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HOMOGENEOUS CHAOS REVISITED

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Homogeneous Chaos Revisited

Let $(\Theta, H, \mathcal{W})$ be an abstract Wiener space. That is: Θ is a separable real Banach space with norm $\|\cdot\|_\Theta$; H is a separable real Hilbert space with norm $\|\cdot\|_H$; $H \subseteq \Theta$, $\|h\|_\Theta \leq C\|h\|_H$ for some $C < \infty$ and all $h \in H$, and H is $\|\cdot\|_\Theta$ - dense in Θ; and $\mathcal{W}$ is the probability measure on $(\Theta, \mathcal{B}_\Theta)$ with the property that, for each $\ell \in \Theta^*$, $\theta \in \Theta \to \langle \ell, \Theta \rangle$ under $\mathcal{W}$ is a Gaussian random variable with mean zero and variance $\|\ell\|_H^2 \equiv \sup\{\langle \ell, h\rangle^2 : h \in H \text{ with } \|h\|_H = 1\}$. Let $\{\ell^k : k \in Z^+\} \subseteq \Theta^*$ be an orthonormal basis in H; set $A = \{\alpha \in N^{Z^+} : |\alpha| = \sum_{k\in Z^+} \alpha_k < \infty\}$; and, for $\alpha \in A$, define

$$H_\alpha(\theta) = \prod_{k\in Z^+} H_{\alpha_k}(\langle \ell^k,\theta\rangle) \ , \ \theta \in \Theta \ ,$$

where

$$H_m(\xi) = (-1)^m e^{\xi^2/2} \frac{d^m}{d\xi^m} (e^{-\xi^2/2}) \ , \quad m \in N \text{ and } \xi \in R^1 \ .$$

Then, $\{(\alpha!)^{-1/2} H_\alpha : \alpha \in A\}$ is an orthonormal basis in $L^2(\mathcal{W})$. Moreover, if, for $m \in N$,

$$Z^{(m)} \equiv \overline{\text{span}\{H_\alpha : |\alpha| = m\}}^{L^2(\mathcal{W})} \ ,$$

then: $Z^{(m)}$ is independent of the particular choice of the orthonormal basis $\{\ell^k : k \in Z^+\}$; $Z^{(m)} \perp Z^{(n)}$ for $m \neq n$; and $L^2(\mathcal{W}) = \bigoplus_{m=0}^{\infty} Z^{(m)}$. These facts were first proved by N. Wiener [6] and constitute the foundations on which his theory of <u>homogeneous chaos</u> is based.

The purpose of the present article is to explain how, for given $\Phi \in L^2(\mathcal{W})$, one can compute the orthogonal projection $\Pi_{\underset{\sim}{Z}^{(m)}}\Phi$ of Φ onto $Z^{(m)}$. In order to describe the procedure, it will be necessary to describe the elementary Sobolev theory associated with $(\Theta, H, \mathcal{W})$. To this end, let Y be a separable real Hilbert space and set $\mathcal{P}(Y) = \operatorname{span}\{\mathcal{H}_\alpha y : \alpha \in \mathcal{A} \text{ and } y \in Y\}$. Then $\mathcal{P}(Y)$ is dense in $L^2(\mathcal{W}; Y)$. Next, for $m \in \mathcal{N}$ and $\Phi \in \mathcal{P}(Y)$, define $\theta \to D^m\Phi(\theta) \in H^{\otimes m} \otimes Y$ by

$$(D^m\Phi(\theta), h^1 \otimes \ldots \otimes h^m \otimes y)_{H^{\otimes m} \otimes Y} = \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \left(\Phi(\theta + \sum_{j=1}^m t_j h^j), y\right)_Y \Bigg|_{t_1 = \ldots = t_m = 0}$$

for $h^1, \ldots, h^m \in H$ and $y \in Y$. Then D^m maps $\mathcal{P}(Y)$ into $\mathcal{P}(H^{\otimes m} \otimes Y)$ and $D^n = D^m \circ D^{n-m}$ for $0 \leq m \leq n$. Associated with the operator $D^m : \mathcal{P}(Y) \to \mathcal{P}(H^{\otimes m} \otimes Y)$ is its adjoint operator ∂^m. Using the Cameron-Martin formula [1], one can easily prove the following lemma.

(1) <u>Lemma</u>: The operator ∂^m does not depend on the choice of orthonormal basis $\{\ell^k : k \in Z^+\}$, $\mathcal{P}(H^{\otimes m} \otimes Y) \subseteq \operatorname{Dom}(\partial^m)$, and $\partial^m : \mathcal{P}(H^{\otimes m} \otimes Y) \to \mathcal{P}(Y)$. Moreover, if $m \in Z^+$, $K = (k_1, \ldots, k_m)$ $(Z^+)^m$, and $\ell^K = \ell^{k_1} \otimes \ldots \otimes \ell^{k_m}$, then

(2) $$\partial^m \ell^K = \mathcal{H}_{\alpha(K)}$$

where $\alpha(K)$ is the element of $\mathcal{A}$ defined by

$$(\alpha(K))_k = \text{card}\{1 \leq j \leq m : k_j = k\},\ k \in Z^+.$$

In particular, $H^{\otimes m} \subseteq \text{Dom}(\partial^m)$.

Since ∂^m is densely defined, it has a well-defined adjoint $(\partial^m)^*$. Set $W^2_m(Y) = \text{Dom}((\partial^m)^*)$ and use $\|\cdot\|_{W^2_m(Y)}$ to denote the associated graph norm on $W^2_m(Y)$. The following lemma is an easy application of inequalities proved by M. and P. Kree [3].

(2) Lemma: $W^2_m(H^{\otimes m} \otimes Y) \subseteq \text{Dom}(\partial^m)$, $\|\partial^m\Psi\|_{L^2(\mathcal{W};Y)} \leq C_m\|\Psi\|_{W^2_m(h^{\otimes m} \otimes Y)}$, and $\partial^m = ((\partial^m)^*)^*$. Moreover, $P(Y)$ is $\|\cdot\|_{W^2_m(Y)}$-dense in $W^2_m(Y)$. Finally, $W^2_{m+1}(Y) \subseteq W^2_m(Y)$ and $\|\cdot\|_{W^m_2(Y)} \leq C_m\|\cdot\|_{W^{m+1}_2(Y)}$ for all $m \geq 0$.

Warning: In view of the proceeding, the use of D^m to denote its own closure $(\partial^m)^*$ is only a mild abuse of notation. Because it simplifies the notation, this abuse of notation will be used throughout what follows.

Now set $\mathcal{W}^2_{-m}(Y) = \mathcal{W}^2_m(Y)^*$, $m \geq 0$, and $\mathcal{W}^2_\infty(Y) = {}_{m=0}^{\infty}\ \mathcal{W}^2_m(Y)$. Then, when $W^2_\infty(Y)$ is given the Fréchet topology determined by $\{\|\cdot\|_{W^2_m(Y)} : m \geq 0\}$, $(W^2_\infty(Y))^*$ is $W^2_{-\infty}(Y) \equiv {}_{m=0}^{\infty}\ W^2_{-m}(Y)$. Moreover, $L^2(\mathcal{W};Y)$ becomes a subspace of $W^2_{-\infty}(Y)$ when $\Phi \in L^2(\mathcal{W}:Y)$ is identified with the linear functional $\Psi \in W^2(Y) \to E^W[(\Phi,\Psi)_Y]$; and in this way $W^2_\infty(Y)$ becomes a dense subspace of $W^2_{-\infty}(Y)$. Finally, D^m has a unique continuous extension as a map from $W^2_{-\infty}(Y)$ into $W^2_{-\infty}(H^{\otimes m} \otimes Y)$ given by $T \to D^mT$ where $D^mT(\Psi) = T(\partial^m\Psi)$ for $\Psi \in W^m_\infty(H^{\otimes m} \otimes Y)$. In particular, for $T \in W^2_{-\infty}(R^1)$, there is a unique $D^mT(1) \in H^{\otimes m}$ defined by:

(3) $$(D^m T(1),\ h)_{H^{\otimes m}} = T(\partial^m h),\ h \in H^{\otimes m}\ .$$

Note that when $\Phi \in W^2_\infty(R^1)$,

(4) $$D^m\Phi(1) = E^{\mathcal{W}}[D^m\Phi]\ .$$

(5) Theorem: Let $\Phi \in L^2(\mathcal{W})$ be given. Then, for each $m \geq 0$:

(6) $$\Pi_{Z(m)}\Phi = \frac{1}{m!}\,\partial^m(D^m\Phi(1)).$$

Hence,

(7) $$\Phi = \sum_{m=0}^{\infty} \frac{1}{m!}\,\partial^m(D^m\Phi(1))\ .$$

In particular, when $\Phi \in W^2_\infty(R^1)$:

(6') $$\Pi_{Z(m)}\Phi = \frac{1}{m!}\,\partial^m E^{\mathcal{W}}[D^m\Phi]$$

and

(7') $$\Phi = \sum_{m=0}^{\infty} \frac{1}{m!}\,\partial^m\, E^{\mathcal{W}}[D^m\Phi]\ .$$

Proof: Simply observe that, by Lemma (1):

$$\begin{aligned}
\partial^m(D^m\Phi(1)) &= \sum_{K\in(Z^+)^m} E^{\mathcal{W}}[\Phi\partial^m\ \ell^K]\,\partial^m\ \ell^K \\
&= \sum_{|\alpha|=m} \binom{m}{\alpha}\ E^{\mathcal{W}}[\Phi H_\alpha]\,H_\alpha \\
&= m!\ \Pi_{Z(m)}\Phi\ .
\end{aligned}$$

The classic abstract Wiener space is the Wiener space associated with a Brownian motion on R^1. Namely, define $H_1(R^1)$ and

$\Theta(R^1)$ to be, respectively, the completion of $C_0^\infty((0,\infty); R^1)$ with respect to

$$\|\psi\|_{H_1(R^1)} \equiv \left(\int_0^\infty |\psi'(t)|^2\, dt\right)^{1/2}$$

and

$$\|\psi\|_{\Theta(R^1)} \equiv \sup_{t\geq 0} \frac{1}{1+t}|\theta(t)| .$$

Then Wiener's famous existence theorem shows that there is a probability measure on $\Theta(R^1)$ such that $(\Theta(R^1), H_1(R^1), \mathcal{W})$ is an abstract Wiener space. For $(\Theta(R^1), H_1(R^1), \mathcal{W})$, K. Itô [2] showed how to cast Wiener's theory of homogeneous chaos in a particularly appealing form. To be precise, set $\square_m = [0,\infty)^m$; and, for $f \in L^2(\square_m)$, define

$$\int_{\square_m} f\, d^m\theta = \sum_{\sigma\in\Pi_m} \int_0^\infty d\theta(t_m) \int_0^{t_{m-1}} d\theta(t_{m-2}) \cdots \int_0^{t_2} f(t_{\sigma(1)},\ldots,t_{\sigma(m)})\, d\theta(t_1)$$

where Π_m denotes the permutation group on $\{1, \ldots, m\}$ and the $d\theta(t)$-integrals are taken in the sense of Itô. What Itô discovered is that, for given $\Phi \in L^2(\mathcal{W})$, there exists a unique symmetric $f_\Phi^{(m)} \in L^2(\square_m)$ such that

$$\Pi_{Z(m)}\Phi = \frac{1}{m!}\int_{\square_m} f_\Phi^{(m)}\, d^m\theta \tag{8}$$

In order to interpret Itô's result in terms of Theorem (5), let $\{\psi^k : k \in Z^+\} \subseteq C_0^\infty((0,\infty); R^1)$ be an orthonormal basis in $L^2(\square_1)$ and define $\ell^k \in \Theta(R^1)^*$ by $\ell^k(dt) = \left(\int_0^\infty \psi^k(s)ds\right)dt$. Then

$<\ell^k, \theta> = \int_{\Box_1} \psi^k d^1\theta$. Moreover, by using, on the one hand, the generating function for the Hermite polynomials and, on the other hand, the uniqueness of solutions to linear stochastic integral equations (cf. H. P. McKean [5]), one finds that for $K \in (k_1, \ldots, k_m) \in (Z^+)^m$:

$$\int_{\Box_m} h^k \, d^m\theta = H_{\alpha(K)}$$

where $\psi^K = \psi^{k_1} \otimes \ldots \otimes \psi^{k_m}$ and $\alpha(K) \in A$ is defined as in Lemma (1). Hence, by Lemma (1):

$$\partial^m \ell^K = \int_{\Box_m} \psi^K \, d^m\theta \,, \quad K \in (Z^+)^m \,. \tag{9}$$

Finally, for $(t_1, \ldots, t_n) \in \Box_m$, define $h_{(t_1,\ldots,t_m)}(s_1, \ldots, s_m) = (s_1 \wedge t_1) \ldots (s_m \wedge t_m)$. Then, for each $h \in H_1(R^1)^{m \otimes m}$, there is a unique $h' \in L^2(\Box_m)$ such that $(h, h_{(t_1,\ldots,t_m)})_{H_1(R^1)^{\otimes m}}$ $\int_0^{t_m} \ldots \int_0^{t_r} h'(s_1, \ldots, s_m) ds_1, \ldots, ds_m$ for all $(t_1, \ldots, t_m) \in \Box_m$

(10) <u>Theorem</u>: Given $\Phi \in L^2(W)$ and $m \geq 1$, then $f_\Phi^{(m)}$ in (8) is $(D^m\Phi(1))'$.

<u>Proof</u>: By (9):

$$\begin{aligned}
\partial^m(D^m\Phi(1)) &= \partial^m\Big(\sum_{K\in(Z^+)^m} (D^m\Phi(1), \ell^K)_{H_1(R^1)^{\otimes m}} \ell^K\Big) \\
&= \sum_{K\in(Z^+)^m} ((D^m\Phi(1))', \psi^K)_{L^2(\Box_m)} \int_{\Box_m} \psi^K \, d^m\theta \\
&= \int_{\Box_m} (D^m\Phi(1))' \, d^m\theta \,.
\end{aligned}$$

Thus, by (6):

$$\Pi_{Z^{(m)}}\Phi = \frac{1}{m!}\int_{\square_m} (D^m\Phi(1))' \, d^m\theta \ .$$

Since $(D^m\Phi(1))'$ is symmetric, the desired identification is now complete.

(11) Remark: It is intuitively clear that the $f_\Phi^{(m)}$ in (8) must be given by $f_\Phi^{(m)}(t_1, \ldots, t_m) = E^{\mathcal{W}}[\Phi\theta'(t_1) \ldots \theta'(t_m)]$. What Theorem (10) says is that a vigorous definition of $E^{\mathcal{W}}[\Phi\theta'(t_1) \ldots \theta'(t_m)]$ is provided by $(D^m\Phi(1))'(t_1, \ldots, t_m)$.

(12) Remark: Given $d > 1$, define $H_1(R^d)$ and $\Theta(R^d)$ by analogy with $H_1(R^1)$ and $\Theta(R^1)$. Then $(\Theta(R^d), H_1(R^d), \mathcal{W})$ becomes an abstract Wiener space when $\mathcal{W}$ is the Wiener measure associated with Brownian motion in R^d. Moreover, an analogous interpretation of $\Pi_{Z^{(m)}}\Phi$ in terms of $D^m\Phi(1)$ can be given in this case as well.

(13) Remark: Theorem (10) is little more than an exercise in formalism unless $\Phi \in W_\infty^2(R^1)$. Fortunately, many interesting functions are in $W_\infty^2(R^1)$. For example, let $\sigma : R^1 \to R^1$ and $b : R^1 \to R^1$ be smooth functions having bounded first derivatives and slowly increasing derivatives of all orders. Define $X(\cdot,x)$, $x \in R^1$, to be the solution to

$$X(T,x) = x + \int_0^T \sigma(X(t,x))d\theta(t) + \int_0^T b(X(t,x))dt, \ T \geq 0.$$

Then, for each $(T,x) \in (0,\infty) \times R^1$, $X(T,x) \in W_\infty^2(R^1)$. In fact, $DX(\cdot,x)$ satisfies:

$$DX(T,x) = \int_0^T \sigma'(X(t,x))\, DX(t,x)\, d\theta(t) + \int_0^T b'(X(t,x))DX(t,x)dt + \int_0^{\cdot\wedge T} \sigma(X(t,x))\, dt;$$

an equation which can be easily solved by the method of variation of parameters. Moreover, $D^m X(T,x)$, $m \geq 2$, can be found by iteration of the preceding.

(14) Remark: In many ways the present paper should be viewed as an outgrowth of P. Malliavin's note [4]. Indeed, it was only after reading Malliavin's note that the ideas developed here occurred to the present author.

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