# TOWARDS A DYNAMIC MODEL OF THE NATIONAL ECONOMY

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Professor L. F. Hamilton
Secretary of the Faculty
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Dear Professor Hamilton:

In accordance with the requirements for graduation, I herewith submit a thesis entitled "Towards a Dynamic Model of the National Economy." I would like to thank Professor Forrester for suggesting the general topic and for helpful discussions.

Sincerely yours,

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ABSTRACT

The development of mathematical models of the national economy and the related problems of describing and explaining business cycles form an important part of modern economics. The economic literature contains many models of the economy. The nature of these models exhibits amazing variety. All the models, however, have the common property that they can in no sense be considered realistic models of the economy - i.e., models which accurately describe the behavior of the real world in some detail. The fact that a great deal of effort has been expended without yielding anything even approaching a realistic model points out quite clearly the difficulty involved in constructing such a model.

In this thesis we would like to review the well known models of the economy in some detail. Both the mathematical development and the intuitive foundations are presented. In studying these models we attempt to indicate how they stand in relation to a realistic model and in particular where they fail to be realistic. We also seek to ascertain what there is of value in the models that might be carried over into the development of a more realistic model. We in addition point out certain attributes which it is felt any realistic model of the economy must possess.

Chapter 1 deals with reasons for constructing models, classification of models, computational problems, and simulation. In this chapter the important concepts of nonequilibrium behavior, continuous succession of steady states, and the nature of true dynamic systems are introduced.

Chapter II presents four well known aggregated models of the economy. These are the Domar, Harrod, Tinbergen, and Klein models. The Domar and Harrod models are very simple differential and difference equation models respectively, which just study the behavior of investment and income for an equilibrium economy. The Tinbergen and Klein models are much more complicated closed models which are usually referred to as statistical models.

Chapter III discusses several differentiated models. These include both the static and dynamic Leontief models. A brief treatment of the Walras equilibrium model is also given. The Dantzig model which applies linear programming to a generalized Leontief model is also treated. The von Neumann model of an economy expanded uniformly at maximum rate is explained also. The application of relaxation phenomena to dynamic models is also treated in this chapter.

Chapter IV discusses the most important work which has been done in economics on the subject of formulating nonequilibrium dynamical equations. This is the work of Professor Samuelson and is concerned with dynamics and the related problem of dynamical stability.

Chapter V summarizes what can be learned from the models which have already been constructed and points out some of the attributes which it is to be expected that any realistic model of the economy will possess. The general conclusion reached is that all of the models thus far developed do not even come close to being able to represent the actual nonequilibrium behavior of the economy. It appears that although there are some general ideas which have evolved in relation to these models which might be usefully applied to a realistic model, there is really little if anything that can be directly carried over into the development of a realistic model.

Thesis Advisor: J.W. Forrester

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# CHAPTER I

## INTRODUCTION

### Models of the National Economy

The construction of mathematical models of the national economy and the related problems of describing and explaining business cycles (refs. 1-4) form an important part of modern economics. These subjects have long been of interest to economists and they have been pursued with especially great vigour since the depression of the 1930's. It is not difficult to find models of the national economy which have been formulated. Indeed, the economic literature is replete with such models. Models have been developed which cover the range from the very simplest imaginable involving only a single equation to very complex models involving hundreds of equations. The variety that exists in these models is amazing. There is variety in the underlying assumptions concerning the model, variety in the scope and purpose of the model, and variety in the type of mathematics which arises in the formulation of the models.

The above paragraph might suggest to the uninitiated reader that the accurate representation of the economy in terms of mathematical models has reached a very highly developed state and little remains to be done in this area. Nothing could be farther from the truth. Although a great deal of work has been expended in developing models of the economy, only the first feeble steps have been taken as yet towards developing a model which can in any sense be said to be an accurate representation of the real world. All the models mentioned above, while having great variety in their treatment of the problem, have one property in common - they in no sense yield an accurate and detailed model of the real world.

The fact that more progress has not been made in formulating models of the national economy which are realistic ones merely serves to point out the great difficulty involved in developing such models. A moments reflection will indicate why this is so. When one considers that the behavior of the economy depends on the whims and actions of millions of consumers, on the plans of hundreds of thousands of business men, on the actions of the government, on the nature of each individual industry which makes up the economy, on the agricultural phase of the economy, on the international situation, etc., it immediately becomes apparent that it is a tremendous undertaking to knit all of these together into a model which will give an accurate description of the behavior of the economy as a whole and also of its important segments as well.

First there are problems in finding relations which describe the behavior of consumers and businessmen. There are also problems in finding relations which accurately describe the operations of any single industry, etc. Even after obtaining the laws governing the operation of the economy, another very sizable undertaking looms ahead when it is necessary to gather sufficient data from the real world to accurately evaluate the parameters which must appear in the laws developed. All of the above applied to the relatively simple problem of describing a static economy. The difficulties are compounded when an attempt is made to develop a model which is general enough to describe the behavior of the economy over time. Technological change now becomes important, peoples' ideas and desires change, etc. Consequently we see that it is not at all strange that even though economists have been working on models of the economy for a number of years, no model has yet been developed which even comes close to really describing the actual economy.

In the following pages we will frequently refer to what is called a "realistic" model of the economy. It has already been suggested that the models which have been developed are not too realistic. Furthermore, in the future we will suggest that it is desirable to construct a realistic model of the economy. It is perhaps worthwhile at this point to suggest what is meant by realistic. Clearly the designation realistic is a relative term. Some models which have been formulated are more realistic than others. Also, no model can be realistic to the most minute detail. By a realistic model of the economy we will mean a model that possesses the following properties:

1. Is general enough to describe the economy in some detail. Thus a model which just explains the behavior of a single variable would not be called realistic. This idea of some detail is, of course, a relative statement also.

2. It must include both the production and consumption aspects of the economy and explain the interactions between them.

3. The relations between the variables must be developed by a consideration of the internal structure of the system and must be capable of describing the system under all possible modes of behavior.

4. Nothing must appear in the model which does not have some correspondence in the real world. In particular, there must not appear variables which cannot be measured, the model must not imply decisions or circumstances which do not correspond to actuality, and time lags and leads must be properly accounted for.

Reasons for Constructing Models of the Economy

Perhaps the most obvious reason for developing models of the national economy is to gain an understanding of just how the economy functions. A real understanding of the workings of the national economy would, of course, be of great benefit to the country as a whole. It would be most useful to the government in formulating its policies if the results could be predicted with considerable accuracy, it would be of use to business firms, and of course to the consuming public as well. If the nature of business cycles was thoroughly understood, it could be hoped that with sufficient education, the government, industry, and consumers would learn to act in such a way that these cycles could be largely if not completely eliminated while still maintaining a free economy. At the present time there is not even agreement among economists as to what suitable policies and actions should be, and hence it is not strange that the government, industry, and consumers may not act in a way to head off inflation, slumps, or other undesirable conditions.

Certain of the models which have been developed abstract from the whole economy just one or two variables which are felt to be of crucial importance. These models then postulate a hypothetical economy and show how the variables selected must behave in such an economy. While these models are of importance in pointing up the importance of certain variables, they fall far short of giving any realistic picture of the entire economy. Indeed, the model may be so hypothetical that the relations derived between the variables of interest may not even have any application to the real world.

Other models of the economy proceed along entirely different lines. They simply select a number of variables whose behavior is to be explained. The variables are not independent, but some variables depend on others of the variables. It is assumed the relations between the variables are linear with the coefficient to be determined by the data. Enough relations between the variables are found so that the number of variables is the same as the number of equations. In addition to the variables to be explained certain other "exogeneous" variables will appear. These determine the behavior of the system through time. The constants which appear in the equations are determined by statistical correlation methods. Such a model is then said to explain the behavior of the economy. Even if these models did succeed in giving the correct time behavior of the variables being explained, which they really don't seem to succeed in doing very well, they could not in actuality be said to explain the behavior of the economy. If we are to understand the operations of the economy it is necessary to have more than a set of equations. We must know why the equations are what they are. One could make an argument for the idea that if we have a set of equations which completely and accurately represents the behavior of the economy, then this is sufficient and it is unnecessary to know why the equations are what they are. The counter argument is, first of all, that it is most unlikely that a correct set of equations could ever be obtained without some understanding of what the actual mechanisms underlying the operation of the economy are. The second counter argument is that presumably these equations will need to be changed from time to time and this will not be easy to do without understanding. A final argument is, that without understanding, to make any decision at all on the economy, a tremendous computation will need to be carried out to see what the implications of this decision are.

Not all models of the economy are developed with the purpose of explaining its behavior. Some models, for example, are developed to be what might be planning models. With such models, one can with use of an assumed bill of goods which consumers will demand, determine what the production from each segment of industry in the economy must be to meet this bill of goods. Such models are quite useful to the government for planning purposes, especially in wartime when the recourses of the economy are strained and it is desired to produce as much as possible for the war effort. Some of these planning models go even farther and select the industries which will produce and those which will not according to some optimization criterion.

Quite clearly, these planning models are not of too much help in actually explaining the workings of a free economy. They would be of much more value in a socialist or collectivist state where the government can make most of the important decisions.

Some models are developed to exhibit special properties of hypothetical economies without having any immediate application to the real world whatever. For example, one model was developed to study the conditions under which an economy could grow at maximum rate with each sector of the economy expanding at the same uniform rate. Other models have been developed to study the conditions for a stable economy - one for which the behavior of the economy would be such as to return to its original path of development after being subjected to random shocks. Thus we see that models of the national economy have been constructed with many purposes in mind. In the following chapters examples will be given of each of the above types of models.

Types of Models of the Economy

Models of the economy can be classified in many ways. They can be grouped according to their purpose, for example, as was done in the preceeding section. We would in the following sections like to consider other methods of classification. Models may also be classified according to the mathematics used, according to whether they are static or dynamic (i.e., whether they explicitly involve time or not), whether they are aggregated or differentiated, etc. Any model of the national economy can be classified under each of the above headings. By studying the various ways of classifying models we will not only uncover the important aspects of any model of the economy, but we will be able to deduce certain requirements which any models which pretends to actually be a representation of the real world must meet.

Aggregated and Differentiated Models

Any model of the national economy can arbitrarily be classified as an aggregated or differentiated model. As the name implies, aggregated models deal only with aggregate variables such as national income, total investment, total savings, etc. The model attempts only to explain the behavior of these aggregate variables. No attempt is made to break down the productive capacity of the economy into various subdivisions which are more or less homogeneous in nature. Neither is there any attempt made at subdivision on the consumption side of the picture. Consumers are essentially assumed to behave as a single unit. It is immediately obvious that an aggregated model does not allow a great deal of detail concerning the workings of the economy to be included in the model. Since aggregate quantities only, appear in the model it is to be expected that such models can only in a very rough way be a true representation of the real world.

Differentiated models, on the other hand, do attempt to subdivide industry and consumers into smaller more homogeneous units. In the differentiated models thus far constructed, the main emphasis has been on breaking down industrial production rather than on breaking down the consumption side. All industrial production, for example, would be broken down into say the steel industry, chemical industry, auto industry, etc. In this way detail concerning the nature of the individual industry can be incorporated into the model. Variables which might appear in such a model would be production for each industry, etc. The aggregate quantities dealt with in the aggregated models can also be obtained in the differentiated models by summing over the industries. It would be expected that a differentiated model would do a better job of explaining the behavior of the aggregate variables than would an aggregated model.

It is quite clear that aggregated and differentiated are only relative terms. An aggregated model might have some differentiation - such as breaking production up into consumers goods and capital goods. On the other hand a differentiated model must have some aggregation for otherwise every single individual establishment in the economy would need to be separately accounted for. It seems reasonably clear that any model which will actually explain the operations of the economy will need to be a differentiated model. How differentiated it needs to be depends on how much detail one is attempting to include. When a certain segment of the economy is being studied it may be satisfactory to have considerable differentiation within this segment and not so much in other segments. In any event, it seems that in even the most realistic model of the national economy there will need to be some aggregation. As will be seen later, there are many problems involved in finding a suitable aggregation scheme.

Open and Closed Models

Models of the national economy can also be classified as being open or closed. By an open model we will mean one which treats only the productive aspects of the economy and assumes that the demand or consumption requirements are given and known. Such models cannot interrelate production, employment, and consumption to give a complete picture of the economy. It is assumed that consumer demands are known and the behavior of the productive side of the economy while striving to meet these demands is studied. There are also open models which involve prices. These models usually take wages or the value added by a given industry to be fixed and known. The model then studies the behavior of the final prices of the goods produced. The planning models of the economy always tend to be open models since they are interested in studying how industrial production or prices will behave when a given set of net requirements or wage rates are given. It cannot be expected that open models can give a complete description of the real world economy. They in a sense only treat half the problem - they do not interrelate production and consumption or prices and wages.

Closed models as the name implies attempt to treat the economy insofar as possible as a closed system. This means that both the supply and demand sides of the economy are included and interrelated. In such a model one must hope to determine both consumption and production (and/or both prices and wages). A closed model cannot be completely closed. If it was, then the future of the economy would be completely determined - and this is something we would certainly not like to believe is true of the actual economy. The variables which are completely determined by the closed model are called the endogenous variables. The variables which lie outside the closed system and are assumed completely independent of it are called exogenous variables. These exogenous variables completely determine the notion of the closed system (i.e., all the endogenous variables) through time.

The concept of a closed system is then a relative thing. How closed it is depends on how many exogeneous variables there are. In the real world, it is hard to think of very many variables which are completely independent of the behavior of the economy. The forces of nature (weather, etc.) are the only apparent ones. To a certain extent, however, it would appear that innovation and the development of radically new ideas and processes are at least in part (although certainly not completely) outside the system. A realistic model of the economy can then be expected to have some exogeneous variables, but in a complete model this number should not be too large. It would seem desirable that a really useful model would have the property that it could be opened at certain places and some variables could be considered exogeneous in order to be able to study the behavior of the system under different sorts of conditions.

Static and Dynamic Models

Static models are those in which time is not involved in an essential way. Dynamic models involve time explicitly. Static models can thus describe the economy only at a single instant of time or over some short interval of time (perhaps a year). Dynamic systems on the other hand, have built into them the provisions for allowing changes with time. The difference between static and dynamic models leads to a fundamental difference in the mathematics that must be employed in developing the model. This will be discussed below. Systems which are truly dynamic must involve the variables at two different points in time. Static models can involve time, but in these models it appears only as a parameter and not as an independent variable. Any model which is static cannot give an accurate representation of the real world through time; a realistic model must be dynamic in nature. Closely allied with the idea of static and dynamic models are the concepts of equilibrium and nonequilibrium behavior which we will now consider.

## Equilibrium and Nonequilibrium

Static models of the economy are always equilibrium models. As in physics or thermodynamics, we can think of equilibrium as a state in which there are no unbalanced forces acting on the system. Equilibrium is essentially a static and timeless concept. A chemical reaction has reached equilibrium when there are no net changes in composition with time (there are no net changes, but individual molecules are imagined to always be changing). A bridge is in equilibrium under the forces acting on it and does not change with time. Under certain circumstances a system can be in equilibrium and still be changing with time. For example, a ball rolling along a frictionless table top under the action of no forces is in equilibrium. It moves with constant velocity. This is a very degenerate case of dynamics, however.

A system which is truly dynamic moves under the action of unbalanced forces. This would seem to be true whether economic dynamics or the mechanics of physical systems is being discussed. Very few models of the economy allow for the existence of unbalanced forces. Most of them assume that at each instant of time there is a balance of forces. Thus more general modes of behavior of the economy cannot be treated.

This idea of assuming a balance of forces at each instant of time can be approached in another way. Consider the diffusion equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = k \frac{\partial T}{\partial t}$$

This equation represents, for example, the unsteady state flow of a compressible gas through a porous medium. The above equation is truly dynamic and represents the flow under unbalanced forces. Instead of bringing in the time in this dynamic way, we can imagine a system for which at each instant of time the forces will be balanced, but the magnitude of the forces changes with time. In other words, we bring in the time only through the boundary conditions say $T(x_o,y) = f_1(t)$; $T(x_1,y) = f_2(t)$. Then the diffusion equation simply becomes Laplace's equation

$$\nabla^2 T = 0$$

An investigation of the conditions under which $\partial T/\partial t$ can be dropped from the diffusion equation shows that this can be done if changes at the boundary are transmitted to the interior with a velocity which is essentially infinite with respect to the diffusion velocity and if disturbances on the boundary are propogated to the interior without distortion of their amplitude. In other words the approximation holds if there are no time lags and if there is no distortion of signals or information which originates on the boundary. A flow in which time appears only in the boundary conditions is called a continuous succession of steady states.

The above ideas can be applied to models of the national economy. A model which assumes equilibrium at each instant of time (no unbalanced forces), I will call a continuous succession of steady states model. Essentially (in part) what the model assumes is that there are no time lags in the system and information is transmitted without distortion. The continuous succession of steady states models, in a rough way, have equilibrium assumptions such as having supply equal to demand at each instant of time, full employment of each instant of time, or expected investment equaling realized investment. In these equilibrium models, mistakes are never made. Industry produces just what is going to be demanded, precisely the right amount of investment is made, etc.

It should be obvious by now that a continuous succession of steady states model cannot be expected to yield an accurate description of the real world. The actual economy seems by no means to be a continuous succession of steady states. If we abandon the continuous equilibrium assumption, then it is necessary to formulate the dynamical laws which represent the motion of the economy under the action of unbalanced forces. This important question has not received very much attention from economists as yet. I will call a situation under which unbalanced forces are at work a nonequilibrium condition. It seems likely that the problem of determining nonequilibrium behavior of the economy will be at least as difficult as that of the laws of nonequilibrium Thermodynamics. Any realistic model must be able to describe nonequilibrium behavior. It must, as a single example, be able to describe the readjustment process when there is overproduction in some industry.

Mathematics Involved

Models of the economy can finally be classified according to the type of mathematics used in the model. The models we will consider in the following two chapters show that quite a variety of mathematics can appear. There is clearly a very close connection between the nature of the model as it could be described according to the points we discussed above and the mathematics employed.

Some of the models just involve a set of simultaneous linear equations (or more generally, nonlinear equations). Such models cannot be dynamic.

They are then either static or continuous succession of steady states models. Another model is a linear programming model. Such a model is clearly a planning model since optimization is involved. We would not expect a realistic model of the economy to be a linear programming model (or nonlinear programming model). There is no reason to believe that some invisible hand is continually guiding the economy in such a way as to maximize or minimize some function (such as utility, perhaps). Other models involve a single differential equation of a system of differential equations, while finally some models involve a single difference equation or a set of difference equations.

As is well known, differential equations are used to represent dynamical nonequilibrium behavior in mechanics. It might thus be expected that differential equations would be a natural means of expressing the relations between the variables in a realistic model of the economy. It must be recognized, however, that there is a certain lack of continuity in the economy. Decisions on investment, etc. are not made continuously, but at discrete times. When aggregated over a large number of such decisions, however, the behavior comes rather close to being continuous.

Another means of representing dynamical behavior is by means of difference equations. Difference equations change the variables on the basis of a finite time step whereas differential equations induce continuous changes. If one imagines any variable written in terms of its Fourier integral, the only difference between describing a variable in terms of a difference equation and a differential equation is in the representation of the high frequencies w in the Fourier integral

$$x(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} F(w) e^{-iwt}\, dw \qquad (1.1)$$

The difference equation using a time interval $\Delta t$ cannot allow any frequencies higher than $1/2\Delta t$. The differential equation allows all frequencies to appear. The very high frequencies are of no importance in the economy since because of the semi discreteness changes do not take place too rapidly. Hence either difference equations or differential equations should be satisfactory for representing the behavior of the economy.

It should be noted, however, that it is to be expected that the problem will be solved on a computer. In order to do this the differential equations must be converted into difference equations. Hence, for this reason, if for no other, it would seem wise to begin the formulation in terms of difference equations and avert the trouble of converting differential equations to difference equations. There is in addition a possible danger if someone not familiar with the system makes the conversion because there are many ways of representing a given set of differential equations as difference equations and the answer can be dependent on which representation is chosen.

Time lags will be very important in a realistic model of the economy since neither information nor materials can be transmitted from one point to another in zero time. The proper representation of time lags is then important in any model of the economy. In general, there are two possible modes for represanting time lags -- these are discrete and continuously distributed time lags. The time lags introduced in all the models that will be studied are of the discrete variety. A discrete time lag is one where a variable at time t depends on the value of the same or a different variable at time t', i.e. for example:

$$I_t = k\, J_{t'} \quad .$$

As a more concrete example, suppose that it takes precisely three days to process an order. Then if $P_t$ is the number of orders that are finished processing on day t and $R_t'$ is the number of orders received on day t', then

$$P_t = R_{t-3}$$

If a certain order is shipped on day t and received on day t', then there is a discrete time delay of t'-t in shipment. This can be represented diagramatically as shown in Fig. 1.1.

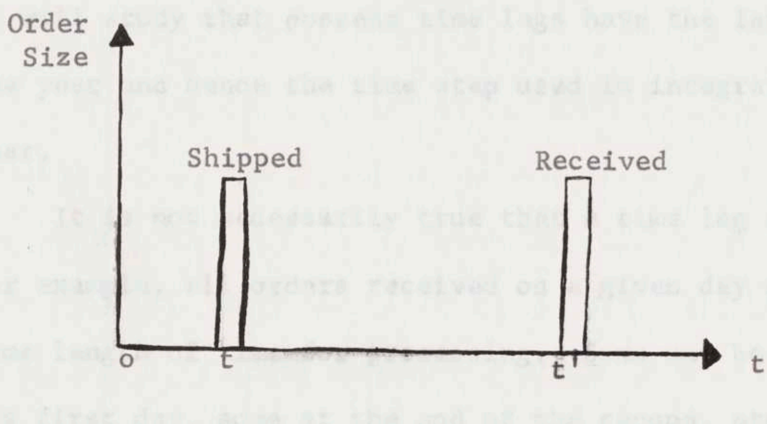



Figure 1.1

In considering time delays, it can be imagined that the input variable I goes into a black box which causes some sort of delay and yields an output 0. It is convenient to imagine that the black box operates on the input to yield the output. This can be represented symbolically as

$$0 = \frac{1}{\underset{\sim}{D}} I \qquad (1.2)$$

where $\underset{\sim}{D}$ is a symbolic operator. We have written the relation in the above form because it is often more convenient to consider the relationship between the variables in the form

$$\underset{\sim}{D}\,0 = I \qquad (1.3)$$

If there is a discrete time lag so that $O_t = I_{t'}$, then $\underset{\sim}{D}$ is just a shift operator $\underset{\sim}{D}\, O_{t'} = O_t$.

When the time delays are discrete as discussed above, then the difference equations with a proper choice of the time unit can have the form

$$B_t = f(B_{t-1}, B_{t-2}, \ldots)$$

The smallest time lag then forms a natural time step in the integration of the difference equations. No frequencies higher than $\frac{1}{2\Delta t}$, $\Delta t$ being the smallest time lag will appear. Most of the economic models which we will study that possess time lags have the lags being multiples of one year and hence the time step used in integrating the models is one year.

It is not necessarily true that a time lag needs to be discrete. For example, all orders received on a given day may not require the same length of time for processing. Some may be finished at the endof the first day, some at the end of the second, etc. Similarly, if a number of orders are simultaneously shipped to several destinations, they need not arrive everywhere at the same time. The pattern of receipts might then look something like that shown in Fig. 1.2.

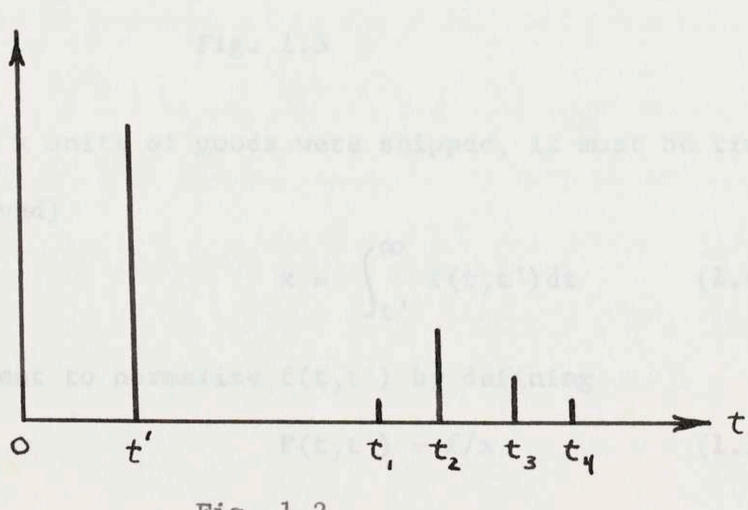



Fig. 1.2

Thus the time delay is not fixed but varies depending on where the goods are shipped.

If now we are not interested in differentiating between the destinations to which goods are shipped we can imagine receipt of the goods in terms of a distributed time lag. Instead of discrete destinations we might imagine a large number so that the distributed time lag can be thought of as a density function $f(t,t')$ with $f(t,t')dt$ being the quantity of goods shipped on $t'$ and received between $t$ and $t+dt$. The continuous case would then look something like that shown in Fig. 1.3.

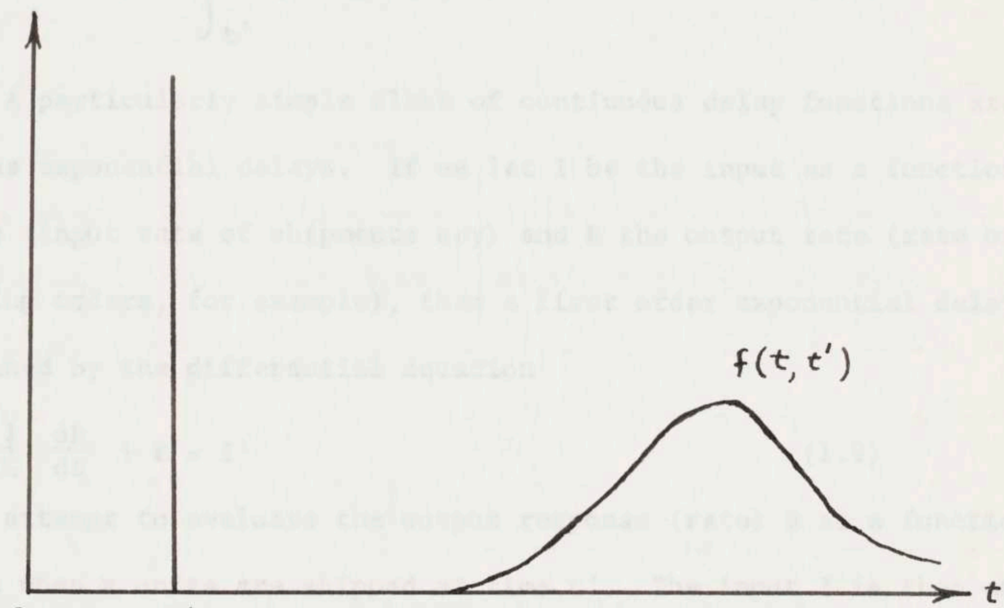



Fig. 1.3

If a total of x units of goods were shipped, it must be true that(assuming all are received)

$$x = \int_{t'}^{\infty} f(t,t')dt \qquad (1.4)$$

It is convenient to normalize $f(t,t')$ by defining

$$F(t,t') = f/x \qquad (1.5)$$

and hence

$$\int_{t'}^{\infty} F(t,t')dt = 1 \qquad (1.6)$$

Suppose we are shipping at a constant rate R(t'). The fraction that will take t-t' to arrive is R(t') F(t,t')dt. The number in transit that will take time t-t' to arrive is R(t')(t-t')Fdt. Consequently the total number in transit will be

$$Q = R(t') \int_{t'}^{\infty} (t-t')\ Fdt \qquad (1.7)$$

Let us define $1/\lambda$ as the mean delay time so that

$$Q = R/\lambda$$

Then from (1.7)

$$1/\lambda = \int_{t'}^{\infty} (t-t')\ F(t,t')dt \qquad (1.8)$$

A particularly simple class of continuous delay functions are known as exponential delays. If we let I be the input as a function of time (input rate of shipments say) and R the output rate (rate of receiving orders, for example), then a first order exponential delay is defined by the differential equation

$$\frac{1}{\lambda}\frac{dR}{dt} + R = I \qquad (1.9)$$

Let us attempt to evaluate the output response (rate) R as a function of time when x units are shipped at time t'. The input I is then an impulse and can be represented by the Dirac delta function

$$I(t) = x\ \delta(t-t') \quad ; \quad \delta(t-t') = \begin{cases} 0 & t < t' \\ 0 & t > t' \end{cases} \qquad (1.10)$$

$$x \int_{t'-}^{t'+} \delta(t-t')dt = x$$

For $t<t'$, $R=0$. For $t>t'$, $I=0$ and we must solve

$$\frac{dR}{dt} + \lambda R = 0$$

or $$R = R(t')e^{-\lambda(t-t')}$$

To evaluate $R(t')$ we integrate (1.9) from $t'-$ to $t'+$. This gives because of (1.10) and since $R(t'-) = 0$

$$\frac{1}{\lambda} R(t'+) = x = \frac{1}{\lambda} R(t')$$

so

$$R(t) = x\lambda e^{-\lambda(t-t')} \tag{1.11}$$

If a unit quantity is the input then

$$R = \lambda\, e^{-\lambda(t-t')} \tag{1.12}$$

This is the response of first order exponential delay to a unit impulse. Note that

$$\lambda \int_{t'}^{\infty} (t-t') e^{-\lambda(t-t')} dt = + \int_{0}^{\infty} e^{-\lambda\eta} d\eta = \frac{1}{\lambda}$$

Hence $\lambda$ is the mean time delay.

Consider now the case where I, the input is some arbitrary function of t. Taking the Laplace transform of (1.9) we have

$$\bar{R} = \frac{1}{(\frac{1}{\lambda} s + 1)} \quad \bar{I}$$

and using the inverse Laplace transform

$$R(t) = \int_{0}^{t} \lambda e^{-\lambda(t-t')} I(t')dt' \tag{1.13}$$

Thus if the response to a unit impulse function is known, the response to any input can be found according to (1.13).

In addition to first order delays we can formulate what are known as nth order exponential delays. To do this we just feed the output from one black box representing a first order delay into another one representing a first order delay. Mathematically the output $R_n$ from an n order delay with the same $\lambda$ in each delay to a unit impulse is

$$\frac{1}{\lambda} \frac{dR_n}{dt} + R_n = R_{n-1} \tag{1.14}$$

$$\frac{1}{\lambda} \frac{dR_1}{dt} + R_1 = \delta(t-t')$$

The solution to the general linear first order equation can be written

$$R_n = c\, e^{-\lambda t} + \lambda\, e^{-\lambda t} \int R_{n-1} e^{\lambda t}\, dt \tag{1.15}$$

By induction we easily see that $R_n$ for a unit impulse at $t'=0$ is

$$R_n = \frac{\lambda^n}{n!}\, t^n\, e^{-\lambda t} \tag{1.16}$$

since

$$\frac{dR_n}{dt} = \frac{\lambda^n}{(n-1)!}\, t^{n-1}\, e^{-\lambda t} - \frac{\lambda^{n+1}}{n!}\, t^n\, e^{-\lambda t}$$

and consequently (1.14) does hold. For an n order delay it is customary to use as the mean delay time $\frac{1}{n}$ times that for a first order delay, i.e. $\lambda_n = n\lambda$.

There is an especially useful relation between the output rate R for a first order delay and the quantity Q in the delay. Note that

$$Q = \int_o^t (I - R)\,dt \tag{1.17}$$

Hence from (1.9)

$$\frac{1}{\lambda}\int_o^t \frac{dR}{dt}\, dt = \frac{1}{\lambda}\, R(t) = Q(t) \tag{1.18}$$

since by assumption $R(0) = 0$. Consider now a second order delay. We can write

$$\frac{dR_2}{dt} = 2\lambda[R_1 - R_2] \quad ; \quad \lambda_2 = 2\lambda$$

Now

$$Q(t) = \int_o^t (I - R_2)\,dt = \int_o^t (I - R_1) + \int_o^t (R_1 - R_2) = K_1 + K_2$$

and

$$R_1 = 2\,\lambda\, K_1 = 2\lambda[Q - K_2] = 2\lambda[Q - \frac{R_2}{2\lambda}]$$

so

$$\frac{dR_2}{dt} = 2\lambda\,[2\lambda Q - 2R_2] = 4\lambda[\lambda Q - R_2] \tag{1.19}$$

Similar type formulas can be developed for higher order delays. Equations (1.18) and (1.19) are especially useful in actually working with delays. Unlike discrete delays, continuous delays are more closely related to differential equations rather than difference equations. In order to properly represent continuous delays in terms of difference equations there is no natural time step. Instead it is essential to make the time step small enough so that terms of high enough frequency can be included to yield an accurate representation of the delay.

In a realistic model of the national economy it is quite possible that both discrete and continuous time lags will be important. When there is considerable aggregation individual discrete time lags are smoothed out to yield an essentially continuous time lag distribution. Under certain circumstances, however, discrete time lags may still be important.

It should be observed that just because a system involves differential or difference equations, that does notimply that the system is truly dynamic. For example, if in some continuous succession of steady states model two variables are to be equal at each instant of time, i.e.

$$S(t) = D(t)$$

then it must be true that the time derivatives are equal also

$$\frac{dS}{dt} = \frac{dD}{dt}$$

In this way differential equations (or difference equations) can appear.

In mechanics, the basic equations are second order differential equations. The coordinates correspond to levels (such as inventories) in economics and the velocities to rates. Thus in a realistic model of the economy we would expect the forces at work to directly affect the second derivative of the levels and hence give second order equations.

This is true. However, in economic systems it is possible to immediately carry out one integration and express the change in levels in terms of the rates. In mechanics this would be equivalent to

$$x_2 - x_1 = \int_1^2 v\,dt$$

The reason we can carry out one integration immediately is that forces in the economy act to change rates and instead of requiring a finite time to induce the change can do so essentially instantaneously. For example, one can essentially instantaneously change shipping rates. Production rates cannot be changed immediately in general and this situation corresponds more directly to mechanics. In this case the actual time required to induce a production rate change can be important.

In summary of the above discussion, we can say that the nature of the assumptions which enter into any model of the economy are very closely related to the type of mathematics used. We saw that it is reasonable to expect that a realistic model of the economy would involve a set of differential or difference equations. It was also observed that time lags both continuous and discrete time lags can be important.

<u>Use of Computers</u>

Until as recently as four or five years ago economists were severely limited in the type of model they could formulate if there was any hope that the model could be solved. This led to the formulation of rather simple and naive models. In addition, the models were almost always linearized, i.e. to yield linear equations, or linear differential or difference equations. It was recognized, of course, that the models were oversimplified and that the linearized form was not necessarily a good approximation. However, it was impossible to obtain any concrete results from the model unless such approximations were made and hence they were made.

With the advent of high speed digital computers it has become possible to solve, in minutes, problems of such a complexity that a few years ago the problem of solution was considered hopelessly difficult. The development of these computers has thus permitted the construction of much more complex and realistic models than was heretofore possible. The equations can be made as nonlinear as necessary and with sufficient time available the computer can obtain the answer. It does not seem that the development of computers has stimulated a vigorous new approach to the problems of dynamic models of the economy on the part of economists which might have been expected.

While present day computers have allowed a tremendous increase in the complexity of problems which can be solved, one must be careful to note that there are still many many problems which lie beyond the ability of present day machines to solve. One such problem almost definitely seems to be a comprehensive dynamic model of the economy. If the economy was broken down into just one hundred industrial subunits with ten or twenty difference equations for each industry, this would yield 1000-2000 simultaneous difference equations. We have not even included the breakdown on the consumption side. The solution of such a problem is quite a ways beyond the capabilities of present generation computers. However, the rate of advance of computer technology is very rapid and it is not unlikely that by the time a very general model of the economy has been formulated, a computer will be available which can handle it.

Simulation

Simulation is to a certain extent a new word for a very old idea. The old idea is that mathematical equations can often be developed to describe the behavior of a physical system. The new idea associated with the word simulation is closely connected with the use of computers.

If we have a complex physical system, one possible way of studying its behavior is to actually build the system and operate it under realistic conditions. When the behavior of the system can be accurately represented by a system of equations (differential, integral, or whatever) then another way of studying the system is to have a computer actually simulate the behavior of the system by solving the equations. This procedure has been used, for example, in simulating flights of guided missiles. There are equations for the inertial guidance, for the auto-pilot, the computer, etc. By simulating flights on a computer it can be determined whether or not the autopilot is actually stable, etc. Such a procedure is much cheaper than actual flights and can be carried out under more closely controlled conditions. Simulation is a perfectly good way of studying the problem, if the equations are correct. The results are no better than the accuracy of the equations.

What realistic models of the economy attempt to do is simulate the behavior of the economy through the use of mathematical equations. Naturally planning models, equilibrium models, etc. do not attempt to simulate the behavior of the actual economy. In using a realistic model of the economy one would actually be simulating the behavior of the actual economy for a given set of conditions using the computer.

There is a certain danger in these simulation methods using a computer. It is very easy to reach a situation where the computer is printing out reams of data which require an array of people just to classify or plot. Much of this data turns out actually to be worthless and this could have easily been seen ahead of time by a person who had a little physical insight into the problem. There is the tendency with simulation to let the computer do all the work, rather than having one attempt to gain an insight into what the equations actually mean and from this be able to qualitatively predict what will happen under given conditions. As Prof. Gilliland has pointed out some of the greatest gains in chemical engineering have come about because someone in computing something by hand had enough insight to see some general relationship. With a computer printing out millions of numbers it is very easy to overlook such general relationships because one gets lost in the mass of data.

Feedback and Control Systems

Another recent development in electrical engineering and mechanical engineering which is becoming of interest in economics is servo-mechanism theory. The most important concept in servomechanism theory is that of feedback. Feedback occurs in a system where there is what might be called a closed loop. A change in the output of the system is through the closed loop used to induce a further change in the system. Feedback can be classified as positive or negative. If we are, for example, attempting to control some variable, then if we use the deviation between the actual value of the variable and the desired value as an input to reduce the error we have negative feedback. Negative feedback has a stabilizing influence. With positive feedback any error is fed back to yield a bigger error. Positive feedback can lead to an explosive situation. An example of positive feedback in the economy is the wage-price spiral. Any increase in wages tends to increase prices which tends to induce another increase in wages, etc.

The uses of servomechanism theory in economics has been pointed out in a number of references (refs. 5, 6, for example). As far as the development of a model of the national economy goes, it appears that the greatest contribution of servomechanism theory will be to help in the understanding of how feedback influences the behavior of the economy. Since we are not trying to design an economy, servomechanism design principles will not be of great use. Furthermore even if we were attempting to see how the economy could be stabilized, we would expect the relations to be nonlinear and the state of the art for nonlinear servo theory is not very highly advanced as yet.

Industrial Dynamics and the Model of Interest

As Professor Forrester has pointed out (ref. 7), the survival and health of an individual firm is very closely associated with the economy as a whole. Consequently in assisting firms to make better decisions and to operate more successfully, it is desirable to have a model which accurately simulates the behavior of the real world. Models which are planning models or equilibrium models are not so much of interest as a model which represents in considerable detail the behavior of the actual economy. In our previous discussion we have developed some properties which must be possessed by any model which is at all realistic. These properties include the following which the model must contain:

1. It must be a differentiated model.
2. It must be a closed model.
3. It must be a dynamic and nonequilibrium model.

In our later discussion more properties will be developed which the model might be expected to possess. We have also seen that in all probability the model will be able to be expressed as a set of simultaneous difference equations (with, perhaps, some integral equations appearing).

Our cursory examination of models of the economy has made it rather clear already that the task of formulating such a model is quite a formidable one. There is needed a detailed understanding of the operations of all industries, and understanding of how businessmen make decisions, of why consumers act as they do, etc. The dynamical laws represented nonequilibrium behavior must be found. Enough equations must be developed to give a closed system. After the formulation is completed, there will be large data gathering problems, and mathematical problems concerned with determining the best values of the parameters. Then, finally, there will be the rather large task of coding the system for a computer and checking out the code. In all likelihood, the initial formulation will not be entirely correct and a rather long period of trial and correction will be needed. Such an undertaking is something which clearly cannot be completed in a few months or even a few years. It will require a number of people working many years to complete the task.

The Road Ahead

In order to better understand what economists have done in the formulation of models of the national economy we will devote the next three chapters to a review of the most important methods and theory. As has already been indicated, there are many more models in existence than could be covered in a work of reasonable length. The number of very well known models is relatively small and many of these will be discussed. An attempt will be made to give an intuitive description of the model as well as to present it in mathematical terms.

In Chapter 2, aggregated models of the economy will be treated. The models discussed are the following:

1. Domar
2. Harrod
3. Tinbergen
4. Klein

While all the models are aggregated, they fall into two distinct groups. The Domar and Harrod models are very simple models involving just a single differential or difference equation respectively. They are equilibrium models and only attempt to discuss the behavior of two variables -- income andinvestment. The Tinbergen and Klein models are much more complicated and involve a rather large set of simultaneous difference equations. They are closed models and do allow for nonequilibrium behavior. These models employ the methods of statistical multiple correlation analysis to determine the many parameters.

In Chapter 3 differentiated models are considered. The models treated are the following:

1. Walras' Model
2. Static input - output system (open and closed)
3. Dynamic input - output system
4. Dantzig's model
5. von Neumann's model
6. Relaxation phenomena

The Walras model is a static equilibrium model involving a set of simultaneous functional equations. The static input-output system is a planning model which requires the solution of a set of simultaneous linear equations to find the production or prices for each industry. The dynamic model is also a planning model which is represented by a system of first order differential equations. Dantzig's model is a linear programming (planning model). Von Neumann's model is essentially a game theory model and studies a special aspect to equilibrium growth. Relaxation phenomena do not represent a special model of the economy but instead represent one particular phase of difficulty which can arise in any realistic differentiated (or even aggregated) model. It is illustrated by application to a simple dynamic input-output model.

Chapter 4 discusses some of the efforts which have been made to treat what is here called nonequilibrium behavior. This chapter also mentions the stability problem. It is based mostly on the work of Professor Samuelson.

Chapter 5 presents a summary of what seems to be needed in a realistic model of the economy. The requirements are drawn from the results of studying the models of the economy that have been developed and what we know about the real world.

# CHAPTER 2

## AGGREGATED MODELS OF THE NATIONAL ECONOMY

### Introduction

In this chapter we would like to discuss several well known models of the national economy which are very highly aggregated in nature. By highly aggregated it is meant that the models deal only with such aggregate quantities as national income, total investment, etc. No attempt is made to break down the productive capacity of the economy into various segments and to consider the interactions of these segments as well as their individual influences on the whole economy. Because of their aggregative nature, it is quite clear that very little detail concerning the operation of the economy can be included in the models.

The models to be discussed take various forms--one is represented by a differential equation, another by a difference equation, and another by a set of simultaneous difference equations. All of the models are dynamic, but two of them are equilibrium type models representing a continuous succession of steady states. Two other models actually represent true dynamic behavior and allow for non-equilibrium behavior. However, in the way in which they are formulated, it is not easy to discern just what the dynamic laws are which are implied in the models. All the models have the property that they are linear models -- i.e. they involve either linear differential or linear difference equations. Two of the models are statistical in nature and rely on the methods of correlation analysis to determine the many parameters which appear in the model. The other models have only a very small number of parameters and hence do not lean heavily on the use of statistics.

We will begin by studying the Domar model of the national economy which is perhaps the simplest of all the models to be considered.

## The Domar Model

### Introduction

The Domar model of the national economy (refs 8-10) concerns itself with the problem of determining how investment must change with time if a full employment economy is to be maintained. He postulates a relation between the rate of change of productive capacity and the investment rate and from this deduces how investment must grow with time. This model contains only a very few variables which are actually of interest in describing the behavior of the economy and it attempts to shed light on only one very special problem--that of investment in a full implyment economy. It is in no sense a model which can "explain" the workings of any actual economy.

Domar's model of the national economy is a differential equation model. This immediately implies the assumption that all changes take place continuously in time and that decisions on such things as investment are being made and altered continuously. For such a highly aggregated model, this is probably not a really bad assumption.

The following assumptions are also made explicitly:

1. There is a constant general price level

2. No time lags are present

3. Savings and investment refer to the income of the same period.

4. Both are net, ie. over and above depreciation

5. Depreciation is measured with respect to replacement costs

6. There exists a numerical quantity known as productive capacity P which is a measurable concept.

Mathematical Formulation

Equilibrium is defined by Domar as follows: The economy will be said to be in equilibrium when its productive capacity P equals its national income Y. It is assumed that employment is a function of the ratio of national income to productive capacity. Thus we see that here equilibrium growth means a full employment economy.

It will be assumed that investment I = I(t) is dollars per year of net investment at time t. The following quantity is then defined

$$\sigma = \frac{dP/dt}{I} \tag{2.1}$$

It will be assumed that $\sigma$ is a constant. $\sigma$ is a measure of rate of change of productive capacity per unit investment rate. The dimensions of $\sigma$ are reciprocal time.

Now for equilibrium, the national income Y must be equal to the productive capacity P, i.e.

$$Y = P \tag{2.2}$$

Thus to maintain equilibrium

$$\frac{dY}{dt} = \frac{dP}{dt} \tag{2.3}$$

(This is just the continuous succession of steady states idea). Domar now goes on to relate the change in national income to the change in investment. This just relies on the familiar multiplier theory i.e.

$$\frac{dY}{dt} = \frac{1}{\alpha}\frac{dI}{dt}, \quad 0 < \alpha \leq 1 \tag{2.4}$$

$\alpha$ will be assumed constant. $\alpha$ is just the marginal propensity to save. Combining (2.1), (2.3), and (2.4) we get

$$\frac{dI}{dt} = \sigma\alpha I \tag{2.5}$$

or

$$I(t) = I(t_o) \exp \sigma\alpha (t-t_o) \tag{2.6}$$

The interesting result is then obtained that net investment must grow at an exponential rate to maintain full employment.

There are several important things with regard to (2.6) which Domar does not mention. If national income at all times is only a fraction of productive capacity i.e.

$$Y = \beta P \quad , \quad 0 < \beta < 1 \tag{2.7}$$

then

$$I(t) = I(t_o) \exp \alpha\beta\sigma (t-t_o) \tag{2.8}$$

Since (2.7) implies that we do not have full employment, we see that even so we still obtain an exponential growth law if employment is not to decrease further.

Let us now depart from the Domar model itself and consider what happens if the equilibrium conditions are not satisfied. According to Domar, if the growth is not exponential, then a continually greater and greater fraction of the productive capacity will be left unused. If the rate of growth is greater than the equilibrium rate, then there will be chronic shortages of goods. There is something rather strange in this exponential growth law. If productive capacity is to grow exponentially, this means that either the working force is growing exponentially, the productivity per man is growing exponentially, or some proper combination of these is taking place. Actually there is no reason at all why such a phenomena should occur. The question is then to point out where the difficulty is. Suppose we know that because of population growth limitations, and the fact that worker productivity is not increasing exponentially that the maximum P as function of time ($P_m(t)$) is known. Then for equilibrium

$$Y(t) = P_m(t) \tag{2.9}$$

and by the multiplier

$$\frac{dI}{dt} = \alpha \frac{dPm}{dt} \tag{2.10}$$

or

$$I - I_o = \alpha \int_{t_o}^{t} \frac{dP_m}{dt}\, dt = \alpha\, [P_m(t) - P_m(t_o)] \tag{2.11}$$

This equation determines the maximum investment that can be absorbed by the economy. It means that for the case being discussed (2.1) breaks down. In other words, the amount invested cannot depend on the rate of change of productive capacity as a direct proportion. Looked at another way, if no people are available, one can try and invest all the money he wants and the productive capacity will not change. It then seems to me that this situation causes a breakdown in the assumption of a constant price level. If more money for investment is available than there is manpower to carry on the work, prices will tend to rise and inflation will set in. We thus see that the Domar model has some important assumptions in it which need not be realized in the actual economy.

We have covered above the most important result of the Domar model. Domar does introduce some additional mathematical developments and for sake of completeness these will now be included. A new quantity capital (K) is defined as the sum of all net investments i.e. capital is essentially all the capital holdings of the economy (buildings, equipment, etc.)

$$K = K(t_o) + I_o e^{-rt_o} \int_{t_o}^{t} e^{rt}\, dt = K_o + \frac{I_o e^{-rt_o}}{r} \left(e^{rt} - e^{rt_o}\right) \qquad (2.12)$$

where we assume I grows exponentially but not necessarily at the equilibrium rate

$$I = I_o \exp\ r(t-t_o) \qquad (2.13)$$

and r is not necessarily the equilibrium value of $\alpha\sigma$. It is now assumed that marginal quantities are equal to the average so that

$$\frac{I}{Y} = \alpha\ , \quad \frac{P}{K} = S \qquad (2.14)$$

S is a new quantity defined by the above and is assumed to be a constant. Then

$$Y = \frac{1}{\alpha} \quad I_o \exp r(t-t_o) \tag{2.15}$$

$$\frac{Y}{K} = \frac{I_o}{\alpha} \exp r(t-t_o) \Big/ K_o + \frac{I_o}{r} [\exp r(t-t_o) -1] \tag{2.16}$$

Then

$$\lim_{t \to \infty} \frac{Y}{K} = \frac{r}{\alpha} \tag{2.17}$$

and

$$\lim_{t \to \infty} \frac{Y}{P} = \frac{r}{\alpha S} \tag{2.18}$$

For $\sigma = S$ (2.19)

$$\lim_{t \to \infty} \frac{Y}{P} = \frac{r}{\alpha\sigma} = \theta \tag{2.20}$$

$\theta$ is called the coefficient of utilization. When the economy grows at the equilibrium rate $\theta = 1$ and productive capacity is fully utilized. When r falls below $\alpha\sigma$ a fraction of capacity $1 - \theta$ is gradually left unused. We essentially proved this same result in a much more simple way when we showed that if a fraction $\beta$ of the productive capacity was to be used then it was still necessary to have an exponential growth law. This additional analysis has shown what the limiting ratio of income to capital, and productive capacity to capital will be even when the growth is not an equilibrium growth.

These results are not nearly as interesting as the equilibrium growth law, however. It can be noted, incidentally, that the final limit is independent of the initial value $Y_o/P_o$. Thus $\Theta$ can be approached either from above or from below. It is not necessary that this system start from equilibrium. Secondly we note that (2.19) is equivalent to making the marginal equal the average since

$$\frac{dP}{dK} = \frac{dP/dt}{dK/dt} = \frac{dP/dt}{I} = \sigma = \frac{P}{K} = S \tag{2.21}$$

Domar finally considers the case where $\sigma < S$. He then claims that $I(S-\sigma)/S$ can be looked upon as capital losses which are not taken into account in calculating income and investment. Capital is now defined as investment minus capital losses i.e. from (2.21)

$$\frac{dK}{dt} = I - \frac{I(S-\sigma)}{S} = I\frac{\sigma}{S} \tag{2.22}$$

or

$$K = K_o + I_o \frac{\sigma}{rS} [\exp r(t-t_o) - 1] \tag{2.23}$$

Here again it follows that

$$\lim_{t \to \infty} \frac{Y}{K} = \frac{r}{\alpha}\frac{S}{\sigma} \quad ; \quad \lim_{t \to \infty} \frac{Y}{P} = \frac{r}{\alpha\sigma} \tag{2.24}$$

This completes our mathematical study of the Domar model. His main result was the exponential growth of investment and national income as a condition for equilibrium. We saw that this might not be consistent with population growth or the increase in worker productivity.

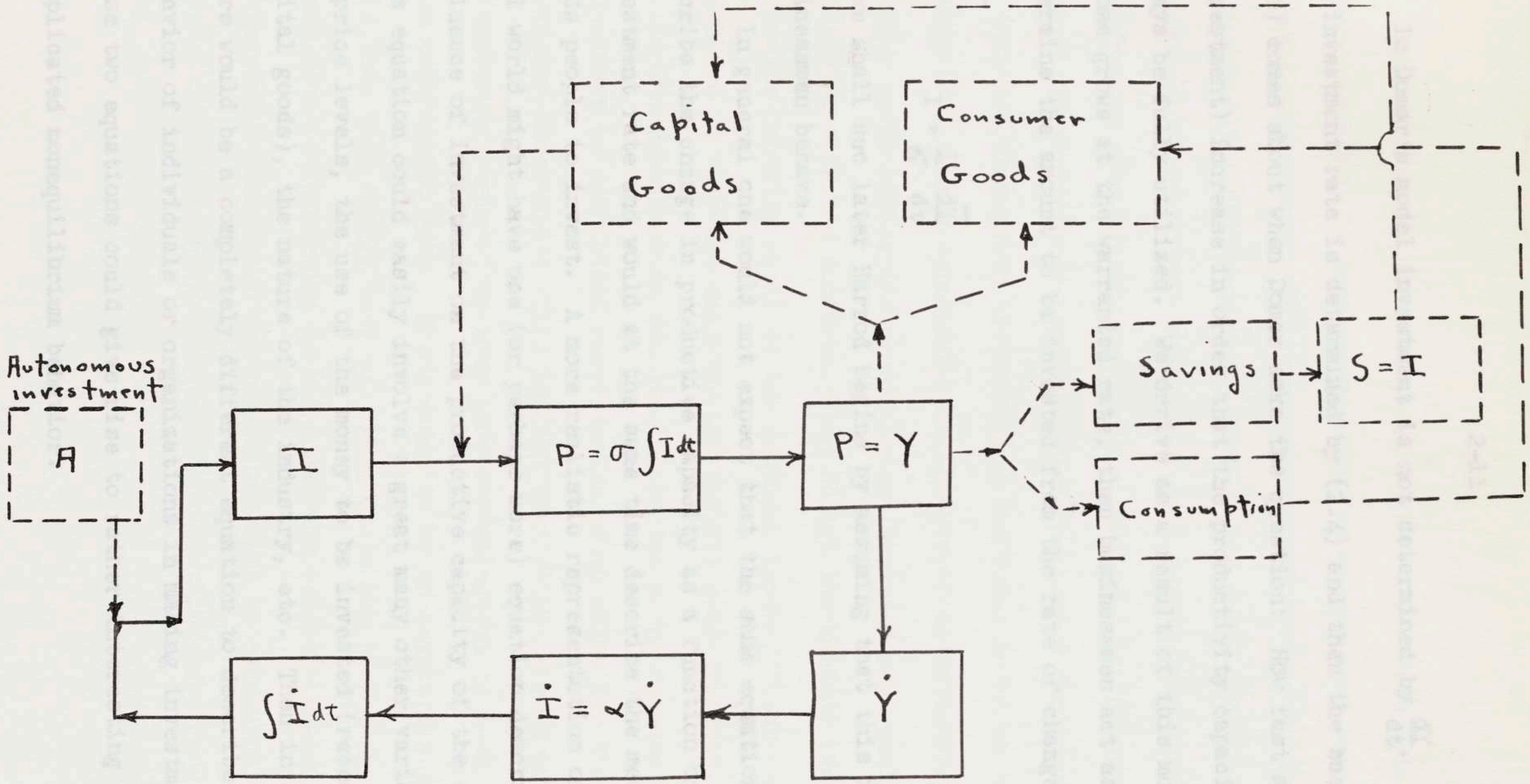
The implications of the last part of his analysis are not so interesting it seems as the first part.

In Figure 2.1 is given a block diagram of the Domar model. It illustrates the closed loop nature of the model. This is our interpretation--not Domar's.

<u>Discussion of the Domar Model</u>

The importance of any economic model quite frequently lies not in the equations themselves but rather in the economic interpretation of the equations. This is largely true of the Domar model. Clearly, by any stretch of the imagination, the equations cannot be considered to accurately describe the behavior of the economy. It is only in the interpretation of the equations that one can hope to gain some insight into the workings of the economy as a whole. As a matter of fact, there are a number of interpretations which can be given to the equations which have been set down. Here we shall present Domar's interpretation. Later, in connection with the Harrod model, we shall present Harrod's interpretation--which differs from Domar's.

To Domar (2.1) is a physical law stating that a certain rate of investment I will bring about a certain rate of change in the productive capacity. That is, investment influences not the productive capacity directly but the rate of change of productive capacity. Naturally it is not hard to question the reality of his proposed physical law. Qualitatively it certainly seems reasonable that something of this sort is true, however. In some vague sort of way technological change and other neglected factors might be accounted for by making $\sigma$ a function of t.




Block Diagram for Domar Model

Fig. 2.1.

In Domar's model investment is not determined by $\frac{dY}{dt}$. Instead the investment rate is determined by (2.4) and then the basic equation (2.5) comes about when Domar asks the question: How fast must income (investment) increase in order that the productivity capacity will always be fully utilized. We derive as a result of this model that if income grows at the warranted rate, then businessmen act as if they determine the amount to be invested from the rate of change of income i.e.

$$I = \frac{1}{\sigma} \frac{d\bar{Y}}{dt}$$

As we shall see later Harrod begins by assuming that this is the way businessmen behave.

In general one would not expect that the same equation would describe the change in productive capacity as a function of the investment rate and would at the same time describe the mechanism which leads people to invest. A more realistic representation of the real world might have one (or perhaps more) equation describing the influence of investment on the productive capacity of the economy. This equation could easily involve a great many other variables such as price levels, the use of the money to be invested (research or capital goods), the nature of the industry, etc. Then in addition there would be a completely different equation to describe the behavior of individuals or organizations in making investment decisions. These two equations could give rise to rather interesting and complicated nonequilibrium behavior.

While the Domar model in no sense is an accurate model of the national economy, it does focus attention on one very important variable--investment. It gives a result which seems to be at least in part qualitatively correct. That is, in order for the economy to be healthy, growing, and maintaining full employment, investment should be an ever increasing function of time. If investment starts to drop off the economy always seems to be headed for trouble.

Another assumption hidden within the Domar model is that the economy will always correctly make the proper distribution between consumer goods and capital goods which is needed for equilibrium growth. If this was not true, even though the economy had the productive capacity equal to the national income, there would be unbalances in both the consumer and capital goods markets.

As was mentioned in the discussion the model assumes that there will always be sufficient labor available to allow for the exponentially increasing investment. This implies that either worker productivity, the number of workers, or some proper combination of both is growing at an exponential rate. There seems to be no reason why in the real world that this should be true. When it is not true then the model breaks down (or according to the model the economy breaks down). It was suggested above that the breakdown might be evidenced by an increase in prices which violates the assumption of a constant price level.

## Harrod's Model

### Formulation

We now wish to consider another model proposed by Harrod (refs. 11-15). This is really very similar to the Domar model. The Harrod model is based on a difference equation rather than a differential equation, however. This means that decisions are not being continuously altered in time, but instead are made at times which are separated by equal time intervals. We will see that the results are essentially the same as those given by the Domar model.

The equilibrium assumption for this model is slightly different from the Domar model. Harrod assumes that equilibrium is given by

$$S(t) = I(t) \tag{2.25}$$

where $S(t)$ is the intended and realized net savings for the period $t$ and $I(t)$ is the intended investment. This relation is not so straight forward as it seems at first glance and its understanding requires a brief explanation of ex ante and ex post values. The income $Y$ for any period is broken up into consumption and savings and hence we can write

$$Y = C + S$$

This $Y$ is also the value of the goods produced. These goods are either sold as consumer goods, or investment goods, or are not sold and hence appear as inventories (perhaps unintended investment on the part of the manufacturer). Consequently, we can also write

$$Y = C + I$$

and hence at the end of the period or ex post savings is equal to investment as definitional

$$S = I$$

This is not what (2.25) means, however, as we shall now see.

At the beginning of the period, i.e. ex ante, there is no reason at all why savings should be equal to investment. In other words intended savings need not equal intended investment. The decisions to save and invest are often made by different people. When they are not equal ex ante then the economy will automatically cause unintended savings or investment. What Harrod really says is that intended savings are always realized. Then if the economy is to be in equilibrium, intended investment must be the same as realized investment or intended investment must be equal to realized savings.

Harrod next assumes that savings are a fixed fraction of income i.e.

$$S(t) = \alpha Y(t) \tag{2.26}$$

where $\alpha$ is a known constant.

The crucial assumption in this model is the one which relates intended investment to national income. It is a relation which purports to describe how investors decide on their intended investment. This relation suggests that investment decisions are based on the rate of change of national income. Mathematically the expression for intended investment becomes

$$\sigma I(t) = Y(t) - Y(t-1) \tag{2.27}$$

where $\sigma$ is a constant. This relation says that intended (ex ante) investment for the period t will be proportional to the difference between income for the period and income for the previous period. The relation (2.27) has come to be known as the accelerator relation.

We can now combine (2.25), (2.26), (2.27) to give

$$\sigma\alpha\, y(t) = y(t) - y(t-1) \tag{2.28}$$

or

$$(1 - \sigma\alpha)\, y(t) - y(t-1) = 0 \tag{2.29}$$

This is a linear homogeneous difference equation of first order. The solution is of the form

$$y(t) = ca^t \tag{2.30}$$

Substituting into (2.29) gives

$$[(1 - \sigma\alpha)\, a - 1]a^t = 0 \tag{2.31}$$

or

$$a = {}^1/_{1 - \sigma\alpha} \tag{2.32}$$

Then

$$y(t) = y_o\left(\frac{1}{1 - \sigma\alpha}\right)^t \tag{2.33}$$

since at $t = 0$, $y = y_o$. It should be remembered that t is not continuous but takes on discrete values. Since

$$I(t) = \alpha\, y(t) \tag{2.34}$$

we have

$$I(t) = I_o\left(\frac{1}{1 - \sigma\alpha}\right)^t \tag{2.35}$$

If we define

$$M = 1/1 - \sigma\alpha \tag{2.36}$$

We can then illustrate graphically the behavior of (2.33) for the various possibilities of M. These are shown in Figures (2.2)-(2.5) on pages 2-17 and 2-18.

Having seen how Y behaves for various values of M, it is now desirable to decide what the economic meaning of the various cases is. It will be recalled that $M = (1 - \sigma\alpha)^{-1}$. Equation (2.27) does not require that $\sigma$ be either greater than or less than 1. On the other hand $\alpha$ is expected to be between 0 and 1. If $\sigma\alpha > 1$ and $M < 0$ then Y oscillates between positive and negative values. This is not permissible if Y is measured in absolute terms. It could have economic significance, however, if Y was measured relative to some level $Y_o$. When $\sigma\alpha < 1$ then $M > 1$ and an explosively increasing Y is obtained. In order to have $0 < M < 1$ it is necessary that $\alpha$ or $\sigma$ be negative. If people saved less, absolutely as income went up, for example, then $\alpha$ could be negative. Thus it is also possible to leave $0 < M < 1$ and have Y decrease with time. It will be observed that the solutions to the difference equation were richer than the solution to the differential equation of the Domar model.

We thus see that the Harrod model also yields an exponentially increasing national income in the normal growth situation. This model is based on the same sort of equilibrium ideas as the Domar model.

Fig 2.2

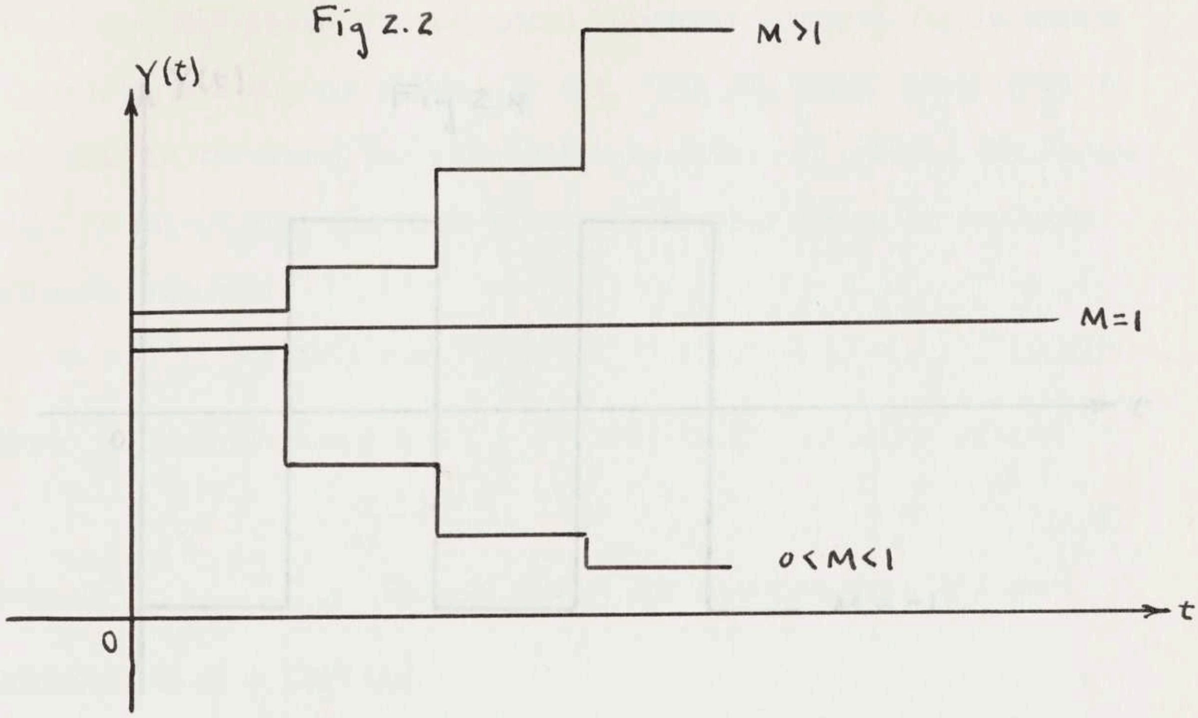
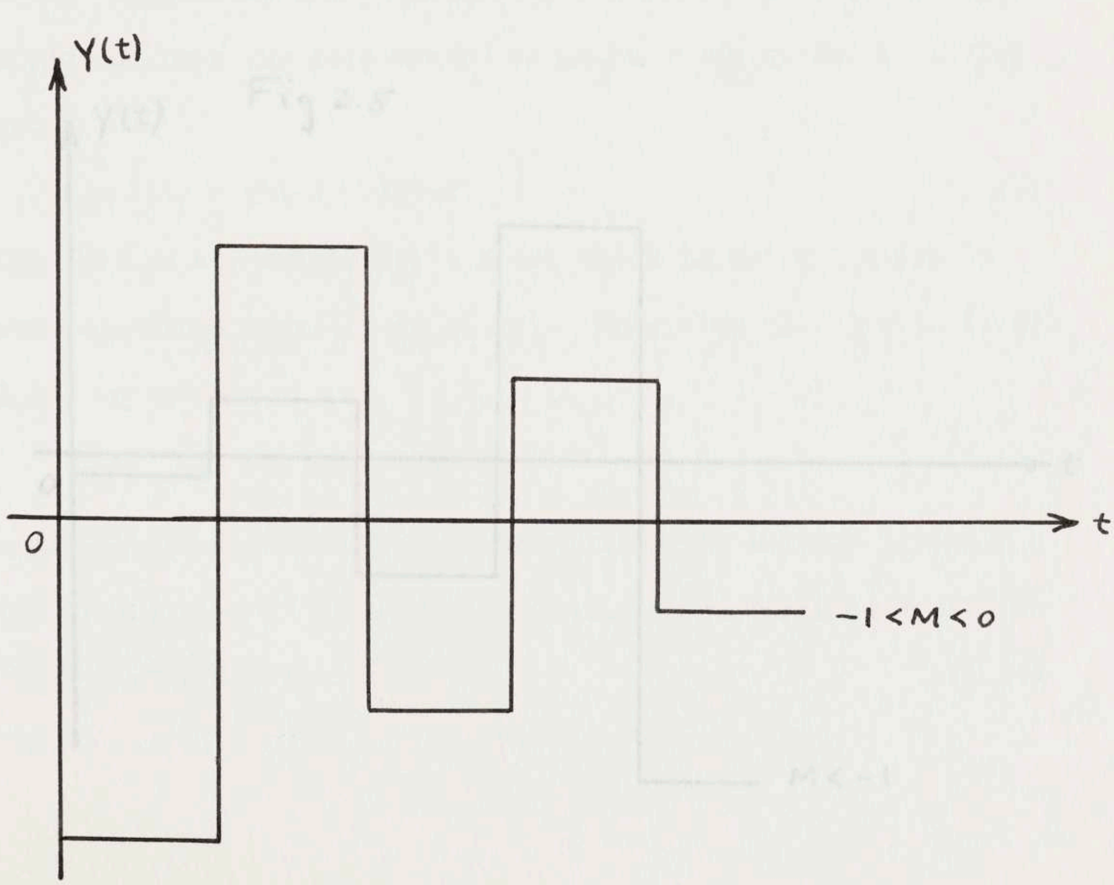



Fig 2.3



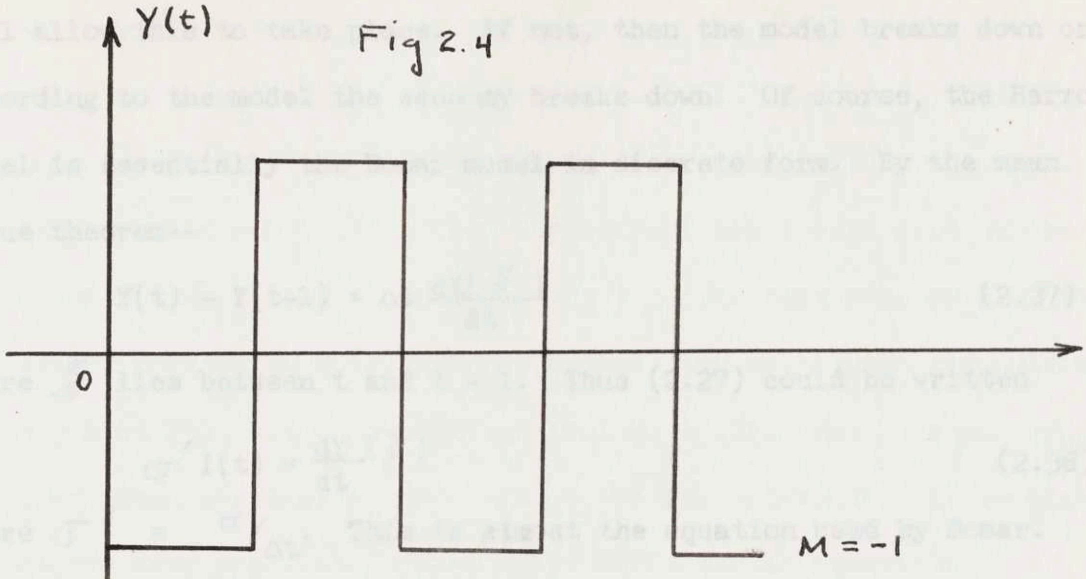



Fig 2.4

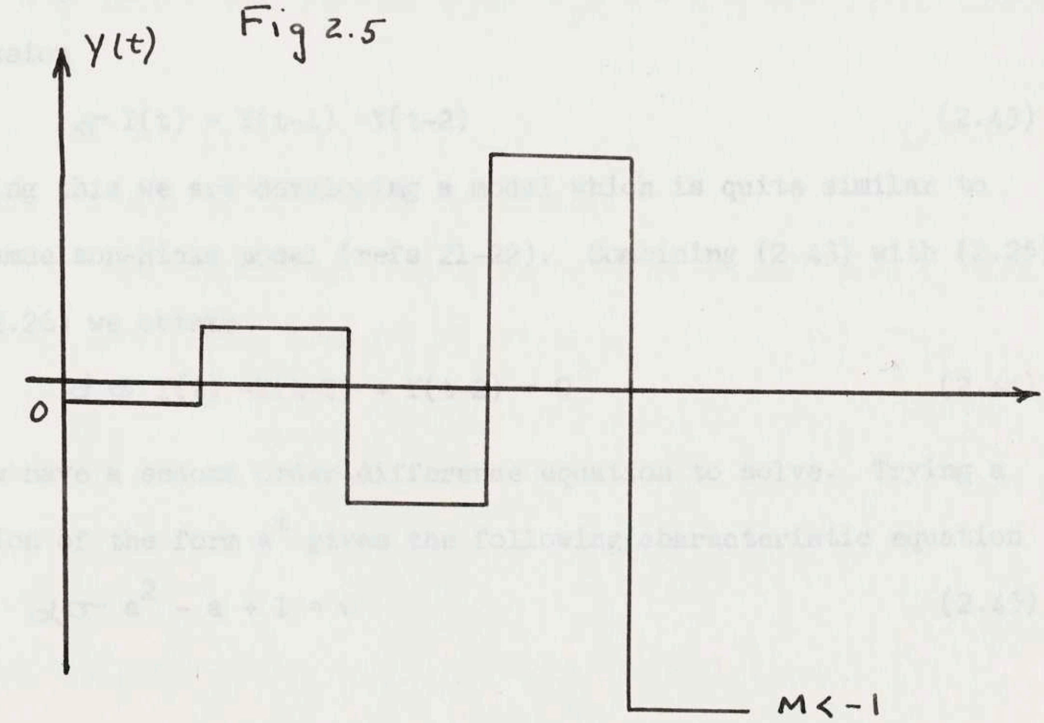



Fig 2.5

It assumes that population increases or worker productivity increases will allow this to take place. If not, then the model breaks down or according to the model the economy breaks down. Of course, the Harrod model is essentially the Domar model in discrete form. By the mean value theorem--

$$Y(t) - Y(t-1) = \Delta t \frac{dY(\xi)}{dt} \qquad (2.37)$$

where $\xi$ lies between t and t - 1. Thus (2.27) could be written

$$\sigma' I(t) = \frac{dY}{dt}(\xi) \qquad (2.38)$$

where $\sigma' = \sigma/_{\Delta t}$. This is almost the equation used by Domar.

Introduction of a Time Lag

As Harrod formulated his model, investment for the period t depended on Y(t) - Y(t-1). However, in reality, the investor will not know Y(t). Suppose we then construct a new model in which we introduce a time lag and have the accelerator relation given by the following expression

$$\sigma I(t) = Y(t-1) - Y(t-2) \qquad (2.43)$$

In doing this we are developing a model which is quite similar to the Samuelson-Hicks model (refs 21-22). Combining (2.43) with (2.25) and (2.26) we obtain

$$\alpha \sigma Y(t) - Y(t-1) + Y(t-2) = 0 \qquad (2.44)$$

We now have a second order difference equation to solve. Trying a solution of the form $a^t$ gives the following characteristic equation

$$\alpha \sigma a^2 - a + 1 = 0 \qquad (2.45)$$

or $$a \pm = \frac{1}{2\alpha\sigma} \pm \frac{1}{2\alpha\sigma} \sqrt{1 - 4\alpha\sigma} \qquad (2.46)$$

The general solution is of the form

$$y(t) = c_+ a_+^t + c_- a_-^t \qquad (2.47)$$

Let us now investigate what kinds of solutions we can have. Suppose that $\alpha\sigma < .25$. Then the radical has a real root and we end up with both $a_+$ and $a_-$ positive and $> 1$. In this case we have exponentially increasing solutions which look just about like the case for $M > 1$ in Fig. 2.2. Now suppose that $\alpha\sigma = .25$. Here our method of solution given above gives only one of the two solutions. As we might expect from the theory of ordinary differential equations

$$t\, a^t$$

should be a solution. This is easily seen to be true since the characteristic equation is

$$\sigma\alpha t\alpha^2 - (t-1)a + t-2 = 0 \quad t(\sigma\alpha a^2 - a+1) + a - 2 = 0 \qquad (2.48)$$

This is true since whene $\sigma\alpha = .25$, $a_\pm = 2$. Thus the general solution is

$$y(t) = a^t[c_+ + c_- t] \qquad (2.49)$$

which is also exponentially increasing.

A very interesting case arises when $\alpha\sigma > .25$. Then $a_\pm$ have imaginary parts, i.e. they are of the form

$$a_\pm = e \mp id \qquad (2.50)$$

A complex number can also be written in polar form, i.e.

$$a_\pm = R(\cos\theta \pm i \sin\theta) \qquad (2.51)$$

$$R = (e^2 + d^2)^{\frac{1}{2}} \ ; \ \theta = \tan^{-1}\frac{d}{e} \qquad (2.52)$$

Thus the solution for imaginary roots can be written

$$y(t) = c_+R^t\,(\cos\theta + i\sin\theta)^t + c_-R^t(\cos\theta - i\sin\theta)^t \qquad (2.53)$$

$$= c_+R^t(\cos t\,\theta + i\sin t\,\theta) + c_-R^t(\cos t\,\theta - i\sin t\,\theta)$$

$$= R^t[(c_+ + c_-)\cos t\,\theta + i\,(c_+ - c_-)\sin t\,\theta]$$

or

$$y(t) = R^t[A\cos t\,\theta + B\sin t\,\theta] \qquad (2.54)$$

Now for

$$.25 < \sigma\,\alpha < .5$$

$$R > 1$$

and one has y(t) increasing exponentially but with oscillation. A plot of y(t) may look like that shown in Fig. 2.6

For

$$.5 < \alpha\,\sigma$$

$$R < 1$$

and y(t) is decreasing with oscillation. y(t) may then look like that shown in Fig. 2.7.

From our new model of the national economy we have obtained essentially the same results as those given by the more simple Harrod model. The only difference lies in the values of $\alpha\,\sigma$ for which changes from one type of behavior to another occur. We now might ask what the physical interpretation of the change from growth to oscillation. Essentially what the results say is that if savings are too large a fraction of income or if income changes too much for a unit investment, the economy will be unstable. We have thus seen that introducing a time lag did not bring about any funds that were new.

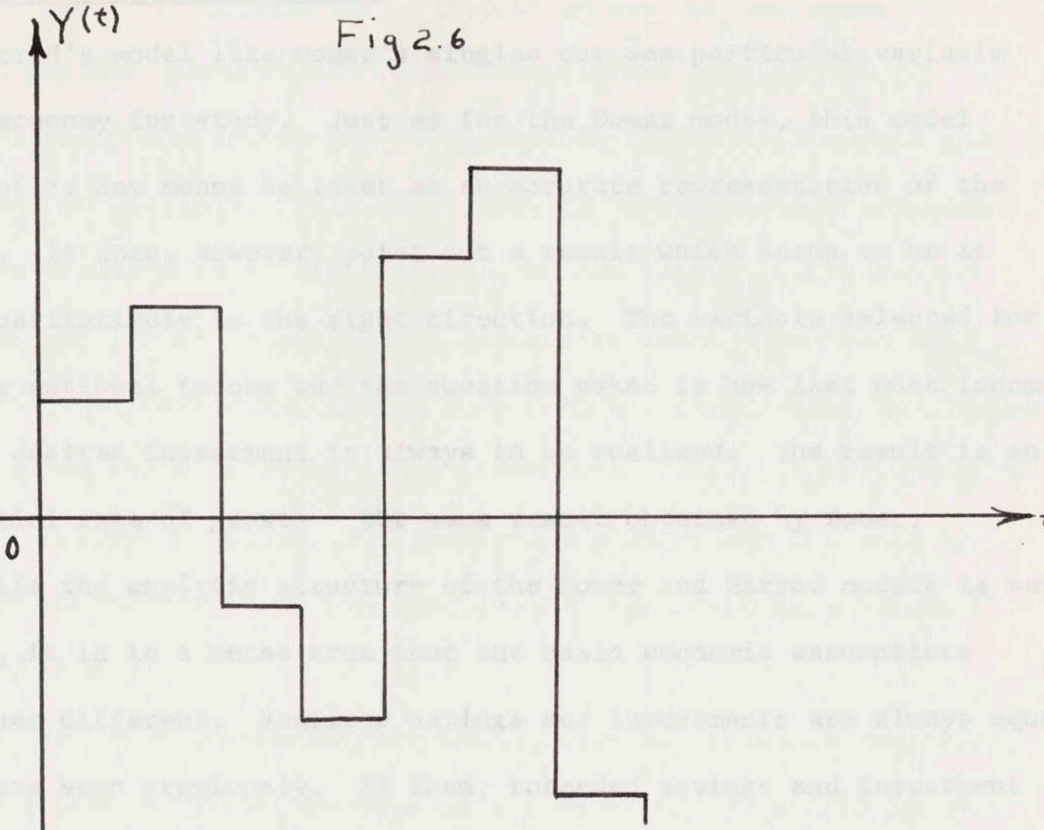


Fig 2.6

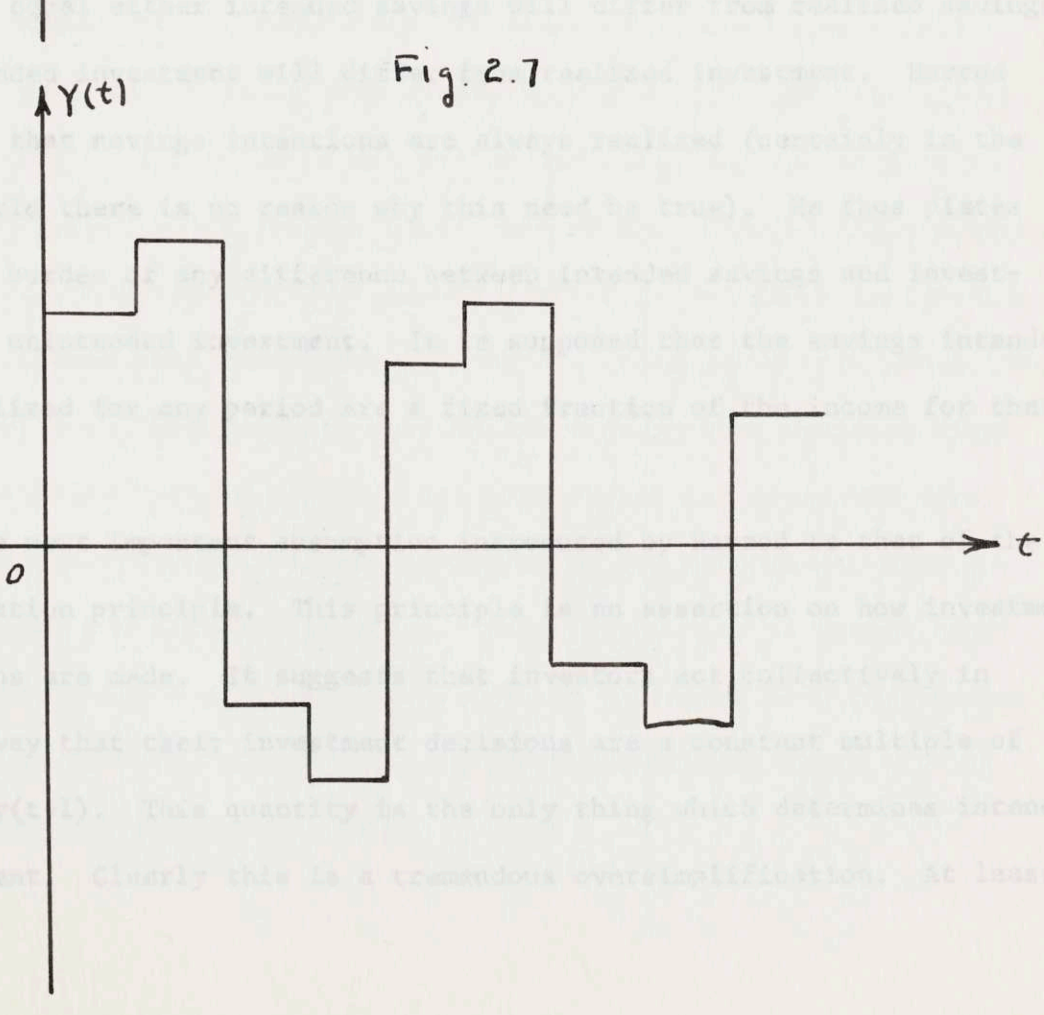


Fig. 2.7

Discussion of the Harrod Model

Harrod's model like Domar's singles out one particular variable in the economy for study. Just as for the Domar model, this model could not by any means be taken as an accurate representation of the economy. It does, however, point out a result which seems to be at least qualitatively in the right direction. The variable selected for study is national income and the question asked is how fast must income grow if desired investment is always to be realized. The result is an exponential rate of growth - the same result obtained by Domar.

While the analytic structure of the Domar and Harrod models is very similar, it is in a sense true that the basic economic assumptions are rather different. Realized savings and investments are always equal as we have seen previously. If then, intended savings and investment are not equal either intended savings will differ from realized savings or intended investment will differ from realized investment. Harrod assumes that savings intentions are always realized (certainly in the real world there is no reason why this need be true). He thus places all the burden of any difference between intended savings and investment on unintended investment. It is supposed that the savings intended and realized for any period are a fixed fraction of the income for that period.

The next important assumption introduced by Harrod is that of the acceleration principle. This principle is an assertion on how investment decisions are made. It suggests that investors act collectively in such a way that their investment decisions are a constant multiple of $y(t) - y(t-1)$. This quantity is the only thing which determines intended investment. Clearly this is a tremendous oversimplification. At least in a qualitative way, however, the greater the rate of change of national income, the more one expects investment to be.

The equilibrium assumption introduced in this model is that desired investment is realized. This is, of course, a continuous succession of steady states type of equilibrium. We can rather easily see intuitively why the model yields the results it does. The realized investment will be a fixed fraction of income. If the realized investment is to be the same as intended investment there must be a certain rate of increase of income. Hence as income increases the rate of change of income must also increase and the result is an exponential growth of income.

When the economy departs from the equilibrium behavior the Harrod model no longer holds and we have no way of saying what will happen. Harrod assumes that if the rate of increase of income becomes less than the equilibrium rate, then enterpreneurs will become pessimistic because they overproduced in one period and will keep the rate of increase of output below the warranted rate so that things will get worse and worse. Conversely Harrod assumes that if income grows faster than the equilibrium rate then there will be ever increasing underproduction. In general then, Harrod feels that if the economy ever deviates from the equilibrium rate of growth, then things will get worse and worse and furthermore it will be very difficult to get the economy back to the straight and narrow path again. Thus in the view of Harrod equilibrium growth is not stable; the slightest shock can send the economy on the road to ruin. There are no built in mechanisms which will in time straighten things out again.

It would be thought that a more realistic model would be obtained if a discrete time lag of one period was introduced into the investment equation since investors really can't use the income of the present period in their decisions. Some might feel that a continuous time lag distribution would be more appropriate. This may or may not be so in a model so simple as time. A discrete time lag of one period was introduced and it was found that the resulting curves were qualitatively the same as for the model without a time lag. Thus qualitatively the time lag did not change the behavior at all. There was, however, an important change in the value of $\alpha\sigma$ for the point of changeover to oscillation. Without the time lag in order to obtain oscillation it had to be true that $\sigma\alpha > 1$. In the model with a time lag $\sigma\alpha$ needs only be greater than .25 to have oscillation. To have y decrease exponentially with time it is still necessary to have $\alpha\sigma < 0$. Thus while a time lag does not change the qualitative behavior it changes the points at which there is a change from one type of behavior to another.

If income is to actually increase at the equilibrium rate then it must be true that the accelerator relation which determines how investors will know which to invest must also be the relation which tells how productive capacity changes for a given amount of investment. Here again, as in the Domar model, we have the same relation describing the technological and psychological aspects of investment.

## Tinbergen's Model

### Introductory Remarks

The Tinbergen model of the U.S. economy (1919-1932) is a statistical model. It was carried out under the auspicies of the League of Nations (ref. 23) and is the most pretentious attempt yet made at applying statistical correlation methods in the testing of business-cycle theories. No claim was made by Tinbergen that the model could be used for predictive purposes. It was set up simply to "explain" the behavior of the American economy in the period 1919-1932. The model is an aggregated model and deals only in aggregate quantities. No atttempt was made to break down the economy on the productive side into different industries or on the consumption side into various types of consumer groups. The most interesting thing about the model is that both the supply and demand sides of the economy are considered. In all the other models, except Klein's, the demand is either assumed given as a function of time or is ignored completely. These are then only partial models of the national economy since they do not explain the connections between supply and demand. Tinbergen on the other hand does relate what consumers spend to their income and their income to what is produced, etc. so that all variables are related to other variables in the system. In this way a closed system is obtained which allows a fairly thorough test of the gross behavior of the economy in terms of the basic relations between the variables which are postulated.

The general procedure in developing a statistical model is to begin with certain functional relationships between the pertinent variables. These functions involve certain unknown parameters which are to be determined statistically from known data. A typical functional relationship might be of the form

$$y = f(\vec{x} \mid \vec{\alpha}) \qquad (2.55)$$

$\vec{x}$ is a vector of variables and $\vec{\alpha}$ is a vector of parameters to be determined. From sets of observations we have sets of data $(y', \vec{x})$, the number of such sets usually being considerably larger than the number of variables to be determined. If the deviations are defined by

$$\epsilon = y' - y = y' - f(\vec{x} \mid \vec{\alpha}) \qquad (2.56)$$

the usual statistical procedure (although by no means the only conceivable one) is to determine $\vec{\alpha}$ so that the sums of the squares of the deviations is minimized.

In theory this process could be carried out for an arbitrary function f. Practical computational difficulties usually require, however, that f be separable, i.e.

$$f(\vec{x} \mid \vec{\alpha}) = \Sigma\, g_i(x_i \mid \vec{\alpha}_i) \qquad (2.57)$$

and furthermore that each $g_i$ be a polynomial in $\alpha_i$, i.e.

$$g_i(x_i \mid \vec{\alpha}_i) = \sum_j \alpha_{ij} x_i^j \qquad (2.58)$$

Even a form which is this complicated is usually too difficult to handle and one ends up assuming a linear relation between the variables

$$y = \Sigma\, \alpha_i x_i = \vec{\alpha} \cdot \vec{x} \qquad (2.59)$$

It is important to point out several things about statistical models. The first has to do with the way in which the initial variables are chosen. A choice can be made from the use of some econometric model or they can be determined simply from trial and error to see what gives the best fit. Similarly the functional relations between the variables may be the result of theoretical reasoning, empirical observation, or just a wild guess. As was mentioned above, the relations between the variables are often taken to be linear just to simplify the computations. Criticisms are often leveled at statistical models on the basis that there was no real model building. Instead some guess was made as to the variables involved, a linear relation was assumed between then and then the constants were determined statistically. Certainly there is room to criticize such models but it is really not the statistical part that deserves criticism. As was noted above statistical methods can be used to determine the parameters in any model. The use of statistics implies nothing about the linearity of the functional relations in the model, the manner in which the relations were derived, or anything else. The parameters in any model must be determined in one way or another and it seems that the use of statistical methods would prove to be an excellent way to determine them if it is not possible to compute the values from theoretical arguments.

Tinbergen, in order to simplify the numerical computations does approximate all the functional equations involving the variables by linear relations. He does, however, introduce time lags, and the cumulative influence of the variables. Consequently he ends up with a system of linear difference equations, the constant coefficients of which were determined from a statistical analysis of the data. It should be recognized that just because a linear relation between the variables is being assumed, that the time behavior of the variable being predicted is not in general linear. It can be anything, and the relations between the computed and observed values of y often look like those shown in Fig. 2.8

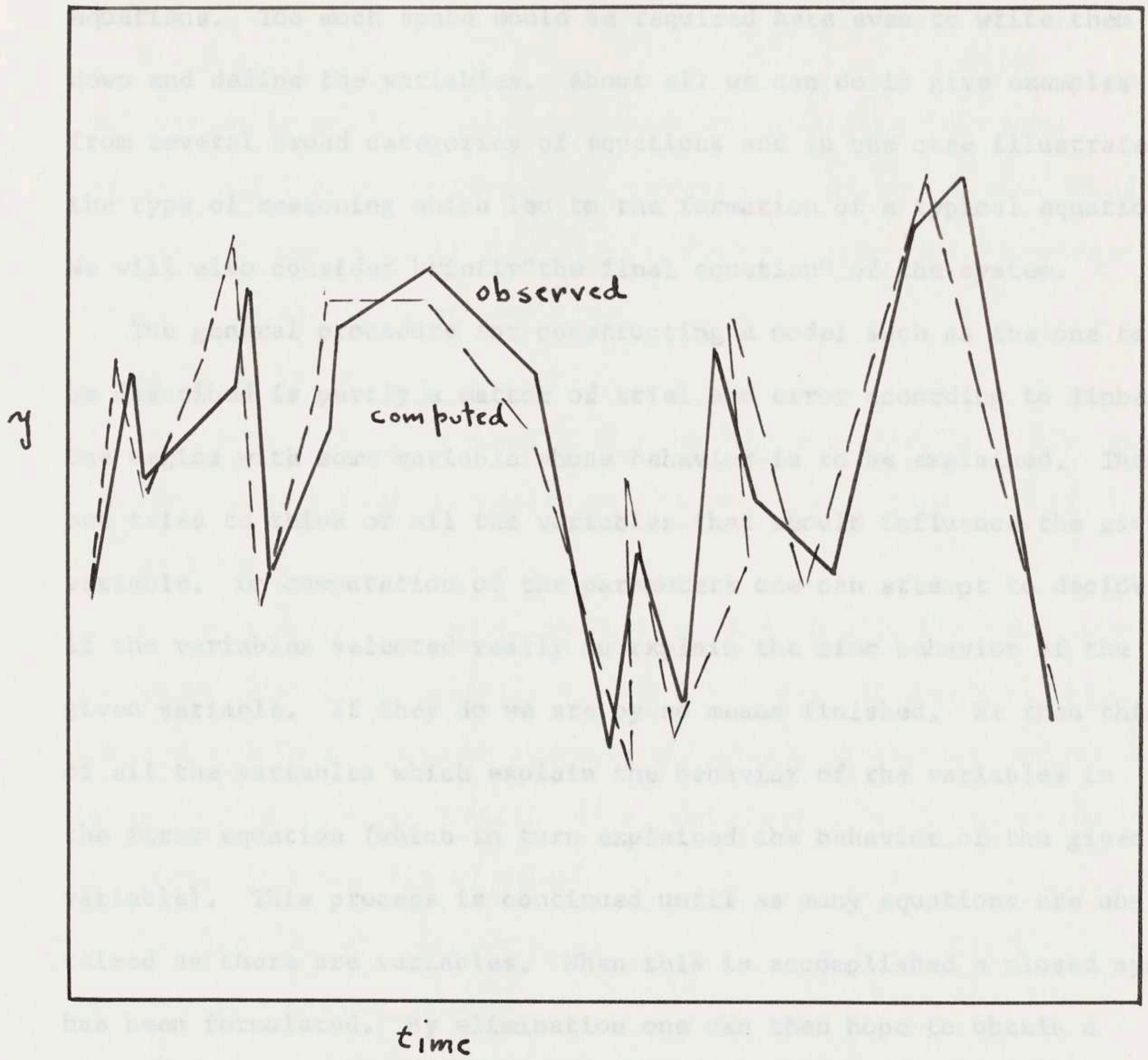



FIGURE 2.8

It will not be possible to present the Tinbergen model in any detail since Tinbergen himself requires about 200 pages to do so. In total his model involves about 69 simultaneous linear difference equations. Too much space would be required here even to write them down and define the variables. About all we can do is give examples from several broad categories of equations and in one case illustrate the type of reasoning which led to the formation of a typical equation. We will also consider briefly "the final equation" of the system.

The general procedure for constructing a model such as the one to be described is partly a matter of trial and error according to Tinbergen. One begins with some variable whose behavior is to be explained. Then one tries to think of all the variables that should influence the given variable. On computation of the parameters one can attempt to decide if the variables selected really do explain the time behavior of the given variable. If they do we are by no means finished. We then think of all the variables which explain the behavior of the variables in the first equation (which in turn explained the behavior of the given variable). This process is continued until as many equations are obtained as there are variables. When this is accomplished a closed system has been formulated. By elimination one can then hope to obtain a difference equation which involves only the variable whose behavior it was originally of interest to predict, known constants, and perhaps some function of time. The above is the procedure actually carried out by Tinbergen.

Simple Example of a Closed Model

It might be helpful to illustrate the construction of a closed model in a very simple case since it will not be possible to show the actual model in complete detail. This example is one given by Tinbergen. Suppose that the value $V_t$ of investment goods produced during the period t depends in a linear way on profits one time period earlier $z_{t-1}$, i.e.

$$V_t = \beta\, z_{t-1} \qquad (2.60)$$

Both variables can be imagined measured as deviations from some "normal" and $\beta$ is a constant to be determined. By statistical analysis we find the best value of $\beta$ to be .2. Further assume that consumption outlay $U_t$ is the total of

a) total wages $L_t$

b) a term $\epsilon_1 z_{t-1}$ indicating that profits are only partially consumed and that the marginal propensity to consume is $\epsilon_1$. There is also a lag of one period.

c) a term $\epsilon_2(z_{t-1} - z_{t-2})$ indicating that speculative gains also influence consumption outlay. Speculative gains are supposed to be proportional to the rate of increase of profits since share prices are assumed to be a linear function of $z_t$ and since a lag is again assumed to exist. Consequently we obtain

$$U_t = L_t + \epsilon_1 z_{t-1} + \epsilon_2(z_{t-1} - z_{t-2}) \qquad (2.61)$$

Statistical analysis indicates that $\epsilon_1 = .4$, $\epsilon_2 = 1$. Finally we have a definitional equation which indicates how profits are computed

$$z_t = U_t + V_t - L_t \qquad (2.62)$$

We now have a complete system since there are three variables $U_t, V_t, L_t$ to be eliminated and there are three equations which can be used to perform the elimination. Using the value of $U_t - L_t$ from (2.61) in (2.62) and (2.60) for $V_t$ we set

$$z_t = (\beta + \epsilon_1 + \epsilon_2)\, z_{t-1} - \epsilon_2 z_{t-2} \qquad (2.63)$$

as the final equation. This is a second order difference equation and consequently z can exhibit a cyclical behavior.

If no time lags were introduced into the system, the speculative term would disappear and the system of equations would become

$V_t = \beta z_t$

$U_t = L_t + \epsilon_1 z_t$

$z_t = U_t + V_t - L_t$

Elimination gives

$z_t(1 - \epsilon_1 - \beta) = 0$

and since $(1 - \epsilon_1 - \beta) \neq 0$, $z_t = 0$, i.e., the system always shows the same normal value of the profits ($z_t$ was a deviation from normal value). No cycles would occur unless the extra-economic data determining the normal levels showed cycles.

The Tinbergen Model

With the above introduction, let us now discuss the actual Tinbergen model of the U.S. economy. The first set of equations are those which are purely definitional in nature - as for example, in our simple illustration, the definition of profit. It is not true that all these definitional relationships involve the variables in a linear manner. When they do not Tinbergen simplifies them by linear relations. As an example, by definition it is true that

$$A = .0156\, c^i n + \frac{c}{m_{Lb}} B^i \tag{2.64}$$

where A is the assets held by individuals, $c^i$ is the number of shares of stocks held by individuals (nominal value), n is the average share price, $B^i$ is the nominal value of bonds held by individuals and $m_{Lb}$ is the bond yield and finally c is a constant. The factor .0156 must be included since the absolute price level of shares in terms of nominal value was 1.56 in 1926, when the share price index stood at 100. This relation is not linear, but Tinbergen approximates it by the following linear relationship

$$A = 1.50\ c^{i} + .90\ B^{i} + .84\ n - 18.0\ m_{Lb} \qquad (2.65)$$

The constants were determined to give a good fit to the data. Just what the theoretical justification for such a changeover is does not seem to clear. However, the linearized approximation does fit the data of the period very well.

The next set of equations involve the demand equations for goods and services. A typical equation in this set is the one explaining consumers' outlay. We will here go through the "reasoning" which led to the development of this equation.

Tinbergen indicates that the following variables, by a priori reasoning, seem to be of importance in explaining consumption

a) Wages and salaries ($L_w + L_s$)

b) Urban non-workers income E

c) Capital gains G

d) The rate of increase in farm prices $p^f - p^f_{-1}$ as an indication of speculative profits, which are not included in E but may nevertheless have influenced consumption (agricultural prices have been selected because they are sensitive to speculative influences).

e) Some measure of the degree of inequality of income distribution for which Pareto's $\alpha$ will be used. It might be injected here that Pareto's law is

$$N_x = A\,x^{-\alpha}$$

where N is the number of persons with income $>$ x and x = income (as shown by tax returns).

f) Cost of living p

g) A trend, standing for slow changes in habits, population growth and changes in population structure.

It is suggested by Tinbergen that some of the variables, especially E might be lagged. It is to be expected that the signs of all regression coefficients except the trend must be positive, i.e. an increase in any of the variables should increase consumer outlays. This is immediately obvious for E and G. For p, the theoretical possibility exists of a negative influence. A negative influence model imply, however, an elasticity of total consumption which is larger than one, and this is not pleasing to any economist. An increase in $\alpha$ means a decrease in concentration and consequently it is logical to expect a positive influence of $\alpha$ on consumption.

On determining the regression coefficients it turned out that those corresponding to $\alpha$ and a lagged value of E were negative instead of being positive as expected. For this reason $\alpha$ and $E_{-1}$ do not appear in the final equation. Tinbergen tried six different combinations of variables. Within each set he was able to determine several different values for the regression coefficients because some of the variables showed a very high intercorrelation. He eliminated a number of cases because the marginal propensity to consume was greater than unity. After some more considerations he finally chose the equation

$$U' - E_F' = .95\ (L_W + L_S) + .77E + .28\ G + .05\ (p^f - p^f_{-1}) + .03p + .37t$$

where $U'$ = total consumption expenditure, $E_F'$ = farmer's consumption expenditure. We went through all this discussion not only to show how Tinbergen obtained his results, but also to show the tremendous amount of work that was required just to develop one of the equations.

The next set of equations are the supply or price equations for goods and services. A typical equation here is one explaining the wage rate. The equation finally obtained is

$$\ell + .75\,\ell_{+1} = .52\ (u + v) + .67p + .89t \qquad (2.67)$$

is the wage rate, u = quantity of consumption goods and services produced, v = volume of production of investment goods, p = cost of living.

The final set of equations describes demand and supply in the money and capital markets. A typical equation here is that explaining the supply of bonds. It is

$$B = -1.7\, m_{Lb} + .07t^2 + 4.3t$$

This was obtained by integration of two equations giving the changes in corporate and government bonds. Tinbergen points out that this is almost a trend and hence replaces it by

$$B = 4.88t. \tag{2.68}$$

Having set up these equations, Tinbergen then had a closed system for the endogenous variables. The system was not causal, however, in the sense that it was completely determined by the initial conditions. There were certain variables, which we shall discuss later, which had to be considered to be outside the system but which did have an important influence on the time behavior of the model.

The Final Equation

The next problem after development of the closed system of equations is to attempt the elimination process to obtain the "final" equation. There is, of course, considerable freedom as to what variable will be the one to appear in the final equation. In any event, when there are so many equations, it is a tremendous task to carry out the elimination. It was not possible to carry out the elimination directly. Tinbergen was forced to make further simplifications. He dropped all the trend terms (this means that the results are then deviations from some straight line) and he dropped the cumulants (integrated variables). The final equation he obtained is then

$$.445\, z^c = .177\, z^c_{-1} - .098\, z^c_{-2} + .007\, z^c_{-3} + .012\, z^c_{-4} - .135\, h_{-2} - .077\, h_{-3}$$
$$-.305\, h_{-4} + .74\,(A\,u + P) - .40\,(A\,u + P)_{-1} - .822\, f + .315\, f_{-1} \qquad (2.69)$$

which can be simplified to read

$$z^c_t = e_1 z^c_{t-1} + e_2\, z^c_{t-2} + e_3\, z^c_{t-3} + e_4\, z^c_{t-4} + f(t) \qquad (2.70)$$

This is a fifth order linear, non-homogenous difference equation with constant coefficients in the variable $z^c$ (net income of corporations). The driving function f(t) prevents the value of $z^c$ from being completely determined by the initial conditions. This driving function contains the exogenous variables which help determine the behavior of the system. We can write

$$f(t) = AU + HO + F + R \qquad (2.71)$$

$$AU = 1.66\ (A\ u + P)_t - .90\ (A\ u + P)_{t-1} \tag{2.72}$$

and represents influences coming from changes in the gold stock A U and in the autonomous component P (Federal Reserve Banks holdings of securities,etc.)

$$HO = -.303\ h_{t-z} - .173\ h_{t-3} - .685\ h_{t-4} \tag{2.73}$$

and depends on the number of houses in existence two to four years previous to the period under consideration. Because of its influence on the actual building volume, this number acts on the present value of $z_c$.

F represents those factors, chiefly climatic, which change crops; i.e.

$$F = -1.847\ f_t + .708\ f_{t-1} \tag{2.74}$$

where f is the agricultural supply available for the United States market.

Finally Tinbergen throws everything else in R and says it is an agglomerate of a non-discernible multitude of disturbances which, each in itself, seem far less important then the three types mentioned above, but when taken together may still be important.

The function f(t) introduces no difficulties when the economy is being studied in retrospect for then f(t) is known. However, if an attempt is being made to use the model for predicting future behavior of the economy we run into some difficulties. It is necessary to know f(t) for the future periods of interest. However, in general, we cannot expect to know f(t) for the future (at least with any great accuracy) and consequently even if the model was absolute truth we would run into some difficulties in applying it. Tinbergen did not run into these problems since he was not so brash as to attempt to use the model for predictive purposes.

<u>Summary of the Tinbergen Model</u>

The Tinbergen model is perhaps the most complete model of the national economy ever presented. It attempted to present a closed system for the endogenous variables. There were certain exogenous variables which were considered to be outside the system, but which did have a definite influence on the time behavior of the model. Consequently, the future behavior of the system was not completely determined by the initial conditions.

This model of the national economy can be represented as a set of simultaneous linear difference equations. The unknown parameters in each equation were determined by statistical multiple regression analysis using known data over the period of interest. All relations among the variables were taken to be linear relations. Economic theory was introduced only to suggest what variables might be important in explaining the behavior of a given variable. No attempt was made to study what the actual functional relationship among the variables should be. Quite frequently a trend (a term containing time) was introduced to make things come out better.

Even though the Tinbergen model is quite pretentious it is still an aggregated model of the economy. No attempt was made to break down the productive capacity of the economy into somewhat more homogeneous units such as steel, autos, chemicals, etc. Any detailed model would, of course, require such a breakdown.

It is then possible to question how well the Tinbergen model represents the real world on several accounts. First, even though an appeal was made to economics to determine what variables were important we can still question whether they are really the correct variables and further whether all the important variables were included. Even more serious is the assumption of linear relationships between all the variables. This may or may not be a good assumption, but certainly it was not proved to be good simply because by using certain regression coefficients the calculated values approximated reasonably well the observed values of the variables. A good theory would indicate not only what variables are important but also the functional relationship between the variables. One can also question the introduction of trends to help make things work out right. In doing this no indication is given as to just what factors bring about the trend. Finally one can question the aggregation introduced and the lack of provision for technological change, etc. (except for trends). Needless to say, technological change is not easy to predict, but it cannot be ignored. Tinbergen is somewhat justified in omitting this since he was not interested in a predictive model.

The Tinbergen model is in some sense or other dynamic, and hence allows for nonequilibrium behavior. Supply is a function of certain variables in the model, consumer expenditures depend on certain variables, and investment depends on a group of variables. However, there is nothing in the model which implies the equilibrium condition that supply exactly equals demand Unfortunately the relations between the variables were not developed from an attempt to understand the inner workings of the system, it is not at all clear that the nonequilibrium behavior allowed for in the model has any real meaning. In fact, one might even question whether it is ever possible to represent the real world in such a highly aggregated form. If the nonlinear feedback effects are important, it is unlikely that one can obtain even a reasonably accurate representation of the system simply by attempting to fit curves to the variables representing the overall behavior. It is also uncertain as to whether or not the exogenous variables have too much influence on the system. If they completely determine its behavior, then the model would not seem to be of great value.

## The Klein Model

### Introduction

The Klein model of the U.S. economy (ref.24) is another statistical type model. Again it is a closed model except for certain exogenous variables. It, like Tinbergen's model, is an aggregated model, which includes both the supply and demand aspects of the system. As with the Tinbergen model, it is assumed that all relations between the variables are linear relations. Statistical multiple regression analysis is used to determine the unknown parameters by fitting to known data. The only difference between the two models lies in the variables selected and the number of difference equations in the final system. Klein's model is attempting to "explain" the economy for the period 1929-1952, which is, of course, a different period than that to which Tinbergen's model was applied. The Klein model is an attempt to revise and extend the Tinbergen model to apply to the period following World War II. It is also an extension of Klein's earlier model (ref.25) of the U.S. economy which covered the period 1921-1941.

One of the main differences between the Klein model and the Tinbergen model is that Klein feels that disposable income is the crucial variable in the consumption equation, not just income as in Tinbergen's model. Disposable income requires the introduction of taxes which did not appear in the Tinbergen model. Instead of speculative profits, Klein substitutes liquid assets. Also Klein introduces specifically the number of workers, which did not appear in the Tinbergen model. Nothing concerning the construction industry appears explicitly in the Klein model as it did in Tinbergen's. Finally, since the Klein model is not so detailed as that of Tinbergen, or because it applies to a different period, nothing appears about the gold supply, total long term debt short claims, etc.

<u>The Complete Model</u>

Since the Klein model does not have nearly as many equations as the Tinbergen model we can present all the equations. We will not attempt to present all the reasoning which led to the development of the equations, however. In general the reasoning reduces simply to trying to justify the use of the variables selected. The model contains twenty simultaneous difference equations. They are

Consumption Equation:

$$C_t = \alpha_0 + \alpha_1 (W_1 + W_2 - T_w)_t + \alpha_2 (P - S_p - T_p)_t + \alpha_3 (A - T_A)_t$$

$$+ \alpha_4 C_{t-1} + \alpha_5 (L_1)_{t-1} + \alpha_6 (N_p)_t + U_{1t} \qquad (2.75)$$

$C$ = consumer expenditures in 1939 dollars

$W_1$ = deflated private employee compensation

$W_2$ = deflated government employee compensation

$W_1 + W_2 - T_w$ = deflated disposable employee compensation

$P$ = deflated non-wage non-farm income

$S_p$ = deflated corporate savings

$P = S_p - T_p$ - deflated disposable non-wage non-farm income

$A$ = deflated farm income

$A - T_A$ = deflated disposable farm income

$L_1$ = deflated end-of-year liquid assets (currency, bank deposits, saving and loan shares, and U.S. government bonds) held by people

$N_p$ = number of persons in the U.S.

$U_1$ = random disturbance.

This equation can be compared with the Tinbergen counterpart given by (2.66)

The Investment Equation:

$$I_t = \beta_o + \beta_1 (P + X + D - T_p - T_A)_t + \beta_2 (P + A + D - T_p - T_A)_{t-1}$$
$$+ \beta_3 (i_L)_{t-1} + \beta_4 K_{t-1} + \beta_5 (L_2)_{t-1} + U_{2t} \tag{2.76}$$

$I$ = gross private domestic capital formation in 1939 dollars

$D$ = capital consumption charges in 1939 dollars (depreciation)

$i_L$ = average yield on corporate bonds

$K$ = end-of-year stock of private capital in 1939 dollars

$L_2$ = deflated end-of-year liquid assets held by enterprises

The Corporate Savings Equation:

$$(S_p)_t = \gamma_o + \gamma_1 (P_c - T_c)_t + \gamma_2 (P_c - T_c - S_p)_{t-1} + \gamma_3 B_{t-1} + U_{3t} \tag{2.77}$$

$P_c$ = deflated corporate profits

$T_c$ = deflated corporate income taxes

B = deflated end-of-year corporate surplus

The Relation between Corporate Profits and Non-wage Non-farm Income

The relation between corporate profits and non-wage non-farm income is taken to be

$$(P_c)_t = \delta_0 + \delta_1 P_t + \delta_2 P_{t-1} + U_{4t} \tag{2.78}$$

The Depreciation Equation

$$D_t = \epsilon_0 + \epsilon_1 (K_t + K_{t-1}) + \epsilon_3 (Y + T + D - W_2)_t + U_{5t} \tag{2.79}$$

$Y + T + D - W_2$ = private gross national product in 1939 dollars

Demand for Labor Equation:

$$(W_1)_t = \zeta_0 + \zeta_1 (Y + T + D - W_2)_t + \zeta_2 (Y + T + D - W_2)_{t-1} + \zeta_3 t + U_{6t} \tag{2.80}$$

t = time trend

The Production Function:

$$(Y + T + D - W_2)_t = \eta_0 + \eta_1 \left[ h(N_W - N_G) + N_E + N_F \right]_t + \eta_2 \quad K_t + K_{t-1}$$
$$+ \eta_3 t + U_{7t}$$

h = index of hours worked per person per year

$N_W$ = number of wage and salary earners

$N_G$ = number of government employees

$N_E$ = number of non-farm entrepreneurs

$N_F$ = number of farm operators

The Labor Market Adjustment Equation:

$$\omega_t - {}_{t-1} = \theta + \theta\,(N - N_W - N_E - N_F)_t + \theta_2\,(P_{t-1} - P_{t-2}) + \theta_3 t + U_{8t} \tag{2.82}$$

$\omega$ = index of hourly wages

$N$ = number of persons in the labor force

$N - N_{\omega} - N_E - N_F$ = unemployment in number of persons

$p$ = general price index

The Import Demand Equation:

$$(F_I)_t = \tau_0 + \tau_1 \left[(W_1 + W_2 + P + A - T_W - T_P - T_A)\frac{p}{p_I}\right]_t + \tau_2 (F_I)_{t-1} + U_{9t} \tag{2.83}$$

$F_I$ = imports of goods and services in 1939 dollars

$p_I$ = index of prices of imports

$W_1 + W_2 + P + A - T_W - T_P - T_A$ = deflated disposable income plus corporate savings

The Agricultural Income Determination Equation:

$$\left(A\frac{p}{pa}\right)_t = x_0 + x_1 \left[(W_1 + W_2 + P - S_P - T_W - T_P)\frac{p}{p_A}\right]_t + x_2 \left[(W_1 + W_2 + P - S_P - T_W - T_P)\frac{p}{pa}\right]_{t-1} + x_3 \left(\frac{p}{pa}\right)_t + x_4 (F_A)_t + U_{10t} \tag{2.84}$$

$p_A$ = index of agricultural prices

$F_A$ = index of agricultural exports

$W_1 + W_2 + P - S_P - T_W - T_P$ = deflated disposable non-farm income.

The Relationship between Agricultural and Nonagricultural Prices

If the government is successful in its price support policy, there will tend to be a relationship between agricultural and nonagricultural prices. Hence the following relationship is assumed:

$$(p_A)_t = \lambda_0 + \lambda_1 p_t + \lambda_2 (p_A)_{t-1} + u_{11t} \tag{2.85}$$

The Household Liquidity Preference Equation

Following Keynes, the authors recognize that liquid assets may be classed into two parts: 1) transactions balances proportional to income and 2) idle balances. The transactions balances consist of currency and checking accounts. The idle balances (speculative portion) are assumed to depend only on interest rates. If $M_t$ is the transactions balance and $(W_1 + W_2 + P + A - T_W - T_p - S_p - T_A)_t$ is the deflated disposable income, then it is assumed that

$$\frac{M_t}{(W_1 + W_2 + P + A - T_W - T_P - S_P - T_A)_1} = \mu_1 \text{ (a constant)}$$

Then

$$(L_1)_t - \mu_1 (W_1 + W_2 + P + A - T_W - T_P - S_P - T_A)_t$$

is a measure of idle funds. Hence the final relation assumed is

$$(L_1)_t - \mu_1 (W_1 + W_2 + P + A - T_W - T_P - S_P - T_A)_t = \mu_0 \left[ (i_L)_t - i_L^o \right]^{\mu_2} + \mu_{12t} \tag{2.86}$$

$i_L$ = average yield rate on corporate bonds in percent

$i_L^o$ = minimum possible interest rate

The Business Liquidity Preference Equation

$$(L_2)_t - \gamma_1 (W_1)_t - \gamma_0 + \gamma_2 (p_t - p_{t-1}) + \gamma_3 (i_s)_t + \gamma_4 (L_2)_{t-1} \qquad (2.87)$$

The Relation between Long and Short Term Interest Rates

$$(i_L)_t = \zeta_0 + \zeta_1 (i_s)_{t-3} + \zeta_2 (i_s)_{t-5} + U_{14t} \qquad (2.88)$$

The Money Market Adjustment Equation

$$\frac{(i_s)_t - (i_s)_{t-1}}{(i_s)_{t-1}} = O_o + O_1 R_t + U_{15t} \qquad (2.89)$$

$R_t$ = excess reserves of banks

Definitions and Accounting Identities

$$C_t + I_t + G_t + (F_E)_t - (F_I)_t = Y_t + T_t + D_t \qquad (2.90)$$

G = government expenditures for goods and services in 1939 dollars.

$F_E$ = exports of goods and services in 1939 dollars

T = deflated indirect taxes less subsidies

$$(W_1)_t + (W_2)_t + P_t + A_t = Y_t \qquad (2.91)$$

$$h_t \frac{t}{p_t} (N_W)_t = (W_1)_t + (W_2)_t \qquad (2.92)$$

$$K_t - K_{t-1} = I_t - D_t \qquad (2.93)$$

$$B_t - B_{t-1} = (S_P)_t \qquad (2.94)$$

<u>Properties and Prediction</u>

Klein points out that from the purists' point of view only variables resulting from natural forces, such as climate and the weather could legitimately be considered exogenous variables. He decided to classify all variables in the public sector as exogenous. These include all the tax variables and government employee compensation and government expenditures. Variables determined by forces outside the country are also considered to be exogenous. These include exports of farm products and of goods and services. Variables having to do with the population and the number of people in the working force are considered exogenous, too. Finally, the index of hours worked, the excess reserves of banks, and the time trend are consdered exogenous. All the other variables are endogenous and are determined by solving the closed system. The exogenous variables prevent the system from being deterministic and serve to move the system through time.

While we indicated at the beginning that the model contained a set of simultaneous linear difference equations, it is worthwhile to point out that not all the equations are completely linear. Equations (2.83),(2.84), (2.86),(2.89),(2.92) are not quite linear, but with some slight changes they could be forced into a linear form just as Tinbergen did with a number of relations. Klein makes a point of the non-linear relations and the fact that the final equations for forecasting $\bar{Y}$ and p are non-linear. These equations are

$$\bar{Y}_t\,(a_o + a_1 p_t) + a_2 + a_3\,\frac{1}{p_t} + a_4\,p_t = 0 \qquad (2.95)$$

$$\frac{1}{p_t}\,(b_o + b_1\,\frac{1}{p_t} + b_2\,\bar{Y}_t)(b_3 + b_4\,\frac{1}{p_t} + b_5\,\bar{Y}_t) + b_6 + b_7\,\frac{1}{p_t} + b_8\,\bar{Y}_t = 0$$

The equations were not derived directly from the model, but involve additional assumptions concerning taxes, etc.

Klein felt that their model could be used for the purpose of forecasting economic activity, and a considerable amount of work was done in testing the model. Two different approaches were used for these tests. One approach was to wait until the information concerning the exogenous variables became available for the previous year and then compute the endogenous variables. These could then be compared with the values actually observed. The other approach, which is the one that is predictive in the true sense of the word, attempted to estimate the values of the exogenous variables for the coming period and using these estimates calculate the values of the endogenous variables.

The first approach is clearly not really predictive. The period being studied must be past before the model can actually be used since it is necessary to know the actual values of the exogenous variables. Using the model in this way eliminates one source of error, however, since correct values of the exogenous variables are used. When the model is used in the true predictive sense and it is necessary to use estimated values of the exogenous variables, then the answers obtained may be incorrect, either because the model is not accurate or because the values of the exogenous variables were not estimated correctly. The model could give an absolutely precise representation of the behavior of the economy and yet incorrect answers would be obtained because the exogenous variables were not correctly predicted. Consequently accurate prediction with the model not only requires that the model correctly represent the economy but also that the exogenous variables be correctly extrapolated.

The Klein model, all in all, could hardly be called an unqualified success. In fact, the results were in many ways quite disappointed. For example, in a number of cases the parameters were not large relative to their standard errors, residuals were rather large and by no means completely random. These discouraging aspects of the model showed up in its predictive ability. For example, the earlier Klein model (ref.25) was tested by Christ (26) for the year 1948 against two naive models. One of these naive models simply assumed no change in the variables from 1947. The other naive model assumed that the change from 1947-1948 was the same as from 1946-1947. Both of these simple models overall did a better job than the Klein model in "predicting" the results for 1948. Thus in a real sense the Klein model can be said to have failed. The later Klein model did not fare much better. The results were not too good, even for prediction in retrospect; i.e., after the values of the exogenous variables were known. Even more discouraging was the fact that it seemed that as each new year's data came in it was necessary to competely recompute all the parameters since important changes took place in them. This of course completely eliminates any possibility that the model could be used even for moderately long range prediction.

The same criticisms that were leveled against the Tinbergen model can be carried over to the Klein model. One can question the choice of variables, the assumption of linearity, the introduction of time trends, etc. In addition, if the non-linear feedback loops are important enough, it may not be possible to describe the system under all conditions, no matter what type of relations between the variables are assumed with a curve fitting process is used to determine the constants. It is not necessarily true that it is impossible, however. Similarly, to mention another objection which was not indicated in connection with the Tinbergen model, if distributed time lags are more important than discrete time lags, then,as was indicated in Chapter 1, the time step must be small enough to include the high enough frequency terms to represent the time lags correctly. Under such circumstances using the time lag for the time step may be much too big. This may or may not be a valid objection against the Klein model also. Perhaps smaller time steps would be needed.

All the problems encountered in the Klein model serve to point up the extreme difficulty incurred in attempting to devise a model which accurately simulates the national economy. The task of devising such a model will be a great one. None of the models considered thus far come anywhere close to meeting this goal. None of the models studied in the following chapter will either. However, the differentiation of industries introduced in the models to be studied in the next chapter will, it seems, be certain to be a part of any model which accurately represents the real world.

# CHAPTER 3

# DIFFERENTIATED MODELS OF THE NATIONAL ECONOMY

## Introduction

In this chapter some well known differentiated models of the national economy will be studied. By differentiated models we will mean models in which the productive capacity of the economy is broken down into various sectors. Each sector is supposed to represent a fairly homogeneous group of products. For example we might divide the economy into steel, chemical, aircraft, etc. production sectors. Obviously there is essentially no limit to how far this process can be carried. In theory one could go so far as to treat every organization separately. Clearly, with this sort of breakdown, a lot more detail can be put into the national economy than was possible with the highly aggregated models. Since each segment of the economy will tend to use products made by the other segments it is possible to study how changes in some one segment will influence the others and the whole economy.

The models to be considered here will be both static and dynamic, i.e. some will consider changes with time while others will not. All the models will be linear in some sense or other. Some of these models make use of mathematics which until a few years ago was not well known. The models will involve certain aspects of game theory, linear programming, and relaxation phenomena. All the models involve some notion of equilibrium existing. This equilibrium may be static or a continuous succession of steady states.

## Walras Model

We will begin with a very brief discussion of a model formulated by Walras (refs. 28-30) in 1874. This model is only of historical interest and hence we give only the bare outlines to illustrate how it helped influence the development of later models.

Walras assumed that there were n factors of production which could be classified into the three categories of land, labor, and capital. These factors of production were combined to form m commodities. They were to be consumed in the same period in which they were produced. Let $X_i$ be the total supply of productive factor i and $a_{ij}$ the quantity of productive factor i needed for one unit of commodity j. Then let $y_j$ be the quantity of each commodity desired. Thus it must be true that

$$\sum_j a_{ij} y_j \leq X_i \qquad i=1, .., n$$

or in matrix notation

$$A \vec{y} \leq \vec{X} \tag{3.1}$$

where A is the matrix $\| a_{ij} \|$. Walras in his formulation assumed that strict equality held. The above is then a system of n inequalities in the m variables $y_j$. The $X_i$, however, are unknown as well as the $y_j$. Let $P_i$ be the unit price of factor i and $p_j$ the unit price of commodity j. Then we must have

$$\sum a_{ij} P_i \leq p_j \quad , j=1, .., m$$

or in matrix form

$$A' \vec{P} \leq \vec{p} \tag{3.2}$$

where A' is the transpose of A. Again Walras in his formulation assumed that the equality held. This is a system of m inequalities in the n variables $P_i$. The $p_j$, however, are also unknown.

If we assume the equalities hold in the above expressions, we have n+m equations in 2m+2n variables. Walras now assumed that there were n equations of the form

$$y_j = F_j(P_1, \ldots P_n, p_1, \ldots p_m), \quad j=1,\ldots,m \qquad (3.3)$$

and n equations of the form

$$X_i = G_i(P_1, \ldots P_n, p_1, \ldots p_m), \quad i=1,\ldots,n \qquad (3.4)$$

These equations relate the supply of the productive factors and the demand for the commodities to the prevailing prices of both. We now have 2m+2n equations in 2m+2n unknowns. One might then hope for a unique solution. We will not attempt to consider as Wald did the conditions under which a unique solution exists. In actuality Walras took the price of one commodity to be unity and hence ended up with 2m+2n-1 equations and unknowns. It seems unnecessary to do this; however, it does simplify things a little.

We will not attempt to discuss any further the Walras model. It is interesting, however, to compare it to the well known Leontief model which we will now consider in some detail.

The great contribution of the Walras model lay in the formulation for the first time of a set of equations representing the general equilibrium of the economy. From these equilibrium equations the quantities of the productive factors used, the amounts of the commodites produced, the prices, and wage rates could, in principle, be determined.

## The Static Input-Output System

### Introduction

A very interesting and much used model of the national economy was formulated in the 1930's by Professor Wassily Leontief of Harvard (ref. 9, 28, 31-32, 36). This model is often referred to under the title of the input-output system. The model we will consider here is a static model. It does not consider the change of the economy with time. We will later present a dynamic generalization of the input-output model. There are two forms of the static model that we will consider. These are called the open system and the closed system.

Before considering the open and closed systems we will give a brief introduction to the philosophy of the model. The input-output model visualizes an economy in which the productive capacity can be broken down into segments called industries. Each commodity in the economy is produced by a different industry. Each industry uses only one process of production. The productive process of every industry is one in which inputs are combined in fixed proportions to produce a commodity. These inputs to any industry are produced by other industries except for the inputs which come from outside the economy. It is the object of the model, knowing certain production coefficients to find the output of each industry. In other words the model makes the following assumptions:

1. The economy can be broken down into a number of industries. Each industry uses only one method of production and produces a single product.

2. The ratio of the output of an industry to any input factor is a constant.

In addition to these, another important assumption is also introduced:

3. Everything that is produced is used in the period and exactly enough is produced to meet demand. This is the equilibrium condition used for the Leontief models. Other more specific assumptions will arise later. We will now consider in some detail the open model of the national economy. This model has found greater applicability than the closed model.

The Open Model

The open model is one in which we assume that we are given a bill of goods which indicates the net (consumer) demand for each product. It is desired to compute how much of each product must be produced to meet this bill of goods and to satisfy interindustry transfers. Let $X_i$ be the total output of industry i and let $Y_i$ be the final net demand for the product of industry i. This final demand includes everything but the quantity of product i used by other industries. That is $Y_i$ includes consumer demand, government demand, quantity needed for foreign trade, etc. $X_i$ and $Y_i$ can be expressed in any units desired; they are generally expressed in monetary units, however.

Now let $x_{ij}$ be the sales of industry i to industry j. It can also be thought of as the number of units of i sold to industry j. This will depend on the physical dimensions used. We then introduce Leontief's assumption that input is a fixed proportion of output. This is written mathematically as

$$\frac{x_{ij}}{X_j} = a_{ij} \tag{3.5}$$

The quantity $a_{ij}$ is assumed to be constant and must also be known to solve the problem.

We next introduce the equilibrium assumption that everything produced is used and enough is produced to meet demand. Thus the total output of industry i must be the final net demand plus the amounts sold to other industries. This can be expressed as the following set of equations

$$
\begin{aligned}
&X_1 - x_{11} - x_{12} - \ldots - x_{1n} = Y_1 \\
&-x_{21} + X_2 - x_{22} - x_{23} \ldots - x_{2n} = Y_2 \\
&\vdots \\
&-x_{n1} - x_{n2} \ldots\ldots + X_n - x_{nn} = Y_n
\end{aligned}
\tag{3.6}
$$

The $x_{ii}$ represent the amounts used by the industry i of its own product.

The assumption concerning the nature of the production process as given by (3.5) can now be entered into equations (3.6). The new set of equations is

$$
\begin{aligned}
&X_1(1-a_{11}) - a_{12}X_2 - \ldots - a_{1n}X_n = Y_1 \\
&-a_{21}X_1 + X_2(1-a_{22}) - \ldots - a_{2n}X_n = Y_2 \\
&\vdots \\
&-a_{n1}X_1 - a_{n2}X_2 \ldots\ldots + X_n(1-a_{nn}) = Y_n
\end{aligned}
\tag{3.7}
$$

This can be written in matrix form as

$$(I - A)\vec{X} = \vec{Y} \tag{3.8}$$

where

$$A = \| a_{ij} \| , \quad I = \| \delta_{ij} \| \tag{3.9}$$

I is just the identity matrix.

Now if the matrix I - A is nonsingular (we know it must be square since the number of products is the same as the number of industries) there is a unique solution to the set of simultaneous equations given by

$$\vec{X} = (I - A)^{-1} \vec{Y} \tag{3.10}$$

The properties of the matrix I-A have been studied thoroughly and many learned theorems have been proved. We shall not discuss these here but instead refer the reader to the book edited by Morgenstern (28 ). If $A_{ij}$ are the elements of $(I-A)^{-1}$, then we can write the following set of equations:

$$X_1 = \sum_j A_{1j} Y_j \tag{3.11}$$

$$\vdots$$

$$X_n = \sum_j A_{nj} Y_j$$

The total production of each industry i is then given as a linear combination of the $Y_j$, which are the known final net demands. This model is called an open model because we are free to specify the bill of goods representing final demands. The answers obtained will depend on the bill of goods specified.

A similar type of model can be set up to find the prices (accounting) of each product. If we think of the $a_{ij}$ previously defined as the number of units of i required to produce one unit of j we can write

$$P_j = \sum_{i=1}^{n} a_{ij} P_i + R_j \quad (j=1,2,\ldots n) \tag{3.12}$$

where $P_j$ is the unit price of j and $R_j$ is the value added by the industry per unit output. Thus we can write the above in matrix form as:

$$(I - A')\overrightarrow{P} = \overrightarrow{R} \tag{3.13}$$

where A' is the transpose of A. Now since I = I', we can write

$$(I - A)'\overrightarrow{P} = \overrightarrow{R}$$

or $$\overrightarrow{P} = [(I - A)^{-1}]'\overrightarrow{R} \tag{3.14}$$

since for any matrix B,

$B'^{-1} = B^{-1'}$.

$$\text{Hence} \quad P_j = \sum_{i=1}^{n} A_{ij} R_i \qquad j=1,..n \qquad (3.15)$$

where the $A_{ij}$ are the quantities defined previously. The $R_j$ can be broken up to include the value added by labor $L_i$ and the non labor value added $C_i$. Thus

$$R_i = L_i + C_i \qquad (3.16)$$

Uses of the Open Model

We might now consider briefly how the open Leontief model can be used. Before any use can be made it is necessary to collect data to determine the coefficients of production. This is often done by setting up a table listing the sales of each industry to every other industry. After finding the total production of each industry the coefficients $a_{ij}$ can be calculated using Equation 3.5 . It is clear that the construction of such a table is a tremendous undertaking. For a subdivision of the economy into 200 industries a total of 40,000 interindustry sales figures are needed. Some of the entries are zero and others are of negligible magnitude. Others can be found quite easily and accurately from the Census of Manufactures and similar sources. Other figures, however, are very difficult to track down. The difficulty lies in the fact that many private firms do not keep a record of purchases according to industrial classifications.

As was mentioned at the beginning, if a bill of goods is specified the total outputs can be computed. Leontief tried this by back calculating. Using coefficients of production calculated from a 1939 input-output table he calculated the total outputs for 1929 using the known bill of goods for 1929. This was a ten equation model. The calculated results compared quite well with the actual values for 1929. This showed that the coefficients of production had not changed much in that time. The above is only one use of the model, however. From the input-output table constructed to calculate the coefficients of production one can also find the amount of labor needed to turn out one unit of each product. Then when the total outputs are computed for a given bill of goods one can also compute what employment in the economy will be. Finally, using (3.15), (3.16) one can study the influences of wage rate changes on prices assuming that all the change is reflected in the price. In particular, for example, one can study how a wage increase in the steel industry will affect the prices in all other industries.

## The Closed Model

We will next briefly discuss Leontief's closed model of the economy. As before we divide the economy into a number of sectors called industries. Now, however, one industry will be households. The output of the household industry will be the services of labor and the inputs from other industries will public consumption. The government will also be considered an industry whose product is services and which requires the products of other industries to perform these services. Foreign trade can also be considered an industry whose inputs are exports and whose product is imports.

We now have a closed system. There is no bill of goods representing net demand which must be specified. Thus, now the amount produced minus the distribution to all other industries must be zero. We then have the following set of equations:

$$X_1 - a_{11}X_1 - a_{12}X_2 - \dots - a_{1n}X_n = 0$$
$$-a_{21}X_1 + X_2 - a_{22}X_2 - a_{23}X_3 \dots a_{2n}X_n = 0 \qquad (3.17)$$
$$\vdots$$
$$-a_{n1}X_1 - a_{n2}X_2 - \dots + X_n - a_{nn}X_n = 0$$

where as before

$$a_{ij} = \frac{x_{ij}}{X_j}$$

Again we can write the above equations in matrix form as

$$(I - A)\ \overrightarrow{X} = \overrightarrow{0} \qquad (3.18)$$

This is a set of n simultaneous equations in n unknowns which is homogeneous. Since the rank of the augmented matrix will always have the same rank as the matrix of the coefficients this system of equations will always have a solution. It will only have a solution $\overrightarrow{x} \neq 0$, however, if the determinant $|I - A|$ vanishes. We will not attempt to consider when this will be true. It should be noted that if $\overrightarrow{x}$ is a solution $\lambda\ \overrightarrow{x}$ is also a solution for any constant $\lambda$. Hence the solution will not be unique. Really only the ratios of the variables are determined. We will not give any further consideration to this closed model.

## The Aggregation Problem

In attempting to use either an open or closed Leontief model to make some actual calculations about the national economy, it is clear that a considerable amount of aggregation of industries will need to be carried out to keep the number of equations to a manageable size (even 200 equations). The question then arises as to how the method of aggregation influences the answers obtained. It is quite obvious that different methods of aggregation could give radically different answers.

We first note that the method of aggregation must be selected to suit the purpose at hand. A method of aggregation which makes available the required information for one purpose may be useless for another. In input-output models it is generally desirable to have a fine breakdown on items that are likely to create bottlenecks. If only a crude subdivision of such items is used, solutions can be obtained which appear mathematically feasible but which are not feasible economically because of lack of substitutability between the various components of the aggregates. For example, an increased level of activity in the steel industry might require greatly increased coal production. If this increase was represented in the input-output table by some aggregate fuel figure it would be difficult to ascertain the problem of increasing the coal production. It then becomes clear that if one is looking for bottlenecks in the economy, the method of aggregation can have an important influence on the answers obtained. In general, it seems true that highly aggregated systems tend to overestimate the attainable levels of production and consumption for the economy.

In general, there are two basic criteria that can be used for aggregation. The first is substitutability. If the products are complete substitutes then obviously no problem arises in aggregating them. However, this rarely occurs. For example, oil may be a good substitute for coal in heating homes but this is not so for many industrial processes. Another basis for aggregation is complementarity. If the products concerned always bear a fixed relation to each other then they may be aggregated. This could lead to vertical aggregation of some industries. One can run into price difficulties here, however, if prices of the various components tend to move in opposite directions.

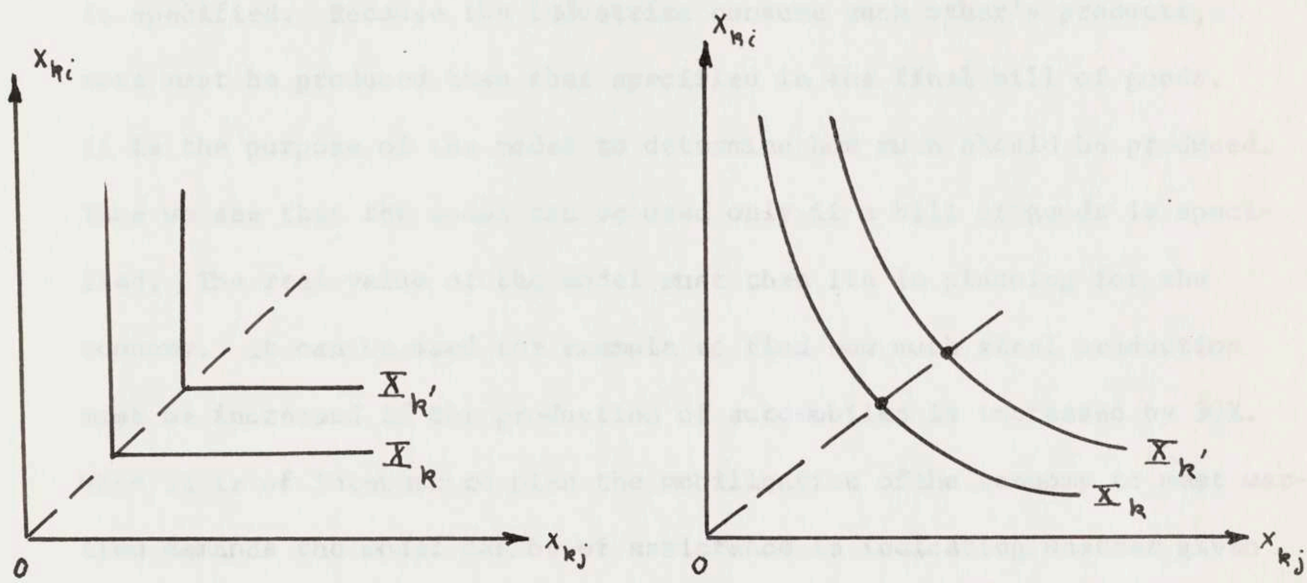
Another suggested method of aggregation is based on aggregating industries with similar production processes. It is claimed that this method of aggregation helps deal with technological change. Presumably all products using the same process will be influenced by technological change in the same way. Often, however, technological change brings about new products. Hence, there seem to be many difficulties involved in aggregation on the basis of productive processes and even its strongest point (simplifying investigation of technological change) may not be too valid.

There are many other difficulties involved in aggregating firms with different accounting systems and inventory valuation schemes. Giant companies which produce a tremendous variety of products also present special difficulties. We will not consider the aggregation problem any further since our only purpose was to point out the numerous difficulties which are encountered when trying to decide on an aggregation method.

<u>Allowable Production Functions for Leontief Models</u>

It is often assumed that since the Leontief models require that $a_{ij} = x_{ij} / X_j$ that the production function must be such that the inputs occur in fixed proportions, i.e. that curves something like those in Fig. (3.1) exist. However, Samuelson (22) has pointed out that this need not be true at all. Indeed curves like those shown in Fig. (3.1) are perfectly admissible. This means that substitutability is allowable. Samuelson points out in Fig. (3.1) only the black circles are actually observed. Samuelson bases his argument on the assumption of a competitive economy. He assumes also that labor is the only primary factor or non-produced good. All desirable substitutions have been made by the competitive market and no variation in the composition of final output or in

Figure 3.1




the total quantity will give rise to price change or substitution. To carry out the proof that $a_{ij}$ is a constant even in the more general case he had to assume a production function which was homogeneous of first order, i.e.

$$X_j = f_j(x_{j1} x_{j2} \dots x_{jn+1}) = m f_j\left(\frac{x_{j1}}{m}, \dots, \frac{x_{jn+1}}{m}\right)$$

where $x_{jn+1}$ refers to labor. We will not take the space to prove Samuelson's result. It is interesting to note, however, that the Leontief model is not quite so restrictive as one might actually suppose.

<u>Discussion of Static Input-Output Models</u>

Inasmuch as these models are completely static and do not involve time or any other elements of a dynamic system so we cannot discuss them with reference to dynamic models of the economy. We may ask, however, just where their greatest usefulness lies and what are the limitations of these models.

As has been indicated previously the input-output system visualizes an economy with n industries, each of which produces one product by one process. There is no substitution. In the open system a bill of goods is specified. Because the industries consume each other's products, more must be produced than that specified in the final bill of goods. It is the purpose of the model to determine how much should be produced. Thus we see that the model can be used only if a bill of goods is specified. The real value of the model must then lie in planning for the economy. It can be used for example to find how much steel production must be increased if the production of automobiles is increased by 30%. When it is of interest to plan the mobilization of the economy to meet wartime demands the model can be of assistance in indicating whether given military procurement programs are feasible, given available amounts of labor and raw materials. This planning aspect of the Leontief models has led to some opposition to them from industry - based on the fear of a planned economy. Indeed research funds for work on input-output analysis have been cut probably for this reason.

There are inherent in the model a number of approximations which are serious even when the model is used as discussed above. First of all there is the assumption of no substitution. One industry makes each product by only one production process. This is clearly not true in the actual economy. Any number of different industries can make a given product by radically different production processes. Similarly one giant company can produce thousands of different products. The assumption of the use of inputs in fixed proportions (or more generally a competitive economy with a homogeneous production function) is an important assumption. If only a relatively small range of production is being considered and if the constants are determined for this range, then this approximation may not be a bad one. Inventories can cause a little trouble even in this static model. If some production is to be for inventory then this production can presumably be included in the bill of goods. If some of the needed production is going to be taken from inventory, then in some cases this might be subtracted from the bill of goods. The problem of aggregation is of course a very important one also when the model is to be applied to the real world. As was indicated previously, misleading answers can be obtained by the improper choice of an aggregation scheme. All in all, whether or not the above assumptions prevent one from obtaining fruitful results depends upon the use which is being made of the model.

The closed model is not too interesting at the present time because it does not determine the absolute level of activity of the economy. Only the ratios of all the variables can be found.

The dual problem to finding the quantities produced is the problem of finding the prices. The same matrix A appears both in finding the quantities and the prices. The prices so determined need not have any relationship to actual prices. This depends on how the value added is treated. If the value added includes some profit, then the prices presumably can be made to resemble actual prices. It is easiest just to call the prices obtained from the model "accounting" prices and not to worry too much about their actual absolute value. The differences in price are more important. As has already been suggested, the price model, can be useful in studying how all prices in the economy might be changed if there was a wage increase in the steel industry - assuming that the increase was completely passed on.

In conclusion, then, we might say that the input-output system finds its greatest use in planning type operations. It seems to be of little help at all in "explaining" the actual operation of the economy over any given period of time. Certain assumptions prevent the model from even giving an accurate representation of a static economy.

Before passing on to the dynamic generalizations we would like to make a reference to a linear programming problem which has an interesting connection to the static Leontief models. Let us consider the transportation problem. Suppose we have a homogeneous product to be shipped in amounts $a_1 \ldots a_m$ from m origins and received in amounts $b_1,\ldots,b_n$ at n destinations where $\Sigma a_i = \Sigma b_i$. Let $x_{ij}$ be the amount shipped from i to j. Then we can write

$$a_i - x_{i1} - x_{i2} - \ldots - x_{in} = 0 \ (i=1,\ldots,m) \qquad (3.19)$$

and

$$b_j - x_{1j} - x_{2j} - \ldots - x_{mj} = 0 \ (j=1,\ldots,n) \qquad (3.20)$$

These equations remind us quite a bit of a Leontief input output model in closed form. The $x_{ij}$ correspond to the amount of i's output used by industry j. The second set of equations corresponds essentially to the price equations where things are summed the other way around. Now suppose $c_{ij}$ is the cost of shipping a unit from i to j. It is of interest to find the $x_{ij}$ which will minimize the total cost

$$\sum_{ij} c_{ij} x_{ij} \qquad (3.21)$$

This is a linear programming problem. In a sense it is going the other way on a Leontief model and is finding the optimum production coefficients. More precisely the interpretation is as follows: Let there be m products and n industries. Let $x_{ij}$ be the amount of product i made by industry j. It is assumed each industry can make all products. Equations correspond to specifying the total amount of each product to be made. Equations

(3.19), (3.20) specify that each industry must operate at some specified level of activity. It is then desired to find how much of each product each industry should make in order that the total manpower required is a minimum. This is a very crude example of how linear programming can enter into the study of Leontief models. In the dynamic generalizations we will see more realistic applications of linear programming. The above example is more realistic if one thinks of it as just a segment of the economy say the chemical industry where every big company can produce every important chemical. Then each industry is a company.

## A Dynamic Leontief Model

### Mathematical Formulation

Leontief (refs. 9, 31, 32) has also proposed a dynamic extention of his input-output system. He does this by taking inventories into account. It is assumed that the inventory of good i held by industry j is dependent on the rate of production of good j. As before we have

$$x_{ij} = a_{ij} X_j \tag{3.22}$$

There is now added the relation

$$S_{ij} = b_{ij} X_j \tag{3.23}$$

where $S_{ij}$ is the stock of good i held by industry j. We will start by discussing the open model. Again let the vector $\overrightarrow{Y}$ represent the net demands. Now, however, an industry can either use its inventory or buy the good from the industry which manufactures it. Leontief here assumes, just as he did for the static case, that an industry produces only enough to meet the final net demand plus the demand of other industries. Now $X_j$ refers to production per unit time. On the other hand $S_{ij}$ is the total inventory. The rate of change of the inventory is

$$\dot{S}_{ij} = b_{ij} \dot{X}_j = b_{ij} \frac{dX_j}{dt} \tag{3.24}$$

is we assume that changes take place continuously. It should be noted that the units of $b_{ij}$ are stock of i per unit production rate of j. This might be in money terms or in actual units of the product. The equation representing the production rate of industry i must be

$$X_i = Y_i + \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} \dot{S}_{ij} \qquad (3.25)$$

or

$$X_i = Y_i + \sum_{j=1}^{n} a_{ij} X_j + \sum_{j=1}^{n} b_{ij} \dot{X}_j \qquad (3.26)$$

For the whole economy then, we have n such equations. It forms a system of coupled, linear, first order, ordinary differential equations.

If we define the matrices

$$A = \left\| a_{ij} \right\| , B = \left\| b_{ij} \right\| \qquad (3.27)$$

and the vectors

$$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} ; \quad \dot{\vec{X}} = \begin{pmatrix} \dot{X}_1 \\ \vdots \\ \dot{X}_n \end{pmatrix} ; \quad \vec{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \qquad (3.28)$$

The system of equations can be written in matrix form

$$(I - A) \vec{X} - B \dot{\vec{X}} = \vec{Y} \qquad (3.29)$$

In order to solve these equations it must be assumed that the bill of goods $\vec{Y}$ is known as a function of time. Suppose we define

$$\bar{A} = I - A \qquad (3.30)$$

The transient part of the solution is then found by solving the homogeneous equations

$$\bar{A} \vec{X} - B \dot{\vec{X}} = \vec{0} \qquad (3.31)$$

The solutions will be of the form

$$\vec{X} = \vec{k} e^{\lambda t} \qquad (3.32)$$

Substituting this into (3.31) gives

$$\bar{A} \vec{k} e^{\lambda t} - \lambda B \vec{k} e^{\lambda t} = \vec{0} \qquad (3.33)$$

or since $e^{\lambda t} \neq 0$

$$(\bar{A} - \lambda B)\vec{k} = \vec{0} \qquad (3.34)$$

This is a set of n homogeneous simultaneous linear equations in n unknowns. In order that there be solutions other than $\vec{k} = \vec{0}$ it is necessary and sufficient that

$$|\bar{A} - \lambda B| = 0 \qquad (3.35)$$

i.e., the determinant of the coefficients must vanish. This then gives an nth order polynomial in the $\lambda$'s. There will be n roots (not necessarily distinct). Associated with each $\lambda n$ there will be a $\vec{k}_n$. The vector $\vec{k}_n$ is only determined up to a scalar multiplier, i.e. $\alpha \vec{k}_n$ would also satisfy the equations. Then a general solution to the set of homogeneous equations is of the form

$$\vec{X} = \sum_{i=1}^{n} c_i \vec{k}_i \, e^{\lambda_i t} \qquad (3.36)$$

where the $c_i$ are arbitrary constants which can be determined by the initial conditions. The solution to the complete set of equations is found by adding a particular solution to the above transient solution. Suppose that $\vec{f}$ (t) satisfies

$$\bar{A}\vec{f}(t) - B\dot{\vec{f}}(t) = \vec{Y}(t)$$

Then $\vec{f}$ (t) is a particular solution and the general solution to (3.29) is

$$\vec{X} = \vec{f}(t) + \sum_{i=1}^{n} c_i \vec{k}_i \, e^{\lambda_i t} \qquad (3.37)$$

If some of the $\lambda_i$ are complex then $\vec{X}$ can exhibit oscillations. If the real parts of the $\lambda_i$ are not zero, the oscillations will either explode or die out. If the real part of $\lambda_i = 0$, then oscillations of constant magnitude can be maintained. When all the real parts of the $\lambda_i$ are negative, then as time goes on the transient solution dies out and $\vec{X}(t) \longrightarrow \vec{f}(t)$. In the circumstance where any $\lambda_i$ has a real part which is $> 0$, then $\vec{X}$ (each component of $\vec{X}$) will increase exponentially regardless of what $\vec{Y}(t)$ does. Such a situation would not make much sense in economic terms, other than pointing out that the system is unstable.

The closed dynamic Leontief model can just be represented by the homogeneous system of equations (since there are no net demands)

$$\bar{A}\,\vec{X} - B\,\dot{\vec{X}} = \vec{0} \tag{3.38}$$

We have already obtained the solution to this system of equations. It is just the transient solution which we considered previously. Again as $t \to \infty$ only the term with $\lambda_n$ having the largest real part will be important. Hence we see that all $X_i$ will ultimately move in fixed proportions, i.e.

$$\frac{X_i}{X_j} = \frac{k_{ri}}{k_{rj}} \tag{3.39}$$

where $\lambda_r$ has the greatest real part.

<u>Discussion of the Dynamic Leontief Model</u>

Leontief attempted to make a dynamic model out of the static input-output system by including inventories. He assumed that the inventory that industry j held of product i was proportional to the production rate of industry j. The static equilibrium assumption was then replaced by a continuous succession of steady states equilibrium. He now assumed that the bill of goods representing the exogeneous rate of consumption to be known as a function of time. The equilibrium assumption then took the form that at each instant of time the production rate of any industry was just the sum of the exogeneous rate of consumption, the rate of consumption by other industries, and the rate of change of inventories. Since the inventories are assumed proportional to the production rate, the rate of change of the inventories must bring in a time derivative of the production rate. Hence the original set of n simultaneous linear equations for the static input-output system here becomes a system of n coupled first order linear differential equations. If the bill of goods is known as a function of time, then these equations can be solved to give the production rate as a function of time.

Having given a description of the dynamic Leontief model, we might now ask how good a model is it of the actual workings of the national economy. Clearly the answer is not very good. We should hasten to point out, however, that here again the main purpose of the model is for use in planning purposes. It would not be expected to explain the operations of the economy in the same sense that a closed model would. All the assumptions that existed in the static model also appear in the dynamic model - no substitution, fixed ratios of inputs, etc. It might be well to investigate in more detail the way inventories are treated. First of all it seems unlikely that the assumption that the inventory stock held by a company of the products of other industries is proportional to the production rate of the given industry. This does not seem to represent very well the situation in the real world where the behavior of inventories does not seem to even approximately obey any such simple rule. The rule might be expected to hold more closely if a time lag was introduced. Really the model does not use the fact that inventories are proportional to production rates. All it uses is the assumption that the rate of change of inventories is proportional to the rate of change of the production rate. In a very rough qualitative way we would expect inventories to increase as the production rate increased and conversely, but the assumption of strict proportionality does not seem realistic.

It might be noted that in this inventory structure investment in new buildings, equipment, etc. can be included after a fashion. If one of the industries is the construction industry and another is something like heavy machinery, then if another industry holds inventories of these it is holding plant and equipment. Here one runs into even more difficulty. It is not possible to invest and disinvest in plant and equipment with every little change in the production rate. This points up the fact that investment in inventories, buildings, etc. might behave differently when production is rising than when production is falling. This brings up the notion of what we will later call relaxation phenomena. At a turning point in production, the law determining the inventory rate may change.

No provision, as such, is made for technological change in the dynamic Leontief model. If the technological coefficients $a_{ij}, b_{ij}$ where functions of time then it would be possible to include, at least in part, technological change. This would then make the system of equations nonlinear. Technological changes which completely changed production processes or introduced completely new products cannot be accounted for.

In this system it is possible to have oscillations appearing which are independent of the bill of goods. If the system is stable these oscillations will die out quickly. Presumably they appear only when the system starts out, i.e. when the economy is initially set up. They should not appear in studying the actual behavior of the economy over a given time period. What one is really interested in is the steady state solution to the system since the transient part should not be present in a mature economy. The steady state solution will exhibit oscillations only if the net bill of goods does.

## Dantzig's Model

### Introduction

Dantzig (ref. 37) has developed a variation on the Leontief open model which can be used both for static and dynamic situations. The main feature of Dantzig's model is that it allows for substitutability. That is, there is more than a single industry producing a given product. Since there will be more industries than product, we no longer obtain a set of n equations in n unknowns to be solved for the n outputs. Now there will be a system of m equations (m products) in n unknowns (n industries). The problem now is not only to find the outputs but also to ascertain which industries will produce a given product. To obtain a unique solution it is now necessary to introduce some decision criterion. Dantzig takes this to be the maximization or minimization of some linear functional (such as the total labor used). With this formulation the problem can be reduced to a linear programming problem. We are trying to maximize or minimize a linear functional subject to a set of linear equalities as constraints. While the Dantzig model can be applied to either static or dynamic situations we will discuss it from the point of view of its being a dynamic model.

Dantzig's model can be considered to be a fairly general dynamic Leontief system with discrete time periods in which (a) alternative substitute activities are allowed, (b) a bill of goods is given over time, and (c) the unknown quantities of activities satisfying the system are to be determined so as to minimize or maximize a specified linear objective function. The model has two features (1) the general Leontief aspect which assumes that there is one and only one output for each activity and (2) a one way flow aspect for which inputs for production in the nth time period can occur only from production in that time period and previous time periods.

In the optimal solution certain activities appear in positive amounts and others in zero quantities. Once it is known which quantities are positive and which are zero, the actual numbers can be obtained by direct solution of the equations. Hence, the fundamental problem is to select the activities which appear in positive quantities. It turns out that (a) the selection of these activities can be made independent of the bill of goods and (b) there is a local optimization property which means that selection of activities for the first k time periods can be made without consideration of the nature of activities producing in time periods later than k.

Mathematical Formulation

Let us now set up the model in detail. The derivation will be carried out in somewhat more detail than Dantzig gives. In order to understand the procedure, however, it is necessary to know something of the simplex method of linear programming. We assume that there will be r time periods $t_1 \ldots t_{r_1}$. These do not need to be of the same length. We let $X_{ji}$ be the amount of some product j produced in time period i. Now for the cases to be considered any industry j in time period i can have inputs from time period i and all previous time periods. Thus in any given time period there must be not only enough production to meet the final net demand (the bill of goods) and to supply other industries for time period i but also enough to supply the industries in periods i+1, ..., r which need inputs from period i. Now we need not assume that the number of industries and products is the same for each time period. We will assume that there are $m_i$ products and $n_i$ industries in time period i. Then a typical equation for product j in time period i can be written

$$\sum_{k=1}^{n_i} a_{jk}^{(i)} X_{ki} + \sum_{k=1}^{n_{i+1}} a_{jk}^{(i+1)} X_{ki+1} + \ldots + \sum_{k=1}^{n_e} a_{jk}^{(r)} X_{kr} = b_j^{(i)} \qquad (3.40)$$

where $b_j^{(i)}$ is the net demand for j in time period i. The first sum represents the production necessary to meet net demand and to supply other industries for time period i. The second sum is the production necessary to supply industries in time period i+1, etc. Until the final sum is the production necessary to supply industries in the final time period r. Now in time period i several industries can produce product j. Hence $a_{jk}^{(i)} > 0$ for industries producing j and $a_{jk}^{(i)} \leq 0$ for all others. A positive a means something is produced of j and a negative a means something is purchased. Then $a_{jk}^{(i+1)} \leq 0, \ldots a_{jk}^{(r)} \leq 0$ for all j,k. It should be noted that it is not necessarily true that in time periods i+1, ...,r that there will even be any product j. This may or may not be true. Some products may disappear while new products appear as the time periods change. For all the products in time period i we can then write the following matrix equation

$$A_{ii}\vec{X}_i + A_{ii+1}\vec{X}_{i+1} + \ldots + A_{in}\vec{X}_n = \vec{b}_i \qquad (3.41)$$

$A_{ii}$ is an $m_i \times n_i$ matrix, $A_{ii+1}$ is an $m_i \times n_{i+1}$ matrix, etc. $\vec{X}_i$'s are the following column vectors:

$$\vec{X}_i = \begin{pmatrix} X_{1i} \\ X_{2i} \\ \vdots \\ X_{ni} \end{pmatrix} ; \quad \vec{X}_{i+1} = \begin{pmatrix} X_{1i+1} \\ X_{2i+1} \\ \vdots \\ X_{ni+1} \end{pmatrix} ; \quad \vec{X}_n = \begin{pmatrix} X_{1n} \\ X_{2n} \\ \vdots \\ X_{nn} \end{pmatrix} \qquad (3.42)$$

We are assuming in the above that an industry need not produce for any time period after $t_n$. Thus for the time period $t_r$, we just have the simple form

$$A_{rr}\vec{X}_r = \vec{b}_r \qquad (3.43)$$

It should be remembered that for all time periods there can be more industries than there are products. Hence some of the X's can be zero. Our total system of equations can then be set down. The system will take the following form:

$$\begin{aligned} A_{11}\vec{X}_1 + A_{12}\vec{X}_2 + \ldots\ldots\ldots\ldots + A_{1r}\vec{X}_r &= \vec{b}_1 \\ A_{22}\vec{X}_2 + \ldots\ldots\ldots\ldots + A_{2r}\vec{X}_r &= \vec{b}_2 \\ A_{33}\vec{X}_3 + \ldots\ldots + A_{3r}\vec{X}_r &= \vec{b}_3 \\ A_{rr}\vec{X}_r &= \vec{b}_r \end{aligned} \tag{3.44}$$

The equations take on this triangular form because there is nothing produced in time period i for the time periods which preceeded i. This can all be combined into one large matrix A as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \ldots\ldots\ldots & A_{1r} \\ 0 & A_{22} & A_{23} & \ldots\ldots\ldots & A_{2r} \\ 0 & 0 & A_{33} & \ldots\ldots\ldots & A_{34} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \ldots\ldots\ldots & A_{rr} \end{pmatrix} \tag{3.45}$$

This is really a partitioned matrix since the elements $A_{ij}$ are themselves matrices. The size of the matrix is $m_1 + m_2 + \ldots + m_r$ columns. We can then write our complete system of equations as

$$A\vec{X} = \vec{b} \tag{3.46}$$

This system of equations is called the constraint system. It is desired to maximize or minimize some function of the form

$$\sum_{i=1,j=1}^{n_i,e} c_{ij}X_{ij} \tag{3.47}$$

This functional may represent the amount of labor used, the total cost, etc.

The model outlined above is precisely a linear programming problem. It could be solved directly by the simplex method. However, such a model would in general have a tremendous number of equations in it. The task of solving it by the simplex method could indeed become a prodigious one. Thus it is desirable to have some more simple way of handling the problem. We will now show how to solve the problem. It is first desirable to review very briefly the simplex method. The simplex method deals with maximizing a form $\Sigma\, c_i x_i$ subject to a set of linear constraints which can be written

$$A\vec{x} = \vec{b} \; ; \; A = (\vec{P}^{(1)} \; \ldots \; \vec{P}^{(n)}) \tag{3.48}$$

where A is an mxn matrix with column vectors $P^{(i)}$. Any m linearly independent $\vec{P}^{(i)}$'s will form a basis for the space. Call the matrix of these vectors B. B is an mxm matrix and is nonsingular. Let the variables associated with the columns of B be $x_1, \ldots, x_m$. If variables $x_{m+1} \ldots x_n$ are set equal to zero a basic solution is obtained. It is a familiar property of a linear programming problem that the optimal solution will never have more than m, x values different from zero, i.e. the optimal solution will be a basic solution. To decide if the initial solution given by $\vec{x} = B^{-1}\vec{b}$ is the optimal compute

$$\vec{y}_j = B^{-1}\vec{P}^{(j)} \tag{3.49}$$

for each $\vec{P}^{(j)}$ not in the basis. Then find

$$\vec{c} \cdot \vec{y}_j - c_j \tag{3.50}$$

$\vec{c}$ is the vector formed from the "prices" of the variables in the basis and $c_j$ is the price associated with $\vec{P}^{(j)}$. It will be assumed we are trying to maximize $\Sigma\, c_i x_i$. Then if (3.50) is $\geqslant 0$ for every j the solution is optimal. If we are minimizing $\Sigma\, c_i x_i$, if (3.50) is $\leqslant$ for every j the solution is optimal. If the solution is not optimal the vector and variable j are put into the basis which make (3.50) the most negative for maximization or the most positive for minimization. It is also possible to compute which vector should come out of the basis. We will not need this for the present problem, however.

Let us now consider the problem of interest. We want to make full use of the peculiar nature of the matrix structure, i.e. its triangular form. It can first be noted that all we need to do to get a basis is to choose one industry for each product and set the production of all others equal to zero. Now it is a well known property of linear programming that the optimal solution will be a basic solution, i.e. no more than $m_1 + m_2 + \ldots + m_n$. This then means that no more than a single industry will be producing a given product. Furthermore putting a new vector into the basis and taking an old one out just means substituting one activity (industry) for another to make a given product.

In the basis matrix, the diagonal submatrices will be square, i.e. B will be of the form

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ & B_{22} & \cdots & B_{2r} \\ 0 & & \cdots\cdots & \\ & & \cdots\cdots & \\ & & & B_{rr} \end{pmatrix} \tag{3.51}$$

$B_{11}$ is an $m_1 \times m_1$ matrix, etc., $B_{rr}$ is an $m_r \times m_r$ matrix. The off diagonal submatrices need not be square. For example, $B_{12}$ is $m_1 \times m_2$, etc. Now since B is nonsingular, the diagonal submatrices will be nonsingular also. We will assume that every number in the bill of goods is positive. Now any activity vector representing production in the first time period has non zero elements only in the first $m_1$ places. Hence such a vector can be expressed as a linear combination only of columns appearing in $B_{11}$. Thus

$\vec{y}_j = B^{-1}\vec{P}^{(j)}$ for a $\vec{P}^{(j)}$ referring to production in the first time period has elements only in the first $m_1$ places. The scalar product $\vec{c} \cdot \vec{y}$ then only involves the c's corresponding to the columns in $B_{11}$. Thus on applying the criterion (3.50) for whether activity $\vec{P}^{(j)}$ should enter the basis and the activity in the basis producing the same good should come out, we need only concern ourselves with the submatrix $B_{11}$. Thus we can optimize the production in the first time period without concerning ourselves with what goes on in the other time periods. Thus it is clear that once the set of first period activity vectors in the basis has been selected so that

$$\vec{c} \cdot \vec{y}_j \geqslant c_j \text{ for maximization or } \vec{c} \cdot \vec{y}_j \leqslant c_j \text{ for minimization}$$

no other shifts in the basis will ever affect this set.

Turning our attention now to the second or later time periods, it will be observed that any replacements of a column in the basis by an activity vector associated with the second period or later will only eliminate a vector belonging to the same period. The best choice of activity vectors for the second time period can also be made using information derived from the second sub-matrix $A_{22}$. To do this it is necessary to pull a slight trick. The manipulations are as follows: The basis matrix is given by (3.51). We now take the inverse to be of the same form, i.e.

$$B^{-1} = \begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} & \bar{B}_{13} & \cdots & \bar{B}_{1r} \\ & \bar{B}_{22} & \bar{B}_{23} & \cdots & \bar{B}_{2r} \\ & \cdots & \cdots & \cdots & \cdots \\ & & & & \bar{B}_{rr} \end{pmatrix} \qquad (3.52)$$

Now since $B\,B^{-1} = I$ we get

1) $B_{11}\bar{B}_{11} = I$ or $\bar{B}_{11} = B_{11}^{-1}$ (the diagonal submatrices are nonsingular)

2) $B_{22}\bar{B}_{22} = I$ or $\bar{B}_{22} = B_{22}^{-1}$

3) $B_{11}\bar{B}_{12} + B_{12}\bar{B}_{22} = 0$ or $\bar{B}_{12} = - B_{11}^{-1} B_{12}B_{22}^{-1}$ (3.53)

All the other elements of $B^{-1}$ can be generated in this way. If we denote by $\vec{P}_1^{(j)}$ the elements 1, ..., $m_1$ of $\vec{P}^{(j)}$ and $\vec{P}_2^{(j)}$ the elements $m_1+1, .., m_1+m_2$ we can write

$$\vec{y}_j = B^{-1}\vec{P}^{(j)} = \begin{pmatrix} B_{11}^{-1} \vec{P}_1^{(j)} - B_{11}^{-1} B_{12}B_{22}^{-1} \vec{P}_2^{(j)} \\ B_{22}^{-1} \vec{P}_2^{(j)} \\ 0 \\ 0 \end{pmatrix} \quad (3.54)$$

when $\vec{P}^{(j)}$ is a vector associated with production in the second time period. It will be remembered that such a vector has zeros in all places after $m_1 + m_2$. If we denote by $\vec{c}_1$ the first $m_1$ elements of $\vec{c}$ and $\vec{c}_2$ the elements $m_1 + 1, \ldots m_1 + m_2$ of $\vec{c}$, we can write

$$\vec{c} \cdot \vec{y}_j = \vec{c}_1 \cdot B_{11}^{-1} \vec{P}_1^{(j)} - \vec{c}_1 \cdot B_{11}^{-1} B_{12}B_{22}^{-1} \vec{P}_2^{(j)} + \vec{c}_2 \cdot B_{22}^{-1} \vec{P}_2^{(j)}$$

The criterion for optimality then depends on the sign of

$$[c_2 - \vec{c}_1 \cdot B_{11}^{-1} B_{12}] B_{22}^{-1} \vec{P}_2^{(j)} - [c_j - \vec{c}_1 \cdot B_{11}^{-1} \vec{P}_1^{(j)}]$$

This can be written

$$\vec{c}_2' \cdot B_{22}^{-1} \vec{P}_2^{(j)} - c_j' \quad (3.55)$$

It is then obvious that the proper vectors for the optimal basis in the second time period we only need consider a smaller local problem and not the whole inverse. Furthermore the optimization can be made independent of all time periods except the first (note that $B_{11}$ is here taken to be the optimum $B_{11}$ not the initial $B_{11}$ before time period one was optimized.)

One can proceed in the same way and optimize all the time periods locally, i.e., only using information from that time period and the known optimum values from previous time periods. It will be noted that all the optimization has been carried out independent of the bill of goods. This is not a typical property of all linear programming problems. It is true here because first of all the original basis or subsequent basis is feasible whatever be the bill of goods if it is feasible for at least one bill of goods with $b_i > 0$ for all i, and since the activity to be dropped from the basis in a Leontief system is always the substitute activity. Thus there is no problem of which vector to remove from the basis.

Discussion of the Dantzig Model

The Dantzig model differs from the ordinary Leontief model in that it allows for substitution. It does not ultimately provide for more than one industry producing a given product. The industry that will produce each product is determined by maximizing or minimizing some linear form. The other industries which could produce the product will end up producing nothing. This model is then even more of a planned economy type of model than the ordinary Leontief model. By some decision criterion it selects those industries which will produce and those which are inefficient and consequently will not produce. Certainly it would not be expected that any model whose behavior is determined by maximizing or minimizing some function could not be expected to be an accurate model of the actual economy. Of course, this is not the purpose of the Dantzig model.

The behavior of inventories in this model is not governed by any simple rule. Production in a given period can be for use in later periods. The amount made in period i for use in period j depends upon the production in period j and is proportional to it. It is not made clear just what a sensible procedure would be, in the real world, for determining these constants of proportionality. The interesting thing about this model is that we don't know the production in the first time period until we have solved the problem for all time periods.

Most of the comments on the static and dynamic Leontief models also carry over directly to the Dantzig model.

Von Neumann's Model

Introduction

Von Neumann (ref. 9, 28, 38) in 1932 presented a model of an expanding economy. Again this is a linear model. It is also an equilibrium type model and represents an economy expanding uniformly at maximum rate. The assumptions made are: i) there constant returns on invested capital, ii) there is availability of land and labor for unlimited expansion, iii) all goods are made from each other, and iv) all incomes in excess of the needs to maintain labor are reinvested. It is supposed that there are m commodities and n activities for producing these goods ($n > m$). Any given activity can produce or use any or all of the commodities. The questions which the model attempts to answer are i) which activities will be used, ii) what is the relative rate of increase of economic activity, iii) what are the prices, and iv) what is the interest rate?

We next want to proceed with the mathematical background for the von Neumann model. The mathematics used here is very similar to that found in the theory of games. In fact the basic theorem on which this model depends is also the basic theorem in the theory of games. We will alter the usual mathematical presentation somewhat by making more use of the duality theory of linear programming. This considerably simplifes, the presentation. In order to discuss the von Neumann model it is necessary to introduce some of the concepts of activity analysis. We do this first.

Activity Analysis

Many models of the national economy are concerned with allocation of resources. In this allocation it is of interest to optimize some particular objective function. We further require that the quantities and prices be non-negative. This problem is then clearly a programming problem. If linearity is introduced it is a linear programming problem. Activity analysis deals with such linear models. Activity analysis can eliminate one of the objections raised about the Leontief model in that it can allow for substitution. Ultimately only a basic set of activities will be used but they will come out of the solution to the problem. Activity analysis has a more direct bearing on the economics of socialism and collectivist planning than it does for the U.S. economy since in the previous case it is possible to make decisions on who will produce what. It has received considerable attention in this country, however, and does give a number of useful insights into the structure of things.

Activity analysis deals with commodities, activities and technologies. We define each of these concepts in turn.

1. Commodities are the primary resources, intermediate products, and final products which the economy uses.

a. Primary factors (such as labor and raw materials) are not made but are available as resources from outside the system.

b. Intermediate products are those produced and consumed within the system.

c. Final products are those produced within the system and made available to meet the objectives of the system.

We have assumed that there are m commodities. Let $x_r$ be the net output of commodity r; it may be positive, negative, or zero. If $x_r > 0$, the rth commodity is a final product, meeting final requirements. If

$x_r < 0$, r is an input, a factor used up. If $x_r = 0$, r is an intermediate product which is produced and also used up in the system.

2. Activities involve combinations of commodities, some being inputs and some being outputs. The inputs and outputs are all in fixed proportions. An activity for example might be the chemical industry. It uses certain commodities as inputs and produces other commodities. To specify in a quantitative way the chemical industry as an activity we would specify how much of each commodity this industry was using. It might not use any of some commodities. We would like to be able to allow the indsstry to change the level of its operations (always changing things in the same proportion, however). This means that what we really want is what might be called a specification of a unit level of operation for this industry or activity. One way to do this is choose some commodity which would be one of the products of the activity and base everything on one unit of this product. A unit activity can be defined in this way. Any level of operation of the activity is then just a scalar multiple of the unit activity level. A unit level of an activity (say the jth) can then be represented as an m dimensional column vector

$$\vec{A}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} \qquad (3.56)$$

where $a_{ij}$ is the amount of theicommodity needed in activity j to maintain it at a unit level. If $a_{ij} < 0$ the commodity is an input, if $a_{ij} > 0$ it is an output, and if $a_{ij} = 0$ the commodity is not used by the activity except perhaps as an intermediate product.

The linearity assumptions are next introduced. These involve

a) Additivity: Two activities can be added without change in their structural coefficients, i.e.

$$\vec{A}_i + \vec{A}_j = \| a_{ei} + a_{ej} \| \qquad (3.57)$$

b) Level of Activity: The level of an activity is given by a scalar multiple of the unit activity, i.e.

$$x \vec{A}_j = \| x\, a_{ej} \| \qquad (3.58)$$

The above two assumptions imply constant returns to scale both within one activity and in combining activities. Furthermore it is assumed that the level of operation is perfectly divisible.

3. A technology can be considered to consist on n activities. It can be represented by the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad (3.59)$$

Reading across a row of A gives the inputs or outputs of a given commodity in the n activities. Reading down any column gives the inputs or outputs of the m commodities for that activity. The matrix A thus summarizes all the technical possibilities of the economy.

If $\vec{x}$ is the column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ which gives the level of each activity in the technology, then

$$b = A\, x \qquad (3.60)$$

gives the net amounts of each commodity used or produced in the technology when operating at a level $\vec{x}$. Now it is a familiar theorem of convex sets that for all $x_i \geqslant 0$, $\sum_{i=1}^{n} \vec{A}_i x_i$ generates a convex polyhedral cone. If a $\vec{b}$ is specified there is a feasible allocation of resources (i.e. one with $\vec{x} \geqslant \vec{0}$) if and only if $\vec{b}$ is an element of the convex polyhedral cone spanned by the $\vec{A}_i$. The n dimensional space of the $\vec{x}$'s is called the activity space and the m dimensional space of the $\vec{A}_i$ and $\vec{b}$ is called the commodity space. These correspond exactly to the solutions space and requirements space in linear programming.

The technology as a whole is said to be operating at a unit level (in contrast to a unit activity level) if $\sum_{i=1}^{n} x_i = 1$, $x_i \geqslant 0$. The collection of points

$$\vec{\xi} = A\vec{x}, \ \Sigma x_i = 1, \ x_i \geqslant 0$$

is just the convex hull of the $\vec{A}_i$ in the commodity space. The set of all rays $\mu \vec{\xi}$, $\mu \geqslant 0$ generates the same convex polyhedral cone referred to previously.

The distinction between the interior and the boundary of the convex polyhedral cone is important. A point is in the interior if there exists a hypersphere about the point which contains only points of the cone. A point is on the boundary if every hypersphere about the point contains both points belonging to the cone and those not belonging to the cone. A collection of points formed by non-negative linear combinations of m independent activities also forms a convex cone which is a subset of the cone generated by the n activities. This subset is called a facet and is on the boundary of the general cone. Any efficient allocation of production, obtained say by maximizing or minimizing a linear functional, will have an activity level $\vec{x}$ which lies on a facet of the cone.

Activity Analysis for Dynamic Situations

We would here like to generalize the activity analysis theory developed in the previous section to take time into account. We will let $\vec{A}_i(t)$ be the ith activity at time t. The structural coefficients in $\vec{A}_i$ can now change with t. Similarly $A_t$ will be the technology matrix at time t. We will assume that the number of commodities and the number of activities remain fixed over time. The time period of interest will be broken up into a number of discrete time intervals (which may be months or years, for example). It is convenient to break up an activity into a set of inputs at the beginning of the time period and a set of outputs at the end of the time period, i.e. write

$$\vec{A}_i(t) = \vec{B}_i(t) - \vec{D}_i(t) \tag{3.61}$$

In this way the numbers appearing in $\vec{B}_i$ and $\vec{D}_i$ are all positive. $\vec{B}_i$ gives the outputs at the end of time period t (actual time t) while $\vec{D}_i(t)$ gives the inputs at the beginning of time period t (actual time t-1). Then the input for period t is at the activity level $\vec{x}_t$ (actual time t-1)

$$D_t \vec{x}_t$$

and the output of period t-1 at a level $\vec{x}_{t-1}$ (actual time t-1) is

$$B_{t-1}\vec{x}_{t-1}$$

Then assuming that there is no inventory from production in previous periods an amount $D_t\vec{x}_t - B_{t-1}\vec{x}_{t-1}$ must be supplied from outside the system at the beginning of time period t. If $\vec{b}_t$ is the maximum amount of the given commodities that can be supplied at the beginning of any time period, it follows that

$$D_t\vec{x}_t - B_{t-1}\vec{x}_{t-1} \leq \vec{b}_t, \quad \vec{x}_t \geq \vec{0} \tag{3.62}$$

The linear programming problem comes in when it is desired to maximize some given objective function (such as employment) over the entire period, i.e.

$$\max \quad z = \sum_t \vec{c}_t \cdot \vec{x}_t \tag{3.63}$$

The solution to this problem also determines the "prices" since the dual problem is

$$D'_t \vec{p}_t - B'_{t-1}\vec{p}_{t-1} \geq \vec{c}_t, \quad \vec{p}_t \geq \vec{0} \tag{3.64}$$

$$\min \sum_t \vec{b}_t \cdot \vec{p}_t$$

Description of the von Neumann Model

The von Neumann model is not a linear programming model. In (3.62), (3.63) he sets $\vec{b}_t \equiv \vec{0}$, $\vec{c}_t \equiv \vec{0}$ so that the linear programming aspects of the problem disappear. He further assumes that the technology does not change with time (i.e., no technological improvement). Only the activity level changes with time. Thus he obtains the system of inequalities

$$D\,\vec{x}_t - B\,\vec{x}_{t-1} \leq 0 \quad ; \vec{x}_t \geq \vec{0} \tag{3.65}$$

$$D'\,\vec{p}_t - B'\,\vec{p}_{t-1} \geq 0 \quad ; p_t \geq \vec{0} \tag{3.66}$$

The implications of the above are that no commodities are available from outside the system, i.e. it is closed and that there is no limit on the quantities of primary factors such as land, labor, etc. available in the system. Only the ratios of the $x_i(t)$ and $p_i(t)$ can be found. Naturally for such a system the inputs at t do not exceed the outputs of the previous period. There are certain things that one would expect of the dual problems and these are now added. If for any constraint i in (3.65) the $<$ sign holds, then for this commodity i, $p_i=0$, i.e. it is a free good. The $p_i$ are just as in the Leontief model accounting prices. The profits in the system are never positive since the sum paid for inputs (at unit activity) in time period t is at least as great as the sum made (at unit activity) in the previous time period (from (3.66)). If in (3.66) the $>$ sign holds for constraint i, then the ith activity is not used, i.e. is used at a zero level.

Von Neumann makes the assumption that the economy is expanding at constant rate. If we write the expansion factor as

$$\alpha = 1 + \frac{r}{100} \tag{3.66}$$

where is the percentage expansion in each period, then

$$\frac{x_i(t+1)}{x_i(t)} = \alpha \quad \text{all } i \tag{3.67}$$

If i is the interest rate and

$$\beta = 1 + \frac{i}{100} \tag{3.68}$$

then the dual relation is

$$\frac{p_i(t+1)}{p_i(t)} = \beta \quad \text{all } i \tag{3.69}$$

The above can be written $\vec{x}_{t+1} = \alpha\, \vec{x}_t$, $p_{t+1} = \beta\, \vec{p}_t$. Then (3.65),(3.66) become

$$\alpha\, D\, \vec{x}_t \leq B\, \vec{x}_t \tag{3.70}$$

$$\beta\, D'\, \vec{p}_t \geq B'\, \vec{p}_t \tag{3.71}$$

Now let $\vec{p}_* \geq \vec{0}$ be a column vector which is not necessarily a solution to (3.71). However from (3.70)

$$\alpha\, \vec{p}_*'\, D\, \vec{x}_t \leq \vec{p}_*'\, B\, \vec{x}_t \tag{3.72}$$

$$\text{or} \quad \alpha \quad \frac{\vec{p}_*'\, B\, \vec{x}_t}{\vec{p}_*'\, D\, \vec{x}_t} \tag{3.73}$$

Similarly, let $\vec{x}_*' \geq \vec{0}$ be a column vector not necessarily a solution to (3.70). However from the transpose of (3.71)

$$\beta\, \vec{p}_t'\, D\, \vec{x}_* \geq \vec{p}_t'\, B\, \vec{x}_* \tag{3.74}$$

$$\text{or} \quad \beta \geq \frac{\vec{p}_t'\, B\, \vec{x}_*}{\vec{p}_t'\, D\, \vec{x}_*} \tag{3.75}$$

Suppose we define the function $\phi\,(\vec{x}_*\ \ \vec{p}_*)$ is

$$\phi(\vec{x}_*, \vec{p}_*) = \frac{\vec{p}_*'\, B\, \vec{x}_*}{\vec{p}_*'\, D\, \vec{x}_*} \tag{3.76(}$$

Then from (3.73(, (3.75)

$$\phi(\vec{p}_*, \vec{x}) \geq \alpha\ ;\ \phi(\vec{p}, \vec{x}_*) \leq \beta \tag{3.77}$$

We will next evaluate $\phi\,(\vec{p}_t, \vec{x}_t)$ where $\vec{p}_t$, $\vec{x}_t$ are actual solutions of (3.71),(3.70) respectively. Converting (3.70) to a system of equalities by introduction of slack we have

$$\alpha D \vec{x}_t + I \vec{X} = B \vec{x}_t \tag{3.78}$$

where X are slack variables. Then

$$\alpha \vec{p}_t' D \vec{x}_t + \vec{p}_t' \cdot \vec{X} = \vec{p}_t' B \vec{x}_t \tag{3.79}$$

However, from our duality assumptions, if $p_i(t) \neq 0$ then $X_i = 0$ and conversely. Hence

$$\vec{p}_t' \cdot \vec{X} = 0$$

and

$$\alpha = \frac{\vec{p}_t' B \vec{x}_t}{\vec{p}_t' D \vec{x}_t} \tag{3.80}$$

In precisely the same way we find

$$\beta = \frac{\vec{p}_t' B \vec{x}_t}{\vec{p}_t' D \vec{x}_t} = \alpha \tag{3.81}$$

The question which then arises is to whether solutions $\vec{p}_t$, $\vec{x}_t$ actually exist. Von Neumann proved using the Bronwer fixed point theorem of topology. The proof is rather intricate and we will not attempt to reproduce it here. It is really a slight generalization of the fundamental theorem for zero sum two person games.

Since $\phi(\vec{p}, \vec{x}^*)$ can be thought of as the rate of growth for any $\bar{\alpha}$, then $\bar{\alpha} \leq \alpha$ and consequently we see that $\alpha$ is the maximum rate of growth for the economy. Similarly $\beta$ is the minimum rate of interest. We then have here an economy in equilibrium growing at its maximum rate.

Discussion of the von Neumann Model

The von Neumann model of the national economy is a most unusual one indeed. It certainly could not in any sense be considered a realistic model of the economy. Indeed, it is probably the most unrealistic of any of the models considered. It visualizes a closed economy so that there are no inputs from the outside. As with the Leontief models it imagines a group of industries each of which produces only one product. These industries use the inputs in fixed proportions. There is no possibility for technological change. There can be more industries than products, however. In the solution only one industry will produce a given product and the others which can produce that product will operate at a zero level (i.e. do not produce). All production except that needed to maintain labor is reinvested. The economy of von Neumann is then one that is growing uniformly in every sector and is growing at maximum rate.

The economy which is the solution of the von Neumann model has the following additional properties:

a) Free goods: The input of any commodity cannot exceed the output of the previous period. In the event that the input is less than the output, the commodity has a zero price and is called a free good.

b) Unused activities: The prices are such that no activity has a positive profit. If the activity has a negative profit it will be at a zero level and hence is not used.

c) Rate of expansion: The rate of expansion of the economy is the largest possible rate.

d) Rate of interest: The rate of interest is the smallest possible compatible with no positive profits.

e) The equilibrium property: The equilibrium rate of growth has the maximum rate of expansion the same as the minimum rate of interest.

The model while not in any way representing the real world does point out the dual nature of quantity produced and prices. This was important in the development of duality theory in linear programming. The model also indicates that it might in general be very difficult to achieve uniform growth in all sectors of the economy. We will see in our discussion of relaxation phenomena that this is somewhat of a rare circumstance for linear type models.

## Relaxation Phenomena

### Introduction

In our discussion of the dynamic Leontief model it was noted that the assumption that inventories were proportional to the production was not too satisfactory. In particular we noted that it would not be possible to disinvest in buildings and equipment in the same way that it was possible to invest in them. Consequently it is to be expected that the law governing investment on the upswing of a cycle will be different from that on the downswing. Consequently at a turning point in the production history one might expect a discontinuous change from one law to a different one. This idea of a very rapid change from one behavior to another is reminiscient of relaxation oscillations in physics.

We would now like to discuss briefly the applications of relaxation oscillations to the construction of dynamic models of the national economy. The subject has received little attention in economics although it has been intensively studied in engineering during recent years. The material we will discuss on linear relaxation phenomena follows in part the article by Georgescu-Roegen (ref. 36). In his book on non-linear oscillations, Minorsky (ref. 40) defines relaxation oscillations as "those quasi-discontinuous oscillations in which rapid changes between certain levels of a physical quantity occur as the result of the loss of a certain internal equilibrium in the system." In other words there is a point where the physical variable undergoes an essentially discontinuous change from one level to another. An example is the blocking oscillator which is used for the sweep on a cathode ray oscilliscope.

For our economic applications of the subject we will depart quite radically from the above definition. We will assume that there are two different laws which govern the behavior of some economic variable. One law holds under certain circumstances and the other law holds under other circumstances. For example the law governing investment might be one thing when business is expanding and another when it is contracting. We will assume that there is an instantaneous, discontinuous change from use of one law to the use of the other law. In general this will be a periodic phenomena in time. With our formulation it is not necessary that either the variable or its derivative change discontinuously when changing from one law to the other. Here we differ considerably from the classical definition of relaxation phenomena. We are so different that it might not be too wise to apply the name relaxation. However, it has been done so we will follow it. Furthermore, we will assume that the periodic behavior is in the phase, i.e. the scope of the variable not in the actual value of the variable. For example in business cycles we do not expect the national incomes to be exactly repeated in each cycle. Instead $\dot{y}$ is the quantity which is periodic.

It seems that the idea of applying the theory of relaxation phenomena arose when a certain difficulty was encountered with a dynamic model presented at a Harvard Seminar by Leontief in 1941. Leontief considered the following model (the general case of which we have already discussed)

$$\begin{aligned} x_1 &= a_{21}x_2 + b_{11}\dot{x}_1 + b_{21}\dot{x}_2 \\ x_2 &= a_{12}x_1 + b_{12}\dot{x}_1 + b_{22}\dot{x}_2 \end{aligned} \tag{3.82}$$

$\dot{x}_1, \dot{x}_2$ are the derivatives with respect to time of $x_1$, $x_2$. This model has $x_i$ as the gross production in monetary terms.

$a_{21}x_2$ is the amount that must be spent for $x_2$ and $b_{11}\dot{x}_1 + b_{21}\dot{x}_2$ is the amount needed for investment. The model thus includes capital changes say for investment in new buildings or inventory.

The equations can be written

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \tag{3.83a}$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2$$

$$A_{11} = \Delta^{-1}(b_{22} + a_{12}b_{21}); \quad A_{12} = -\Delta^{-1}(b_{21} + a_{21}b_{22})$$

$$A_{21} = -\Delta^{-1}(b_{12} + a_{12}b_{11}); \quad A_{22} = \Delta^{-1}(b_{11} + a_{21}b_{12})$$

$$\Delta = b_{11}b_{22} - b_{12}b_{21}$$

If we assume a solution of the form $x_i = a_i e^{\lambda t}$, $\lambda$ must satisfy the characteristic equation

$$\begin{vmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{vmatrix} = 0 \tag{3.85}$$

The solutions, when the roots are distinct, are then of the form

$$x_1 = A\,e^{\lambda_1 t} + B\,e^{\lambda_2 t} \tag{3.86}$$

$$x_2 = A\,\mu_1 e^{\lambda_1 t} + B\,\mu_2 e^{\lambda_2 t}$$

where $\mu_i$ is a function of the $a_{ij}$ and $b_{ij}$ determined from (3.34)

Starting from some initial conditions Leontief assumed the solution to be given by (3.86) until a turning point with $\dot{x}_1 = 0$ was reached. He then assumed the behavior of the system changed discontinuously to a new law with $b_{11} = 0$, i.e. if $\dot{x}_1 < 0$, there is no disinvestment. Then we get a new solution

$$x_1 = A' e^{\eta_1 t} + B' e^{\eta_2 t} \tag{3.87}$$
$$x_2 = A'\mu_1 e^{\eta_1 t} + B'\mu_2 e^{\eta_2 t}$$

where A' and B' are determined by making $x_1$ and $x_2$ at the turning point (time $t_o$) the same as $x_1$ and $x_2$ determined from (3.86). This law will continue to hold until say $\dot{x}_1$ becomes zero again (or some other similar criterion). Then we go back to the law given by (3.86) except that A and B can now change. The new A and B values are determined in the same way that A' and B' were found. Leontief then asked the question whether it would be possible to predict in a finite number of steps the final outcome of the system. It is this problem which we would like to consider briefly. What we are really asking is what is the asymptotic behavior of $x_1$ and $x_2$. If they approach a unique asymptotic value it would be typical in economics to say that the dynamic system tends toward a unique equilibrium. A case of this type for $x_1$ is illustrated in Fig. (3.2). $x_2$ will show a similar behavior.

Phase Plane Treatment of Relaxation Oscillations

We will now study the behavior of a system of differential equations of the form

$$\dot{x}_1 = P(x_1,x_2); \quad \dot{x}_2 = Q(x_1,x_2) \tag{3.88}$$

Any point for which P, Q are not simultaneously zero is called a regular point and only one integral curve passes through such a point. P = Q = 0 is a singular point. It is not true necessarily, that only a single curve passes through a singular point. We may eliminate the time from (3.88) to give

$$\frac{dx_2}{dx_1} = \frac{Q}{P} \tag{3.89}$$

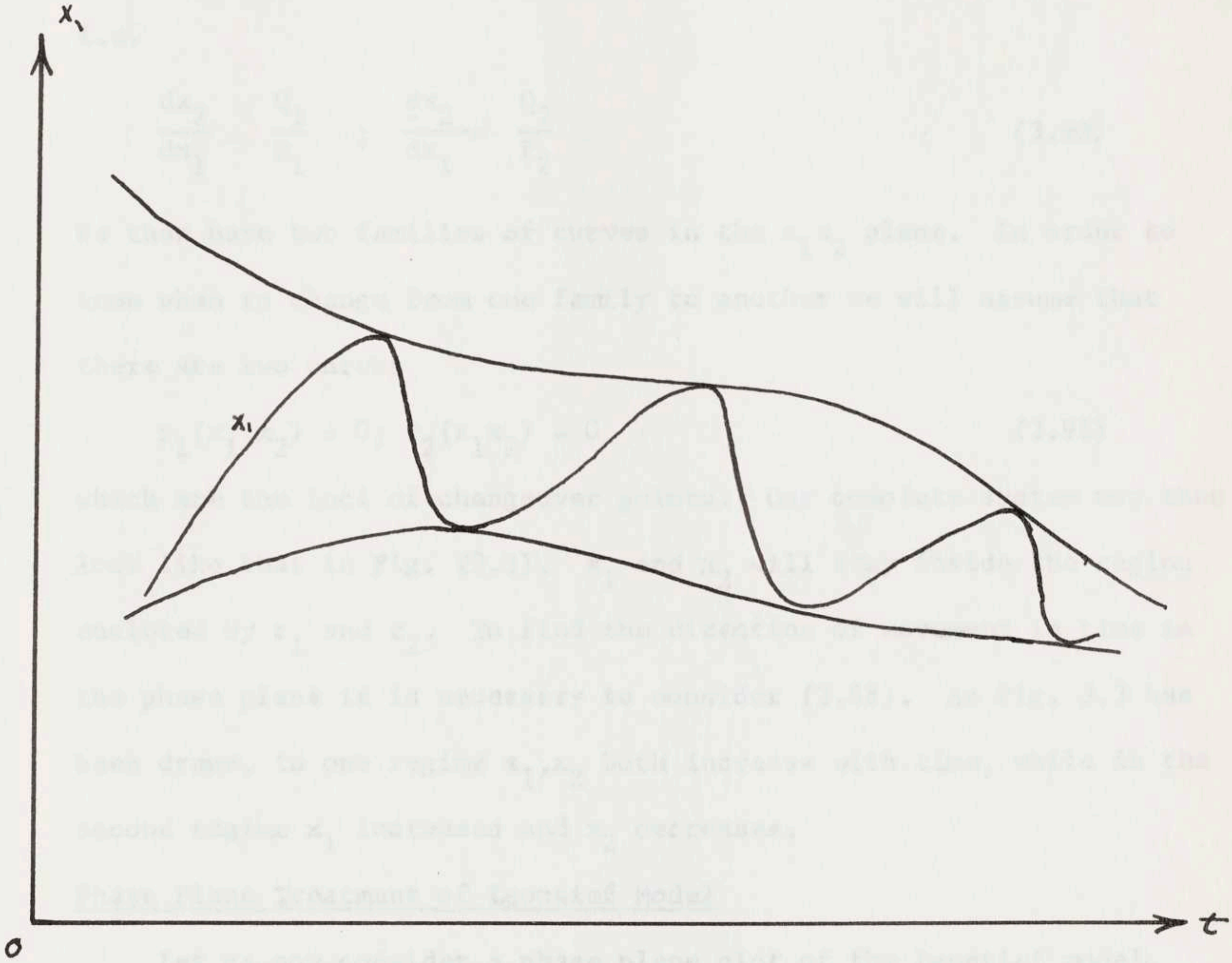



FIGURE 3.2

This is the differential equation for a family of curves in the $x_1x_2$ plane. This plane is called a phase plane and it will be in this plane that we will study the set of equations (3.88). In actuality we assume that there are two independent laws and hence two equations, i.e.

$$\frac{dx_2}{dx_1} = \frac{Q_1}{P_1} \quad ; \quad \frac{dx_2}{dx_1} = \frac{Q_2}{P_2} \tag{3.90}$$

We then have two families of curves in the $x_1x_2$ plane. In order to know when to change from one family to another we will assume that there are two curves

$$r_1(x_1,x_2) = 0; \; r_2(x_1x_2) = 0 \tag{3.91}$$

which are the loci of changeover points. Our complete system may then look like that in Fig. (3.3). $x_1$ and $x_2$ will stay inside the region enclosed by $r_1$ and $r_2$. To find the direction of movement in time in the phase plane it is necessary to consider (3.88). As Fig. 3.3 has been drawn, in one regime $x_1,x_2$ both increase with time, while in the second regime $x_1$ increases and $x_2$ decreases.

<u>Phase Plane Treatment of Leontief Model</u>

Let us now consider a phase plane plot of the Leontief model. Dividing (3.83a) by (3.83b)

$$(A_{21}x_1 + A_{22}x_2)\, dx_1 = (A_{11}x_1 + A_{12}x_2)\, dx_2 \tag{3.92}$$

We will assume $|\Delta| \neq 0$ so that a solution exists. We will further assume that $|a| = 0$ where the matrix A is given by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} b_{22} & b_{21} \\ -b_{12} & -b_{11} \end{pmatrix} \begin{pmatrix} 1 & -a_{21} \\ a_{12} & -1 \end{pmatrix} \tag{3.93}$$

$$|a| = \begin{vmatrix} 1 & -a_{21} \\ a_{12} & -1 \end{vmatrix} = a_{21}a_{12} - 1 = 0 \rightarrow a_{21} = 1/a_{12} \tag{3.94}$$

Equation (3.92) can then be written

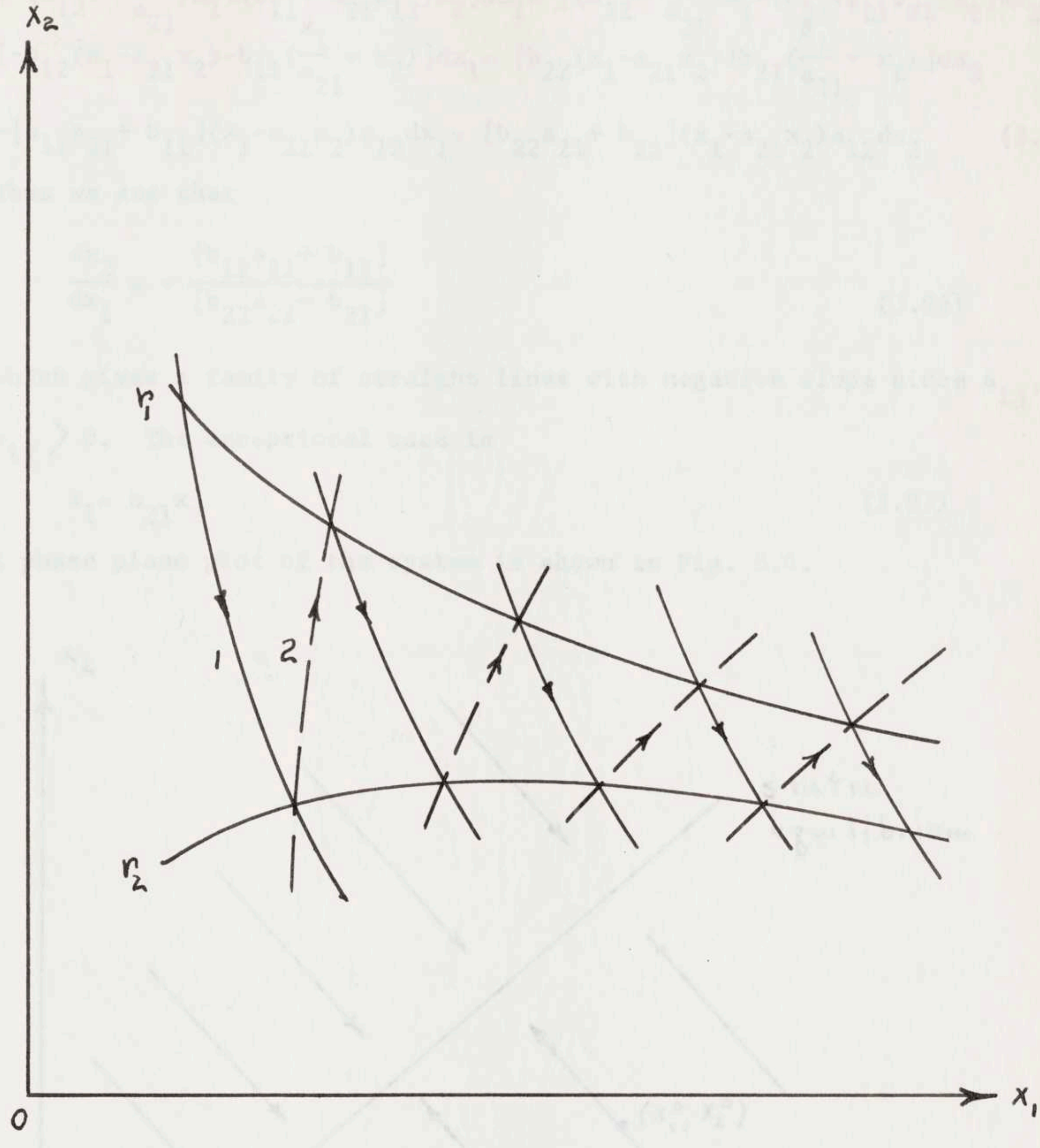



FIGURE 3.3

$$[-(b_{12}+\frac{b_{11}}{a_{21}})x_1+(b_{11}+a_{21}b_{12})x_2]dx_1=[(b_{22}+\frac{b_{21}}{a_{21}})x_1-(b_{21}+a_{21}b_{22})x_2]dx_2$$

$$[-b_{12}(x_1-a_{21}x_2)-b_{11}(\frac{x_1}{a_{21}}-x_2)]dx_1=[b_{22}(x_1-a_{21}x_2)+b_{21}(\frac{x_1}{a_{21}}-x_2)]dx_2$$

$$-[b_{12}a_{21}+b_{11}](x_1-a_{21}x_2)a_{12}dx_1=[b_{22}a_{21}+b_{21}](x_1-a_{21}x_2)a_{12}dx_2 \qquad (3.95)$$

Thus we see that

$$\frac{dx_2}{dx_1}=-\frac{[b_{12}a_{21}+b_{11}]}{[b_{22}a_{22}+b_{21}]} \qquad (3.96)$$

which gives a family of straight lines with negative slope since $a_{ij}$, $b_{ij} > 0$. The exceptional case is

$$x_1=a_{21}x_2 \qquad (3.97)$$

A phase plane plot of the system is shown in Fig. 3.4.

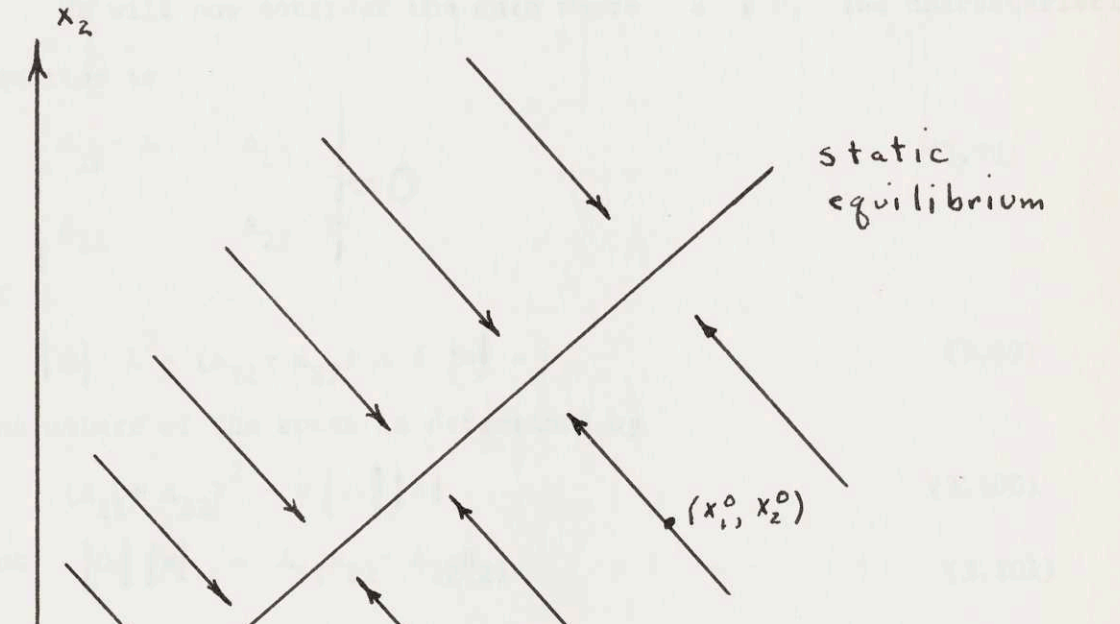



FIGURE 3.4

It will be observed as we cross the line $x_1 = a_{21}x_2$, the direction of movement in time of the system changes (because of the factor $x_1 - a_{21}x_2$ in $\dot{x}_1$ and $\dot{x}_2$). The direction of movement on either side of $x_1 = a_{21}x_2$ depends upon the sign of $|\Delta|$. We have shown in Fig. (3.4) the case where $|\Delta| < 0$. It will now be obvious that $x_1 = a_{21}x_2$ is a line of static equilibrium. If the economic system starts on the line, i.e. $x_1 = a_{21}x_2$, then $\dot{x}_1 = \dot{x}_2 = 0$ and there will be no change with time. If the initial point is not on the line, then the system will be dynamic and will move alone one of the lines of negative slope. The direction of movement will be towards the equilibrium line if $|\Delta| < 0$ and will be unstable and move away from the equilibrium line if $|\Delta| > 0$.

We will now consider the case where $a \neq 0$. The characteristic equation is

$$\begin{vmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{vmatrix} = 0 \qquad (3.98)$$

or

$$|\Delta| \lambda^2 - (A_{11} + A_{22}) \lambda + |a| = 0 \qquad (3.99)$$

The nature of the roots is determined by

$$(A_{11} + A_{22})^2 - 4 |\Delta| |a| \qquad (3.100)$$

But $|\Delta| |a| = A_{11}A_{22} - A_{12}A_{21}$ (3.101)

Thus (3.100) becomes

$$(A_{11} - A_{22})^2 + 4 A_{12}A_{21} \geqslant 0 \qquad (3.102)$$

Now when $|\Delta| > 0$, $|a| > 0$, then from (3.100), (3.102) we see that the roots are of the form $\lambda_2 > \lambda_1 > 0$. It will be useful to note the following general results. When $\dot{x}_1 = 0$, from (3.83a) we obtain the line

$$A_{11}x_1 + A_{12}x_2 = 0 \qquad (3.103)$$

Also if $\dot{x}_1 = \lambda x_1$, $\dot{x}_2 = \lambda x_2$ then from (3.83a), (3.83b)

$$(A_{11}-\lambda)x_1 + A_{12}x_2 = 0 \tag{3.104a}$$

$$(A_{22}-\lambda)x_2 + A_{21}x_1 = 0 \tag{3.104b}$$

For the case of $\lambda_2 > \lambda_1 > 0$, both $x_1, x_2$ increase exponentially. The line of $\dot{x}_1 = 0$ has a positive slope as can be seen from (3.84) and so does the line (3.104a) for the smaller root. The phase plane plot will look something like that of Fig. 3.5. When one gets away from the line $\dot{x}_1 = \lambda_1 x_1$, $\dot{x}_2 = \lambda_2 x_2$, the higher cost $\lambda_2$ comes into play and causes the isodromes to deviate away from the line. A check of the coefficients of each of the exponentials shows that they bend as shown (ignore the dotted lines for the moment; they will be discussed later). To get an exact plot of the isodromes in the phase plane can be quite a task. Usually it is sufficient to know qualitatively the behavior.

If $|\Delta| < 0$, $|a| < 0$, then the roots change sign and are of the form $\lambda_2 < \lambda_1 < 0$. Instead of increasing exponentials we here have decreasing exponentials. The phase plane plot looks the same as Fig. 3.5 except that the direction of movement in time is reversed. In this the economy will always contract towards zero. We can then call the system stable. In the previous case if we are not on the special line $x_1$ or $x_2$ will be driven to zero and in such a case it is reasonable to say that the economy is unstable.

When $|\Delta| < 0$, $|a| > 0$, then $\lambda_2 < 0 < \lambda_1$ and one of the exponentials is increasing and the other decreasing. Again only for $\lambda$, does the line $\dot{x}_i = \lambda x_i$ lie in the first quadrant. The phase plane plot then looks something like that shown in Fig. 3.6. In this case we always tend towards the same equilibrium and the system can be called stable.

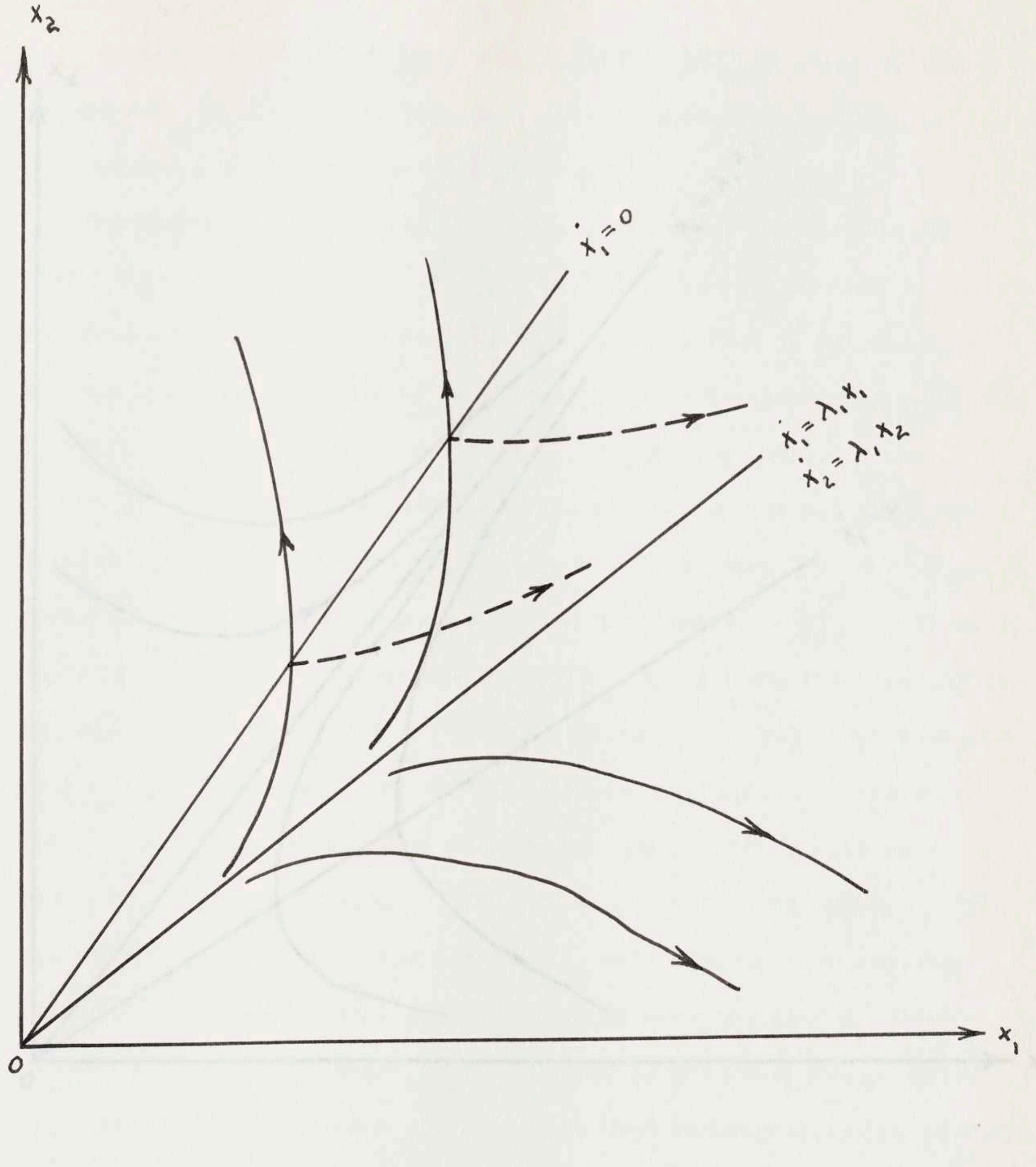



FIGURE 3.5

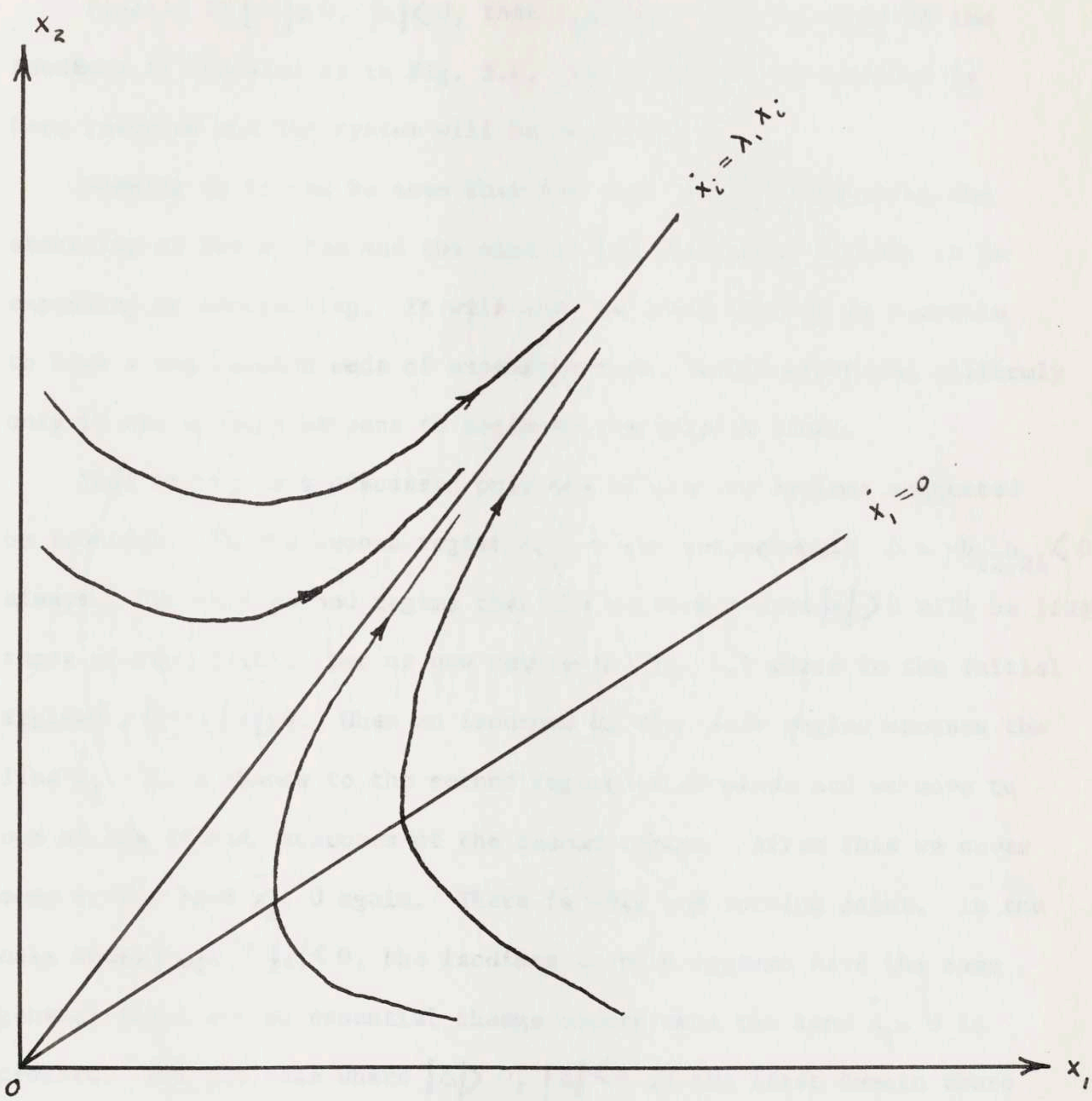



FIGURE 3.6

Finally if $|\Delta| > 0$, $|a| < 0$, then $\lambda_1 < 0 < \lambda_2$ and the shape of the isodimes is the same as in Fig. 3.6. The direction of movement is here reversed and the system will be unstable.

Summing up it can be seen that the sign of $|\Delta|$ determining the stability of the system and the sign of $|a|$ determines whether it is expanding or contracting. It will also be noted that it is possible to have a von Neumann mode of expansion (each sector expanding uniformly only if the economy happens to begin on the special line).

Thus far we have discussed only one of the two regimes suggested by Leontief. In the second regime $b_{11} = 0$ and consequently $\Delta = -b_{12}b_{21} < 0$ always. For this second regime then the isodromes when $|a| > 0$ will be like those of Fig. (3.6). Let us now return to Fig. 3.5 where in the initial regime $|\Delta| > 0$, $|a| > 0$. When an isodrome of the first regime crosses the line $\dot{x}_1 = 0$, a change to the second regime takes place and we move to one of the dashed isodromes of the second regime. After this we never come to the line $\dot{x}_1 = 0$ again. There is only one turning point. In the case where $|\Delta| < 0$, $|a| < 0$, the isodimes in both regimes have the same general shape and no essential change occurs when the line $\dot{x}_1 = 0$ is crossed. For the case where $|\Delta| > 0$, $|a| < 0$ in the first domain there is a change when the line $\dot{x}_1 = 0$ is crossed and geometrically the situation looks like that in Fig. 3.7. Here again only one turning point is possible. In general, then, we see that in this Leontief model there can be only a single turning point.

## Phase Plane Plots in General

In the previous section we presented some aspects of the use of relaxation phenomena in economics. We presented the approach used by Georgescu-Roegen, except that it was necessary to carry out the development

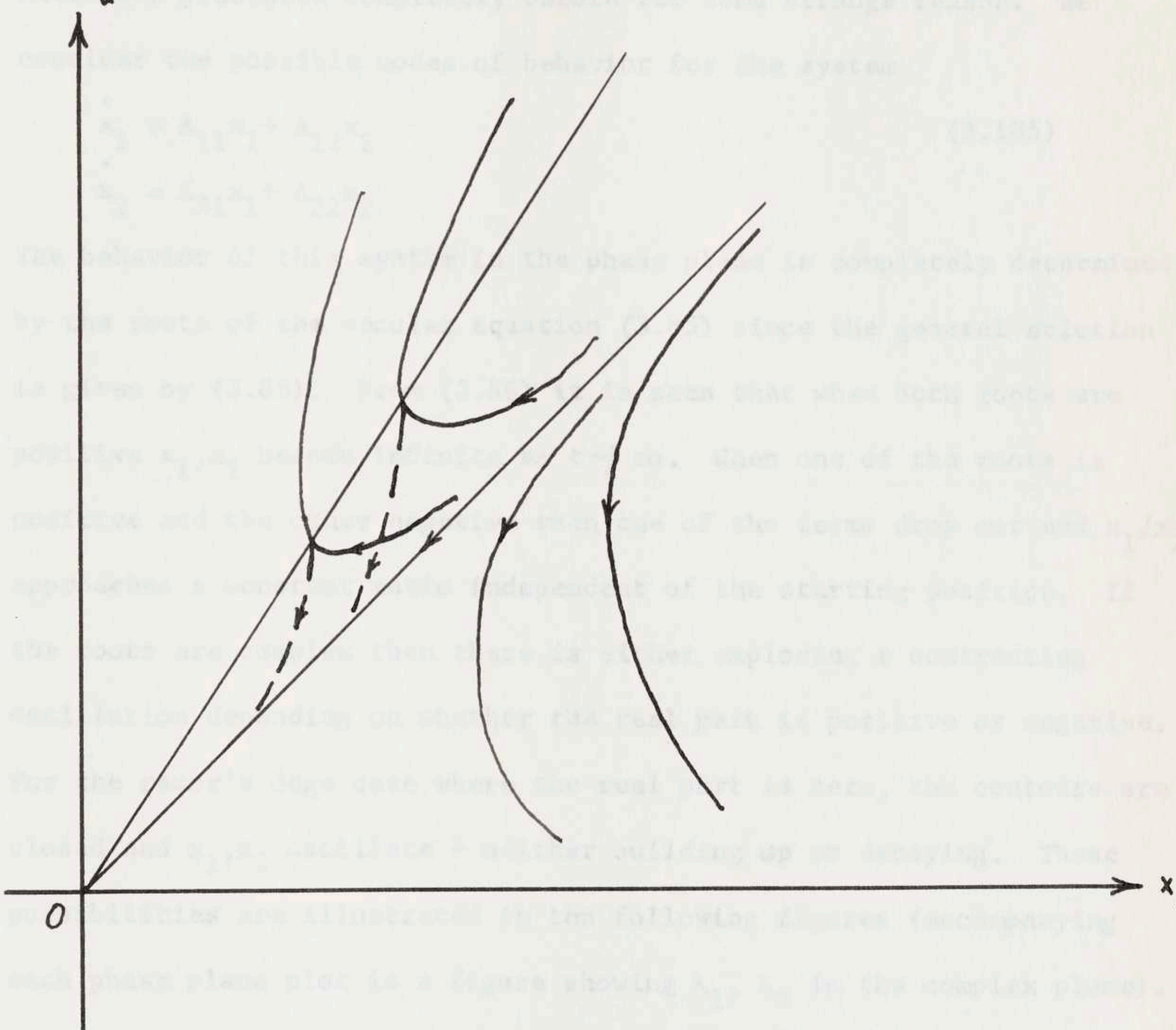



FIGURE 3.7

in more detail for reasons of clarity.We would here like to present all the possibilities for simple phase plane plots. These do not seem to have been presented completely before for some strange reason. We consider the possible modes of behavior for the system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 \qquad (3.105)$$
$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2$$

The behavior of this system in the phase plane is completely determined by the roots of the secular equation (3.85) since the general solution is given by (3.86). From (3.86) it is seen that when both roots are positive $x_1, x_2$ become infinite as $t \to \infty$. When one of the roots is positive and the other negative then one of the terms drop out and $x_1/x_2$ approaches a constant ratio independent of the starting position. If the roots are complex then there is either exploding r contracting oscillation depending on whether the real part is positive or negative. For the razor's edge case where the real part is zero, the contours are closed and $x_1, x_2$ oscillate - neither building up or decaying. These possibilities are illustrated in the following figures (accompanying each phase plane plot is a figure showing $\lambda_1$, $\lambda_2$ in the complex plane). The following figures indicate the possible motions of a dynamic linear economy. Naturally things need not be centered at the origin but could instead be centered at some point $(x_1^o, x_2^o)$. When relaxation phenomena are present then there can be a change from one type of motion to another. The methods of analysis illustrated here cannot be carried over directly to the n good case. However, in that more general case it would be expected that qualitatively the same sort of behavior would be exhibited.

Fig. 3.8

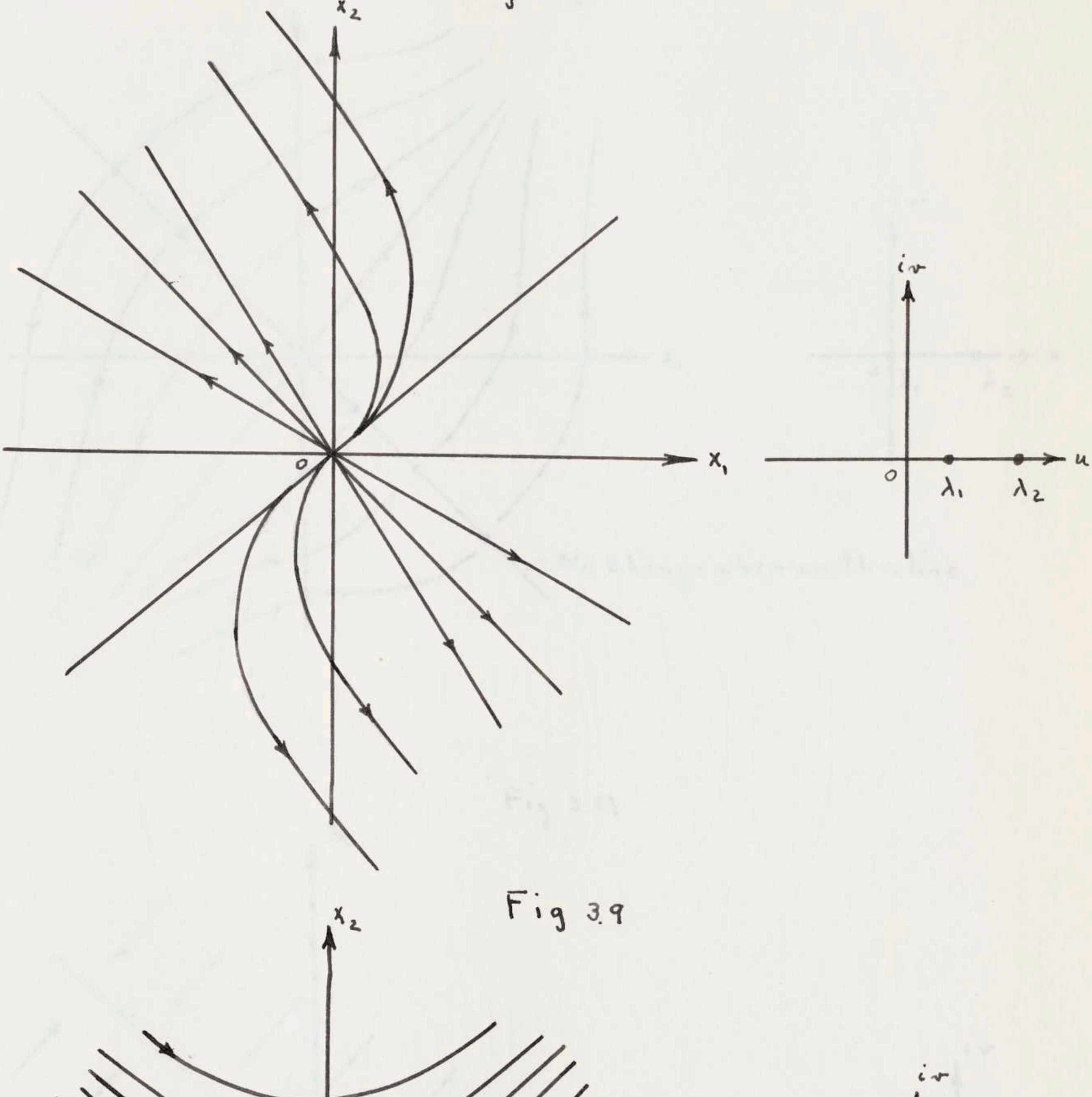



Fig 3.9

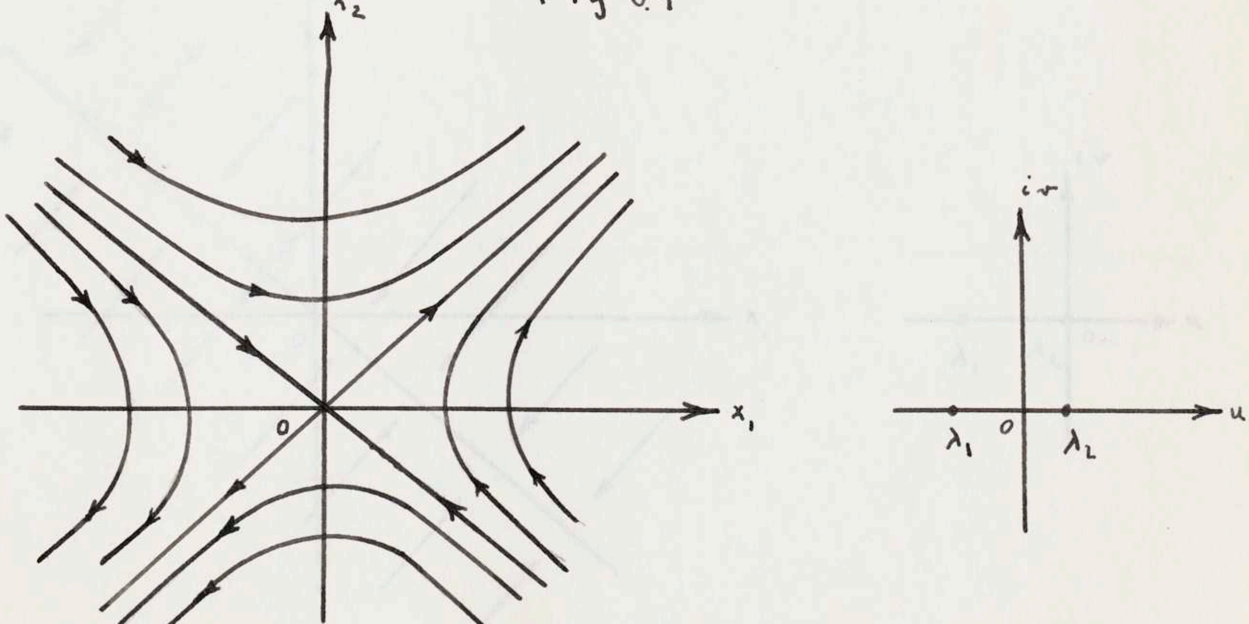
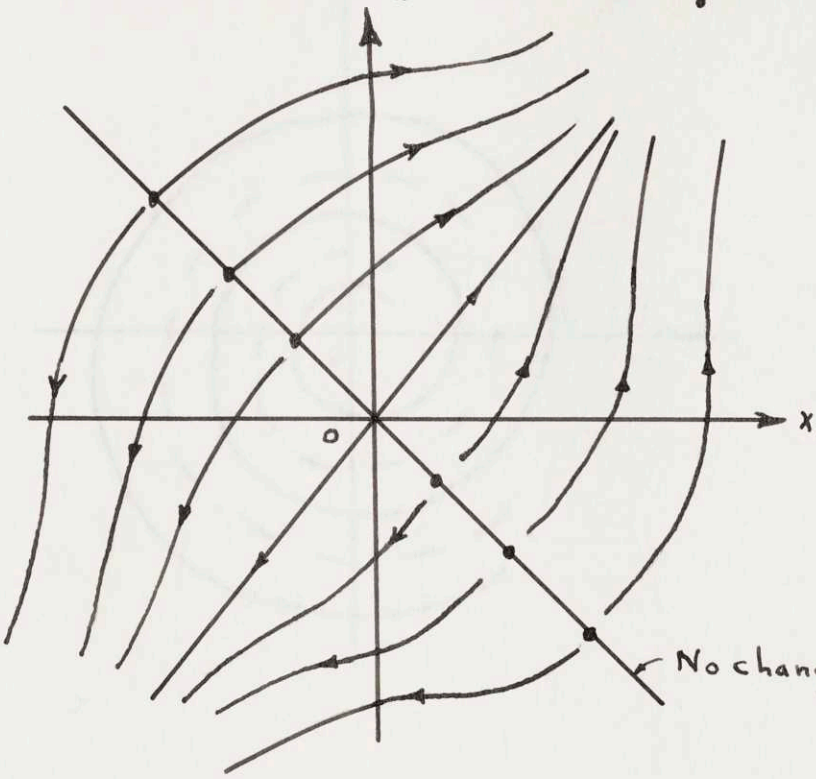


Fig 3.10



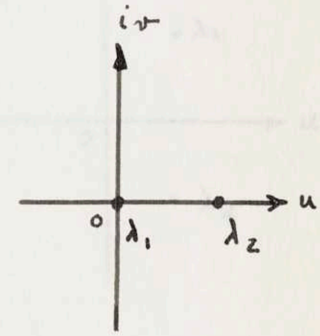
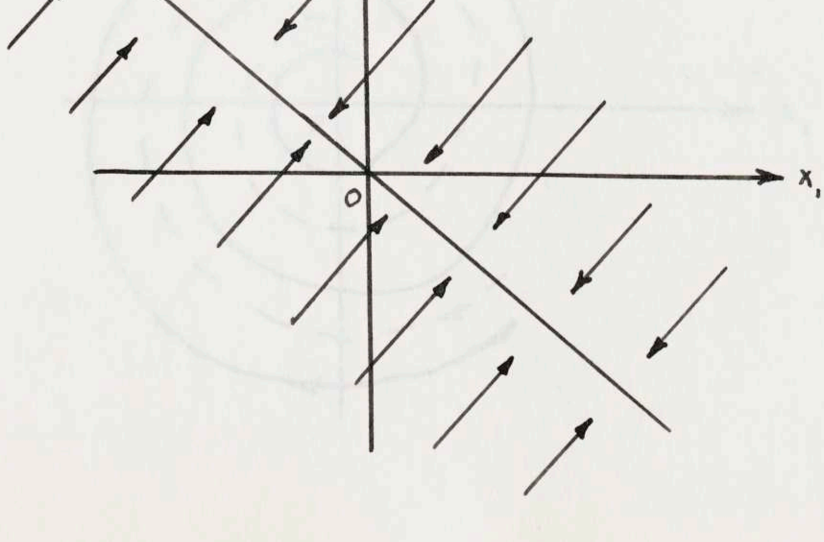




Fig 3.11



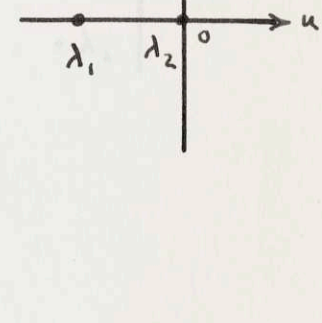
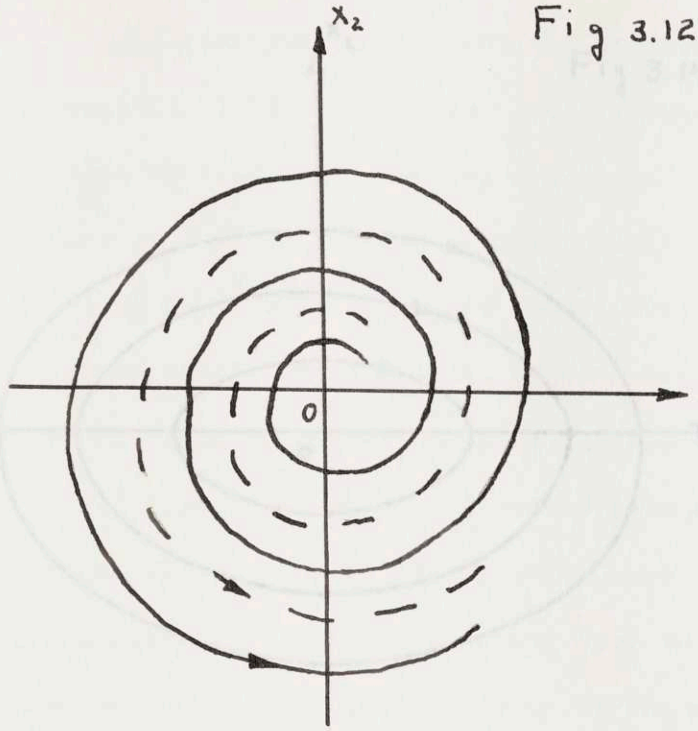



Fig 3.12



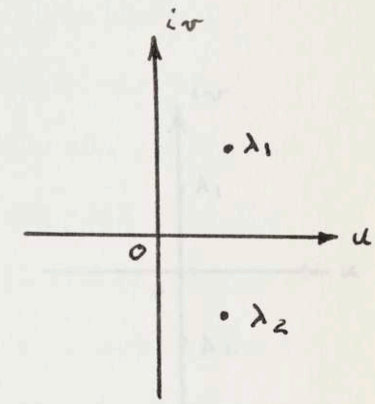
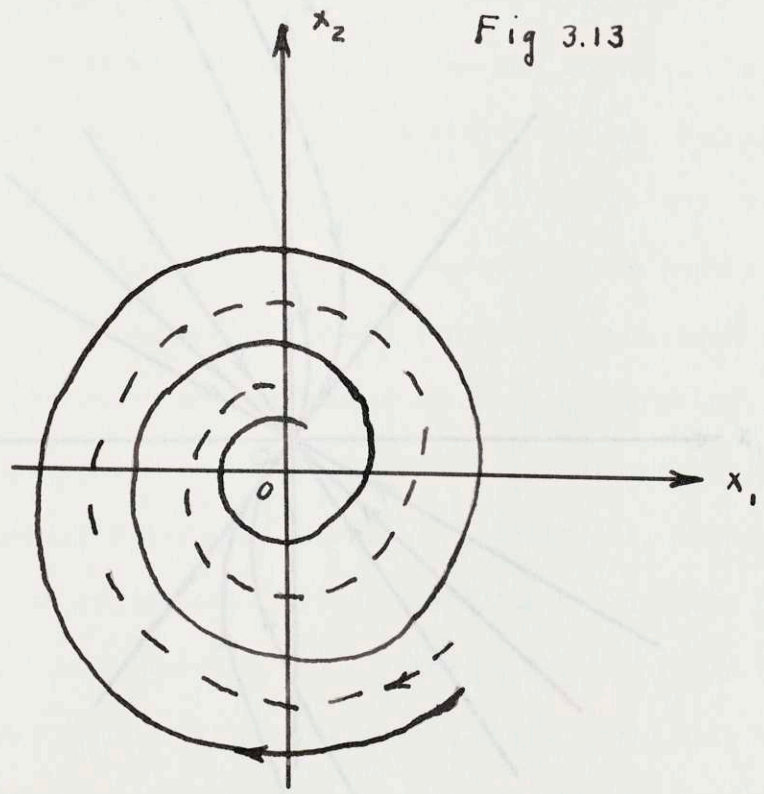




Fig 3.13



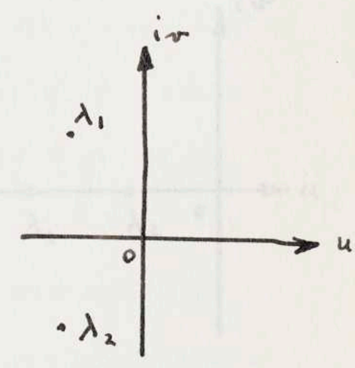



Fig 3.14

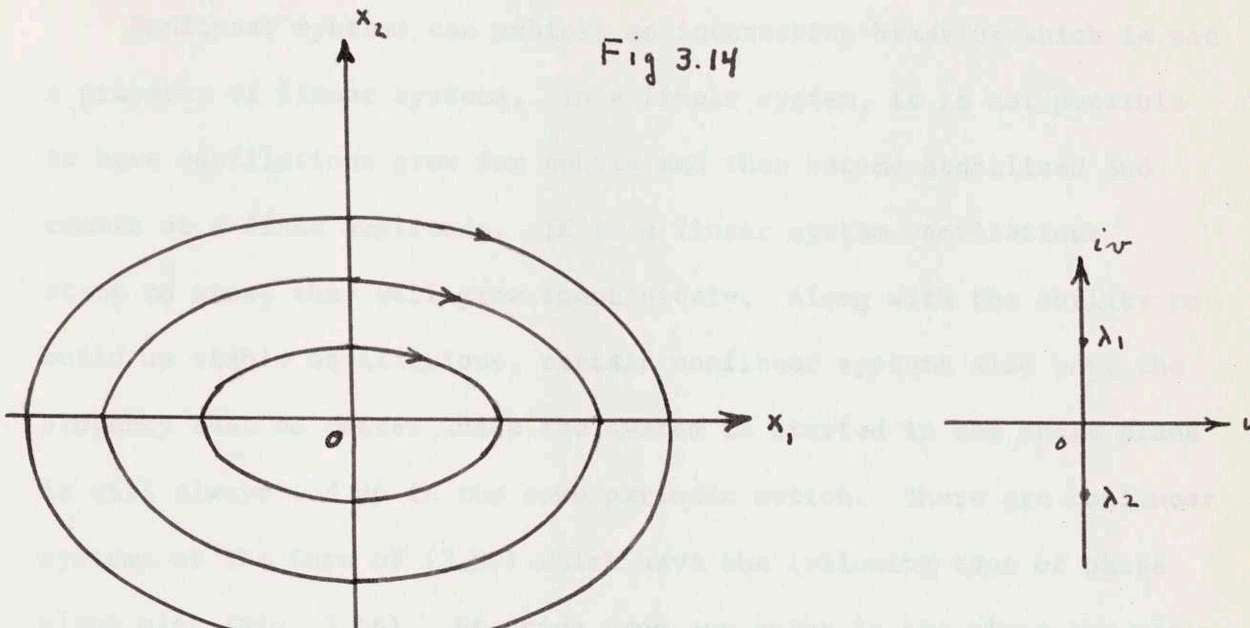
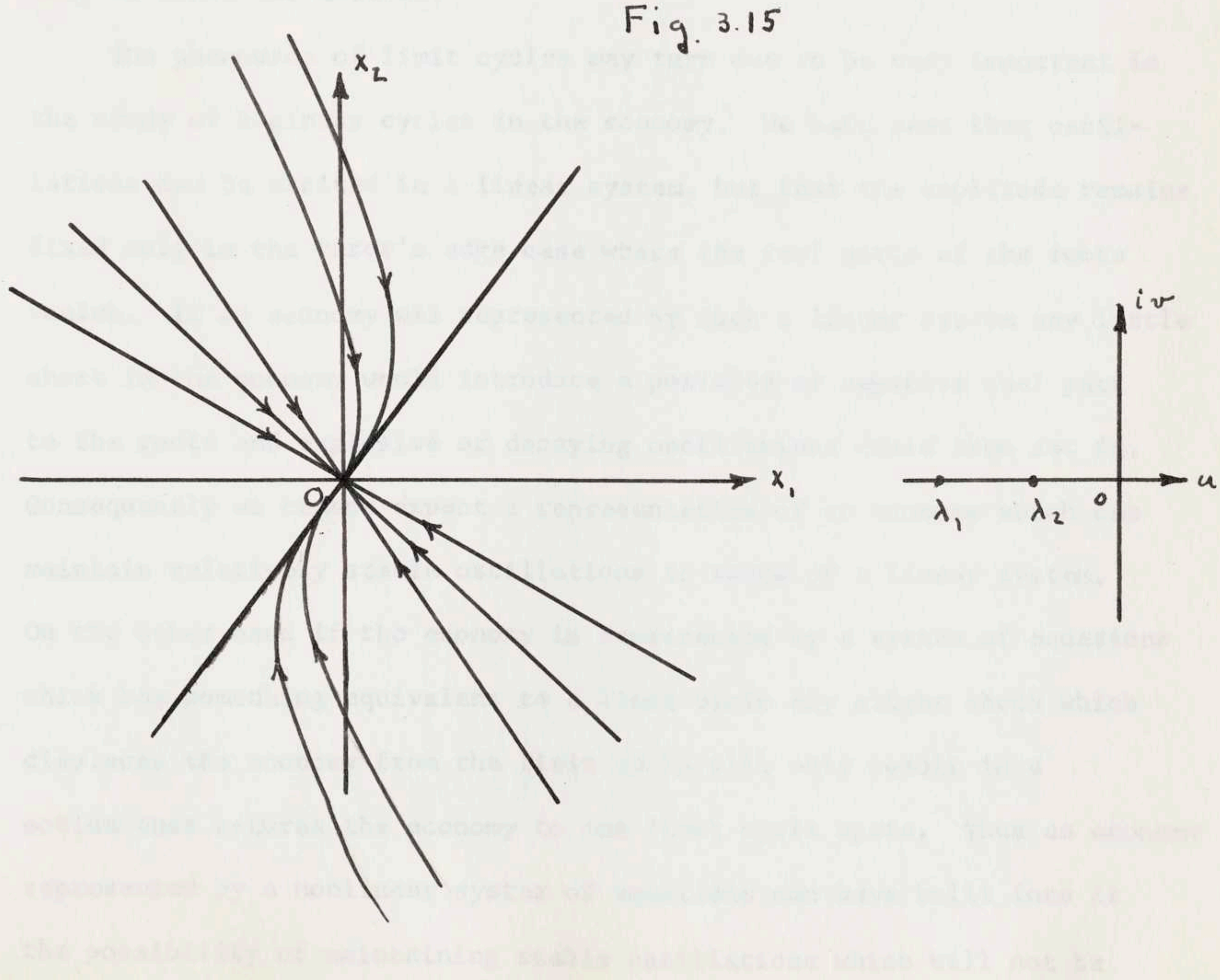



Fig. 3.15



Limit Cycles

Nonlinear systems can exhibit an interesting behavior which is not a property of linear systems. In a linear system, it is not possible to have oscillations grow for awhile and then become stabilized and remain at a fixed amplitude. If in a linear system oscillations start to grow, they will grow indefinitely. Along with the ability to build up stable oscillations, certain nonlinear systems also have the property that no matter where the system is started in the phase plane it will always end up in the same periodic motion. There are nonlinear systems of the form of (3.88) which have the following type of phase plane plot (Fig. 3.16). Starting from any point in the plane the motion will be such as to finally end up on the closed curve shown. The limiting closed curve is called a limit cycle and is characteristic only of nonlinear motions.

The phenomena of limit cycles may turn out to be very important in the study of business cycles in the economy. We have seen that oscillations can be excited in a linear system, but that the amplitude remains fixed only in the razor's edge case where the real parts of the roots vanish. If an economy was represented by such a linear system any little shock in the economy would introduce a positive or negative real part to the roots and explosive or decaying oscillations would then set in. Consequently we cannot expect a representation of an economy which can maintain relatively stable oscillations in terms of a linear system. On the other hand if the economy is represented by a system of equations which has something equivalent to a limit cycle any slight shock which displaces the economy from the limit cycle will only result in a motion that returns the economy to the limit cycle again. Thus an economy represented by a nonlinear system of equations can have built into it the possibility of maintaining stable oscillations which will not be

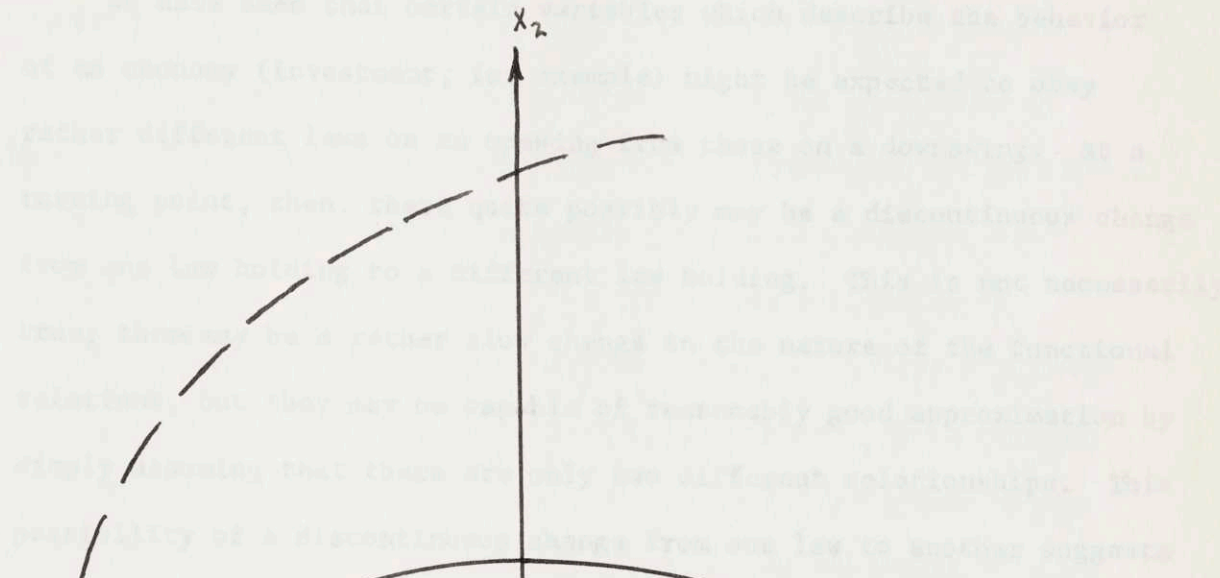



FIGURE 3.16

influenced by slight shocks.

Summary of Relaxation Phenomena Discussion

We have seen that certain variables which describe the behavior of an economy (investment, for example) might be expected to obey rather different laws on an upswing from those on a downswing. At a turning point, then, there quite possibly may be a discontinuous change from one law holding to a different law holding. This is not necessarily true; there may be a rather slow change in the nature of the functional relations, but they may be capable of reasonably good approximation by simply assuming that there are only two different relationships. This possibility of a discontinuous change from one law to another suggests that a study of relaxation oscillations - a familiar subject in mechanics - can be of help in studying the problem. By studying certain simple linear systems in the phase plane it is possible to get an intuitive feeling for the rather wide variety of possible motions that a dynamic economy may have. This approach cannot be directly generalized to a system with a large number of variables, but qualitatively one would expect similar sorts of behavior.

In our discussion we have seen that we cannot expect to be able to realistically represent an economy capable of sustained oscillations in terms of a linear system, since a slight shock would cause the oscillations to become explosive or to decay. On the other hand an economy represented by certain types of nonlinear equations can exhibit sustained oscillations and also have the property that a displacement of the system will always return it to the original motion. The closed limiting curve in the phase plane is called a limit cycle.

CHAPTER 4

# NONEQUILIBRIUM BEHAVIOR

## Introduction

As has already been indicated, very little has been done by economists towards developing laws which treat dynamic and nonequilibrium behavior of the economy. Economists have been almost entirely concerned with equilibrium behavior of the continuous succession of steady states variety. Recently Professor Samuelson undertook a study of dynamic behavior. He was, in actuality, not interested in formulating dynamic laws for their own sake. He was really concerned with a form of equilibrium also. His concern lay with the stability of the system i.e. whether the system when displaced from equilibrium, and hence subjected to unbalanced forces, would return again to the equilibrium state as time approached infinity. If this was true the system was called dynamically stable.

In an initial formulation of a realistic model of the economy, the main concern will be that of finding the dynamical laws which represent the nonequilibrium behavior. After all this has been done, then it may become of interest to investigate what means exist for stabilizing the economy and for removing insofar as possible the tendency for business cycles to exist and persist. We present Professor Samuelson's work (refs 41-44) because it does represent an attempt to formulate dynamical laws. The model is certainly a long way from representing the real world, but it is a step in the right direction.

Professor Samuelson's work was concerned with the clearing of markets which can arise when demand is less than supply or vice versa. This is in reality only one of the many places where nonequilibrium will appear in the economy.

Simplest Model

Consider a case in which there are only two goods. Let us define net excess demand = consumption - production. We take $x_1$, $x_2$ to be the excess demands for goods 1 and 2. When $x_1 = x_2 = 0$, the system is in equilibrium. It is assumed that there is a budget constraint which gives the total amount to be spent on goods 1 and 2. Consequently if $p_1$ and $p_2$ are the units prices of 1 and 2, the budget equation places the following restriction on the excess demands

$$p_1 x_1 + p_2 x_2 = 0 \quad \text{or} \quad x_1 + \frac{p_2}{p_1} x_2 = 0 \tag{4.1}$$

Whenever there is a shift in the demand for good 2 there must be a compensating shift in the demand for good 1. Thus we only need consider the excess demand for good 2 and the price ratio $^{p_2}/_{p_1}$ .

We will assume that we are dealing with an economy where when there is an excess demand (positive or negative) there will be a change in the price ratio from the equilibrium value. The curve might look like that shown in Figure 4.1. Thus for any excess demand we can say what the price ratio is. The dynamical question is, however, if an excess demand appears, how will the price ratio and hence the excess demand change with time. The only way to answer this question

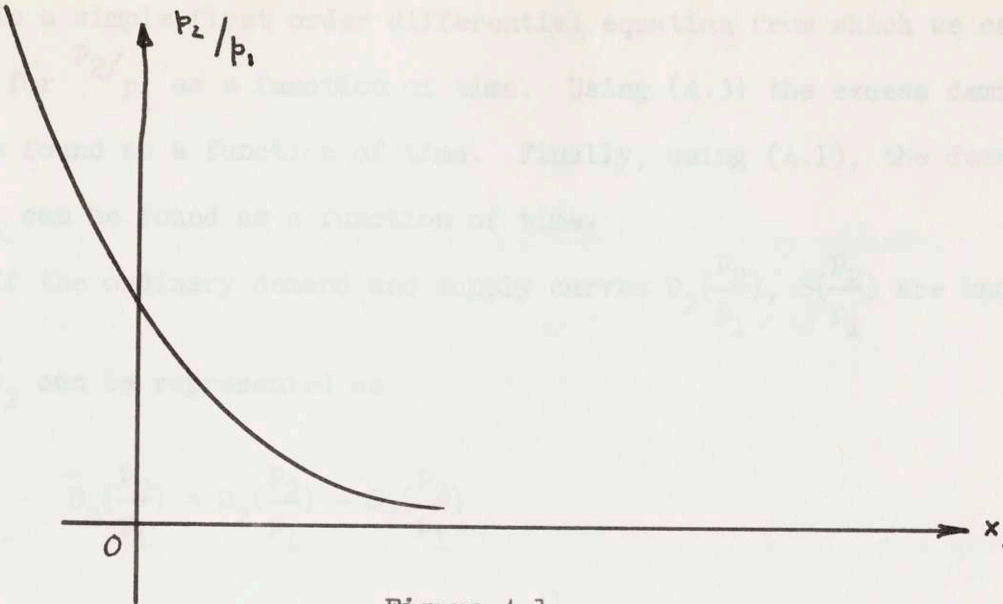



Figure 4.1

in the real world is to find out. For our model, we are free to choose any sort of dynamical law that we care to.

The very simple dynamic law which Professor Samuelson suggested is the following.

$$\frac{d({}^{p_2}/_{p_1})}{dt} = kx_2 \text{ , } k > 0 \tag{4.2}$$

The rate of change of the price ratio is just proportional to the excess demand, and for positive excess demands the price ratio will start to increase with time. The curve given in Figure 4.1 can be represented as

$$x_2 = \bar{D}_2(\frac{p_2}{p_1}) \tag{4.3}$$

Thus (4.2) becomes

$$\frac{d({}^{p_2}/_{p_1})}{dt} = k\bar{D}_2\ (\frac{p_2}{p_1}) \tag{4.4}$$

This is a simple first order differential equation from which we can solve for ${}^{p_2}/_{p_1}$ as a function of time. Using (4.3) the excess demand can be found as a function of time. Finally, using (4.1), the demand for $x_1$ can be found as a function of time.

If the ordinary demand and supply curves $D_2(\frac{p_2}{p_1})$, $S(\frac{p_2}{p_1})$ are known then $\overline{D}_2$ can be represented as

$$\overline{D}_2(\frac{p_2}{p_1}) = D_2(\frac{p_2}{p_1}) - S_2(\frac{p_2}{p_1}) \qquad (4.5)$$

Professor Samuelson then goes on to study the stability of this system. Suppose we let $({}^{p_2}/_{p_1})_o$ be the price ratio at equilibrium when $x_2 = 0$. If we define the deviation from equilibrium as

$$P = \frac{p_2}{p_1} - (\frac{p_2}{p_1})_o \qquad (4.6)$$

then if we expand (4.5) about $({}^{p_2}/_{p_1})_o$ and keep only the first two terms we have

$$\overline{D}_2(\frac{p_2}{p_1}) = \left\{ D_2'[(\frac{p_2}{p_1})_o] - S_2'[(\frac{p_2}{p_1})_o] \right\} P \qquad (4.7)$$

where $D_2'$ and $S_2'$ are the slopes of the demand and supply curves, Equation (4.4) can then be written

$$\frac{dP}{dt} = k[D_{2_o}' - S_{2_o}'] P \qquad (4.8)$$

or

$$P = P_o \exp [k(D_{2_o}' - S_{2_o}')]\, t \tag{4.8}$$

If then P is to $\rightarrow$ o as t $\rightarrow \infty$ or the price ratio is to approach

$(^{p_2}/_{p_1})_o$ we see that it is necessary and sufficient that

$$D_{2_o}' < S_{2_o}' \tag{4.9}$$

.

We normally think of supply and demand curves as having this property.

The Multiple Market Case

The same general ideas can be extended to the n good case. If we define excess demand as before, then in this more general case we have

$$x_i = \bar{D}_i\left(\frac{p_2}{p_1}, \dots, \frac{p_n}{p_1}\right) \qquad i = 2, \dots, n \tag{4.10}$$

The value of $x_1$ is determined from the budget constraint

$$\sum_{i-1}^{n} x_i p_i = 0 \tag{4.11}$$

For equilibrium

$$\vec{x} = \vec{0} = \begin{pmatrix} x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \tag{4.12}$$

The simplest dynamical equations are

$$\frac{d\left(^{p_i}/_{p_1}\right)}{dt} = k_i x_i \qquad i=2, \ldots, n \tag{4.13}$$

The change in $^{p_i}/_{p_1}$ is assumed to depend in the above simple way only the excess demand for that good. In general, we might expect a more complicated relationship in which the price rate of change might depend on the excess demands for all the goods. In matrix vector notation the system of dynamical equations can be written

$$\frac{d \dfrac{\vec{p}}{p_1}}{dt} = K \,\vec{\bar{D}}\left(\frac{\vec{p}}{p_1}\right) \tag{4.14}$$

$$K = \begin{pmatrix} k_2 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix} ; \quad \frac{\vec{p}}{p_1} = \begin{pmatrix} ^{p_2}/_{p_1} \\ \vdots \\ ^{p_n}/_{p_1} \end{pmatrix} ; \quad \vec{\bar{D}} = \begin{pmatrix} \bar{D}_2 \\ \vdots \\ \bar{D}_n \end{pmatrix} \tag{4.15}$$

This is a set of n - 1 first order (in general nonlinear) differential equations to solve for the price ratios as a function of time. Knowing these price ratios as a function of time, the excess demands can also be found as a function of time.

Testing the stability of such a system in the large is quite a difficult task in general. An indication of the behavior in the large can be found by studying the behavior very near the equilibrium point i.e. using the linearized approximation. By Taylor's theorem for n variables we have expanding about the equilibrium point and retaining only terms through order P.

$$\bar{D}_i(\frac{\vec{p}}{p_1}) = \sum \frac{\partial \bar{D}_i}{\partial p_{j/p_1}} P_j = \sum \bar{D}_{i_j} P_j$$

or

$$\vec{D} = \bar{D}\vec{P} \quad ; \quad \bar{D} = \| \bar{D}_{i_j} \| \tag{4.16}$$

We thus obtain for the linearized approximation

$$\frac{d\vec{P}}{dt} = K\bar{D}\vec{P} \tag{4.17}$$

Trying a solution

$$\vec{P} = \vec{k} e^{\lambda t} \tag{4.18}$$

We obtain a necessary and sufficient condition that $\vec{k} \neq \vec{0}$ is that the $\lambda$ must satisfy

$$| KD - \lambda I | = 0 \tag{4.19}$$

A sufficient condition that the system be dynamically stable in the small is that the real parts of each root $\lambda$ is negative. Then all the exponentials will decay with time and $\vec{P} \to \vec{0}$ as $t \to \infty$.

In general the dynamical behavior of even the linearized approximations can be quite complex. When relaxation phenomena were considered, the possible modes of dynamic behavior for such a system involving only two variables was studied. The reader can review Figures 3.8 - 3.15 for possible modes of behavior. Naturally, when there are n variables, the situation is even more complex.

## Summary

The analysis of dynamic behavior presented above is perhaps the most advanced which has been carried out in economics. Certainly it is a far cry from what would be needed to represent the dynamic nonequilibrium behavior of the economy. Only the simplest imaginable dynamic equations were written down for a highly artificial situation. Here, as always seems to be the case in economics, the interest was directed at understanding the dynamic stability of equilibrium and not with formulating dynamic laws to represent nonequilibrium behavior. It seems that much could be gained if economists could be persuaded to give less attention to equilibrium and more attention to nonequilibrium behavior.

The above analysis did, perhaps, point out something of general interest. That is, in any dynamic situation it might be expected that both prices and quantities will be involved and interrelated. We might expect this to be true in any realistic dynamic model of the economy. In general we might expect prices, quantities produced, quantities demanded, etc. all to appear in the dynamical equations which represent nonequilibrium behavior. The above study also brings up the question of how the equations representing the dynamic behavior of the economy will ever be determined. This is a hard question to answer. Certainly it will not be easy to determine these equations. It is not too feasible simply to try and sit down and guess what they might be. On the other hand, since the relationships are very complex, it is not easy to study data (even if enough data were available) and discern the proper relationships. It seems that it will be possible to develop these relationships only by observing how businessmen act and by questioning them on how they make decisions, and doing the same thing with consumers, and the government as well. Perhaps in this way a skeletal structure of a dynamic system can be developed. In all probability the finer details can be filled in only after considerable testing over an extended period of time.

Finally, we might note several more things about Professor Samuelson's analysis. The same approach could perhaps be used to discuss the dynamic behavior of wage rates in a market where wage rates had some freedom to change. This simple analysis could be applied in a number of places but everywhere it would stand the same criticism of not being realistic. It might be noted that the discussion of Professor Samuelson is not really completely dynamic because he assumes that the demand curves are independent of time. If they were allowed to be functions of time, this would, of course, complicate the mathematical analysis considerably.

CHAPTER 5

# CONCLUSIONS

## Review

In the previous chapters we have reviewed a number of models of the national economy. These models varied widely in their nature and scope. None of them, however, came even close to what we have defined as a realistic representation of the actual economy. Many of the models do not even attempt to build a realistic model of the economy. Some of them are planning models, others investigate the properties of hypothetical economies. The statistical models are the only ones which even come vaguely close to our definition of realistic. They also fall far short of yielding realism. They fail in that they are too aggregated and in that they do not attempt to develop relations between the variables which are based on the actual structure of the system. Without more differentiation than they have, it is not at all clear that a realistic model could be developed even if they did attempt to more adequately represent the functional relations between the variables. There is the possibility with nonlinear systems that one can never deduce the actual nature of what is going on inside the system simply by observing what goes in and comes out.

The construction of a realistic model of the economy is clearly a very difficult task and this explains why more progress has not been made in this direction. Until recently economists had to form very naive models if there was any hope of solving the model, since computers have only recently become available. The biggest gap lies in the fact that essentially nothing has been done in the study of the behavior of the economy under unbalanced forces. We reviewed Professor Samuelson's work which is the most advanced work that has been done on this subject. It was seen to be for from what will actually be needed to accurately describe nonequilibrium behavior.

Our conclusion from studying these models is that most of the task of developing a realistic model of the economy lies in the future. There is really very little that can be synthesized from existing models to form a firm foundation for developing the type of model that will be useful in industrial dynamics.

Attributes of a Realistic Model

In Chapter 1 several properties which a realistic model of the economy should possess were brought to light. First, we noted that a realistic model should be a differentiated model both with respect to consumption and production. It cannot be hoped that the effects of changes in one sector of production say on the whole economy can be determined unless the model is sufficiently differentiated. Naturally, there must always be some aggregation and this brings up the problem, already briefly discussed, of the best procedures for aggregation. There does not seem to be any single best aggregation procedure and consequently as the model is developed decisions will need to be made as to what is the best way for the purpose at hand. This differentiated model to be realistic will need to include many variables--the total number could easily be in the thousands.

It was also suggested that the model must be closed, i.e. it must relate supply and demand so that the behavior of the whole economy will be simulated. Naturally there will be some exogenous variables, but these should be at a minimum since in the real world it is difficult to think of too many variables which are completely independent of the behavior of the economy. On the other hand, it would be desirable to be able to "cut open" the model in various places and treat certain variables as exogenous. In this way studies could be made which would demonstrate how the economy would behave if the temporarily exogenous variables moved in some given way.

Naturally any realistic model of the economy must be a dynamic model capable of representing the motion of the economy under unbalanced forces. There is no reason in the world to believe that the actual economy behaves as a continuous succession of steady states. In fact, all the evidence overwhelmingly suggests that precisely the opposite is true. Neither is there any reason to believe that the actual economy is dynamically stable. The model must be able to represent the most general type of nonequilibrium behavior. To be able to determine the equations which describe the most general modes of nonequilibrium behavior, it is necessary to really understand the fundamental nature and structure of the economy. This essentially means that the structure must be built up from a study of the internal workings of the system. This would not be so essential if only equilibrium behavior was to be considered. Such behavior requires only a much more superficial understanding of the actual operation of the economy.

Ordinary difference equations were proposed as the most natural mathematical instrument for developing the general model of the economy. This is especially true since a computer must be used to obtain solutions. It was indicated that differential equations would in all probability serve just as well as difference equations; however, they would need to be converted to difference equations to be solved on a computer and consequently one might as well start out with difference equations. As was indicated previously, time lags will be important in any realistic model. All the models studied involved only discrete time lags. In a realistic model there will be considerable aggregation over firms and individuals (even though the model is differentiated). Under such circumstances, the time lags approach a continous delay distribution rather than a discrete lag. The means of treating continous delays was indicated in Chapter 1. Both discrete and continuous lags may appear in a realistic model.

Several other points have appeared in our discussion of the models and related subjects. First of all, it seems reasonable that what we have called relaxation phenomena could appear in any realistic model. That is there may be practically discontinuous changes resulting from a loss of internal equilibrium which will cause the system to shift from obeying one dynamical law to obeying a different dynamical law. For example the law determining the investment of some industry on the upswing might be completely different from that in the downswing, and an almost discontinuous change from one to the other may take place at a turning point. In general most must allow for such behavior.

From the very brief discussion in Chapter 4 of what has been done on nonequilibrium behavior, the introduction of the dynamical behavior of prices in relation to other important variables is something which appears likely to generalize to a realistic model. It is to be expected that prices will appear in the dynamical equations as well as quantities and that prices at different points in time will be involved. Unlike the Leontief system which separates prices and quantities it seems unlikely that such a separation will be possible in a realistic model. Prices will not be determined independently of production unless for the particular industry there is some mechanism for fixing prices. Even in such a case where these prices are known, it is to be expected that production will depend on these prices. Thus the whole complex of wages, prices, consumption, and production will appear and it does not seem likely that the system will break up in such a way as to allow any of these to be determined completely independently of the others.

The models which have been studied also point out one or two things it would be most unfortunate to try and incorporate into any general realistic model. First of all, it will be recalled that both of the statistical models, which we have considered the most realistic of all the models, as well as practically all the other models expressed all the variables in money terms. This seems completely unnecessary and in fact could easily be quite a hinderance in any large model in trying to set up equations representing the actual structure of the system. If we are dealing with the production of automobiles, we want to speak of the number of automobiles produced and not the dollar value of the production. As soon as a factor is thrown in to convert to dollar value, we have changed the nature of the equations since the dollar value depends on the price level. The only way to really treat variables in such a way that the real world can be accurately simulated is to express these variables in what might be called their "natural units".

Even worse than expressing everything in dollar units is the practice employed in the statistical models of reducing all the dollar amounts to some standard year in order to eliminate price changes. It would just be unthinkable in any realistic model involving many variables to use some standard year. There is no reason at all why this needs to be true. Equations which really represent dynamic behavior must be able to account for price changes automatically without the need for reducing everything to some standard year. Clearly there is nothing sacred about any one year and there is no reason why it needs to be accepted as an all time standard. The above is not an argument against price deflation as such. In the statistical models price deflation could have been taken care of, rather than by referring everything to a base year, through expressing things in terms of changes in the variables rather than in the absolute value of the variables. In this way the ratio of the price index for the present and past period could be used to convert things to proper terms. The use of price deflation, of course, means that we do not have a model which is general enough to work in terms of actual prices. It would seem desirable to try and develop the latter type of model.

In addition to the attributes already considered several more can be added. A realistic model must treat correctly the flow of information--delays, distortion effectiveness, etc.; the flow of materials, production, distribution, etc.; the flow of money--wages, sales, etc. (prices act as conversion factors between money flow and material flow); and finally, of course, it must treat correctly the flow of labor. The model must be able to knit together the actions of the government, business, agriculture, and consumers to form a unified whole.

The Variables

The variables that should appear in a realistic model are the variables which appear in the real world. Many variables which are used in the models we have studied do not seem appropriate for a realistic model. The actual variables of interest are quantities produced, sales made, wages paid, etc. In this way one should be able to avoid the pitfalls of making mistakes in accounting identities, or in computing the gross national product, or in deciding whether investment is autonomous or ex post or ex ante, etc. Things like the multiplier which relates change in investment to change in income, etc. will not appear. By dealing with variables which actually appear in the real world and are actually measurable, many difficulties can be avoided.

The Dynamical Equations

As has been indicated many times in the above discussion, the dynamical equations must be able to represent the actual behavior of the system under unbalanced forces. To develop these equations it will be necessary to start with the basic unit of interest whether it be a firm or a consumer and develop the equations from these basic building blocks. We must ask just what are the decisions that are made in the real world and how are they actually made. We must determine the influence of time lags on the basic unit. Then the influence of other sectors of the economy in the basic unit must be found. They by adding up and interrelating the basic units we can hope to develop a realistic model of the whole economy.

It is not too important to spend a great deal of time arguing whether or not the results of the analysis will be linear or nonlinear. We must accept whatever the real world indicates the actual situation to be. Some of the equations may turn out to be linear while others will be nonlinear. Certainly it will not make any difference to the computer one way or another. In general we would expect nonlinearities to appear in a number of places. These nonlinearities can have an important influence on the behavior of the economy. At such a time the understanding of these nonlinearities will become most important. At the present time, however, it need not be a matter of great concern as to what the precise nature of these nonlinearities will be since it is known that the computer won't care.

Uncoupling the Equations

We have already indicated that a set of simultaneous difference equations should provide an adequate mathematical representation of a realistic model. It should be pointed out that simultaneous means nothing about the way the variables appear in the equations. Simultaneous equations are just a set of equations which all hold at the same time. Certainly there will be a set of equations which hold at the same time. In the system of equations for the national economy, we do not expect in general we can solve one equation for all time disregarding the others. In other words, there is some coupling between the equations. Usually we cannot advance to a new time until we have solved all the equations one or two time steps back (how far back this is depends on how the equations are set up).

It is important and interesting to see just how we might expect the equations to be coupled together. Now decisions can be based only on what has happened in the past. They cannot be based on the values of any variables at the instant the decision is being made. In other words, what is happening elsewhere in the economy at the precise instant that a decision is being made cannot influence the decision. Information for decision making comes from events which have already transpired in time. This means that the variables which determine the value of a variable at a given time are all values at some previous time. Consequently, in any equation there will never appear two variables at the new time which are unknown. Because of the nature of the decision-making process, it is then possible to uncouple the equations at each time step and solve each one independently of the others. All must be solved, in general, however, before one can go on to the next time step. The equations are uncoupled at each time step, but not for all time.

On the basis of the above arguments it might be felt that in evaluating the variables at time t, a given variable at time t will appear in only one equation. By the reasoning given on decisions above this is true. However, under certain conditions, this may not be true. It seems that it will always be true that the equations can be solved sequentially, but it may turn out that a variable at the new time will occur in several equations. This could happen in the following way: Suppose the time step was something like a month. Consider consumer expenditures. The consumer expenditures, if paychecks come once a week or twice a month will then depend on wages received in between the times at which something like expenditures are computed. In such a situation the computation of expenditures might most accurately be obtained by taking an average of total monthly wages for the previous period and total monthly wages for the present period. This would bring in wages at t. This is precisely the reason that the values of the variables for the present period frequently appear in the statistical models. The time step (usually being one year) is so large that it is a better approximation to use the present values (values at the end of the present year) rather than the values at the end of the previous year. The statistical models do not seem to be completely uncoupled i.e. the equations cannot be solved entirely sequently to determine the variables. They are almost completely uncoupled, however. It would only take a slight (and perhaps more realistic) modification of one or two equations to make them uncoupled.

The Role of Statistics

In general we should expect that the behavior of our model will not be completely determined even after we have specified the initial conditions and the exogenous variables as a function of time. Naturally we would like to make it as nearly completely determined as possible, but there are certain events which cannot be determined completely. One cannot be absolutely sure how many automobiles will be purchased or what the distribution between the makes will be. Instead one may just have an estimated probability distribution. To take care of this randomness which can creep in, a Monte Carlo method can be used by which uncertain events are determined by using a random number table. It is not easy to say just how important such random fluctuations will be. However, at the national economy level, where there will be considerable aggregation, they should tend to smooth out and become less important than say at the firm level.

The use of statistics may find some additional use in the development of the model, however. In statistical thermo-dynamics, statistical methods are used to derive from the macroscopic laws of motion, the thermodynamic behavior of the macroscopic system. Statistics is used not out of necessity but out of practicality. It is in theory possible to solve the Schrodinger equation for all the particles in the system and in this way derive the macroscopic behavior. This is in reality an essentially impossible task, however, Hence, statistics is used to easily obtain the gross properties. Something similar to this may have to be done in the model of the national economy. It is not possible to treat each consumer (or grocery store) individually. To obtain the equations for the gross behavior something like statistical mechanics may have to be used.

Then statistics may enter into the determination of the parameters in the equations. Whenever possible it is desirable to determine each parameter directly from theoretical arguments. If each parameter cannot be determined individually, but instead some must be determined simultaneously from the data, then statistical methods can be of help in doing this.

<u>Technological</u> <u>Change</u> <u>Innovation</u>, <u>Research</u>

In any dynamic model technological change presents a real problem. It is not possible to predict in advance what radical changes innovation might bring. On the other hand, technological change and innovation are not completely exogenous variables since they depend on the amount spent on research. It does not seem possible to predict what research will produce. However, it may be possible to incorporate into the system the possibility of having the equations automatically change themselves with the proper time lag as new things are developed in the research laboratories. In other words, once we know that something is developed at the research stage, it may be possible to build into the system the capability of taking this new discovery and actually introduce it into the system hence modifying the system. Given a new discovery we may be able to have the system modify itself and thus save the formulators the trouble of continually modifying the system with each new technological change. There is probably a limit to how much the system can modify itself and hence for really radical changes the model may need to be revised by the formulators.

The Future

The results of this work point out rather clearly to the fact that the task of formulating a realistic model of the national economy still lies in the future. There is no question about the fact that such a model would be of immense value. However, the development of such a model will be no easy task. The models which have been constructed yield little in the way of a solid foundation on which to begin. However, the techniques developed during and since the war in simulation and feedback systems, along with modern computers will provide considerable help. In any event, it might easily require a fairly large number of people working perhaps a decade to accomplish the task.

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