \). Then \( h \) is evaluation at a point of \( K \), proving \( \Psi(A(K)) = K \).

**Theorem 3.** Let \( K \) be a compact subset of a complex manifold \( M \) of dimension \( m \geq 2 \). Let \( U \) be open in \( M \) and \( f_1, ..., f_k \) in \( H^0(U, \mathcal{O}_U^M) \), \( k \ll n \), and suppose \( V = \{ f_1 = 0, ..., f_k = 0 \} \) intersects \( \partial MW \) in an open subset of \( V \), and \( V_n = \{ f_1 = \varepsilon_n, ..., f_k = \varepsilon_n \} \) intersects \( \text{int} K \cap U \) in an open subset of \( V_n \). Suppose also \( \{dF = 0\} \cap V \) is nowhere dense in \( V \), where \( F(z) = (f_1(z), ..., f_k(z)) \).
Then no point of $V$ is a strong boundary point of $A^o(K)$.

Proof: Let $x$ be in $[dF\neq 0]$. Then there is a neighborhood $W$ of $x$ and functions $z_1, \ldots, z_m$ in $H^o(W, \mathcal{O}_W^M)$ such that $z_1(x) = 0$, $z_{m-i+1} = f_i$, $1 \leq i \leq k$, and $z_1, \ldots, z_m$ form a coordinate system for $W$. We may suppose $W = \{ |z_j| < 1 \}$. Now $[z_1 = 0] \subset K, m-k < i \leq m$. Let $\omega$ be the inverse to the map $x \rightarrow (z_1(x), \ldots, z_m(x))$. For $g$ in $C(K)$ let $g_0(w_1, \ldots, w_{m-k}) = g(\omega(w_1, \ldots, w_{m-k}, 0, \ldots, 0)$, and $g_n(w_1, \ldots, w_{m-k}) = g(\omega(w_1, \ldots, w_{m-k}, \epsilon_n, \ldots, \epsilon_n)$. Then $g_n \rightarrow g_0$ uniformly on compact subsets of $P^{m-k}$. For, given $\eta > 0$, since $K$ is compact and $g$ is in $C(K)$, there is a $\delta > 0$ such that

for $x, y$ in $K$, $d(x, y) < \delta$ implies $|g(x) - g(y)| < \eta$. ($d$ = distance in a metric on a neighborhood of $K$ in $M$). Since $z_1, \ldots, z_m$ form a coordinate system for $W$, there is an $r > 0$, such that

for $x, y$ in $K$, $|z_1(x) - z_1(y)| < r$, $1 \leq i \leq m$ implies $|g(x) - g(y)| < \eta$.

Let $N$ be such that $\epsilon_n < r$ for $N \leq n$. Then

$|g(\omega(w_1, \ldots, w_{m-k}, 0, \ldots, 0) - g(\omega(w_1, \ldots, w_{m-k}, \epsilon_n, \ldots, \epsilon_n)| < \eta$,

for $(w_1, \ldots, w_{m-k})$ in $P^{m-k}$, thus $\|g_0 - g_n\| < \eta$. Now let $g$ be in $A^o(K)$. Then $g$ is in $H^o(\text{int}K, \mathcal{O}_{\text{int}K}^M)$. Since $V_n$ is a subvariety of $\text{int}K \cap W$, then $g|_{V_n} = g_n \circ \omega^{-1}$ is in $H^o(V_n, \mathcal{O}^V_{V_n})$. Then $g_n$ is in $H^o(P, \mathcal{O}_P^{m-k})$. Thus since $g_n \rightarrow g$ uniformly, then $g_0$ is also in $H^o(P, \mathcal{O}_P^{m-k})$ or $g_0 \circ \omega^{-1} = g|_V$ is in $H^o(V, \mathcal{O}_{V\cap W}^V)$.

Thus every $g$ in $A^o(K)$ is a holomorphic function on
the variety $V$ at the points $[dF \neq 0]$. But $[dF = 0]$ is a nowhere dense variety on $V$, therefore, by the lemma stated below, $g$ does not take a maximum on $V$, for, since $V$ is of dimension greater than one it is not compact.

Thus no function in $A^0(K)$ takes a maximum in $V$, thus no point of $V$ is a strong boundary point for $A^0(K)$.

For the algebra $A(K)$, this theorem is trivial, for in this case, functions holomorphic on a neighborhood of $V$ is dense in $A(K)$, and all these functions are automatically holomorphic on $V$.

Lemma 1. Let $V$ be a noncompact variety in a complex manifold $M$, and let $V_1$ be a nowhere dense subvariety on $V$. Suppose $f$ is continuous on $V$, and $f |_{V-V_1}$ is holomorphic. Then $f$ does not take a maximum on $V$.

This follows since $f$ can be lifted as a holomorphic function on the normal model for $V$ (see definition 17, Chapter Three) and then the maximum modulus principle for varieties applies (see [3], vol. 2).
CHAPTER TWO
REINHARDT DOMAINS

1. Introduction

In this chapter we will study the algebra $A(K)$ where $K$ is a Reinhardt compactum. The space of maximal ideals and the Silov boundary of $A(K)$ will be characterized in terms of the mapping $(z_1, \ldots, z_n) \rightarrow (\log|z_1|, \ldots, \log|z_n|)$. The relation between the Silov boundary of $A(K)$ and the topological boundary of $K$ is further explored in the case that $K$ is a complete Reinhardt domain. More particularly the sphere and the polycylinder, being extreme cases, are contrasted with respect to $(S(A(K)), \Gamma(A(K)))$-maximality.

Def 8. A Reinhardt set $S$ in $C^n$ is a set satisfying the following condition:

$z = (z_1, \ldots, z_n)$ is in $S$ implies $[(e^{i\theta}z_1, \ldots, e^{i\theta}z_n) ; \quad 0 \leq \theta \leq 2\pi, \quad i=1, \ldots, n]$ is contained in $S$.

$R$ is complete Reinhardt if and only if $z = (z_1, \ldots, z_n)$ in $S$ implies $[(t_1z_1, \ldots, t_nz_n) ; \quad |t_i| \leq 1, \quad i=1, \ldots, n]$ is contained in $S$. 
Df 9. A (complete) Reinhardt compactum (domain)
is a connected (complete) Reinhardt set which is compact
(open).

Let $Q^n$ be the $n$-dimensional vector space over the
reals. Let $\overline{Q}^n$ be $Q^n$ with the hyperplanes $x_i = -\infty, 1 \leq i \leq n,$
added, i.e., $\overline{Q}^n$ is a space containing $Q^n$ homeomorphic
to $[x_1 \geq 0, \ldots, x_n \geq 0]$ such that the homeomorphism
restricted to $Q^n$ is $(x_1, \ldots, x_n) \rightarrow (e^{x_1}, \ldots, e^{x_n}).$

Df 10. Let $L: C^n \rightarrow \overline{Q}^n$,

$$L(z_1, \ldots, z_n) = (\log |z_1|, \ldots, \log |z_n|).$$

Remark: 1) $R$ is a Reinhardt set implies that $\text{int} R,$
$\partial R, \overline{R},$ are also Reinhardt. Further $\text{int} R = L^{-1}\text{int} L(R),$
$\partial R = L^{-1}\partial L(R)$.

2. The Polycylinder

Let $P^n = [(z_1, \ldots, z_n) \in C^n; |z_1| \leq 1, 1 \leq i \leq n].$
$P^n$ is a complete Reinhardt compactum. We assume known
that $A(P^n)$ is the same as $A^o(P^n),$ and is the closure on
$P^n$ of the polynomials, and is also the set of functions
continuous on $T^n = [(z_1, \ldots, z_n) ; |z_1| = 1, 1 \leq i \leq n]$ whose
$(k_1, \ldots, k_n)$th Fourier coefficient is zero whenever
some $k_i$ is negative. Further, $S(A(P^n)) = P^n$ and
$\Gamma(A(P^n)) = T^n[2, 14].$ In any event, these facts will
come out of the considerations of complete Reinhardt
domains in sections 4 and 5. For $n = 1,$ $A(P)$ is a
maximal subalgebra of $C(T),$ but for $n > 1,$ this is no
longer true. The method of proof given by Hoffman and
Singer is to first show that A(P) is (P,T)-maximal and then to show that for any larger subalgebra B of C(T), either S(B) ∩ P or B is dense in C(T). The latter fact is not true for n > 1; the subalgebra B of functions continuous on T whose restriction to 

\[ [z_1 = 1, \ldots, z_{n-1} = 1] \] has negative Fourier coefficients zero is a proper subalgebra of C(T) containing A. Here S(B) = T ∪ \{z_1 = 1, \ldots, z_{n-1} = 1, |z_n| ≤ 1\}. We prove now that this is precisely what fails, in the case of n > 1, for in fact A(P^n) is always (P^n,T^n)-maximal. In fact,

**Theorem 4.** A(P^n) is (P^n,T^n)-maximal.

Proof: Suppose B is a closed subalgebra of C(T^n) containing A, and S(B) ∩ P^n. More precisely, every function in B extends uniquely to P^n preserving norm. We prove by induction on n that B = A(P^n).

The case n = 1 is just part of the maximality theorem for the unit disc in C^1.

We assume that A(P^{n-1}) is (P^{n-1},T^{n-1})-maximal.

Let |w_1| = 1, then D = \{(z_1, \ldots, z_n) in P^n; z_1 = w_1\} = \{(w_1,z_2,\ldots,z_n); |z_1| ≤ 1, 2 ≤ i ≤ n\} is conformal to the n-1 dimensional polycylinder under the map \( \pi: z \rightarrow (z_2, \ldots, z_n) \). Thus the algebra \( A^o = \{f \circ w; f in A(P^{n-1})\} \) is (D,T^o)-maximal, \( T^o = \{(w_1,z_2,\ldots,z_n); |z_1| = 1, 2 ≤ i ≤ n\} \). Since D ⊆ D, obviously \( A^o \) is (D,T^o)-maximal. Let B|_D be the restriction of B to D.
1) \( S(B|_{T^0}) \supset D \). Since \( S(B) \supset \mathbb{P}^n \), \( S(B) \supset D \). We have only to prove that for \( z \) in \( D \), \( |f(z)| \leq |f|_{T^0} \) for all \( f \) in \( B \); then evaluation at \( z \) is a continuous homomorphism of \( B_{T^0} \), so \( z \in S(B|_{T^0}) \). Suppose on the contrary that there exists an \( f \) in \( B \) and a \( z^0 \) in \( D \) such that \( \|f\|_{T^0} \leq |f(z^0)| \), i.e., we may suppose

\[
|f(w_1, e^{i\theta_1}, \ldots, e^{i\theta_n})| < 1 - \varepsilon < 1 < |f(z^0)|
\]

for all \( \theta_1, \ldots, \theta_n \). Then there is a neighborhood \( U \) of \( w_1 \) such that

\[
|f(z_1, e^{i\theta_1}, \ldots, e^{i\theta_n})| < 1 - \varepsilon \text{ for all } z_1 \text{ in } U.
\]

Let \( g(z) = \frac{1}{2}(e^{i\arg w_1} + z_1) \). Then \( |g(w_1, z_2, \ldots, z_n)| = 1 \) for all \( (z_2, \ldots, z_n) \) and \( |g(z_1, \ldots, z_n)| < 1 \) for \( z_1 \neq w_1 \). Then there is an \( \eta > 0 \) such that for \( z_1 \) not in \( U \),

\[
|g(z_1, \ldots, z_n)| < 1 - \eta.
\]

Choose \( N \) so that \((1-\eta)^N < \frac{1-\varepsilon}{\|f\|_{T^0}}\), and let \( h = g^N f \). Then \( h \) is in \( B \) and

a) \( |h(z^0)| = |g(z^0)^N f(z^0)| = |g(z^0)|^N |f(z^0)| > |f(z^0)| > 1 - \varepsilon \), since \( z^0 \) is in \( D \).

b) for \( e^{i\theta} \) in \( U \), \( |h(e^{i\theta}, e^{i\theta_1}, \ldots, e^{i\theta_n})| =
\]

\[
|g(e^{i\theta}, e^{i\theta_1}, \ldots, e^{i\theta_n})|^N |f(e^{i\theta}, \ldots, e^{i\theta_n})| < 1 \cdot (1 - \varepsilon).
\]

c) for \( e^{i\theta} \) not in \( U \), \( |h(e^{i\theta}, \ldots, e^{i\theta_n})| \leq
\]

\[
|g(z_1, e^{i\theta_1}, \ldots, e^{i\theta_n})|^N |f(e^{i\theta}, \ldots, e^{i\theta_n})| \leq (1 - \eta)^N \|f\| < 1 - \varepsilon.
\]

Therefore \( \|h\|_{T^0} < |h(z^0)| \), contradicting \( B \) is a sub-algebra of \( C(T^n) \) and \( z^0 \) in \( S(B) \). Thus we have \( S(B|_{T^0}) \supset D \).

Let \( B^0 \) be the closure of \( B|_{T^0} \) in the sup norm on \( T^0 \). Then \( B^0 \subset C(T^0) \), and \( S(B^0) \supset D \). (It is easy, in fact, using the function \( g \) to prove that \( B_{T^0} \) is actually closed).
2) \( \mathbb{B}^0 \subset \mathbb{A}^0 \). For if \( f \) is in \( \mathbb{A}^0 \), then \( f = g^0 \nu \), \( g \) in \( \mathbb{A}(\mathbb{P}^{n-1}) \). Extend \( f \) to all of \( \mathbb{P}^n \) by \( \tilde{f}(z_1, \ldots, z_n) = g(z_2^{*}, \ldots, z_n^{*}) \), then \( \tilde{f} \) is in \( \mathbb{C}(\mathbb{P}^n) \); further since \( \nu \) is holomorphic, \( g \) holomorphic on \( \text{int} \mathbb{P}^{n-1} \) implies \( \tilde{f} \) holomorphic on \( \text{int} \mathbb{P}^n \). Thus \( \tilde{f} \) is in \( \mathbb{A}(\mathbb{P}^n) \subset \mathbb{B} \), and \( \tilde{f} \bigg|_{\mathbb{I}_0} \) is in \( \mathbb{B} \bigg|_{\mathbb{I}_0} \subset \mathbb{B}^0 \). Thus \( \mathbb{A}^0 \subset \mathbb{B}^0 \). By the induction assumption then \( \mathbb{B}^0 = \mathbb{A}^0 \), so, since \( \mathbb{B} \bigg|_{\mathbb{I}_0} \supset \mathbb{A}^0 \), \( \mathbb{B} \bigg|_{\mathbb{I}_0} = \mathbb{A}^0 \).

3) We can prove the same for any \( w_j \), \( |w_j| = 1 \), \( 1 \leq j \leq n \). Thus, for any \( g \) in \( \mathbb{B} \), \( g \big|_{(z_j = w_j)} \) is analytic in the \( n-1 \) polycylinder \( (z_j = w_j) \cap \mathbb{P}^n \). Let \( g \) have the Fourier series

\[
\sum_{k_1, \ldots, k_n} a_{k_1} \ldots k_n e^{ik_1 \theta_1} \ldots e^{ik_n \theta_n}.
\]

Then \( g \big|_{(z_1 = w_1)} \) has the Fourier series

\[
\sum_{k_1, \ldots, k_n} \left( \sum_{k_1} a_{k_1} \ldots k_1 w_1^{k_1} \right) e^{ik_1 \theta_1} \ldots e^{ik_n \theta_n},
\]

and, since it is analytic,

\[
\sum_{k_1, \ldots, k_n} a_{k_1} \ldots k_1 w_1^{k_1} = 0 \quad \text{if} \quad k_1 < 0 \quad \text{for any} \quad i \neq 1.
\]

This is true for all \( |w_1| = 1 \), thus, by uniqueness of Fourier series, we must have

\[
a_{k_1} \ldots k_1 = 0 \quad \text{for all} \quad k_1 \quad \text{if some} \quad k_1 < 0, \quad i \neq 1.
\]

Thus \( a_{k_1} \ldots k_n = 0 \) whenever \( k_1 < 0, \ i \neq 1 \). Similarly, by considering \( g \big|_{(z_n = w_n)} \) we can prove \( a_{k_1} \ldots k_n = 0 \) whenever \( k_i < 0, \ i \neq n \). Together we have \( a_{k_1} \ldots k_n = 0 \) whenever \( k_i < 0, \ i \neq n \). Then (1) has only positive terms, so represents an analytic function. Thus \( g \) is in \( \mathbb{A}(\mathbb{P}^n) \).

Thus \( \mathbb{B} \subset \mathbb{A}(\mathbb{P}^n) \), so \( \mathbb{B} = \mathbb{A}(\mathbb{P}^n) \).
This theorem uses the fact that \( \partial P^n \) consists of varieties to great advantage; in fact, it is the crux of the proof. In the next section we shall show that for the unit ball \( B \), whose topological boundary is the Silov boundary of \( A(B) \), \( A(B) \) is not \( (S(A(B),A(B))) \)-maximal.

I have made an attempt to find all maximal algebras containing \( A(P^n) \), but with no success. Given any one-dimensional variety \( V \) in a neighborhood of \( P^n \) such that \( V \cap \partial P^n \subset T^n \) and bounds a domain of \( V \) in \( P^n \), we define \( A^V \) to be the set of all functions \( f \) in \( C(T^n) \) such that \( f|_{V \cap T^n} \) extends to a function holomorphic in \( V \cap T^n \). We shall prove in Chapter IV that \( A^V \) is contained in a unique maximal algebra; and is maximal if \( V \) is a manifold. Thus to every such variety corresponds a maximal subalgebra of \( C(T^n) \) containing \( A(P^n) \). Among these are to be found the maximal algebras obtained from the Fourier transforms of \( L^1(G_+) \); \( G_+ \) a maximal semigroup of \( \mathbb{I}^n \) (\( \mathbb{I} = \text{integers} \)) containing \([(k_1, \ldots, k_n); k_i \geq 0] \). Whether or not these are all the maximal algebras containing \( A(P^n) \) is not known.

3. The Unit Ball

Let \( B \) be the unit ball in \( \mathbb{C}^n \); \( B = \{z; |z_1|^2 + \ldots + |z_n|^2 \leq 1\} \). As in the case of the polycylinder the following facts are well known, and in any event come out of the theorems in sections 4 and 5 on complete Reinhardt domains.
1) \( A(B) = \{ f \text{ in } C(B); \text{ f is holomorphic on } \text{int } B \} = \) closure of polynomials on \( B \).

2) \( S(A(B)) = B \) and \( \Gamma(A(B)) = \partial B \). Further, every boundary point is a strong boundary point.

Let \( CP^n \) be projective \( n \)-space, \( \pi: C^n - 0 \to CP^n \) the natural projection. For \( s \) in \( CP^n \), let \( \pi^{-1}(s) = C_s = \) the complex line in 'the direction \( s \)'. Let \( D_s = C_s \cap B, \Gamma_s = C_s \cap \partial B \). \( D_s \) is a disc, \( \Gamma_s \) is the circle bounding \( D_s \). Now for any \( s^o \) in \( CP^n \), there is a neighborhood \( U \) of \( s^o \) and a map \( m^o: U \to \partial B \) such that the map \( \varphi^o; (t,s) \to tm^o(s) \) is a homeomorphism of \( U \times C^1 \) and \( \pi^{-1}(U); \) in particular \( \varphi^o: [|t| = 1] \times U \leftrightarrow \pi^{-1}(U) \cap \partial B \) is a homeomorphism. Thus, in \( \pi^{-1}(U) \) we have a coordinate system, i.e., we can write \( z = (t,s) \) for \( x \) in \( U \). We shall call \( U \) a coordinate neighborhood of \( s \). The mapping \( (t,s) \leftrightarrow t \) is a conformal map of \( C_s \) onto the plane, \( D_s \) onto the unit circle. Let \( A(D_s) \) be the algebra \( A(\{ |t| \leq 1 \}) \) removed to \( D_s \) by this mapping. Let \( M = \{ f \text{ in } C(B); f \big|_{D_s} \text{ is in } A(D_s) \} \). It is proven by Hoffman and Singer \[15\] that \( M \) is a closed subalgebra of \( C(\partial B) \) and \( S(M) = B, \Gamma(M) = \partial B \) (in the case \( n = 2 \), but their proof depends only of the facts mentioned here, so applies in general). Obviously \( M \not\subset A(B) \), so \( A(B) \) is not \((B, \partial B)\)-maximal.

**Theorem 5.** \( M \) is \((B, \partial B)\)-maximal.

Proof: Let \( G \) be a subalgebra of \( C(\partial B) \), \( G \supset M \), and \( S(G) \supset B \). We will show first that for all \( f \) in \( G \),
Let $U$ be a coordinate neighborhood of $s$, so points in $\pi^{-1}(U)$ have coordinates $(\tau, \sigma)$, $\tau$ in $C^1$, $\sigma$ in $U$. Suppose $z = (t, s)$ is in $\text{int}D_\Sigma$ and is such that $t \neq 0$, and there is an $f$ in $G$ with

$$||f||_{\Sigma} < r < 1 = f(z)$$

By choosing a power of $f$ we may assume $r < |t|$. Let $V$ be a neighborhood of $s$ in $U$ such that $||f||_{\pi^{-1}(V) \cap \partial B} < |t|$; by continuity such a neighborhood exists. Let $h$ be in $C(\mathbb{C}P^n)$ such that $h(s) = 1$, $h = 0$ off $V$, $||h|| = 1$, and write $\tilde{h}(\tau, \sigma) = h(\sigma)$. Define $\tilde{z}$ in $C(\pi^{-1}(V) \cap B)$,

$$\tilde{z}(\tau, \sigma) = \tau.$$ Then $g = \tilde{h} \tilde{z} f$ is in $G$ since $\tilde{h} \tilde{z} |_{D_\Sigma} = h(s) \cdot t |_{D_\Sigma}$ is a constant multiple of a function in $A(D_\Sigma)$ if $s$ is in $V$, and is identically zero if $s$ is not in $V$. Then $\tilde{h} \tilde{z}$ is in $M$, so $\tilde{h} \tilde{z} f$ is in $G$. But we have

a) $|g(z)| = |h(s) \cdot t \cdot f(z)| = |t|$, 
b) in $\pi^{-1}(U) \cap \partial B$: $|g(\tau, \sigma)| = (h(\sigma)|\partial f(\tau, \sigma)| < 1 \cdot |t|$, 
c) in $\partial B - \pi^{-1}(U)$, $g(\tau, \sigma) = 0$, therefore

$$|g(z)| > ||g||_{\partial B}$$

contradicting the fact that $\Gamma(G) = \partial B$.

Thus $||f||_{D_\Sigma} - 0 = ||f||_{\Sigma}$, so by continuity $||f||_{D_\Sigma} = ||f||_{\Sigma}$. Thus every point of $D_\Sigma$ is a homomorphism of $G|_{\Sigma}$, so $G|_{\Sigma} \neq 0(\Sigma)$. But $G|_{\Sigma} \supset A(D_\Sigma)$, so by maximality,

$$G|_{\Sigma} = A(D_\Sigma).$$ Thus for any $f$ in $G$, $f|_{D_\Sigma}$ is in $A(D_\Sigma)$, therefore $f$ is in $M$. Thus $G = M$.

We will now describe all superalgebras of this algebra $M$. This is made easy because of the fibering of $B$ as circles of $\mathbb{C}P^n$. Let $N$ be the subalgebra of
\( C(\partial B) \) of all functions whose restriction to any \( \Gamma_s \) is analytic. Then \( S(N) \) has the same structure as \( \partial B \), but with discs replacing the circles as fibers. \( M \) is the set of functions in \( N \) which identify the centers of all these discs. Roughly speaking, a superalgebra of \( M \) is an algebra of functions continuous on \( \partial B \) which are analytic on some of the circles \( \Gamma_s \), and whose restrictions to the corresponding centers is a subalgebra of \( C(S) \). This will become more precise in the following more general considerations.

Let \( S, X \) be compact Hausdorff spaces, \( \pi: X \to S \) a projection, i.e., \( \pi \) is continuous and open. Suppose \( S \) is covered by open sets \( U^i \) for each of which there exists a homeomorphism \( \varphi^i: \pi^{-1}(U^i) \to U^i \times \{ |t| = 1 \} \), and if \( s \) is in \( U^i \cap U^j \), \( \varphi^i \pi^{-1}(s) \circ \varphi^{-1}(s) : \{ |t| = 1 \} \to \{ |t| = 1 \} \) is the restriction of an analytic function to \( \{ |t| = 1 \} \).

We write \( \pi^{-1}(s) = \Gamma_s \) and in a 'coordinate neighborhood' \( \pi^{-1}(U^i) \) we will write \( x = (t, s) \), meaning \( \varphi^i(x) = (t, s) \).

Let \( A(\Gamma_s) = f \circ \varphi^{-1}_s; f \text{ in } A(\{ |t| = 1 \}) \) = the algebra of analytic functions on the unit disc. Then \( A(\Gamma_s) \) is a maximal subalgebra of \( C(\Gamma_s) \). Let \( N = \{ f \text{ in } C(X); f|_{\Gamma_s} \text{ in } A(\Gamma_s) \} \).

**Theorem 6.** Let \( B \) be a subalgebra of \( C(X) \) containing \( N \). Either

1) \( B = C(X) \) or

ii) there exists a closed subset \( F \subseteq S \) such that
\[ B \mid \mathcal{G}_s = \mathcal{A}(\mathcal{G}_s) \text{ for } s \text{ in } F \text{ and } B \mid \mathcal{G}_s = \mathcal{C}(\mathcal{G}_s) \text{ for } s \text{ not in } F. \] Any \( f \) in \( \mathcal{C}(X) \) such that \( f \mid \mathcal{G}_s \) is in \( \mathcal{A}(\mathcal{G}_s) \) for \( s \) in \( F \), is in \( B \).

In case (ii) the space of maximal ideals \( S(B) \) is obtained by replacing the fibers \( \mathcal{G}_s \) over \( F \) by discs \( D_s \) so that the set \( \mathcal{G}_s, s \text{ not in } F \) \( \cup \) \( D_s, s \text{ in } F \) is homeomorphic to \( X \), and \( S(B) \) is topologized so that the restriction to each fiber is the usual topology and \( \pi \) is still continuous and open.

Proof: For \( s \) in \( S \), let \( B_s = B \mid \mathcal{G}_s \). Now \( \mathcal{A}(\mathcal{G}_s) \subset B_s \subset \mathcal{C}(\mathcal{G}_s) \), and \( \mathcal{A}(\mathcal{G}_s) \) is maximal, so either \( B_s = \mathcal{A}(\mathcal{G}_s) \) or \( B_s \) is dense in \( \mathcal{C}(\mathcal{G}_s) \). Let \( F = \{ s \in S; B_s = \mathcal{A}(\mathcal{G}_s) \} \).

\( F \) is closed.

Suppose \( s_n \) in \( F \), \( s_n \to s \). Let \( U \) be a coordinate neighborhood for \( s \), and suppose \( \{ s_n \} \subset U \). Let \( f \) in \( B \) and write

\[ f_n(t) = f(t, s_n) \text{ for } s_n \text{ in } U. \]

For any \( x \) in \( \mathcal{G}_s \), there is a neighborhood \( V \) of \( x \) such that for \( y, z \) in \( V \), \( |f(y) - f(z)| < \varepsilon \). Cover compact \( \mathcal{G}_s \) by finitely many such \( V_1, \ldots, V_n \), and let \( U^o = \cap_{k=1}^n \mathcal{U}_k(V_1) \cdots \mathcal{U}_k(V_n) \). \( U^o \) is a neighborhood of \( s \); let \( N \) such that \( n > N \) implies \( s_n \) is in \( U^o \). Then for any \( t \) and \( n > N \),

\[ |f(t, s_n) - f(t, s)| < \varepsilon, \text{ or} \]

\[ |f_n(t) - f(t, s)| < \varepsilon \text{ or} \]

\[ \sup |f_n(t) - f(t, s)|_{|t| = 1} < \varepsilon. \]

Thus the sequence \( f_n(t) \) converges uniformly to \( f(t, s) \).
Since \( f_n = f \big|_{\Gamma_s^n} \) is in \( A(\Gamma_s^n) \), \( f_n \) is analytic on \( |t| = 1 \), therefore so is \( f(t,s) = f \big|_{\Gamma_s} \). Therefore \( f \big|_{\Gamma_s} \) is in \( A(\Gamma_s) \), proving \( B_s = A(\Gamma_s) \), and thus \( s \) is in \( F \).

We now compute \( S(B) \). Let \( h \) be a non-zero homomorphism of \( B \). For \( g \) in \( C(S) \), define \( \tilde{g}(x) = g(nx) \). Then \( \tilde{g} \) is in \( C(X) \), further \( \tilde{g} \big|_{\Gamma_s} \) is constant so is in \( A(\Gamma_s) \), thus \( g \) is in \( N \). Define \( h' \) on \( C(S) \),

\[
    h'(g) = h(\tilde{g}).
\]

Since \( h(1) = 1 \), and \( l = \tilde{l} \), \( h'(l) = 1 \), so \( h' \) is a non-zero homomorphism of \( C(S) \), therefore there is a point \( s \) in \( S \) such that for all \( g \) in \( C(S) \), \( h(g) = g(s) \). We prove now \( h \) is a continuous homomorphism on \( B_s \), i.e.,

\[
    |h(f)| \leq \|f\|_{\Gamma_s^n} + \varepsilon
\]

(for \( \|f\|_{\Gamma_s^n} \) is a continuous function of \( s \)). Let \( g \) be in \( C(S) \), \( g(s) = 1 = \|g\|_S \), and \( \|g\|_{S-U} < \frac{\varepsilon}{f} \). Then

\[
    \|\tilde{g}f\|_{S-U} < \varepsilon, \quad \|\tilde{g}f\|_{S-U}(U) \leq \|\tilde{g}f\|_{n-U} < \|f\|_{\Gamma_s^n} + \varepsilon,
\]

thus \( \|\tilde{g}f\| \leq \|f\|_{\Gamma_s^n} + \varepsilon \). But

\[
    |h(f)| = |g(s)h(f)| = |h(\tilde{g}f)| \leq \|\tilde{g}f\| \leq \|f\|_{\Gamma_s^n} + \varepsilon.
\]

This for any \( \varepsilon \), thus \( |h(f)| \leq \|f\|_{\Gamma_s^n} \). This for all \( f \) in \( B \). Thus if \( f \big|_{\Gamma_s} = g \big|_{\Gamma_s} \), \( \|f-g\|_{\Gamma_s} = 0 \), so \( h(f) = h(g) \), and thus \( h \) is a well-defined continuous homomorphism on \( B_s \).

Then, if \( s \) is in \( F \), \( B_s = A(\Gamma_s) \). Let \( D_s = S(A(\Gamma_s)) \), then \( D_s \) is homeomorphic to the unit disc, and \( \partial D_s = \Gamma_s \). In this case \( h \) is evaluation at a point of \( D_s \).
If \( s \) is not in \( F \), \( \overline{E}_s = \mathcal{C}(\Gamma_s) \), so \( h \) is evaluation at a point of \( \Gamma_s \). Thus \( S(B) \) is contained in the set described in the statement of the theorem. But, if \( h \) is a homomorphism of \( B_s \) for any \( s \), then it surely is a homomorphism of \( B \), by defining

\[
h(g) = h(g|_{\Gamma_s}) \quad \text{for any } g \text{ in } B.
\]

Thus \( S(B) \) is precisely the set described.

Now \( B_s \) is actually a closed subalgebra of \( \mathcal{C}(\Gamma_s) \). This is true if \( s \) is in \( F \) for then \( B_s = A(\Gamma_s) \), a closed algebra. Otherwise this follows from a general Banach algebra theorem, since \( \Gamma_s \) is a hull, and the quotient norm and the sup norm on \( \Gamma_s \) coincide; but the proof here is easy, so we will give it.

Let \( f_n \) be in \( B \) and suppose \( f_n|_{\Gamma_s} \) converges, we may assume \( \|f_n - f_{n+1}\|_{\Gamma_s} < 2^{-n} \). Then there is a neighborhood \( U \) of \( s \) such that \( \|f_n - f_{n+1}\|_{\Gamma(U)} < 2^{-n} \). Let \( h \in \mathcal{C}(S) \), \( h(s) = 1 \), \( h = 0 \) off \( U \), and \( \|h\| = 1 \). Then for \( u_n = \widetilde{h}(f_n - f_{n+1}) \), \( \|u_n\| < 2^{-n} \). Thus \( \sum u_n = u \) is in \( B \). Now \( u_n|_{\Gamma_s} = f_n - f_{n+1}|_{\Gamma_s} \) so \( \sum u_n|_{\Gamma_s} = \sum (f_n - f_{n+1})|_{\Gamma_s} = (f_1 - f_{N+1})|_{\Gamma_s} \). Therefore \( f_{N+1}|_{\Gamma_s} = f_1|_{\Gamma_s} - \sum u_n|_{\Gamma_s} \), then \( \lim f_{N+1}|_{\Gamma_s} = f_1 - u|_{\Gamma_s} \), so \( \lim f_n|_{\Gamma_s} \) is in \( B_s \).

Now let \( f \) in \( \mathcal{C}(X) \) such that \( f|_{\Gamma_s} \) is analytic if \( s \) is in \( F \). Then \( f \) is in \( B \). For each \( s \) in \( S \) there is an \( f_s \) in \( B \) such that \( f_s = f \) on \( \Gamma_s \). Then there is a neighborhood \( U_s \) of \( s \) such that \( |f_s - f| < \varepsilon \) on \( \pi^{-1}(U_s) \).

Choose finitely many \( U_{s_1}, \ldots, U_{s_n} \) covering \( S \) and let \( h_1, \ldots, h_n \) be a partition of unity relative to this
covering. Then \(|\sum \tilde{h}_i f_{s_i} - f| < \varepsilon \). For, given \( x \) in \( X \), if \( \tilde{h}_i(x) \neq 0 \), we know \( x \) is in \( \pi^{-1}(U_{s_i}) \), so \( |f_{s_i}(x) - f(x)| < \varepsilon \). But \( \sum \tilde{h}_i(x)f_{s_i}(x) \) is in the convex set spanned by the \( f_{s_i}(x) \), so is within \( \varepsilon \) of \( f(x) \). Thus \( f \) is approximable on \( X \) by functions in \( B \); since \( B \) is closed, then \( f \) is in \( B \).

Thus the set of closed superalgebras of \( N \) is in one-one correspondence with the closed subsets \( F \) of the base \( S \). We denote by \( B(F) \) the algebra corresponding to the closed set \( F \), that is,

\[
B(F) = \{ f \in C(X); f|_{\Gamma_s} \text{ is in } A(\Gamma_s) \text{ for } s \in F \}.
\]

Obviously, \( B(S) \) is the given algebra \( N \), and \( B(\text{empty set}) = C(X) \). From the theorem, then \( N \) has the space of maximal ideals \( S(N) \supseteq X \), and there is a projection \( \pi:S(N)\rightarrow S \) which extends \( \pi:X\rightarrow S \) such that \( \pi^{-1}(s) = D_s \) is a disc and \( \partial D_s = X\cap \pi^{-1}(s) = \Gamma_s \).

**Lemma 2.** Let \( K \) be a closed subset of \( S(N) \) such that \( \pi:K\rightarrow S \) is a homeomorphism. Then \( K \) is a hull of an ideal in \( N \).

**Proof:** Given \( x^0 \) in \( S(N) \), \( x^0 \) not in \( K \), we have to show that there exists an \( f \) in \( N \) such that \( f = 0 \) on \( K \) and \( f(x^0) \neq 0 \). Let \( \pi(x^0) = s^0 \), and let \( U \) be a coordinate neighborhood of \( s \), so that \( \pi^{-1}(U) \) has coordinates \((t,s)\). \( \pi:K\cap \pi^{-1}(U) \rightarrow U \) is a homeomorphism, so there exists a map \( \varphi:U \rightarrow [|t| = 1] \) such that \( K = \{(\varphi(s),s); s \in U\} \). Define \( f \) in \( C(\pi^{-1}(U)) \), \( f(t,s) = t - \varphi(s) \) and \( h \) in \( C(S) \), \( h(s^0) = 1 \), \( h = 0 \) off \( U \).
Then $\tilde{h}f$ is in $C(X)$, and $\tilde{h}f|_{\Gamma_g} = h(s)[t - \varphi(s)]$ is in $A(\Gamma_g)$, so $\tilde{h}f$ is in $N$. If $x$ is in $K$, $\pi(x)$ not in $U$, then $\tilde{h}f(x) = 0$. If $x$ is in $K\cap U^{-1}(U)$, $x = (\varphi(s), s)$, so $\tilde{h}f(x) = h(s)[\varphi(s) - \varphi(s)] = 0$, thus $\tilde{h}f = 0$ on $K$. But $x^0 = (t^0, s^0)$ is not in $K$, so $t^0 \neq \varphi(s^0)$, therefore $\tilde{h}f(x^0) \neq 0$. Thus $K$ is a hull.

**Lemma 3.** Let $C = [\tilde{h}; \ h \in C(S)]$. Then $N = M \cdot C$, where $K$ is as in Lemma 2, and $M = [f \in N; f$ is constant on $K]$.

**Proof:** Let $f$ be in $N$. Write $\varphi: S \to S(N)$, $\varphi(s) = s$, $\pi \varphi = $ identity. $f(x) = f(x) - f(\varphi(s)x) + f(\varphi(s)x)$. Then $f - f \circ \varphi \circ \pi$ is in $M$ and $f \circ \varphi \circ \pi$ is in $C$. If $g$ is in $M \cdot C$, $g|_{D_s}$ is constant, so $g(x) = g(\varphi(s))$ for all $x$ in $D_s$. But $g|_K$ is constant, so $g(x) = g|_K$ for all $x$. Thus $M \cdot C =$ constants.

Now $S(M)$ is the identification space of $S(N)$ with $K$ identified to a point. If $S = \mathbb{CP}^n$, then $M, N$ are precisely the algebras previously denoted by $M, N$ respectively, where here $K =$ set of centers of the discs $D_s$. Given any closed $F \subset S$, the algebra $B(F)$ is a superalgebra of $N$, and $S(B(F))$ is a subspace of $S(N)$; more precisely, $S(B(F)) = \bigcup_{s \in F} D_s \cup \bigcup_{s \notin F} D_s$.

**Theorem 7.** Let $K$ be a closed subset of $\text{int}S(N)$ such that $\pi: K \to S$ is a homeomorphism. Let $\varphi: S \to K$, $\pi \circ \varphi = $ identity. Let $B \subset C(X)$ be a superalgebra of $M =$ $[f \in N; f(\text{constant})]$. Then there is a closed
subset $F$ of $S$ and a not necessarily separating subalgebra $A$ of $C(F)$ such that
\[ B = \{ f \in B(F); f|_{K^m(F')} \text{ is in } A \} . \]

Proof: Let $B'$ be the subalgebra of $C(X)$ generated by $B$ and $C'$. Then, since $B \supseteq N$, we have, by Lemma 3 that $B' \supseteq N$, so there is an $F$ such that $B' = B(F)$. As in Theorem 6, $S(B(F))$ is a subspace of $S(N)$ consisting of discs over $F$, circles over $S-F$.

1) Now, for any $s$ in $S$, $B|_s = B'|_s$, since $B'$ is generated by $B$ and $C$, and $C|_s = \text{constants}$. Then, if $f$ is in $C(X)$ and $f|_s$ is in $A(\Gamma_s)$ for $s$ in $F$ and $f|_{K^m(F')} = 0$, then $f$ is in $B$.

For $B|_s = B'|_s = C(\Gamma_s)$ for $s$ not in $F$. Thus there is an $f_s$ in $B$ such that
\[ f_s|_s = \left( \frac{t - q(s)}{1 - q(s)t} \right) f|_s, \]
so there is a neighborhood $U_s$ of $s$ contained in a coordinate neighborhood such that
\[ \|f_s - \left( \frac{t - q(s)}{1 - q(s)t} \right) f\|_{\pi^{-1}(U_s)} < \varepsilon \]
where $(t, s)$ are taken as coordinates in $\pi^{-1}(U_s)$.

If $s$ is in $\Gamma_s$, there is a function $f_s$ in $M$ such that $f_s|_s = f|_s$ and $f_s|_K = 0$. Thus there is a neighborhood $U_s$ of $s$ such that
\[ \|f_s - f\|_{\pi^{-1}(U_s)} < \varepsilon . \]

Let $U_1, \ldots, U_n$ be chosen from these $\{U_s\}$ covering $S$, and $f_1, \ldots, f_n$ the corresponding functions. Let $h_1, \ldots, h_n$ be a partition of unity relative to the covering $U_1, \ldots, U_n$. Suppose $U_1, \ldots, U_k$ come from points $s$ not in $F$.
U_k+1, ..., U_n from points in F.

Now \( \tilde{h}_i \left( \frac{t-q(s)}{1-q(s)^t} \right) \) is in \( M, 1 \leq i \leq k \), so \( \tilde{h}_i \left( \frac{t-q(s)}{1-q(s)^t} \right) f_1 \) is in \( B \). If \( i > k \), \( \tilde{h}_i f_1 \) is in \( M \), since \( f_1 \big|_K = 0 \). Then

\[
 g = \sum_{i=1}^{k} \tilde{h}_i \left( \frac{t-q(s)}{1-q(s)^t} \right) f_1 + \sum_{i=k+1}^{n} \tilde{h}_i f_1 \] is in \( B \).

We show \( \|g - f\| < \varepsilon \).

If \( \tilde{h}_i(x) \neq 0 \), then \( x \) is in \( \pi^{-1}(U_i) \). Suppose \( i > k \).

Then

\[
|f_1(x) - f(x)| < \varepsilon .
\]

If \( i \leq k \),

\[
|f_1(x) - \left( \frac{t-q(s)}{1-q(s)^t} \right) f(x)| < \varepsilon , \quad (x = (t,s)), \quad \text{so}
\]

\[
\left| \frac{t-q(s)}{1-q(s)^t} f_1(x) - f(x) \right| < \varepsilon , \quad \text{also, since}
\]

\[
\left| \frac{t-q(s)}{1-q(s)^t} \right| = 1 , \quad \text{for } x \text{ in } \partial D_s .
\]

Since \( g(x) \) is in the convex set spanned by these values,

\[
|g(x) - f(x)| < \varepsilon .
\]

This for all \( x \) in \( X \), so \( \|g - f\| < \varepsilon \), and thus since \( B \) is closed, \( f \) is in \( B \).

ii) Let \( A = \left\{ g \in C(F) \mid \text{there exists } f \in B \text{ such that } g(s) = f(q(s)) \right\} \).

Now \( A = \left\{ g \in C(F) \mid \text{there exists } \tilde{g} \in B_0 C, g = G\big|_{F} \right\} . \) (2)

Surely \( A \) contains this set of functions, for if \( \tilde{g} \) is in \( B_0 C \), then \( \tilde{g}(x) = G(q^s(x)) \), and \( G\big|_{F}(s) = \tilde{g}(q(s)) = g(s) \). Let \( g \) be in \( A \); \( g(s) = f(q(s)) \), \( s \) in \( F \), with \( f \) in \( B \).

Let \( \tilde{G} \) be in \( C(S) \), \( \tilde{G}\big|_{F} = g \). Then \( f - \tilde{G} \) is in \( B \). For if \( s \) is in \( F \), \( (f-\tilde{G})\big|_{F} \) is in \( A\big|_{F} \), and \( (f-\tilde{G})\big|_{K^{p-n}} \cdot (F) = 0 \).

Thus since \( f \) is in \( B \); \( f - (f-\tilde{G}) \) is also in \( B \), so (2) is proven.

iii) \( A \) is closed. For \( B_0 C \) is a closed subalgebra
of $B$. If $g$ is in $A$, and $g = G|_F$, $\widetilde{G}$ is in $B \otimes C$, and also $g = G'|_F$, $G'$ in $C(S)$, then $\widetilde{G} - G'$ is in $B$, since $\widetilde{G} - G'|_{\Gamma_s} = 0$ for $s$ in $F$, and thus since $\widetilde{G}$ is in $B$, $G' = \widetilde{G} - (\widetilde{G} - G')$ is in $B$. Thus $g$ in $A$ implies any extension of $g$ as a continuous function on $S$ is in $B$. Thus if $g_n$ is in $A$, $g_n \to g$ in the sup norm on $F$, there are extensions $G_n$ such that $G_n$ converges on $S$ to a function $G$ in $C(S)$. Then $G$ is in $C \otimes B$, since the $G_n$ are in $B \otimes C$, so that $g = G|_F$ is in $A$.

iv) Now let $f$ in $B(F)$; $f|_{K_0 \pi^{-1}(F)}$ is in $A$. Let $\widetilde{G}$ in $B \otimes C$, $G|_F = f|_{K_0 \pi^{-1}(F)}$. Then $f - \widetilde{G}|_{\Gamma_s}$ is in $A(\Gamma_s)$ if $s$ is in $F$, and $f - \widetilde{G}|_{K_0 \pi^{-1}(F)} = 0$. Thus $f - G$ is in $B$. But $G$ is in $B$, implying that $f$ is in $B$. Thus the theorem is proven.

In particular if $F = S$, the class of algebras between $M$ and $B(F) = N$ are in one-one correspondence with the subalgebras of $S$, and the class of superalgebras of $M$ is in one-one correspondence with the class of pairs $(F, A)$, $F$ a closed subset of $S$, $A$ a closed subalgebra of $C(F)$.

Let $B$ be a superalgebra of $M$, $B$ corresponds to $(F, A)$. Then if $F'$ is a proper subset of $B$, the algebra $B'$ corresponding to $(F', A|_{F'})$ is a strictly larger algebra than $B'$; for if $s$ is in $F - F'$, $B'|_{\Gamma_s} = C(\Gamma_s)$ while $B|_{\Gamma_s} = A(\Gamma_s)$. Thus if $B$ is a maximal algebra of $M$, $F$ is a single point $s$ in $S$, and then $A$ can only be the constants, so the maximal algebras of $M$ are
contained among the algebras $B_{[s]}$ for $s$ in $S$. But each such is a maximal algebra of $C(X)$; for if $B' \supset B_{[s]}$ it is a superalgebra of $N$, so $B' = B(F)$. But we must have $F \subset [s]$. If $F = [s]$, $B' = B_{[s]}$, if $F = \emptyset$, $B' = C(X)$. Thus we have

Corollary 1. The class of maximal algebras containing $M$ is precisely the class of maximal algebras containing $N$, and is $[B_{[s]}; s \in S]$. $N$ is the intersection of all maximal algebras containing it; $M \neq N$, so $M$ is not.

That $N$ is the intersection of all the $B_{[s]}$ is evident from the definition of $N$.

In the case where $S = \mathbb{CP}^n$, and $X$ is the boundary of the unit ball $B$ in $n$-space, there are many algebras between $M$ and $A(B)$ whose space of maximal ideals contains the unit ball. In fact let $U \subset \text{int}B$ be a domain; then the subalgebra of $M$ of functions holomorphic on $U$ is such an algebra. The analogous situation in the case of the polycylinder occurs when we take $S = T^{n-1}$, the $n-1$ dimensional torus. In this case $M = \{ f \in C(\{ |z_1| = 1, \ldots, |z_n| = 1 \}) \text{ such that } f \text{ restricted to the set } [(z_1, \ldots, z_n) \in T^n; z_j/z_n = e^{i\theta}(\theta_1, \ldots, \theta_{n-1}) \text{ in } T^{n-1}] \text{ is analytic for all } (\theta_1, \ldots, \theta_{n-1}) \}$. It is easy to see that $S(M) = \{(z_1, \ldots, z_n) \in \mathbb{P}^n; |z_j/z_n| \leq 1\}$, a proper subset of the polycylinder.
4. Reinhardt compacta

Let \( R \) be a Reinhardt compactum. For \( u = (u_1, \ldots, u_n) \) in \( R \), \( u_i \geq 0 \), let \( T_u = \{ (e^{i\theta}u_1, \ldots, e^{i\theta}u_n); 0 \leq \theta \leq 2\pi, 1 \leq i \leq n \} \). Let \( A(R) \) be the uniform closure on \( R \) of the algebra of functions holomorphic in a neighborhood of \( R \). The following is an extension from the case of complete Reinhardt compacta of a theorem of de Leeuw \([17]\):

**Theorem 8.** i) If \( f \) is in \( A(R) \), then, for \( u, v \) in \( R \), we have

\[
\int_{T_u} f(u_1 e^{i\theta}, \ldots, u_n e^{i\theta}) z_1^{-k_1} \ldots z_n^{-k_n} \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_n}{2\pi} = \int_{T_v} f(v_1 e^{i\theta}, \ldots, v_n e^{i\theta}) z_1^{-k_1} \ldots z_n^{-k_n} \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_n}{2\pi}.
\]

for all integers \((k_1, \ldots, k_n)\)

ii) Suppose \( R_n(z_1 \ldots z_n = 0) = \emptyset \). Then if \( f \) is in \( C(R) \) and (3) holds, there exists rational functions \( r_n \), with poles only on \((z_1 \ldots z_n = 0)\) such that \( r_n \rightarrow f \) uniformly on \( R \).

iii) Suppose \( R_n(z_1 = 0) \neq \emptyset, 1 \leq i \leq m, R_n(z_{m+1} \ldots z_n = 0) = \emptyset \). Then, for any \( k_i < 0, 1 \leq i \leq m \), the integrals in (3) are zero.

iv) If \( f \) in \( C(R) \) satisfies (3) and (4), there exist rational functions \( r_n \), with poles only on \((z_{m+1} \ldots z_n = 0)\) such that \( r_n \rightarrow f \) uniformly on \( R \).

v) Suppose \( R_n(z_1 = 0) \neq \emptyset, 1 \leq i \leq n \). Then if any \( k_i < 0 \), the integrals in (3) are zero.
vi) If \( f \) is in \( C(R) \) and satisfies (3) and (5), then \( f \) is uniformly approximable on \( R \) by polynomials.

Proof: 1) Let \( f \) in \( H(R) \), i.e., \( f \) is holomorphic in a neighborhood of \( R \). Then there is a domains \( U \supset R \) such that \( f \) is in \( H(U) \). We may take \( U \) to be Reinhardt since \( R \) is Reinhardt. We can write \( f \) as a Laurent series

\[
\sum a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}
\]

in a neighborhood of any torus \( T_u \subset R \). Since \( U \) is connected the same series expresses \( f \) in all of \( U \). But for any \( T_u \subset U, u > 0 \),

\[
\sum a_{k_1 \ldots k_n} = \int_{T_u} f(u_1 e^{i\theta_1}, \ldots, u_n e^{i\theta_n}) z_1^{-k_1} \ldots z_n^{-k_n} \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_n}{2\pi}
\]

proving i) for \( f \) in \( H(R) \). Then, since \( H(R) \) is dense in \( A(R) \) and \( \int f \, d\text{Haar} \) is continuous on \( C(R) \), i) is proven.

If \( R_\alpha(z_1 = 0) \neq \emptyset \), let \((0, w_2, \ldots, w_n)\) be in \( R \).
If \( f \) is in \( H(R) \), then \( f \) is in \( H([0, e^{i\theta_2}, e^{i\theta_3}, \ldots, e^{i\theta_n}; 0 \leq \theta \leq 2\pi]) \). Let \( f \) have the Laurent series

\[
\sum a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}.
\]

Since \( f \) is analytic in a disc \([z_1, e^{i\theta_2}, e^{i\theta_3}, \ldots, e^{i\theta_n}; |z_1| < \varepsilon]\) (\( \varepsilon \) does not depend on \((\theta_2, \ldots, \theta_n))\), the series

\[
\sum_{k_1 = -\infty}^{\infty} (\sum a_{k_1 \ldots k_n} (e^{i\theta_2})^{k_2} \ldots (e^{i\theta_n})^{k_n}) z_1^{k_1}
\]

has zero coefficients for \( k_1 < 0 \), therefore, if \( k_1 < 0 \),

\[
\sum (a_{k_1 \ldots k_n} w_2^{k_2} \ldots w_n^{k_n}) e^{i k_1 \theta_1} \ldots e^{i k_n \theta_n} = 0
\]
for all \((\theta_2, \ldots, \theta_n)\). This is only possible if
\[
 a_{k_1 \ldots k_n} w_1^{k_1} \ldots w_n^{k_n} = 0 \quad \text{for all} \ (k_1, \ldots, k_n).
\]
If \(w_2 \ldots w_n \neq 0\), this implies \(a_{k_1 \ldots k_n} = 0\) if \(k_1 < 0\).
But if \(w_2 \ldots w_n = 0\), since \(f\) is actually holomorphic in a neighborhood of \((0, w_2, \ldots, w_n)\), we can replace it by a point \((0, w'_2, \ldots, w'_n)\) in \(U\) such that \(w'_2 \ldots w'_n \neq 0\), and use the same argument as above. Then iii) is proven for \(m = 1\) and \(f\) in \(H(R)\); by continuity of \(\int f d\operatorname{Haar}\), it is true for \(A(R)\). Similarly we can prove the same for any \(i\) such that \(R_n(z_1 = 0) \neq \emptyset\), so iii) is proven and also v).

2) Let \(R_n(z_1 \ldots z_n = 0) = \emptyset\), we assume \(f\) is in \(C(R)\) and (3) holds. Then for \(T_u \subset R\), \(f|_{T_u}\) is in \(C(T_u)\) and has a representing Fourier series
\[
f|_{T_u} = \sum a_{k_1 \ldots k_n} e^{ik_1 \theta_1} \ldots e^{ik_n \theta_n} =
\]
\[
= \sum \frac{a_{k_1 \ldots k_n}}{u_1^{k_1} \ldots u_n^{k_n}} z_1^{k_1} \ldots z_n^{k_n}
\]
\[
= \sum A_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n} \quad \text{on} \ T_u.
\]
But by (3), \(A_{k_1 \ldots k_n}\) does not depend on \(T_u\), i.e., \(f|_{T_u}\) has the Fourier series
\[
\sum A_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}
\]
for all \(u\) such that \(T_u \subset R\).
Then there exist rational functions \(r_N\), whose only poles are on \((z_1 \ldots z_n = 0)\) such that \(r_N|_{T_u}\) is the \((N, \ldots, N)\)th \((C, 1)\)-partial sum of \(f|_{T_u}\). Now, by Fejer's
theorem, \( r_N \|_{T_u} \rightarrow f \|_{T_u} \) uniformly, and, more explicitly we can write

\[
|r_N(e^{i\theta_1}u_1, \ldots, e^{i\theta_n}u_n) - f(e^{i\theta_1}u_1, \ldots, e^{i\theta_n}u_n)| \leq \\
\leq \phi(\theta_1, \ldots, \theta_n, \delta) + K_N(\delta) \|f\|_{T_u},
\]

where \( \phi(\theta_1, \ldots, \theta_n, \delta) = \sup \left[ |f(z_1, \ldots, z_n) - f(e^{i\theta_1}u_1, \ldots, e^{i\theta_n}u_n)| : |z_1 - e^{i\theta_1}u_1| < \delta \right] \)

and \( K_N(\delta) \rightarrow 0 \) as \( N \rightarrow \infty \). [12]

Now since \( f \) is uniformly continuous on \( R \), given \( \epsilon > 0 \), there exists a \( \delta \) such that \( |\phi(\theta_1, \ldots, \theta_n, \delta)| < \frac{\epsilon}{2} \) for all \( (e^{i\theta_1}u_1, \ldots, e^{i\theta_n}u_n) \) in \( R \). Then for \( N \) large enough so that \( K_N(\delta) < \frac{\epsilon}{2\|f\|_R} \), we have, for all \( z \) in \( R \),

\[
|r_N(z) - f(z)| < \epsilon.
\]

Thus ii) is proven.

Similarly iv) and vi) are proven, for if there is a \( u \) in \( R \) \( (z_1 = 0) \), we only admit \( f \) in \( C(R) \) such that \( f \|_{T_u} \) has its \((k_1, \ldots, k_n)\)-coefficients zero for \( k_1 < 0 \). Thus the \( r_N \) as defined as in the above will be the required (by iv) or vi)) type of functions, and will be holomorphic in a neighborhood of \( R \). Further, inequality (6) will hold for all \( u \) in \( R \), if some \( u_1 \neq 0 \), for in these cases \( T_u \) is replaced by a torus of smaller dimension, but the preceding estimates still apply.

If \( 0 = (0, \ldots, 0) \) is in \( R \), we need still to verify vi). But we have in the above for \( N \) large enough \( \|r_N - f\| < \epsilon \) on \( R - [0] \). But \( R \) is assumed to be con-
nected and $r_n$ and $f$ are continuous on $R$, so that also 
$$|r_n(0) - f(0)| < \varepsilon,$$
thus 
$$|r_n - f| < \varepsilon$$
on all of $R$, so the theorem is proven.

**Lemma 4.** Let $R$ be a Reinhardt compactum, $R\cap(z_1...z_n=0) = \emptyset$. Then $S(A(R)) = \text{rationally convex hull} \; \tilde{R}$ of $R$.

**Proof:** By theorem 8, $A(R)$ is the uniform closure on $R$ of the rational functions. Thus if $z$ is in $\tilde{R}$, for any rational function $r$, $|r(z)| \leq \|r\|_R$,
so evaluation at $z$ is a homomorphism of $A(R)$. Thus $S(A(R)) \supset \tilde{R}$. Let $h$ be in $S(A(R))$, and let $z^* = (h(z_1),...,h(z_n))$. Then $h(z_1) = z_1^*; h(z_1^{-1}) = \frac{1}{z_1^*},$
and for $r$ any rational functions, since $r$ is a polynomial in $z_1$ and $z_1^{-1}, 1 \leq i \leq n$, we have $h(r) = r(z^*)$.
Thus $h$ is evaluation at $z^*$, since the rational functions are dense in $A(R)$, and since $|r(z^*)| = |h(r)| \leq \|r\|_R$
for all rational functions $z^*$ is in $R$.

**Theorem 9.** Let $R$ be a Reinhardt compactum such that $R\cap(z_1...z_n=0) = \emptyset$. Let $H = \text{convex hull of } L(R)$.
Then $S(A(R)) = L^{-1}(H)$.

**Proof:** First, if $R$ is holoconvex, then $L(R)$ is convex. Let $p^1 = (u_1,\ldots,u_n), p^2 = (v_1,\ldots,v_n)$ be in $L(R)$. Let $\Lambda$ be the line segment joining $p^1$ and $p^2$
given by:
$$\sum_{j=1}^{n} a_{ij} x_j = c_i, \; i = 1,\ldots,n-1 \; \; \; u_1 \leq x_1 \leq v_1.$$
Let $V$ be the one-dimensional variety
$$V(\prod_{j=1}^{n} z_j^{a_{ij}} - e^c i = 0, \; 1 \leq i \leq n-1)$$
(even if the $a_{ij}$ are not rational, $V$ is, off $(z_1 ... z_n=0)$
locally the zeros of holomorphic functions. Let $T_\Lambda = \bigcup_{p \in T_\Lambda} T_p$, where for $p = (x_1, \ldots, x_n)$, $T_p$ is the torus through $(e^{x_1}, \ldots, e^{x_n})$. Then $T_\Lambda \cap V$ is a compact set, with interior on $V$, and $\partial(T_\Lambda \cap V) \subset T_p \cup T_{p'}$.

Now by theorem 8, the rational functions with poles on $z_1 \ldots z_n = 0$ is dense in $A(R)$, and $T_\Lambda \cap V \cap (z_1 \ldots z_n = 0) = \emptyset$. Thus if $r$ is such a rational function $r|_{T_\Lambda \cap V}$ is holomorphic on $T_\Lambda \cap V$, and thus by the maximum modulus principle, for any $z$ in $T_\Lambda \cap V$,

$$|r(z)| \leq \|r\|_{T_\Lambda \cap V} \leq \|r\|_{T_p \cup T_{p'}} \leq \|r\|_{R},$$

since $T_p \cup T_{p'} \subset R$. Thus evaluation at $z$ is a homomorphism of $A(R)$, so $z$ is in $S(A(R)) = R$, by assumption.

Then $L(z)$ is in $L(R)$. Now for any $p = (x_1, \ldots, x_n)$ in $\Lambda$, there is a $z$ in $T_\Lambda \cap V$, i.e., $z = (e^{x_1}, \ldots, e^{x_n})$ such that $L(z) = p$. Thus $\Lambda \subset L(R)$, proving $L(R)$ is convex.

Now let $R$ be any Reinhardt compactum,

$R \cap (z_1 \ldots z_n = 0) = \emptyset$, $S(A(R)) = \tilde{R}$, the rational convex hull of $R$, by lemma 4. Then $A(R) = A(\tilde{R})$, and $S(A(\tilde{R})) = \tilde{R}$, thus $\tilde{R}$ is holoxconvex, so by the above $L(\tilde{R})$ is convex. $L(\tilde{R}) \supset L(R)$, so $L(\tilde{R}) \supset H = \text{convex hull of } L(R)$. Then $\tilde{R} \supset L^{-1}(H)$.

Now suppose $p$ is not in $H$. Then, since $H$ is convex, and compact, we can find a hyperplane $P: \Sigma a_i x_i = c$ through $p = (u_1, \ldots, u_n)$ such that $P \cap H = \emptyset$. We may assume $a_i$ are rational and by multiplication by a common denominator that the $a_i$ are integers. Further
we may assume $\Sigma_{i=1}^n x_i < c$ on $H$. For $g = \prod z_i^{a_i}, \|g\|_R < e^c$ and $e^c = g(e^{u_1}, \ldots, e^{u_n})$. Then $(e^{u_1}, \ldots, e^{u_n})$ is not in $S(A(R)) = \tilde{R}$. This implies $p$ is not in $L(R)$, for $R$ must also be a Reinhardt domain: for if $f$ is in $A(R)$, define $f^\theta = f(e^{ie\theta} z_1, \ldots, e^{ie\theta} z_n)$. Since $R$ is Reinhardt, $f^\theta$ is in $A(R)$ also, and the map $f \rightarrow f^\theta$ is an automorphism of $A(R)$. Thus if $h: A(R) \rightarrow \mathbb{C}$ is a homomorphism, so also is $h^\theta(f) = h(f^\theta)$. If $h = \text{evaluation at } (z_1, \ldots, z_n), h^\theta = \text{evaluation at } (e^{ie\theta} z_1, \ldots, e^{ie\theta} z_n)$, so this point also is in $\tilde{R}$. Thus $\tilde{R}$ is Reinhardt. Thus we have $L(S(A(\tilde{R}))) = H$, so $S(A(R)) = L^{-1}(H)$.

**Theorem 10.** Let $R$ be any Reinhardt compactum. Suppose $R_n(z_1 = 0) \neq \emptyset, 0 \leq i \leq m \leq n$ ($m = 0$ means $R_n(z_1, \ldots, z_n = 0) = \emptyset$), and $R_n(z_{m+1}, \ldots, z_n = 0) = \emptyset$.

1. $A(R) = \text{uniform closure on } R \text{ of polynomials in } z_1, \ldots, z_n, z_{m+1}^{-1}, \ldots, z_n^{-1}$;

2. $S(A(R)) = L^{-1}$ (closure in $\mathbb{Q}$ of $[x_1, \ldots, x_n]$; there is $(u_1, \ldots, u_n)$ in the convex hull of $L(R)$ such that $x_1 \leq u_1, \ldots, x_m \leq u_m, x_{m+1} = u_{m+1}, \ldots, x_n = u_n$) $\cup \left( \bigcup_{i=1}^m S(A(R_n(z_1 = 0))) \right)$.

3. If $m = n, S(A(R))$ is a complete Reinhardt compactum,

iv) If $m < n$, and $R = \overline{\text{Int} R}$, $A(R) = C(R) \cap H(\text{Int} R)$.

Proof: 1) is just theorem 8. Suppose $m < n$ and $R = \overline{\text{Int} R}$; let $f$ by in $C(R) \cap H(\text{Int} R)$. Then for $T_u$,

$T_v \subset \text{Int} R$, (3) of theorem 8 holds, and thus (3) holds for all $T_u \subset \text{Int} C R$ by continuity (since the integrands re-
main bounded). Let $\Sigma a_{k_1 \cdots k_m} z_1^{k_1} \cdots z_n^{k_n}$ be the Laurent series representing $f$. We prove $a_{k_1 \cdots k_m} = 0$ for $k_i < 0, i \leq m$. Since $g = z_{m+1}^{-k_{m+1}} \cdots z_n^{-k_n}$ is in $C(R) \cap H(\text{int}R)$, we have $fg$ in $C(R)$ and has the term $a_{k_1 \cdots k_m} z_1^{-k_1} \cdots z_n^{-k_n}$ in its Laurent series. Thus since $\text{int}R \cap (z_1 = 0) \neq \emptyset$, $fg$ remains bounded as $z$ in $R$ converges to $(z_1 = 0)$. This is not possible unless $a_{k_1 \cdots k_m} = 0$ for $k_i < 0$. Thus $f$ satisfies (3) and (4) of theorem 8, so is in $A(R)$. Thus iv) is proven.

Now we find $S(A(R))$. Write the right side of ii) as
$$L^{-1}(H) \cup \tilde{S}(A(R_0(z_1 = 0))).$$
(8)
We prove $H = L(S(A(R)) - (z_1 \cdots z_n = 0))$. First, that $L^{-1}(\text{convex hull of } L(R)) \subset S(A(R))$ is evident from the proof in the first part of theorem 9. Secondly, let $u = (u_1, \ldots, u_n)$ be in the convex hull of $L(R)$ and $x_i \leq u_i, 1 \leq i \leq m$. Every rational function $r$ with poles only on $(z_{m+1} \cdots z_n = 0)$ is such that
$$|r(e^{x_1}, \ldots, e^{x_m}, e^{u_{m+1}}, \ldots, e^{u_n})| \leq \sup [|r(e^{u_1+i\theta}, \ldots, e^{u_{m+1}+i\theta}, e^{u_{m+1}}, \ldots, e^{u_n})|, (\theta, \ldots, \theta) \text{ in } T_m]$$
for $r(z_1, \ldots, z_m, e^{u_{m+1}}, \ldots, e^{u_n})$ is a polynomial in $(z_1, \ldots, z_m)$. Thus for any $f$ in $A(R)$,
$$|f(e^{x_1}, \ldots, e^{x_m}, e^{u_{m+1}}, \ldots, e^{u_n})| \leq \sup [|f(t, e^{u_{m+1}}, \ldots, e^{u_n})|; t \text{ in } T(u_1, \ldots, u_n)]$$
$$\leq \|f\|_{S(A(R))} = \|f\|_{S_A},$$
so $(x_1, \ldots, x_m, u_{m+1}, \ldots, u_n)$ is in $L(S(A(R)))$. Thus
$H \subset L(S(A(R)))$, since the latter must be closed in $\mathbb{Q}^n$.

Conversely, if $u$ is not in $H \cap \mathbb{Q}^n$, since $H$ is convex, there is a line

$$L: \sum_{j=1}^{n} a_{ij} x_j = c_i, \quad i = 1, \ldots, n-1$$

through $u$ such that

$$\sum_{j=1}^{n} a_{ij} x_j < c_i, \quad i = 1, \ldots, n-1 \text{ on } H. \quad (9)$$

We may suppose the $a_i$ are integers. By definition of $H$, there are points $v = (v_1, \ldots, v_n)$ in $H$ with $v_i$, $1 \leq i \leq m$ arbitrarily negatively large. Thus if any $a_{rs} < 0$, $s \leq m$, there would be a point $v$ in $H$ such that $\sum_{j=1}^{n} a_{rj} v_j > c_r$, a contradiction. Then $a_{rs} \geq 0, s \leq m$, so $g_r = \prod_{j=1}^{n} z_j^{a_{rj}}$ is in $A(R)$, and from (9)

$$|g(e^{u_1}, \ldots, e^{u_n})| > \|g\|_R,$$

since $H \supset L(R)$.

Thus $(e^{u_1}, \ldots, e^{u_n})$ is not in $S(A(R))$, so $u$ is not in $L(S(A(R)))$.

Now if $u = (u_1, \ldots, u_n)$ is not in $\mathbb{Q}^n$, nor in $H$; since we are only interested in $S(A(R)) \cap (z_1 \ldots z_m \neq 0)$, we can assume $u_1, \ldots, u_m > -\infty$. Then, since $u$ is not in $\mathbb{Q}^n$ some $u_j = -\infty, j > m$. But $z_j^{-1}$ is in $A(R)$ and is infinite at $(e^{u_1}, \ldots, e^{u_n})$, so $(e^{u_1}, \ldots, e^{u_n})$ is not in $S(A(R))$, therefore $u$ is not in $L(S(A(R)))$. Thus (8) is proven.

Now let $w = (0, w_2, \ldots, w_n)$ be in $S(A(R))$ and not in $L^{-1}(h)$, we prove evaluation at $w$ is a homomorphism of $A(R \cap (z_1 = 0))$.

Let $r$ be any rational function in $A(R \cap (z_1 = 0))$, then $r$ is a polynomial in $z_1, \ldots, z_n, z_{m+1}^{-1}, \ldots, z_n^{-1}$ and
\[ |r(w)| \leq \|r\|_{R}. \] But since \( r(z_1, \ldots, z_n) = r(0, z_2, \ldots, z_n) \), we have \( \|r\|_{R} = \|r\|_{\pi R} \) where \( \pi(z_1, \ldots, z_n) = (0, z_2, \ldots, z_n) \).

Then \( \pi R \subseteq (L^{-1}(H) \cap (z_1 = 0)) \cup (R(z_1 = 0)) \). Since \( H \) is convex; \( L^{-1}(H) \) is a holoconvex set, thus also is \( L^{-1}(H) \cap (z_1 = 0) \). Since \( w \) is not in \( L^{-1}(H) \), there is a rational function \( g' \) such that \( g'(w) = 1 \) and

\[ \|g\|_{L^{-1}(H)} < \left[ 2 \|r\|_{L^{-1}(H)} \right]^{-1} \|r\|_{\pi R}, \]

where \( \pi R(z_1 = 0) \). Let

\[ g(z_1, \ldots, z_n) = g'(0, z_2, \ldots, z_n). \] Then \( g(w) = 1 \) and

\[ \|g\|_{L^{-1}(H)} \leq \|g\|_{L^{-1}(H)}. \] Then

\[ |r(w)| = |g(r(w))| \leq \|g\|_{\pi R} \leq \|g\|_{L^{-1}(H)}, \]

but

\[ \|g\|_{L^{-1}(H)} < \frac{2\|r\|_{\pi R}}{\|r\|_{L^{-1}(H)}} \|r\|_{L^{-1}(H)} < \frac{1}{2} \|r\|_{\pi R}, \]

so we have, for any rational \( r \),

\[ \|r(0, w_2, \ldots, w_n)\| \leq \|g\|_{\pi R} \|r\|_{\pi R}. \]

Thus evaluation at \( w \) is a continuous homomorphism of \( A(\pi R(z_1 = 0)) \) so is in \( S(A(\pi R(z_1 = 0))) \).

Then \( S(A(R)) = \left( S(A(R)) \right) \cap \left( \bigcup_{z_1} S(R(z_1 = 0)) \right) \cap \left( \bigcup_{z_1} S(A(\pi R(z_1 = 0))) \right) \).

But obviously the last set above is contained in \( S(A(R)) \), so ii) of theorem 10 is proven.

iii) If \( m = n \), then \( A(R) = \) closure of the polynomials on \( R \), so \( S(A(R)) = \) polynomial convex hull of \( R \).

Now if \( z \) is in \( S(A(R)) \), since \( S(A(R)) \) is Reinhardt \( T_z \subseteq S(A(R)) \). Let \( P_z = \left\{ (t_1 z_1, \ldots, t_n z_n); |t_1| \leq 1 \right\} \).

Then for any polynomial \( p \),

\[ \|p\|_{P_z} = \|p\|_{T_z} \leq \|p\|_{R}. \]

Thus every point of \( P_z \) is a continuous homomorphism
of \( A(R) \), so \( P_z < S(A(R)) \); this for all \( z \) in \( S(A(R)) \),
so \( S(A(R)) \) is complete Reinhardt.

In the case \( m = n \), i.e., \( R (z_1 = 0) \neq \emptyset \), \( 1 \leq i \leq m \),
it is not necessarily true that \( A(R) = C(R) \cap H(\text{int}R) \).
For instance, let \( R = [(z,w); |z| \leq |w| \leq 1] \). From
what we have proven, \( S(A(R)) = \mathbb{P}^2 = [(z,w); |z| \leq 1,
|w| \leq 1] \), so \( A(R) = A(\mathbb{P}^2) \). Then, since \( \frac{z_i}{w} = g \) is not
in \( A(\mathbb{P}^2) \), it is not in \( A(R) \). But \( g \) is in \( H(\text{int}R) \), and
is continuous on \( R \). We need only check continuity
at \((0,0)\) for that is the only point of \( R \) at which \( g \)
is not defined continuously. Let \( (z,w) \) be in \( R \), then
\( z = tw; |t| \leq 1, g(z,w) = t^a \frac{w^a}{w} = t^aw \), so for all
\((z,w) \) in \( R \), \( |g(z,w)| \leq |w| \). Thus, defining \( g(0,0) = 0 \), we see that \( g \) is in \( C(R) \).

We now use the above knowledge to give a refinement
of a well known result on holomorphic convexity
of Reinhardt domains \([7,8,9]\). If \( D \) is a domain in \( \mathbb{C}^n \),
let \( \overline{D} \) represent its envelop of holomorphy. We will
show that there is a domain of holomorphy \( U \subset \mathbb{C}^n \), \( U \supset \overline{D} \),
such that every function holomorphic on \( D \) extends to a
function holomorphic on \( U \). Then \( U = \overline{D} \). For \( X \) a set
in \( \mathbb{Q}^n \), let \( \mathcal{X}^o \) be its convex hull in \( \mathbb{Q}^n \). Loosely speaking
we will have \( \overline{D} = L_{-1}(\mathcal{X}^o) \), for Reinhardt domains \( D \).

**Theorem 11.** Let \( D \) be a Reinhardt domain, \( \overline{D} \) its
envelop of holomorphy.

i) \( D \cap (z_1 \cdots z_n = 0) = \emptyset \) implies \( \overline{D} = L_{-1}(\text{int}(\text{L}(D)^o)) \).

ii) \( D \cap (z_1 = 0) \neq \emptyset \), \( 1 \leq i \leq m \leq n \), \( D \cap (z_{m+1} \cdots z_n = 0) \neq \emptyset \)
implies \( D = \text{int}(L_{-1}(\text{L}(D)^o)) - (z_{m+1} \cdots z_n = 0) \).
Proof: 1) Write $L(D) = \bigcup_{n} \mathcal{O}_n$, $\mathcal{O}_n$ an increasing sequence of domains such that $\overline{\mathcal{O}_n}$ is compact and contained in $D$. By theorem 10, if $f$ is in $H(D)$, $f$ is in $A(L^{-1}(\overline{\mathcal{O}_n}))$, so $f$ extends to a function holomorphic on $L^{-1}(\text{int}\overline{\mathcal{O}_n})$. This for all $n$, thus $f$ extends to a function holomorphic on $L^{-1}(\text{int}\mathcal{O}_n)$. But $\bigcup_{n} \text{int}\mathcal{O}_n$ is just the convex hull of $L(D)$. Thus every $f$ in $H(D)$ extends to a function holomorphic on $L^{-1}(\text{convex hull of } L(D))$.

2) Let $U$ be a convex domain in $\mathbb{Q}^n$. Then $L^{-1}(U) = r$ is a domain of holomorphy:

Since $U \subset \mathbb{Q}^n$, $R(z_1 \ldots z_n = 0) = \emptyset$. We prove $R$ is holoconvex. Let $K$ be compact, $K \subset R$. We may assume $K$ is Reinhardt, since every compact subset of $R$ is contained in a Reinhardt compact subset of $R$. Then $L(K)$ is compact, therefore $S(A(K)) = L^{-1}(L(K)^c)$. Now, since the rational functions are dense in $A(K)$, and $H(D)$

> [rational functions] so also is $H(D)$ dense in $A(K)$; thus $S(A(K))$ is the $H(D)$-convex hull of $K$. Now $L(R) = U$ is convex, $U \supset L(K)$, so $U \supset L(K)^c$. Thus $R = L^{-1}(U) \supset L^{-1}(L(K)^c) = S(A(K))$. Thus $S(A(K))$ is a compact subset of $R$, proving $S$ is holoconvex.

3) Thus if $L^{-1}(L(D)^c) \supset D$, it is the envelop of holomorphy of $K$. This is the case if $D \cap (z_1 \ldots z_n = 0) = \emptyset$, so 1) is proven.

4) Suppose $D \cap (z_i = 0) \neq \emptyset$, $1 \leq i \leq m \leq n$, $D \cap (z_{m+1} \ldots z_n = 0) = \emptyset$. Let $\omega^1$ be in $D$, $\omega^1 = 0$. Write
D' = \text{int}L^{-1}(L(D)^c) \cup D. Then intD' = \text{int}L^{-1}(L(D)^c). For L^{-1}(L(D)^c) contains D - (z_1\ldots z_n = 0), so is dense in D. Also D' = \tilde{D}, since every f in H(D) already extends to H(D'). We prove \tilde{D}' \cap \{z_1 = 0\} - (z_{m+1}\ldots z_n = 0\} \subseteq \text{int}D' \cap \{z_1 = 0\} - (z_{m+1}\ldots z_n = 0\} for i \leq m. We may take i = 1.

Let \( w = (0, w_2, \ldots, w_n) \) be in intD', \( w_{m+1}\ldots w_n \neq 0 \). We prove that for f in H(D'), f extends as a function holomorphic in a neighborhood of w. Since \( w \) is in intD' there is a polycylinder \( P(w, r) \subset \text{int}D', P(w, r) = \{z; |z_i - w_i| < r\} \). If \( w' \) is in \( P(w, r) \cap (z_1\ldots z_n \neq 0) \), then L(\( w' \)) is in intL(D') = intL(D)^c = L(D)^c, since L(D)^c is convex. Thus \( w' \) is in D'. Let K be a Reinhardt compactum in D' containing \( [w' = (w_1, \ldots, w_n)]; |w_i' - w_i| = r, 1 \leq i \leq m, |w_{i+1} - w_i| \leq r, m+1 \leq i \leq n \}

and \( \omega_i, 1 \leq i \leq m \). Since \( Kz_{2i} = 0 \) \neq \emptyset, 1 \leq i \leq m, S(A(K)), contains all points L^{-1}(x) such that \( x_i \leq y_i, 1 \leq i \leq m \) and \( (y_1, \ldots, y_m, x_{m+1}, \ldots, x_n) \) in L(K). But L(\( w' \)) is in L(K), so all \( (z_1, \ldots, z_m, w_{m+1}, \ldots, w_n) \) with \( |z_i| \leq |w_i|, 1 \leq i \leq m \) is in S(A(K)). Thus

\( [(z_1, \ldots, z_n); |z_i| \leq r/2, 1 \leq i \leq m, |z_{i+1} - w_i| \leq r/2, m+1 \leq i \leq n] \)

is contained in S(A(K)), so that S(A(K)) contains a neighborhood of w. Now if f is in H(D'), then f is in A(K), so f is in H(intS(A(K))), and thus f extends to a function holomorphic in a neighborhood of w.

Thus 4) is proven.

5) \( D^0 = \text{int}(L^{-1}(L(D)^c)) \) is a domain of holomorphy.
If \( L(D)^c \) contains \((x_1, \ldots, x_n)\) with \( x_1 \) arbitrarily small, it is easy to see, since \( L(D)^c \) is connected and convex, that if \((u_1, \ldots, u_n)\) is in \( L(D)^c \), then so is every \((y_1, \ldots, y_n)\) such that \( y_j = u_j \) for \( j \neq 1 \), and \( y_1 \leq u_1 \) \( (10) \)

Now let \( K \) be compact in \( D^o \). If \( L(K) \subsetneq Q^n \), then \( S(A(K)) = \subsetneq L^{-1}(L(K)^c) \subsetneq L^{-1}(L(D)^c) \), and \( L(K)^c \) is compact so \( K \) is compact in \( D^o \). Suppose \((0, z_2, \ldots, z_n)\) is in \( K \). Then \( L(S(A(K))) \) is the smallest convex set containing \( K \) which satisfies \((10)\), but since \((0, z_2, \ldots, z_n)\) is in \( K \subset D \), so also does \( L(D)^c \cap Q^n \) satisfy \((10)\), therefore \( L(D) \cap \cap Q^n \subsetneq L(S(A(K))) \cap Q^n \). Thus \( S(A(K)) \cap \cap = \subsetneq L^{-1}(L(D)^c) \). Then there is a neighborhood \( U \) of \( S(A(K)) \) such that \( U \cap \cap \subsetneq L^{-1}(L(D)^c) \). Then \( S(A(K)) \subsetneq \cap \subsetneq \int \cap L^{-1}(L(D)^c) = D^o \). But since the rational functions with poles off \( K^o \) are dense in \( A(K), S(A(K)) \) is \( H(D^o) \)-convex, so the convex hull of \( K \) is thus a compact subset of \( D^o \). Thus \( D^o \) is holomorphic convex and \( 5) \) is proven.

6) Now let \( D \) be a Reinhardt domain, \( D \cap (z_i=0) \neq \emptyset \), \( 1 \leq i \leq m \leq n \), \( D \cap (z_{m+1} \ldots z_n=0) = \emptyset \). Then from the above \( D \cap L^{-1}(L(D)^c) \cap \int L^{-1}(L(D)^c) \cap (z_1=0)-(z_{m+1} \ldots z_n=0) \)

But \( (z_{m+1} \ldots z_n=0) \) is a domain of holomorphy; thus by \( 5) \) so also is \( \int L^{-1}(L(D)^c) \), so their intersection is also a domain of holomorphy and thus must be \( D \).

Theorem 11 is proven.
Corollary 1. If \( D_n(z_1 \ldots z_n=0) = \emptyset \), or if \( D \) is complete Reinhardt, then \( D \) is a domain of holomorphy if and only if \( L(D) \cap \mathbb{Q}^n \) is convex.

Proof: By theorem 11, in the first case \( \tilde{D} = L^{-1}(L(D)^c) \). Thus \( D = \tilde{D} \) if and only if \( L(D)^c = L(D) \).

If \( D \) is complete Reinhardt, \( D = \text{int} \tilde{D} \). For if \( z \) is in \( \text{int} \tilde{D} \) there is a neighborhood \( U \) of \( z \), \( U \subset \tilde{D} \). Then there is a \( w \) in \( U \) such that \( |w_1| > |z_1| \). Since \( w \) is in \( \tilde{D} \), there is a \( w' \) in \( K \) such that \( |w'_1| > |z_1| \) also. But \( D \) is complete Reinhardt, thus \( z \) is in \( D \). Thus if \( L(D) \) is convex we have

\[
\tilde{D} = \text{int} L^{-1}(L(D)^c) = \text{int} L^{-1}(L(D)) = \text{int} \tilde{D} = D,
\]

so \( D \) is a domain of holomorphy. Conversely, if \( D \) is a domain of holomorphy we can write \( D = \bigcup K_n \), where \( K_n \) is \( H(D) \)-convex and a complete Reinhardt compactum, and \( K_n \subset K_{n+1} \). Then \( K_n \) is holoconvex, so \( L(K_n) \) is convex. \( D = \bigcup K_n \) implies \( L(D) \cap \mathbb{Q}^n = \bigcup [L(K_n) \cap \mathbb{Q}^n] \), i.e., \( L(D) \) is an increasing union of convex sets so is convex.

Corollary 2. If \( D_n(z_1=0) \neq \emptyset \), \( 1 \leq i \leq n \), then \( \tilde{D} \) is a complete Reinhardt domain.

Proof: Write \( D = \bigcup K_n \), \( K_n \cap (z_1=0) \neq \emptyset \), \( 1 \leq i \leq n \), and \( K_n \subset K_{n+1} \). Then, by theorem 10, \( S(A(K_n)) \) is a complete Reinhardt compactum, and every \( f \) in \( H(D) \) is in \( A(K) \), so extends to \( H(\text{int} S(A(K_n))) \). Thus \( \tilde{D} \supset D^o = \bigcup \text{int} S(A(K_n)) \). Since \( S(A(K_n)) \) is complete Reinhardt, so is \( \text{int} S(A(K_n)) \), thus so also is \( D^o \). But \( L(S(A(K_n))) \)
is convex, thus \( L(\text{int}S(A(K^n_n))) \) is convex. \( L(D^o) \) is the increasing union of \( L(\text{int}S(A(K^n_n))) \), thus is convex. By corollary 1, \( D^o \) is a domain of holomorphy, so \( \tilde{D} = D^o \).

**Corollary 3.** Let \( D \) be a complete Reinhardt domain. \( D \) is a domain of holomorphy if and only if \( \tilde{D} \) is holoconvex.

**Proof:** We need only prove, by ii) of theorem 10, and corollary 2 above that \( L(D) \) is convex if and only if \( L(\tilde{D}) \) is convex. The implication one way is obvious. Suppose \( L(\tilde{D}) \) is convex, let \( u,v \) be in \( L(D) \). Then there is \( u',v' \) in \( L(D) \) such that \( u'_1 > u_1, v'_1 > v_1 \), since \( L(D) \) is open. Then for any \( w \) on the line segment between \( u \) and \( v \) there is a \( w' \) on the line segment between \( u' \) and \( v' \) such that \( w'_1 > w_1 \). Now \( w' \) is in \( L(\tilde{D}) \), since \( u', v' \) are in \( L(D) \) \( L(\tilde{D}) \), and \( L(\tilde{D}) \) is convex. Then there is a \( w^o \) in \( L(D) \) such that \( w^o_1 > w_1 \) also. But \( D \) is complete Reinhardt, thus \( w \) is in \( L(D) \). Thus \( L(D) \) is convex.

We note that this statement is not true if we drop the assumption of completeness, for \( \{ 0 < |z| < |w| < 1 \} \) is a domain of holomorphy, but as we have seen (page 45), its closure \( \{ 0 \leq |z| \leq |w| \leq 1 \} \) is not holoconvex. But the implication the other way is always true:

**Theorem 12.** Let \( K \) be a holoconvex compact set in \( \mathbb{C}^n \). Then \( \text{int}K = D \) is a domain of holomorphy.

**Proof:** Suppose \( D \) is not a domain of holomorphy,
and \( D \neq \emptyset \). Then there is a manifold \( M \), a nonsingular mapping \( \varphi : M \to \mathbb{C}^n \), a biholomorphic mapping \( \varphi \) of \( D \) onto a proper open subset \( G \) of \( M \). Further, for all \( f \) in \( H(D) \), there is an \( F \) in \( H(M) \) such that \( f = F \circ \varphi \). Finally \( \varphi \circ \varphi = I \), the identity mapping of \( D \) into \( \mathbb{C}^n \) [10].

Let \( m \) be in \( \partial G \); since \( \varphi \) is nonsingular, there is a connected neighborhood \( U \) of \( m \) such that \( \varphi \mid U \) is a biholomorphic map onto a neighborhood \( U' \) of \( \varphi(m) = z \). Since \( \varphi \circ \varphi = I \), \( z \) is in \( \partial D \). We may shrink \( U \) so that \( U' \) is a polycylinder \( P(z, r) \) about \( z \). Now let \( z^0 \) be in \( U' \setminus K \), such exists, for if not \( K \) is dense in \( U' \), so \( K \cap U' \), implying \( z \) is in \( \text{int} K = D \), contradicting \( z \) is in \( \partial D \). Let \( m^0 = \varphi^{-1}(z^0) \). We will prove that \( m^0 \) determines a homomorphism of \( A(K) \) which is not evaluation at any point of \( K \).

For \( f \) in \( H(D) \), let \( \tilde{f} \) in \( H(M) \) such that \( f = \tilde{f} \circ \varphi \). The map \( f \to \tilde{f} \) is one-one since \( D \neq \emptyset \). Define \( h: A(K) \to \mathbb{C}^1 \),

\[
h(f) = \left. \tilde{f} \right|_{\text{int} K}(m^0).
\]

If \( g \) is in \( H(D) \) is such that \( \tilde{g}(m^0) \neq \tilde{g}(m) \) for all \( m \) in \( G \), then \( [\tilde{g}(m) - \tilde{g}(m^0)]^{-1} \) is in \( H(G) \), but not in \( H(M) \), a contradiction since every \( f \) in \( H(G) \) extends to \( H(M) \). Thus for \( f \) in \( A(K) \), \( h(f) \) is a value of \( f \) taken at some point of \( D \), so \( |h(f)| \leq \|f\|_K \). Thus \( h \) is a continuous homomorphism of \( A(K) \).

Now, if \( \tilde{z}_1 \) extends the coordinate function \( z_1 \) to \( M \), \( \tilde{z}_1 \circ \varphi = z_1 \), so since \( \varphi = \varphi^{-1} \) in \( U \cap G \), \( \tilde{z}_1 = z_1 \circ \varphi \) in
UnG, but \( \text{UnG} \neq \emptyset \), and \( U \) is connected, thus \( \tilde{z}_1 = z_1 \) in \( U \). Thus \( \tilde{z}_1(m^o) = z_1(\psi(m^o)) = z_1(z^o) = z_1^o \). But \( z^o \) is not in \( K \), so for any point in \( K \), there is an \( i \) such that \( z_1^o \neq w_1 \). Then \( \tilde{z}_1(m^o) \neq z_1(w) \), so \( h(z_1) \neq z_1(w) \), and thus \( h \) is not evaluation at \( w \). Then \( h \) is a homomorphism of \( A(K) \) not in \( K \), so \( S(A(K)) \neq K \), and thus \( K \) cannot be holoconvex.

In the case of Reinhardt domains, whenever we had \( K \) holoconvex and \( \text{int}K \neq \emptyset \), it followed that \( A(K) = C(K) \cap H(\text{int}K) \).

\[ (11) \]

Is this always true? Also, what are the conditions on a bounded domain of holomorphy which will ensure that its closure is holoconvex? These are still unanswered questions.

We finally note that the assumption \( \text{int}K \neq \emptyset \) is essential in case (11), even if \( K \) is complete Reinhardt, i.e., let \( K = P^n_1(z_1 \ldots z_n = 0) \). Then \( K \) has no interior, but yet \( A(K) \) is not \( C(K) \), in fact \( K \) is the algebra denoted by \( M \) in section 3, where here the base space \( S \) is \([1, 2, \ldots, n]\).

As theorem 10 has demonstrated a Reinhardt compactum \( R \) is holoconvex if and only if \( \overline{\text{int}}R \) and \( R \cap (z_1 = 0) \) are holoconvex. Since \( R \cap (z_1 = 0) \) is a Reinhardt domain in a lower dimension, and because of its independence from \( \overline{\text{int}}R \) we shall now only consider holoconvex Reinhardt compacta satisfying \( R = \overline{\text{int}}R \), to avoid irrelevant complications.
Theorem 13. Let \( R \) be a holoconvex Reinhardt compactum, \( R = \text{Int} R \). Then, writing \( \Gamma(A(R)) \) as the Silov boundary of \( A(R) \), we have

1) \( \Gamma(A(R)) \cap (z_1 \ldots z_n \neq 0) = L^{-1}(\text{extreme points of } L(R)) \).

2) If \( z \) is in \( \partial R - \Gamma(A(R)) \), there is a variety through \( z \) on \( \partial R \), i.e., there is a neighborhood of \( z \), a variety \( V \) such that \( z \) is in \( V \cap U < \partial R \).

Proof: Let \( u^* = (u_1^*, \ldots, u_n^*) \) be an extreme point of \( L(R) \), \( U \) any neighborhood of \( u^* \), \( \bar{U} \cap L(z_i = 0) = \emptyset \), \( 1 \leq i \leq n \).

Then \( u^* \) is not in the convex hull of \( L(R) - U \), so there is a hyperplane \( \Sigma a_i x_i = c \) through \( u^* \) such that \( \Sigma a_i x_i < c \) on \( L(R) - U \). We can change \( P \) a bit so that the \( a_i \) are rational, and thus by multiplication by a common denominator we may suppose the \( a_i \) are integers. Let \( f(z) = \prod_{i=1}^{n} z_i^{a_i} \).

If \( f \) were not holomorphic on \( R \), i.e., it has a pole on \( R \), there would be some \( a_j < 0 \), and \( R \cap (z_j = 0) \neq \emptyset \). But \( R \) is holoconvex, \( R = \text{Int} R \), therefore there are \( z \) in \( R \) with \( z_j \) arbitrarily small, i.e., there are \( x \) in \( L(R) \) with \( x_j \) arbitrarily close to \( -\infty \). Now \( \Sigma a_i x_i \leq c \) on \( L(R) - U \); but since \( a_j < 0 \), we can make \( \Sigma a_i x_i > c \) for some \( x \) in \( L(R) \), arbitrarily close to \( x_j = -\infty \), thus we cannot have \( x \) in \( L(R) - U \), so \( x \) is in \( U \). Thus \( \bar{U} \cap (z_j = 0) \neq \emptyset \), a contradiction. Then \( f \) is in \( H(R) \), further, \( f(e^{u_1^*}, \ldots, e^{u_n^*}) = e^c \), but for \( z \) in \( R - L^{-1}(U) \), \( |f(z)| < e^c \). Thus \( (e^{u_1^*}, \ldots, e^{u_n^*}) \) is in \( \Gamma(A(R)) \), and since \( \Gamma(A(R)) \) is a Reinhardt set, \( \Gamma(A(R)) \) contains \( L^{-1}(u^*) \). Thus \( \Gamma(A(R)) \) contains \( L^{-1}(\text{extreme points of } L(R)) \).
Now let \( u^o \) be in \( L(R) \), \( u^o \) is not an extreme point.

Now, if \( u^o \) is in \( \text{int} L(R) \), we know \( L^{-1}(u^o) \subseteq \text{int} R \), so, by the maximum modulus principle, \( L^{-1}(u^o) \cap (A(R)) = \emptyset \), thus we may assume \( u^o \) is in \( L(R) \).

\( u^o \) is not an extreme point, thus we can write
\[ u^o = t e^1 + (1-t)e^8, e^1, e^8 \text{ in } L(R), \quad 0 < t < 1. \]
Since \( L(R) \) is convex, the whole segment \( S: t e^1 + (1-t)e^8; \)
\[ 0 < t < 1 \]
is contained in \( L(R) \), and in fact in \( \partial L(R) \), since \( u^o \) is in \( \partial L(R) \). Let \( \mathcal{L}: \sum_{i=1}^{n-l} \alpha_{ij} x_j = c_1, \quad 1 \leq i \leq n-l \)
be the line containing this segment. Again, since \( L(R) \) is convex and \( S \subset \partial L(R) \), there are lines parallel to \( \mathcal{L} \) and arbitrarily close to \( \mathcal{L} \) which have segments \( S' \) contained in \( \text{int} L(R) \). Thus there is a sequence \( \varepsilon_n \to 0 \) such that for the lines \( \mathcal{L}_n: \sum_{i=1}^{n-l} \alpha_{ij} x_j = c_1 - \varepsilon_n \) and
\( U \) a neighborhood of \( \mathcal{L}, \mathcal{L}_n \cap U \) is a segment contained in \( \text{int} L(R) \).

Let \( V \) be the variety
\[ V[\sum_{i=1}^{n-l} \alpha_{ij} x_j = e^c_1, \quad 1 \leq i \leq n-l], \]
\[ V_n = V[\sum_{i=1}^{n-l} \alpha_{ij} x_j = e^c_1 - \varepsilon_n, \quad 1 \leq i \leq n-l]. \]
Even if the \( \alpha_{ij} \) are irrational, on \( z \neq 0; \sum_{i=1}^{n-l} \alpha_{ij} x_j \) can be defined locally as a holomorphic function, so \( V, V_n \) are varieties in a neighborhood \( U \) of \( (e^{u^o_1}, ..., e^{u^o_n}) = z^o \). Besides this, \( V \cap \partial R \) is open on \( V \), and \( V_n \cap \text{int} R \) is open on \( V_n \). Thus by theorem 3, no point of \( V \cap \partial R \) is a strong boundary point, so \( z^o \) is not a strong boundary point; and since the set of strong boundary points is also a Reinhardt set, no point of \( T_{z^o} \) is a strong boundary point. Now there is a neighborhood \( U \) of \( z^o \) such that
$L(U) \cap (\text{extreme points of } L(R))$ is empty. Thus, by the same argument no $z$ in $U$ can be a strong boundary point.

But $\Gamma(A(R))$ is metric, so $[\text{strong boundary points}]$ is dense in $\Gamma(A(R))$. Thus $\Gamma(A(R)) \cap U$ is empty, thus $z^*$ is not in $\Gamma(A(R))$. Thus $T_{z^*} \cap \Gamma(A(R))$ is empty. Thus i) is proven, and ii) is proven for $z$ in $(z_1 \ldots z_n \neq 0)$.

Let now $w$ in $\bigcup_{z_1=0} U \cap A(R)$, we have yet to prove that if $w$ is not in $\Gamma(A(R))$, there is a variety of dimension greater than 0 through $w$ on $\partial R$. The proof is by induction on $n$.

The case $n = 1$ is trivial, for every boundary point is a Silov boundary point.

Suppose $n \geq 2$. If $w = (0, w_2, \ldots, w_n)$ is not a Silov boundary point of $R \cap (z_1=0)$, then there is a variety $V$ of dimension $\geq 1$ through $w$ in $R \cap (z_1=0)$ (the boundary of $R$ as a subset of $\{z_1=0\}$) by the induction hypothesis. But then also $V \subset \partial R$.

We suppose now that $w$ is a Silov boundary point of $R \cap (z_1=0)$. Let $P^1(w, \frac{1}{n}) = [z; \ |z_2| < \frac{1}{n}, \ldots, |z_n| < \frac{1}{n}]$, $U_n = (0, z_2, \ldots, z_n)$ in $P^1(w, \frac{1}{n})$. By assumption, since the rational functions are dense in $A(R(z_1=0))$, there is a rational function $r$ of $z_1, \ldots, z_n$ such that $r|_{U_n} = 1$ and $|r| < \varepsilon$ in $R \cap (z_1=0) - U_n$. Then $r$ extends, by $\tilde{r}(z_1, \ldots, z_n) = r(z_2, \ldots, z_n)$ to a rational function in $C^n$ such that $|\tilde{r}|_{P_n} = 1$ and $|\tilde{r}| < \varepsilon$ in $R - P_n$ (since $R \cap (z_1=0) \neq \emptyset$, if $(z_1, \ldots, z_n)$ is in $R$, then also is $(0, z_2, \ldots, z_n)$ by theorem 10. Thus $R \cap (z_1=0)$ is precisely the projection...
of R on \((z_1=0)\). Then, there must be a point \(z^n\) in 
\(P^1(w, \frac{1}{n}) \cap \Gamma(A(R))\). Let \(W\) be a fixed neighborhood of \(w\) 
such that \(W \cap \Gamma(A(R)) = \emptyset\). Then \(z^n\) is in \(\Gamma(A(R)) - W\), 
which is compact. Thus there is a point \(z^o\) in \(\Gamma(A(R)) - W\), 
a limit point of the \(z^n\). Since \(z^n\) is in \(P^1(w, \frac{1}{n})\), 
\(z^n_i \to w_i\) for \(i \geq 2\). Thus \(z^o = (z^o_1, w_2, \ldots, w_n)\). Now 
every \((z_1, w_2, \ldots, w_n)\) such that \(|z_1| \leq |z^o_1|\) is thus in 
\(R\), so that the set \([|z_1| \leq |z^o_1|, z_i=w_i, 2 \leq i \leq n] = V\) 
is a one dimensional variety containing \(w\), contained 
in \(R\), and in fact, it is contained in \(\partial R\), for if any 
point belonged to \(\text{int} R\), so also must \(w\).

5. Complete Reinhardt Compacta in \(C^a\)

In this section \(R\) will always represent a complete 
Reinhardt compactum in \(C^a\). We will show that there 
is in general no other qualification for a closed sub-
set of \(R\) to be the Silov boundary of \(A(R)\) than that it be Reinhardt. We will first give a parametrization 
of the boundary of the unit ball, the 3-sphere, \(S^3\), 
which will simplify the discussion. Let \(I\) represent 
the closed interval \([0, \frac{\pi}{2}]\) and \(T = \{t \mid |t| = 1\}\). Define 
\(\omega: I \times T \times T \to S^3\) by

\[ \omega(\theta, t_1, t_2) = (t_1 \cos \theta, t_2 \sin \theta). \]

Define \(\pi: S^3 \to I\) as the projection \(\pi(t_1 \cos \theta, t_2 \sin \theta) = \theta\).
If \(\theta\) is in \(I\), let \(T_\theta = T(\cos \theta, \sin \theta)\), i.e., \(T_\theta\) is the 
torus through the point \((\cos \theta, \sin \theta)\). Let \(T_\theta\) be the 
polycylinder determined by \(T_\theta\). If \(R\) is a complete
Reinhardt compactum there is a function \( \xi \) on \( I \) such that \( R = \left\{ \xi(\theta, t_1, t_2); \ \xi \leq \xi(\theta) \right\} \). Also, since \( \Gamma(A(R)) \) must be Reinhardt, \( \pi(\Gamma(A(R))) = F \) is closed on \( I \), and 
\[ \Gamma(A(R)) = \pi^{-1}(F), \text{ i.e., } \Gamma(A(R)) = \left\{ \xi(\theta)\omega(\theta, t_1, t_2); \ \theta \in F \right\}. \]

**Theorem 14.** For \( R \) any closed subset of \( I \), there is a holoconvex complete Reinhardt compactum \( R \) in \( C^2 \) such that \( \pi(\Gamma(A(R))) = F \).

**Proof:** Let \( R = \) the monomial convex hull of \( \omega(FxTxT) \), i.e., \( R = \left\{ (w_1, w_3); |w_1 n_1 w_3| \leq \|z_1 n_1 z_3 n_3\|^\omega(FxTxT) \right\} \) for all \( n_1 \neq 0, n_3 \neq 0 \).

If \( (s_1, s_3) \) is in \( R \), so is any \( (t_1 s_1, t_2 s_3) \) since 
\[ |(t_1 s_1) n_1 (t_2 s_3) n_3| \leq |s_1 n_1 s_3 n_3| = \|z_1 n_1 z_3 n_3\|^T(s_1, s_3). \]

Thus \( R \) is a complete Reinhardt compactum. Now to say a Reinhardt compactum \( R \) is monomially convex is to say \( L(R) \) is convex. For if \( p^0 = x^0, y^0 \) is not in \( L(R) \), then \( (e^{x^0}, e^{y^0}) \) is not in \( R \), so there is a \( z_1 n_1 z_3 n_3 \) such that 
\[ |\exp(n_1 x^0 + n_3 y^0)| > \|z_1 n_1 z_3 n_3\|^R, \text{ i.e., } n_1 x^0 + n_3 y^0 > n_1 x + n_3 y \text{ for all } (x, y) \text{ in } L(R). \]

Thus if \( p^0 \) is not in \( L(R) \), there is a line through \( p^0 \) not intersecting \( L(R) \), proving \( L(R) \) is convex since it is connected.

Furthermore \( L(R) \) is the convex hull of \( L(FxTxT) \). Thus \( R \) is holoconvex. Further \( R \subset \text{unit ball, } B, \text{ so } A(R) \supset A(B). \) Since every point of \( FxTxT \) is a Silov boundary point for \( A(B) \), it is so for \( A(R), \text{ thus } \Gamma(A(R)) \supset \text{extreme points of } FxTxT, \text{ and } \Gamma(A(R)) \subset A(R). \)

On the other hand \( L(FxTxT) \supset \text{extreme points of }
L(R) = L(\Gamma(A(R))) by theorem 13, so F\times T and \Gamma(A(R)) coincide on \(z_1z_2 \neq 0\). Thus we have \(\Gamma(A(R)) = (F\times T) \cap (z_1z_2 = 0)\), thus if \(0, \frac{\pi}{2}\) are in F, we have \(\Gamma(A(R)) = F\times T\).

If 0 is not in F, let \(\varepsilon' = \inf\{\varepsilon \in F\}, \varepsilon' > 0\).

Thus \(\|z_1\|_{F \times T} = \cos \varepsilon'\). Now the monomial \(z_1\) separates all points \([z; |z_1| > \cos \varepsilon']\) from R, whereas every point in \(P_{\varepsilon'}\) is in r. Thus if \(\pi(z^0) = 0\), i.e., \(z^0 = 0\), and \(z^0\) is in \(\gamma R\), \(z^0\) lies on the variety \(z_1 = \cos \varepsilon \arg z^0\) on \(\gamma R\), so cannot be a Silov boundary point. Similarly if \(\frac{\pi}{2}\) is not in f, no point of \(z_1 = 0\) can be in \(\Gamma(A(R))\), thus we have \(\Gamma(A(R)) = F\times T\).

We consider now the relation of the size of the Silov boundary and the question of \((Y,X)\)-maximality. The extreme cases are here considered, and they seem to indicate that the class of algebras larger than \(A(R)\) with space of maximal ideals R increases as the Silov boundary increases. For instance, the only complete Reinhardt compactum with Silov boundary a single torus is the polycylinder. For, in this case \(L(R)\) is convex; if \((u,v)\) is in \(L(R)\), then all \((x,y) x \leq u, y \leq v\) is in \(L(R)\), and \(L(R)\) has only one extreme point. This is only possible if \(L(R)\) is the third quadrant determined by a horizontal line \(y = v^0\), and a vertical line \(x = u^0\). Then \(R\) is the polycylinder \([z_1, z_2); |z_1| \leq e^{u^0},
|z_2| \leq e^{v^0}\]. In this case we have (theorem 4) that \(R\) is \((R, \Gamma(A(R)))\)-maximal. At the other extreme we have:
Theorem 15. Let $R$ be a holoconvex complete Reinhardt compactum. If under the projection $\pi: C^2-0 \rightarrow I$ (defined page 56) $\pi(\Gamma) = \Gamma(A(R))$, contains an open set, then $R$ is not $(R, \Gamma)$-maximal; in fact there is a subalgebra $B$ of $G(\Gamma)$ such that (i) $B \not\supset A$ (ii) $\overline{r(B)} = \Gamma$ (iii) $S(B) = S(A)$.

The idea of the proof depends only on the circularity of $R$, but we shall need the previously given characterization (theorem 13) of the Silov boundary points, which we only have for Reinhardt compacta.

Proof: Let $\pi^0(z_1, z_2) = \frac{z_1}{z_2}$ be the map of $C^2-0$ onto the Riemann sphere $S$. $\pi^0(\Gamma)$ contains an open set. For $\pi(\Gamma)$ contains an open set, and since $\Gamma(A(R))$ is Reinhardt, $\Gamma = \pi^0(\Gamma)$ contains an open set. $\pi^0$ is an open map, so $\pi^0(\Gamma)$ contains an open set. Since we shall no longer use the projection $\pi: C^2-0 \rightarrow I$, we will drop the superscript in $\pi^0$ and let $\pi(z_1, z_2) = \frac{z_1}{z_2}$.

Let $V$ be an open disc contained in $\pi(\Gamma)$. We may assume $\omega$(the image under $\pi$ of $z_2=0$) is the center of $V$. Let $H = \{f \text{ in } C(S); f \text{ is analytic on } S-V\}$; the properties of $H$ are well known, $S(H) = S, \overline{r(H)} = \overline{V}$.

For $f$ in $C(S)$, the map $\pi(f) = f \circ \pi$ defines a function $\widetilde{f}$ in $C^2-0$; also if $f$ is analytic on an open set $U$, $\widetilde{f}$ is analytic on $\pi^{-1}(U)$. Let $\tilde{H} = \{\widetilde{f}; f \text{ in } H\}$. Then every function in $\tilde{H}$ is bounded and continuous in $C^2-0$ and is holomorphic in $(C^2-0)-\pi^{-1}(\overline{V})$. 
Let now \( I = \{ g \in A(R); g(0) = 0 \} \), and let \( B \) be the algebra of functions defined on \( R \) which are uniform limits of polynomials of the form
\[
\sum \tilde{f}_1 g_1 + g_0, \quad \tilde{f}_1 \text{ in } \tilde{H}, \ g_1 \text{ in } I.
\]

\( B \) has the following properties:

i) every \( f \) in \( B \) is in \( C(R) \), and \( f(0) = 0 \).

ii) for \( s \) in \( S \), writing \( D_s = R\pi^{-1}(s) \), \( f \) in \( B \) is analytic on \( \text{int}D_s \).

iii) every \( f \) in \( B \) is analytic on \( \text{int}(R\pi^{-1}(S-V)) \).

iv) \( B \not\subseteq I \). For if \( f \) is in \( I; f=0 \) on an open set implies \( f \) is identically zero. But for \( h \) in \( H \), \( h=0 \) on an open set \( U \subseteq \bar{U} \subseteq V \), \( \tilde{h}z_1 \) is in \( B \) and vanishes on \( \pi^{-1}(U) \), an open subset of \( R \).

v) \( S(B) = \mathbb{R} - 0 \) (the following proof is just the proof used by Hoffman and Singer in [16]).

Let \( h \) in \( S(B) \). Define \( \varphi \) on \( H \), \( \varphi(f) = \frac{h(fg)}{h(g)} \), \( g \) is in \( B \) and \( h(g) \neq 0 \). Then \( \varphi(f) \) is a nonzero homomorphism on \( H \). First we note that if \( h(g') \neq 0 \), then \( \varphi(f) = \frac{h(fg')}{h(g')} \).

For
\[
\frac{h(fg')}{h(g')} = \frac{h(fg')h(g)}{h(g)h(g)} = \frac{h(fg')}{h(g)} \frac{h(g)}{h(g')} = \varphi(f)
\]

Now if \( f^1, f^2 \) are in \( H \),
\[
\varphi(f^1f^2) = \frac{h(f^1f^2g)}{h(g)} = \frac{h(f^1f^2g)h(g')}{h(g)h(g')} = \frac{h(f^1f^2g'g)}{h(g)h(g')}
\]
\[
= \frac{h(f^1g)h(f^2g')}{h(g)h(g')} = \varphi(f^1)\varphi(f^2).
\]

Thus we can write
\[
\varphi(f) = \frac{h(fg)}{h(g)} = f(s_h), \quad s_h \text{ in } S.
\]
Since I \subset B, h|_I is a homomorphism of I. Now for any f in H, g in I, h(\tilde{fg}) = f(s_h)h|_I(g). This is true if h(g) \neq 0, by the above. But if h(g) = 0, we know there is a k in B such that h(k) \neq 0, and thus
\[ h(\tilde{fg}) = \frac{h(\tilde{fg})h(k)}{h(k)} = \frac{h(\tilde{fg})h(k)}{h(k)} = \frac{h(\tilde{fk})}{h(k)}h(g) = 0. \]
Thus h(\tilde{f}_I g + g_\circ) = \Sigma f_1(s_h)h|_I(g_1) + h|_I(g_\circ), \quad (12)
and so h is completely determined by s_h and h|_I, and in fact, h|_I = 0 implies h is 0, so h|_I is a nonzero homomorphism of I, i.e., a point w_h in R-0 (since \( S(A(R)) = R \) and I is an ideal of A(R) whose hull is \([0]\)).

Now in fact also s_h is determined by w_h, in fact s_h = \pi(w_h), which will finish the proof that \( S(B) = R-0 \), since, if this is the case, then f(s_h) = \tilde{f}(w_h) and thus for all f in B, h(\tilde{f}) = \tilde{f}(w_h) by (12). Suppose, on the contrary that s_h \neq \pi(w_h). We may assume that neither of them is \( \alpha \). Now let f(s) = s-s_h, then \( \tilde{f}(z_1, z_2) = \frac{z_1}{z_2} - s_h \). Now, writing w_h = (w_1, w_2), by assumption w_2 \neq 0; let g(z_1, z_2) = \frac{z_2}{w_2}, then g is in I and \( \tilde{fg} = (z_1-s_h z_2)/w_2 \) is in I. Thus we have, on the one hand
\[ h(\tilde{fg}) = \tilde{fg}(w_h) = \frac{1}{w_2} (w_1 - s_h w_2) = \frac{w_1}{w_2} - s_h \neq 0, \]
and on the other hand,
\[ h(\tilde{fg}) = f(s_h)g(w_h) = 0 \cdot 1 = 0, \] a contradiction.

(vi) \( \Gamma(B) = \Gamma(I) = \Gamma(A) \).

That \( \Gamma(I) = \Gamma(A) \) is clear. \( \Gamma(B) \supset \Gamma(I) \) since \( B \supset I \), and they have the same maximal ideals. Further \( \Gamma(B) \) is contained in R, since every f in B is analytic on at least a complex line through any interior point.
Now suppose $z$ is in $\partial R - \Gamma(I)$, then $z$ is in $\partial R - \Gamma(A(R))$.

Now since $R$ is Reinhardt, by theorem 13, there is a neighborhood $U$ of $z$ and functions $f_1, \ldots, f_k$ in $H(U)$ such that $z$ is in $W = V(f_1 = 0, 1 \leq i \leq k) \subset R$ and $W_n = V(f_1 = 0, l \leq i \leq k) \subset \mathbb{D} \cap R$. Now by definition of $B$, if $\pi(U)\bar{\mathbb{V}} = \emptyset$, every $f$ in $B$ is holomorphic on $U$. As in theorem 3 then, $f|_W$ must be holomorphic, so that $z$ is not a strong boundary point. This for all $z$ in $\partial R - \Gamma(I)$, so that there are no strong boundary points in $R - \Gamma(I)$, an open set. Since the set of strong boundary points is dense in $\Gamma(B)$, we have $\Gamma(B) \subset \Gamma(I)$.

Thus $\Gamma(B) = \Gamma(A)$.

Now let $B^0 = B$ with constants adjoined. Then $\Gamma(B^0) = \Gamma(B) = \Gamma(A)$, and $S(B^0)$ is the one point compactification of $S(B) = R - C$, thus $S(B^0) = R$. $B^0 = B$ with constants adjoined $\supset I$ with constants adjoined $= A(R)$. But also $B^0$ contains a function which vanishes on an open set, but no function of $A(R)$ vanishes on an open set, thus $B^0 \not\subset A(R)$.
CHAPTER THREE
ONE-DIMENSIONAL VARIETIES

1. Introduction

The proof of theorem 4, i.e., the \((P^n, T^n)\)-maximality of the polycylinder has two parts: 1) to show that every function in a larger algebra with the same space of maximal ideals is actually holomorphic on the varieties in the boundary of \(P^n\), 2) to show that any \(f\) holomorphic on all these varieties extends to all of \(P^n\) as a holomorphic function. An attempt to generalize theorem 4 to analytic polyhedra, i.e., compact sets \(P\) of the form \(\{z \in U; |f_1(z)| \leq 1; f_1\) holomorphic in \(U\}\), would have to overcome these two problems. In the case of \(P^n\) the first step was accomplished by induction, since the varieties on the boundary of \(P^n\) were just copies of \(P^{n-1}\), the second step was accomplished by Fourier series arguments. In the analytic polyhedron the varieties on the boundary need not be manifolds, thus a more general maximality theorem is required. It is the purpose of this chapter to prove this maximality theorem for one-dimensional varieties; although the problem has the metamorphosis here de-
scribed, the discussion goes far beyond the problem of the 'first step' in generalizing theorem 4. As for the 'second step', that of extension from \( \mathcal{A}P \) to all of \( P \), whose interest transcends the problem of generalizing theorem 4, I have no solution to offer.

Before proceeding to consider varieties in general, we note that in the case where there is no pathology at all, I have a proof of theorem 4; in fact it is proven that \( P \) is equivalent to a polycylinder under a biholomorphic map.

**Lemma 5.** Let \( R \) be a Riemann surface, \( f \) analytic on \( R \). Suppose \( K = \{ |f| \leq 1 \} \neq \emptyset \) is compact, \( \{ |f| < 1 \} \) is connected and \( f' \neq 0 \) on \( \{ |f| < 1 \} \). Then the map \( f: K \to \{ |z| \leq 1 \} \) is a homeomorphism, and \( f: \{ |f| < 1 \} \to \{ |z| < 1 \} \) is conformal.

**Proof:** Write \( U = \{ |f| < 1 \} \) and \( D = \{ |z| < 1 \} \), we prove \( f: U \to D \). \( f(U) \) is open in \( D \), we prove it is also closed. Let \( z = \lim z_n, z_n \in f(U), z \in D \); so there exists \( w_n \in U \) such that \( f(w_n) = z_n \). Since \( \bar{U} = \{ |f| \leq 1 \} \) is assumed compact, there is a \( w \) in \( \bar{U} \) such that \( w_n \to w \) for some subsequence \( w_{n_k} \). By continuity \( f(w_{n_k}) \to z \), so \( f(w) = z \), and thus \( |f(w)| < 1 \), i.e., \( w \) is in \( U \). Then \( f(U) \) is a neighborhood of \( z \), so \( z \) is in \( f(U) \). Then \( f(U) \) is open and closed in \( D \), since \( D \) is connected and \( f(U) \) nonempty, \( f(U) = D \).

Thus \( U \) is a covering space of \( D \). For let \( z \in D \); since \( \bar{U} \) is compact, there exist finitely many
w_1, \ldots, w_n in U such that f(w_j) = z. Let U_1, \ldots, U_n be disjoint neighborhoods of w_1, \ldots, w_n respectively; since f' = 0, f is locally conformal, so we could pick U_1, \ldots, U_n so that f:U_j \rightarrow f(U_j) is conformal. Let V be a disc about z contained in \bigcap_{j=1}^n f(U_j) such that f^{-1}(V) \subset \bigcup_{j=1}^n U_j. Such a V exists. For if not, then for every V, neighborhood of z there is a w_v in \overline{U_1 \cup U_2} such that f(w_v) is in V. The set \{w_v; V a neighborhood of z\} has a limit point w in \overline{U_1 \cup U_2}, but by continuity we must have f(w) = z, so w must be some w_j, a contradiction. Thus such a V exists. Let V_1, \ldots, V_n be the inverse images of V at each w_j respectively. Then f^{-1}(V) = \bigcap_{j=1}^n V_j, the V_j are disjoint and f:V_j \rightarrow V conformally. But D is simply connected, thus f:U \rightarrow D one-one. Now also f:|f| = 1 one-one onto |z| = 1 also. For let w_n in U such that f(w_n) = (1-\frac{1}{n})e^{i\theta}; by compactness a subsequence converges to some w in \overline{U}, and by continuity f(w) = e^{i\theta}.

Now suppose f(w_1) = e^{i\theta} = f(w_2). Let V_1, V_2 be disjoint neighborhoods of w_1, w_2. Then V_1 \cap U, V_2 \cap U are disjoint subsets of U, f is one-one on U, so f(V_1 \cap U) and f(V_2 \cap U) are also disjoint. But f(V_1 \cap U) = f(V_1) \cap D and f(V_1) \cap f(V_2) = W is a neighborhood of e^{i\theta}. Thus we have a neighborhood W of e^{i\theta} which is disjoint from \{|z| < 1\}, an impossibility. Thus f:K \rightarrow \{|z| < 1\} is one-one and onto, so is a homeomorphism.

**Theorem 16.** Let f_1, \ldots, f_n be holomorphic in a domain \Omega in a manifold M of dimension n. Suppose K =
If \( |f_j| \leq 1, 1 \leq j \leq n \) is compact in \( \mathcal{A} \) and \( \text{int} K \) is connected and nonempty. Let \( F: \mathcal{A} \rightarrow \mathbb{C}^n, F(m) = (f_1(m), \ldots, f_n(m)) \), and suppose \( dF \neq 0 \) on \( K \). Then \( F:K \rightarrow \mathbb{C}^n \) is a homeomorphism and \( F: \text{int} K \rightarrow \text{int} \mathbb{C}^n = D^n \) is biholomorphic.

Proof: First of all \( dF \neq 0 \) on \( K \) implies \( dF \neq 0 \) in a neighborhood of \( K \). For the set \( \{dF = 0\} = \{ \frac{\partial f_1, \ldots, \partial f_n}{\partial z_1, \ldots, \partial z_n} = 0 \} \) in any coordinate system \((U:z_1, \ldots, z_n)\).

Thus \( \{dF = 0\} \) is a variety in \( \mathcal{A} \). Since \( \{dF = 0\}_{\text{int}K} = \phi \), if \( \{dF = 0\}_{\text{int}K} \neq \phi \), it is a compact variety, and thus must be of dimension 0 [3]. But this is impossible for \( n \geq 2 \), since locally \( \{dF = 0\} \) is given by the vanishing of a single function so must be of dimension \( n-1 \).

Thus we can prove theorem 16 assuming \( dF \neq 0 \) on a neighborhood of \( K \), which we take to be \( \mathcal{A} \), and proceed by induction. The case \( n = 1 \) is proven in lemma 5, thus we assume the theorem for the case \( n-1 \) and prove it for \( n \).

First we prove \( F: \text{int} K \rightarrow \mathbb{C}^n \) is onto.

Let \( (z_1^0, \ldots, z_n^0) = z^0 \) be in \( \mathbb{C}^n \). Now there is a \( z^1 = (z_1^1, \ldots, z_n^1) \) in \( F(U) \), consider the variety \( V(f_1 = z_1^1) \). Now \( V \) has no singular points in a neighborhood of its intersection with \( K \). For there \( f_1, \ldots, f_n \) are local coordinates, so \( f_2, \ldots, f_n \) define a local coordinate map of \( V(f_1 = z_1^1) \) onto a neighborhood in \( \mathbb{C}^{n-1} \), therefore \( V \circ z \) is an \( (n-1) \) dimensional manifold, \( K_0 V \) is compact.
and $f_2, \ldots, f_n$ satisfy all the hypotheses on $V(f_1 = z^1)$, so by the induction hypothesis they define a conformal map of each component of $\text{int}K \cap V$ onto $D^{n-1}$, i.e., there is a $w$ in $V$ such that $f_j = z^j$ for $j \geq 2$, so $F(w) = (z_1, z_2, \ldots, z_n)$. Now look at $V(f_1 = z_1, \ldots, f_n = z_n)$, the same arguments show that there is a $w^*$ on this variety such that $f_1(w^*) = z^1$. Thus $F(w^*) = z^*$, so $F$ is onto.

Now for $z^*$ in $\text{int}P^n$ we prove $F^{-1}(z^*) \cap \text{int}K$ consists of finitely many points. For the variety $V(f_1 = z^1, \ldots, f_n = z_n)$ does not intersect $K$, so has a compact component in $\text{int}K$, which must then be $0$-dimensional and in particular, consists of finitely many points.

The rest of the proof proceeds just as in lemma 5, i.e., we prove $F: \text{int}K \rightarrow D^n$ is a covering map, and thus since $D^n$ has a trivial fundamental group $F$ must be one-one. Then, as in lemma 5, we can show $F:K \rightarrow P^n$ is a homeomorphism.

2. Riemann surfaces

In the discussion of the varieties, we will associate with a variety a complex one dimensional manifold; a finite union of Riemann surfaces and we shall deduce the maximality theorem from consideration of these Riemann surfaces. Since we shall be interested in compact sets on the surface, we can replace the original surface by an open set on a compact Riemann
surface. More explicitly, given a compact $K$ on a Riemann surface, $K$ has a neighborhood $U$ of finite genus, i.e., $U$ is conformal to an open subset of a closed Riemann surface $R'$. We will consider $K \subset U$ as subsets of $R'$. Now the Riemann-Roch theorem guarantees the existence of the following functions meromorphic on $R'$ with all poles in $R'-U$:

**Lemma 6.**

1) given $p \neq q$ in $U$, there is an $f$ such that $f(p) \neq f(q)$.

2) given $p$ in $U$ and an integer $N$, there is an $f$ with a zero of order $N$ at $p$, i.e., if $(U, z)$ is a local parameter at $p$, $f$ has the Taylor series in $U$: $f = \sum a_n z^n$ with $a_1 = 0$, $1 \leq N$, $a_{N+1} \neq 0$.

**Proof:**

1) Let $p \neq q$ in $U$; by the Riemann-Roch theorem, there is a meromorphic function $f$ holomorphic on $U$ with a zero of some order $k$ at $p$. If $f(q) \neq 0$, then $f(p) \neq f(q)$. Suppose $f$ has a zero of order $m$ at $q$. Let $g$ be meromorphic on $R'$, with a pole only at $q$ of order, say, $n$. Then $f^n g^k(q) \neq 0$, but $f^n g^k(p) = 0$, so $f^n g^k$ is the required function.

2) Let $f$ be as in 1). By the Riemann-Roch theorem there is a meromorphic function $g$ whose only pole is at $p$ and is of order $m$, where $m$ is prime to $p$, so long as $m$ is large enough. Then there are positive $s$, $t$ such that $sk - tm = 1$. Then $f^s g^t$ is holomorphic on $U$ and has a zero of order 1 at $p$. Thus $(f^s g^t)^N$ has a
zero of order \( N \) at \( p \).

**Lemma 7.** Let \( K \) be a compact proper subset of a compact Riemann surface \( R \). Let \( M = \{ f \text{ meromorphic on } R; \text{ holomorphic in a neighborhood of } K \} \). \( \overline{M} \) = uniform closure of \( M \) on \( K \). Then \( S(\overline{M}) = K \).

The following proof is modeled after a theorem of Arens [1].

**Proof:** By lemma 6, \( M \) separates the points of \( K \), and every point of \( K \) is a continuous homomorphism of \( M \), so \( S(\overline{M}) \supset K \).

Let \( U \) be a neighborhood of \( K \). Let \( h \) be in \( S(\overline{M}) \) \( h \neq \) evaluation at \( r \) in \( K \). Let \( F \) be meromorphic on \( R \) with its only pole in \( U \) a simple pole at \( r \). Now there is a \( g \) in \( M \) such that \( h(g) = 1 \), \( g(r) = 0 \). Then \( gF \) is in \( M \). Let \( t = h(gF) \), then for \( P = F-t \), \( gP \) is in \( M \) and \( h(gP) = h(g(F-t)) = h(gF) - th(g) = t-t \cdot 1 = 0 \).

Let \( r_1, \ldots, r_n \) be the zeros of \( P \) on \( K \) of order \( m_1, \ldots, m_n \) respectively. If \( h \neq \) evaluation at any \( r_j \) there is an \( f_j \) in \( M \), \( h(f_j) = 1 \), \( f_j(r_j) = 0 \). Let \( f = \prod_{j=1}^{n} f_j^{m_j} P^{-1} \). \( f \) is in \( M \), and since \( f(r) = 0 \), \( fP \) is also in \( M \). Thus \( h(\prod_{j=1}^{n} f_j^{m_j}) = \prod_{j=1}^{n} h(f_j)^{m_j} = 1 \), on the one hand, but \( \prod_{j=1}^{n} f_j^{m_j} = fP \), so
\[
h(\prod_{j=1}^{n} f_j^{m_j}) = h(fP) = h(fP)h(g), \quad \text{since } h(g) = 1,
\]
\[
= h(fP_g) = h(f)h(gP) = 0,
\]
a contradiction. Thus \( h \) is evaluation at \( r \) or some \( r_j \), i.e., evaluation at some point of \( K \).
We will use the following three deep theorems concerning $A(K)$, where $K$ is a compact set on a Riemann surface.

**Theorem 17.** (Wermer). Let $\Omega$ be a domain on a compact Riemann surface $R$ bounded by analytic curves $\gamma_1, \ldots, \gamma_n$ such that $K = \Omega \cup \gamma_1 \cup \ldots \cup \gamma_n$ is a compact proper subset of $R$. Let $A = \{ f \in C(K); f \in H(\Omega) \}$. Then $A$ is a maximal subalgebra of $C(\gamma_1 \cup \ldots \cup \gamma_n)$. (The extension from $n = 1$ is due to Royden) [22, 20].

**Theorem 18.** (Runge's Theorem) (Behnke). Let $R$ be a Riemann surface, $U \subset U'$ domains on $R$. Suppose $U$ is simply connected with respect to $U'$, i.e., if $\gamma \subset U$ bounds a domain in $U'$ it bounds a domain in $U$. Then every function holomorphic on $U$ can be approximated uniformly on compact sets by functions holomorphic on $U'$ [4].

**Corollary.** Let $K$ be compact, $U$ a domain containing $K$ such that $\bar{U}$ is compact. If there is a family of domains $U(n) \subset U$, $n$ an integer, satisfying i) $U(n)$ is simply connected with respect to $U(1)$, ii) $U(n) \supset U(n+1)$, iii) $\bigcap_n U(n) = K$. Then every function holomorphic on $U$ can be approximated uniformly on compact subsets by functions in $H(U')$.

**Proof:** Let $f$ be in $A(K)$, $f$ can be uniformly approximated by functions holomorphic in a neighborhood of $K$. If $U$ is a neighborhood of $K$ there is an $n$ such
that \( U(n) \subset U \). If \( f \) is in \( H(\Omega, c^R) \), then \( f \mid_{U(n)} \) is uniformly approximable on \( K \) by functions in \( H(U') \). Thus the corollary is proven.

**Definition 11.** If \( K \subset R \) is compact and there is a domain \( U \supset K \) such that \( K \) is \( H(\Omega, c^R) \)-convex, we call \( K \) a Runge set.

Thus if \( K \) satisfies the hypotheses of the corollary, it is a Runge set.

**Lemma 8.** Let \( K \subset R \), \( K = \Omega \cup \ldots \cup \Omega \); \( \Omega \) is a domain, and \( \gamma_i \) are piecewise analytic arcs. Then \( K \) is a Runge set.

**Proof:** Now \( \gamma K = \gamma \nu \ldots \gamma \nu \); let \( A_1 \) be an annulus about \( \gamma_1 \), and let \( \gamma^1 \) be an arc homologous to \( \gamma_1 \) in \( A_1 \) such that \( \gamma^1 \cap K = \emptyset \). Let \( U^1 \) be the domain bounded by the \( \gamma^1 \) and suppose the \( \gamma^1_i \) are such that \( U^1 \supset U^{i+1} \) and \( \bigcap K = K \). Now \( U^1 \) is simply connected in \( U^1 \). For let \( \gamma \subset U^1 \) bound the domain \( \omega \subset U^1 \). \( \gamma \) is not between \( \gamma^1_i \) and \( \gamma^1 \) for the region between these is disjoint from \( U^1 \). Thus since \( \gamma^1_i \) is not in \( \omega \), neither is \( \gamma^1 \). Thus since \( \omega \cap \gamma^1 = \emptyset \) for all \( i \), we must have \( \omega \subset U^1 \). Then by the corollary, \( K \) is a Runge set.

**Theorem 19.** (Bishop). Let \( \omega \) be an open Riemann surface, \( K \) a compact subset of \( \omega \), \( B \) an algebra of functions analytic on \( \omega \) such that (i) \( K \) is \( B \)-convex, (ii) \( [p \text{ in } K; \ f(p) = f(q) \text{ for all } f \text{ in } B] \cup \ [p \text{ in } K; \ df(p) = 0 \text{ for all } f \text{ in } B] \) (the singular set relative to \( B \), denoted by \( S \)), is finite \( [6] \).

Then there is a positive integer \( N \) such that \( \overline{E} \),
the closure of \( B \) in the uniform topology on \( K \) contains the ideal of \( A \) consisting of those functions in \( A \) which vanish on the singular set \( S \), and vanish to order at least \( N \) at those points of \( S \) interior to \( C \).

We are at present interested in Bishop's theorem when \( B = H^0(\alpha, \sigma^2) \) and \( \alpha \) is a neighborhood of \( K \) whose closure is a proper subset of a compact Riemann surface \( R \). Now if \( K \) is a Runge set there is an \( \alpha \) such that \( H^0(\alpha, \sigma^2) \) is dense in \( A(K) \). Now by lemma 7, since \( S(A(K)) \supset K \), and \( A(K) \ni \mathcal{M} \), we must have \( S(A(K)) = K \).

But this implies \( K \) is \( H^0(\alpha, \sigma^2) \)-convex. For if not there is \( r \) in \( \alpha - K \) such that evaluation at \( r \) is a continuous homomorphism of \( H^0(\alpha, \sigma^2) \) and thus of its closure \( A(K) \), a contradiction. Now, Bishop's theorem applies; here the singular set is empty and we have:

**Corollary.** Let \( K \) be a Runge set in a compact Riemann surface \( R \). There is a neighborhood \( U \) of \( K \) such that \( H^0(U, \sigma_U^R) \) is dense in \( C(K) \cap H(\text{int}K) \), and in particular \( A(K) = C(K) \cap H(\text{int}K) \).

We shall return to consider Bishop's theorem in more detail in Chapter IV.

We prove another theorem now, which shall be used in the consideration of varieties.

**Theorem 20.** Let \( K \) be a proper compact set on a compact Riemann surface \( R \). Let \( x_1 \) be in \( \text{int}K \), \( 1 \leq i \leq k \), and let \( n_1, \ldots, n_k \) be positive integers. Let \( I = [f \in A(K); f \text{ has a zero of order } \geq n_i \text{ at } x_i] \). Let
\[ A^0 = \{ f \text{ in } A(K); f(x) - f(x_1) \text{ has a zero of order } \geq n_1 \text{ at } x_1 \}. \]

Let \( f_i, 1 \leq i \leq k, \) be meromorphic on \( R, \) holomorphic on \( K, \) such that \( f_{1j} \) has a zero of order \( j-1 \) at \( x_1, \) \( 0 \leq j \leq n_1, \) and a zero of at least order \( n_r \) at \( x_1 \) for \( r \neq 1 \) (a zero of order \(-1\) means \( f(x_1) \neq 0 \)). Let \( V \) be the vector space spanned by the \( f_{1j}. \)

Then

i) \( A(K) = V + I, \)

ii) given \( x_1, \ldots, x_e, \) \( 1 \leq e \leq k, \) there is an \( f \) in \( A^0 \cap V \) such that \( f(x_1) = 1, \) for \( 1 \leq i \leq e, \) and \( f(x_1) = 0, \) \( e < i \leq k. \)

Proof: i) Let \( (U_1^2, z^2) \) be a coordinate system at \( x_1. \) Then for \( f \) in \( \mathcal{O}_x, f \) has a Taylor series \( f = \sum_{j=1}^{k} \lambda_{ij}(f)z^j, \) where \( \lambda_{ij} \) is a linear functional on \( \mathcal{O}_x \) (depending on the parameter chosen); in fact \( \lambda_{ij} \) is a linear functional on the class \( M = \{ f \text{ meromorphic on } R, \text{ holomorphic on } K \}. \) By assumption \( \lambda_{ij}(f_{1j}) \neq 0 \) for \( 1 \leq i \leq k, \) \( 1 \leq j \leq n_1, \) and \( \lambda_{rs}(f_{1j}) = 0 \) for \( r \neq 1, \) \( 1 \leq s \leq n_r, \) or \( r = 1 \) and \( s < j. \) Thus the matrix \( M = (\lambda_{rs}(f_{1j})), 1 \leq r, 1 \leq k, 1 \leq s \leq n_r, 1 \leq j \leq n_1, \) is a triangular \( (\Sigma n_1)x(\Sigma n_1) \) matrix all of whose diagonal entries are nonzero. Thus \( M \) is nonsingular. Now for \( f \) in \( A(K), \) let \( (f) \) represent the vector

\[(\lambda_{11}(f), \ldots, \lambda_{1n_1}(f), \lambda_{21}(f), \ldots, \lambda_{2n_1}(f), \ldots, \lambda_{kn_k}(f))\]

Then there is a vector \( (a) = (a_{ij}) \) such that

\[ M(a) = (f) \]
Let \( g = \Sigma a_{ij} f_{ij} \). Then the vector \((g) = M(a) = (f)\), so the vector \((g-f) = 0\), i.e., \( I_{rs}(g-f) = 0\), for \( 1 \leq r < k, 1 \leq s \leq n_r \). Thus \( g-f \) has a zero of order \( \geq n_1 \) at \( x_i \), i.e., \( g-f \) is in \( I \). Thus we have \( f = g + h \), \( g \) in \( V \), \( h \) in \( I \), so \( i \) is proven.

ii) Let \( 1 \leq e \leq k \). Let \( (b) = (b_{ij}) \) be the solution of \( M(b) = (c) \), where \( c_{1o} = 1, 1 \leq i \leq e, c_{ij} = 0 \) for \( i > e \) or \( j \neq 0 \). Let \( h = \Sigma b_{ij} f_{ij} \). Then \( I_{10}(h) = 1 \) if \( i \leq e \), \( I_{rs}(h) = 0 \) for \( r > e \) or \( s \neq 0 \). Then \( h-h(x_i) \) has a zero of order \( \geq n_1 \) at \( x_i \), so \( h \) is in \( A^e \), and \( h(x_i) = 1 \) if \( 1 \leq i \leq e \), \( h(x_i) = 0 \), \( e < i \leq k \).

3. The Normal Model

In the Introduction we defined the sheaf \( \mathcal{E}^V \) of germs of holomorphic functions on a variety in a Stein manifold \( M \). The sheaf is a coherent analytic sheaf [10], and it follows from theorems A and B that if \( f \) is in \( H^\infty(V, \mathcal{O}^V) \), there is an \( F \) in \( H^\infty(M, \mathcal{O}^M) \) such that \( f = F|_V \). We now define a general analytic space (denoted by \( g \)-space in [11]).

Df 12. Let \( \omega: V_1 \rightarrow V_2 \) be a continuous map, \( V_1, V_2 \) varieties in \( M_1, M_2 \) respectively; \( \omega \) is called holomorphic if and only if \( \hat{\omega}(\partial_{x}^V S) \subseteq \mathcal{O}^V_x \) for all \( x \) in \( V_2 \), where \( \hat{\omega}(f) = (f \circ \omega) \). Otherwise stated, \( \omega \) is holomorphic if for \( f \) in \( H^\infty(U, \mathcal{O}^V_2) \), \( U \) open in \( V_2 \), then \( f \circ \omega \) is in \( H^\infty(\omega^{-1}(U), \mathcal{O}^V_1) \).

Df 13. A general analytic space (g.a.s.) is a separable Hausdorff space such that
1) for \( x \) in \( S \), there is a neighborhood \( U_x \) of \( x \) and a homeomorphism \( \omega_x : U_x \rightarrow V_x \subset U \), \( V_x \) a variety in \( U \subset \mathbb{C}^n \) for some \( n \). We call \((U_x, \omega_x)\) the local parameter at \( x \).

11) given \( x, y \) in \( S \), \((U_x, \omega_x), (U_y, \omega_y)\) local parameters such that \( U_x \cap U_y \neq \emptyset \), then \( \omega_x \circ \omega_y^{-1} : V_x \cap \omega_y(U_x) \rightarrow V_x \cap \omega_x(U_y) \) is biholomorphic.

Df 14. If \( S \) is connect g.a.s. we define its dimension: for \( x \) in \( S \), \((U_x, \omega_x)\) local parameter, \( \omega_x : U_x \rightarrow V_x \): let \( \dim S = \dim V_x \). This is well-defined.

Df 15. Let \( S \) be a g.a.s., \( U \) open contained in \( S \), \( f : U \rightarrow \mathbb{C} \). \( f \) is holomorphic if and only if for all \( x \) in \( U \), \((U_x, \omega_x)\) a local parameter, \( f \circ \omega_x^{-1} : V_x \cap \omega_x(U) \rightarrow \mathbb{C} \) is holomorphic. Let \( \mathcal{O}^S \) represent the sheaf of germs of holomorphic functions on \( S \).

Df 16. \( x \) in \( S \) is a regular point if and only if for \((U_x, \omega_x)\) a local parameter, \( \omega_x(x) \) is a regular point of \( V_x \).

This too is well-defined. Letting \( S_{\text{reg}} = \{ \text{regular points of } S \} \), we know \( S_{\text{reg}} \) is dense in \( S \), \( S_{\text{sing}} = S - S_{\text{reg}} \) is a g.a.s. of strictly smaller dimension. Thus if \( S \) is one dimensional, \( S - S_{\text{reg}} \) is a discrete set of points.

Df 17. Let \( S \) be a g.a.s. The normal model of \( S \) is a pair \((R, \pi)\) such that

1) \( R \) is a locally compact, locally connected Hausdorff space,
ii) \(\pi: R \to S\) is continuous, proper and onto.

iii) \(\pi^{-1}(s)\) is finite for all \(s\) in \(S\) (in fact \(\pi^{-1}(s) = x_1, \ldots, x_k(s)\) where \(k(s)\) is the number of irreducible components of \(S\) at \(x\)).

iv) \(\pi^{-1}(S_{\text{sing}})\) separates no connected set in \(R\), and is nowhere dense.

v) \(\pi: R - \pi^{-1}(S_{\text{sing}}) \to S - S_{\text{sing}}\) is a homeomorphism.

The facts we shall need concerning the normal model are the following:

i) the normal model exists and is unique up to a homeomorphism commuting with \(\pi\), i.e., if \((R', \pi')\) is another normal model, there is a homeomorphism \(\omega: R \leftrightarrow R'\) such that \(\pi \circ \omega = \pi'\).

ii) the normal model \(R\) can be made into a general analytic space in a unique way so that \(\pi\) is holomorphic, and \(\mathcal{O}_R^R\) is integrally closed for all \(r\) in \(R\).

iii) Define the following presheaf on \(S\): for \(U\) open on \(S\), let \(\mathcal{O}(U) = H^0(\pi^{-1}(U), \mathcal{O}_R^R)\). Let \(\mathcal{O}\) represent the sheaf defined by the collection of modules \([\mathcal{O}(U); U\) an open subset of \(S]\). The stalk \(\mathcal{O}_s\) is isomorphic to \(\mathcal{O}_R^R x_1 \ldots x_k\) where \(\pi^{-1}(s) = [x_1, \ldots, x_k]\), and is the integral closure of \(\mathcal{O}_s^S\) (\(\mathcal{O}_s^S\) is embedded in \(\mathcal{O}_s\) in a natural way, for \(\pi(H^0(U, \mathcal{O}_s^S))\) is a submodule of \(H^0(\pi^{-1}(U), \mathcal{O}_R^R) = \mathcal{O}(U)\)). Let \(\mathcal{Y}_s = [u \in \mathcal{O}_s^S; u \text{ is a sub-}

u\hat{\mathcal{O}}_{\mathcal{O}^S_\mathcal{S}}, \text{ then } \mathcal{V}_{\mathcal{S}} \neq (0), \text{ and the sheaf } \mathcal{V}; \mathcal{V}_{\mathcal{S}} = \mathcal{V}_{\mathcal{S}} \text{ is a coherent subsheaf of } \mathcal{O}^S_\mathcal{S}.

All these facts are to be found stated in [11], their proofs go back to Oka [18]. We are going to prove below that for a one dimensional g.a.s. the normal model is a one-dimensional manifold (not necessarily connected, i.e., a union of Riemann surfaces.)

Finally we shall need two theorems of Remmert and Grauert (the version of the first that we want is Oka's; a proof can be found in Baily's notes [3].

Theorem 21. (Abbildungssatz). Let \( \omega: S_1 \rightarrow S_2 \) be a proper holomorphic map, \( S_1, S_2 \) are general analytic spaces. Suppose also that \( \omega^{-1}(s_2) \) is finite for all \( s_2 \) in \( S_2 \). Then \( \omega(S_1) \) is a variety in \( S_2 \) [19].

Theorem 22. (Remmert and Grauert). Let \( V \) be a variety, \( U \) an open set in \( V \) and \( f_n \) a sequence of cross-sections of \( \mathcal{O}^V \) over \( U \), which converge uniformly on compact subsets of \( U \). Then \( f_n \rightarrow f \) is in \( H^0(U, \mathcal{O}^V) \) [11].

We will now restrict our attention to a one dimensional connected general analytic space \( S \).

Theorem 23. The normal model \( R \) of \( S \) is a one-dimensional manifold.

Proof: By the uniqueness expressed in ii) above, we need only show we can make \( R \) a one-dimensional manifold so that \( \pi \) is holomorphic, since \( \mathcal{O}^M_x \) is always
integrally closed for $M$ a manifold. Now, we know that $R^{-1}(S_{\text{sing}})$ is a one-dimensional manifold biholomorphic to $S_{\text{reg}}$. $S_{\text{sing}}$ is a zero dimensional variety on $S$, i.e., a discrete set of points, thus $\pi^{-1}(S_{\text{sing}})$ is also discrete. Let $x$ be in $\pi^{-1}(S_{\text{sing}})$, $\pi(x) = s$. Then there is an irreducible branch $M$ of $R$ at $s$ in a neighborhood $U$ of $s$ containing no other singular point, and a neighborhood $U'$ of $x$ such that $(U', \pi)$ is the normal model for $M$. We now use the following lemma:

**Lemma 9.** Let $V$ be an irreducible one-dimensional variety in $\mathbb{C}^n$ at $0$, suppose $0$ is the only singular point of $V$. Then there is a Riemann surface $R$, a homeomorphism $\zeta: R \rightarrow V_0(\{z_1 < \varepsilon\})$, such that $\zeta|_{R - \varepsilon^{-1}(0)}$ is biholomorphic.

Proof: In a neighborhood of $0$, by the Nullstellensatz for a suitable choice of coordinates we can write $V = V(f_1, \ldots, f_{n-1})$ where

$$
\begin{align*}
f_1(z_1, \ldots, z_n) &= g_1(z_1, z_2) = z_2 + \sum \alpha_1(z_1)z_2 \quad \text{where the } g_j \\
f_j(z_1, \ldots, z_n) &= z_j - \frac{g_j(z_1, z_2)}{D(z_1)}, \text{ where the } g_j \text{ are distinguished polynomials, and } D(0) = 0.
\end{align*}
$$

Let $R$ be the Riemann surface covering $\{z_1 < \varepsilon\}$ of the algebraic function defined by $g_1(z_1, z_2) = 0$, and $\pi^o: R \rightarrow (\{z_1 < \varepsilon\})$ the projection. Let $x$ in $R$, $\pi^o x \neq 0$, Then $x$ correspond to a certain branch $\varphi_x$ of this algebraic function. We define $\zeta: R \rightarrow V_0(\{z_1 < \varepsilon\})$

$$
\begin{align*}
\zeta(x) &= (\pi^o x, \varphi_x(\pi^o x), \frac{g_2(\pi^o x, \varphi_x(\pi^o x))}{D(\pi^o x)}, \ldots, \frac{g_n(\pi^o x, \varphi_x(\pi^o x))}{D(\pi^o x)}) \\
\zeta(x) &= 0 \text{ if } \pi^o x = 0.
\end{align*}
$$
If $\pi^1$ is the projection of $V$ onto $(|z_1| < \varepsilon)$ it is obvious that $\pi^0 = \pi^1 \circ \xi$.

$\xi$ is one-one. For if $\varphi x = \varphi y$, then $\pi^0 x = \pi^0 y$,
$\varphi x (\pi^0 x) = \varphi y (\pi^0 y) = \varphi y (\pi^0 x)$ which implies that $\varphi x = \varphi y$, or $x = y$.

$\xi$ is biholomorphic on $R - \varepsilon^{-1}(0)$. Let $\pi^0 x \neq 0$, then for $U_x$ small enough neighborhood of $x$, $(U_x, \pi^0)$ is a local parameter at $x$. Similarly, since $\pi x$ is in $V_{reg}$, $(U_x, \pi^1)$ is a local parameter at $x$ for $U_x$ small enough. But since $\pi^0 = \pi^1 \circ \xi$, $\xi$ also is biholomorphic in a sufficiently small neighborhood of $x$.

Now, since $s$ is the only singular point in a neighborhood of $M$, and $M$ is irreducible as $s$, then $\pi: U' \leftrightarrow M$ is a homeomorphism. By definition, there is a neighborhood $U$ of $s$ in $M$ and a biholomorphic map $\omega: U \leftrightarrow V$, a variety through $0$ in some $\mathbb{C}^n$. By lemma 9, there is an $\varepsilon > 0$, a Riemann surface $R$ and a homeomorphism $\xi: R \leftrightarrow V \cap (|z_1| < \varepsilon)$ such that $\xi|_{R - \varepsilon^{-1}(0)}$ is biholomorphic. Then $\xi^{-1} \omega \circ \pi: (\omega \circ \pi)^{-1} \cap (|z_1| < \varepsilon) = W \leftrightarrow R$ is a homeomorphism. Let $(W, \xi^{-1} \omega \circ \pi)$ be a local parameter at $x$. This for all $x$ in $\pi^{-1}(S_{sing})$, and for $x$ not in $\pi^{-1}(S_{sing})$, let $W$ be a neighborhood of $x$ disjoint from $\pi^{-1}(S_{sing})$, and let $(W, \pi)$ be a local parameter at $x$.

Then this system of local coordinates defines $R$ as a complex manifold only if they match up right. We only need check the case of $(W; \pi)$ defined at a regular point and $W^\circ, \xi^{-1} \omega \circ \pi)$ at a singular point. Now $W^1 \cap W^2$ is disjoint from $\pi^{-1}(S_{sing})$, so $\pi$ is biholomorphic, $\omega$ is always biholomorphic, and since $\pi(W^1 \cap W^2) \cap S_{sing}$ is...
empty, 0 is not in \( \omega(\pi(W^n W^m)) \), so \( \epsilon^{-1} \) is biholomorphic on \( \omega(\pi(W^n W^m)) \). Thus \( \epsilon^{-1} \circ \omega \circ \eta^{-1} : \pi(W^n W^m) \rightarrow \epsilon^{-1} \circ \omega \circ \eta(W^n W^m) \) is biholomorphic. Thus \( R \) is defined as a complex manifold; let \( \mathcal{O}^m \) be the sheaf of germs of holomorphic functions determined by this structure.

Now, finally, \( \pi \) is holomorphic. Let \( f \) be in \( H^0(U, \mathcal{O}^S) \). We have to prove \( f \circ \pi \) is in \( H^0(\epsilon^{-1}(U), \mathcal{O}^m) \). Now since \( \pi(R) = \epsilon^{-1}(S_{\text{sing}}) \) is biholomorphic, \( f \circ \pi \) is holomorphic off \( \epsilon^{-1}(S_{\text{sing}}) \), a discrete set, and is continuous on \( U \). Thus, since \( (R, \mathcal{O}^m) \) is a manifold, \( f \circ \pi \) is in \( H^0(\epsilon^{-1}(U), \mathcal{O}^m) \).

If \( x \) is a singular point, \( \pi^{-1}(x) = (r_1, \ldots, r_n) \) and for \( U \) a neighborhood of \( x \), there are neighborhoods \( U_i \) of \( r_i \) such that \( \pi : \bigcup_{i=1}^n U_i \rightarrow U \) and \( \pi : \bigcup_{i=1}^n U_i - (r_1, \ldots, r_n) \rightarrow U - x \) one-one onto. We call the \( \pi(U_i) \) the sheets of \( S \) at \( x \). Note that \( \pi(U_i) \) is a g.a.s. and \( U_1 \) is its normal model. If \( x \) is in \( \pi(U_i)_{\text{reg}} \), then \( \pi : U_i \rightarrow \pi(U_i) \) is a biholomorphic map. If \( x \) is in \( \pi(U_i)_{\text{sing}} \), then \( \pi \) is just holomorphic. This type of point we call a branch point of \( \pi(U_i) \) or a branch point of \( S \) if \( \pi^{-1}(x) \) is just one point (\( x \) is an irreducible point). If \( \pi^{-1}(x) \) is more than one point we call \( x \) an identification point.

Let \( x \) be a branch point of \( S \), then \( \pi : R \rightarrow S \) is one-one holomorphic from a neighborhood of \( \pi^{-1}(x) = r \) to a neighborhood of \( x \) in \( S \). For the moment we replace these neighborhoods by \( R \) and \( S \) respectively and we assume \( S \) is a variety \( V \) in \( \mathbb{P}^n \). Then \( \hat{\pi}(\mathcal{O}_x^V) \) is a subring of \( \mathcal{O}_x^R \). In fact we have:
Lemma 10. $\hat{\mathcal{H}}(\partial_x^V)$ is precisely the set of germs of functions which can be uniformly approximated in a neighborhood of $r$ by polynomials in the $\hat{\mathcal{H}}(z_1), \ldots, \hat{\mathcal{H}}(z_n)$. Further, $\hat{\mathcal{H}}(\partial_x^V)$ contains a power of the maximal ideal of $\mathcal{O}_x^R$.

Proof: Let $f$ be in $\hat{\mathcal{H}}(\partial_x^V)$, then $f = g \cdot \pi$, $g$ in $\mathcal{O}_x^V$. Then for a sufficiently small polycylinder $P^n$ about $x$, there is an $h$ in $H(P^n)$ such that $g = h \restriction_V$. But then $h$ is approximable by polynomials in compact neighborhoods of $x$, so also is $g$; thus $g \cdot \pi = f$ is approximable by polynomials in the $z_j \cdot \pi = \hat{\mathcal{H}}(z_j)$ in a neighborhood of $r$. Conversely, if $f$ is in $\mathcal{O}_x^R$, in a neighborhood of $r$, $f \cdot \pi^{-1}$ is defined, since $\pi$ is one-one. If $f$ is approximable by polynomials in the $\hat{\mathcal{H}}(z_j)$, $f \cdot \pi^{-1}$ is approximable in a neighborhood of $x$ on $V$ by polynomials in the $z_j$. But by theorem 22, $\mathcal{O}_x^V$ is closed in this topology, so $f \cdot \pi^{-1}$ is in $\mathcal{O}_x^V$, implying $f = f \cdot \pi^{-1} \cdot \pi$ is in $\hat{\mathcal{H}}(\partial_x^V)$.

Now by iii) on page 76, there is an ideal $\mathfrak{c}$ of $\mathcal{O}_x^V$ such that $\mathfrak{c} \cdot \mathcal{O}_x^R \subset \mathcal{O}_x^V$. But $\mathfrak{c} \cdot \mathcal{O}_x^R$ is an ideal of $\mathcal{O}_x^R$, thus $\mathcal{O}_x^V$ contains an ideal $I$ of $\mathcal{O}_x^R$, and $I \neq \mathcal{O}$, since $\mathfrak{c} \neq \mathcal{O}$. But the only ideals of $\mathcal{O}_x^R$ are the powers of the maximal ideal. Let $g$ be in $I$, suppose $g$ has a zero of order $n$ at $r$; $g'$ has a zero of order $\geq n$. Then $g'/g$ is in $\mathcal{O}_x^R$, so $g' = g \frac{g'}{g}$ is also in $I$. Thus $I$ contains $M^n$ ($M = \text{maximal ideal}$). Let $n^o = \min\{n; I \supset M^n\}$. Then $I = M^{n^o}$, for if not then there is a $g$ in $I$ with a zero of order $m < n^o$, but this implies, as we have seen, $I \supset M^n$, contradicting the definition of $n^o$. 
Corollary. If $x$ is a branch point of $S$, $\pi: R \to S$, $\pi(r) = x$, then there exists an $N$ such that, if $f$ is analytic in a neighborhood of $r$ and vanishes at $r$ with order $\geq N$, then $f$ is in $\hat{\pi}(\mathfrak{c}_x^S)$.

Now we look at the local behavior at an identification point; so we may assume we have maps $\pi_i: \{ |z| < 1 \} \to V_i \subset \mathbb{R}^n$, $\pi_i(0) = 0$, and $V_1, \ldots, V_k$ are the irreducible components of a variety $V$ containing $0$. Let $T = \{ (\mathbf{f}_1, \ldots, \mathbf{f}_k); \mathbf{f}_j \in \partial^1 \}$ such that there exists $g$ in $\mathfrak{c}_o^0, g(V_j \circ \pi_j = f_j)$.

Lemma 11. There are integers $N_i \geq 0$ such that if $f_j$ is in $\partial^1_o$ and vanishes at $0$ with order $\geq N_j$, then $(f_1, \ldots, f_k)$ is in $T$.

Proof: Write $V_i = V(p_1^i, \ldots, p_{n-1}^i)$. Since $V_j \cap V_i = \{ 0 \}$, there is $p_j^i$ in $I(V_j)$ such that $p_j^i |_{V_i}$ has its only zero at $0$. Let $n_{ij}$ be the order of vanishing of $p_j^i |_{V_i} \circ \pi_i$ at $0$. According to the above corollary there is an integer $n_{i1}$ such that if $f$ is in $\partial^1_o$ and vanishes at $0$ with order $\geq n_{i1}$, then $f$ is in $\mathfrak{c}_i (\mathfrak{c}_o^n)$. Let $N_i = \sum_j n_{ij}$. These are the required integers.

Let $(f_1, \ldots, f_k)$ be as in the statement of the lemma. Then $f_i = f_1_{\sum j \neq i} p_j^i |_{V_i} \circ \pi_i^{-1}$ is in $\partial^1_o$ and vanishes with order $\geq n_{i1}$, so there is a $g_i$ in $\mathfrak{c}_i^n$ such that $g_i |_{V_i} = f_i$, or, what is the same $f_i = (g_i \bigcap j \neq i (p_j^i |_{V_i}) \circ \pi_i$.
But now, for $m \neq 1$, $g_m \bigcap j \neq m p_j^i |_{V_i}$ vanishes on $V_i$, so we have
\[ f_i = (\sum_{m=1}^k g_m \bigcap j \neq m p_j^i) \circ \pi_i. \]
Then $g = \sum_{m} g_{m} \prod_{j=1}^{i} p_{j}$ is in $\Theta_{0}^{n}$, $g \cdot \pi_{i} = f_{i}$, proving $(f_{1}, \ldots, f_{n})$ is in $T$.

**Lemma 12.** $T = \{ (f_{1}, \ldots, f_{k}) ;$ there is $t$ in $C$ and for all $j$, $f_{j} - t$ is in the maximal ideal of $\Theta_{0}^{j}$ if and only if there is a coordinate system $z_{1}, \ldots, z_{n}$ at $0$ such that $V = \bigcup_{i} V_{i}$; $V_{1}$ is the $z_{1}$-axis.

Proof: Suppose such a coordinate system existed.

Let $(f_{1}, \ldots, f_{k})$ be as described, with $t = 1$. Now since $V_{j}$ is the $z_{j}$-axis, $f_{j}(z_{j}) = g_{j}(0, \ldots, z_{j}, \ldots, 0)$ where $g_{j}$ is in $\Theta_{0}^{n}$ and is constant in the direction normal to $V_{j}$. Let $g = \prod_{j=1}^{k} g_{j}$. Then

$$g \big|_{V_{i}} = \prod_{j=1}^{k} g_{j}(0, \ldots, 0, z_{1}, 0, \ldots, 0) = g_{1}(0, \ldots, 0, z_{1}, 0, \ldots, 0) \prod_{j=1}^{k} g_{j}(0) = f_{1}(z_{1}),$$

thus $(f_{1}, \ldots, f_{k})$ is in $T$.

Now we assume $T$ is as described; if $f_{1}, \ldots, f_{k}$ are in $\Theta_{0}^{n}$, and $f_{1}(0) = f_{j}(0)$, then there is a $g$ in $\Theta_{0}^{n}$ such that $g \big|_{V_{j}} = f_{j}$.

First we note that each $V_{j}$ is regular at the origin. For if not, then $\hat{\pi}_{j}(z_{1}), \ldots, \hat{\pi}_{j}(z_{n})$ have derivative $0$ at the origin, since every function in $\Theta_{0}^{V_{j}}$ has $0$ derivative at the origin. But by assumption there is a $g_{j}$ in $\Theta_{0}^{n}$ such that $g_{j} \big|_{V_{j}} = z \cdot \pi_{j}$; $g_{j} \big|_{V_{i}} = 0$, $i \neq j$. Then $\hat{\pi}_{j}(g_{j})$ does not have zero derivative, so we must have that $V_{j}$ is regular at the origin.

Then $V_{j}$ has a well defined tangent $t_{j}$ at the origin such that for $h$ in $\Theta_{0}^{n}$, $dh(t_{j})|_{0} = d\hat{\pi}_{j}(h)(0)$, thus $d g_{j}(t_{j})|_{0} = 1$, $d g_{j}(t_{1})|_{0} = 0$. Thus the differentials
$d_{g_1}, \ldots, d_{g_k}$ are linearly independent. In particular $k \leq n$ necessarily.

Let $I$ be the ideal of germs of functions vanishing on $V$. Then $d_{g_1}, \ldots, d_{g_k}$, together with $[df; f \in I]$ span the cotangent space at $0$. For if not, then there is a tangent vector $t$ which is annihilated by all these. Let $h$ be in $\mathcal{C}^n_0$ such that $dh(t) = 1$, $h(0) = 0$. We assume fixed a neighborhood $U$ of $0$ in $\mathcal{C}^n$ in which $g_1, \ldots, g_k$, $h$ are holomorphic, and $g_j(U) \subset g_j(U \cap V_j)$. Since $g_j|_{V_j}$ is an open map, such a $U$ exists. $\hat{n}_j(h)$ is a holomorphic function on $g_j(U)$, so we can write $\hat{n}_j(h) = f_j(z)$ on $g_j(U)$. But then $f_j \circ g_j$ is holomorphic on $U$, and $h|_{V_j} = f_j \circ g_j|_{V_j}$. Further $0 = h(0) = f_j \circ g_j(0) = f_j(0)$. Then $h = \Sigma f_j \circ g_j$ is in $I$. For let $x$ be in $V \cap U$, say $x$ is in $V_i$. Then $h(x) = \Sigma f_j \circ g_j(x) = h|_{V_i}(x) - \Sigma f_j \circ g_j|_{V_i}(x)$

$$= f_i \circ g_i(x) - \Sigma f_j \circ g_j|_{V_i}(x) = \Sigma f_j(0) = 0$$

since $g_j|_{V_i} = 0$. Then $h = \Sigma f_j \circ g_j + f$, and $f$ is in $I$, so $dh = \Sigma f_j dg_j + df$, so $l = dh(t) = 0$, a contradiction.

Thus, since $d_{g_1}, \ldots, d_{g_k}$, $[df; f \in I]$ span, and $d_{g_1}, \ldots, d_{g_k}$ are linearly independent, there exist $g_{k+1}, \ldots, g_n$ in $I$ such that $d_{g_1}, \ldots, d_{g_n}$ span the cotangent space. Then $g_1, \ldots, g_n$ can be taken as a local coordinate system. Further, $[g_1 = 0, i \neq j] \subset V_j$ and is one-dimensional submanifold of a neighborhood of $0$, so must coincide with $V_j$. Thus $V_j$ is an axis of this coordinate system.
We now consider $K$ a compact set on a connected one-dimensional general analytic space $S$, with $(R, \pi)$ as normal model. Let $\tilde{K} = \pi^{-1}(K \cap S_{\text{reg}})$; $\tilde{K}$ is a compact set on $R$. Because $\pi|_{R \cap \pi^{-1}(S_{\text{sing}})}$ is a biholomorphism and $S_{\text{sing}}$ is discrete, $K \cap S_{\text{reg}}$ is a finite point set, which we shall refer to as $K_{\text{sing}}$, and $\tilde{K} \cap \pi^{-1}(S_{\text{sing}})$ is biholomorphic to $K \cap S_{\text{reg}}$, and since $\pi^{-1}(\text{point})$ is a finite set, $\tilde{K}$ differs from $\pi^{-1}(K \cap S_{\text{reg}})$ by a finite point set. Thus if $K = \text{int} \tilde{K}$, so also $\tilde{K} = \text{int} \tilde{K}$; if $\text{int} K$ is connected, $\tilde{K}$ is connected, if $K$ is bounded by finitely many piecewise analytic curves, so also is $\tilde{K}$. As usual, let $A(\tilde{K})$ be the uniform closure on $\tilde{K}$ of the algebra of functions analytic in a neighborhood of $\tilde{K}$, and $A(K)$ the uniform closure on $K$ of the algebra of cross sections of $\sigma^S$ in a neighborhood of $K$. Now the components of $R$ are Riemann surfaces, and given any $s$ in $S$, there is a neighborhood $U$ of $s$ such that $\pi^{-1}(U)$ intersects only finitely many of these components. Thus since $K$ is compact, $\pi^{-1}(K)$ intersects only finitely many of these components. $\pi^{-1}(K) \supset \tilde{K}$, so we can write $\tilde{K} = \tilde{K}_1 \cup \cdots \cup \tilde{K}_n$ where $\tilde{K}_i$ is a compact subset of a Riemann surface $R_i$, a component of $R$. Now if we take $U$ to be a small enough neighborhood of $K$, so that $\pi^{-1}(U) R_i$ is compact on $R_i$, then $\pi^{-1}(U) R_i$ can be considered as an open subset of a compact Riemann surface, $R'_i$. We now replace $S$ by $U$, and $R$ by $\pi^{-1}(U) R_1'$. We have the following theorem:
Theorem 24. Let $K$ be a compact subset of a connected one-dimensional g.a.s., $S$. Then there are finitely many compact Riemann surfaces $R_1, \ldots, R_n$, $\tilde{K}_i \subset U_i \subset R_i$, $\tilde{K}_i$ compact, $U_i$ open, and a map $\pi_i: U_i \to U$, a neighborhood of $K$ such that

1) letting $R = \bigcup U_i$, $\pi: R \to U, \pi|_{U_i} = \pi_i$, then $(R, \pi)$ is the normal model for $U$.

2) $\bigcup \tilde{K}_i = \pi^{-1}(K \cap S_{\text{reg}})$, $K = \bigcup \tilde{K}_i$

3) let $K \cap \text{Sing} = \{s_1, \ldots, s_m\}$. Let $\pi^{-1}(s_i) = \tilde{K}_i = \{x_{i1}, \ldots, x_{ik_i}\}$. There exist integers $n_{ij}$ such that $A(K) \ni \{f \circ \pi^{-1} ; f \text{ analytic in a neighborhood of } \tilde{K}, \ f(x_{ip}) = f(x_{iq}) \text{ for all } p, q, 1 \leq i \leq n; f(x) - f(x_{ij}) \text{ has a zero of at least order } n_{ij} \text{ at } x_{ij}\}$.

4) In particular $A(K) \ni \{f \circ \pi^{-1} ; f \text{ in } I\}$, where $I$ is an ideal of $A(\tilde{K})$ with finite hull; in fact hull $I = \pi^{-1}(K \cap \text{Sing})$.

Proof: 1) and 2) are already proven above. As for 3) the integers $n_{ij}$ are just those given by lemma 11 applied at the point $s_i$. That is, if $\pi^{-1}(s_i) = \{x_{i1}, \ldots, x_{ik_i}\}$, then for $S_j$ a branch of $S$ at $s_i$, let $n_{ij}$ be the integer given by lemma 11, so that if $f_j$ is in $e^{R}_{x_{ij}}$ and vanishes at $x_{ij}$ with order $\geq n_{ij}$, then there is a $g$ in $e^{n}_{s_i}$ (where we now consider a neighborhood of $s_i$ as embedded in $C^n$) such that $g \circ \pi = f_j$. Now let $f$ be in $H^{0}(V, e^{R}_{V})$, $V$ a neighborhood of $K$, suppose $f$ is as described in 3). Since $f$ identifies the points which $\pi$ identifies, $f \circ \pi^{-1}$ is a contin-
uous function on $K$. Since $S_{\text{sing}}(K)$ is finite, and $S_{\text{sing}}$ is discrete, there is a neighborhood $W$ of $K$ such that $W \cap S_{\text{sing}} = K \cap S_{\text{sing}}$; we can take $V = \pi^{-1}(W)$. Then $g = f \circ \pi^{-1}$ is a continuous function on $W$, and since $\pi|_{\text{rel}}(S_{\text{sing}})$ is biholomorphic, for all $s$ in $S_{\text{reg}}$ we have $g_s$ in $\mathcal{O}_S$. If $s = s_i$, then by lemma 11, since $f$ is as described in iii), there is a $G$ in $\mathcal{O}_s^C$ such that $G|_{S_i} = g - g(s_i)|_{S_i}$, so $g = g + g(s_i)|_{S}; G + g(s_i)$ is in $\mathcal{O}_s^n$, so $g$ is in $\mathcal{O}_s^n$. Thus $g$ is in $\mathcal{H}^W(W, \mathcal{O}_s)$, so that $g|_{S} = f \circ \pi^{-1}|_{S}$ is in $A(K)$. Thus iii) is proven.

Letting $H(K) = \{f \text{ holomorphic in a neighborhood of } K\}$, then by iii), $A(K) \supset [f \circ \pi^{-1}; f \text{ in } I]$ where $I = \{f \text{ in } H(K); f(x_{ij}) = 0, f \text{ has a zero of order } n_{ij}, \text{at } x_{ij}\}$ is an ideal of $H(K)$. Now $H(K)$ is dense in $A(K)$. Thus $\overline{I}$ (closure of $I$ in $A(K)$) is also an ideal of $A(K)$, and since $A(K)$ is closed, $A(K) \supset [f \circ \pi^{-1}; f \text{ in } I]$. Now $\text{hull} \overline{I} = \{x_{ij}\}$. Surely each $x_{ij}$ is in $\text{hull} \overline{I}$. On the other hand if $x$ is in $K$, $x \neq x_{ij}$, as in the proof of lemma 7, there is an $f_{ij}$ in $H(K)$ such that $f_{ij}(x) \neq 0, f_{ij}(x_{ij}) = 0$. Then $f = \prod f_{ij}^{\sup n_{ij}}$ is in $I$ and $f(x) = 0$, so $x$ is not in $\text{hull} \overline{I}$.

**Theorem 25.** $S(A(K)) = K$.

**Proof:** Since $A(K)$ is an algebra of continuous functions on $K$, evaluation at any point of $K$ is a continuous homomorphism of $A(K)$. We prove $A(K)$ separates point of $K$. Let $s, t$ in $K$, $s$ in $S_{\text{reg}}$. Let $x = \pi^{-1}(s)$. There is a function $f_{tk}$ in $H(K)$ such that $f_{tk}(x) \neq 0,$
\(f_{tk}(y_k) = 0, \) for all \(y_k\) in \(\pi^{-1}(t)\). There is an \(f_{ij}\) in \(H(\tilde{K})\) such that \(f_{ij}(x) \neq 0, f_{ij}(a_{ij}) = 0\). Then

\[ f = \left[ \prod f_{tk}^{-1} \right]_{i}^{j} \text{ is in } I \] \(f(x) \neq 0, f(\pi^{-1}t) = 0.\)

Thus \(g = f \cdot \pi^{-1}\) is in \(A(K)\) and \(g(s) \neq g(t).\)

Suppose now \(s = s_1, t = s_2.\) Let \(K^o\) be a compact set such that \(\tilde{K} \subset \text{int} K^o\) and \(\pi^{-1}(s_{\text{sing}}) \cap K^o = \pi^{-1}(s_{\text{sing}}) \cap \tilde{K}.\)

By ii) of theorem 20, there is an \(f\) in \(A(K^o), f(x_{1j}) = 1,\) \(f(x_{1j}) = 0 \) if \(i \neq 1,\) and \(f(x) - f(x_{1j})\) has a zero of order at least \(n_{1j}\) at \(x_{1j}.\) Then \(f\) is in \(H(\tilde{K})\) and by iii) of theorem 24, \(g = f \cdot \pi^{-1}\) is in \(A(K).\) But \(g(s) = 1, g(t) = 0.\) Thus \(A(K)\) separates points.

Let \(h\) in \(S(A(K)),\) suppose \(h\) is not evaluation at any \(s_1, \ldots, s_n.\) Then there is an \(f^o\) in \(A(K)\) such that \(f^o(s_j) = 0, 1 \leq j \leq n,\) and \(h(f^o) = 1.\) For \(f\) in \(\bar{I},\) let

\[ \tilde{h}(f) = h(f \cdot \pi^{-1}). \] \(\tilde{h}\) is a nonzero continuous homomorphism on \(\bar{I},\) since \([f^o \cdot \pi]^N,\) where \(N = \sup n_{ij},\) is in \(\bar{I},\) and

\[ \tilde{h}((f^o \cdot \pi)^N) = h((f^o)^N) = 1. \] But now, by lemma 7,

\[ S(A(\tilde{K})) = \tilde{K}\) and \(\text{hull} \bar{I} = [x_{1j}],\) so \(S(\bar{I}) = \bar{K} - [x_{1j}].\)

Then \(\tilde{h}(f) = f(x)\) for some \(x\) in \(K, x \neq x_{1j} .\) Thus \(\pi x\)

is a regular point of \(S.\) Let \(g\) be in \(A(K),\) we prove

\[ h(g) = g(\pi x). \] Then, since \(f^o \cdot \pi\) is in \(I,\)

\[ g(f^o \cdot \pi) = (g \cdot \pi)(f^o \cdot \pi) \) is in \(I,\) so that

\[ h(g) = h(g)h(f^o) = h(gf^o) = \tilde{h}(gf^o) = (gf^o)(x) = \]

\[ g(\pi x)(f^o \cdot \pi)(x) = g(\pi x) \tilde{h}(f^o \cdot \pi) = g(\pi x).\]

Then \(S(A(K)) = K.\)

**Theorem 26.** Suppose \(K\) is a Runge set (i.e., there is a neighborhood \(U\) of \(K\) such that \(K = R^o(U, \mathcal{E}^S_U)\)-con-
vex), then $\tilde{K}$ is a Runge set. The latter condition is enough to guarantee that $K$ is a Runge set and that $A(K) = C(K) \cap H^0(\text{int} \tilde{K}, \mathcal{O}_U^S)$. 

Proof: If $K$ is a Runge set, there is a $U \subset S$ such that $U$ contains $K$ and $K$ is $H^0(U, \mathcal{O}_U^S)$-convex. Then $\tilde{K}$ is a Runge set (or at least $\tilde{K}_1 \subset R_i$ is a Runge set on each Riemann surface $R_i$). If $K \subset V \subset U$, then obviously $K$ is also $H^0(V, \mathcal{O}_V^S)$-convex, so we may assume $U$ is so small that $\pi^{-1}(U) \cap (\mathcal{O}_U^S) = \pi^{-1}(\mathcal{O}_V^S)$. Then if $x$ is in $\tilde{U}_1 = \pi^{-1}(U) \cap \tilde{R}_1$, but not in $\tilde{K}_1$, then $\pi x$ is not in $K$, so there is an $f$ in $H^0(U, \mathcal{O}_U^S)$ such that $|f(\pi x)| > \|(f|_K)\|_K$. Then $f \cdot \pi$ is holomorphic on $\tilde{U}_1$, and $|(f \cdot \pi)(x)| > \|f\|_{\tilde{K}_1}$. Thus $\tilde{K}_1$ is $H^0(\tilde{U}_1, \mathcal{O}_U^S)$-convex. Then by the corollary to theorem 19, $A(K_1) = C(K_1) \cap H(\text{int} \tilde{K}_1)$, so $A(K) = C(K) \cap H(\text{int} \tilde{K})$, (since $A(K) = \Sigma A(K_1)$, for $\tilde{K}$ is the disjoint union of the $\tilde{K}_1$.

Now, let $K \subset \text{sing} = \{s_1, \ldots, s_n\}$, let $s_1, \ldots, s_n$ be in $\tilde{K}$, $s_{e+1}, \ldots, s_n$ in $\text{int} \tilde{K}$. Let $\pi^{-1}(s_i) = \{x_{ij}, 1 \leq j \leq k_i\}$, and $n_{ij}$ the integer corresponding to $s_i$ as in the above. Let $I = \{f \in H(\tilde{U}) \backslash f \text{ has a zero of at least order } n_{ij} \text{ at } x_{ij}\}$. Now $\tilde{K}$ is $I$-convex. For $I$ is an ideal in $H(\tilde{U})$ and since $\tilde{K}$ is $H(\tilde{U})$-convex, the closure $A(\tilde{K}, \tilde{U})$ of $H(\tilde{U})$ on $\tilde{K}$ is $A(\tilde{K})$, and $S(A(\tilde{K})) = \tilde{K}$. $I$ is an ideal of $A(\tilde{K})$, so $S(I)$ is a subspace of $\tilde{K}$. But if $\tilde{K}$ is not $I$-convex, there is an $x$ in $\tilde{U} \cap \tilde{K}$ such that $|f(x)| < \|f\|_K$ for all $f$ in $I$, so evaluation at $x$ extends to a homomorphism of $I$, not evaluation at any
point of \( \tilde{K} \); since \( I \) separates points of \( U \) (but for \( \text{hull } C(\tilde{K}) \)), contradicting \( S(\tilde{I}) \subset \tilde{K} \). Since \( I \subset \{ f \circ \pi; f \text{ in } \mathcal{H}^0(\pi(U), \mathfrak{S}) \} \), then \( K \) is \( \mathcal{H}^0(\pi(U), \mathfrak{S}) \)-convex.

Now, by Bishop's theorem (theorem 19), \( \tilde{I} \) contains \( \{ f \text{ in } C(K) \cap \mathcal{H}(\text{int} K) \text{ such that } f \text{ vanishes on } \pi^{-1}(S_{\text{sing}}) \} \) and has a zero of order \( \geq N \) at \( x_{ij}, i > e \}, \) for some integer \( N \). Let \( \mathcal{J} \) be the ideal of \( A(\tilde{K}), \mathcal{J} = \{ f \text{ in } A(\tilde{K}); f(x) - f(x_{ij}) \text{ has a zero of order } N \text{ for } i > e \} \).

Now let \( f \) be in \( C(K) \cap \mathcal{H}(\text{int} K, \mathfrak{S} \text{ int} K) \) such that \( f \) vanishes on \( S_{\text{sing}}, \) then \( f \circ \pi \) is in \( C(\tilde{K}) \cap \mathcal{H}(\text{int} \tilde{K}) = A(\tilde{K}), \) since \( \tilde{K} \) is a Runge set. By theorem 20, considering only the points \( x_{ij}, i > e \), we can write \( f \circ \pi = g + h, \) \( g \) is meromorphic on \( \mathbb{R} \), holomorphic on \( \tilde{K} \), and \( h \) is in \( \mathcal{J} \). We can arrange in fact, that \( h \) is in \( \tilde{I} \) by adding and subtracting a meromorphic function in \( \mathcal{J} \) which takes the values \( g(x_{ij}) \) at \( x_{ij}, i \leq e \); then \( h \) vanishes on \( \pi^{-1}(S_{\text{sing}}) \). Since \( g \) is holomorphic in a neighborhood of \( \tilde{K}, \) we can write \( g = g_1 + g_2, \) where \( g_1, g_2 \) are meromorphic functions, holomorphic on \( \tilde{K}, \) such that \( g_1 \) vanishes with order \( n_{ij} \) at \( x_{ij}, i \leq e, \) and \( g_2 \) is in \( \mathcal{J}, \) i.e., let \( g_2 \) be in \( \mathcal{J} \) and \( g_2 \) have the same power series up to the \( n_{ij} \)th term as \( -g \) as \( x_{ij} \) (in some local coordinate system) for all \( i \leq e. \) This is just another linear space problem as in theorem 20. Also \( g_1, g_2 \) vanish on \( \pi^{-1}(S_{\text{sing}}) \). Thus we have \( f \circ \pi = g_1 + (g_2 + h); g_1 \) vanishes of order \( n_{ij} \) at \( x_{ij}, i \leq e, \) and \( g_2 + h \) is in \( \mathcal{J} \) and vanishes on \( \pi^{-1}(S_{\text{sing}}) \). Thus \( g_2 + h \) is in \( \tilde{I} \) and
\[ F^{-1}; \text{F in } \mathcal{F} \] CA(K), so \((g_a + h) \pi^{-1}\) is in A(K), so also is in \(C(K) \cap H^o(\text{int}K, \mathcal{O}^S_{\text{int}K})\). Thus \(g_1 \pi^{-1} = f - (g_a + h) \pi^{-1}\) is also in \(C(K) \cap H^o(\text{int}K, \mathcal{O}^S_{\text{int}K})\). But \(g_1\) is meromorphic on \(\mathbb{R}\), and holomorphic on \(\bar{K}\), so is holomorphic in a neighborhood \(W\) of \(\bar{K}\). Thus \(g_1 \pi^{-1}\) is in \(H^o(\pi(W), \mathcal{O}^S_{\pi(W)})\). But for \(x_{ij}, i \leq e, g_1\) vanishes of order \(n_{ij}\), so by lemma 11, \([g_1 \pi^{-1}]_{s_i} \) is in \(\mathcal{O}^S_{s_i}\). This, for all \(i \leq e\), but for \(i > e, g_1 \pi^{-1}\) is in \(\mathcal{O}^S_{s_i}\) since \(s_i\) is in \(\text{int}K\). Thus \(g_1 \pi^{-1}\) is in \(H^o(\pi(W), \mathcal{O}^S_{\pi(W)})\) \(\pi(W)\) is a neighborhood of \(K\), so \(g_1 \pi^{-1}\) is in \(A(K)\). \((g_a + h) \pi^{-1}\) is in \(A(K)\) also, thus \(f = g_1 \pi^{-1} + (g_a + h) \pi^{-1}\) is in \(A(K)\).

Since \(\mathcal{K} \mathcal{O}^S_{\text{sing}} = [s_1, \ldots, s_e]\) is finite, given any complex numbers \(t_1, \ldots, t_e\), there is a \(g\) in \(A(K)\) such that \(g(s_i) = t_i\). Thus if \(f\) is in \(C(K) \cap H^o(\text{int}K, \mathcal{O}^S)\), there is a \(g\) in \(A(K)\) such that \(g(s_i) = f(s_i)\). Thus \(f-g\) is in \(C(K) \cap H^o(\text{int}K, \mathcal{O}^S)\) and vanishes on \(\mathcal{K} \mathcal{O}^S_{\text{sing}}\), so by the above is in \(A(K)\). Thus \(f = (f-g) + g\) is also in \(A(K)\). Theorem 26 is proven.

4. The Maximaliy Theorem

Theorem 27. Let \(S\) be a connected one-dimensional g.a.s., let \(K\) be compact, \(K = \Omega \cup \gamma_1 \cup \ldots \cup \gamma_n\), \(\Omega\) a domain, \(\gamma_1, \ldots, \gamma_n\) piecewise analytic curves.

A) \(A(K)\) is a maximal subalgebra of \(C(\mathcal{K})\) if and only if \(\text{int}K = \Omega\) is a manifold.

B) \(A(K)\) is \((K, \mathcal{K})\)-maximal if and only if
1) intK has no branch points

ii) for z in intK an identification point, there is a coordinate system \((U_z, \omega_z)\) such that \(\omega_z: U_z \to V\), a variety in int\(P^n\) = open polycylinder, and \(V = \bigcup_{j=1}^{n} (z_j\text{-axis}) = \bigcup_{j=1}^{n} V(z_j = 0, 1 \neq j)\).

C) \(A(K)\) is contained in precisely one maximal algebra \(M\), and if \(B\) is a subalgebra of \(C(\partial K)\), \(B \supset A(K)\), then \(B\) is dense in \(C(\partial K)\) or \(B \subset M\).

Proof: Let \(R\) be the normal model for a neighborhood \(U\) of \(K\), \(\pi: R \to U\) the projection, so that \(R = \bigcup_{n} U_n\), where \(U_n\) is a domain on a compact Riemann surface \(R_n\).

Let \(\tilde{K}\) be as previously defined. Then \(\tilde{K}\) is compact, and \(\tilde{K} = \tilde{\kappa} \cup \tilde{\gamma}_1 \cup \ldots \cup \tilde{\gamma}_n\) where \(\tilde{\kappa}\) is open and \(\tilde{\gamma}_i\) are piecewise analytic curves, \(\tilde{\kappa} = \tilde{\gamma}_1 \cup \ldots \cup \tilde{\gamma}_n\).

1) We first prove the necessity of A). Suppose \(A(K)\) is a maximal subalgebra of \(C(\partial K)\). Let \([s_1, \ldots, s_n]\) = \(K_0 S_{\text{sing}}\), and \(\pi^{-1}(s_1) = [x_{11}].\) Let \(B = \{ f|_{\tilde{\kappa} \cup \tilde{\gamma}_1 \cup \ldots \cup \tilde{\gamma}_n} \); \(f\) in \(A(\tilde{K})\), \(f(x_{1j}) = f(x_{1k})\) for \(s_1\) in \(\partial K\). Then \(B\) is a closed subalgebra of \(C(\partial K)\), and since \(\pi\) is holomorphic, \(B \supset A(K)\). Further, since every \(f\) in \(B\) has a holomorphic extension into \(\tilde{\kappa}\), \(B \neq C(\partial K)\). Thus, since by assumption \(A(K)\) is maximal, \(B = A(K)\). Now, for \(x \neq y\) in \(intK\), there is a meromorphic function \(f\) on \(R\), holomorphic on \(\tilde{K}\), such that \(f(x_{1j}) = 0\) for \(s_1\) in \(\partial K\) and \(f(x) \neq f(y)\) (see theorem 20). Thus, since \(f \circ \pi^{-1}\) is in \(B = A(K)\), there is a \(g\) in \(A(K)\) such that \(g \circ \pi = f\). Thus \(g(\pi x) \neq g(\pi y)\), so \(\pi x \neq \pi y\). Thus \(\pi: \tilde{\kappa} \to \Omega\) is...
one-one, thus is a homeomorphism, so has no identification points. Now let be in intK, \( \pi' (s) = x \) and \( g \) in \( A(\tilde{K}) \) such that \( g(x_{ij}) = 0, s_1 \) in \( \partial K, g(x) = 0 \) and \( dg(x) \neq 0 \). Then \( g \circ \pi^{-1} \) is in \( B = A(K) \), thus there is an \( f \) in \( A(K) \) such that \( f \circ \pi = g \). Now for some neighborhood \( \tilde{U} \) of \( x, g: \tilde{U} \rightarrow D \), a disc with center the origin, so \( F: \pi(\tilde{U}) = U \rightarrow D \) is a holomorphic homeomorphism.

But \( f \) is biholomorphic in \( U \). For let \( u \) be in \( \partial^S_t, t \in U \), then \( \hat{\pi}(u) \) is in \( \partial^R_{\pi^{-1}(t)} \), and thus since \( g \) is biholomorphic, \( \hat{\pi}^{-1}(\hat{\pi}(u)) \) is in \( \partial^S_{\hat{\pi}^{-1}(t)} \), i.e., \( f(u) \) is in \( \partial^S_{f(t)} \). Then \((U,f)\) is a local parameter at \( s \) mapping \( U \) onto a manifold; this for all \( s \) in \( r \), so \( r \) must be a manifold.

2) We now prove the sufficiency of \( A \), i.e., we assume \( r \) is a manifold, Then \( \cap S_{sing} = \emptyset \). Let \([s_1, \ldots, s_n] = K \cap S_{sing} = \lambda K \cap S_{sing} \), and \( \pi^{-1}(s_i) = [x_{ij}] \).

Since \( K \) is a Runge set, by theorem 26, \( A(K) = C(K) \cap H^0(\text{int} K, \partial^S) \), and also \( \tilde{K} \) is a Runge set, so \( A(\tilde{K}) = C(\tilde{K}) \cap H^0(\text{int} K, \partial^R) \), which is, by theorem 17, a maximal subalgebra of \( C(\partial K) \). Then by corollary 2 of theorem 1, \([f \in A(\tilde{K}); f(x_{ij}) = f(x_{ik}), 1 \leq i \leq n] \) is a maximal subalgebra of \( C(Y) \), where \( Y = \partial \tilde{K} \) with these points \([x_{i1}, \ldots, x_{in}] \) identified for \( 1 \leq i \leq n \). But then \( Y \) is homeomorphic to \( \partial K \) via \( \pi^{-1}: Y \rightarrow K \), so \([f \circ \pi^{-1}; f \in A(\tilde{K}), f(x_{ij}) = f(x_{ik}), 1 \leq i \leq n] \) is a maximal subalgebra of \( C(\partial K) \). But this is just \( A(K) \). For surely it contains \( A(K) \). Conversely, if \( f \) is in \( A(\tilde{K}) \), and \( f(x_{ij}) = f(x_{ik}), 1 \leq i \leq n \), then \( f \circ \pi^{-1} \) is in \( C(K) \).
Further, \( \pi: \tilde{\Omega} \leftrightarrow \Omega \) is biholomorphic, (since \( \Omega \subset S_{\text{reg}} \)) so \( f \circ \pi^{-1} \mid_{\tilde{x}_2} \) is in \( H(\Omega) \). But \( A(K) = C(K) \cap H(\Omega) \), then \( f \circ \pi^{-1} \mid_{\tilde{x}_2} \) is in \( A(K) \). Thus \( A(K) \) is a maximal subalgebra of \( C(\Omega) \).

3) We now prove part B) of theorem 27. Let \( s_1, \ldots, s_n \) be the singular points of \( S \) on \( K \) and \( \pi^{-1}(s_i) = \{ x_{ij} \} \). Suppose \( \{ s_1, \ldots, s_e \} \subset \text{int} K, \{ s_{e+1}, \ldots, s_n \} \not\subset K \).

Then \( A^1 = \{ f \in A(\tilde{\Omega}); f(x_{ij}) = f(x_{ik}), x_{ij}, x_{ik} \in \tilde{\Omega}, i = e+1, \ldots, n \} \) is a maximal subalgebra of \( C(\tilde{\Omega}) \) since \( A(\tilde{\Omega}) \) is maximal and \( A^1 \) just identifies the points of \( \pi^{-1}(s_i), i > e+1 \) (theorem 1, corollary 2). Let \( A^2 = \{ f \in A(\tilde{\Omega}); f(x_{ij}) = f(x_{ik}) \text{ for all } i, j, k \} \). Then \( A^2 \) is the subalgebra of \( A^1 \) of all \( f \) in \( A^1 \) which are constant on finitely many finite point sets. Since any finite point set is a hull we can apply corollary 1 of theorem 1 to obtain

Lemma 13. \( A^2 \) is maximal among all subalgebras of \( C(\Omega) \) having \( K \) as space of maximal ideals.

Now let us suppose \( A(K) \) satisfies the conditions of the theorem. We want to prove \( A^0 = \{ f \circ \pi; f \in A(K) \} = A^2 = \{ f \in A(\tilde{\Omega}); f(x) = f(y) \text{ whenever } \pi x = \pi y \} \). Then we will be through for \( \pi \) is a homeomorphism of \( K \) with \( \Gamma(A^2) \) and \( K \) with \( S(A^2) \), so whatever holds for \( A^2 \) will hold for \( A^0 \). Surely \( A^0 \subset A^2 \). Let \( f \) be in \( A^2 \). Then \( f = g \circ \pi \), \( g \) in \( C(K) \). Let \( s \) be in \( \text{int} K \), we prove \( \{ g \}_s \) is in \( \mathfrak{S}_s \). Letting \( \pi^{-1}(s) = \{ x_1, \ldots, x_t \} \), we have, for neighborhoods \( U_1 \) of \( x_1 \), \( \pi(\bigcup_{i=1}^{t} U_1) = U \), a neighborhood of \( s \). Letting \( f_i = f \mid_{U_1} \), we see that \( (f_1, \ldots, f_t) \) is in \( T \) (as defined in lemma 11), according to the condi-
tion (ii), so there is a $g'$ in $\mathcal{O}^S$ such that $f = g' \circ \pi$ in a neighborhood of $x_i$ for all $i$. But then $g = g'$ in a neighborhood of $s$, so $[g]_s = [g']_s$ is in $\mathcal{O}^S_s$.

Thus $g$ is in $H^0(\text{int} K, \mathcal{O}^S_{\text{int} K})$, so $A^0 \subset A^g$, so $A^0 = A^g$.

4) Now suppose $A(K)$ is $(K, \mathcal{O}^g)$-maximal. Then by lemma 13 we must have $A^g = A^s$.

Let $s$ be in $\text{int} K$, $\pi^{-1}(s) = [x_1, \ldots, x_t]$. Let $U_1$ be a neighborhood of $x_1$. $U_1 \cap U_j = \emptyset, i \neq j$, and $U = \pi(\bigcup_{i} U_i)$.

We may assume $U \subset C^n$ and $s \circ U = V$, a one-dimensional variety in the polycylinder $P^n$ in $C^n$, with 0 as its only singular point (now $s = 0$), since some neighborhood of $s$ is biholomorphic to a variety. $V$ has the irreducible components $V_1 = \pi(U_1)$. We will find functions $g_i$ in $\mathcal{O}_0^n$ such that $g_i|_{V_j} = 0, i \neq j$ and $g_i|_{V_i} \circ \pi^{-1}$ is a local parameter in a neighborhood of $x_i$. Then, as in the proof of lemma 12, we will obtain the desired result.

First of all there is an $f$ in $A^n (f(x_{1j}) = 0$ for all $i, j$) such that $df(x_1) \neq 0$ (by theorem 20). Then $f \circ \pi^{-1}$ is in $A(K)$. Thus as in 1) of this theorem, this makes $s$ a regular point of $\pi(U_1)$. Thus $V_1$ is regular at 0, so has a well-defined tangent vector $t_1$. Further, since there exists an $f$ in $A^n$ such that $df(x_1) \neq 0$, $df(x_j) = 0$ for $j \neq 1$, then, for $f = \pi(g)$, $dg(t_1) \neq 0$, $dg(t_j) = 0, j \neq 1$, so the varieties $V_1, V_j$ are nontangent and, further, the vectors $t_1, \ldots, t_k$ are linearly independent. Thus $k \leq n$.

We have to show, for all $i$ that there is a $g_i$ in $I(\bigcup_{j \neq i} V_j)$ such that $dg_i(t_1) \neq 0$. We prove this by
Lemma 14. Given regular one-dimensional varieties $V_1, \ldots, V_k$ at $0$ in $\mathbb{C}^n$ with tangents $t_1, \ldots, t_k$ which are linearly independent, then for any $m \leq k$, there exists $g_i, i = 1, \ldots, k$, $g_i$ in $\Theta^n_0$ such that

1) $dg_i(t_1) \neq 0$

2) $g_i|_{V_j} = 0$ for $j = 1, \ldots, m, j \neq i$.

Proof: (by induction on $m$).

$m=1$. Then we take $i \neq 1$, $V_1$ is a regular variety, so there is a coordinate system $(z_1, \ldots, z_n)$ such that $V_1 = V(z_2, \ldots, z_n)$. Since $dz_1, \ldots, dz_n$ span the cotangent space, $(dz_2, \ldots, dz_n)$ can annihilate only one tangent vector in a basis. They all annihilate $t_1$, so, for any $i$ there is a $z_r$ such that $dz_r(t_1) \neq 0$. Take $g_i = z_r$.

Now suppose the lemma is true for all $i$ and $m-1$.

Given $i$, we prove there is a $g_i$, $dg_i(t_1) \neq 0$, $g_i|_{V_j} = 0$ for $j=1, \ldots, m$. We may suppose $i > m$, for the induction hypothesis applies if $i \leq m$. By assumption there is $f_m$ such that $df_m(t_m) \neq 0$, $f_m|_{V_j} = 0$, $j=1, \ldots, m-1$. We must have also $df_m(t_1) = 0$ (since $df_n$ annihilates all but one vector in a basis). We also have $f_1$ such that $df_1(t_1) \neq 0$. $f_1|_{V_j} = 0$, $j=1, \ldots, m-1$.

Now $f_1|_{V_m}$ is an analytic function of $f_m|_{V_m}$ since $f_m|_{V_m}$ is a local parameter. Thus in a neighborhood $U$ of $0$ we can write $f_1|_{V_m} = h(f_m|_{V_m})$, $h$ in $H^0(U, \mathcal{O}_{V_m})$. Pick a $W$ contained in $U$ such that $f_m(W) \subseteq f_m(U \cap V_m)$; such a $W$ exists. Then $h \circ f_m$ is in $H^0(W, \mathcal{O}_W)$. Write $g_1 = f_1 - h \circ f_m$. Then

$$dg_1(t_1) = df_1(t_1) - h'(0)df_m(t_1) = df_1(t_1) \neq 0,$$
Thus the lemma is proven and the necessity in part B follows.

5) As we have observed, \([f \circ \pi; f \text{ in } A(K)]\) contains an ideal of \(A(\tilde{\alpha})\) whose hull is a finite point set. Further, \([f \circ \pi^{-1}; f \text{ in } A^1] = M\), for \(A^1\) as defined above is a maximal subalgebra of \(C(\alpha_1)\) and contains \(A(K)\). Then \(A(K)\) contains an ideal of \(M\) whose hull is a finite point set. Thus by corollary 3 to theorem 1, if \(B \subset C(\alpha_1)\), \(B \supset A(K)\), then either \(B = C(S(B))\), or \(B \subset M\). In the former case, since \(A(K)\) separates points on \(\alpha_1\), so does \(B\), so \(S(B) = \alpha_1\). Thus theorem 27 is proven.
1. Introduction

Theorem 28. (Wermer [24]). Let $M$ be a Riemann surface, $\Gamma$ a simple closed analytic curve on $M$ such that $\Gamma$ is the boundary of a region $D$ with $D\cup\Gamma$ compact. Let $f, g$ be holomorphic in a neighborhood of $D\cup\Gamma$, $df$ does not vanish on $\Gamma$ and $f, g$ together separate points on $\Gamma$. Then there is a finite subset $T$ of $D\cup\Gamma$ and an integer $n$ such that if $h$ is in $A(D\cup\Gamma)$, $h$ vanishes at each point of $T$ with order no less than $n$, then $h$ is approximable on $D\cup\Gamma$ by functions in the algebra $\{f, g\}$ of polynomials in $f$ and $g$.

In the appendix of [24] he extends this to finitely many functions. The crucial feature of this theorem is to prove that $D\cup\Gamma$ is $\{f, g\}$-convex; with this assumption the proof depends only on theorems in Several Complex Variables of a general nature. As we have seen, theorem 19 (Bishop) replaces $D\cup\Gamma$ by any compact set $K$, assumes $K$ is $\{f, g\}$-convex and proves that any $f$ in $C(K)\cap H(\text{int}K)$ with the required zeros at interior points is in the
closure of \{f, g\}. The main point here is the approximation theorem, i.e., that \{f, g\} is dense in an ideal of \(C(K)\cap H(\text{int} K)\). We will prove this theorem of Wermer under the assumption that the functions are meromorphic on \(M\), and holomorphic on \(K\). We will use the following two facts:

**Lemma 15.** Let \(V\) be a variety in \(\mathbb{P}^n = [z; |z_j| < 1]\), \(K \subset V\) a compact \(H^o(V, \mathcal{O}^V)\)-convex set. Then \(K\) is a polynomial convex.

**Proof:** Let \(x\) not in \(K\). If \(x\) is not in \(V\), there is an \(f\) in \(H^o(\mathbb{P}^n, \mathcal{O}^n)\) such that \(f(x) = 1, f|_V = 0\). If \(x\) is in \(V\), there is an \(f\) in \(H^o(V, \mathcal{O}^V)\) such that \(f(x) = 1\) and \(\|f\|_K < \varepsilon\). But, by theorems A and B, \(f = g|_V\); \(g\) in \(H^o(\mathbb{P}^n, \mathcal{O}^n)\), so \(g(x) = 1\) and \(\|g\|_K < \varepsilon\). Thus \(K\) is \(H^o(\mathbb{P}^n, \mathcal{O}^n)\)-convex, but since \(\mathbb{P}^n\) is polynomial convex, \(K\) is also polynomial convex.

**Lemma 16.** Let \(S\) be a connected set in \(\mathbb{P}^n\) satisfying the following condition:

for all \(x\) in \(S\), there is a neighborhood \(U_x\) of \(x\) such that \(S \cap U_x\) is a variety in \(U_x\) (i.e., \(S\) is locally a variety). Suppose \(K \subset S\), as a subset of \(\mathbb{P}^n\) is compact and polynomial convex. Then for any \(f\) in \(H^o(U \cap S, \mathcal{O}^S_{U \cap S})\), \(U\) a neighborhood of \(K\), \(f|_K\) is uniformly approximable by polynomials.

**Proof:** For \(x\) in \(K\), there is a \(U_x\) such that \(S \cap U_x\) is a variety in \(U_x\). Cover \(K\) by finitely many \(U_1, \ldots, U_n\). Let \(V_j\) open, \(\overline{V}_j\) compact, contained in \(U_j\) and \(V_1, \ldots, V_n\)
cover $K$. Then $\bigcup_{j=1}^{n} S_{n}V_{j}$ is compact. Let $K' = \bigcup_{j=1}^{n} S_{n}V_{j}$. Then $K' \subset S$, $K'$ is compact and $\text{int}K' \supset K$. By the polynomial convexity, we can find an analytic polyhedron $P = \left\{ |p_{j}| < 1; j = 1, \ldots, k, p_{j} \text{ polynomials} \right\}$ such that $K \subset P \subset S \subset \text{int}K'$. Then $P \cap S$ is closed in $P$ (for $K'$ is closed, and $K' \cap P = S \cap P$). Thus since $P \cap S$ is locally a variety, $P \cap S$ is a variety in $P$. Every $f$ in $H^{0}(P \cap S, [\frac{P \cap S}{P}])$ then is the restriction of $g$ in $H^{0}(P, [\frac{P \cap S}{P}])$ to $P \cap S$ by theorems A and B. Since $P$ is polynomial convex, then, $g|_{K} = f|_{K}$ is approximable by polynomials.

2. Wermer's Theorem for meromorphic functions

Lemma 17. Let $M$ be a compact Riemann surface, $\gamma$ a domain on $M$ bounded by an analytic curve $\gamma$. Let $K = \bigcup_{\gamma}$, then for any proper open set $U \ni K$, $K$ is $H(U)$-convex.

Proof: By corollary 1 to theorem 18, since we know that $S(A(K)) = K$, we have only to define a sequence of open sets $U(n)$, $U(n) \supset U(n+1)$, such that $\bigcap_{n} U(n) = K$, and $U(n)$ is simply connected with respect to $U$. Let $\omega$ be an annullus around $\gamma$; i.e., there is an analytic mapping $\omega$ of a neighborhood $\mathcal{A}$ of $\gamma$ onto $[1-\varepsilon < \{ \omega \} < 1+\varepsilon]$, such that $\omega(\gamma) = \left\{ |z| = 1 \right\}$. Suppose $\omega \mathcal{A} - U = \emptyset$. Let $\gamma_{n}$ be the curve $\omega^{-1}( |z| = 1 + \varepsilon \frac{1}{n+1})$, and $U(n)$ the domain bounded by $\gamma_{n}$ containing $K$. Then $\bigcap_{n} U(n) = K$, and $U(n) \supset U(n+1)$, and $U(n) \subset U$. Let $\gamma \subset U(n)$ bound a domain $W$ in $U$. $W \cap U(n)$ is nonempty and open in $W$. It is also
closed in \(W\). For if \(x_n\) in \(W \cap U(n)\), \(x_n \to x\), and \(x\) is in \(W\) but \(x\) is not in \(U(n)\), it must be in \(\partial U(n)\). But

\[\partial U(n) = \bigcap_n \neq \emptyset\]

is disjoint from \(W\). For suppose \(W \cap \bigcap_n \neq \emptyset\), then also \(W \cap (M - U(n)) \neq \emptyset\), for \(W\) is open. But \(M - U(n)\) is connected and never intersects \(\partial W\), so is contained in \(W\), and thus contained in \(U\). But \(U(n) \subseteq U\) also, so \(U = M\), contradicting the assumption. Thus, since \(x\) cannot be in \(\partial U(n)\), it must be in \(U(n)\), thus \(W \cap U(n)\) is closed. But \(W\) is connected, so \(W \cap U(n) = W\), i.e., \(W \subseteq U(n)\). Thus \(U(n)\) is simply connected with respect to \(U\).

**Theorem 29.** Let \(M\) be a compact Riemann surface, \(K = \Omega \cup \gamma\), \(\Omega\) a domain and \(\gamma\) an analytic curve. Let \(f_1, \ldots, f_n\) be meromorphic in \(M\) and holomorphic on \(K\) such that

\[N = \{p \in M;\ \text{there is } q \in M \text{ such that } f_j(p) = f_j(q) \text{ for all } j = 1, \ldots, n\}\]

if finite. Let \(\mathcal{A}^o = \text{uniform closure on } K \text{ of the algebra of polynomials in } f_1, \ldots, f_n\). Then \(\mathcal{A}^o\) contains an ideal of \(K\).

Proof: We may assume \(||f_j||_K < 1\) for \(j = 1, \ldots, n\). Let \(R = \{m \in M; (f_j(m)) < 1\}; \bar{R}\) is compact since the \(f_j\) are meromorphic on \(M\). Define \(F: R \to \mathbb{P}^n = [z; |z_j| < 1, 1 \leq j \leq n]\),

\[F(m) = (f_1(m), \ldots, f_n(m)).\]

1) \(F\) is holomorphic and it is proper. For let \([m_n]\) in \(F^{-1}(A)\), a compact in \(\mathbb{P}^n\). \(\bar{R}\) is compact, so there exists an \(m\) in \(\bar{R}\) such that \(m_n \to m\). Then \(f_i(m)\) is a
limit point of \([f_1(m_n)]\) and since \([f_1(m_n)] = \{n_1F(m_n)\}\) is contained in a compact subset of the unit disc, we have \(|f_1(m)| < 1\). Therefore, \(F(m) = (f_1(m), \ldots, f_n(m))\) is defined and is in \(P\); since \(A\) is compact, \(F(m)\) is in \(A\). Therefore \(m\) in \(F^{-1}(A)\); and \(F(A)\) is compact.

Then by Remmert's Abbildungssatz (theorem 21; Remmert has proven this without the finiteness condition [19]), \(F(R) = S\) is a variety in \(P^n\).

2) \((R, F)\) is the normal model for \(S\).

Let \(s\) be in \(S\). Let \(F^{-1}(s) = [x_1, \ldots, x_k]\). Each \(x_i\) has a neighborhood \(U_i\) in \(S\) such that \(\overline{U_i} \cap \mathbb{N}\) is empty, thus \(F\) is one-one on \(\overline{U_i}\), and \(\overline{U_i}\) is compact. Thus \(F: U_i \rightarrow F(U_i)\) is a homeomorphism. Since \(\bigcup_{i} [df_1 = 0]\) is discrete, we may also assume that except possible for \(x_i\), some \(df_1 \neq 0\) at every point of \(U_i\). Thus \(F \mid_{U_i-x_i}\) is a holomorphic map of a connected set onto \(F(U_i)-s\), so \(F(U_i)-s\) is a connected submanifold of \(S_{\text{reg}}\), and thus must be an open set contained in one irreducible branch \(S_i\) of \(S\) at \(s\). Thus \(F(U_i)\) is a neighborhood of \(s\) on \(S_i\). Then \(\bigcup_i F(U_i)\) is a neighborhood of \(s\). For suppose \(s_n \rightarrow s\), \(s_n\) in \(S\). Let \(x_n\) be in \(R\) such that \(F(x_n) = s_n\), then by the properness of \(F\), there exists an \(x\) in \(R\) such that \(x_n \rightarrow x\) so \(x\) must be some \(x_i\), since \(F(x) = s\). But then the \(x_n\) from some \(n^*\) on must be in \(U_i\), so \(s_n\) in \(F(U_i)\). Thus \(\bigcup_i F(U_i)\) is a neighborhood of \(s\) on \(S\).

Let \(s\) in \(S_{\text{sing}}\). The branches of \(S\) at \(s\) are just the \(F(U_i)\) described above, then \(s\) is an identification
point if and only if $F^{-1}(s)$ is more than one point. If $F^{-1}(s) = x$, then we must have here $df_j(x) = 0$, $1 \leq j \leq n$, for if not then $F$ is biholomorphic in a neighborhood of $x$, which would make $S$ a manifold at $s$. Conversely if $s$ in $S_{\text{reg}}$, then $dz_j(s) \neq 0$ for some $j$, i.e., $z_j$ is one-one in a neighborhood of $x$, implying $df_j(x) \neq 0$. Thus $F^{-1}(S_{\text{sing}}) = N \cap [df_j = 0, 1 \leq j \leq n]$, which is a finite set (since $\overline{R}$ is compact). Now, for $x$ not in $F^{-1}(S_{\text{sing}})$, we have that some $df_j \neq 0$, and $F$ is a homomorphism in a neighborhood of $x$ onto a neighborhood of $F(x)$; it follows that $F$ is biholomorphic in a neighborhood of $x$. Thus $R - F^{-1}(S_{\text{sing}})$ is biholomorphic. Thus $(R, F)$ is the normal model for $S$.

Now by lemma 17, $D$ is $H^\infty(R, \mathcal{O}^R)$-convex. It follows that $K^\circ = F(K)$ is also $H^\infty(S, \mathcal{O}^S)$-convex as in theorem 26 (since as we have seen in chapter III $H^\infty(S, \mathcal{O}^S)$ contains an ideal of $H^\infty(R, \mathcal{O}^R)$). Now, by lemma 15, $K^\circ$ is polynomial-convex.

Now, as in chapter III, $[f \circ F; f$ in $A(K^\circ)]$ contains an ideal $I$ of $A(K)$. If $f$ is in $H^\infty(U, \mathcal{O}^S_U)$, $U$ a neighborhood on $S$ of $K$, then by lemma 16, there are polynomials $p_n$ such that $p_n|_{K^\circ} \rightarrow f|_{K^\circ}$. Thus $p_n \circ F \rightarrow f \circ F$ on $K$, but $p_n \circ F = P_n(f_1, \ldots, f_n)$ is in $A^\circ$. Thus $f \circ F$ is in $A^\circ$, and finally $I \subset A^\circ$.

Corollary. If in addition we assume that $f_1, \ldots, f_n$ separate the points of $K$ and for all $m$ in $K$, there is an
such that \( df_j(w) \neq 0 \), then \( A^0 = A(K) \).

Proof: In this case there is a neighborhood \( U \) of \( K \) such that \( F_U \) is biholomorphic. Then \( \hat{F}(H^\alpha(F(U), \partial^S) = H^\alpha(U, \partial^R) \). Since \( f \) in the former is approximable on \( K \) by polynomials in \( f_1, \ldots, f_n \), then \( A^0 \) is dense in \( H^\alpha(U, \partial^R) \). The latter is dense in \( A(K) \), thus \( A^0 = A(K) \).

3. Extension of Wermer's theorem

Lemma 18. \( K \) is a compact subset of a Riemann surface.

\[ f_1, \ldots, f_n \] are holomorphic in a neighborhood \( U \) of \( K \), and \( \omega: U \rightarrow \mathbb{C}^n \), \( \omega(r) = (f_1(r), \ldots, f_n(r)) \). \( S \) is locally a variety, and \( K \) is convex with respect to polynomials in \( f_1, \ldots, f_n \). Then \( \omega(K) \) is polynomial convex.

Proof: Let \( U \supset V \supset V \supset K \), \( V \) a domain, \( \overline{V} \) compact. Then \( \omega(\overline{V}) \) is compact. For \( x \) in \( \partial V \), there is a \( p_x(f_1, \ldots, f_n) \) such that \( |p_x(x)| > \|p_x\|_K \), \( p_x \) a polynomial. Then \( |p_x(\omega(x))| > \|p_x\|_{\omega(K)} \).

Thus for every \( z \) in \( \partial \omega(\overline{V}) \), there is a polynomial \( p_z \) such that \( |p_z(z)| > \|p_z\|_{\omega(K)} \). By compactness, there are polynomials \( p_1, \ldots, p_k \) such that \( \omega(\overline{V}) \subset \{ |p_j| < 1, j=1, \ldots, k \} \cap S \supset \omega(K) \).

If we add proper multiples of the coordinate functions, we can say that there is an analytic polyhedron \( P = \{ |p_j| < 1; p_j \text{ polynomials} \} \) such that \( K \subset P \) and \( \partial \omega(P) \) is a variety in \( P \). \( \partial \omega(P) \) is closed in \( P \), for it is the intersection of a closed set, \( \omega(\overline{V}) \), with \( P \). Now \( \omega(H^\alpha(SP, \partial^S_{SP})) \) contains the polynomials in \( f_1, \ldots, f_n \), so by assumption, \( \omega(K) \) is \( H^\alpha(SP, \partial^S_{SP}) \)-convex. Now lemma 15 applies, proving \( \omega(K) \) is polynomial convex.

Theorem 30. Let \( f_1, \ldots, f_n \) be functions holomorphic
in an annulus $|z| = 1 = \Gamma$. Suppose $f_1, \ldots, f_n$ separate the points of $\Gamma$, $df_1$ is never zero on $\Gamma$, and the algebra $A^o = \text{uniform closure on } \Gamma$ of polynomials in $f_1, \ldots, f_n$ is not $C(\Gamma)$. Then there is a connected set $S \subset \mathbb{C}^n$, a closed analytic curve $\gamma$, bounding an open connected subset $\alpha$ of $S$, and a homeomorphism $\varphi: \Gamma \rightarrow \gamma$ such that

1) $\alpha$ is locally a variety
2) $A^o = [f \circ \varphi; f \text{ in } A(\alpha \cup \gamma)]$.

Proof: Under the assumptions Wermer has proven: there exists a Riemann surface $R$, a biholomorphism $\psi: \alpha \rightarrow \beta \subset R$, $\varphi'(\Gamma) = \gamma'$, an analytic curve bounding a domain $\alpha'$ such that for $A^o_\beta = [f \circ \varphi^{-1}; f \text{ is in } A^o]$, $A^o_\beta \subset A(\alpha' \cup \gamma')$ and $A^o_\beta \supset I$, $I$ an ideal of $A_\alpha(\alpha' \cup \gamma')$ [23, 24, 25]. Let $\alpha' \cup \gamma' = U$, and define $\psi: U \rightarrow \mathbb{C}^n$, $\psi(r) = (f_1(r), \ldots, f_n(r))$ $(f_j = f \circ \varphi^{-1})$. By Remmert's Abbildungsatz [19] $\psi(U) = S$ is locally a variety, i.e., a general analytic space. Let $\gamma = \psi(\gamma')$, $\alpha = \psi(\alpha')$; then $\alpha = \psi^{-1}$ is biholomorphic. If $p$ is a polynomial in $f_1, \ldots, f_n$, $p \circ \varphi^{-1}$ is a polynomial in $z_1, \ldots, z_n$, and thus is in $A(\alpha \cup \gamma)$.

Since $\alpha' \cup \gamma'$ is convex with respect to $\Theta$, it is convex with respect to $A^o_\beta$. But that implies that $\alpha \cup \gamma$ is polynomially convex by lemma 17, so by lemma 16, every $f$ in $A(\alpha \cup \gamma)$ is approximable on $\alpha \cup \gamma$ by polynomials, so that $f \circ \varphi$ is in $A^o$.

The essential fact in the above proof is that $\alpha' \cup \gamma'$ is $A^o$-convex, thus with the same proof (i.e.,
Remmert's Abbildungssatz and lemmas 16 and 17) we have:

**Corollary.** Let $K$ be a compact set on a Riemann surface $M$, $A_0$ a subalgebra of $A(K)$ generated by $f_1, \ldots, f_n$ in $H(K)$. Suppose $S(A^0) = K$. Then there exists a set $S \subset \mathbb{C}^n$ which is locally a variety, and $\psi: U \to S$, $U$ a neighborhood of $K$ such that $A^0 = \{ f \circ \psi ; f \text{ in } A(\varphi(K)) \}$. 
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BIOGRAPHICAL NOTE

The author attended the College of the City of New York from 1952 to 1956, receiving his B.S. degree in 1956. He then attended the Massachusetts Institute of Technology from 1956 to 1959, receiving his M.S. degree in 1957. He was a teaching assistant at the Massachusetts Institute of Technology for the academic year 1957 - 1958, and was a General Electric fellow for the year 1958-1959.