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PRODUCT BMO, LITTLE BMO AND RIESZ COMMUTATORS IN THE BESSEL SETTING

XUAN THINH DUONG, JI LI, YUMENG OU, BRETT D. WICK, AND DONGYONG YANG*

ABSTRACT. In this paper, we study the product BMO space, little bmo space and their connections with the corresponding commutators associated with Bessel operators studied by Weinstein, Huber, and by Muckenhoupt–Stein. We first prove that the product BMO space in the Bessel setting can be used to deduce the boundedness of the iterated commutators with the Bessel Riesz transforms. We next study the little bmo space in this Bessel setting and obtain the equivalent characterization of this space in terms of commutators, where the main tool that we develop is the characterization of the predual of little bmo and its weak factorizations. We further show that in analogy with the classical setting, the little bmo space is a proper subspace of the product BMO space. These extend the previous related results studied by Cotlar–Sadosky and Ferguson–Sadosky on the bidisc to the Bessel setting, where the usual analyticity and Fourier transform do not apply.

1. INTRODUCTION

The study of commutators of multiplication operators with Calderón–Zygmund operators has its roots in complex function theory and Hankel operators. This was later extended to the case of general Calderón–Zygmund operators by Coifman, Rochberg and Weiss [CRW], who showed that the space of bounded mean oscillation introduced by John and Nirenberg is characterized by a family of commutators:

$$\|b\|_{\text{BMO}(\mathbb{R}^n)} \approx \max_{1 \leq j \leq n} \|[M_b, R_j]\|_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)}$$

where R_j is the j th Riesz transform. Results of this type have then been extended by Uchiyama to handle spaces of homogeneous type under certain assumptions on the measures and to show that a single Hilbert transform (Riesz transform) actually characterizes BMO [Uch]. These results were further extended to the multiparameter setting showing that the product BMO space of Chang and Fefferman can also be characterized by iterated commutators (see Hilbert transform in [FL] and Riesz transforms in [LPPW]) and little bmo by the boundedness of two commutators (see Hilbert transform in [FS] and Riesz transforms in [DLWY3]). The analysis here is intimately connected to the underlying space \mathbb{R}^n and to the fact that the Riesz transforms are connected to a particular differential operator, the Laplacian.

In 1965, B. Muckenhoupt and E. Stein in [MSt] introduced harmonic analysis associated with Bessel operator Δ_λ , defined by setting for suitable functions f ,

$$\Delta_\lambda f(x) := \frac{d^2}{dx^2} f(x) + \frac{2\lambda}{x} \frac{d}{dx} f(x), \quad \lambda > 0, \quad x \in \mathbb{R}_+ := (0, \infty).$$

The related elliptic partial differential equation is the following “singular Laplace equation”

$$(1.1) \quad \Delta_{t,x}(u) := \partial_t^2 u + \partial_x^2 u + \frac{2\lambda}{x} \partial_x u = 0$$

studied by A. Weinstein [W], and A. Huber [Hu] in higher dimensions, where they considered the generalised axially symmetric potentials, and obtained the properties of the solutions of this equation, such as the extension, the uniqueness theorem, and the boundary value problem for certain domains. In [MSt] they developed a theory in the setting of Δ_λ which parallels the classical

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one associated to the standard Laplacian, where results on $L^p(\mathbb{R}_+, dm_\lambda)$ -boundedness of conjugate functions and fractional integrals associated with Δ_λ were obtained for $p \in [1, \infty)$ and $dm_\lambda(x) := x^{2\lambda} dx$.

We also point out that Haimo [H] studied the Hankel convolution transforms $\varphi\sharp_\lambda f$ associated with the Hankel transform in the Bessel setting systematically, which provides a parallel theory to the classical convolution and Fourier transforms. It is well-known that the Poisson integral of f studied in [MSt] is the Hankel convolution of the Poisson kernel with f , see [BDT]. Since then, many problems in the Bessel context were studied, such as the boundedness of the Bessel Riesz transform, Littlewood–Paley functions, Hardy and BMO spaces associated with Bessel operators, A_p weights associated with Bessel operators (see, for example, [K, AK, BFBMT, V, BFS, BHN, BCFR, YY, DLWY, DLWY2, DLMWY] and the references therein).

The aim of this paper is to study the product BMO and little bmo spaces via Riesz commutators in the Bessel setting. In particular, the two main results we obtain can be seen as the analogs in the Bessel setting of the corresponding results in the classical setting. Notably in our proof we bypass the use of analyticity and Fourier transform since they are not applicable in this Bessel operator setting. We first show that the product BMO space in the Bessel setting can be used to prove the boundedness of the iterated commutators with the Bessel Riesz transforms. We next study the little bmo space in this Bessel setting and obtain the equivalent characterization of this space in terms of commutators. We further show, again in analogy with the classical setting, that the little bmo space is a proper subspace of the product BMO space.

To be more precise, for every interval $I \subset \mathbb{R}_+$, we denote it by $I := I(x, t) := (x - t, x + t) \cap \mathbb{R}_+$. The measure of I is defined as $m_\lambda(I(x, t)) := \int_{I(x, t)} y^{2\lambda} dy$. And recall that the Riesz transform $R_{\Delta_\lambda}(f)$ is defined as follows

$$(1.2) \quad R_{\Delta_\lambda}(f)(x) := \int_{\mathbb{R}_+} -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta f(y) dm_\lambda(y).$$

In the product setting $\mathbb{R}_+ \times \mathbb{R}_+$, we define $d\mu_\lambda(x_1, x_2) := dm_\lambda(x_1) \times dm_\lambda(x_2)$ and $\mathbb{R}_\lambda := (\mathbb{R}_+ \times \mathbb{R}_+, d\mu_\lambda(x_1, x_2))$. We denote by $R_{\Delta_\lambda, 1}$ the Riesz transform on the first variable and $R_{\Delta_\lambda, 2}$ the second.

The first main result of this paper is the upper bound of the iterated Riesz commutators $[[b, R_{\Delta_\lambda, 1}], R_{\Delta_\lambda, 2}]$ in terms of product BMO space $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$. For the definition of $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$ we refer to Definition 2.5 in Section 2.

Theorem 1.1. *Let $b \in \text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$. Then we have*

$$(1.3) \quad \|[[b, R_{\Delta_\lambda, 1}], R_{\Delta_\lambda, 2}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \leq C \|b\|_{\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)}.$$

For simplicity we only state the result for the case of two iterations; though the proof we provide works just as well for any number of parameters.

The proof strategy we employ to show this result is now the standard way to prove upper bounds for commutator estimates, see for example [LPPW, LPPW2] and [DO] in the Euclidean setting. We express the Riesz transforms as averages of Haar shift type operators and then study the boundedness of the commutator with each Haar shift. These can be broken into paraproduct operators for which the boundedness follows by the BMO assumption. The main novelty in this proof is that we actually demonstrate a more general result by showing that a version of the above Theorem holds in product spaces of homogeneous type $X_1 \times X_2$ in terms of the product BMO space $\text{BMO}(X_1 \times X_2)$ (for the definition, we refer to Section 2, see also Definition 2.6 in [DLWY]). We provide a statement of the main result in this direction as follows, which will be proved in Section 2.

Theorem 1.2. *Let (X_i, ρ_i, μ_i) be a space of homogeneous type. Let T_i be the Calderón–Zygmund operator on X_i and let $b \in \text{BMO}(X_1 \times X_2)$. Then we have*

$$\|[[b, T_1], T_2]\|_{L^2(X_1 \times X_2, \mu_1 \times \mu_2) \rightarrow L^2(X_1 \times X_2, \mu_1 \times \mu_2)} \leq C \|b\|_{\text{BMO}(X_1 \times X_2)}.$$

For precise definitions of the product spaces of homogeneous type, the product BMO space, and Calderón–Zygmund operators, we refer to Section 2, see also [HLW]. Since we have that \mathbb{R}_λ is a space of homogeneous type, it is clear that Theorem 1.1 follows from the above theorem as a corollary.

The second main result of this paper is characterization of the little bmo space associated with Δ_λ , $\text{bmo}(\mathbb{R}_\lambda)$, which is the space of functions satisfying the following definition.

Definition 1.3. A function $b \in L^1_{loc}(\mathbb{R}_\lambda)$ is in $\text{bmo}(\mathbb{R}_\lambda)$ if

$$(1.4) \quad \|b\|_{\text{bmo}(\mathbb{R}_\lambda)} := \sup_{R \subset \mathbb{R}_+ \times \mathbb{R}_+} \frac{1}{\mu_\lambda(R)} \iint_R |b(x_1, x_2) - m_R(b)| d\mu_\lambda(x_1, x_2) < \infty,$$

where

$$(1.5) \quad m_R(b) := \frac{1}{\mu_\lambda(R)} \iint_R b(x_1, x_2) d\mu_\lambda(x_1, x_2)$$

is the mean value of b over the rectangle R .

One can easily observe that this norm is equivalent to the following norm:

$$\|b\|_{\text{bmo}(\mathbb{R}_\lambda)} \approx \max \left\{ \sup_{x \in \mathbb{R}_+} \|b(x, \cdot)\|_{\text{BMO}_{\Delta_\lambda}(\mathbb{R}_+, dm_\lambda)}, \sup_{y \in \mathbb{R}_+} \|b(\cdot, y)\|_{\text{BMO}_{\Delta_\lambda}(\mathbb{R}_+, dm_\lambda)} \right\};$$

namely these functions are uniformly in $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_+, dm_\lambda)$ in each variable separately. This leads to the following characterization of $\text{bmo}(\mathbb{R}_\lambda)$:

Theorem 1.4. *Let $b \in L^2_{loc}(\mathbb{R}_\lambda)$. The following conditions are equivalent:*

- (i) $b \in \text{bmo}(\mathbb{R}_\lambda)$;
- (ii) *The commutators $[b, R_{\Delta_\lambda, 1}]$ and $[b, R_{\Delta_\lambda, 2}]$ are both bounded on $L^2(\mathbb{R}_\lambda)$;*
- (iii) *There exist $f_1, f_2, g_1, g_2 \in L^\infty(\mathbb{R}_\lambda)$ such that $b = f_1 + R_{\Delta_\lambda, 1}g_1 = f_2 + R_{\Delta_\lambda, 2}g_2$ and moreover,*

$$\|b\|_{\text{bmo}(\mathbb{R}_\lambda)} \approx \inf \left\{ \max_{i=1,2} \left\{ \|f_i\|_{L^\infty(\mathbb{R}_\lambda)}, \|g_i\|_{L^\infty(\mathbb{R}_\lambda)} \right\} \right\},$$
where the infimum is taken over all possible decompositions of b ;
- (iv) *The commutator $[b, R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}]$ is bounded on $L^2(\mathbb{R}_\lambda)$.*

The proof of the equivalence between (i) and (ii) in this theorem, relies on a recent new result obtained by a subset of authors in [DLWY], which shows that in the one parameter setting $b \in \text{BMO}(\mathbb{R}_+, dm_\lambda)$ if and only if the commutator $[b, R_{\Delta_\lambda}]$ is a bounded operator on $L^2(\mathbb{R}_+, dm_\lambda)$.

Moreover, the proof of the equivalence between (i) and (iv) extends the result of Ferguson–Sadosky [FS] to the Bessel setting, where no analyticity or Fourier transform is available. We prove this characterization by understanding a certain weak factorization of the predual of $\text{bmo}(\mathbb{R}_\lambda)$. To obtain this, we first define the little Hardy space $h^{1,\infty}(\mathbb{R}_\lambda)$ in terms of $(1, \infty)$ -rectangular atoms with a one-parameter version of cancellation. However, it is less direct to see how the duality works by using only $(1, \infty)$ -rectangular atoms. We also introduce the $(1, q)$ -rectangular atoms for $1 < q < \infty$, and then prove that $h^{1,\infty}(\mathbb{R}_\lambda)$ can be characterised equivalently by $(1, q)$ -rectangular atoms. Then, by using the $(1, 2)$ -rectangular atoms, the duality of $h^{1,\infty}(\mathbb{R}_\lambda)$ with $\text{bmo}(\mathbb{R}_\lambda)$ follows from the standard argument, see for example [CW77] (see also [J, Section II, Chapter 3]). This factorization particularly uses key estimates on the kernel of the Riesz transforms, especially the lower bound conditions, which was studied in [BFBMT] and refined recently by the subset of authors [DLWY]; these estimates are essentially different from the standard Riesz transforms on \mathbb{R}^n . We point out that the characterizations of the little Hardy space in terms of $(1, q)$ -rectangular atoms are new even when we refer back to the classical case of Ferguson–Sadosky [FS].

Finally as a corollary of the characterization of $\text{bmo}(\mathbb{R}_\lambda)$ in Theorem 1.4 and the Fefferman–Stein type decomposition of $\text{BMO}(\mathbb{R}_\lambda)$ as proved in [DLWY2], we show that:

Corollary 1.5. *$\text{bmo}(\mathbb{R}_\lambda)$ is a proper subspace of $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$, i.e.,*

$$\text{bmo}(\mathbb{R}_\lambda) \subsetneq \text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda).$$

Again, this is in analogy with the corresponding results in the Euclidean setting. Containment of the spaces follows from property (iii) and a similar characterization of product BMO in this setting. The fact that it is a proper containment follows from a simple construction. These results, as well as corollaries about the relevant factorizations, can be found in Section 3.

A natural question that arises from this work is whether the space $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$ can be characterized by the iterated commutators:

$$\|[[b, R_{\Delta_\lambda,1}], R_{\Delta_\lambda,2}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \approx \|b\|_{\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)}.$$

As evidence for this we point out that in the case of one parameter this result was answered by a subset of the authors in [DLWY]; and it was shown that the space $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$ can indeed be characterized by the commutator. We also point out that using the methods of Section 3 it is possible to obtain a lower bound on the iterated commutator in terms of a “rectangle $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$ ”. While we would like to return to this characterization in subsequent work, we want to point out some challenges with obtaining the lower bound. The analogous proof in the Euclidean spaces, [FL, LPPW], uses key properties of the Fourier transform, the Riesz/Hilbert transforms and wavelets. Some of these tools do not translate well to the setting at hand and instead a new proof seems to be needed.

2. UPPER BOUND OF ITERATED COMMUTATOR $[[b, T_1], T_2]$

In this section we prove Theorem 1.2, which extends the main result of [DO] to spaces of homogeneous type introduced by Coifman and Weiss [CW77]. We first recall some necessary notation and definitions on spaces of homogeneous type, including the product Calderón–Zygmund operators and product BMO space on space of homogeneous type as well as some fundamental tools such as the Haar basis and representation theorem, which will be crucial to the proof of Theorem 1.2.

2.1. Preliminaries. By a quasi-metric we mean a mapping $\rho: X \times X \rightarrow [0, \infty)$ that satisfies the axioms of a metric except for the triangle inequality which is assumed in the weaker form

$$(2.1) \quad \rho(x, y) \leq A_0(\rho(x, z) + \rho(z, y)) \quad \text{for all } x, y, z \in X$$

with a constant $A_0 \geq 1$.

We define the quasi-metric ball by $B(x, r) := \{y \in X : \rho(x, y) < r\}$ for $x \in X$ and $r > 0$. We say that a nonzero measure μ satisfies the *doubling condition* if there is a constant C_μ such that for all $x \in X$ and $r > 0$,

$$(2.2) \quad \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty.$$

We recall that (X, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss [CW77] if d is a quasi-metric and μ is a nonzero measure satisfying the doubling condition.

We also denote the product space

$$(2.3) \quad X_1 \times X_2 := (X_1, d_1, \mu_1) \times (X_2, d_2, \mu_2),$$

where for each $i := 1, 2$, the space (X_i, d_i, μ_i) is a space of homogeneous type, with the coefficient $A_{0,i}$ for the quasi-metric d_i as in (2.1) and with the coefficient C_{μ_i} for the measure μ_i as in (2.2), respectively.

We now recall the BMO and product BMO spaces on general spaces of homogeneous type. The case of one parameter is the following, expected definition.

Definition 2.1. A locally integrable function f is in $\text{BMO}(X)$ if and only if

$$(2.4) \quad \|f\|_{\text{BMO}(X)} := \frac{1}{\mu(B)} \int_B |f(x) - f_B| d\mu(x) < \infty,$$

where $f_B := \mu(B)^{-1} \int_B f(y) d\mu(y)$, and B is any quasi-metric ball in X .

For the case of product BMO we need to introduce wavelets on spaces of homogeneous type. To begin with, recall the set $\{x_\alpha^k\}$ of *reference dyadic points* as follows. Let δ be a fixed small positive parameter (for example, as noted in [AH, Section 2.2], it suffices to take $\delta \leq 10^{-3}A_0^{-10}$). For $k = 0$, let $\mathcal{X}^0 := \{x_\alpha^0\}_\alpha$ be a maximal collection of 1-separated points in X . Inductively, for $k \in \mathbb{Z}_+$, let $\mathcal{X}^k := \{x_\alpha^k\} \supseteq \mathcal{X}^{k-1}$ and $\mathcal{X}^{-k} := \{x_\alpha^{-k}\} \subseteq \mathcal{X}^{-(k-1)}$ be δ^k - and δ^{-k} -separated collections in \mathcal{X}^{k-1} and $\mathcal{X}^{-(k-1)}$, respectively.

As shown in [AH, Lemma 2.1], for all $k \in \mathbb{Z}$ and $x \in X$, the reference dyadic points satisfy

$$(2.5) \quad d(x_\alpha^k, x_\beta^k) \geq \delta^k \quad (\alpha \neq \beta), \quad d(x, \mathcal{X}^k) = \min_\alpha d(x, x_\alpha^k) < 2A_0\delta^k.$$

Also, taking $c_0 := 1$, $C_0 := 2A_0$ and $\delta \leq 10^{-3}A_0^{-10}$, we see that c_0 , C_0 and δ satisfy $12A_0^3C_0\delta \leq c_0$ in [HK, Theorem 2.2]. By applying Hytönen and Kairema's construction ([HK, Theorem 2.2]). We conclude that there exists a set of dyadic cubes $\{Q_\alpha^k\}_{k \in \mathbb{Z}, \alpha \in \mathcal{X}^k}$ associated with the reference dyadic points $\{x_\alpha^k\}_{k \in \mathbb{Z}, \alpha \in \mathcal{X}^k}$. We call the reference dyadic point x_α^k the *center* of the dyadic cube Q_α^k . We also identify with \mathcal{X}^k the set of indices α corresponding to $x_\alpha^k \in \mathcal{X}^k$. We now denote the system of dyadic cubes as

$$\mathcal{D} := \bigcup_k \mathcal{D}_k, \quad \text{with } \mathcal{D}_k := \{Q_\alpha^k : \alpha \in \mathcal{X}^k\}.$$

Note that $\mathcal{X}^k \subseteq \mathcal{X}^{k+1}$ for $k \in \mathbb{Z}$, so that every x_α^k is also a point of the form x_β^{k+1} . We denote $\mathcal{Y}^k := \mathcal{X}^{k+1} \setminus \mathcal{X}^k$ and relabel the points $\{x_\alpha^k\}_\alpha$ that belong to \mathcal{Y}^k as $\{y_\alpha^k\}_\alpha$.

Definition 2.2 ([HLW]). We define the product BMO space $\text{BMO}(X_1 \times X_2)$ in terms of wavelet coefficients by $\text{BMO}(X_1 \times X_2) := \{f \in (\mathring{\mathcal{G}}_{1,1})' : \mathcal{C}(f) < \infty\}$, with the quantity $\mathcal{C}(f)$ defined as follows:

$$(2.6) \quad \mathcal{C}(f) := \sup_\Omega \left\{ \frac{1}{\mu(\Omega)} \sum_{\substack{R=Q_{\alpha_1}^{k_1} \times Q_{\alpha_2}^{k_2} \subset \Omega, \\ k_1, k_2 \in \mathbb{Z}, \alpha_1 \in \mathcal{Y}^{k_1}, \alpha_2 \in \mathcal{Y}^{k_2}}} |\langle \psi_{\alpha_1}^{k_1} \psi_{\alpha_2}^{k_2}, f \rangle|^2 \right\}^{1/2},$$

where Ω runs over all open sets in $X_1 \times X_2$ with finite measure.

Here we point out that the notation $(\mathring{\mathcal{G}}_{1,1})'$ in the definition above denotes the space of distributions in the product setting $X_1 \times X_2$. We recall the test function and distribution spaces, and the one-parameter version of which was defined by Han, Müller and Yang [HMY1, HMY2], and then the product version by Han, Li and Lu [HLL], where the extra reverse doubling conditions of the underlying measures are required. Here we cite the definition of test functions and distributions in both the one-parameter setting and product setting in [HLW], where there is no extra assumptions on the quasi-metric and doubling measure. Moreover, the notation ψ_α^k , $\alpha \in \mathcal{Y}^{k_1}$, denotes the orthonormal basis on general spaces of homogeneous type (X, d, μ) constructed by Auscher and Hytönen (see [AH] Theorem 7.1).

Next we recall the definition for Calderón–Zygmund operators on spaces of homogeneous type and the representation theorems for these Calderón–Zygmund operators.

A continuous function $K(x, y)$ defined on $X \times X \setminus \{(x, y) : x = y\}$ is called a *Calderón–Zygmund kernel* if there exist constant $C > 0$ and a regularity exponent $\varepsilon \in (0, 1]$ such that

- (a) $|K(x, y)| \leq CV(x, y)^{-1}$;
- (b) $|K(x, y) - K(x, y')| + |K(y, x) - K(y, x')| \leq C \left(\frac{d(y, y')}{d(x, y)} \right)^\varepsilon V(x, y)^{-1}$ if $d(y, y') \leq \frac{d(x, y)}{2A_0}$.

Above $V(x, y) := \mu(B(x, d(x, y)))$. The smallest such constant C is denoted by $|K|_{CZ}$. We say that an operator T is a *singular integral operator* associated with a Calderón–Zygmund kernel K if the operator T is a continuous linear operator from $C_0^\eta(X)$ into its dual such that

$$\langle Tf, g \rangle = \int_X \int_X g(x)K(x, y)f(y)d\mu(y)d\mu(x)$$

for all functions $f, g \in C_0^\eta(X)$ with disjoint supports. Here $C_0^\eta(X)$ is the space of all continuous functions on X with compact support such that

$$\|f\|_{C_0^\eta(X)} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\eta} < \infty.$$

The operator T is said to be a *Calderón–Zygmund operator* if it extends to be a bounded operator on $L^2(X)$. If T is a Calderón–Zygmund operator associated with a kernel K , its operator norm is defined by $\|T\|_{CZ} = \|T\|_{L^2 \rightarrow L^2} + |K|_{CZ}$.

We now recall the explicit construction in [KLPW] of a Haar basis $\{h_Q^u : Q \in \mathcal{D}, u = 1, \dots, M_Q - 1\}$ for $L^p(X)$, $1 < p < \infty$, associated to the dyadic cubes $Q \in \mathcal{D}$ as follows. Here $M_Q := \#\mathcal{H}(Q) = \#\{R \in \mathcal{D}_{k+1} : R \subseteq Q\}$ denotes the number of dyadic sub-cubes (“children”) the cube $Q \in \mathcal{D}_k$ has.

Theorem 2.3 ([KLPW]). *Let (X, ρ) be a geometrically doubling quasi-metric space and suppose μ is a positive Borel measure on X with the property that $\mu(B) < \infty$ for all balls $B \subseteq X$. For $1 < p < \infty$, for each $f \in L^p(X)$, we have*

$$f(x) = \sum_{Q \in \mathcal{D}} \sum_{u=1}^{M_Q-1} \langle f, h_Q^u \rangle_{L^2(X)} h_Q^u(x),$$

where the sum converges (unconditionally) both in the $L^p(X)$ -norm and pointwise μ -almost everywhere.

We now recall the decomposition of a Calderón–Zygmund operator T into dyadic Haar shifts, (see for example [Hy, NRV, NV]).

Theorem 2.4. *Let T be a Calderón–Zygmund operator associated with a kernel K . Then it has a decomposition: for $f, g \in C_0^\eta(X)$,*

$$(2.7) \quad \langle g, Tf \rangle_{L^2(X)} = c(\|T\|_{2 \rightarrow 2} + |K|_{CZ}) E_w \sum_{m, n=0}^{\infty} \tau(m, n) \langle g, S_w^{m, n} f \rangle_{L^2(X)},$$

where E_w is the expectation operator with respect to the random variable w , \mathcal{D}_w is the random dyadic system, $S_w^{m, n}$ is a dyadic Haar shift with parameters m, n on \mathcal{D}_w defined as follows

$$S_w^{m, n}(f)(x) = \sum_{L \in \mathcal{D}_w} \sum_{\substack{I \in \mathcal{D}_w, I \subset L \\ g(I)=g(L)+m}} \sum_{i=1}^{M_I-1} \sum_{\substack{J \in \mathcal{D}_w, J \subset L \\ g(J)=g(L)+n}} \sum_{j=1}^{M_J-1} a_{L, I, J} \langle h_I^i, f \rangle_{L^2(X)} h_J^j(x)$$

with

$$|a_{L, I, J}| \leq \frac{\sqrt{\mu(I)} \sqrt{\mu(J)}}{\mu(L)} \quad \text{and} \quad \tau(m, n) \leq C \delta^{m+n},$$

where δ is the small positive number satisfying $\delta \leq 10^{-3} A_0^{-10}$ with A_0 the constant in (2.1).

With these tools at hand, we note that the idea and approach of the proof of Theorem 1.2 is similar to the main result of [DO]. For the sake of clarity, we provide an outline of the proof in the following two subsections.

2.2. The one parameter case: $[b, T]$, $b \in \text{BMO}(X)$. To begin with, we derive a decomposition of the one-parameter commutator $[b, T]$ into basic paraproduct type operators.

Theorem 2.5. *Let $b \in \text{BMO}(X)$, $f \in C_0^\eta(X)$, and T be a Calderón–Zygmund operator. Then,*

(i) *for a cancellative dyadic shift $S_w^{m, n}$, $[b, S_w^{m, n}]$ can be represented as a finite linear combination of operators of the form*

$$(2.8) \quad S_w^{m, n}(B_k(b, f)), B_k(b, S_w^{m, n} f)$$

where $k \in \mathbb{Z}$, $0 \leq k \leq \max(m, n)$ and the total number of terms is bounded by $C(1 + \max(m, n))$ for some universal constant C ;

(ii) for a noncancellative dyadic shift $S_\omega^{0,0}$ with symbol a , $[b, S_\omega^{0,0}]f$ can be represented as a finite linear combination of operators of the form

$$(2.9) \quad S_\omega^{0,0}(B_0(b, f)), B_0(b, S_\omega^{0,0}f), \tilde{B}_0(b, S_\omega^{0,0}f), P(b, a, f), P^*(b, a, f)$$

and the total number of terms is bounded by a universal constant.

The paraproduct like operators in the above theorem are defined as the following. The generalized dyadic paraproduct

$$(2.10) \quad B_k(b, f) := \sum_I \sum_{i'=1}^{M_{I^{(k)}}-1} \sum_{i=1}^{M_I-1} \langle b, h_{I^{(k)}}^{i'} \rangle_{L^2(X)} \langle f, h_I^i \rangle_{L^2(X)} h_I^i h_{I^{(k)}}^{i'},$$

where $I^{(k)}$ denotes the k -th dyadic ancestor of I . Observe that when $k = 0$, this is the classical paraproduct

$$(2.11) \quad \tilde{B}_0(b, f) := \sum_I \sum_{i=1}^{M_I-1} \langle b, h_I^i \rangle_{L^2(X)} \langle f, h_I^0 \rangle_{L^2(X)} h_I^0 h_I^i.$$

And the trilinear operator

$$(2.12) \quad P(b, a, f) := \sum_I \sum_{i=1}^{M_I-1} \langle b, h_I^i \rangle_{L^2(X)} \langle f, h_I^i \rangle_{L^2(X)} h_I^i h_I^i \sum_{J, J \subsetneq I} \sum_{j=1}^{M_J-1} \langle a, h_J^j \rangle_{L^2(X)} h_J^j,$$

with P^* being understood as the adjoint of P with b and a fixed. The important property of the above operators is that they are uniformly bounded on L^2 with BMO symbols.

Lemma 2.6. *Given $a, b \in \text{BMO}(X)$ and $k \geq 0$, we have*

$$\|B_k(b, f)\|_{L^2(X)} \lesssim \|b\|_{\text{BMO}(X)} \|f\|_{L^2(X)},$$

$$\|\tilde{B}_0(b, f)\|_{L^2(X)} \lesssim \|b\|_{\text{BMO}(X)} \|f\|_{L^2(X)},$$

and

$$\|P(a, b, f)\|_{L^2(X)} \lesssim \|a\|_{\text{BMO}(X)} \|b\|_{\text{BMO}(X)} \|f\|_{L^2(X)}.$$

The lemma is well-known for $\tilde{B}_0(b, f)$, which is the classical paraproduct. For $B_k(b, f)$, $k \geq 1$ and for $P(b, a, f)$, the boundedness follows from adaptations and modifications of [DO, Lemma 3.6 and 3.7] to the spaces of homogeneous type. The relevant properties of spaces of homogeneous type here are the orthogonality of the Haar bases $\{h_J^j\}_{J,j}$, H^1 -BMO duality, dyadic square function characterization of dyadic H^1 and the John-Nirenberg inequality.

The proof of Theorem 2.5 follows essentially the same strategy of [DO, Theorem 3.2]. Unlike the Euclidean setting, where associated with any $Q \in \mathcal{D}$ are a fixed number of Haar functions that are constant on each child (of the same measure) of Q , in spaces of homogeneous type, there are M_Q Haar functions h_Q^u for any $Q \in \mathcal{D}$ and the measure of each child of Q can be different. Fortunately, by closely examining the argument in [DO], one observes without much difficulty that the only properties of the Haar systems it relies on are the martingale structure:

$$(2.13) \quad \sum_{J: I \subsetneq J} \sum_{j=1}^{M_J-1} \langle f, h_J^j \rangle_{L^2(X, \mu)} h_J^j h_I^i = \langle f, h_I^0 \rangle_{L^2(X, \mu)} h_I^0 h_I^i.$$

and the fact that the dyadic cubes in \mathcal{D} are properly nested. We omit the details of the proof.

In particular, Theorem 2.5, together with Lemma 2.6 and the representation of Calderón-Zygmund operators by Haar shifts (Theorem 2.4), implies almost immediately the upper bound of the commutator $[b, T]$ in spaces of homogeneous type:

$$(2.14) \quad \|[b, T]\|_{L^2(X) \rightarrow L^2(X)} \leq C \|b\|_{\text{BMO}(X)},$$

which recovers the upper bound result of [CRW, KL, BC]. More importantly, Theorem 1.2 follows from iterating Theorem 2.5 and BMO estimates of certain bi-parameter paraproduct like operators, which we explain in the next subsection.

2.3. The iterated case: $[[b, T_1], T_2]$. Applying the representation theorem (Theorem 2.4) in both variables, one could obtain Theorem 1.2 by proving for any $f \in C_0^\eta(X_1 \times X_2)$ that

$$(2.15) \quad \left\| \sum_{m_1, m_2, n_1, n_2=0}^{\infty} \tau(m_1, n_1) \tau(m_2, n_2) [[b, S_1^{m_1, n_1}], S_2^{m_2, n_2}] f \right\|_{L^2(X_1 \times X_2)} \\ \lesssim \|b\|_{BMO(X_1 \times X_2)} \|f\|_{L^2(X_1 \times X_2)}.$$

By an iteration of Theorem 2.5, one can represent $[[b, S_1^{m_1, n_1}], S_2^{m_2, n_2}]$ as a finite linear combination of basic operators which are essentially tensor products of the operators B_k , \tilde{B}_0 and P in the one-parameter setting as in (2.10), (2.11) and (2.12), and the total number of terms is no greater than $C(1 + \max(m_1, n_1))(1 + \max(m_2, n_2))$. Estimate (2.15) then follows from the uniform boundedness of these operators which we conclude in Lemma 2.7 below. The proof of Theorem 1.2 is thus complete.

More precisely, we need to consider the following paraproduct like operators in the bi-parameter setting (to condense notation that we omit the subscript $L^2(X_1 \times X_2)$ on the inner products). To begin with, we let $a, b \in BMO(X_1 \times X_2)$, $a^1 \in BMO(X_1)$ and $a^2 \in BMO(X_2)$. The generalized bi-parameter dyadic paraproduct

$$B_{k,l}(b, f) := \sum_I \sum_{i'=1}^{M_{I^{(k)}}-1} \sum_{i=1}^{M_I-1} \sum_J \sum_{j'=1}^{M_{J^{(l)}}-1} \sum_{j=1}^{M_J-1} \langle b, h_{I^{(k)}}^{i'} \otimes h_{J^{(l)}}^{j'} \rangle \langle f, h_I^i \otimes h_J^j \rangle h_I^i h_{I^{(k)}}^{i'} \otimes h_J^j h_{J^{(l)}}^{j'}.$$

Parallel to (2.11), we also have

$$\tilde{B}_{k,l}^{(1)}(b, f) := \sum_I \sum_{i'=1}^{M_{I^{(k)}}-1} \sum_J \sum_{j'=1}^{M_{J^{(l)}}-1} \sum_{j=1}^{M_J-1} \langle b, h_{I^{(k)}}^{i'} \otimes h_{J^{(l)}}^{j'} \rangle \langle f, h_I^0 \otimes h_J^j \rangle h_I^0 h_{I^{(k)}}^{i'} \otimes h_J^j h_{J^{(l)}}^{j'}, \\ \tilde{B}_{k,l}^{(2)}(b, f) := \sum_I \sum_{i'=1}^{M_{I^{(k)}}-1} \sum_{i=1}^{M_I-1} \sum_J \sum_{j'=1}^{M_{J^{(l)}}-1} \langle b, h_{I^{(k)}}^{i'} \otimes h_{J^{(l)}}^{j'} \rangle \langle f, h_I^i \otimes h_J^0 \rangle h_I^i h_{I^{(k)}}^{i'} \otimes h_J^0 h_{J^{(l)}}^{j'}, \\ \tilde{B}_{k,l}^{(3)}(b, f) := \sum_I \sum_{i'=1}^{M_{I^{(k)}}-1} \sum_J \sum_{j'=1}^{M_{J^{(l)}}-1} \langle b, h_{I^{(k)}}^{i'} \otimes h_{J^{(l)}}^{j'} \rangle \langle f, h_I^0 \otimes h_J^0 \rangle h_I^0 h_{I^{(k)}}^{i'} \otimes h_J^0 h_{J^{(l)}}^{j'}.$$

The trilinear operator

$$PP(b, a, f) := \sum_I \sum_{i=1}^{M_I-1} \sum_J \sum_{j=1}^{M_J-1} \langle b, h_I^i \otimes h_J^j \rangle \langle f, h_I^i \otimes h_J^j \rangle h_I^i h_I^i \otimes h_J^j h_J^j. \\ \sum_{I_1: I_1 \subsetneq I} \sum_{i'=1}^{M_{I_1}-1} \sum_{J_1: J_1 \subsetneq J} \sum_{j'=1}^{M_{J_1}-1} \langle a, h_{I_1}^{i'} \otimes h_{J_1}^{j'} \rangle h_{I_1}^{i'} \otimes h_{J_1}^{j'},$$

where all the Haar functions are cancellative. And the new mixed type trilinear operators

$$BP_k(b, a^2, f) := \sum_I \sum_{i'=1}^{M_{I^{(k)}}-1} \sum_{i=1}^{M_I-1} \sum_J \sum_{j=1}^{M_J-1} \langle b, h_{I^{(k)}}^{i'} \otimes h_J^j \rangle \langle f, h_I^i \otimes h_J^j \rangle h_I^i h_{I^{(k)}}^{i'} \otimes h_J^j h_J^j. \\ \sum_{J_1: J_1 \subsetneq J} \sum_{j'=1}^{M_{J_1}-1} \langle a^2, h_{J_1}^{j'} \rangle_2 h_{J_1}^{j'},$$

$$\begin{aligned}
 \tilde{B}P_k(b, a^2, f) &:= \sum_I \sum_{i'=1}^{M_I(k)-1} \sum_J \sum_{j=1}^{M_J-1} \langle b, h_{I(k)}^{i'} \otimes h_J^j \rangle \langle f, h_I^0 \otimes h_J^j \rangle h_I^0 h_{I(k)}^{i'} \otimes h_J^j h_J^j. \\
 &\quad \sum_{J_1: J_1 \subsetneq J} \sum_{j'=1}^{M_{J_1}-1} \langle a^2, h_{J_1}^{j'} \rangle_2 h_{J_1}^{j'}, \\
 PB_l(b, a^1, f) &:= \sum_I \sum_{i=1}^{M_I-1} \sum_J \sum_{j'=1}^{M_{J(l)}-1} \sum_{j=1}^{M_J-1} \langle b, h_I^i \otimes h_{J(l)}^{j'} \rangle \langle f, h_I^i \otimes h_J^j \rangle h_I^i h_I^i \otimes h_J^j h_{J(l)}^{j'}. \\
 &\quad \sum_{I_1: I_1 \subsetneq I} \sum_{i'=1}^{M_{I_1}-1} \langle a^1, h_{I_1}^{i'} \rangle_1 h_{I_1}^{i'}, \\
 P\tilde{B}_l(b, a^1, f) &:= \sum_I \sum_{i=1}^{M_I-1} \sum_J \sum_{j'=1}^{M_{J(l)}-1} \langle b, h_I^i \otimes h_{J(l)}^{j'} \rangle \langle f, h_I^i \otimes h_J^0 \rangle h_I^i h_I^i \otimes h_J^0 h_{J(l)}^{j'}. \\
 &\quad \sum_{I_1: I_1 \subsetneq I} \sum_{i'=1}^{M_{I_1}-1} \langle a^1, h_{I_1}^{i'} \rangle_1 h_{I_1}^{i'}.
 \end{aligned}$$

Lemma 2.7. *Given $a, b \in \text{BMO}(X_1 \times X_2)$, $a^1 \in \text{BMO}(X_1)$ and $a^2 \in \text{BMO}(X_2)$, we have*

$$\|PP(b, a, f)\|_{L^2(X_1 \times X_2)} \lesssim \|b\|_{\text{BMO}(X_1 \times X_2)} \|a\|_{\text{BMO}(X_1 \times X_2)} \|f\|_{L^2(X_1 \times X_2)}$$

and the same for $PP_1(b, a, f)$, which denotes the partial adjoint of PP in the first variable with respect to the third input function; moreover, for $k, l \geq 0$, we have

$$\|B_{k,l}(b, f)\|_{L^2(X_1 \times X_2)} \lesssim \|b\|_{\text{BMO}(X_1 \times X_2)} \|f\|_{L^2(X_1 \times X_2)}$$

and the same for $\tilde{B}_{k,l}^{(1)}(b, f)$, $\tilde{B}_{k,l}^{(2)}(b, f)$ and $\tilde{B}_{k,l}^{(3)}(b, f)$;

$$\|BP_k(b, a^2, f)\|_{L^2(X_1 \times X_2)} \lesssim \|b\|_{\text{BMO}(X_1 \times X_2)} \|a^2\|_{\text{BMO}(X_2)} \|f\|_{L^2(X_1 \times X_2)}$$

and the same for $\tilde{B}P_k(b, a^2, f)$;

$$\|PB_l(b, a^1, f)\|_{L^2(X_1 \times X_2)} \lesssim \|b\|_{\text{BMO}(X_1 \times X_2)} \|a^1\|_{\text{BMO}(X_1)} \|f\|_{L^2(X_1 \times X_2)}$$

and the same for $P\tilde{B}_l(b, a^1, f)$.

The above result can be derived similarly as in [DO, Lemmas 4.1, 4.2, and 4.5], therefore we omit most of the details. We point out that a key fact that is crucial is the following multi-parameter John-Nirenberg inequality in the homogeneous setting. The multiparameter John-Nirenberg inequality was first introduced in [CF, Section III] for the product BMO space defined via the wavelet basis (see also [Tao, Proposition 4.1] for dyadic product BMO on $\mathbb{R} \times \mathbb{R}$ defined via Haar basis). We note that this John-Nirenberg inequality also holds with the Haar system in the setting of space of homogeneous type. For the details, we refer to [CF, pp.199–200] and omit it here.

Lemma 2.8. *Given $b \in \text{BMO}(X_1 \times X_2)$ and $p \in (1, \infty)$, there holds*

$$(2.16) \quad \left\| \left(\sum_{R=I \times J \subset \Omega} \sum_{i=1}^{M_I-1} \sum_{j=1}^{M_J-1} |\langle b, h_I^i h_J^j \rangle|^2 \frac{\chi_R}{\mu(R)} \right)^{1/2} \right\|_{L^p(X_1 \times X_2)} \leq C \|b\|_{\text{BMO}(X_1 \times X_2)} \mu(\Omega)^{1/p}.$$

3. PROOF OF THEOREM 1.4

3.1. Proof of (i) \iff (ii). Suppose that $b \in \text{bmo}(\mathbb{R}_\lambda)$. Then we know that for any fixed $x_2 \in \mathbb{R}$, $b(x_1, x_2)$ as a function of x_1 is in the standard one-parameter $\text{BMO}(\mathbb{R}_+, dm_\lambda)$, a symmetric result holds for the roles of x_1 and x_2 interchanged. Moreover, we further have that

$$(3.1) \quad \|b\|_{\text{bmo}(\mathbb{R}_\lambda)} \approx \sup_{x_1 \in \mathbb{R}_+} \|b(x_1, \cdot)\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} + \sup_{x_2 \in \mathbb{R}_+} \|b(\cdot, x_2)\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)},$$

where the implicit constants are independent of the function b .

Next, we recall a recent result by a subset of the authors [DLWY], where they obtained that

$$(3.2) \quad \|b\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \approx \|[b, R_{\Delta_\lambda}]\|_{L^2(\mathbb{R}_+, dm_\lambda) \rightarrow L^2(\mathbb{R}_+, dm_\lambda)},$$

where $\text{BMO}(\mathbb{R}_+, dm_\lambda)$ is the standard one-parameter BMO space on $(\mathbb{R}_+, dm_\lambda)$.

Combining (3.1) and (3.2), we obtain that

$$\begin{aligned} \|b\|_{\text{bmo}(\mathbb{R}_\lambda)} &\approx \sup_{x_1 \in \mathbb{R}_+} \|[b(x_1, \cdot), R_{\Delta_\lambda, 2}]\|_{L^2(\mathbb{R}_+, dm_\lambda) \rightarrow L^2(\mathbb{R}_+, dm_\lambda)} \\ &\quad + \sup_{x_2 \in \mathbb{R}_+} \|[b(\cdot, x_2), R_{\Delta_\lambda, 1}]\|_{L^2(\mathbb{R}_+, dm_\lambda) \rightarrow L^2(\mathbb{R}_+, dm_\lambda)}, \end{aligned}$$

which implies that (i) \iff (ii).

3.2. Proof of (i) \iff (iii). From [BDT], we know that $H^1(\mathbb{R}_+, dm_\lambda)$ can be characterized via Bessel Riesz transforms, i.e., $f \in H^1(\mathbb{R}_+, dm_\lambda)$ if and only if $f, R_{\Delta_\lambda}(f) \in L^1(\mathbb{R}_+, dm_\lambda)$, and

$$\|f\|_{H^1(\mathbb{R}_+, dm_\lambda)} \approx \|f\|_{L^1(\mathbb{R}_+, dm_\lambda)} + \|R_{\Delta_\lambda}(f)\|_{L^1(\mathbb{R}_+, dm_\lambda)}.$$

Then by the duality of $H^1(\mathbb{R}_+, dm_\lambda)$ with $\text{BMO}(\mathbb{R}_+, dm_\lambda)$, and following the same approach as in [FS], we obtain the following decomposition for $\text{BMO}(\mathbb{R}_+, dm_\lambda)$:

$b \in \text{BMO}(\mathbb{R}_+, dm_\lambda)$ if and only if there exist $f, g \in L^\infty(\mathbb{R}_+, dm_\lambda)$ such that

$$(3.3) \quad b = f + R_{\Delta_\lambda}g.$$

Moreover,

$$\|b\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \approx \inf\{\|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} + \|g\|_{L^\infty(\mathbb{R}_+, dm_\lambda)}\},$$

where the infimum is taken over all possible decompositions of b as in (3.3). As a consequence, the argument (i) \iff (iii) follows from (3.1) and (3.3).

3.3. Proof of (i) \iff (iv).

3.3.1. Proof of (i) \implies (iv). We point out that the proof of the upper bound of $[b, R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}]$ follows directly from the property of $\text{bmo}(\mathbb{R}_\lambda)$ and the L^2 boundedness of the Bessel Riesz transforms $R_{\Delta_\lambda, 1}$ and $R_{\Delta_\lambda, 2}$.

To see this, for $b \in \text{bmo}(\mathbb{R}_\lambda)$, we remark that

$$[b, R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}] = R_{\Delta_\lambda, 1}[b, R_{\Delta_\lambda, 2}] + [b, R_{\Delta_\lambda, 1}]R_{\Delta_\lambda, 2}.$$

Then based on (3.1) and the result of [DLWY], we know that

$$\begin{aligned} &\|[b, R_{\Delta_\lambda, 2}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} + \|[b, R_{\Delta_\lambda, 1}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \\ &\lesssim \sup_{x_1 \in \mathbb{R}_+} \|b(x_1, \cdot)\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} + \sup_{x_2 \in \mathbb{R}_+} \|b(\cdot, x_2)\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \\ &\lesssim \|b\|_{\text{bmo}(\mathbb{R}_\lambda)}. \end{aligned}$$

Then, denote by Id_1 and Id_2 the identity operator on $L^2(\mathbb{R}_+, dm_\lambda)$ for the first and second variable, respectively. We further have

$$[b, R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}] = (R_{\Delta_\lambda, 1} \otimes \text{Id}_2) \circ [b, R_{\Delta_\lambda, 2}] + [b, R_{\Delta_\lambda, 1}] \circ (\text{Id}_1 \otimes R_{\Delta_\lambda, 2}),$$

where we use $T_1 \circ T_2$ to denote the composition of two operators T_1 and T_2 . Thus, we obtain that

$$\begin{aligned} &\|[b, R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \\ &= \|(R_{\Delta_\lambda, 1} \otimes \text{Id}_2) \circ [b, R_{\Delta_\lambda, 2}] + [b, R_{\Delta_\lambda, 1}] \circ (\text{Id}_1 \otimes R_{\Delta_\lambda, 2})\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \\ &\leq \|(R_{\Delta_\lambda, 1} \otimes \text{Id}_2) \circ [b, R_{\Delta_\lambda, 2}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} + \|[b, R_{\Delta_\lambda, 1}] \circ (\text{Id}_1 \otimes R_{\Delta_\lambda, 2})\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \\ &\leq \|R_{\Delta_\lambda, 1}\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \|[b, R_{\Delta_\lambda, 2}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \\ &\quad + \|[b, R_{\Delta_\lambda, 1}]\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \|R_{\Delta_\lambda, 2}\|_{L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda)} \\ &\lesssim \|b\|_{\text{bmo}(\mathbb{R}_\lambda)}, \end{aligned}$$

which implies (i) \implies (iv).

3.3.2. *Proof of (i) \Leftarrow (iv).* We begin with some preliminaries.

Proposition 3.1 ([DLWY]). *The Riesz kernel $R_{\Delta_\lambda}(x, y)$ satisfies:*

- (i) *There exist $K_1 > 2$ large enough and a positive constant $C_{K_1, \lambda}$ such that for any $x, y \in \mathbb{R}_+$ with $y > K_1 x$,*

$$(3.4) \quad R_{\Delta_\lambda}(x, y) \geq C_{K_1, \lambda} \frac{x}{y^{2\lambda+2}}.$$

- (ii) *There exist $K_2 \in (0, 1)$ small enough and a positive constant $C_{K_2, \lambda}$ such that for any $x, y \in \mathbb{R}_+$ with $y < K_2 x$,*

$$(3.5) \quad R_{\Delta_\lambda}(x, y) \leq -C_{K_2, \lambda} \frac{1}{x^{2\lambda+1}}.$$

- (iii) *There exist $K_3 \in (0, 1/2)$ small enough and a positive constant $C_{K_3, \lambda}$ such that for any $x, y \in \mathbb{R}_+$ with $0 < y/x - 1 < K_3$,*

$$R_{\Delta_\lambda}(x, y) \geq C_{K_3, \lambda} \frac{1}{x^\lambda y^\lambda} \frac{1}{y - x}.$$

Definition 3.2. Suppose $q \in (1, \infty]$. A q -atom on \mathbb{R}_λ is a function $a \in L^q(\mathbb{R}_\lambda)$ supported on a rectangle $R \subset \mathbb{R}_\lambda$ with $\|a\|_{L^q(\mathbb{R}_\lambda)} \leq \mu_\lambda(R)^{\frac{1}{q}-1}$ and satisfying the cancellation property

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} a(x_1, x_2) d\mu_\lambda(x_1, x_2) = 0.$$

Let $Atom_q(\mathbb{R}_\lambda)$ denote the collection of all such atoms.

Definition 3.3. Suppose $q \in (1, \infty]$. The atomic Hardy space $h^{1,q}(\mathbb{R}_\lambda)$ is defined as the set of functions of the form

$$(3.6) \quad f = \sum_i \alpha_i a_i$$

with $\{a_i\}_i \subset Atom_q(\mathbb{R}_\lambda)$, $\{\alpha_i\}_i \subset \mathbb{C}$ and $\sum_i |\alpha_i| < \infty$. Moreover, $h^{1,q}(\mathbb{R}_\lambda)$ is equipped with the norm $\|f\|_{h^{1,q}(\mathbb{R}_\lambda)} := \inf \sum_i |\alpha_i|$ where the infimum is taken over all possible decompositions of f in the form (3.6).

For these little Hardy spaces, we first have the following conclusion.

Theorem 3.4. *Let $q \in (1, \infty)$. Then the spaces $h^{1,q}(\mathbb{R}_\lambda)$ and $h^{1,\infty}(\mathbb{R}_\lambda)$ coincide with equivalent norms.*

We first recall the following Whitney type covering lemma from [CW77].

Lemma 3.5. *Suppose $U \subsetneq \mathbb{R}_+ \times \mathbb{R}_+$ is an open bounded set and $\tilde{C} \in [1, \infty)$. Then there exists a sequence of cubes $\{Q_j\}_j$ satisfying*

- (i) $U = \cup_j Q_j = \cup_j \tilde{C}Q_j$;
- (ii) *there exists a positive constant M such that no point of $\mathbb{R}_+ \times \mathbb{R}_+$ belongs to more than M of the balls $\tilde{C}Q_j$, which is called as the M -disjointness of $\{\tilde{C}Q_j\}_j$;*
- (iii) $3\tilde{C}Q_j \cap (\mathbb{R}_+ \times \mathbb{R}_+ \setminus U) \neq \emptyset$ for each j .

Now we establish a useful lemma which is a variant of [CW77, Lemma (3.9)]. To this end, we recall the strong maximal function defined by setting, for all $(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\mathcal{M}_s f(x_1, x_2) := \sup_{R \ni (x_1, x_2)} \frac{1}{\mu_\lambda(R)} \int_R |f(y_1, y_2)| d\mu_\lambda(y_1, y_2).$$

It is already known that \mathcal{M}_s is bounded on $L^p(\mathbb{R}_\lambda)$, with $p \in (1, \infty)$.

Lemma 3.6. *If $f \in L^1_{\text{loc}}(\mu_\lambda)$ has support in $R_0 := I_0 \times J_0$ centered at (x_0^1, x_0^2) , then there exists a positive constant C_1 such that*

$$U^\alpha := \{(x_1, x_2) \in \mathbb{R}_\lambda : \mathcal{M}_s f(x_1, x_2) > \alpha\} \subset 3R_0$$

whenever $\alpha \in (C_1 m_{R_0}(|f|), \infty)$, where $m_R(f)$ is as in (1.5).

Proof. We only need to prove that, if $\alpha \in (C_1 m_{R_0}(|f|), \infty)$, then $\mathbb{R}_\lambda \setminus (3R_0) \subset \mathbb{R}_\lambda \setminus U^\alpha$.

For any $x := (x_1, x_2) \notin 3R_0$, we have $|x_1 - x_0^1| \geq |I_0|$ and $|x_2 - x_0^2| \geq |J_0|$. Then it is easy to show that, for any rectangle $R \ni (x_1, x_2)$ satisfying $|I| \leq |I_0|$ or $|J| \leq |J_0|$, $R \cap R_0 = \emptyset$. Then

$$\mathcal{M}_s f(x) = \sup_{I \ni x_1, |I| \geq |I_0|} \sup_{J \ni x_2, |J| \geq |J_0|} \frac{1}{m_\lambda(I)m_\lambda(J)} \int_I \int_J |f(y_1, y_2)| d\mu_\lambda(y_1, y_2).$$

For any rectangle $R := I \times J \ni (x_1, x_2)$ such that $|I| \geq |I_0|, |J| \geq |J_0|$ and $R \cap R_0 \neq \emptyset$, it is easy to see that $R_0 \subset 3R$. This, together with $\text{supp}(f) \subset R_0$ implies that

$$\begin{aligned} \frac{1}{\mu_\lambda(R)} \int_R |f(y_1, y_2)| d\mu_\lambda(y_1, y_2) &\leq \frac{\mu_\lambda(R_0)}{\mu_\lambda(R)} \frac{1}{\mu_\lambda(R_0)} \int_{R_0} |f(y_1, y_2)| d\mu_\lambda(y_1, y_2) \\ &\leq \frac{\mu_\lambda(3R)}{\mu_\lambda(R)} m_{R_0}(|f|) \leq C_1 m_{R_0}(|f|). \end{aligned}$$

Thus, we have $\mathcal{M}_s f(x) \leq C_1 m_{R_0}(|f|)$. Moreover, if $\alpha > C_1 m_{R_0}(|f|)$, then $\alpha > \mathcal{M}_s f(x_1, x_2)$, that is, $(x_1, x_2) \notin U^\alpha$, which completes the proof of Lemma 3.6. \square

Proof of Theorem 3.4. We have observed that $h^{1,\infty}(\mathbb{R}_\lambda) \subset h^{1,q}(\mathbb{R}_\lambda)$ for $q \in (1, \infty)$. Thus, we only need to establish the converse. We do so by showing that for any $(1, q)$ -atom a with $\text{supp}(a) \subset R_0$, $b := \mu_\lambda(R_0) \cdot a$ has an atomic decomposition $b = \sum_{i=0}^\infty \alpha_i b_i$, where each $b_i, i \in \mathbb{Z}_+$, is a $(1, \infty)$ -atom and

$$\sum_{i=0}^\infty |\alpha_i| \lesssim 1.$$

We show this by induction. In order to state the inductive hypothesis we first introduce some necessary notation.

For each positive integer n , let \mathbb{N}^n denote the n -fold Cartesian product of the natural numbers \mathbb{N} , $\mathbb{N}^0 := \{0\}$. We write i_n to represent a general element of \mathbb{N}^n . The inductive hypothesis we establish is the following one:

There exists a collection of rectangles $\{R_{i_l}\}, i_l \in \mathbb{N}^l$ for $l \in \{1, 2, \dots\}$, such that, for each $n \in \mathbb{N}$,

$$(3.7) \quad b = \sum_{l=1}^{n-1} \sum_{i_l \in \mathbb{N}^l} M C_\lambda \alpha^{l+1} \mu_\lambda(3R_{i_l}) a_{i_l} + \sum_{i_n \in \mathbb{N}^n} h_{i_n} =: G_n + H_n,$$

where $p \in (1, q)$, $\alpha \in (1, \infty)$ is large enough which depends on p, q and is to be fixed later, C_λ satisfies for any rectangle $R \subset \mathbb{R}_\lambda$, $\mu_\lambda(9R) \leq C_\lambda \mu_\lambda(R)$, and

- (I) a_{i_l} is a $(1, \infty)$ -atom supported in $3R_{i_l}$, $l \in \{1, 2, \dots, n-1\}$, $i_l \in \mathbb{N}^l$;
- (II) $\cup_{i_n \in \mathbb{N}^n} R_{i_n} \subset \{x \in \mathbb{R}_\lambda : \mathcal{M}_{s,p} b(x) > \alpha^n / 2\}$, where $p \in (1, q)$ and $\mathcal{M}_{s,p}(f) := [\mathcal{M}_s(|f|^p)]^{1/p}$;
- (III) $\{3R_{i_l}\}$ is an M^l -disjoint collection;
- (IV) the function h_{i_n} is supported in R_{i_n} for each $i_n \in \mathbb{N}^n$;
- (V) $\int_{\mathbb{R}_\lambda} h_{i_n}(x) d\mu_\lambda(x) = 0$ for each $i_n \in \mathbb{N}^n$;
- (VI) $|h_{i_n}(x)| \leq |b(x)| + 2C_\lambda^{1/p} \alpha^n \chi_{R_{i_n}}(x)$ for each $i_n \in \mathbb{N}^n$, where $\chi_{R_{i_n}}$ is the characteristic function of R_{i_n} ;
- (VII) $[m_{R_{i_n}}(|h_{i_n}|^p)]^{1/p} \leq 2C_\lambda^{1/p} \alpha^n$ for each $i_n \in \mathbb{N}^n$.

We begin with proving that

$$(3.8) \quad I_p := \frac{1}{\mu_\lambda(R_0)} \sum_{n=1}^\infty \sum_{i_n \in \mathbb{N}^n} M C_\lambda \alpha^{n+1} \mu_\lambda(3R_{i_n}) \lesssim 1.$$

Indeed, from (III), (II), $b = \mu_\lambda(R_0)a$ and the boundedness of $\mathcal{M}_{s,p}$ from $L^q(\mathbb{R}_\lambda)$ to $L^{q,\infty}(\mathbb{R}_\lambda)$, we deduce that

$$\begin{aligned}
 (3.9) \quad \sum_{i_n \in \mathbb{N}^n} \mu_\lambda(3R_{i_n}) &\leq C_\lambda M^n \mu_\lambda \left(\bigcup_{i_n \in \mathbb{N}^n} R_{i_n} \right) \\
 &\leq C_\lambda M^n \mu_\lambda(\{x \in \mathbb{R}_\lambda : \mathcal{M}_{s,p}b(x) > \alpha^n/2\}) \\
 &\lesssim C_\lambda M^n 2^q \alpha^{-nq} \|b\|_{L^q(\mathbb{R}_\lambda)}^q \\
 &\lesssim C_\lambda M^n 2^q \alpha^{-nq} \mu_\lambda(R_0).
 \end{aligned}$$

This fact implies that

$$\mathbb{I}_p \lesssim MC_\lambda \sum_{n=1}^{\infty} \alpha^{n+1} C_\lambda M^n 2^q \alpha^{-nq} \approx MC_\lambda^2 \alpha 2^q \sum_{n=1}^{\infty} (\alpha^{1-q} M)^n \lesssim 1,$$

if α is large enough such that $\alpha^{1-q} M < 1$, which gives (3.8).

By (IV), (VII), Hölder's inequality and (3.9), we obtain

$$\begin{aligned}
 \int_{\mathbb{R}_\lambda} |\mathbb{H}_n(x)| d\mu_\lambda(x) &\leq \sum_{i_n \in \mathbb{N}^n} \int_{\mathbb{R}_\lambda} |h_{i_n}(x)| d\mu_\lambda(x) \\
 &\leq 2C_\lambda^{1/p} \alpha^n \sum_{i_n \in \mathbb{N}^n} \mu_\lambda(R_{i_n}) \\
 &\lesssim 2C_\lambda^{1/p} \alpha^n C_\lambda M^n 2^q \alpha^{-nq} \|b\|_{L^q(\mathbb{R}_\lambda)}^q \\
 &\lesssim (M\alpha^{1-q})^n \|b\|_{L^q(\mathbb{R}_\lambda)}^q.
 \end{aligned}$$

This, together with $q > 1$, shows that \mathbb{G}_n converges to b in $L^1(\mu)$. Then the representation (3.7) holds true in $L^1(\mathbb{R}_\lambda)$.

Let us show that the hypothesis is valid for $n = 1$. Let

$$U^\alpha := \{(x_1, x_2) \in \mathbb{R}_\lambda : \mathcal{M}_{s,p}b(x_1, x_2) > \alpha\}.$$

Observe that $m_{R_0}(|b|) \leq 1$. By this and Lemma 3.6, we find that $U^\alpha \subset 3R_0$ provided $\alpha^p > C_1$ therein. Moreover, U^α is a bounded open set. By the boundedness of $\mathcal{M}_{s,p}$ from $L^q(\mathbb{R}_\lambda)$ to $L^{q,\infty}(\mathbb{R}_\lambda)$, we conclude that there exists a positive constant $C_{p,q}$ such that,

$$\mu_\lambda(U^\alpha) \leq C_{p,q} \alpha^{-q} \|b\|_{L^q(\mathbb{R}_\lambda)}^q \leq C_{p,q} \alpha^{-q} \mu_\lambda(R_0).$$

If $\alpha^q > C_{p,q}$, then $\mu_\lambda(U^\alpha) < \mu_\lambda(R_0) < \infty$. We see that, $\mathbb{R}_\lambda \setminus U^\alpha$ can not be empty. Applying Lemma 3.5 with $\tilde{C} = 3$ therein, we obtain a sequence of rectangles (cubes actually) $\{R_i\}_i$ satisfying (i) through (iii) therein. Let $\chi_i := \chi_{R_i}$,

$$\eta_i(x) := \begin{cases} \frac{\chi_i(x)}{\sum_k \chi_k(x)}, & \text{if } x \in U^\alpha; \\ 0, & \text{otherwise,} \end{cases}$$

$$g_0(x) := \begin{cases} b(x), & \text{if } x \notin U^\alpha; \\ \sum_i m_{R_i}(\eta_i b) \chi_i(x), & \text{if } x \in U^\alpha \end{cases}$$

and

$$h_i(x) := \eta_i(x)b(x) - m_{R_i}(\eta_i b)\chi_i(x)$$

for all $x \in \mathbb{R}_+ \times \mathbb{R}_+$. It follows that $b = g_0 + \sum_i h_i$. For almost every $x \notin U^\alpha$, we see that

$$|g_0(x)| = |b(x)| \leq \mathcal{M}_{s,p}b(x) \leq \alpha.$$

If $x \in U^\alpha$, by the Hölder inequality, (ii) and (iii) of Lemma 3.5 and the definition of U^α , we obtain

$$\begin{aligned}
(3.10) \quad |g_0(x)| &\leq \sum_i \frac{1}{\mu_\lambda(R_i)} \int_{R_i} |\eta_i(y)b(y)| d\mu_\lambda(y) \chi_i(x) \\
&\leq \sum_i \frac{\mu_\lambda(9R_i)}{\mu_\lambda(R_i)} \left[\frac{1}{\mu_\lambda(9R_i)} \int_{9R_i} |b(y)|^p d\mu_\lambda(y) \right]^{1/p} \chi_i(x) \\
&\leq \sum_i C_\lambda \alpha \chi_i(x) \\
&\leq MC_\lambda \alpha.
\end{aligned}$$

Combining these two estimates, we conclude that, for almost every $x \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$(3.11) \quad |g_0(x)| \leq MC_\lambda \alpha.$$

We have seen that $U^\alpha \subset 3R_0$ and that, for $x \notin U^\alpha$, $g(x) = b(x)$. By $\text{supp}(b) \subset 3R_0$, we conclude that $\text{supp}(g) \subset 3R_0$. Also,

$$(3.12) \quad \text{supp}(h_i) \subset R_i$$

and

$$(3.13) \quad \int_{\mathbb{R}_\lambda} h_i(x) d\mu_\lambda(x) = 0$$

for any i . Since $\{R_i\}_i$ are M -disjoint, we have

$$\begin{aligned}
(3.14) \quad \sum_i \|h_i\|_{L^1(\mathbb{R}_\lambda)} &\leq 2 \sum_i \|\eta_i b\|_{L^1(\mathbb{R}_\lambda)} \leq 2 \sum_i \int_{R_i} |b(x)| d\mu_\lambda(x) \\
&\leq 2M \int_{U^\alpha} |b(x)| d\mu_\lambda(x) \\
&\leq 2M \mu_\lambda(R_0).
\end{aligned}$$

Observe that $\int_{\mathbb{R}_\lambda} g_0(x) d\mu_\lambda(x) = 0$. Thus,

$$(3.15) \quad a_0 := g_0 / (MC_\lambda \alpha \mu_\lambda(3R_0))$$

is a $(1, \infty)$ -atom supported in $3R_0$, and we have

$$b = MC_\lambda \alpha \mu_\lambda(3R_0) a_0 + \sum_i h_i.$$

This shows (I).

Now observe that

$$\bigcup_i R_i = U^\alpha = \{x \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{M}_{s,p} b(x) > \alpha\} \subset \{x \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{M}_{s,p} b(x) > \alpha/2\}.$$

This shows (II).

Since $0 \leq \eta_i \leq 1$, arguing as in (3.10), we obtain

$$\begin{aligned}
|h_i(x)| &\leq |\eta_i(x)b(x)| + |m_{R_i}(\eta_i b)| \chi_i(x) \\
&\leq |b(x)| + [m_{R_i}(|b|^p)]^{1/p} \chi_i(x) \\
&\leq |b(x)| + C_\lambda^{1/p} \alpha \chi_i(x).
\end{aligned}$$

Thus, (VI) holds true. From this together with the definition of U^α and Lemma 3.5 (iii), we further deduce that

$$\begin{aligned}
[m_{R_i}(|h_i|^p)]^{1/p} &\leq [m_{R_i}(|b|^p)]^{1/p} + C_\lambda^{1/p} \alpha \\
&\leq \left[\frac{\mu_\lambda(9R_i)}{\mu_\lambda(R_i)} m_{9R_i}(|b|^p) \right]^{1/p} + C_\lambda^{1/p} \alpha \\
&\leq 2C_\lambda^{1/p} \alpha,
\end{aligned}$$

which implies (VII). Moreover, (III) is a consequence of Lemma 3.5(ii), and (IV) holds true by (3.12) and (V) holds true by (3.13). This shows that the induction holds true for $n = 1$.

We now assume that the hypothesis holds true for n and show that it is also valid for $n + 1$. Let

$$U_{i_n}^\alpha := \{x \in \mathbb{R}_\lambda : \mathcal{M}_{s,p} h_{i_n}(x) > \alpha^{n+1}\}.$$

By (IV) for n , we have $\text{supp}(h_{i_n}) \subset R_{i_n}$. Moreover, it follows, from (VII) for n , provided $\alpha^p > 2^p C_1 C_\lambda$, that

$$C_1 m_{R_{i_n}} (|h_{i_n}|^p) \leq C_1 C_\lambda (2\alpha^n)^p < \alpha^{(n+1)p}.$$

By Lemma 3.6, we see that

$$(3.16) \quad U_{i_n}^\alpha = \left\{x \in \mathbb{R}_\lambda : \mathcal{M}_s (|h_{i_n}|^p)(x) > \alpha^{(n+1)p}\right\} \subset 3R_{i_n}.$$

Let rectangles $\{R_{i_n,k}\}_k$ be a Whitney covering of $U_{i_n}^\alpha$. From (i) and (ii) of Lemma 3.5 and (3.16), it follows that

$$\bigcup_k 3R_{i_n,k} = U_{i_n}^\alpha \subset 3R_{i_n}$$

and $\{3R_{i_n,k}\}_k$ is M -disjoint. Since, from (III) for n , we know $\{3R_{i_n}\}_{i_n}$ are M^n -disjoint, it follows that the totality of rectangles (cubes) in the family $\{3R_{i_n,k}\}_{k,i_n}$ are M^{n+1} -disjoint. This establishes (III) for $n + 1$.

We now put

$$g_{i_n}(x) := \begin{cases} h_{i_n}(x), & \text{if } x \notin U_{i_n}^\alpha; \\ \sum_k m_{R_{i_n,k}} (\eta_k^{i_n} h_{i_n}) \chi_{R_{i_n,k}}(x), & \text{if } x \in U_{i_n}^\alpha \end{cases}$$

and

$$h_{i_n,k} := \eta_k^{i_n} h_{i_n} - m_{R_{i_n,k}} (\eta_k^{i_n} h_{i_n}) \chi_{R_{i_n,k}},$$

where

$$\eta_k^{i_n}(x) := \chi_{R_{i_n,k}}(x) / \sum_k \chi_{R_{i_n,k}}(x)$$

for $x \in U_{i_n}^\alpha$, and is 0 if $x \notin U_{i_n}^\alpha$. If $x \in U_{i_n}^\alpha$, then

$$\begin{aligned} |g_{i_n}(x)| &\leq \sum_k \left| m_{R_{i_n,k}} (\eta_k^{i_n} h_{i_n}) \chi_{R_{i_n,k}}(x) \right| \\ &\leq \sum_k \frac{\mu_\lambda(9R_{i_n,k})}{\mu_\lambda(R_{i_n,k})} \frac{1}{\mu_\lambda(9R_{i_n,k})} \int_{9R_{i_n,k}} |h_{i_n}(y)| d\mu_\lambda(y) \chi_{R_{i_n,k}}(x) \\ &\leq MC_\lambda \alpha^{n+1}, \end{aligned}$$

while if $x \notin U_{i_n}^\alpha$, then

$$|g_{i_n}(x)| = |h_{i_n}(x)| \leq \mathcal{M}_{s,p} h_{i_n}(x) \leq \alpha^{n+1}.$$

In any case, we have

$$\|g_{i_n}\|_{L^\infty(\mu)} \leq MC_\lambda \alpha^{n+1}.$$

Since the support of h_{i_n} is within $R_{i_n} \subset 3R_{i_n}$ and $U_{i_n}^\alpha \subset 3R_{i_n}$, it follows that the support of g_{i_n} is included in $3R_{i_n}$. Moreover, $\int_{\mathbb{R}_\lambda} h_{i_n,k}(x) d\mu_\lambda(x) = 0$ (which shows that property (V) is valid for $n + 1$). By an argument used in the estimate for (3.14), it is easy to see that

$$\sum_k \|h_{i_n,k}\|_{L^1(\mathbb{R}_\lambda)} \leq 2M \|h_{i_n}\|_{L^1(\mathbb{R}_\lambda)}.$$

It then follows from this that

$$h_{i_n} = g_{i_n} + \sum_k h_{i_n,k}$$

is valid in $L^1(\mu)$ and $\int_{\mathbb{R}_\lambda} g_{i_n}(x) d\mu_\lambda(x) = 0$.

Let

$$a_{i_n} := g_{i_n} / \{MC_\lambda \alpha^{n+1} [\mu_\lambda(3R_{i_n})]\}.$$

Then a_{i_n} is a $(1, \infty)$ -atom supported in the rectangle $3R_{i_n}$. From this, we deduce that (3.7) holds true for $n+1$ and so does (I). Property (IV) is trivially true. Moreover, by the definition of $h_{i_n, k}$, (VI) for n and Lemma 3.5(iii), we conclude that

$$\begin{aligned} |h_{i_n, k}(x)| &\leq \left\{ |h_{i_n}(x)| + \left[C_\lambda \frac{1}{\mu_\lambda(9R_{i_n, k})} \int_{9R_{i_n, k}} |h_{i_n}(x)|^p d\mu_\lambda(x) \right]^{1/p} \right\} \chi_{R_{i_n, k}}(x) \\ &\leq \left\{ |b(x)| + 2C_\lambda^{1/p} \alpha^n + C_\lambda^{1/p} \alpha^{n+1} \right\} \chi_{R_{i_n, k}}(x) \\ &\leq \left\{ |b(x)| + 2C_\lambda^{1/p} \alpha^{n+1} \right\} \chi_{R_{i_n, k}}(x) \end{aligned}$$

if $\alpha > 2$. This establishes (VI) for $n+1$.

On the other hand, by the definitions of $h_{i_n, k}$ and $U_{i_n}^\alpha$, we have

$$\begin{aligned} [m_{R_{i_n, k}}(|h_{i_n, k}|^p)]^{1/p} &\leq 2 [m_{R_{i_n, k}}(|h_{i_n}|^p)]^{1/p} \\ &\leq 2 [C_\lambda m_{9R_{i_n, k}}(|h_{i_n}|^p)]^{1/p} \\ &\leq 2C_\lambda^{1/p} \alpha^{n+1}, \end{aligned}$$

which shows (VII).

Finally, from (VI) for n , we deduce that

$$\mathcal{M}_{s, p}(h_{i_n})(x) \leq \mathcal{M}_{s, p}(b)(x) + 2C_\lambda^{1/p} \alpha^n.$$

Thus, if $x \in U_{i_n}^\alpha$, then

$$\alpha^{n+1} < \mathcal{M}_{s, p}(h_{i_n})(x) \leq \mathcal{M}_{s, p}(b)(x) + 2C_\lambda^{1/p} \alpha^n.$$

It follows that, if $\alpha > 4C_\lambda^{1/p}$, then $\alpha^{n+1}/2 < \mathcal{M}_{s, p}(b)(x)$. Thus,

$$\bigcup_{i_n, k} R_{i_n, k} = \bigcup_{i_n} \bigcup_k R_{i_n, k} \subset \bigcup_{i_n} U_{i_n}^\alpha \subset \{x \in \mathbb{R}_\lambda : \mathcal{M}_{s, p}(b)(x) > \alpha^{n+1}/2\}$$

and (II) is valid for $n+1$. This finishes the proof of Theorem 3.4. \square

Based on Theorem 3.4, we now denote by $h^1(\mathbb{R}_\lambda)$ the little Hardy space, and we have the following result on the duality of $h^1(\mathbb{R}_\lambda)$ with $\text{bmo}(\mathbb{R}_\lambda)$.

Theorem 3.7. *The predual of $\text{bmo}(\mathbb{R}_\lambda)$ is $h^1(\mathbb{R}_\lambda)$.*

Proof. The duality of $h^{1,2}(\mathbb{R}_\lambda)$ with $\text{bmo}(\mathbb{R}_\lambda)$ follows from a standard argument, see for example [CW77] (see also [J, Section II, Chapter 3]). Hence, by Theorem 3.4, the predual of $\text{bmo}(\mathbb{R}_\lambda)$ is $h^{1,\infty}(\mathbb{R}_\lambda)$. \square

Our main result of this section is the following.

Theorem 3.8. *For every $f \in h^1(\mathbb{R}_\lambda)$, there exist sequences $\{\alpha_j^k\}_j \in \ell^1$ and functions $g_j^k, h_j^k \in L^\infty(\mathbb{R}_\lambda)$ with compact support, such that*

$$(3.17) \quad f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k)$$

in the sense of $h^1(\mathbb{R}_\lambda)$, where $\Pi(g, h)$ is the bilinear form defined as

$$(3.18) \quad \Pi(g, h) := g \cdot R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}(h) - h \cdot \widetilde{R_{\Delta_\lambda, 1}} \widetilde{R_{\Delta_\lambda, 2}}(g),$$

where $\widetilde{R_{\Delta_\lambda, 1}}$ and $\widetilde{R_{\Delta_\lambda, 2}}$ are the adjoints of $R_{\Delta_\lambda, 1}$ and $R_{\Delta_\lambda, 2}$, respectively.

Moreover, we have that

$$\|f\|_{h^1(\mathbb{R}_\lambda)} \approx \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \alpha_j^k \right| \left\| g_j^k \right\|_{L^2(\mathbb{R}_\lambda)} \left\| h_j^k \right\|_{L^2(\mathbb{R}_\lambda)} \right\},$$

where the infimum is taken over all representations of f in the form (3.17) and the implicit constants are independent of f .

To prove Theorem 3.8, we study the property of the bilinear form $\Pi(f, g)$ as defined in (3.18), which connects to the commutator $[b, R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}]$.

Proposition 3.9. *For every $g, h \in L^\infty(\mathbb{R}_\lambda)$ with compact support, the bilinear form $\Pi(g, h)$ is in $h^1(\mathbb{R}_\lambda)$ with the norm satisfying*

$$(3.19) \quad \|\Pi(g, h)\|_{h^1(\mathbb{R}_\lambda)} \leq C \|g\|_{L^2(\mathbb{R}_\lambda)} \|h\|_{L^2(\mathbb{R}_\lambda)}.$$

Proof. First, it is clear that for every $g, h \in L^\infty(\mathbb{R}_\lambda)$ with compact support, the bilinear form $\Pi(f, g)$ is in $L^1(\mathbb{R}_\lambda)$ with compact support and satisfies

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} \Pi(g, h)(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) = 0.$$

Moreover, for $b \in \text{bmo}(\mathbb{R}_\lambda)$ and for every $g, h \in L^\infty(\mathbb{R}_\lambda)$ with compact support, we have

$$(3.20) \quad \begin{aligned} \left| \langle b, \Pi(g, h) \rangle_{L^2(\mathbb{R}_\lambda)} \right| &= \left| \langle b, g R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2} h - h \widetilde{R_{\Delta_\lambda, 1}} \widetilde{R_{\Delta_\lambda, 2}} g \rangle_{L^2(\mathbb{R}_\lambda)} \right| \\ &= \left| \langle [b, R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}] f, g \rangle_{L^2(\mathbb{R}_\lambda)} \right| \\ &\lesssim \|b\|_{\text{bmo}(\mathbb{R}_\lambda)} \|f\|_{L^2(\mathbb{R}_\lambda)} \|g\|_{L^2(\mathbb{R}_\lambda)}, \end{aligned}$$

This, together with the duality result as in Theorem 3.7, implies that for every $g, h \in L^\infty(\mathbb{R}_\lambda)$ with compact supports, the bilinear form $\Pi(f, g)$ is in $h^1(\mathbb{R}_\lambda)$. Moreover, the $h^1(\mathbb{R}_\lambda)$ norm of $\Pi(f, g)$ satisfies (3.19). In fact, we point out that from the fundamental fact as in [Gra, Exercise 1.4.12 (b)], we have

$$\|\Pi(g, h)\|_{h^1(\mathbb{R}_\lambda)} \approx \sup_{b: \|b\|_{\text{bmo}(\mathbb{R}_\lambda)} \leq 1} |\langle b, \Pi(g, h) \rangle_{L^2(\mathbb{R}_\lambda)}|,$$

which, together with (3.20), immediately implies that (3.19) holds. \square

Next, we provide the following approximation to each $h^{1,\infty}(\mathbb{R}_\lambda)$ atom via the bilinear form defined as in (3.18).

Theorem 3.10. *Let ϵ be an arbitrary positive number. Let $a(x_1, x_2)$ be an ∞ -atom as defined in Definition 3.2. Then there exist two functions $f, g \in L^\infty(\mathbb{R}_\lambda)$ with compact supports and a constant $C(\epsilon)$ depending only on ϵ such that*

$$\|a - \Pi(f, g)\|_{h^1(\mathbb{R}_\lambda)} < \epsilon,$$

where $\|f\|_{L^2(\mathbb{R}_\lambda)} \|g\|_{L^2(\mathbb{R}_\lambda)} \leq C(\epsilon)$.

To prove Theorem 3.10, we first provide a technical lemma as follows.

Lemma 3.11. *Let $R := I(x_{0,1}, r_1) \times I(x_{0,2}, r_2)$ and $\tilde{R} := I(y_{0,1}, r_1) \times I(y_{0,2}, r_2)$ be two rectangles in $\mathbb{R}_+ \times \mathbb{R}_+$ with $r_1 \leq \min\{x_{0,1}, y_{0,1}\}$ and $r_2 \leq \min\{x_{0,2}, y_{0,2}\}$. Moreover, assume that $|x_{0,1} - y_{0,1}| \geq 4r_1$ and $|x_{0,2} - y_{0,2}| \geq 4r_2$.*

Let $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ with $\text{supp } f \subseteq R \cup \tilde{R}$. Further, assume that

$$|f(x_1, x_2)| \lesssim \tilde{C}_1 \chi_R(x_1, x_2) + \tilde{C}_2 \chi_{\tilde{R}}(x_1, x_2)$$

and that f has a mean value zero property:

$$(3.21) \quad \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) = 0.$$

Then there exists a positive constant C independent of $x_{0,1}, x_{0,2}, y_{0,1}, y_{0,2}, r_1, r_2, \tilde{C}_1$ and \tilde{C}_2 such that

$$\|f\|_{h^1(\mathbb{R}_\lambda)} \leq C \left(\log_2 \frac{|x_{0,1} - y_{0,1}|}{r_1} + \log_2 \frac{|x_{0,2} - y_{0,2}|}{r_2} \right) \left(\tilde{C}_1 \mu_\lambda(R) + \tilde{C}_2 \mu_\lambda(\tilde{R}) \right).$$

Proof. Suppose f satisfies the conditions as stated above. We will show that f has an atomic decomposition as the form in Definition 3.3. To see this, we first define two functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ by

$$\begin{aligned} f_1(x_1, x_2) &= f(x_1, x_2), (x_1, x_2) \in R; & f_1(x_1, x_2) &= 0, (x_1, x_2) \in \mathbb{R}^2 \setminus R, & \text{and} \\ f_2(x_1, x_2) &= f(x_1, x_2), (x_1, x_2) \in \tilde{R}; & f_2(x_1, x_2) &= 0, (x_1, x_2) \in \mathbb{R}^2 \setminus \tilde{R}. \end{aligned}$$

Then we have that $f = f_1 + f_2$ and that

$$|f_1(x_1, x_2)| \lesssim \tilde{C}_1 \chi_R(x_1, x_2) \quad \text{and} \quad |f_2(x_1, x_2)| \lesssim \tilde{C}_2 \chi_{\tilde{R}}(x_1, x_2).$$

Define

$$\begin{aligned} g_1^1(x_1, x_2) &:= \frac{\chi_{2R}(x_1, x_2)}{\mu_\lambda(2R)} \iint_R f_1(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2), \\ f_1^1(x_1, x_2) &:= f_1(x_1, x_2) - g_1^1(x_1, x_2), \\ \alpha_1^1 &:= \|f_1^1\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2R). \end{aligned}$$

Then we claim that $a_1^1 := (\alpha_1^1)^{-1} f_1^1$ is a rectangle atom as in Definition 3.2. First, it is direct that a_1^1 is supported in $2R$. Moreover, we have that

$$\begin{aligned} &\int_{\mathbb{R}_+ \times \mathbb{R}_+} a_1^1(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) \\ &= (\alpha_1^1)^{-1} \int_{\mathbb{R}_+ \times \mathbb{R}_+} (f_1(x_1, x_2) - g_1^1(x_1, x_2)) dm_\lambda(x_1) dm_\lambda(x_2) \\ &= (\alpha_1^1)^{-1} \left(\int_{\mathbb{R}_+ \times \mathbb{R}_+} f_1(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) - \int_{\mathbb{R}_+ \times \mathbb{R}_+} f_1(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) \right) \\ &= 0 \end{aligned}$$

and that

$$\|a_1^1\|_{L^\infty(\mathbb{R}_\lambda)} \leq |(\alpha_1^1)^{-1}| \|f_1^1\|_{L^\infty(\mathbb{R}_\lambda)} = \frac{1}{\mu_\lambda(2R)}.$$

Thus, a_1^1 is an ∞ -atom as in Definition 3.2. Moreover, we have

$$\alpha_1^1 = \|f_1^1\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2R) \leq \|f_1\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2R) + \|g_1^1\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2R) \lesssim \tilde{C}_1 \mu_\lambda(R),$$

where the implicit constant depends only on λ . We now have

$$f_1(x_1, x_2) = f_1^1(x_1, x_2) + g_1^1(x_1, x_2) = \alpha_1^1 a_1^1 + g_1^1(x_1, x_2).$$

For $g_1^1(x_1, x_2)$, we further write it as

$$g_1^1(x_1, x_2) = g_1^1(x_1, x_2) - g_1^2(x_1, x_2) + g_1^2(x_1, x_2) =: f_1^2(x_1, x_2) + g_1^2(x_1, x_2)$$

with

$$g_1^2(x_1, x_2) := \frac{\chi_{4R}(x_1, x_2)}{\mu_\lambda(4R)} \iint_R f_1(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2).$$

Again, we define

$$\alpha_1^2 := \|f_1^2\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(4R) \quad \text{and} \quad a_1^2 := (\alpha_1^2)^{-1} f_1^2,$$

and following similar estimates as for a_1^1 , we see that a_1^2 is an ∞ -atom as in Definition 3.2 with

$$\|a_1^2\|_{L^\infty(\mathbb{R}_\lambda)} \leq \frac{1}{\mu_\lambda(4R)} \quad \text{and} \quad \alpha_1^2 \lesssim \tilde{C}_1 \mu_\lambda(R),$$

where the implicit constant depends only on λ .

Then we have

$$f_1(x_1, x_2) = \sum_{i=1}^2 \alpha_1^i a_1^i + g_1^2(x_1, x_2).$$

Continuing in this fashion we see that for $i \in \{1, 2, \dots, i_0\}$,

$$f_1(x_1, x_2) = \sum_{i=1}^{i_0} \alpha_1^i a_1^i + g_1^{i_0}(x_1, x_2),$$

where for $i \in \{2, \dots, i_0\}$,

$$\begin{aligned} g_1^i(x_1, x_2) &:= \frac{\chi_{2^i R}(x_1, x_2)}{\mu_\lambda(2^i R)} \iint_R f_1(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2), \\ f_1^i(x_1, x_2) &:= g_1^{i-1}(x_1, x_2) - g_1^i(x_1, x_2), \\ \alpha_1^i &:= \|f_1^i\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2^i R) \quad \text{and} \\ a_1^i &:= (\alpha_1^i)^{-1} f_1^i. \end{aligned}$$

Here we choose i_0 to be the smallest positive integer such that

$$i_0 \geq \log_2 \frac{|x_{0,1} - y_{0,1}|}{r_1} + \log_2 \frac{|x_{0,2} - y_{0,2}|}{r_2}.$$

Moreover, for $i \in \{1, 2, \dots, i_0\}$, we have

$$\begin{aligned} \alpha_1^i &\leq \|f_1^i\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2^i R) \leq (\|g_1^{i-1}\|_{L^\infty(\mathbb{R}_\lambda)} + \|g_1^i\|_{L^\infty(\mathbb{R}_\lambda)}) \mu_\lambda(2^i R) \\ &\leq \mu_\lambda(2^i R) \left(\frac{1}{\mu_\lambda(2^{i-1} R)} \iint_R |f_1(y_1, y_2)| dm_\lambda(y_1) dm_\lambda(y_2) \right. \\ &\quad \left. + \frac{1}{\mu_\lambda(2^i R)} \iint_R |f_1(y_1, y_2)| dm_\lambda(y_1) dm_\lambda(y_2) \right) \\ &\lesssim \mu_\lambda(2^i R) \frac{1}{\mu_\lambda(2^{i-1} R)} \|f_1\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(R) \\ &\lesssim \tilde{C}_1 \mu_\lambda(R), \end{aligned}$$

where the implicit constant depends only on λ .

Following the same steps, we also obtain that for $i \in \{1, 2, \dots, i_0\}$,

$$f_2(x_1, x_2) = \sum_{i=1}^{i_0} \alpha_2^i a_2^i + g_2^{i_0}(x_1, x_2),$$

where for $i \in \{2, \dots, i_0\}$,

$$\begin{aligned} g_2^i(x_1, x_2) &:= \frac{\chi_{2^i \tilde{R}}(x_1, x_2)}{\mu_\lambda(2^i \tilde{R})} \iint_{\tilde{R}} f_2(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2), \\ f_2^i(x_1, x_2) &:= g_2^{i-1}(x_1, x_2) - g_2^i(x_1, x_2), \\ \alpha_2^i &:= \|f_2^i\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2^i \tilde{R}) \quad \text{and} \\ a_2^i &:= (\alpha_2^i)^{-1} f_2^i. \end{aligned}$$

Similarly, for $i \in \{1, 2, \dots, i_0\}$, we have

$$\alpha_2^i \lesssim \tilde{C}_2 \mu_\lambda(\tilde{R}).$$

Combining the decompositions above, we obtain that

$$f(x_1, x_2) = \sum_{j=1}^2 \sum_{i=1}^{i_0} \alpha_j^i a_j^i + g_j^{i_0}(x_1, x_2).$$

We now consider the tail $g_1^{i_0}(x_1, x_2) + g_2^{i_0}(x_1, x_2)$. To handle that, consider the rectangle \overline{R} defined as

$$\overline{R} := I\left(\frac{x_{0,1} + y_{0,1}}{2}, (2^{i_0} + 1)r_1\right) \times I\left(\frac{x_{0,2} + y_{0,2}}{2}, (2^{i_0} + 1)r_2\right).$$

Then, it is clear that $R \cup \tilde{R} \subset \bar{R}$, and that $2^{i_0}R, 2^{i_0}\tilde{R} \subset \bar{R}$. Thus, we get that

$$\frac{\chi_{\bar{R}}(x_1, x_2)}{\mu_\lambda(\bar{R})} \iint_{\bar{R}} f_1(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2) + \frac{\chi_{\bar{R}}(x_1, x_2)}{\mu_\lambda(\bar{R})} \iint_{\bar{R}} f_2(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2) = 0.$$

Hence, we write

$$\begin{aligned} g_1^{i_0}(x_1, x_2) + g_2^{i_0}(x_1, x_2) &= \left(g_1^{i_0}(x_1, x_2) - \frac{\chi_{\bar{R}}(x_1, x_2)}{\mu_\lambda(\bar{R})} \iint_{\bar{R}} f_1(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2) \right) \\ &\quad + \left(g_2^{i_0}(x_1, x_2) - \frac{\chi_{\bar{R}}(x_1, x_2)}{\mu_\lambda(\bar{R})} \iint_{\bar{R}} f_2(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2) \right) \\ &=: f_1^{i_0+1} + f_2^{i_0+1}. \end{aligned}$$

We now define

$$\begin{aligned} \alpha_1^{i_0+1} &:= \|f_1^{i_0+1}\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2^{i_0+1}R), \quad \alpha_2^{i_0+1} := \|f_2^{i_0+1}\|_{L^\infty(\mathbb{R}_\lambda)} \mu_\lambda(2^{i_0+1}\tilde{R}) \\ a_1^{i_0+1} &:= (\alpha_1^{i_0+1})^{-1} f_1^{i_0+1} \quad \text{and} \quad a_2^{i_0+1} := (\alpha_2^{i_0+1})^{-1} f_2^{i_0+1}. \end{aligned}$$

Again we can verify that $a_1^{i_0+1}$ is an ∞ -atom as in Definition 3.2 with

$$\|a_1^{i_0+1}\|_{L^\infty(\mathbb{R}_\lambda)} = \frac{1}{\mu_\lambda(2^{i_0+1}R)}.$$

Moreover, we also have

$$\alpha_1^{i_0+1} \lesssim \tilde{C}_1 \mu_\lambda(R).$$

Similarly, $a_2^{i_0+1}$ is an ∞ -atom as in Definition 3.2 with

$$\|a_2^{i_0+1}\|_{L^\infty(\mathbb{R}_\lambda)} = \frac{1}{\mu_\lambda(2^{i_0+1}\tilde{R})},$$

and we also have

$$\alpha_2^{i_0+1} \lesssim \tilde{C}_1 \mu_\lambda(\tilde{R}).$$

Thus, we obtain that

$$f(x_1, x_2) = \sum_{j=1}^2 \sum_{i=1}^{i_0+1} \alpha_j^i a_j^i,$$

which implies that $f \in h^1(\mathbb{R}_\lambda)$ and

$$\begin{aligned} \|f\|_{h^1(\mathbb{R}_\lambda)} &\leq \sum_{j=1}^2 \sum_{i=1}^{i_0+1} \alpha_j^i \\ &\leq C \left(\log_2 \frac{|x_{0,1} - y_{0,1}|}{r_1} + \log_2 \frac{|x_{0,2} - y_{0,2}|}{r_2} \right) \left(\tilde{C}_1 \mu_\lambda(R) + \tilde{C}_2 \mu_\lambda(\tilde{R}) \right). \end{aligned}$$

Therefore, we finish the proof of Lemma 3.11. \square

Proof of Theorem 3.10. Suppose a is an atom of $h^1(\mathbb{R}_\lambda)$ supported in a rectangle

$$R := I(x_{0,1}, r_1) \times I(x_{0,2}, r_2),$$

as in Definition 3.2. Observe that if $r_1 > x_{0,1}$, then $I(x_{0,1}, r_1) = (x_{0,1} - r_1, x_{0,1} + r_1) \cap \mathbb{R}_+ = I(\frac{x_{0,1}+r_1}{2}, \frac{x_{0,1}+r_1}{2})$. Therefore, without loss of generality, we may assume that $r_1 \leq x_{0,1}$, and similarly assume that $r_2 \leq x_{0,2}$. Let K_2 and K_3 be the constants appeared in (ii) and (iii) of Proposition 3.1 respectively, and $K_0 > \max\{\frac{1}{K_2}, \frac{1}{K_3}\} + 1$ large enough. For any $\epsilon > 0$, let \tilde{M} be a positive constant large enough such that $\tilde{M} \geq 100K_0$ and $\frac{\log_2 \tilde{M}}{\tilde{M}} < \epsilon$.

We now consider the following four cases.

Case (a): $x_{0,1} \leq 2\tilde{M}r_1, x_{0,2} \leq 2\tilde{M}r_2$.

In this case, let $y_{0,1} := x_{0,1} + 2\widetilde{M}K_0r_1$ and $y_{0,2} := x_{0,2} + 2\widetilde{M}K_0r_2$ and

$$\widetilde{R} := I(y_{0,1}, r_1) \times I(y_{0,2}, r_2).$$

Then for $i = 1, 2$,

$$(1 + K_0)x_{0,i} \leq y_{0,i} \leq (1 + 2\widetilde{M}K_0)x_{0,i}.$$

Define

$$(3.22) \quad g(x_1, x_2) := \chi_{\widetilde{R}}(x_1, x_2)$$

and

$$(3.23) \quad h(x_1, x_2) := -\frac{a(x_1, x_2)}{\widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_{0,1}, x_{0,2})}.$$

We first point out that by the fact that $y_i/x_{0,i} > K_2^{-1}$ for any $y_i \in I(y_{0,i}, r_i)$, $i = 1, 2$, and Proposition 3.1 (ii), we see that

$$(3.24) \quad \begin{aligned} & \left| \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_{0,1}, x_{0,2}) \right| \\ &= \left| \int_{y_{0,1}-r_1}^{y_{0,1}+r_1} \widetilde{R}_{\Delta_\lambda, 1}(y_1, x_{0,1}) dm_\lambda(y_1) \int_{y_{0,2}-r_2}^{y_{0,2}+r_2} \widetilde{R}_{\Delta_\lambda, 1}(y_2, x_{0,2}) dm_\lambda(y_2) \right| \\ &\gtrsim \int_{y_{0,1}-r_1}^{y_{0,1}+r_1} \frac{1}{y_1} dy_1 \int_{y_{0,2}-r_2}^{y_{0,2}+r_2} \frac{1}{y_2} dy_2 \sim \frac{r_1}{y_{0,1}} \frac{r_2}{y_{0,2}} \sim \frac{1}{\widetilde{M}^2}. \end{aligned}$$

Then from the definitions of g and h above, we have

$$\|g\|_{L^2(\mathbb{R}_\lambda)} = \mu_\lambda(\widetilde{R})^{\frac{1}{2}}$$

and

$$\|h\|_{L^2(\mathbb{R}_\lambda)} = \frac{1}{\left| \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_{0,1}, x_{0,2}) \right|} \|a\|_{L^2(\mathbb{R}_\lambda)} \leq \frac{\mu_\lambda(R)^{-\frac{1}{2}}}{\left| \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_{0,1}, x_{0,2}) \right|}.$$

Thus, from (3.24), we have that

$$\|g\|_{L^2(\mathbb{R}_\lambda)} \|h\|_{L^2(\mathbb{R}_\lambda)} \lesssim \widetilde{M}^2 \mu_\lambda(\widetilde{R})^{\frac{1}{2}} \mu_\lambda(R)^{-\frac{1}{2}} \lesssim \widetilde{M}^2 \left(\frac{y_{0,1}^{2\lambda} r_1 y_{0,2}^{2\lambda} r_2}{x_{0,1}^{2\lambda} r_1 x_{0,2}^{2\lambda} r_2} \right)^{\frac{1}{2}} \lesssim \widetilde{M}^{2+2\lambda}.$$

Now, write

$$\begin{aligned} & a(x_1, x_2) - \Pi(g, h)(x_1, x_2) \\ &= \left(a(x_1, x_2) + h(x_1, x_2) \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_1, x_2) \right) - g(x_1, x_2) \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(h)(x_1, x_2) \\ &=: w_1(x_1, x_2) + w_2(x_1, x_2). \end{aligned}$$

Moreover, we define

$$D_1 := \frac{m_\lambda(I(y_{0,1}, r_1))}{m_\lambda(I(x_{0,1}, r_1)) m_\lambda(I(x_{0,1}, |y_{0,1} - x_{0,1}|)) m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|))}$$

and

$$D_2 := \frac{1}{m_\lambda(I(x_{0,1}, |y_{0,1} - x_{0,1}|)) m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|))}.$$

First, consider w_1 . Observe that $\text{supp } w_1 \subseteq R$ and

$$|w_1(x_1, x_2)| = |a(x_1, x_2)| \frac{\left| \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_{0,1}, x_{0,2}) - \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_1, x_2) \right|}{\left| \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_{0,1}, x_{0,2}) \right|}.$$

Then as $(x_1, x_2) \in R$, we can estimate

$$\left| \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_{0,1}, x_{0,2}) - \widetilde{R}_{\Delta_\lambda, 1}\widetilde{R}_{\Delta_\lambda, 2}(g)(x_1, x_2) \right|$$

$$\begin{aligned}
&= \left| \int_{\tilde{R}} \left[\widetilde{R_{\Delta\lambda,1}}(x_{0,1}, y_1) \widetilde{R_{\Delta\lambda,2}}(x_{0,2}, y_2) - \widetilde{R_{\Delta\lambda,1}}(x_1, y_1) \widetilde{R_{\Delta\lambda,2}}(x_2, y_2) \right] dm_\lambda(y_1) dm_\lambda(y_2) \right| \\
&\lesssim \int_{\tilde{R}} \left[\frac{|x_1 - x_{0,1}|}{|y_1 - x_{0,1}| m_\lambda(I(x_{0,1}, |y_1 - x_{0,1}|)) m_\lambda(I(x_{0,2}, |y_2 - x_{0,2}|))} \right. \\
&\quad \left. + \frac{|x_2 - x_{0,2}|}{|y_2 - x_{0,2}| m_\lambda(I(x_{0,2}, |y_2 - x_{0,2}|)) m_\lambda(I(x_1, |y_1 - x_1|))} \right] dm_\lambda(y_1) dm_\lambda(y_2) \\
&\lesssim \mu_\lambda(\tilde{R}) \left[\frac{r_1}{|y_{0,1} - x_{0,1}| m_\lambda(I(x_{0,1}, |y_{0,1} - x_{0,1}|)) m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|))} \right. \\
&\quad \left. + \frac{r_2}{|y_{0,2} - x_{0,2}| m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|)) m_\lambda(I(x_1, |y_{0,1} - x_1|))} \right].
\end{aligned}$$

Combining the above estimates, (3.24), and the definition of w_1 immediately gives:

$$\begin{aligned}
|w_1(x_1, x_2)| &\lesssim \widetilde{M}^2 \|a\|_{L^\infty(R)} \mu_\lambda(\tilde{R}) \left[\frac{r_1 m_\lambda(I(y_{0,1}, r_1))}{|y_{0,1} - x_{0,1}| m_\lambda(I(x_{0,1}, |y_{0,1} - x_{0,1}|)) m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|))} \right. \\
&\quad \left. + \frac{r_2 m_\lambda(I(y_{0,2}, r_2))}{|y_{0,2} - x_{0,2}| m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|)) m_\lambda(I(x_1, |y_{0,1} - x_1|))} \right] \chi_R(x_1, x_2) \\
&\lesssim \left[\frac{m_\lambda(I(y_{0,1}, r_1))}{m_\lambda(I(x_{0,1}, r_1)) m_\lambda(I(x_{0,1}, |y_{0,1} - x_{0,1}|)) m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|))} \right. \\
&\quad \left. + \frac{m_\lambda(I(y_{0,2}, r_2))}{m_\lambda(I(x_{0,2}, r_2)) m_\lambda(I(x_{0,2}, |y_{0,2} - x_{0,2}|)) m_\lambda(I(x_1, |y_{0,1} - x_1|))} \right] \chi_R(x_1, x_2) \\
&\lesssim D_1 \chi_R(x_1, x_2).
\end{aligned}$$

Now, consider $w_2(x_1, x_2)$. Note that

$$w_2(x_1, x_2) = \frac{1}{\widetilde{R_{\Delta\lambda,1}} \widetilde{R_{\Delta\lambda,2}}(g)(x_{0,1}, x_{0,2})} \chi_{\tilde{R}}(x_1, x_2) R_{\Delta\lambda,1} R_{\Delta\lambda,2}(a)(x_1, x_2).$$

Clearly, $\text{supp } w_2 \subseteq \tilde{R}$. Furthermore, using the mean value zero property of $a(x_1, x_2)$, we have:

$$\begin{aligned}
R_{\Delta\lambda,1} R_{\Delta\lambda,2}(a)(x_1, x_2) &= \int_R \left(R_{\Delta\lambda,1}(x_1, y_1) R_{\Delta\lambda,2}(x_2, y_2) - R_{\Delta\lambda,1}(x_1, x_{0,1}) R_{\Delta\lambda,2}(x_2, x_{0,2}) \right) \\
&\quad \times a(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2).
\end{aligned}$$

Then following similar estimates as in w_1 above, we have

$$\begin{aligned}
|w_2(x_1, x_2)| &\lesssim \widetilde{M}^2 \|a\|_{L^\infty(R)} \mu_\lambda(\tilde{R}) \left[\frac{r_1 m_\lambda(I(x_{0,1}, r_1))}{|x_1 - x_{0,1}| m_\lambda(I(x_{0,1}, |x_1 - x_{0,1}|)) m_\lambda(I(x_{0,2}, |x_2 - x_{0,2}|))} \right. \\
&\quad \left. + \frac{r_2 m_\lambda(I(x_{0,2}, r_2))}{|x_2 - x_{0,2}| m_\lambda(I(x_{0,2}, |x_2 - x_{0,2}|)) m_\lambda(I(x_{0,1}, |x_1 - x_{0,1}|))} \right] \chi_{\tilde{R}}(x_1, x_2) \\
&\lesssim \left[\frac{1}{m_\lambda(I(x_{0,1}, |x_1 - x_{0,1}|)) m_\lambda(I(x_{0,2}, |x_2 - x_{0,2}|))} \right. \\
&\quad \left. + \frac{1}{m_\lambda(I(x_{0,2}, |x_2 - x_{0,2}|)) m_\lambda(I(x_{0,1}, |x_1 - x_{0,1}|))} \right] \chi_{\tilde{R}}(x_1, x_2) \\
&\lesssim D_2 \chi_{\tilde{R}}(x_1, x_2).
\end{aligned}$$

Combining the estimates of w_1 and w_2 , we can conclude that $a - \Pi(f, g)$ has support contained in

$$R \cup \tilde{R}$$

and satisfies

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} (a(x_1, x_2) - \Pi(f, g)(x_1, x_2)) dm_\lambda(x_1) dm_\lambda(x_2) = 0.$$

Then, from Lemma 3.11, we have

$$\begin{aligned}
 \|a - \Pi(f, g)\|_{h^1(\mathbb{R}_\lambda)} &\lesssim \left(\log_2 \frac{|x_{0,1} - y_{0,1}|}{r_1} + \log_2 \frac{|x_{0,2} - y_{0,2}|}{r_2} \right) \left(D_1 \mu_\lambda(R) + D_2 \mu_\lambda(\tilde{R}) \right) \\
 &\lesssim \left(\log_2 \frac{|x_{0,1} - y_{0,1}|}{r_1} + \log_2 \frac{|x_{0,2} - y_{0,2}|}{r_2} \right) \left(\frac{r_1}{|x_{0,1} - y_{0,1}|} + \frac{r_2}{|x_{0,2} - y_{0,2}|} \right) \\
 &\lesssim \frac{\log_2 \tilde{M}}{\tilde{M}} \\
 &\lesssim \epsilon.
 \end{aligned}$$

Case (b): $x_{0,1} > 2\tilde{M}r_1$, $x_{0,2} \leq 2\tilde{M}r_2$.

In this case, let $y_{0,1} := x_{0,1} - \frac{\tilde{M}r_1}{K_0}$ and $y_{0,2} := x_{0,2} + 2\tilde{M}K_0r_2$ and

$$\tilde{R} := I(y_{0,1}, r_1) \times I(y_{0,2}, r_2).$$

We also let g and h be the same as in (3.22) and (3.23), respectively.

Then $\frac{2K_0-1}{2K_0}x_{0,1} < y_{0,1} < x_{0,1}$. For every $y_1 \in I(y_{0,1}, r_1)$, from the facts that $K_0 > \max\{\frac{1}{K_2}, \frac{1}{K_3}\} + 1$ and $M \geq 100K_0$, we have

$$0 < \frac{x_{0,1}}{y_1} - 1 < K_3.$$

To continue, for the first variable, we use Proposition 3.1 (iii) and the fact that $y_1 \sim y_{0,1} \sim x_{0,1}$ for any $y_1 \in I(y_{0,1}, r_1)$; and for the second variable, we use Proposition 3.1 (ii) the fact that $y_2/x_{0,2} > K_2^{-1}$ for any $y_2 \in I(y_{0,2}, r_2)$. Then we see that

$$\begin{aligned}
 (3.25) \quad &\left| \widetilde{R_{\Delta_{\lambda,1}}} \widetilde{R_{\Delta_{\lambda,2}}}(g)(x_{0,1}, x_{0,2}) \right| \\
 &= \left| \int_{y_{0,1}-r_1}^{y_{0,1}+r_1} \widetilde{R_{\Delta_{\lambda,1}}}(y_1, x_{0,1}) dm_\lambda(y_1) \int_{y_{0,2}-r_2}^{y_{0,2}+r_2} \widetilde{R_{\Delta_{\lambda,1}}}(y_2, x_{0,2}) dm_\lambda(y_2) \right| \\
 &\gtrsim \int_{y_{0,1}-r_1}^{y_{0,1}+r_1} \frac{1}{x_{0,1}^\lambda y_{0,1}^\lambda} \frac{1}{x_{0,1} - y_1} dm_\lambda(y_1) \int_{y_{0,2}-r_2}^{y_{0,2}+r_2} \frac{1}{y_2} dy_2 \\
 &\sim \int_{y_{0,1}-r_1}^{y_{0,1}+r_1} \frac{1}{x_{0,1} - y_{0,1}} dy_1 \frac{r_2}{y_{0,2}} \sim \frac{1}{\tilde{M}^2}.
 \end{aligned}$$

Thus, from (3.25), we have that

$$\|g\|_{L^2(\mathbb{R}_\lambda)} \|h\|_{L^2(\mathbb{R}_\lambda)} \lesssim \tilde{M}^2 \left(\frac{y_{0,1}^{2\lambda} r_1 y_{0,2}^{2\lambda} r_2}{x_{0,1}^{2\lambda} r_1 x_{0,2}^{2\lambda} r_2} \right)^{\frac{1}{2}} \lesssim \tilde{M}^{2+\lambda}.$$

Then to estimate $a(x_1, x_2) - \Pi(g, h)(x_1, x_2)$, we define w_1 and w_2 to be the same as in Case (a). And following the same estimates as in Case (a), we obtain that

$$w_1(x_1, x_2) \lesssim D_1 \chi_R(x_1, x_2) \quad \text{and} \quad w_2(x_1, x_2) \lesssim D_2 \chi_{\tilde{R}}(x_1, x_2).$$

Then, the fact that $\|a - \Pi(f, g)\|_{h^1(\mathbb{R}_\lambda)} \lesssim \epsilon$ now immediately follows from Lemma 3.11 and the argument in Case (a).

Case (c): $x_{0,1} \leq 2\tilde{M}r_1$, $x_{0,2} > 2\tilde{M}r_2$.

In this case, let $y_{0,1} := x_{0,1} + 2\tilde{M}K_0r_1$ and $y_{0,2} := x_{0,2} - \frac{\tilde{M}r_2}{K_0}$ and

$$\tilde{R} := I(y_{0,1}, r_1) \times I(y_{0,2}, r_2).$$

We also let g and h be the same as in (3.22) and (3.23), respectively. Then, by handling the estimates symmetrically to Case (b), we obtain that

$$(3.26) \quad \left| \widetilde{R_{\Delta_{\lambda,1}}} \widetilde{R_{\Delta_{\lambda,2}}}(g)(x_{0,1}, x_{0,2}) \right| \gtrsim \frac{1}{\tilde{M}^2},$$

which gives

$$\|g\|_{L^2(\mathbb{R}_\lambda)} \|h\|_{L^2(\mathbb{R}_\lambda)} \lesssim \widetilde{M}^{2+\lambda}.$$

Again we obtain that $\|a - \Pi(f, g)\|_{h^1(\mathbb{R}_\lambda)} \lesssim \epsilon$.

Case (d): $x_{0,1} > 2\widetilde{M}r_1$, $x_{0,2} > 2\widetilde{M}r_2$.

In this case, let $y_{0,1} := x_{0,1} - \frac{\widetilde{M}r_1}{K_0}$ and $y_{0,2} := x_{0,2} - \frac{\widetilde{M}r_2}{K_0}$ and

$$\widetilde{R} := I(y_{0,1}, r_1) \times I(y_{0,2}, r_2).$$

We also let g and h be the same as in (3.22) and (3.23), respectively. Then for $i = 1, 2$, $\frac{2K_0-1}{2K_0}x_{0,i} < y_{0,i} < x_{0,i}$. For every $y_i \in I(y_{0,i}, r_i)$, from the facts that $K_0 > \max\{\frac{1}{K_2}, \frac{1}{K_3}\} + 1$ and $\widetilde{M} \geq 100K_0$, we have

$$0 < \frac{x_{0,i}}{y_i} - 1 < K_3.$$

To continue, we use Proposition 3.1 (iii) and the fact that $y_i \sim y_{0,i} \sim x_{0,i}$ for any $y_i \in I(y_{0,i}, r_i)$ for $i = 1, 2$. Then we see that

$$(3.27) \quad \left| \widetilde{R_{\Delta_\lambda, 1}} \widetilde{R_{\Delta_\lambda, 2}}(g)(x_{0,1}, x_{0,2}) \right| \\ \gtrsim \int_{y_{0,1}-r_1}^{y_{0,1}+r_1} \frac{1}{x_{0,1}^\lambda y_{0,1}^\lambda} \frac{1}{x_{0,1} - y_1} dm_\lambda(y_1) \int_{y_{0,2}-r_2}^{y_{0,2}+r_2} \frac{1}{x_{0,2}^\lambda y_{0,2}^\lambda} \frac{1}{x_{0,2} - y_2} dm_\lambda(y_2) \\ \sim \int_{y_{0,1}-r_1}^{y_{0,1}+r_1} \frac{1}{x_{0,1} - y_{0,1}} dy_1 \int_{y_{0,2}-r_2}^{y_{0,2}+r_2} \frac{1}{x_{0,2} - y_{0,2}} dy_2 \sim \frac{1}{\widetilde{M}^2}.$$

Thus, from (3.27), we have that

$$\|g\|_{L^2(\mathbb{R}_\lambda)} \|h\|_{L^2(\mathbb{R}_\lambda)} \lesssim \widetilde{M}^2 \left(\frac{y_{0,1}^{2\lambda} r_1 y_{0,2}^{2\lambda} r_2}{x_{0,1}^{2\lambda} r_1 x_{0,2}^{2\lambda} r_2} \right)^{\frac{1}{2}} \lesssim M^2.$$

Again we obtain that $\|a - \Pi(f, g)\|_{h^1(\mathbb{R}_\lambda)} \lesssim \epsilon$. \square

Proof of Theorem 3.8. We first point out that from (3.19), for every $g, h \in L^\infty(\mathbb{R}_\lambda)$ with compact support,

$$\|\Pi(g, h)\|_{h^1(\mathbb{R}_\lambda)} \lesssim \|g\|_{L^2(\mathbb{R}_\lambda)} \|h\|_{L^2(\mathbb{R}_\lambda)}.$$

Based on this upper bound, for every $f \in h^1(\mathbb{R}_\lambda)$ having the representation (3.17) with

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}_\lambda)} \|h_j^k\|_{L^2(\mathbb{R}_\lambda)} < \infty,$$

we have that

$$\|f\|_{h^1(\mathbb{R}_\lambda)} \lesssim \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \left\| \Pi(g_j^k, h_j^k) \right\|_{h^1(\mathbb{R}_\lambda)} \lesssim \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}_\lambda)} \|h_j^k\|_{L^2(\mathbb{R}_\lambda)},$$

which gives that

$$\|f\|_{h^1(\mathbb{R}_\lambda)} \lesssim \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}_\lambda)} \|h_j^k\|_{L^2(\mathbb{R}_\lambda)} : f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\}.$$

It remains to show that for every $f \in h^1(\mathbb{R}_\lambda)$, f has a representation as in (3.17) with

$$(3.28) \quad \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}_\lambda)} \|h_j^k\|_{L^2(\mathbb{R}_\lambda)} : f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\} \lesssim \|f\|_{h^1(\mathbb{R}_\lambda)}.$$

To this end, assume that f has the following atomic representation $f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1$ with $\sum_{j=1}^{\infty} |\alpha_j^1| \leq \tilde{C}_0 \|f\|_{h^1(\mathbb{R}_\lambda)}$ for certain absolute constant $\tilde{C}_0 \in (1, \infty)$. We show that for every $\epsilon \in (0, \tilde{C}_0^{-1})$ and every $K \in \mathbb{N}$, f has the following representation

$$(3.29) \quad f = \sum_{k=1}^K \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) + E_K,$$

where

$$(3.30) \quad \sum_{j=1}^{\infty} |\alpha_j^k| \leq \epsilon^{k-1} \tilde{C}_0 \|f\|_{h^1(\mathbb{R}_\lambda)},$$

and $E_K \in h^1(\mathbb{R}_\lambda)$ with

$$(3.31) \quad \|E_K\|_{h^1(\mathbb{R}_\lambda)} \leq (\epsilon \tilde{C}_0)^K \|f\|_{h^1(\mathbb{R}_\lambda)},$$

and $g_j^k \in L^2(\mathbb{R}_\lambda)$, $h_j^k \in L^2(\mathbb{R}_\lambda)$ for each k and j , $\{\alpha_j^k\}_j \in \ell^1$ for each k satisfying that

$$(3.32) \quad \|g_j^k\|_{L^2(\mathbb{R}_\lambda)} \|h_j^k\|_{L^2(\mathbb{R}_\lambda)} \lesssim C(\epsilon)$$

with the absolute constant $C(\epsilon) = \tilde{M}^{2+2\lambda}$, where M is the constant in the proof of Theorem 3.10 satisfying $\tilde{M} \geq 100K_0$ and $\frac{\log_2 \tilde{M}}{M} < \epsilon$.

In fact, for given ϵ and each a_j^1 , by Theorem 3.10 we obtain that there exist $g_j^1 \in L^2(\mathbb{R}_\lambda)$ and $h_j^1 \in L^2(\mathbb{R}_\lambda)$ with

$$\|g_j^1\|_{L^2(\mathbb{R}_\lambda)} \|h_j^1\|_{L^2(\mathbb{R}_\lambda)} \lesssim C(\epsilon)$$

and

$$\|a_j^1 - \Pi(g_j^1, h_j^1)\|_{h^1(\mathbb{R}_\lambda)} < \epsilon.$$

Now we write

$$f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 = \sum_{j=1}^{\infty} \alpha_j^1 \Pi(g_j^1, h_j^1) + \sum_{j=1}^{\infty} \alpha_j^1 [a_j^1 - \Pi(g_j^1, h_j^1)] =: M_1 + E_1.$$

Observe that

$$\|E_1\|_{h^1(\mathbb{R}_\lambda)} \leq \sum_{j=1}^{\infty} |\alpha_j^1| \|a_j^1 - \Pi(g_j^1, h_j^1)\|_{h^1(\mathbb{R}_\lambda)} \leq \epsilon \tilde{C}_0 \|f\|_{h^1(\mathbb{R}_\lambda)}.$$

Since $E_1 \in h^1(\mathbb{R}_\lambda)$, for the given \tilde{C}_0 , there exists a sequence of atoms $\{a_j^2\}_j$ and numbers $\{\alpha_j^2\}_j$ such that $E_1 = \sum_{j=1}^{\infty} \alpha_j^2 a_j^2$ and

$$\sum_{j=1}^{\infty} |\alpha_j^2| \leq \tilde{C}_0 \|E_1\|_{h^1(\mathbb{R}_\lambda)} \leq \epsilon \tilde{C}_0^2 \|f\|_{h^1(\mathbb{R}_\lambda)}.$$

Again, we have that for given ϵ , there exists a representation of E_1 such that

$$E_1 = \sum_{j=1}^{\infty} \alpha_j^2 \Pi(g_j^2, h_j^2) + \sum_{j=1}^{\infty} \alpha_j^2 [a_j^2 - \Pi(g_j^2, h_j^2)] =: M_2 + E_2,$$

and

$$\|g_j^2\|_{L^2(\mathbb{R}_\lambda)} \|h_j^2\|_{L^2(\mathbb{R}_\lambda)} \lesssim C(\epsilon) \quad \text{and} \quad \|a_j^2 - \Pi(g_j^2, h_j^2)\|_{h^1(\mathbb{R}_\lambda)} < \frac{\epsilon}{2}.$$

Moreover,

$$\|E_2\|_{h^1(\mathbb{R}_\lambda)} \leq \sum_{j=1}^{\infty} |\alpha_j^2| \|a_j^2 - \Pi(g_j^2, h_j^2)\|_{h^1(\mathbb{R}_\lambda)} \leq (\epsilon\tilde{C}_0)^2 \|f\|_{h^1(\mathbb{R}_\lambda)}.$$

Now we conclude that

$$f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 = \sum_{k=1}^2 \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) + E_2,$$

Continuing in this way, we deduce that for every $K \in \mathbb{N}$, f has the representation (3.29) satisfying (3.32), (3.30), and (3.31). Thus letting $K \rightarrow \infty$, we see that (3.17) holds. Moreover, since $\epsilon\tilde{C}_0 < 1$, we have that

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \leq \sum_{k=1}^{\infty} \epsilon^{-1} (\epsilon\tilde{C}_0)^k \|f\|_{h^1(\mathbb{R}_\lambda)} \lesssim \|f\|_{h^1(\mathbb{R}_\lambda)},$$

which implies (3.28) and hence, completes the proof of Theorem 3.8. \square

Proof of (i) \Leftarrow (iv). Suppose that $b \in L_{loc}^2(\mathbb{R}_\lambda)$. Assume that $[b, R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}]$ is bounded on $L^2(\mathbb{R}_\lambda)$.

From the definition of $h^1(\mathbb{R}_\lambda)$, given $f \in h^1(\mathbb{R}_\lambda)$, there exists a number sequence $\{\lambda_j\}_{j=1}^{\infty}$ and atoms $\{a_j\}_{j=1}^{\infty}$ such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the series converges in the $h^1(\mathbb{R}_\lambda)$ norm and $\|f\|_{h^1(\mathbb{R}_\lambda)} \approx \sum_{j=1}^{\infty} |\lambda_j|$. Hence, we have that $f_N := \sum_{j=1}^N \lambda_j a_j$ tends to f as $N \rightarrow +\infty$ in the $h^1(\mathbb{R}_\lambda)$ norm, which implies that $h^1(\mathbb{R}_\lambda) \cap L_c^\infty(\mathbb{R}_\lambda)$ is dense in $h^1(\mathbb{R}_\lambda)$, where recall that $L_c^\infty(\mathbb{R}_\lambda)$ is the subspace of $L^\infty(\mathbb{R}_\lambda)$ consisting of functions with compact support in $\mathbb{R}_+ \times \mathbb{R}_+$.

Now for $f \in h^1(\mathbb{R}_\lambda) \cap L_c^\infty(\mathbb{R}_\lambda)$, from Theorem 3.8, we choose a weak factorization of f such that

$$(3.33) \quad f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k)$$

in the sense of $h^1(\mathbb{R}_\lambda)$, where the sequence $\{\alpha_j^k\} \in \ell^1$ and the functions g_j^k and h_j^k are in $L_c^\infty(\mathbb{R}_\lambda)$ satisfying

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \|g_j^k\|_{L^2(\mathbb{R}_\lambda)} \|h_j^k\|_{L^2(\mathbb{R}_\lambda)} \lesssim \|f\|_{h^1(\mathbb{R}_\lambda)}.$$

From the definition of bilinear form Π as in (3.18), we see that $\Pi(g_j^k, h_j^k)$ is in $L^2(\mathbb{R}_\lambda)$ with compact support.

Since $f \in h^1(\mathbb{R}_\lambda) \cap L_c^\infty(\mathbb{R}_\lambda)$, we see that f is in $L^2(U)$, where we use the set U to denote the support of f . Hence,

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} b(x_1, x_2) f(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2)$$

is well-defined, since $b \in L_{loc}^2(\mathbb{R}_\lambda)$ and hence in $L^2(U)$.

We now define

$$b_i(x_1, x_2) = b(x_1, x_2) \chi_{\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : |b(x_1, x_2)| \leq i\}}(x_1, x_2), \quad i = 1, 2, \dots$$

It is clear that $b_i(x_1, x_2) \rightarrow b(x_1, x_2)$ as $i \rightarrow \infty$ in the sense of $L^2(U)$. And then we have

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} b(x_1, x_2) f(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) = \lim_{i \rightarrow \infty} \int_{\mathbb{R}_+ \times \mathbb{R}_+} b_i(x_1, x_2) f(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2).$$

Next, for each $i = 1, 2, \dots$, we have that

$$\begin{aligned}
 & \int_{\mathbb{R}_+ \times \mathbb{R}_+} b_i(x_1, x_2) f(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) \\
 &= \int_{\mathbb{R}_+ \times \mathbb{R}_+} b_i(x_1, x_2) \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k)(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) \\
 &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \int_{\mathbb{R}_+ \times \mathbb{R}_+} b_i(x_1, x_2) \Pi(g_j^k, h_j^k)(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) \\
 &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \langle b_i, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)}
 \end{aligned}$$

since b_i is in $L^\infty(U)$ and hence is in $\text{bmo}(\mathbb{R}_\lambda)$, (3.33) holds in $h^1(\mathbb{R}_\lambda)$ and each $\Pi(g_j^k, h_j^k)$ is in $h^1(\mathbb{R}_\lambda)$ as showed in Proposition 3.9.

As a consequence, we obtain that

$$\begin{aligned}
 (3.34) \quad |\langle b, f \rangle_{L^2(\mathbb{R}_\lambda)}| &= \lim_{i \rightarrow \infty} \left| \int_{\mathbb{R}_+ \times \mathbb{R}_+} b_i(x_1, x_2) f(x_1, x_2) dm_\lambda(x_1) dm_\lambda(x_2) \right| \\
 &\leq \lim_{i \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| |\langle b_i, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)}| \\
 &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lim_{i \rightarrow \infty} |\alpha_j^k| |\langle b_i, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)}|,
 \end{aligned}$$

where the equality above holds since all the terms are non-negative. Next, since $b_i(x_1, x_2) \rightarrow b(x_1, x_2)$ as $i \rightarrow \infty$ in the sense of $L^2(V)$ and $\Pi(g_j^k, h_j^k)$ is in $L^2(V)$ with V the support of $\Pi(g_j^k, h_j^k)$, we have that

$$\lim_{i \rightarrow \infty} \langle b_i, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)} = \langle b, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)},$$

which implies that

$$\lim_{i \rightarrow \infty} |\langle b_i, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)}| = |\langle b, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)}|.$$

This, together with (3.34), yields that

$$\begin{aligned}
 |\langle b, f \rangle_{L^2(\mathbb{R}_\lambda)}| &\leq \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\alpha_j^k| |\langle b, \Pi(g_j^k, h_j^k) \rangle_{L^2(\mathbb{R}_\lambda)}| \\
 &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \cdot \left| \left\langle g_j^k, [b, R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}] h_j^k \right\rangle_{L^2(\mathbb{R}_\lambda)} \right|,
 \end{aligned}$$

which is further bounded by

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \left\| g_j^k \right\|_{L^2(\mathbb{R}_\lambda)} \left\| [b, R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}] h_j^k \right\|_{L^2(\mathbb{R}_\lambda)} \\
 &\leq \left\| [b, R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}] : L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda) \right\| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \left\| g_j^k \right\|_{L^2(\mathbb{R}_\lambda)} \left\| h_j^k \right\|_{L^2(\mathbb{R}_\lambda)} \\
 &\lesssim \left\| [b, R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2}] : L^2(\mathbb{R}_\lambda) \rightarrow L^2(\mathbb{R}_\lambda) \right\| \|f\|_{h^1(\mathbb{R}_\lambda)}.
 \end{aligned}$$

Then by the fact that $\{f \in h^1(\mathbb{R}_\lambda) : f \text{ has compact support}\}$ is dense in $h^1(\mathbb{R}_\lambda)$, and the duality between $h^1(\mathbb{R}_\lambda)$ and $\text{bmo}(\mathbb{R}_\lambda)$ (see Theorem 3.7), we finish the proof. \square

Proof of Corollary 1.5. Suppose $b \in \text{bmo}(\mathbb{R}_\lambda)$. Then based on (iii) of Theorem 1.4, we obtain that there exist $f_1, f_2, g_1, g_2 \in L^\infty(\mathbb{R}_\lambda)$ such that $b = f_1 + R_{\Delta_\lambda, 1}g_1 = f_2 + R_{\Delta_\lambda, 1}g_2$ and moreover, $\|b\|_{\text{bmo}(\mathbb{R}_\lambda)} \approx \inf \left\{ \max_{i=1,2} \{ \|f_i\|_{L^\infty(\mathbb{R}_\lambda)}, \|g_i\|_{L^\infty(\mathbb{R}_\lambda)} \} \right\}$ where the infimum is taken over all possible decompositions of b .

We now show that b is also in $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$. To see this, we recall the recent result of decomposition of $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$ obtained in [DLWY2].

Theorem 3.12 ([DLWY2]). *The following two statements are equivalent.*

- (i) $\varphi \in \text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$;
- (ii) *There exist $h_i \in L^\infty(\mathbb{R}_\lambda)$, $i = 1, 2, 3, 4$, such that*

$$\varphi = h_1 + R_{\Delta_\lambda, 1}(h_2) + R_{\Delta_\lambda, 2}(h_3) + R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}(h_4).$$

Back to the proof, we now choose $h_1 = f_1$, $h_2 = g_1$, $h_3 = h_4 = 0$. Then it is easy to see that

$$b = h_1 + R_{\Delta_\lambda, 1}(h_2) + R_{\Delta_\lambda, 2}(h_3) + R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}(h_4),$$

which implies that $b \in \text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$.

Similarly, we can also choose $h_1 = f_2$, $h_3 = g_2$, $h_2 = h_4 = 0$. Combining these two choices, we further obtain that

$$\|b\|_{\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)} \lesssim \|b\|_{\text{bmo}(\mathbb{R}_\lambda)},$$

which implies that

$$\text{bmo}(\mathbb{R}_\lambda) \subset \text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda).$$

Next we prove that $\text{bmo}_{\Delta_\lambda}(\mathbb{R}_\lambda)$ is a proper subspace of $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$. To see this, we let K_3 be the constant in (iii) of Proposition 3.1. Since $R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}$ is a product Calderón–Zygmund operator on \mathbb{R}_λ and hence it is bounded from $L^\infty(\mathbb{R}_\lambda)$ to $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$ (see [HLL]). Then, it is direct that the following function

$$(3.35) \quad b(x_1, x_2) := R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}(\chi_{(1,2) \times (1,2)})(x_1, x_2)$$

is in $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$.

Next we claim that this function $b(x_1, x_2)$ is not in $\text{bmo}(\mathbb{R}_\lambda)$. To see this, we first note that $b(x_1, x_2)$ can be written as

$$b(x_1, x_2) = R_{\Delta_\lambda}(\chi_{(1,2)})(x_1)R_{\Delta_\lambda}(\chi_{(1,2)})(x_2).$$

We now verify that $R_{\Delta_\lambda}(\chi_{(1,2)})(x_1)$ is not in $L^\infty(\mathbb{R}_+, dm_\lambda)$. In fact, by Proposition 3.1, for every $\delta > 0$ small enough and $x_1 \in (1 - \delta, 1)$, we choose $\epsilon = 2\delta$. Then we have

$$\begin{aligned} R_{\Delta_\lambda}(\chi_{(1,2)})(x_1) &= \int_1^2 R_{\Delta_\lambda}(x_1, y)y^{2\lambda}dy \geq \int_{x_1+\epsilon}^{(1+K_3)x_1} R_{\Delta_\lambda}(x_1, y)y^{2\lambda}dy \\ &\geq \int_{x_1+\epsilon}^{(1+K_3)x_1} C_{K_3, \lambda} \frac{1}{x_1^\lambda y^\lambda} \frac{1}{y-x_1} y^{2\lambda} dy \\ &\gtrsim \int_{x_1+\epsilon}^{(1+K_3)x_1} \frac{1}{y-x_1} dy \\ &= \ln(y-x_1) \Big|_{x_1+\epsilon}^{(1+K_3)x_1} \\ &= \ln(K_3x_1) - \ln \epsilon \\ &= \ln(K_3x_1) - \ln(2\delta). \end{aligned}$$

Then it is direct that when $\delta \rightarrow 0^+$, $R_{\Delta_\lambda}(\chi_{(0,1)})(x_1)$ is unbounded around the interval $(1 - \delta, 1)$.

Hence, for the function $b(x_1, x_2)$ defined as in (3.35), when we fix x_1 , $b(x_1, x_2)$ as a function of x_2 is in $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_+, dm_\lambda)$. However, it is not uniform for the variable x_1 . \square

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REFERENCES

- [AK] K. F. Andersen and R. A. Kerman, Weighted norm inequalities for generalized Hankel conjugate transformations, *Studia Math.*, **71** (1981/82), 15–26.
- [AH] P. Auscher and T. Hytönen, Orthonormal bases of regular wavelets in spaces of homogeneous type, *Appl. Comput. Harmon. Anal.* **34** (2013), 266–296.
- [BCFR] J. J. Betancor, A. Chicco Ruiz, J. C. Fariña and L. Rodríguez-Mesa, Maximal operators, Riesz transforms and Littlewood-Paley functions associated with Bessel operators on BMO, *J. Math. Anal. Appl.*, **363** (2010), 310–326.
- [BDT] J. J. Betancor, J. Dziubański and J. L. Torrea, On Hardy spaces associated with Bessel operators, *J. Anal. Math.* **107** (2009) 195–219.
- [BFBMT] J. J. Betancor, J. C. Fariña, D. Buraczewski, T. Martínez and J. L. Torrea, Riesz transform related to Bessel operators, *Proc. Roy. Soc. Edinburgh Sect. A*, **137** (2007), 701–725.
- [BFS] J. J. Betancor, J. C. Fariña and A. Sanabria, On Littlewood-Paley functions associated with Bessel operators, *Glasg. Math. J.*, **51** (2009), 55–70.
- [BHNV] J. J. Betancor, E. Harboure, A. Nowak and B. Viviani, Mapping properties of functional operators in harmonic analysis related to Bessel operators, *Studia Math.*, **197** (2010), 101–140.
- [BC] M. Bramanti and M. C. Cerutti, Commutators of singular integrals on homogeneous spaces, *Boll. Un. Mat. Ital. B (7)* **10** (1996), 843–883.
- [CF] S.-Y.A. Chang and R. Fefferman, A continuous version of duality of H^1 with BMO on the bidisc, *Annals of Math.* **112** (1980), 179–201.
- [CRW] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. of Math. (2)* **103** (1976), 611–635.
- [CW77] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.
- [DO] L. Dalenc and Y. Ou, Upper bound for multi-parameter iterated commutators, *Publ. Mat.* **60** (2016), 191–220.
- [DLMWY] X. T. Duong, J. Li, S. Mao, H. Wu and D. Yang, Compactness of Riesz transform commutator associated with Bessel operators, *J. Anal. Math.*, to appear.
- [DLWY] X. T. Duong, J. Li, B. D. Wick and D. Yang, Hardy space via factorization, and BMO space via commutators in the Bessel setting, *Indiana Univ. Math. J.*, to appear.
- [DLWY2] X. T. Duong, J. Li, B. D. Wick and D. Yang, Characterizations of product Hardy spaces in Bessel setting, arXiv:1606.03500.
- [DLWY3] X. T. Duong, J. Li, B. D. Wick and D. Yang, Commutators, little bmo and weak factorization, *Annales de l'Institut Fourier.*, to appear.
- [FS] S. H. Ferguson and C. Sadosky, Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures, *J. Anal. Math.* **81** (2000), 239–267.
- [FL] S. H. Ferguson and M. T. Lacey, A characterization of product BMO by commutators, *Acta Math.*, **189** (2002), 143–160.
- [Gra] L. Grafakos, Classical Fourier Analysis, Second Edition, Graduate Texts in Mathematics, **249**, Springer, 2008.
- [H] D. T. Haimo, Integral equations associated with Hankel convolutions, *Trans. Amer. Math. Soc.*, **116** (1965), 330–375.
- [HLL] Y. Han, J. Li, and C.-C. Lin, Criteria of the L^2 boundedness and sharp endpoint estimates for singular integral operators on product spaces of homogeneous type, accepted by *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 2015.
- [HLL] Y. Han, J. Li and G. Lu, Multiparameter Hardy space theory on Carnot-Carathéodory spaces and product spaces of homogeneous type, *Trans. Amer. Math. Soc.* **365** (2013), 319–360.
- [HLW] Y. Han, J. Li, and L.A. Ward, Hardy space theory on spaces of homogeneous type via orthonormal wavelet bases, *Applied and Computational Harmonic Analysis*, to appear.
- [HMY1] Y. Han, D. Müller and D. Yang, Littlewood-Paley characterizations for Hardy spaces on spaces of homogeneous type, *Math. Nachr.* **279** (2006), 1505–1537.
- [HMY2] Y. Han, D. Müller and D. Yang, A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces, *Abstr. Appl. Anal.* **2008**, Article ID 893409, 250 pp.
- [Hu] A. Huber, On the uniqueness of generalized axially symmetric potentials, *Ann. Math.*, **60**, (1954), 351–358.

- [Hy] T. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, *Ann. of Math. (2)* **175** (2012), 1473–1506.
- [HK] T. Hytönen and A. Kairema, Systems of dyadic cubes in a doubling metric space, *Colloq. Math.* **126** (2012), 1–33.
- [J] J. L. Journé, Calderón-Zygmund Operators, Pseudodifferential Operators And the Cauchy Integral of Calderón. Lecture Notes in Mathematics, 994. Springer-Verlag, Berlin, 1983.
- [KLPW] A. Kairema, J. Li, C. Pereyra and L.A. Ward, Haar bases on quasi-metric measure spaces, and dyadic structure theorems for function spaces on product spaces of homogeneous type, *J. Funct. Anal.*, **271** (2016), 1793–1843.
- [K] R. A. Kerman, Boundedness criteria for generalized Hankel conjugate transformations, *Canad. J. Math.*, **30** (1978), 147–153.
- [KL] S. G. Krantz and S.-Y. Li, Boundedness and compactness of integral operators on spaces of homogeneous type and applications. I, *J. Math. Anal. Appl.* **258** (2001), 629–641.
- [LPPW] M. Lacey, S. Petermichl, J. Pipher and B. D. Wick, Multiparameter Riesz commutators, *Amer. J. Math.*, **131** (2009), 731–769.
- [LPPW2] M. Lacey, S. Petermichl, J. Pipher and B. D. Wick, Iterated Riesz commutators: a simple proof of boundedness, Harmonic analysis and partial differential equations, *Contemp. Math.*, **505** (2010), 171–178.
- [MSt] B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.*, **118** (1965), 17–92.
- [NRV] F. Nazarov, A. Reznikov, and A. Volberg, The proof of A_2 conjecture in a geometrically doubling metric space, *Indiana Univ. Math. J.* **62** (2013), 1503–1533.
- [NV] F. Nazarov, A. Reznikov, and A. Volberg, A simple sharp weighted estimate of the dyadic shifts on metric spaces with geometric doubling, *Int. Math. Res. Not. IMRN* (2013), 3771–3789.
- [Tao] T. Tao, Dyadic product H^1 , BMO, and Carleson's counterexample, Short stories, <http://www.math.ucla.edu/~tao/preprints/harmonic.html>.
- [Uch] A. Uchiyama, The factorization of H^p on the space of homogeneous type, *Pacific J. Math.* **92** (1981), 453–468.
- [V] M. Villani, Riesz transforms associated to Bessel operators, *Illinois J. Math.*, **52** (2008), 77–89.
- [W] A. Weinstein, Discontinuous integrals and generalized potential theory, *Trans. Amer. Math. Soc.* **63** (1948) 342–354.
- [YY] Da. Yang and Do. Yang, Real-variable characterizations of Hardy spaces associated with Bessel operators, *Anal. Appl. (Singap.)*, **9** (2011), 345–368.

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