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HIGHER CRITICAL POINTS IN AN ELLIPTIC FREE BOUNDARY PROBLEM

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Abstract. We study higher critical points of the variational functional associated with a free boundary problem related to plasma confinement. Existence and regularity of minimizers in elliptic free boundary problems have already been studied extensively. But because the functionals are not smooth, standard variational methods cannot be used directly to prove the existence of higher critical points. Here we find a nontrivial critical point of mountain pass type and prove many of the same estimates known for minimizers, including Lipschitz continuity and nondegeneracy. We then show that the free boundary is smooth in dimension 2 and prove partial regularity in higher dimensions.

1. Introduction

In this paper we consider a superlinear free boundary problem related to plasma confinement (see, e.g., [10, 15, 17, 26, 27, 28]). Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, and define the functional

$$J(v) = \int_\Omega \left[\frac{1}{2}\,|\nabla v|^2 + Q_p(x,v)\right] dx$$

with

$$Q_p(x,v) = \chi_{\{v>1\}}(x) - \frac{1}{p}\,(v-1)_+^p$$

for $2 < p < \infty$ if $N = 2$ and for $2 < p < 2N/(N-2)$ if $N \geq 3$. We seek a nonminimizing critical point of this functional in the usual Sobolev space $H^1_0(\Omega)$, the

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closure of $C_0^\infty(\Omega)$ in the norm

$$\|v\|^2 = \int_\Omega |\nabla v|^2 \, dx.$$

The critical point u of J that we construct is Lipschitz continuous in $\bar{\Omega}$. The region

$$\{u > 1\} \subset\subset \Omega$$

represents the plasma, and the boundary of the plasma,

$$F(u) := \partial\{x \in \Omega : u(x) > 1\},$$

is the free boundary.

The function u satisfies the following interior Euler-Lagrange equation

$$-\Delta u = (u-1)_+^p, \quad \text{in} \quad \Omega \setminus F(u), \tag{1.1}$$

where $w_\pm = \max\{\pm w, 0\}$ denote the positive and negative parts of w, respectively. The function u also satisfies, in various generalized forms, the free boundary condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2 \quad \text{on} \quad F(u),$$

where $\nabla u^\pm$ are the limits of ∇u from $\{u > 1\}$ and $\{u \le 1\}^\circ$, respectively. The ultimate goal is to show that at most (or all) points, the free boundary is smoooth, and at those points the free boundary condition is satisfied in the ordinary, classical sense.

The assumption $p > 2$ makes the Euler-Lagrange equation superlinear, which helps us to prove existence of a nontrivial mountain pass solution. We also make use of the assumption $p > 2$ in proving important nondegeneracy properties of u that lead to regularity of the free boundary. The upper limitation on p is imposed so that the inclusion from $H_0^1(\Omega)$ to $L^p(\Omega)$ is a compact. The limiting exponent $p = 2N/(N-2)$, $N \ge 3$, is treated in [31].

Our first theorem, Theorem 1.2, says that there is a Lipschitz continuous mountain pass solution to the variational problem. Our second theorem, Theorem 1.4, says that this solution is nondegenerate and satisfies the free boundary condition in the sense of viscosity. Our third theorem, Theorem 1.5, establishes full regularity of the free boundary in dimension 2 and partial regularity in higher dimensions. We believe that these are the first results in the literature to address existence and regularity of higher critical points of free boundary functionals. This paper is an improvement on our preprint [21], which established weaker partial regularity of the free boundary.

For minimizers there is a large literature proving existence and partial regularity of the free boundary. (See, for example, [1, 2, 3, 5, 6, 7, 8, 9, 29, 30] and the references therein). Our results are less general than those for minimizers, which apply to many more classes of potentials than $Q_p(x, v)$. We chose this family of potential functions because we are able to prove that the corresponding functional has a nontrivial mountain pass solution. In addition to being less general, our results give less regularity for the free boundary than is valid for minimizers. We have only proved that our critical point has a smooth free boundary in dimension 2. We conjecture that our results are best possible in the sense that there does exist an axisymmetric mountain pass solution in dimension 3 with a singular free boundary point resembling the example in [1]. In the case of minimizers, the best results to date are that the free boundary is smooth everywhere in all dimensions $N \leq 4$ and has singularities on a closed set of Hausdorff dimension at most $N-5$ in higher dimensions (see [12, 19, 13]).

To formulate our results more precisely, we recall the definition of a mountain pass point.

Definition 1.1 (Hofer [18])**.** We say that $u \in H_0^1(\Omega)$ is a mountain pass point of J if the set $\{v \in U : J(v) < J(u)\}$ is neither empty nor path connected for every neighborhood U of u.

Let
$$\Gamma = \big\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ J(\gamma(1)) < 0\big\}$$
be the class of continuous paths from 0 to the set $\{u \in H_0^1(\Omega) : J(u) < 0\}$, and denote
$$c^* = c^*(\Omega) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} J(u).$$
It will follow from an integration by parts that our mountain pass point u belongs to the Nehari-type manifold
$$\mathcal{M} = \left\{u \in H_0^1(\Omega) : \int_{\{u>1\}} |\nabla u|^2\, dx = \int_{\{u>1\}} (u-1)^p\, dx > 0\right\}.$$
Our first main result is the following.

Theorem 1.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, and J as above. Then*

a) $c^* = c^*(\Omega) > 0$.

b) The functional J has a mountain pass point u satisfying $J(u) = c^*$, and u minimizes $J|_{\mathcal{M}}$. In particular, by part (a) the solution is nontrivial.

c) The function u is Lipschitz continuous on $\bar{\Omega}$ solving the interior Euler-Lagrange equation (1.1)*. Moreover, u solves the free boundary condition in the variational sense of Definition 4.3.*

The following nondegeneracy is the fundamental estimate needed to be able to establish more detailed properties of the free boundary.

Definition 1.3. We say that a continuous function u in $\bar{\Omega}$ is *nondegenerate* if there exist constants r_0, $c > 0$ such that if $x_0 \in \{u > 1\}$ and $r := \operatorname{dist}(x_0, \{u \leq 1\}) \leq r_0$, then $u(x_0) \geq 1 + cr$.

Our other main results are as follows.

Theorem 1.4. *The mountain pass solution u in Theorem 1.2 is nondegenerate in the sense of Definition 1.3 and satisfies the free boundary condition in the sense of viscosity, namely, if there is a ball B tangent to the free boundary and a point $x_0 \in \partial \{u > 1\} \cap \partial B$, then u has an asymptotic expansion of the form*

$$u(x) = \alpha\langle x - x_0, \nu\rangle_+ - \beta\langle x - x_0, \nu\rangle_- + o(|x - x_0|), \quad x \to x_0,$$

with

$$\alpha > 0, \quad \beta \geq 0, \quad \alpha^2 - \beta^2 = 2,$$

where ν is the interior unit normal to ∂B at x_0 if $B \subset \{u > 1\}$ and the exterior unit normal if $B \subset \{u \leq 1\}^\circ$.

Theorem 1.5. *The mountain pass solution u in Theorem 1.2 has a free boundary $\partial\{u > 1\}$ of finite $(N-1)$-dimensional Hausdorff measure that is a C^∞ hypersurface except on a closed set of Hausdorff dimension at most $N-3$. Near the smooth subset of the free boundary, $(u-1)_\pm$ are smooth and the free boundary equation is satisfied in the classical sense. If $N = 2$, then the exceptional set is empty, that is, the free boundary is smooth at every point. In dimension $N = 3$, the free boundary has at most finitely many nonsmooth points.*

The proof of Theorem 1.5 depends on two propositions of independent interest. Define

$$\delta_0 := \operatorname{dist}(\{u > 1\}, \partial\Omega) > 0.$$

Proposition 1.6. *If u is a nondegenerate, Lipschitz continuous interior solution as in* (1.1)*, then there exists a constant $C > 0$ such that whenever $r \leq \delta_0/2$,*

$$\sigma(\partial \{u > 1\} \cap B_r(x_0)) \leq C r^{N-1},$$

where σ denotes $(N-1)$-dimensional Hausdorff measure. In particular, the free boundary $\partial \{u > 1\}$ has finite $(N-1)$-dimensional Hausdorff measure.

Proposition 1.7. *If u is a nondegenerate, Lipschitz continuous interior solution as in* (1.1) *that minimizes $J|_{\mathcal{M}}$, then there is a constant $c > 0$ such that whenever $x_0 \in \partial \{u > 1\}$ and $0 < r \leq \delta_0/2$,*

$$c \leq \frac{\mathcal{L}(\{u > 1\} \cap B_r(x_0))}{\mathcal{L}(B_r(x_0))} \leq 1 - c, \tag{1.2}$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$. Thus, the topological boundary of $\{u > 1\}$ coincides with its measure-theoretic boundary.

Let us point out that the existence of a mountain pass solution is by no means routine due to the lack of smoothness of J. Indeed, J is not even continuous, much less of class C^1. For the functional in which the discontinuous term $\chi_{\{u>1\}}$ is removed, there is no difficulty in applying the mountain pass theorem, as, for example, in Flucher and Wei [15] and Shibata [26].

The outline of the proof of Theorem 1.2 is as follows. In Section 2, we introduce an approximation J_ϵ to the functional J, find associated mountain pass solutions u^ϵ, and prove uniform Lipschitz bounds on these solutions with the help of a uniform estimate of Caffarelli, Jerison, and Kenig [11] (see Proposition 2.8). Along the way, we show that $c^* > 0$ (part (a) of the theorem). In Section 3, we show that a subsequence of u^ϵ converges to a function u that solves the Euler-Lagrange equation in the complement of the free boundary. In Section 4 we show that our putative solution u belongs to the Nehari manifold $\mathcal{M}$ and minimizes J when restricted to $\mathcal{M}$. We also show that $J(u) = c^*$, which ultimately leads to the variational equation for u.

In Section 5 we prove Theorem 1.4 by showing that any Lipschitz continuous minimizer of J on $\mathcal{M}$ solving the interior equation (1.1) is nondegenerate. For minimizers, nondegeneracy is proved using a harmonic replacement. Our proof of nondegeneracy is somewhat different; it depends on $p > 2$ and projection onto the Nehari manifold. The second part of the theorem is a corollary of theorems of Lederman and Wolanski [24], which say that if a singular limit u such as ours is nondegenerate, then it is a viscosity solution. (Note, however, that we obtain a stronger form of viscosity solution because of a further complementary nondegeneracy proved in Proposition 1.7.)

In Section 6 we prove Proposition 1.6, and in Section 7 we prove Proposition 1.7. Both bounds in Proposition 1.7 should be viewed as nondegeneracy estimates. The lower bound by c is an easy consequence of the nondegeneracy of Definition 1.3. The upper bound by $1 - c$ is a new kind of complementary nondegeneracy of the region $\{u \leq 1\}$. In Section 8, we conclude the proof of Theorem 1.5 using a blow-up argument based on the monotonicity formula of G. Weiss described in the appendix, Section 9.

2. Approximate mountain pass solutions

We approximate J by C^1-functionals as follows. Let $\beta : \mathbb{R} \to [0, 2]$ be a smooth function such that $\beta(t) = 0$ for $t \leq 0$, $\beta(t) > 0$ for $0 < t < 1$, $\beta(t) = 0$ for $t \geq 1$, and $\int_0^1 \beta(s)\, ds = 1$. Then set

$$\mathcal{B}(t) = \int_0^t \beta(s)\, ds,$$

and note that $\mathcal{B} : \mathbb{R} \to [0,1]$ is a smooth nondecreasing function such that $\mathcal{B}(t) = 0$ for $t \le 0$, $\mathcal{B}(t) > 0$ for $0 < t < 1$, and $\mathcal{B}(t) = 1$ for $t \ge 1$. For $\varepsilon > 0$, let

$$J_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}\,|\nabla u|^2 + \mathcal{B}\left(\frac{u-1}{\varepsilon}\right) - \frac{1}{p}\,(u-1)_+^p\right] dx, \quad u \in H_0^1(\Omega)$$

and note that J_ε is of class C^1.

If u is a critical point of J_ε, then u is a weak solution of

$$\begin{cases} \Delta u = \dfrac{1}{\varepsilon}\,\beta\left(\dfrac{u-1}{\varepsilon}\right) - (u-1)_+^{p-1} & \text{in } \Omega \\[2ex] u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

and hence also a classical $C^{2,\alpha}$ solution by elliptic regularity theory.

Note that if u is not identically zero, then it is nontrivial in a stronger sense, namely, $u > 0$ in Ω and $\{u > 1\}$ is a nonempty open set. In fact, if $u \le 1$ then it is harmonic in Ω and hence identically zero (since $u = 0$ on $\partial\Omega$). Thus any nonzero u is strictly greater than 1 on an open set. Furthermore, on $\{u < 1\}$, u is the harmonic function with boundary values 0 on $\partial\Omega$ and 1 on $\partial\{u \ge 1\}$, hence strictly positive. (Here we are using the assumption that Ω is connected.)

Let $\Phi \in C_0^1(\Omega, \mathbb{R}^N)$. Then by (2.1),

$$\begin{aligned} &\operatorname{div}\left[\left(\frac{1}{2}|\nabla u|^2 + \mathcal{B}((u-1)/\varepsilon) - \frac{1}{p}(u-1)_+^p\right)\Phi - (\nabla u \cdot \Phi)\nabla u\right] \\ &\quad = \left(\frac{1}{2}|\nabla u|^2 + \mathcal{B}((u-1)/\varepsilon) - \frac{1}{p}(u-1)_+^p\right)\operatorname{div}\Phi - \nabla u(D\Phi)\cdot\nabla u\,. \end{aligned}$$

Hence,

$$\int_\Omega \left[\left(\frac{1}{2}|\nabla u|^2 + \mathcal{B}((u-1)/\varepsilon) - \frac{1}{p}(u-1)_+^p\right)\operatorname{div}\Phi - \nabla u(D\Phi)\cdot\nabla u\right] dx = 0\,. \tag{2.2}$$

This is one form of the critical equation that we will ultimately show is inherited in the limit as $\varepsilon \to 0$ by our mountain pass solution. It is the critical point equation for J_ε with respect to domain variations. Indeed, for sufficiently small t, $x \mapsto x + t\Phi(x)$ is a diffeomorphism of Ω, and the left side of (2.2) is

$$\left.\frac{d}{dt}\right|_{t=0} J_\varepsilon(u(x + t\Phi(x)))\,.$$

Lemma 2.1. *J_ε satisfies the Palais-Smale compactness condition, that is, every sequence $(u_j) \subset H_0^1(\Omega)$ such that $J_\varepsilon(u_j)$ is bounded and $J_\varepsilon'(u_j) \to 0$ in $H_0^1(\Omega)$ norm has a convergent subsequence.*

Proof. We have

$$J_\varepsilon(u_j) = \int_\Omega \left[\frac{1}{2}\,|\nabla u_j|^2 + \mathcal{B}\left(\frac{u_j - 1}{\varepsilon}\right) - \frac{1}{p}\,(u_j - 1)_+^p\right] dx = \mathrm{O}(1) \tag{2.3}$$

and

$$J_\varepsilon'(u_j)\, v_j = \int_\Omega \left[\nabla u_j \cdot \nabla v_j + \frac{1}{\varepsilon}\,\beta\left(\frac{u_j - 1}{\varepsilon}\right) v_j - (u_j - 1)_+^{p-1}\, v_j\right] dx = \mathrm{o}(\|v_j\|), \quad v_j \in H^1_0(\Omega). \tag{2.4}$$

We begin by showing that $\|u_j\|$ is bounded. Write $u_j = u_j^+ + u_j^-$, where $u_j^\pm$ are defined by

$$u_j^+ := (u_j - 1)_+, \quad u_j^- = 1 - (u_j - 1)_-\,.$$

Since $\mathcal{B}$ is bounded (2.3) gives

$$\int_\Omega \left[|\nabla u_j^+|^2 + |\nabla u_j^-|^2 - \frac{2}{p}\,(u_j^+)^p\right] dx \le C < \infty.$$

Taking $v_j = u_j^+$ in (2.4) and using

$$\int_\Omega |v|\, dx \le C\|v\|$$

and the fact that β is bounded, we have

$$\int_\Omega (u_j^+)^p\, dx \le \int_\Omega |\nabla u_j^+|^2\, dx + C\|u_j^+\|.$$

Combining our inequalities gives

$$\left(1 - \frac{2}{p}\right) \|u_j^+\|^2 + \|u_j^-\|^2 \le C\left(\|u_j^+\| + 1\right),$$

which implies that $\|u_j^\pm\|$ are bounded, and hence $\|u_j\|$, is bounded.

Replace u_j by a subsequence (still denoted u_j) that tends weakly to u in $H^1_0(\Omega)$ and such that u_j tends to u in $L^p(\Omega)$ norm and pointwise almost everywhere. Then $J_\varepsilon'(u_j)u \to 0$ and $J_\varepsilon'(u_j)u_j \to 0$ imply that

$$\lim_{j\to\infty} \|u_j\|^2 = \|u\|^2.$$

Finally,

$$\limsup_{j\to\infty} \|u_j - u\|^2 = \limsup_{j\to\infty} (\|u_j\|^2 + \|u\|^2 - 2\langle u_j, u\rangle) = 2\|u\|^2 - 2\langle u, u\rangle = 0$$

□

Since $p < 2N/(N-2)$, the Sobolev imbedding theorem implies

$$J_\varepsilon(u) \ge \int_\Omega \left[\frac{1}{2}\,|\nabla u|^2 - \frac{1}{p}\,|u|^p\right] dx \ge \frac{1}{2}\,\|u\|^2 - C\,\|u\|^p \quad \forall u \in H^1_0(\Omega)$$

for some constant C depending on Ω. Since $p > 2$, then there exists a constant $\rho > 0$ such that

$$\|u\| \le \rho \implies J_\varepsilon(u) \ge \frac{1}{3}\,\|u\|^2.$$

Moreover,

$$J_\varepsilon(u) \le \int_\Omega \left[\frac{1}{2}\,|\nabla u|^2 + 1 - \frac{1}{p}\,(u-1)_+^p\right] dx$$

and hence, again because $p > 2$, there exists a function $u_0 \in H^1_0(\Omega)$ such that $J_\varepsilon(u_0) < 0 = J_\varepsilon(0)$. Therefore, the class of paths

$$\Gamma_\varepsilon = \left\{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0,\ J_\varepsilon(\gamma(1)) < 0\right\}$$

is nonempty and

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon}\ \max_{u \in \gamma([0,1])}\ J_\varepsilon(u) \ge \frac{\rho^2}{3}. \tag{2.5}$$

Lemma 2.2. *J_ε has a (nontrivial) critical point u^ε at the level c_ε.*

Proof. If not, then there exists a constant $0 < \delta \le c_\varepsilon/2$ and a continuous map $\eta : \{J_\varepsilon \le c_\varepsilon + \delta\} \to \{J_\varepsilon \le c_\varepsilon - \delta\}$ such that η is the identity on $\{J_\varepsilon \le 0\}$ by the first deformation lemma (see, e.g., Perera and Schechter [25, Lemma 1.3.3]). By the definition of c_ε, there exists a path $\gamma \in \Gamma_\varepsilon$ such that $\max_{\gamma([0,1])} J_\varepsilon \le c_\varepsilon + \delta$. Then $\widetilde{\gamma} := \eta \circ \gamma \in \Gamma_\varepsilon$ and $\max_{\widetilde{\gamma}([0,1])} J_\varepsilon \le c_\varepsilon - \delta$, contradicting the definition of c_ε. □

Lemma 2.3. *We have $c_\varepsilon \le c^*$. In particular, by* (2.5)*, $c^* > 0$ and Theorem 1.2 (a) holds.*

Proof. Since $\mathcal{B}((t-1)/\varepsilon) \le \chi_{\{t>1\}}$ for all t, $J_\varepsilon(u) \le J(u)$ for all $u \in H^1_0(\Omega)$. So $\Gamma \subset \Gamma_\varepsilon$ and

$$c_\varepsilon \le \max_{u \in \gamma([0,1])} J_\varepsilon(u) \le \max_{u \in \gamma([0,1])} J(u) \quad \forall \gamma \in \Gamma. \quad \square$$

For $0 < \varepsilon \le 1$, u^ε have the following uniform regularity properties.

Lemma 2.4. *There exists a constant $C > 0$ such that, for $0 < \varepsilon \le 1$, $\|u^\varepsilon\| \le C$.*

Proof. By Lemma 2.3,

$$\int_\Omega \left[\frac{1}{2}\,|\nabla u^\varepsilon|^2 + \mathcal{B}\left(\frac{u^\varepsilon - 1}{\varepsilon}\right) - \frac{1}{p}\,(u^\varepsilon - 1)_+^p\right] dx \le c^*$$

and hence

$$\frac{1}{2}\int_\Omega |\nabla u^\varepsilon|^2\,dx \le c^* + \frac{1}{p}\int_{\{v_\varepsilon>0\}} v_\varepsilon^p\,dx, \tag{2.6}$$

where $v_\varepsilon = u^\varepsilon - 1$. Testing (2.1) with $(u^\varepsilon - 1 - \varepsilon)_+$ gives

$$\int_{\{u^\varepsilon>1+\varepsilon\}} |\nabla u^\varepsilon|^2\,dx = \int_{\{v_\varepsilon>\varepsilon\}} v_\varepsilon^{p-1}\,(v_\varepsilon - \varepsilon)\,dx. \tag{2.7}$$

Fix $\lambda > 2/(p-2)$. Multiplying (2.7) by $(\lambda+1)/p\lambda$ and subtracting from (2.6) gives

$$\begin{aligned}\frac{1}{2}\int_{\{u^\varepsilon\le 1+\varepsilon\}} |\nabla u^\varepsilon|^2\,dx + \frac{(p-2)\lambda-2}{2p\lambda}\int_{\{u^\varepsilon>1+\varepsilon\}} |\nabla u^\varepsilon|^2\,dx\\ \le c^* + \frac{1}{p}\int_{\{0<v_\varepsilon\le\varepsilon\}} v_\varepsilon^p\,dx + \frac{1}{p\lambda}\int_{\{v_\varepsilon>\varepsilon\}} v_\varepsilon^{p-1}\left[(\lambda+1)\,\varepsilon - v_\varepsilon\right]dx.\end{aligned} \tag{2.8}$$

The last integral is less than or equal to $\int_{\{\varepsilon<v_\varepsilon<(\lambda+1)\,\varepsilon\}} v_\varepsilon^{p-1}\left[(\lambda+1)\,\varepsilon - v_\varepsilon\right]dx$ and hence (2.8) gives

$$\min\left\{\frac{1}{2}, \frac{(p-2)\lambda-2}{2p\lambda}\right\}\int_\Omega |\nabla u^\varepsilon|^2\,dx \le c^* + \frac{\varepsilon^p\,\mathcal{L}(\Omega)}{p}\left[1 + (\lambda+1)^{p-1}\right].$$

The conclusion follows. □

Lemma 2.5. *There exists a constant $C > 0$ such that, for $0 < \varepsilon \le 1$,*

$$\max_{x\in\Omega} u^\varepsilon(x) \le C.$$

Proof. Since $p < 2N/(N-2)$, we have $N(p-2)/2 < 2N/(N-2)$. Fix $N(p-2)/2 < q < 2N/(N-2)$. Since

$$-\Delta u^\varepsilon = (u^\varepsilon - 1)_+^{p-1} - \frac{1}{\varepsilon}\,\beta\left(\frac{u^\varepsilon - 1}{\varepsilon}\right) \le (u^\varepsilon - 1)_+^{p-1} \le (u^\varepsilon)^{p-1},$$

there exists a constant $C > 0$ such that

$$\|u^\varepsilon\|_\infty \le C\,\|u^\varepsilon\|_q^{2q/(2q-N(p-2))}$$

by Bonforte et al. [4, Theorem 3.1]. Since u^ε is bounded in $L^q(\Omega)$ by the Sobolev imbedding theorem and Lemma 2.4, the conclusion follows. □

By Lemma 2.5, $(u^\varepsilon - 1)_+^{p-1} \le A_0$ for some constant $A_0 > 0$ independent of ε. Let $\varphi_0 > 0$ be the solution of

$$\begin{cases} -\Delta\varphi_0 = A_0 & \text{in } \Omega\\ \varphi_0 = 0 & \text{on } \partial\Omega.\end{cases}$$

Lemma 2.6. *For* $0 < \varepsilon \le 1$*,*

$$u^\varepsilon(x) \le \varphi_0(x) \quad \forall x \in \Omega,$$

in particular, $\{u^\varepsilon \ge 1\} \subset \{\varphi_0 \ge 1\} \subset\subset \Omega$*.*

Proof. Since $\beta(t) \ge 0$ for all t,

$$-\Delta u^\varepsilon \le (u^\varepsilon - 1)_+^{p-1} \le A_0 = -\Delta \varphi_0,$$

so $u^\varepsilon \le \varphi_0$ by the maximum principle. □

Lemma 2.7. *There exists a constant* $C > 0$ *such that, for* $r > 0$ *and* $0 < \varepsilon \le 1$*, if* $B_r(x_0) \subset \Omega$*, then*

$$\max_{x \in B_{r/2}(x_0)} |\nabla u^\varepsilon(x)| \le C/r.$$

Proof. Since $\beta(t) \le 2$ for all t,

$$\Delta u^\varepsilon \le \frac{1}{\varepsilon}\,\beta\left(\frac{u^\varepsilon - 1}{\varepsilon}\right) \le \frac{2}{\varepsilon}\,\chi_{\{|u^\varepsilon - 1| < \varepsilon\}}(x),$$

and since $\beta(t) \ge 0$ for all t,

$$-\Delta u^\varepsilon \le (u^\varepsilon - 1)_+^{p-1} \le A_0,$$

so

$$\pm \Delta u^\varepsilon \le \max\{2, A_0\}\left(\frac{1}{\varepsilon}\,\chi_{\{|u^\varepsilon - 1| < \varepsilon\}}(x) + 1\right).$$

Since u^ε is also uniformly bounded in $L^2(\Omega)$ by Lemma 2.5, the conclusion follows from the following result of Caffarelli, Jerison, and Kenig [11]. □

Proposition 2.8 ([11, Theorem 5.1])**.** *Suppose that* u *is a Lipschitz continuous function on* $B_1(0) \subset \mathbb{R}^N$ *satisfying the distributional inequalities*

$$\pm \Delta u \le A\left(\frac{1}{\varepsilon}\,\chi_{\{|u-1| < \varepsilon\}}(x) + 1\right)$$

for some constants $A > 0$ *and* $0 < \varepsilon \le 1$*. Then there exists a constant* $C > 0$*, depending on* N*,* A*, and* $\int_{B_1(0)} u^2\,dx$*, but not on* ε*, such that*

$$\max_{x \in B_{1/2}(0)} |\nabla u(x)| \le C.$$

3. Limits of mountain pass solutions

Let $\varepsilon_j \searrow 0$, let $u_j = u^{\varepsilon_j}$ be the critical point of J_{ε_j} obtained in Lemma 2.2, and let $c_j = J_{\varepsilon_j}(u_j)$ (an abuse of notation, since this value was previously denoted c_{ε_j}).

Lemma 3.1. *There exists a Lipschitz continuous function u on $\bar{\Omega}$ such that $u \in H^1_0(\Omega) \cap C^2(\bar{\Omega} \setminus \partial\{u > 1\})$, and, for a suitable sequence ε_j,*

(a) $u_j \to u$ uniformly on $\overline{\Omega}$,
(b) $-\Delta u = (u-1)_+^{p-1}$ on $\Omega \setminus \partial\{u > 1\}$,
(c) $u_j \to u$ strongly in $H^1_0(\Omega)$,
(d) $J(u) \le \liminf c_j \le \limsup c_j \le J(u) + \mathcal{L}(\{u = 1\})$.

Moreover, u is nontrivial in the sense that $J(u) + \mathcal{L}(\{u = 1\}) > 0$.

Proof. First we prove (a). The majorant φ_0 of Lemma 2.6 gives a uniform lower bound $\delta_0 > 0$ on the distance from $\{u^\varepsilon \ge 1\}$ to $\partial\Omega$. Thus u^ε is positive, harmonic and bounded by 1 in a δ_0 neighborhood of $\partial\Omega$. It follows from standard boundary regularity theory that u^ε is uniformly bounded in a $\delta_0/2$ neighborhood in, say, C^3 norm. In particular, the family is compact in C^2 norm on this set. By Lemmas 2.5 and 2.7, the family u^ε is uniformly Lipschitz continuous on the compact subset of Ω at distance greater or equal to $\delta_0/2$ from $\partial\Omega$. Finally, by Lemma 2.4, u^ε is uniformly bounded in $H^1_0(\Omega)$. Thus we can choose ε_j so that u_j converges uniformly in $\bar{\Omega}$ to a Lipschitz function u, and so that there is strong converence in C^2 on a $\delta_0/2$ neighborhood of $\partial\Omega$ and, finally, that there is weak convergence of u_j to u in $H^1_0(\Omega)$.

Next we show that u satisfies the interior part of the Euler-Lagrange equation:

$$-\Delta u = (u-1)_+^{p-1} \quad \text{in} \quad \{u \ne 1\}\,.$$

Let $\varphi \in C^\infty_0(\{u > 1\})$. Then $u \ge 1 + 2\,\varepsilon$ on the support of φ for some $\varepsilon > 0$. For all sufficiently large j, $\varepsilon_j < \varepsilon$ and $|u_j - u| < \varepsilon$ by (a). Then $u_j \ge 1 + \varepsilon_j$ on the support of φ, so testing

$$-\Delta u_j = (u_j - 1)_+^{p-1} - \frac{1}{\varepsilon_j}\,\beta\left(\frac{u_j - 1}{\varepsilon_j}\right) \tag{3.1}$$

with φ gives

$$\int_\Omega \nabla u_j \cdot \nabla\varphi\, dx = \int_\Omega (u_j - 1)^{p-1}\,\varphi\, dx.$$

Passing to the limit gives

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx = \int_\Omega (u - 1)^{p-1}\,\varphi\, dx \tag{3.2}$$

since u_j converges to u weakly in $H^1_0(\Omega)$ and uniformly on Ω. Hence u is a distributional (and thus a classical) solution of $-\Delta u = (u-1)^{p-1}$ in $\{u > 1\}$.

A similar argument shows that u satisfies $-\Delta u = 0$ in $\{u < 1\}$. We show next that u is also harmonic in the possibly larger set $\{u \le 1\}^\circ$. Since $\beta \ge 0$, testing (3.1) with any nonnegative $\varphi \in C_0^\infty(\Omega)$ and passing to the limit gives

$$-\Delta u \le (u-1)_+^{p-1} \quad \text{in } \Omega \tag{3.3}$$

in the distributional sense. On the other hand, since u is harmonic in $\{u < 1\}$, $\min(u, 1)$ satisfies the super-mean value property. This implies (see, for instance, [1, Remark 4.2])

$$\Delta \min(u, 1) \le 0$$

in the distributional sense. Combining with (3.3), we find that

$$\Delta u = 0$$

as a distribution on the open set $\{u \le 1\}^\circ$. Thus the same equation holds in the strong sense, and this concludes the proof of (b). (Note that we do not exclude the case of connected components of $\{u \le 1\}^\circ$ on which $u \equiv 1$.)

Since u_j tends weakly to u in $H_0^1(\Omega)$, $\|u\| \le \liminf \|u_j\|$. So to prove (c), it suffices to show that $\limsup \|u_j\| \le \|u\|$. Recall that u_j converges in C^2 norm to u in a neighborhood of $\partial\Omega$ in $\bar{\Omega}$. Let n denote the outer unit normal to $\partial\Omega$. Multiplying (3.1) by $u_j - 1$, integrating by parts, and noting that $\beta((t-1)/\varepsilon_j)\,(t-1) \ge 0$ for all t gives

$$\int_\Omega |\nabla u_j|^2\,dx \le \int_\Omega (u_j - 1)_+^p\,dx - \int_{\partial\Omega} \frac{\partial u_j}{\partial n}\,d\sigma \to \int_\Omega (u-1)_+^p\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\,d\sigma. \tag{3.4}$$

Fix $0 < \varepsilon < 1$. Taking $\varphi = (u - 1 - \varepsilon)_+$ in (3.2) yields

$$\int_{\{u > 1+\varepsilon\}} |\nabla u|^2\,dx = \int_\Omega (u-1)_+^{p-1}\,(u-1-\varepsilon)_+\,dx, \tag{3.5}$$

and integrating $(u - 1 + \varepsilon)_-\,\Delta u = 0$ over Ω gives

$$\int_{\{u < 1-\varepsilon\}} |\nabla u|^2\,dx = -(1-\varepsilon) \int_{\partial\Omega} \frac{\partial u}{\partial n}\,d\sigma. \tag{3.6}$$

Adding (3.5) and (3.6), and letting $\varepsilon \searrow 0$, we find that[1]

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega (u-1)_+^p\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\,d\sigma.$$

This together with (3.4) gives

$$\limsup \int_\Omega |\nabla u_j|^2\,dx \le \int_\Omega |\nabla u|^2\,dx$$

[1] Here we are using the well known fact that $\displaystyle\int_{\{u=1\}} |\nabla u|^2\,dx = 0$.

as desired.

To prove (d), write

$$J_{\varepsilon_j}(u_j) = \int_\Omega \left[\frac{1}{2}\,|\nabla u_j|^2 + \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right)\chi_{\{u \neq 1\}}(x) - \frac{1}{p}\,(u_j - 1)_+^p\right] dx + \int_{\{u=1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx.$$

Since $\mathcal{B}((u_j - 1)/\varepsilon_j)\,\chi_{\{u \neq 1\}}$ converges pointwise to $\chi_{\{u>1\}}$ and is bounded by 1, the first integral converges to $J(u)$ by (a) and (c). Since $0 \le \mathcal{B}(t) \le 1$ for all t,

$$0 \le \int_{\{u=1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx \le \mathcal{L}(\{u = 1\}).$$

(d) follows.

By (d) and (2.5),

$$J(u) + \mathcal{L}(\{u = 1\}) \ge \frac{\rho^2}{3} > 0$$

and hence u is nontrivial. □

4. Critical points on the Nehari manifold

Lemma 4.1. *Every nonzero $v \in C^0(\bar{\Omega}) \cap H_0^1(\Omega)$ that solves $-\Delta v = (v-1)_+^{p-1}$ in $\Omega \setminus \partial\{v > 1\}$ belongs to the Nehari manifold $\mathcal{M}$ and satisfies $J(v) > 0$.*

Proof. As before for u^ε, if $v \le 1$ in Ω, then it is harmonic and hence identically zero. Thus if v is not identically zero, $\{v > 1\}$ is a nonempty open set, where it it satisfies $-\Delta v = (v-1)^{p-1}$. As in the proof of Lemma 3.1 (c), multiplying this equation by $v - 1$ and integrating over the set $\{v > 1\}$ shows that v lies on $\mathcal{M}$. Finally, since $v \in \mathcal{M}$,

$$J(v) = \frac{1}{2}\int_{\{v<1\}} |\nabla v|^2\, dx + \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\{v>1\}} |\nabla v|^2\, dx + \mathcal{L}(\{v > 1\}) > 0. \quad □$$

Proposition 4.2. *We have*

$$c^* \le \inf_{v \in \mathcal{M}} J(v). \tag{4.1}$$

If $v \in \mathcal{M}$ and $J(v) = c^*$, then v is a mountain pass point of J.

Proof. For $v \in H_0^1(\Omega)$, set

$$v^+ = (v-1)_+, \quad v^- = 1 - (v-1)_-; \qquad v = v^- + v^+.$$

Let

$$W = \left\{v \in H_0^1(\Omega) : v^+ \neq 0, \text{ and } v^- \neq 0\right\}$$

Then $\mathcal{M} \subset W$, and for $v \in W$, we define the curve

$$\zeta_v(s) = \begin{cases} (1+s)\, v^-, & s \in [-1,0] \\ v^- + s\, v^+, & s \in (0,\infty), \end{cases}$$

which passes through v at $s = 1$. For $s \in [-1,0]$,

$$J(\zeta_v(s)) = \frac{(1+s)^2}{2} \int_{\{v<1\}} |\nabla v|^2\, dx$$

is increasing in s. There is a discontinuity in J at $s = 0$:

$$\lim_{s \searrow 0} J(\zeta_v(s)) = J(\zeta_v(0)) + \mathcal{L}(\{v > 1\}) > J(\zeta_v(0)).$$

For $s \in (0,\infty)$,

$$J(\zeta_v(s)) = \frac{1}{2} \int_{\{v<1\}} |\nabla v|^2\, dx + \frac{s^2}{2} \int_{\{v>1\}} |\nabla v|^2\, dx - \frac{s^p}{p} \int_{\{v>1\}} (v-1)^p\, dx + \mathcal{L}(\{v > 1\}) \tag{4.2}$$

and

$$\frac{d}{ds}\, J(\zeta_v(s)) = s \left[\int_{\{v>1\}} |\nabla v|^2\, dx - s^{p-2} \int_{\{v>1\}} (v-1)^p\, dx\right].$$

Define

$$s_v = \left[\frac{\displaystyle\int_{\{v>1\}} |\nabla v|^2\, dx}{\displaystyle\int_{\{v>1\}} (v-1)^p\, dx}\right]^{1/(p-2)}. \tag{4.3}$$

Thus, we see that $J(\zeta_v(s))$ increases for $s \in [-1, s_v)$, attains its maximum at $s = s_v$, decreases for $s \in (s_v, \infty)$, and

$$\lim_{s \to \infty} J(\zeta_v(s)) = -\infty. \tag{4.4}$$

For each $v \in \mathcal{M}$, (4.4) implies that we may choose $\bar{s} > 1$ sufficiently large so that $J(\zeta_v(\bar{s})) < 0$. Note that $s_v = 1$. Therefore,

$$\gamma_v(t) = \zeta_v((\bar{s} + 1)\, t - 1), \quad t \in [0,1]$$

defines a path $\gamma_v \in \Gamma$ such that

$$\max_{w \in \gamma_v([0,1])} J(w) = J(\zeta_v(s_v)) = J(v),$$

so $c^* \leq J(v)$. Thus (4.1) holds.

Next, suppose $v \in \mathcal{M}$ and $J(v) = c^*$. Let U be a neighborhood of v. The path γ_v passes through v at $t = 2/(\bar{s}+1) =: \bar{t}$ and $J(\gamma_v(t)) < c$ for $t \neq \bar{t}$. By the continuity of γ_v, there exist $0 < t^- < \bar{t} < t^+ < 1$ such that $\gamma_v(t^\pm) \in U$, in particular, the set $\{w \in U : J(w) < c\}$ is nonempty. If it is path connected, then this set contains a path η joining $\gamma_v(t^\pm)$, and reparametrizing $\gamma_v|_{[0,t^-]} \cup \eta \cup \gamma_u|_{[t^+,1]}$ gives a path in Γ on which $J < c^*$, contradicting the definition of c^*. So the set is not path connected, and v is a mountain pass point of J. This concludes the proof of Proposition 4.2. □

We can now conclude the proof of part (b) of Theorem 1.2. The limit u obtained in Lemma 3.1 belongs to $\mathcal{M}$ by Lemma 3.1 (b) and Lemma 4.1. Hence

$$\inf_{\mathcal{M}} J \leq J(u).$$

By Lemma 3.1 (d), Lemma 2.3, and (4.1), we also have

$$J(u) \leq \liminf c_j \leq \limsup c_j \leq c^* \leq \inf_{\mathcal{M}} J.$$

In all,

$$J(u) = c^* = \inf_{\mathcal{M}} J.$$

Thus, u minimizes J restricted to $\mathcal{M}$, and by Proposition 4.2 it is a mountain pass point of J. By construction u is Lipschitz continuous on $\bar{\Omega}$.

The inequalities of the preceding paragraph also show that

$$\lim_{j\to\infty} c_j = c^*.$$

This property will enable us to take the limit in the variational equations for u_j to show that u is a variational solution in the following sense.

Definition 4.3. A *variational solution* u of the Euler-Lagrange equation for J is a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \left[\left(\frac{1}{2}|\nabla u|^2 + \chi_{\{u>1\}} - \frac{1}{p}(u-1)_+^p \right) \operatorname{div} \Phi - \nabla u (D\Phi) \cdot \nabla u \right] dx = 0\,.$$

for every $\Phi \in C_0^1(\Omega, \mathbb{R}^N)$.

Note first that $c_j \to c^*$ implies $J_{\varepsilon_j}(u_j) \to J(u)$ as $j \to \infty$. Since u_j converges to u uniformly and strongly in $H_0^1(\Omega)$, we obtain

$$\lim_{j\to\infty} \int_\Omega \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx = \mathcal{L}(\{u > 1\}).$$

Hence,

$$\limsup_{j\to\infty} \int_{\{u\leq 1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx \leq \mathcal{L}(\{u > 1\}) - \liminf_{j\to\infty} \int_{\{u>1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx$$

On the other hand, because u_j tends uniformly to u,

$$\liminf_{j\to\infty} \int_{\{u>1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx \ge \mathcal{L}(\{u \ge 1 + \delta\})$$

for every $\delta > 0$. Taking the limit as $\delta \to 0$, we find that

$$\liminf_{j\to\infty} \int_{\{u>1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx \ge \mathcal{L}(\{u > 1\})$$

Therefore,

$$\limsup_{j\to\infty} \int_{\{u\le 1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) dx = 0 \tag{4.5}$$

It follows from this and the dominated convergence theorem that

$$\begin{aligned}\lim_{j\to\infty} \int_{\Omega} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) \operatorname{div}\Phi\, dx &= \lim_{j\to\infty} \int_{\{u>1\}} \mathcal{B}\left(\frac{u_j - 1}{\varepsilon_j}\right) \operatorname{div}\Phi\, dx \\ &= \int_{\{u>1\}} \operatorname{div}\Phi\, dx\end{aligned}$$

This limiting value takes care of the only potentially discontinuous term in the variational equation. The others tend to the appropriate limits because u_j tends uniformly to u and strongly in $H^1_0(\Omega)$. Thus the variational equation for u holds because it is the limit of the variational equation (2.2) for u_j. This concludes the proof of part (c) of Theorem 1.2.

5. Nondegeneracy

In this section we prove our main estimate of nondegeneracy.

Proposition 5.1. *If u is a Lipschitz continuous minimizer of $J|_{\mathcal{M}}$ that satisfies the interior Euler-Lagrange equation* (1.1)*, then u is nondegenerate as in Definition 1.3.*

Proof. For $v \in W$, ζ_v intersects $\mathcal{M}$ exactly at one point, namely, where $s = s_v$, and $s_v = 1$ if $v \in \mathcal{M}$. So we can define a continuous projection $\pi : W \to \mathcal{M}$ by

$$\pi(v) = \zeta_v(s_v) = v^- + s_v\, v^+.$$

Lemma 5.2. *For $v \in W$,*

$$J(\pi(v)) = \frac{1}{2} \int_{\{v<1\}} |\nabla v|^2\, dx + \left(\frac{1}{2} - \frac{1}{p}\right) s_v^2 \int_{\{v>1\}} |\nabla v|^2\, dx + \mathcal{L}(\{v > 1\}).$$

In particular, for $v \in \mathcal{M}$, since $\pi(v) = v$,

$$J(v) = \frac{1}{2} \int_{\{v<1\}} |\nabla v|^2\, dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\{v>1\}} |\nabla v|^2\, dx + \mathcal{L}(\{v > 1\}).$$

Proof. $J(\pi(v))$ is given by (4.2) with $s = s_v$, and

$$s_v^2 \int_{\{v>1\}} |\nabla v|^2 \, dx = s_v^p \int_{\{v>1\}} (v-1)^p \, dx. \quad \square$$

Now consider u Suppose that $B_r(x_0) \subset \{x \in \Omega : u(x) > 1\}$ and $\exists\, x_1 \in \partial B_r(x_0)$ such that $u(x_1) = 1$. Define

$$v(y) = \frac{1}{r}\,(u(x_0 + ry) - 1).$$

Our goal is to show that

$$\alpha := v(0) \geq c > 0.$$

We begin by observing that

$$0 < v(y) = \frac{1}{r}(u(x_0 + ry) - u(x_1)) \leq \frac{L}{r}|x_0 - x_1 + ry| \leq 2L \quad \forall y \in B_1(0), \tag{5.1}$$

where L is the Lipschitz constant of u in $\{u \geq 1\}$, and

$$-\Delta v = r^p v^{p-1} \quad \text{in } B_1(0).$$

Define h by

$$\begin{cases} -\Delta h = r^p v^{p-1} & \text{in } B_1(0) \\[2ex] \quad h = 0 & \text{on } \partial B_1(0). \end{cases}$$

Then $|h| \leq CL^{p-1} r^p$, and applying the Harnack inequality to $v - h + \max h$, there is a constant C depending on N and L such that

$$v(y) \leq C\,(\alpha + r^p) \quad \forall y \in B_{2/3}(0).$$

Take a smooth cutoff function $\psi : B_1(0) \to [0,1]$ such that $\psi = 0$ in $\overline{B_{1/3}(0)}$, $0 < \psi < 1$ in $B_{2/3}(0) \setminus \overline{B_{1/3}(0)}$ and $\psi = 1$ in $B_1(0) \setminus B_{2/3}(0)$, let

$$w(y) = \begin{cases} \min\{v(y), C\,(\alpha + r^p)\,\psi(y)\}, & y \in B_{2/3}(0) \\[1ex] v(y), & \text{otherwise,} \end{cases}$$

and set $z(x) = 1 + rw((x - x_0)/r)$. Since u is a minimizer of $J|_{\mathcal{M}}$,

$$J(u) \leq J(\pi(z)).$$

Since $z^- = u^-$, $z = 1$ in $\overline{B_{r/3}(x_0)}$, and $\{z > 1\} = \{u > 1\} \setminus \overline{B_{r/3}(x_0)}$, Lemma 5.2 implies this inequality can be rewritten as

$$\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\{u>1\}} |\nabla u|^2 \, dx + \mathcal{L}(B_{r/3}(x_0)) \leq \left(\frac{1}{2} - \frac{1}{p}\right) s_z^2 \int_{\{u>1\}} |\nabla z|^2 \, dx.$$

Let $y = (x - x_0)/r$ and define

$$\mathcal{D} := \left\{x \in B_{2r/3}(x_0) : v(y) > C\,(\alpha + r^p)\,\psi(y)\right\}.$$

Because $z = u$ outside $\mathcal{D}$, the last inequality implies

$$s_z^2 \int_{\mathcal{D}} |\nabla z|^2\,dx + \left(s_z^2 - 1\right) \int_{\{u>1\} \setminus \mathcal{D}} |\nabla u|^2\,dx \ge \frac{2p}{p-2}\,\mathcal{L}(B_{1/3}(0))r^N. \tag{5.2}$$

Since $\{z > 1\} = \{u > 1\} \setminus \overline{B_{r/3}(x_0)}$ and $z = 1$ in $\overline{B_{r/3}(x_0)}$,

$$s_z^{p-2} = \frac{\displaystyle\int_{\{z>1\}} |\nabla z|^2\,dx}{\displaystyle\int_{\{z>1\}} (z-1)^p\,dx} = \frac{\displaystyle\int_{\{u>1\}} |\nabla z|^2\,dx}{\displaystyle\int_{\{u>1\}} (z-1)^p\,dx}.$$

Since $z = u$ in $\{u > 1\} \setminus \mathcal{D}$, we have

$$s_z^{p-2} \le \frac{\displaystyle\int_{\{u>1\}} |\nabla u|^2\,dx + \int_{\mathcal{D}} |\nabla z|^2\,dx}{\displaystyle\int_{\{u>1\}} (u-1)^p\,dx - \int_{\mathcal{D}} (u-1)^p\,dx} = \frac{A_1 + \displaystyle\int_{\mathcal{D}} |\nabla z|^2\,dx}{A_1 - \displaystyle\int_{\mathcal{D}} (u-1)^p\,dx},$$

where, since $u \in \mathcal{M}$,

$$A_1 = \int_{\{u>1\}} |\nabla u|^2\,dx = \int_{\{u>1\}} (u-1)^p\,dx.$$

It follows as in (5.1) that $0 < u - 1 < 2Lr$ in $\mathcal{D}$, and $\mathcal{L}(\mathcal{D}) = \mathrm{O}(r^N)$ as $r \to 0$. Therefore

$$\int_{\mathcal{D}} (u-1)^p\,dx = \mathrm{O}(r^{p+N}).$$

It follows that

$$s_z^{p-2} \le 1 + \frac{1}{A_1} \int_{\mathcal{D}} |\nabla z|^2\,dx + \mathrm{O}(r^{p+N}). \tag{5.3}$$

We have

$$\int_{\mathcal{D}} |\nabla z|^2\,dx = C^2\,(\alpha + r^p)^2\,r^N \int_{\{y:x\in\mathcal{D}\}} |\nabla \psi|^2\,dy. \tag{5.4}$$

The right-hand side is $\mathrm{O}(r^N)$ since $0 < \alpha < 2L$ by (5.1). So (5.3) gives

$$s_z^2 \le 1 + \frac{2}{(p-2)A_1} \int_{\mathcal{D}} |\nabla z|^2\,dx + \mathrm{O}(r^{q+N}),$$

where $q = \min\{p, N\} \geq 2$. Using this estimate in (5.2) now gives

$$\frac{1}{r^N} \int_{\mathcal{D}} |\nabla z|^2 \, dx + \mathrm{O}(r^q) \geq 2\, \mathcal{L}(B_{1/3}(0)).$$

In view of (5.4), we find that there are r_0, $c > 0$ such that $r \leq r_0$ implies $\alpha \geq c$, which was our goal. □

Since the mountain pass solution of Theorem 1.2 satisfies the hypotheses of Proposition 5.1, we obtain the first part of Theorem 1.4. The fact that this solution is a viscosity solution now follows from results of Lederman and Wolanski.

We will define a weak viscosity solution is as follows.

Definition 5.3. We say that $u \in C(\Omega)$ satisfies the free boundary condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2$$

in the *weak viscosity sense* if whenever there exist a point $x_0 \in \partial\{u > 1\}$, a ball $B \subset \{u > 1\}$, then either there are $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $\alpha_1^2 \leq 2$ and $\alpha_2^2 \leq 2$ and

$$u(x) = 1 + \alpha_1 \langle x - x_0, \nu\rangle_+ + \alpha_2 \langle x - x_0, \nu\rangle_- + o(|x - x_0|), \quad x \to x_0,$$

with ν the interior normal to ∂B at x_0, or else there are $\alpha > 0$ and $\beta \geq 0$ such that $\alpha^2 - \beta^2 = 2$ and

$$u(x) = 1 + \alpha \, \langle x - x_0, \nu\rangle_+ - \beta \, \langle x - x_0, \nu\rangle_- + \mathrm{o}(|x - x_0|), \quad x \to x_0.$$

Moreover, if the ball $B \subset \{u \leq 1\}$, then the second asymptotic formula (with α and β as above, but with ν the exterior normal to ∂B at x_0) holds.

Denote

$$f_j(x) = -(u_j(x) - 1)_+^{p-1}, \quad f(x) = -(u(x) - 1)_+^{p-1}, \quad x \in \Omega.$$

Since u_j converges uniformly to u, f_j converges uniformly to f. Therefore, u_j solves an equation of the form (2.1) (denoted $E_\varepsilon(f^\varepsilon)$ in the paper of Lederman and Wolanski [24]). Since by Proposition 5.1, u is nondegenerate, Corollaries 7.1 and 7.2 of [24] imply that u satisfies the free boundary condition in the weak viscosity sense. Furthermore, Proposition 1.7, proved below, shows that the case $u > 1$ on both sides of the free boundary (the case of positive α_1 and α_2) is ruled out. This concludes the proof of Theorem 1.4.

6. The free boundary has finite Hausdorff measure

In this section we prove Proposition 1.6. Let u be a Lipschitz, nondegenerate solution to the interior equation (1.1). The outline and most details are the same as the proof of Theorem 3.4 of Caffarelli-Salsa [5]. The only difference is that $u-1$ solves an inhomogeneous equation $\Delta(u-1) = (u-1)_+^{p-1}$ in $\{u-1>0\}$ rather than being harmonic.

Lemma 6.1. *(See (3.4) [5]) There exists a constant $C > 0$ such that whenever $r \le \delta_0/2$, with $\delta_0 = dist(\{u>1\}, \partial\Omega)$, and $\tau > 0$,*

$$\int_{B_r(x_0)\cap\{|u-1|<\tau\}} |\nabla u|^2\, dx \le C\tau r^{N-1}.$$

Proof. Denote by L the Lipschitz constant of u on $\bar{\Omega}$ and by M the maximum of u over $\bar{\Omega}$. (This estimate does not depend on nondegeneracy.)

For $0 < \varepsilon < \tau$, let $u_\varepsilon^\tau = \min\{(u-1-\varepsilon)_+, \tau - \varepsilon\}$. Since $-\Delta u = (u-1)^{p-1}$ in $\{u > 1\}$ and $u_\varepsilon^\tau = 0$ in $\{u \le 1+\varepsilon\}$, we have $-u_\varepsilon^\tau\, \Delta u = (u-1)^{p-1}\, u_\varepsilon^\tau$ in Ω. Integrating this equation over $B_r(x_0)$ gives

$$\int_{B_r(x_0)} \nabla u \cdot \nabla u_\varepsilon^\tau\, dx = \int_{\partial B_r(x_0)} \frac{\partial u}{\partial n}\, u_\varepsilon^\tau\, d\sigma + \int_{B_r(x_0)} (u-1)^{p-1}\, u_\varepsilon^\tau\, dx,$$

where n is the outward unit normal to $\partial B_r(x_0)$. Since $u_\varepsilon^\tau = u-1-\varepsilon$ in $\{1+\varepsilon < u < 1+\tau\}$ and u_ε^τ is constant outside this set,

$$\int_{B_r(x_0)} \nabla u \cdot \nabla u_\varepsilon^\tau\, dx = \int_{B_r(x_0)\cap\{1+\varepsilon<u<1+\tau\}} |\nabla u|^2\, dx \to \int_{B_r(x_0)\cap\{1<u<1+\tau\}} |\nabla u|^2\, dx$$

as $\varepsilon \searrow 0$. We also have

$$\left| \int_{\partial B_r(x_0)} \frac{\partial u}{\partial n}\, u_\varepsilon^\tau\, d\sigma \right| \le L\tau\sigma(\partial B_r) \le c_N \tau L r^{N-1},$$

and

$$\int_{B_r(x_0)} (u-1)^{p-1}\, u_\varepsilon^\tau\, dx \le c_N \tau M^{p-1} r^N.$$

So for a constant C depending only on L, M and the diameter of Ω,

$$\int_{B_r(x_0)\cap\{1<u<1+\tau\}} |\nabla u|^2\, dx \le C\tau r^{N-1}.$$

Since u is harmonic in $\{u < 1\}$, a similar argument gives the same bound for the integral over $\{1 - \tau u < 1\}$. The conclusion follows since $\nabla u = 0$ a.e. on the set $\{u = 1\}$. □

Lemma 6.2. *(See Lemma 1.10 of* [5]*) There exist constants $r_0, \lambda > 0$ such that whenever $x_0 \in \{u > 1\}$ and $r := dist\,(x_0, \{u \leq 1\}) \leq r_0$, there is a point $x_1 \in \partial B_r(x_0)$ satisfying*

$$u(x_1) \geq 1 + (1 + \lambda)\,(u(x_0) - 1).$$

Proof. Suppose not. Then there are sequences $\lambda_j \searrow 0$ and $x_j \in \{u > 1\}$ with $r_j := \operatorname{dist}(x_j, \{u \leq 1\}) \to 0$ such that

$$\max_{x \in \partial B_{r_j}(x_j)} u(x) < 1 + (1 + \lambda_j)\,(u(x_j) - 1).$$

Since u is nondegenerate, we may assume that $u(x_j) \geq 1 + c r_j$ for some constant $c > 0$. Noting that $B_{r_j}(x_j) \subset \{u > 1\}$ and $\exists\, x'_j \in \partial B_{r_j}(x_j)$ such that $u(x'_j) = 1$, set

$$v_j(y) = \frac{1}{r_j}\,(u(x_j + r_j y) - 1), \quad y_j = \frac{1}{r_j}\,(x'_j - x_j).$$

Then $v_j \in C(\overline{B_1(0)}) \cap C^2(B_1(0))$ satisfies

$$-\Delta v_j = r_j^p\, v_j^{p-1} \quad \text{in } B_1(0), \tag{6.1}$$

$$\max_{y \in \partial B_1(0)} v_j(y) < (1 + \lambda_j)\, v_j(0), \tag{6.2}$$

$$v_j(0) \geq c, \quad v_j(y_j) = 0. \tag{6.3}$$

We have

$$0 \leq v_j(y) = \frac{1}{r_j}\,(u(x_j + r_j y) - u(x'_j)) \leq \frac{L}{r_j}\,|x_j - x'_j + r_j y| \leq 2L \quad \forall y \in \overline{B_1(0)},$$

$$|v_j(y) - v_j(z)| = \frac{1}{r_j}\,|u(x_j + r_j y) - u(x_j + r_j z)| \leq L\,|y - z| \quad \forall y, z \in \overline{B_1(0)},$$

$$\int_{B_1(0)} |\nabla v_j(y)|^2\, dy = r^{-N} \int_{B_{r_j}(x_j)} |\nabla u(x)|^2\, dx \leq N \alpha_N L^2,$$

so, for suitable subsequences, v_j converges weakly in $H^1(B_1(0))$ and uniformly on $\overline{B_1(0)}$ to some Lipschitz continuous function v, and y_j converges to some point $y_0 \in \partial B_1(0)$. For any $\varphi \in C_0^\infty(B_1(0))$, testing (6.1) with φ gives

$$\int_{B_1(0)} \nabla v_j \cdot \nabla \varphi\, dx = r_j^p \int_{B_1(0)} v_j^{p-1}\, \varphi\, dx,$$

and passing to the limit as $r_j \to 0$ gives

$$\int_{B_1(0)} \nabla v \cdot \nabla \varphi\, dx = 0.$$

So v is harmonic in $B_1(0)$. By (6.2),

$$\max_{y \in \partial B_1(0)} v(y) \leq v(0),$$

and hence v is constant by the maximum principle. On the other hand,

$$v(0) \geq c > 0 = v(y_0)$$

by (6.3), which is impossible when v is constant. □

The rest of the proof follows [5] with no change. From the preceding lemma, a chaining argument carried out in Theorem 1.9 and Lemma 3.3 of [5] gives the following:

Lemma 6.3. *There exist constants $0 < r_0 \leq \delta_0$ and $\gamma > 0$ such that whenever $x_0 \in \partial \{u > 1\}$ and $0 < r \leq r_0$, there is a point $x \in B_r(x_0) \setminus B_{r/2}(x_0)$ satisfying $u(x) \geq 1 + \gamma r$, in particular,*

$$\sup_{x \in B_r(x_0)} u(x) \geq 1 + \gamma r.$$

Next, at the beginning of the proof of Theorem 3.4 [5], the following lemma is deduced from Lemma 6.3:

Lemma 6.4. *There exist constants $0 < r_0 \leq \delta_0$ and $\kappa > 0$ such that whenever $x_0 \in \partial \{u > 1\}$ and $0 < r \leq r_0$,*

$$\int_{B_r(x_0)} |\nabla u|^2 \, dx \geq \kappa r^N.$$

The rest of the proof of Proposition 1.6 is a covering argument, exactly as in Theorem 3.4 of [5].

7. Nondegeneracy of the non-plasma phase $\{u \leq 1\}$

Now we turn to the proof of Proposition 1.7, which says that not only $\{u > 1\}$ but also $\{u \leq 1\}$ has significant measure near each topological boundary point. The measure-theoretic boundary $\partial_* E$ of a measurable set $E \subset \mathbb{R}^N$ is defined as the set of $x \in \mathbb{R}^N$ such that for all $r > 0$,

$$\mathcal{L}(E \cap B_r(x)) > 0 \quad \text{and} \quad \mathcal{L}(E^c \cap B_r(x)) > 0.$$

Evidently, the measure-theoretic boundary $\partial_* E$ is a subset of the topological boundary ∂E. The proposition is a quantitative, scale-invariant estimate showing that the topological boundary is contained in the measure-theoretic boundary.

Lemma 7.1. *Let $B = B_1(0)$ be the unit ball in R^N. Let $h \in C(\bar{B})$ be a harmonic function in the ball and such that*

$$h(0) > 0; \quad |h(x) - h(y)| \leq L|x - y|, \quad x, y \in \partial B.$$

Define

$$\varepsilon = \sigma(\partial B \cap \{h \leq 0\}).$$

There exists a constant $C > 0$ depending on dimension and L such that

$$\int_{\{h \leq 0\}} |\nabla h|^2 \, dx \leq C \left[\frac{\varepsilon}{h(0)} \right]^{1/N}.$$

Proof. Since $h(0) > 0$, there is at least one point of ∂B at which h is positive. It follows that

$$h(y) \geq -2L$$

for all $y \in \partial B$ and hence in all of B by the maximum principle.

The Poisson integral formula says

$$h(x) = \frac{1 - |x|^2}{\omega_N} \int_{\partial B} \frac{h(y)}{|x - y|^N} \, d\sigma(y), \quad x \in B,$$

in which $\omega_N = \sigma(\partial B)$. Therefore,

$$\begin{aligned} \int_{y \in \partial B \cap \{h > 0\}} \frac{h(y)}{|x - y|^N} \, d\sigma(y) &\geq \frac{1}{2^N} \int_{\partial B \cap \{h > 0\}} h \, d\sigma \\ &\geq \frac{1}{2^N} \int_{\partial B} h \, d\sigma = \frac{\omega_N h(0)}{2^N}. \end{aligned}$$

For any $\kappa > 0$ and any $x \in B_{1-\kappa}(0)$, we have

$$\int_{\partial B \cap \{h \leq 0\}} \frac{h(y)}{|x - y|^N} \, d\sigma(y) \geq -\frac{2\varepsilon L}{\kappa^N}$$

Choose

$$\kappa = 4(\varepsilon L / \omega_N h(0))^{1/N}.$$

Then for every $x \in B_{1-\kappa}(0)$, $h(x) > 0$, and hence

$$\{h \leq 0\} \subset B \setminus B_{1-\kappa}(0).$$

Denoting the spherical part of the gradient by ∇_θ, we have $|\nabla_\theta h| \leq L$. Using the expansion of h in spherical harmonics, we have

$$\int_{\partial B} |\nabla h|^2 d\sigma \leq 2 \int_{\partial B} |\nabla_\theta h|^2 d\sigma \leq 2L\omega_N$$

(In fact, the best constant is $N/(N-1)$, achieved by linear functions h.) Furthermore, since $|\nabla h|^2$ is subharmonic, for all $r<1$,

$$\int_{\partial B_r} |\nabla h|^2 d\sigma \le \int_{\partial B} |\nabla h|^2 d\sigma.$$

Therefore,

$$\begin{aligned}\int_{\{h\le 0\}} |\nabla h|^2\, dx &\le \int_{1-\kappa}^1 \int_{\partial B_r} |\nabla h|^2\, d\sigma\, dr \\ &\le 2L\omega_N \int_{1-\kappa}^1 dr = 2L\omega_N\kappa \\ &= c_N L \left(\frac{\varepsilon L}{h(0)}\right)^{1/N}\end{aligned}$$

□

We will now deduce a variant of Lemma 6.3.

Lemma 7.2. *There exist constants r_0, $c_0>0$ and $c_1>0$ such that whenever $x_0 \in \partial\{u>1\}$ and $0<r\le r_0$, there is $x_1 \in \partial B_r(x_0)$ such that*

$$u(x_1)-1 \ge c_1 r$$

Moreover,

$$\fint_{\partial B_r(x_0)} (u-1)_+\, d\sigma \ge c_0 r \tag{7.1}$$

Proof. Suppose by contradiction that there is no such c_1. Then $(u(x)-1)_+ \ll r$ on $\partial B_r(x_0)$. Note, in addition, that

$$\Delta(u-1)_+ \ge -(u-1)_+^{p-1} \ge -(Lr)^{p-1} \quad \text{in} \quad B_r(x_0).$$

Consider the barrier function v solving $\Delta v = -(Lr)^{p-1}$ in $B_r(x_0)$, with constant boundary values $v=a\ll r$ on $\partial B_r(x_0)$. Then $(u-1)_+ \le v$, and for sufficiently small r, $v\ll r$ on all of $B_r(x_0)$. But this contradicts Lemma 6.3 which says that there is a point of $B_r(x_0)\setminus B_{r/2}(x_0)$ at which $u-1$ is larger than γr.

Next, take $x_1 \in \partial B_r(x_0)$ as above for which $u(x_1) > 1 + c_1 r$. By Lipschitz continuity, $u(x) > 1 + c_1 r/2$ on $B_{c_1 r/2L}(x_1)$. Thus we have (7.1) for a constant $c_0>0$ depending only on c_1 and L. □

Lemma 7.3. *There exist a positive constants c, ε_0 and C depending on dimension and the Lipschitz constant L such that whenever $x_0 \in \partial\{u>1\}$, $0<r\le r_0$,*

$$\sigma(\partial B_r(x_0)\cap\{u\le 1\}) < \varepsilon r^{N-1} \tag{7.2}$$

for some $\varepsilon < \varepsilon_0$*, and* v *is the harmonic function in* $B_r(x_0)$ *with* $v = u$ *on* $\partial B_r(x_0)$*, we have*

$$\int_{B_r(x_0)} (v-1)_+^p \, dx + \int_{B_r(x_0)} (u-1)_+^p \, dx \leq C r^{p+N}, \tag{7.3}$$

$$\int_{B_r(x_0)} |\nabla v|^2 \, dx \leq \int_{B_r(x_0)} |\nabla u|^2 \, dx - c r^N, \tag{7.4}$$

$$\int_{\{v \leq 1\} \cap B_r(x_0)} |\nabla v|^2 \, dx \leq C \varepsilon^{1/N} r^N. \tag{7.5}$$

Proof. We have $|u-1| \leq Lr$ on $\overline{B_r(x_0)}$, and hence $|v-1| \leq Lr$ by the maximum principle. Thus (7.3) follows.

To prove (7.4), begin by noting that

$$\int_{B_r(x_0)} \left(|\nabla u|^2 - |\nabla v|^2\right) dx = \int_{B_r(x_0)} |\nabla(u-v)|^2 \, dx + 2 \int_{B_r(x_0)} \nabla(u-v) \cdot \nabla v \, dx.$$

Since $u - v \in H_0^1(B_r(x_0))$,

$$\int_{B_r(x_0)} |\nabla(u-v)|^2 \, dx \geq \frac{\lambda_1}{r^2} \int_{B_r(x_0)} (u-v)^2 \, dx,$$

where $\lambda_1 > 0$ is the first Dirichlet eigenvalue of the negative Laplacian in $B_1(0)$, and since $u = v$ on $\partial B_r(x_0)$ and $\Delta v = 0$ in $B_r(x_0)$, an integration by parts gives

$$\int_{B_r(x_0)} \nabla(u-v) \cdot \nabla v \, dx = \int_{\partial B_r(x_0)} (u-v) \, \frac{\partial v}{\partial n} - \int_{B_r(x_0)} (u-v) \, \Delta v \, dx = 0$$

Hence

$$\int_{B_r(x_0)} \left(|\nabla u|^2 - |\nabla v|^2\right) dx \geq \frac{\lambda_1}{r^2} \int_{B_r(x_0)} (u-v)^2 \, dx. \tag{7.6}$$

We have

$$|u(x) - v(x)| \geq |v(x_0) - u(x_0)| - |u(x) - u(x_0)| - |v(x) - v(x_0)|. \tag{7.7}$$

Fix $\kappa \in (0, 1/2)$. Furthermore, for

$$|u(x) - u(x_0)| \leq L|x - x_0| \leq L\kappa r, \quad x \in B_{\kappa r}(x_0) \tag{7.8}$$

Since $|v-1| \leq Lr$ on $B_r(x_0)$ and v is harmonic inside the ball, it follows that $|\nabla v| \leq C$ on $B_{\kappa r}(x_0)$, and hence

$$|v(x) - v(x_0)| \leq C\kappa r \quad \text{for all } x \in B_{\kappa r}(x_0). \tag{7.9}$$

Since $u(x_0) = 1$, the mean value property of v implies
(7.10)
$$\begin{aligned} v(x_0) - u(x_0) &= \fint_{\partial B_r(x_0)} (u-1)\, d\sigma = \fint_{\partial B_r(x_0)} (u-1)_+\, d\sigma - \fint_{\partial B_r(x_0)} (u-1)_-\, d\sigma \\ &\ge \left(c_0 - \frac{L\varepsilon}{\omega_N}\right) r \ge \frac{c_0}{2}\, r \end{aligned}$$
by (7.1), (7.2), and since $|u-1| \le Lr$ on $\partial B_r(x_0)$. Combining (7.7)–(7.10) and taking κ sufficiently small gives $|u(x) - v(x)| \ge c_0 r/3$, for all $x \in B_{\kappa r}(x_0)$, Together with (7.6), this yields (7.4).

To prove (7.5), we apply Lemma 7.1 to
$$h(y) = \frac{1}{r}\,(v(x_0 + ry) - 1), \quad y \in B,$$
noting that
$$h(0) = \frac{1}{r}\,(v(x_0) - u(x_0)) \ge \frac{c_0}{2}$$
by (7.10). □

Proof of Proposition 1.7. Let r_0 and $\gamma > 0$ be as in Lemma 6.3, let $x_0 \in \partial \{u > 1\}$, and let $0 < r \le r_0$. Then there is $x_1 \in B_{r/2}(x_0)$ such that $u(x_1) \ge 1 + \gamma r/2$. Let $\kappa = \min \big\{1/2, \gamma/2L\big\}$. Then
$$u(x) \ge u(x_1) - L|x - x_1| > 1 + \left(\frac{\gamma}{2} - L\kappa\right) r \ge 1 \quad \forall x \in B_{\kappa r}(x_1),$$
so the volume fraction of $\{u > 1\}$ in $B_r(x_0)$ of (1.2) is at least κ^N.

If the second inequality in (1.2) does not hold, then for arbitrarily small $\rho, \gamma > 0$, $\exists\, x_0 \in \partial \{u > 1\}$ such that
$$\mathcal{L}(\{u \le 1\} \cap B_\rho(x_0)) < \gamma \rho^N. \tag{7.11}$$
Then
$$\int_{\rho/2}^{\rho} \sigma(\{u \le 1\} \cap \partial B_r(x_0))\, dr < \gamma \rho^N,$$
and hence for some r, $\rho/2 \le r \le \rho$,
$$\sigma(\{u \le 1\} \cap \partial B_r(x_0)) \le 2\gamma \rho^{N-1} \le 2^N \gamma r^{N-1}$$
In other words, inequality (7.2) in Lemma 7.3 holds for $\varepsilon = 2^N \gamma$ and some $r \in (\rho/2, \rho)$.

Let v be as in Lemma 7.3, and let $w = v$ in $B_r(x_0)$ and $w = u$ in $\Omega \setminus B_r(x_0)$. Then

$$\int_\Omega |\nabla w|^2\,dx \le \int_\Omega |\nabla u|^2\,dx - cr^N, \tag{7.12}$$

$$\left| \int_{\{w>1\}} (w-1)^p\,dx - \int_{\{u>1\}} (u-1)^p\,dx \right| \le Cr^{p+N}, \tag{7.13}$$

$$\int_{\{w\le 1\}} |\nabla w|^2\,dx \le \int_{\{u\le 1\}} |\nabla u|^2\,dx + C\varepsilon^{1/N} r^N \tag{7.14}$$

by (7.4), (7.3), and (7.5), respectively.

By (7.11),

$$\begin{aligned} \mathcal{L}(\{w>1\}) &\le \mathcal{L}(\{u>1\} \setminus B_r(x_0)) + \mathcal{L}(B_r(x_0)) \\ &= \mathcal{L}(\{u>1\}) + \mathcal{L}(\{u\le 1\} \cap B_r(x_0)) \\ &\le \mathcal{L}(\{u>1\}) + \gamma\rho^N, \end{aligned}$$

Recalling $r \ge \rho/2$ and $\varepsilon = 2^N\gamma$, we have

$$\mathcal{L}(\{w>1\}) \le \mathcal{L}(\{u>1\}) + \varepsilon r^N. \tag{7.15}$$

Estimate (7.11) also implies

$$\int_{\{u\le 1\}\cap B_r(x_0)} |\nabla u|^2\,dx \le L^2\gamma\rho^N$$

which together with (7.12) gives

$$\int_{\{w>1\}} |\nabla w|^2\,dx \le \int_{\{u>1\}} |\nabla u|^2\,dx - \frac{c}{2}\,r^N \tag{7.16}$$

for sufficiently small γ.

Referring to (4.3), by (7.16) and (7.13),

$$s_w \le \left[\frac{\displaystyle\int_{\{u>1\}} |\nabla u|^2\,dx - \frac{c}{2}\,r^N}{\displaystyle\int_{\{u>1\}} (u-1)^p\,dx - Cr^{p+N}} \right]^{1/(p-2)} \le 1$$

for sufficiently small r since $u \in \mathcal{M}$. Then by Lemma 5.2,

$$J(\pi(w)) \le \frac{1}{2} \int_{\{w<1\}} |\nabla w|^2\,dx + \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\{w>1\}} |\nabla w|^2\,dx + \mathcal{L}(\{w>1\})$$

Finally, using (7.14), (7.15), and (7.16),

$$J(\pi(w)) \leq J(u) + \left[C\varepsilon^{1/N} + \varepsilon - \left(\frac{1}{2} - \frac{1}{p}\right)\frac{c}{2}\right] r^N < J(u)$$

if ε is sufficiently small. This is a contradiction, since $\pi(w) \in \mathcal{M}$ and u minimizes $J|_{\mathcal{M}}$. □

8. Proof of regularity of the free boundary

The purpose of this section is to prove Theorem 1.5. We do this by taking blow-up limits and applying a monotonicity result of G. Weiss.

Consider a boundary point $x_0 \in F(u) = \partial \{u > 1\}$. The Lipschitz continuity of u implies there is a sequence $r_j \to 0$ such that

$$w_j(y) = r_j^{-1}(u(x_0 + r_j y) - 1)$$

converges uniformly on compact subsets of $\mathbb{R}^N$ to a Lipschitz continuous function $W(y)$. We now show that W inherits all the properties we found for u.

Lemma 8.1. *a) The function W is Lipschitz continuous, uniformly in all $\mathbb{R}^N$, and solves the interior Euler-Lagrange equation*

$$\Delta W = 0 \quad in \quad \mathbb{R}^N \setminus \partial \{W > 0\}.$$

b) (nondegeneracy of W) There is $c > 0$ such that for every $r > 0$ and every y_1 such that $B_r(y_1) \subset \{W > 0\}$ we have

$$W(y_1) \geq cr.$$

For every $r > 0$ and every $y_0 \in \partial \{W > 0\}$ there is $y_1 \in B_r(y_0)$ such that

$$W(y_1) \geq cr.$$

c) (locally finite perimeter) There is a constant C such that for every ball B_r of radius $r > 0$,

$$\sigma(B_r \cap \partial \{W > 0\}) \leq C r^{N-1}.$$

d) (nondegeneracy of the phase $\{W \leq 0\}$) For every $r > 0$ and every $y_0 \in \partial \{W > 0\}$,

$$\mathcal{L}(B_r(y_0) \cap \{W \leq 0\}^\circ) \geq cr^N, \quad \mathcal{L}(B_r(y_0) \cap \{W > 0\}) \geq cr^N.$$

e) (viscosity solution) For every $r > 0$, if there is a tangent ball from either side of the free boundary, that is, a ball B_r such that $y_0 \in \partial B_r \cap \partial \{W > 0\}$ and either $B_r \subset \{W > 0\}$ or $B_r \subset \{W \leq 0\}$, then W has an asymptotic expansion as $y \to y_0$ of the form

$$W(y) = \alpha \langle y - y_0, \nu \rangle_+ - \beta \langle y - y_0, \nu \rangle_- + o(|y - y_0|),$$

with $\alpha > 0$, $\beta \geq 0$ and $\alpha^2 - \beta^2 = 2$.

f) (variational solution) W satisfies the variational equation

$$\int_{\mathbb{R}^N} \left[\left(\frac{1}{2} |\nabla W|^2 + \chi_{\{W>1\}} \right) \operatorname{div} \Phi - \nabla w (D\Phi) \cdot \nabla w \right] dx = 0,$$

for every $\Phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

Proof. All of the results except part (f) are proved by methods of Caffarelli described in [6, 7, 8] and [5]. Part (f) is proved the same way as [20] Proposition 4.2.

On any compact subset of $\{W > 0\}$, we have $w_j > 0$ for all sufficiently large j, and therefore W inherits the first nondegeneracy property of part (b) from w_j. Moreover, the equation $\Delta w_j = r_j (w_j)_+^{p-1}$ holds in a fixed neighborhood of the compact set. It follows that w_j belongs to $C^{2,\alpha}$ uniformly on the compact set. Hence a subsequence of w_j converges is C^2 to W. Taking the limit in the equation we find that $\Delta W = 0$ on $\{W > 0\}$. The second nondegeneracy property of (b) follows from the first using the Lipschitz bound and the fact that W is harmonic in the set $\{W > 0\}$.

Denote

$$E_j = \overline{\{w_j > 1\}}, \quad E = \overline{\{W\} > 1}.$$

We claim that for a suitable subsequence

$$E_j \cap \bar{B} \to E \cap \bar{B} \tag{8.1}$$

in Hausdorff distance for every ball $B \subset \mathbb{R}^N$. Choose the subsequence so that $E_j \cap \bar{B}_R$ converges in Hausdorff distance to a compact set K. We wish to show that

$$K = E \cap \bar{B}$$

Indeed, the fact that w_j converges uniformly to W implies $K \supset \{W > 0\} \cap B$ and hence, since K is compact, $K \supset E \cap \bar{B}$.

If $x \notin E$, then we now show that for sufficiently small $\varepsilon > 0$ and large enough j,

$$w_j(y) \le 0 \quad \text{for all } y \in B_\varepsilon(x). \tag{8.2}$$

Choose $\varepsilon > 0$ sufficiently small that

$$B_{2\varepsilon}(x) \cap E = \emptyset.$$

Choose $\delta \ll \varepsilon$. Since $W \le 0$ on $B_{2\varepsilon}(x)$ and w_j tends uniformly to W, for sufficiently large j,

$$w_j(y) \le \delta \quad \text{for all } y \in B_{2\varepsilon}(x)$$

Suppose by contradiction that there is $y_1 \in B_\varepsilon(x)$ such that $w_j(y_1) > 0$. By nondegeneracy (Definition 1.3) $w_j(y_1) \le \delta$ implies there is y_2, $|y_2 - y_1| \le C\delta$ such that $w_j(y_2) \le 0$. Hence there is a point $y_3 \in \partial w_j > 0$ on the segment between y_1 and y_2. By the second form of nondegeneracy, Lemma 6.3, there is a point $y_4 \in B_\varepsilon(y_3)$ for which $w_j(y_4) \ge \gamma\varepsilon$. But $y_4 \in B_{2\varepsilon}(x)$, so this contradicts $\delta << \varepsilon$.

We have just shown in (8.2) that for all sufficiently large j, $B_\varepsilon(x) \cap E_j = \emptyset$. It follows that $x \notin K$, which finishes the proof of (8.1).

Next, note that the same argument says that on compact subsets of E^c (the interior of $\{W \le 0\}$) we have $w_j \le 0$ for sufficiently large j and we can use the equation $\Delta w_j = 0$ to conclude that $\Delta W = 0$ on E^c. This concludes part (a).

By the same argument as Proposition 1.6, we have part (c). In particular,

$$\mathcal{L}(\partial E) = \mathcal{L}(\partial \{W > 0\}) = 0.$$

By uniform convergence of w_j to W we have

$$\mathcal{L}((E \setminus w_j > 0) \cap B) = \mathcal{L}((\{W > 0\} \setminus \{w_j > 0\}) \cap B) \to 0, \quad j \to \infty.$$

for every ball B. On the other hand, we just showed that on every compact subset of $\{W \le 0\}^\circ$, we have $w_j \le 0$ for sufficiently large j. From this and the fact that $\partial \{W \le 0\}$ has zero measure it follows that

$$\mathcal{L}((\{W \le 0\} \setminus \{w_j \le 0\}) \cap B) \to 0, \quad j \to \infty.$$

In all,

$$\chi_{\{w_j > 0\}} \to \chi_{\{W > 0\}} \quad \text{in} \quad L^1(B)$$

Part (d) now follows from the convergence in Hausdorff distance and the corresponding estimates for u in Proposition 1.7.

Next we turn to part (e). It follows from the methods of Caffarelli [9], Caffarelli-Salsa [5], and of Lederman-Wolanski [24] that the limit W is a solution in the weak viscosity sense of Definition 5.3. Moreover, if there is a tangent ball at y_0 from either the $\{W > 0\}$ of the $\{W \le 0\}^\circ$ side, then W has an asymptotic of the form

$$W(y) = \alpha \langle y - y_0, \nu \rangle_+ - \beta \langle y - y_0, \nu \rangle_- + o(|y - y_0|),$$

with $\alpha > 0$. From part (d), we have the additional information that $\mathcal{L}(\{W \le 0\} \cap B_r(y_0)) \ge c r^N$, which rules our the case $\beta < 0$. Thus $\beta \ge 0$, in that case, and the methods of Caffarelli also show that $\alpha^2 - \beta^2 = 2$.

Finally, we demonstrate part (f) by using the variational equation for w_j and applying the dominated convergence theorem. Recall that if K is a compact subset of $\{W > 0\}$, respectively, $\{W \le 0\}^\circ$), then for sufficiently large j, $K \subset \{w_j > 0\}$, respectively $K \subset \{w_j \le 0\}^\circ$. It follows that for large j, w_j is uniformly $C^{2,\alpha}$ on K. Thus taking subsequences, we may assume ∇w_j converges pointwise to ∇W on $\mathbb{R}^N \backslash \partial \{W > 0\}$. Since $\partial \{W > 0\}$ has Lebesgue measure zero, and the compact set K was arbitrary, we can choose the subsequence ∇w_j so that it tends pointwise almost everywhere in $\mathbb{R}^N$ to ∇W. Recall also that on a suitable subsequence, $\chi_{w_j > 0} \to \chi_{W > 0}$ in $L^1(B_r)$ for any $r < \infty$. Since the test function Φ has compact support, the dominated convergence theorem applies. Taking the limit in the variational equation, Definition 4.3, for w_j, we obtain (f).

□

The proof of Theorem 1.5 proceeds by induction on dimension. The first step ($N = 2$) requires relatively few of the conclusions of Lemma 8.1.

Consider any $x_0 \in \partial\{u > 1\}$ and $w_j \to W$ as above. Because w_j is a variational solution, the theorem of Weiss, Corollary 9.2, applies and says that the limit W is homogeneous, $W(ry) = rW(y)$ for all $y \in \mathbb{R}^N$. By Lemma 8.1 (a), W is harmonic in the cones $\{W > 0\}$ and $\{W \le 0\}^\circ$. When $N = 2$, the cones are sectors and by part (b) of the lemma, the only possibility is that for some unit vector ν, and some $\alpha > 0$ and $\beta \ge 0$,

$$W(x) = \alpha\langle x, \nu\rangle_+ - \beta\langle x, \nu\rangle_- .$$

Incidentally, it does sometimes happen that W is strictly positive on both sides of the free boundary for limits of other kinds of non-minimizing critical points, even in dimension 2 (see [20]). But as in our earlier discussion of viscosity solutions in Theorem 1.4, $\beta < 0$ is ruled out by the fact that $\mathcal{L}(B_r(0) \cap \{W \le 0\}) \ge cr^N$.

It follows from the central results in the work on free boundaries of Caffarelli that the free boundary of u is a smooth hypersurface in a neighborhood of x_0. In fact, because w_j tends uniformly to W and w_j is nondegenerate, the free boundary of u is "flat" near x_0. The solution u satisfies the free boundary condition in the viscosity sense, and hence the free boundary is smooth using the "flat implies Lipschitz" and "Lipschitz implies smooth" theorems of Caffarelli [6, 7, 8]; see also [5]. (Those theorems were carried out for zero right hand side, but can be modified to this situation without difficulty because $(u-1)_+^{p-1}$ is zero at the free boundary.) In dimension 3, we follow the inductive method of G. Weiss. Consider the cone

$$\Gamma = \{W > 0\} .$$

Let $y_0 \in \Gamma$, $y_0 \ne 0$. By the uniform Lipschitz bound on W, there is sequence $r_j \to 0$ for which the limit

$$\bar{W}(z) := \lim_{j\to\infty} r_j^{-1} W(y_0 + r_j z)$$

exists and the convergence is uniform on compact subsets of $\mathbb{R}^N$. Since the radial derivative of W is zero, one can show that $y_0 \cdot \nabla\bar{W}(z) \equiv 0$ for all $z \in \mathbb{R}^N \setminus \partial\{\bar{W} > 0\}$. Thus $\bar{W}$ is a two-dimensional solution. Furthermore, since by Lemma 8.1 (e), W is a variational solution, Corollary 9.2 with $Q \equiv 1$ implies that $\bar{W}$ is homogeneous. It follows as in the two-dimensional case that $\bar{W}$ is a planar solution and hence that Γ is smooth near y_0. It then follows that the free boundary of u is flat near every point of a punctured neighborhood of x_0. The free boundary $\partial\{u > 1\}$ is covered by finitely many balls of this type, and the free boundary is smooth except possibly at the centers of these balls. This completes the proof in dimension 3. The bound on the Hausdorff dimension of the singular set in higher dimensions follows by an induction as in G. Weiss [30]. This concludes the proof of Theorem 1.5.

Without using scale-invariance and Weiss monotonicity one can obtain a weaker, qualitative version of the preceding results. Namely, the free boundary is smooth except on a closed set of zero $(N-1)$-dimensional Hausdorff measure.

Recall that Proposition 1.7 implies that the topological boundary $\partial\{u > 1\}$ is the same as the measure-theoretic boundary. Proposition 1.6 implies that $\partial\{u > 1\}$ has finite $(N-1)$-dimensional Hausdorff measure. By the criterion for finite perimeter of Section 5.11 of Evans-Gariepy [14], the set $\{u > 1\}$ has finite perimeter, that is, $\chi_{\{u>1\}}$ is a function of bounded variation. By Lemma 1 Section 5.8 of Evans-Gariepy [14], the reduced boundary of a set of finite perimeter is of full $(N-1)$-dimensional Hausdorff measure in the measure-theoretic boundary. Since, by definition, at every point of the reduced boundary there is a measure-theoretic normal, we may apply Theorem 9.2 of Lederman and Wolanski [24]) saying that the free boundary is a $C^{1,\alpha}$-surface in a neighborhood of each point for which there is a measure-theoretic normal. (This theorem applies with the same hypotheses as the theorem about viscosity solutions, namely, that u_j tends uniformly to u and u is nondegenerate.) Thus the set of points where the free boundary is smooth is an open set of full $(N-1)$ Hausdorff measure in the free boundary.

9. Appendix: Weiss monotonicity

For completeness, we state and prove the monotonicity formula of G. Weiss in the form used here.

Proposition 9.1. *[G. Weiss] Suppose that w is a Lipschitz continuous function in the unit ball $B \subset \mathbb{R}^N$. Let $Q \in C^\alpha(B)$ for some $\alpha > 0$ be such that $Q(0) = 0$ Suppose that w satisfies the variational free boundary equation*

$$\int_B \left[\frac{1}{2}|\nabla w|^2 + Q(x)\chi_{\{w>0\}}\right] \operatorname{div}\Phi \, dx - \int_B \nabla w D\Phi \cdot \nabla w \, dx = 0$$

for every $\Phi \in C_0^\infty(B, \mathbb{R}^N)$. Suppose further that $w(0) = 0$ and $w\Delta w$ is well-defined as a distribution and satisfies

$$|w\Delta w| \le C|x|^\alpha.$$

Denote

$$\psi(r) = r^{-N} \int_{B_r} \left[\frac{1}{2}|\nabla w|^2 + \chi_{\{w>0\}}\right] dx - \frac{1}{2} r^{-1-N} \int_{\partial B_r} w^2 \, d\sigma$$

Then for every $0 < r_0 < r_1 \le 1$,

$$\psi(r_1) - \psi(r_0) = \int_{r_0<|x|<r_1} (x \cdot \nabla w - w)^2 \frac{dx}{|x|^{N+2}} + O(r_1^\alpha)$$

Proof. The proof is close to the one in G. Weiss [29]. Consider a test function $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ that is Lipschitz continuous and compactly supported in B. By taking convolution with a smooth approximate identity, we can find a sequence of test functions in $C_0^\infty(B, \mathbb{R}^N)$ whose gradients tend pointwise almost everywhere to $\nabla\Phi$. Thus, by the dominated convergence theorem, the variational equation is valid for Φ.

We will use the family of Lipschitz continuous test functions $\Phi_\varepsilon(x) = \eta_\varepsilon(x)x$, where

$$\eta_\varepsilon(x) = \begin{cases} 1 & |x| \le r - \varepsilon \\ \frac{r-|x|}{\varepsilon} & r - \varepsilon \le |x| \le r \\ 0 & r \le |x| \le 1. \end{cases}$$

We have

$$\frac{\partial}{\partial x_j}\Phi^i_\varepsilon(x) = \eta_\varepsilon(x)\delta_{ij} - \frac{x_i x_j}{\varepsilon|x|}\chi_{\{r-\varepsilon<|x|<r\}},$$

and hence

$$\operatorname{div}\Phi_\varepsilon(x) = N\eta_\varepsilon(x) - \frac{|x|}{\varepsilon}\chi_{\{r-\varepsilon<|x|<r\}}.$$

By Fubini's theorem, for almost every fixed r, $0 < r < 1$, $\nabla w(ry)$, $y \in \partial B$, is a well-defined function in $L^2(\partial B)$. Furthermore, let $\phi \in L^\infty(\mathbb{R})$ have compact support and integral equal to 1. The vector-valued maximal theorem implies that for almost every r,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int \phi((s-r)/\varepsilon)\nabla w(sy)\,ds = \nabla w(ry)$$

in $L^2(\partial B)$ norm. Therefore, we can take the limit as $\varepsilon \to 0$ in the variational formula of the hypothesis with test function Φ_ε to find for almost every r,

$$\begin{aligned} 0 = &\int_{B_r}\left[N\left(\frac{1}{2}|\nabla w|^2 + Q(x)\chi_{\{w>0\}}\right) - |\nabla w|^2\right]dx \\ &- r\int_{\partial B_r}\left(\frac{1}{2}|\nabla w|^2 + Q(x)\chi_{\{w>0\}}\right)d\sigma + r\int_{\partial B_r}\left(\frac{x\cdot\nabla w}{|x|}\right)^2 d\sigma. \end{aligned}$$

Integrating by parts, and using $|w\Delta w| \le C|x|^\alpha$,

$$\int_{B_r}|\nabla w|^2\,dx = \int_{\partial B_r} w\frac{x\cdot\nabla w}{|x|}\,d\sigma + O(r^{N+\alpha})$$

Combining these two equation, multiplying by $-r^{-N-1}$ and using $|Q-1| \le C|x|^\alpha$, we have

$$-Nr^{-N-1}\int_{B_r}\left(\frac{1}{2}|\nabla w|^2 + \chi_{\{w>0\}}\right)dx + r^{-N}\int_{\partial B_r}\left(\frac{1}{2}|\nabla w|^2 + \chi_{\{w>0\}}\right)d\sigma$$
$$= -r^{-N-1}\int_{\partial B_r} w\frac{x\cdot\nabla w}{r}\,d\sigma + r^{-N-2}\int_{\partial B_r}(x\cdot\nabla w)^2\,d\sigma + O(r^{-1+\alpha}).$$

On the other hand, for almost every r, $0<r<1$,

$$\frac{d}{dr}\left(r^{-2}\int_{y\in\partial B} w(ry)^2\,d\sigma\right)$$
$$= -2r^{-3}\int_{y\in\partial B} w(ry)^2\,d\sigma + r^{-2}\int_{y\in\partial B} 2w(ry)\,y\cdot\nabla w(ry)\,d\sigma$$
$$= -2r^{-2-N}\int_{x\in\partial B_r} w^2\,d\sigma + 2r^{-1-N}\int_{x\in\partial B_r} w\frac{x\cdot\nabla w}{r}\,d\sigma$$

Thus, ψ is absolutely continuous, and for almost every r, $0<r<1$,

$$\psi'(r) = -Nr^{-N-1}\int_{B_r}\left(\frac{1}{2}|\nabla w|^2 + \chi_{w>0}\right)dx + r^{-N}\int_{\partial B_r}\left(\frac{1}{2}|\nabla w|^2 + \chi_{w>0}\right)d\sigma$$
$$+ r^{-2-N}\int_{\partial B_r} w^2\,d\sigma - r^{-1-N}\int_{\partial B} w\frac{x\cdot\nabla w}{r}\,d\sigma$$
$$= r^{-N-2}\int_{\partial B_r}(x\cdot\nabla w - w)^2\,d\sigma + O(r^{-1+\alpha}).$$

Integrating in r finishes the proof of the proposition. □

Corollary 9.2. *Suppose that w is as in Proposition 9.1. If r_j tends to zero and*

$$\frac{1}{r_j}w(r_jx) \to W(x)$$

uniformly on compact subsets of $\mathbb{R}^N$. Then W is homogeneous of degree 1:

$$W(rx) = rW(x)$$

for all $x\in\mathbb{R}^N$.

Proof. From the proposition, we have

$$\int_{\{|x|<1\}}(x\cdot w - w)^2\frac{dx}{|x|^{N+2}} < \infty.$$

It follows that

$$\int_{\{|x|<r\}}(x\cdot w - w)^2\frac{dx}{|x|^{N+2}} \to 0 \quad \text{as} \quad r\to 0.$$

Hence, for fixed $0 < a < b < \infty$, as $r_j \to 0$,

$$\int_{a<|y|<b} (y \cdot \nabla w_j(y) - w_j(y))^2 \frac{dy}{|y|^{N+2}} = \int_{ar_j<|x|<br_j} (x \cdot \nabla w(x) - w(x))^2 \frac{dx}{|x|^{N+2}} \to 0.$$

Thus the sequence $y \cdot \nabla w_j - w_j$ tends to zero in L^2 norm on $a < |y| < b$. A subsequence tends weakly to $y \cdot \nabla W - W$, showing that $y \cdot \nabla W - W = 0$ weakly in L^2. In particular, W is homogeneous of degree 1 as a distribution on $0 < |y| < \infty$. Since W is Lipschitz continuous, it is also homogeneous in the ordinary sense. □

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