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THE INTERPRETATION OF COORDINATES IN GENERAL RELATIVITY

by

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Abstract

The Interpretation of Coordinates
in
General Relativity

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This paper is a study of the role of space-time coordinates in General Relativity. It will first be recalled how, in general, any attempt to attach a "real" meaning to coordinates in four-space is incorrect. The "reality" of the coordinates in describing General Relativity will be shown to exist only through certain locally measured quantities.

A general scheme for attaching a meaning to the coordinates will be developed and illustrated. An explicit analysis of the interpretation of the rotating coordinate system will be made in the light of this scheme.

Thesis supervisor: Philip Morrison
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Introduction

This paper may be considered to have two main themes. The first is a discussion of the role of coordinates in General Relativity and their interpretation. The second is a complete and explicit analysis, in the light of our discussion, of one example - the rotating coordinate system.

The Need for a "Curved Space"

We first accept the validity of Einstein's Special Theory of Relativity and its formalism. An event is described by the place where it occurred and the time it occurred. Thus an event in a certain material particle is defined by the three coordinates of that particle and the time when the event occurs. If we use as coordinates of an event the quantities $x,y,z,\tau = ict$ (This is one coordinate convention, others being possible.), then we may consider x,y,z,τ, as components of a vector in four-dimensional space (a "four-vector"). According to Special Relativity, the sum of the squares of these components (the square of the "length" of the four-vector), $x^2 + y^2 + z^2 + \tau^2$, does not change under rotations of the four-dimensional system of coordinates, which include transformations from one inertial reference system to another (i.e. Lorentz transformations) as a special case.

Before discussing the specific framework of General Relativity, we point out that the definition of an event is actually independent of any field theory and may be understood before any such theory is proposed. (We are of course assuming, at least, that the words "place" and "time" are understood.[1])

The two basic principles of General Relativity are the

Principle of Equivalence and the Principle of Covariance. The Principle of Equivalence, as verified to a high degree of accuracy by Eötvos[2] and Dicke[3], states that the properties of a particle's motion in a noninertial system are locally the same as those in an inertial system in the presence of a gravitational field. In other words, a noninertial reference system is locally equivalent to a certain gravitational field. The Principle of Covariance states that the general laws of physics can be expressed in a form which is independent of the choice of a coordinate system.

The Principle of Covariance has often been misinterpreted; we therfore consider it more carefully here. It was first pointed out by Kretschmann[4] that any physical law can be put into a covariant form; hence the adoption of the principle imposes no necessary restrictions on the nature of the law - we can make the theory covariant without increasing its physical content. (As an example, Maxwell's equations in an inertial frame can be written in a well-known covariant form without enlarging its physical content in inertial frames.) However, the Principle of Covariance does have its use. In searching for the appropriate laws of physics we want to immediately eliminate unsuspected assumptions that could arise from the use of any particular coordinate system. By expressing everything in covariant form,

as can be done, we make no use of a particular coordinate system.

The previous discussion showed the implications (or lack of them) in adopting the Principle of Covariance. We can also give a more physical motivation for adopting the principle. As emphasized by Einstein, the laws of physics are to be regarded as a codification of the results of experimental observations. The recording of these observations (in space and time) is, of course, conveniently carried out with the help of some system of space-time coordinates. Nevertheless, the actual physical behavior can be in no way affected by the coordinate system used, which may be introduced by the experimenter in any way he chooses. We therefore feel that the laws of physics should be expressed in a form which makes no reference to any particular coordinate system.

Having examined the basic principles of General Realativity, we can proceed to see what they imply about space and time.

Consider an empty space I, and set up a Euclidean coordinate system in I. We can set up a cartesian system of axes in I (x,y,z) and label coordinate points along the three axes, using say a rigid finite straight rod laid out along the axes. By Special Relativity, for one observer

the rod has the same length anywhere in I. If $\underline{l}$ is the length of the rod at any arbitrary position in space, we would obviously have

$$l^2 = (x_1-x_2)^2+ (y_1-y_2)^2+ (z_1-z_2)^2$$

or

$$dl^2 = dx^2 + dy^2 + dz^2$$

for an infinitesimally small length.

We can measure time in I by using one clock to set up a unit of time, which by Special Relativity, ticks at the same rate no matter where the observer is standing in I (that is , the proper-time interval at all points in I is the same).

Now consider another system I' rotating uniformly with respect to I. Let us take the axis of rotation as the z axis and let us confine our attention to the x-y plane. We assume that we may apply the deductions from Special Relativity to non-uniform motion so long as we restrict ourselves to such a small region of space and time that in it the motion may be considered as uniform and rectilinear (i.e. "use very, very short measuring rods and take quick measurements"). Let an observer in I lay out his measuring rod to measure the radius r, and circumference c, of a circle in I. He will obviously find that $c = 2\pi r$. Now let an observer at rest in I' measure the circumference c' and radius r' of the same

circle with the same measuring rod. If he measures the distance between two points along the radius of the circle with the measuring rod, he will find

$$dr = dr' \quad ,$$

for th velocity of the measuring rod relative to I is perpendicular to the direction of the rod and thus does not give rise to any Lorentz contraction along the rod. On the other hand, if we lay our (infinitesimal) measuring rod along the circumference c' of the circle, the rod will have a velocity relative to I in its own direction and it will therefore be Lorentz contracted relative to its length lying in I. By successively laying out the measuring rod, the observer in I' will be able to measure r' and c'. It is clear what will happen here: Both observers will measure the same radius,

$$r = r' \quad ,$$

but the observer in I', because of the shortening of the rod in I' relative to I will have to set the rod against the circumference a greater number of times than the observer in I did in his measurement of the circumference. Therefore the ratio of the circumference c' to the radius r' in I' will not be 2π , but

$$c'/r' > 2\pi \quad .$$

This relation, obtained by means of measurement with a

standard measuring rod at rest in I', deviates from the results obtained in Euclidean geometry.

Looking at time in I and I', we also find a difference of interpretation. We know we can place standard clocks everywhere in I and synchronize them by light signals from a standard clock at some point in I. We could now define the time in I' by means of standard clocks inserted at rest everywhere in this system and set according to the standard clocks in I, for example, by putting them to zero at the moment when the clocks in I with which they coincide show zero. However, a standard clock at some point in I' has a velocity relative to I and will therefore be retarded relative to the clocks in I. The further out the clocks are in I', the greater will they be retarded relative to I. Therefore clocks at rest in I' do not run synchronously.

We have found that, for a rotating system, both space and time are "distorted". We find that Euclidean geometry no longer applies in I' and that our notion of time must be revised.

What applied to the rotating system as a special case, remains valid for arbitrarily moving systems. If an observer be imagined at any point, then the other points of the system will for a short space-time interval execute a uniform translational motion relative to him, but which varies from

point to point. For him, a unit measuring rod will have a different length, a clock a different speed, from point to point.

Since any arbitrary moving system, be it accelerated or in rotation, corresponds at different points according to the Principle of Equivalence to different gravitational fields, varying in general from point to point, exactly the same considerations will apply to arbitrary non-homogeneous gravitational fields. The units of length and time, considered from one particular point as reference, are different at different points of the gravitational field.

In the Special Theory of Relativity, we considered space and time as coordinates in a general "four-space". Our development of General Relativity has not introduced any concepts that force us to abandon the notion of a space-time continuum. However, we now recognize that our space-time must be more general than the Euclidean four-space found in Special Relativity for inertial frames. We must now talk about a general "curved space-time".

Looking once again at our example of the rotating system I', we recognize that if another system I'' were set up, rotating with respect to I but at a different angular velocity than I', the space and time relations in I'' with respect to I would be different than the relations in I' with

respect to I. Similarly for different arbitrarily moving systems. In other words different gravitational fields will lead to different curved space-times: This is the really significant new concept in General Relativity - the geometry is determined from the theory and not given a priori. It is the distribution of matter that determines the "curvature" of space-time.

The Meaning of Coordinates and a "Curved Space"

Coordinates

We see now, that General Relativity forces us to concern ourselves with a general "curved" space-time. Let us look first at the role of coordinates in space-time. General Relativity tells us that the nature of space-time (henceforth referred to as a "space") is determined by the distribution of matter. The "shape " of the space is an intrinsic part of the space; it does not depend on our choice of a coordinate system, for the way we choose to label the points in space can in no way affect the space itself. The coordinates act only to catalog all our points in space. We may choose our coordinate system any way we please; sometimes a particular choice makes the solution of our equations simpler and "physically clearer" (a vague phrase, which will be intensely analyzed and attacked as we go on). Of course we must limit our choice of coordinate system so that each point in the space is labelled uniquely; for example, we must choose coordinates to be monotonically increasing so that no amiguity can arise in the labelling of any two points, etc. The real property of a space is its "shape", and we must investigate this further.

We are generally concerened with a four dimensional space

with coordinate systems that can be transformed to other systems by

$$\bar{x}^i = f^i(x^0, x^1, x^2, x^3)$$

$$x^j = g^j(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) \quad ,$$

where f and g are continuous and one-to-one. As we know from Special Realativity, the interval between two infinitesimally close points in Euclidean four-space, in terms of Cartesian coordinates is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

We want a general expression for ds that is good for our general curved space. By various arguments[5] one can show that our geometry can be taken to be Riemannian. A Riemannian space is defined as a collection of points in which the interval between two infinitesimally close points, the metric, is given by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \quad ,$$

the summation convention being used, where the $g_{\mu\nu}$ are functions of the coordinates x^μ and the $g_{\mu\nu}$ are assumed to be symmetric,

$$g_{\mu\nu} = g_{\nu\mu} \quad .$$

The Special Relativity metric above ("Lorentz metric") may be written as

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

This is our well-known Euclidean flat space.

We wish to write everything in a form that is independent of any coordinate system, and this leads us to develop the tensor algebra. (We will assume some knowledge of the tensor algebra, and only explicitly develop relations that are pertinent to our discussion.)

We note that ds^2 is obviously a scalar, since it is the distance between two infinitesimally close points in the space and this distance is obviously independent of the choice of a coordinate system to label the points.

Curvature

In order to attach a meaning to a "curved space", we want to find some criterion of the space that "distinguishes flat from curved." Obviously, since this is a criterion of the space itself, we demand it to be at least in a covariant form (if not an invariant), since the choice of a coordinate system to label the points in space cannot affect the actual space.

Consider a general curved space, S. A covariant vector

A_μ

A_i in S transforms according to the formula

$$A_i = \frac{\partial \bar{x}^k}{\partial x^i} \bar{A}_k \quad . \qquad (1)$$

The differential dA_i of A_i transforms as

$$dA_i = \frac{\partial \bar{x}^k}{\partial x^i} d\bar{A}_k + \bar{A}_k d\frac{\partial \bar{x}^k}{\partial x^i}$$

$$= \frac{\partial \bar{x}^k}{\partial x^i} d\bar{A}_k + \bar{A}_k \frac{\partial^2 \bar{x}^k}{\partial x^i \partial x^l} dx^l \quad .$$

Thus dA_i does not, in general, transform at all like a vector (the same also applies, of course, to the differential of a contravariant vector dA^i). This is due to the fact that dA_i is the difference of vectors located at different (infinitesimally separated) points of space; at different points in space vectors transform differently, since the coefficients in the transformation formula (1) is a function of the coordinates.

In order to obtain a differential of a vector that is itself a vector, the two infinitesimally separated vectors that are to be subtracted from each other must be located at the same point in space. We must somehow translate one of the vectors to the point where the second is located, and then determine the difference of two vectors which now refer to the same point in space. We impose on the translation the metric requirement that the scalar product of two vectors

be invariant under translation. In particular, the length of a vector will then remain unchanged under the translation, as in the case with Cartesian coordinates in a Euclidean space (i.e. the components do not change); in this case the difference coincides with the ordinary differential dA_i , as we must have. Such a translation in a Euclidean space with Cartesian coordinates is precisely the translation of a vector parallel to itself. This operation is therefore known as "parallel displacement".

In curvillinear coordinates in both curved and flat spaces, the components of a vector A^i would be expected to change under a parallel displacement. This change δA^i, depends linearly on the values of the components of A^i themselves, since the sum of two vectors must transform according to the same law as each of the constituents. Therefore

$$\delta A^i = - \Gamma^i_{kl} A^k \, dx^l$$

where the Γ^i_{kl} , called Christoffel symbols, are found to have the form

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) \tag{2}$$

and to transform according to

$$\Gamma^i_{kl} = \bar{\Gamma}^m_{np} \frac{\partial x^l}{\partial \bar{x}^m} \frac{\partial \bar{x}^n}{\partial x^k} \frac{\partial \bar{x}^p}{\partial x^l} + \frac{\partial^2 \bar{x}^m}{\partial x^k \partial x^l} \frac{\partial x^l}{\partial \bar{x}^m} \tag{3}$$

(Similarly, it is found that

$$\delta A_i = \Gamma^{k}_{il} A_k \, dx^l \quad . \qquad (4)\ )$$

From (3) one can show that there always exists a coordinate systemin which all the $\Gamma^{i}_{kl} = 0$ at some point.

Let us now parallel displace a vector A_k along an infinitesimal closed path, and see if there is any change in the vector after it is so displaced. We write the change ΔA_k as

$$\Delta A_k = \oint \Gamma^{i}_{kl} A_i \, dx^l \qquad (5)$$

from Eq. (4), where the integral is taken over the closed path. We apply the four-dimensional generalization of Stoke's theorem[7]

$$\oint A_i \, dx_i = \int df_{ki} \frac{\partial A_i}{\partial x_k}$$

$$= \frac{1}{2} \int df_{ik} \left(\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \right)$$

where df_{ik} is the element of area enclosed in the curve. Therefore (5) gives

$$\Delta A_k = \frac{1}{2} \left[\frac{\partial (\Gamma^{i}_{km} A_i)}{\partial x^l} - \frac{\partial (\Gamma^{i}_{kl} A_i)}{\partial x^m} \right] \Delta f^{lm}$$

where Δf^{lm} is the infinitesimal area enclosed by the contour,

$$\Delta A_k = \frac{1}{2}\left[\frac{\partial \Gamma^{i}_{km}}{\partial x^l} A_i - \frac{\partial \Gamma^{i}_{kl}}{\partial x^m} A_i + \Gamma^{i}_{km}\frac{\partial A_i}{\partial x^l} - \Gamma^{i}_{kl}\frac{\partial A_i}{\partial x^m}\right] \Delta f^{lm} \qquad (6)$$

To first order accuracy, which is sufficient for the infinitesimal contour, we can regard the components of A_i at points inside the contour as being uniquely determined by their values on the contour by Eq. (4), i.e.

$$\frac{\partial A_i}{\partial x^l} = \Gamma^{k}_{il} A_k$$

Substituting in (6) we get

$$\Delta A_k = \frac{1}{2} R^{i}_{klm} A_i \Delta f^{lm}$$

where R^{i}_{klm} is a tensor of the fourth rank, called the curvature tensor, given by

$$R^{i}_{klm} = \frac{\partial \Gamma^{i}_{km}}{\partial x^l} - \frac{\partial \Gamma^{i}_{kl}}{\partial x^m} + \Gamma^{i}_{nl}\Gamma^{n}_{km} - \Gamma^{i}_{nm}\Gamma^{n}_{kl} \qquad (8)$$

Also, for a parallel displacement, $\Delta(A^k A_k) = 0$,which yields

$$\Delta A^k = -\frac{1}{2} R^{k}_{ilm} A^i \Delta f^{lm} \qquad (9)$$

From (7) and (9) it follows that for any finite closed path,

the vanishing of $R^{\hat{}}_{klm}$ guarantees that an arbitrary vector will not change under a parallel displacement.

For a Euclidean space, the curvature tensor is clearly zero, since

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \\ & & & -1 \end{pmatrix}$$

in Cartesian coordinates; if coordinate transformations are carried out in such a space, the transformed components of $R^{\hat{}}_{klm}$ will still vanish because of their tensor character. Therefore a necessary condition for the space to be flat is that $R^{\hat{}}_{klm} = 0$.

Let us see if $R^{\hat{}}_{klm} = 0$ is a sufficient condition for a flat space; that is, if $R^{\hat{}}_{klm} = 0$, is the space Euclidean? If $R^{\hat{}}_{klm}$ vanishes everywhere, then a Lorentz metric may be generated by parallel displacement of a Cartesian set of axes to all of space because this is now a unique operation, independent of the path. Since the metric was a Lorentz metric at the starting point, it is a Lorentz metric everywhere.

Therefore $R^{\hat{}}_{klm} = 0$ is a necessary and sufficient condition for a space to be Euclidean (flat). A space which has $R^{\hat{}}_{klm} \neq 0$ is said to be "curved".

Note that the vanishing or non-vanishing of the

curvature tensor tells us about an intrinsic property of the space - its vanishing or non-vanishing is not dependent on the choice of a coordinate system. (Though its numerical value at the same point may be different in two different coordinate systems, this is not the condition chosen to give us the nature of the space.) Note too that even though there always exists a coordinate system in every space which will be Cartesian at some point (that is, such that all the $\Gamma^{\hat{\imath}}_{\hat{k}\hat{l}} = 0$ at that point), this does not imply that $R^{\hat{\imath}}_{\hat{k}\hat{l}\hat{m}} = 0$ at that point, since the derivatives of $\Gamma^{\hat{\imath}}_{\hat{k}\hat{l}}$ do not become zero along with the $\Gamma^{\hat{\imath}}_{\hat{k}\hat{l}}$.

Having derived a "curvature tensor" and having related it to a curved (non-Euclidean) space, we wish to be sure that this corresponds to our accepted notion of "curved".

Consider a spherical surface. If the sphere is of radius a , its metric is well known to be

$$ds^2 = -a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

which gives the non-vanishing metric terms, with

$$x^1 = \theta \qquad x^2 = \phi$$

as

$$g_{11} = -a^2 \qquad g_{22} = -a^2\sin^2\theta$$

$$g^{11} = -\frac{1}{a^2} \qquad g^{22} = \frac{-1}{a^2 \sin^2\theta}$$

$$g = a^4 \sin^2\theta$$

The non-vanishing Γ^{λ}_{kl} are

$$\Gamma^{1}_{22} = -\frac{1}{2}\sin 2\theta \qquad \Gamma^{2}_{21} = \Gamma^{2}_{12} = \cot\theta$$

The covariant curvature tensor is defined as

$$R_{iklm} = g_{in} R^{n}_{klm}$$

and has the following symmetry properties,

$$R_{iklm} = -R_{kilm}$$

$$R_{iklm} = -R_{ikml}$$

$$R_{iklm} = R_{lmik}$$

Also $R^{i}_{klm} + R^{i}_{mkl} + R^{i}_{lmk} = 0$.

The above realtions show that there is only one independent component of R_{iklm}, which may be written as

$$R_{1221} = -a^2 \sin^2\theta$$

We define the tensor R_{ik} as

$$R_{ik} = g^{lm} R_{likm} = R^{l}_{ilk} \quad .$$

R_{ik} in this case has two non-zero components

$$R_{11} = -1 \qquad R_{22} = -\sin^2\theta \quad .$$

The "scalar curvature" R ,

$$R = g^{ik} R_{ik}$$

is found to be

$$R = g^{11} R_{11} + g^{22} R_{22} = \frac{2}{a^2} \quad .$$

Now the Gaussian curvature of a sphere of radius a is known to be $\frac{1}{a^2}$ and the last result shows that R is proportional to the Gaussian curvature. This result turns out to be a particular example of a general theorem[6]: In a space with two dimensions, i.e. an ordinary surface, R_{iklm} has only one independent component and $\frac{R}{2}$ (with the proper sign choice for the independent component) is equal to the Gaussian curvature of the surface (the reciprocal of the product of the principal radii of curvature).

We now see that the notion of the "curvature" tensor

is tied-up with our intuitive notion of curvature and curved space, for the two dimensional case. As we go to N-dimensional spaces, we naturally lose our feeling for curvature. But we can still say that, as in the two dimensional case, the non-vanishing of $R^{\hat{\cdot}}_{\cdot\cdot\cdot}$ is a necessary and sufficient condition for the space to be "curved".

The necessity for carefully establishing an intrinsic criterion for a curved space will become clear after we discuss the reality of coordinates.

The Unreality of Coordinate Systems

We discussed above the role of coordinates, i.e. that they are labels for every point in space - a system to catalog all our points (events). In order to be able to refer back to our points if we wish, we set up a systematic labelling scheme. We can draw lines everywhere in space, and attach numbers to the points on the line: each point in space will then be identified in the catalog by the intersection of the lines there. This is our usual, normal approach to labelling points. For instance, if we wish to label all the points on a flat disk, we can set up a system of polar coordinates, where we draw straight lines from the center of the disk to the edge. We can take a standard ruler and mark off these lines to act as our "radial coordinate", r. We can then set up our usual "angular coordinate" θ , in the usual way. We know from Euclidean geometry, and can measure on the disk, that the relation between any circumference c on the disk (r = constant, θ goes from 0 to 2π) to its radius is $c = 2\pi r$. We have a real meaning for our (r, θ) coordinate system, used to label all points on the disk. If someone says, "This is the r coordinate.", we know exactly what he means.

We now maintain that in General Relativity, a real, descriptive meaning cannot be given, in general, to any

coordinate system used to describe the space-time continuum. In other words, in a general curved space, a coordinate which does have actual different values at different points cannot be interpreted or understood in the sense of the meaning given to a coordinate in a Euclidean space. (In the tone of the preceeding paragraph, if someone in this space says, "This is the r coordinate.", we do not know at all what he means.)

Let us reexamine the rotating system I' , described above. In this system, we still talked about radial directions and circumferences of circles, but we saw that $c' > 2\pi r'$. Let us try to imagine a polar coordinate system on a disk in I' . As we go out along an r' , far from the center of the disk we would expect another radial line that was close to our radial line near the center to be far away from us now. But we know by Special Relativity that, as v_θ approaches c, the two radial lines will get infinitesimally close to each other. It is clear that our usual notions of r and θ cannot be preserved in this space.

Turning to the time coordinate t' ; setting up a uniform time coordinate in I' does not correspond to the "ticking" of real clocks at every point in I, as was shown above.

Since the (r', θ', z', t') system does not have its usual interpretation in I' , perhaps we can find another, better coordinate system that does have the same interpretation in I' as in an inertial system. Such a coordinate system cannot exist by the Principle of Covariance; for, the curvature of the space is an intrinsic part of the space, determined by the distribution of matter, independent of any coordinate system chosen to label the space points. The choice of one or another coordinate system does not attribute any new or different physical properties to the space. One coordinate system in the space will be just as uninterpretable in the large in terms of its Euclidean counterpart (if it has one) as another.

In light of these facts, we should slightly clarify our previous discussion of the rotating system I' . For a massless circle, we compared the ratio of circumference to radius for the same circle in both I' and in an inertial system I. Instead of saying that the observer in I' measured r' , it would be more correct to say he measured the distance of a taut string tied at the center and at a point on the circle = l' ; this makes the measurements in I and I' measurements of the same thing. Even in this wording (where we do not claim that the observer's measurement of l' necessarily corresponds to his "radial

coordinate" r') it is true that $c'/l' > 2\pi$ (and that in the flat space ($\omega = 0$) l' would actually correspond to r), so the Euclidean relation in the flat space does not hold here and our space is still shown to be curved.

We now examine the most well-known solution of Einstein's equations, the Schwarzschild solution, to show the lack of reality of the coordinates. The Schwarzschild line element is given in terms of (r, θ, ϕ, t) as [7]

$$ds^2 = c^2\left(1 - \frac{2r_0}{r}\right)dt^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - \frac{dr^2}{1 - \frac{2r_0}{r}}$$

where $r_0 \equiv \frac{GM}{c^2}$.

Using this metric, the equations of motion for a free particle can be written down and solved [9]. The orbit of a particle in terms of r, θ, and ϕ is usually discussed in the same way as one discusses the orbit of a particle in terms of r, θ, and ϕ in the classical Kepler case (i.e. while the equations of motion and subsequent solutions are different for the two cases, r, θ, and ϕ are talked about as if they had the same meaning in both cases). Such a description is incorrect. As an illustration, we will show that the r here does not have all the properties of the Euclidean radius vector, for example, to be equal both to the distance from the center and to the length of the

circumference divided by 2π .

The element of spatial distance is [8]

$$dl^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{dr^2}{1-\frac{2r_0}{r}}$$

The distance from the center along r is

$$\int_0^r dl = \int_0^r \frac{dr}{\sqrt{1-\frac{2r_0}{r}}}$$

which is $> r$ (for $r \neq \infty$).

The circumference of the "circle" drawn through the given point (with center at the origin) is

$$C = r\int_0^{2\pi} d\theta = 2\pi r$$

Consequently we see that the ratio of the circumference of a circle to its radius is less than 2π .

The time coordinate too does not have the properties of a Euclidean uniform t ; for the proper-time interval of a clock at some r ($\neq \infty$) in the space is

$$d\tau = \sqrt{1-\frac{2r_0}{r}}\, dt = f(r) < dt \quad .$$

Note that if r_0 is very small, or r very large, the line element approaches the flat space case with r, θ, ϕ, and t taking on their "real" connotation. Thus we find it

convenient to have picked our four arbitrary coordinates to have names r, θ, ϕ, and t, even though that is not what they really are - the line element in this form agrres very closely with our intuitive notion of space, which is based mainly on Euclidean geometry.

We see that any valid attempt at a derivation of the Schwarzschild line element can only be made on the basis of symmetry arguments. The meaning of the coordinates cannot play any part in the derivation.

The choice of one coordinate labelling over another may lead to simpler-looking mathematical equations and may possibly lead to equations that even resemble flat space equations, but it should be understood (although many times it is not) that this does not lead to a simpler meaning of the equations or the space itself.

The Interpretation of Coordinates

We have shown the "non-reality" of coordinates in a general curved space. But we have to have some foothold on the space, and particles moving in it - hence the need for coordinates. We must therefore see what interpretation we can give to coordinate systems in general space-time. Fundamentally, is it possible for an observer, moving on his world line in a general curved space, to attach a real meaning to the coordinate system he uses to (a) write down his equations and (b) label his points in space? The only way an observer can attach a "real" meaning is in terms of measurables; merely conceptual descriptions are no longer valid.

Let us examine what measurables are possible. In general, a space contains mass points (which is equivalent to the geometry of the space) and free test particles and observers (which, for simplicity, are usually assumed to have no effect on the curvature of the space). It is obvious that our observer cannot use the velocity of a particle moving in the space as a measurable for he has not interpreted "direction" yet. (An observer can measure the velocity of a particle moving along a certain direction, and can label that direction as being along some arbitrary coordinate. However, if he wants to use that

arbitrary coordinate everywhere in space, and if he wants the components of his metric to be written in terms of his arbitrary coordinates, he must make measurements along every coordinate at every point in space - clearly an impossible situation. We are concerned with an observer who has already written down his equations in some coordinate system and wants to know what the coordinates mean. This is the practical problem.) In a space containing particles that do not radiate, the useful measurable is the observer's proper time. This leads us to: A real and complete interpretation of the coordinate system in a general curved space exists if and only if every parameter in the theory can be written in terms of the observer's proper time. (This is a postulate of the present work.) It is interesting that, in a Euclidean space we know that we are able to construct and label our coordinate system by many different schemes (using rigid rulers, taut strings, standard clocks, particle velocities, etc.), one scheme being as interpretable as another. However, in a general curved space, we see that we are limited in our choice of measurables and subsequent schemes.

We must now develop a concrete procedure for interpreting coordinates. We propose the following: Given a general curved space S with its metric, with free

particles moving in S described by the geodesic equations of motion, and a regular coordinate system C (with the metric and equations of motion written in terms of C) - an observer in S can give a real and complete interpretation of C in terms of proper time intervals and proper time-delay intervals(time-delay being the proper time interval it takes for a signal to be sent from the observer to a free particle and return) if the following can be satisfied:

A. Solve the geodesic equations for the propagation of a light ray in S in terms of C;

B. Calculate the proper time delay, between an observer and a test particle at some point in S, only in terms of the coordinates of the observer and the test particle and the parameters found in the metric;

C. Solve the goedesic equations of motion for a free particle in S in terms of C, finding the coordinate motion of the particle as a function of coordinate time and the metric parameters (this will introduce coordinate dependent constants of integration as additional parameters);

D. Using the particles' geodesic equations of motion (including observer), calculate the proper time delay only as a function of the parameters and coordinate time;

E. Evaluate the parameters as a function only of specific proper time intervals and proper time-delay intervals. We

can then write all our coordinates as functions only of proper time measurements.

That this analysis can always be successfully carried out is by no means obvious. Excluding mathematical difficulties, which are not physics' fault, certain geometries (mass distributions), symmetries (or lack of symmetries), time dependences, and/or singularities or other peculiarities may physically prohibit a complete real interpretation of coordinates everywhere. (However, if peculiarities arise, it may be possible, in the light of trying to give a "real" meaning to our space by this scheme, to give a "real" interpretation to these peculiarities; or, at the opposite extreme, we may be forced to modify the theory itself.)

We now see the need for carefully developing a tensor criterion for curvature. Having written our metric and equations in some chosen coordinate system, we can evaluate $R^{\hat{}}_{klm}$ and see if the space is actually curved; this we can do in <u>any</u> coordinate system we have chosen to use. If we find the space to be curved, we know we cannot attach a "flat space meaning" to the coordinates, no matter how similar they, or their equations of motion, resemble the flat space case; rather, we must always consider the description of motion and the space itself in terms of our

operational development of reality, given above.

An examination of the literature of General Relativity since its inception fifty years ago shows that the problem of the interpretation of coordinates has been practically untouched. The two most commonly discussed cases in General Relativity, the rotating coordinate system and the Schwarzschild case, have not been carefully investigated. We will carry through a complete analysis of coordinates for the rotating coordinate system.

Shapiro, in his "fourth test" of General Relativity[10], has investigated the interpretation of coordinates in the Schwarzschild case (however, only as it applies specifically to his problem). Before analyzing the rotating coordinate system in the large, it will be useful to examine a simple case of Shapiro's general problem in order to give a simple illustration of the proposed scheme.

Shapiro[11] considers the Schwarzschild metric due to the sun, with an observer being on the earth (considered as a massless test particle) and another test particle (e.g. Mercury or Venus); both observer E, and test particle P have periodic bounded motion - which greatly simplifies the subsequent analysis. It is assumed that the change in position of E and P between the times of transmission of a signal from E and echo reception may be ignored.

The unit of time chosen is a specified number of oscillations of the observer's atomic clock, the unit of mass is the sun's, and the unit of distance is the coordinate distance of the observer E, from the origin. Three parameters remain to be determined in terms of these units: the coordinate distance a from the origin to P, the speed of light c , and the gravitational constant μ .

The Schwarzschild line element is given as

$$ds^2 = \left(1 - \frac{2r_0}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2r_0}{r}\right)} - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \qquad (1)$$

where $r_0 \equiv \frac{\mu}{c^2}$.

It is well-known that one possible solution for a test particle in a Schwarzschild space is r = constant. This is assumed to be the motion for both E and P ; also that E and P move in coplanar orbits, $\theta = \frac{\pi}{2}$. For r = constant, the coordinate-time orbital period (i.e. elapsed coordinate-time between $\phi = 0$ and 2π) can be shown to be

$$T = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{\mu}} \qquad (2)$$

Since we assume r = constant and define r = 1 for E, we get

for the relation between coordinate time t, and the observer's proper time, τ [7]

$$\tau = \frac{1}{c} \int ds$$

$$= \frac{1}{c} \int [(1-2r_0) c^2 dt^2 - r^2 d\phi^2]^{\frac{1}{2}}$$

$$= \frac{1}{c} \int \left[(1-2r_0) c^2 - r^2 \left(\frac{d\phi}{dt}\right)^2 \right]^{\frac{1}{2}} dt \quad .$$

But we know from the definition of T that

$$\frac{d\phi}{dt} T = 2\pi$$

$$\frac{d\phi}{dt} \frac{2\pi r^{\frac{3}{2}}}{\sqrt{\mu}} = 2\pi$$

$$\therefore \quad \frac{d\phi}{dt} = \frac{\sqrt{\mu}}{r^{\frac{3}{2}}} = \sqrt{\mu} \quad ,$$

for we have defined r = 1 for E.

$$\therefore \quad \tau = \frac{1}{c} \int [(1-2r_0) c^2 - \mu]^{\frac{1}{2}} dt$$

$$= \int \left[1 - 2r_0 - \frac{\mu}{c^2} \right]^{\frac{1}{2}} dt$$

$$\tau = (1 - 3r_0)^{\frac{1}{2}} t$$

since $r_0 \equiv \frac{\mu}{c^2}$.

With the positions of E and P given by $(1, \phi_E)$ and (a, ϕ_P) respectively, Shapiro calculates the proper time delay, for a radar pulse to make a round trip from E to P and back, to be [11]

$$T = \frac{2}{c} r_{EP} + \frac{2r_0}{c} \left\{ 2 \ln \left[\frac{1}{a} \left(\frac{r_{EP} + 1 - a \cos(\phi_P - \phi_E)}{r_{EP} + \cos(\phi_P - \phi_E) - a} \right) \right] \right.$$

$$\left. - \frac{1+a}{r_{EP}} \left[1 - \cos(\phi_P - \phi_E) \right] - \frac{3}{2} r_{EP} \right\} \quad (3)$$

where

$$r_{EP}^2 = 1 + a^2 - 2a \cos(\phi_P - \phi_E)$$

We call inferior conjunction the time at which the proper-time delay is a minimum, and the time between successive inferior conjunctions we call the synodic period. (Note we do not use the classical definition of inferior conjunction - "when the planet is between the earth and the sun", for we cannot be sure what that means

here; instead we give a real and operational definition, in terms of an observer's clock.)

The coordinate-time synodic period can be expressed in terms of the coordinate-time orbital periods of E and P as

$$\frac{1}{T_s} = \frac{1}{T_P} - \frac{1}{T_E}$$

Use of Eq. (2) yields

$$T_s = \frac{2\pi}{\sqrt{\mu}} \frac{a^{\frac{3}{2}}}{1 - a^{\frac{3}{2}}} ,$$

which gives the proper time synodic period as

$$T_s = \left(\frac{2\pi}{\sqrt{\mu}}\right) \frac{a^{\frac{3}{2}}}{1 - a^{\frac{3}{2}}} \left(1 - 3r_0\right)^{\frac{1}{2}} . \qquad (4)$$

Examining Eq.(3), we find that the time delay at inferior conjunction can be written as

$$T_i = \left(\frac{2}{c}\right)(1 - a) - \left(\frac{4r_0}{c}\right)\left[\ln a + \frac{3}{4}(1 - a)\right] . \qquad (5)$$

The time delay at elongation (a point Shapiro defines to be where the sun-earth-planet angle is a maximum) is given by Shapiro approximately as

$$T_e = \frac{2}{c}(1 - a^2)^{\frac{1}{2}} + \frac{4r_0}{c}\left[\ln \frac{1 + (1 - a^2)^{\frac{1}{2}}}{a} - \frac{5}{4}(1 - a^2)^{\frac{1}{2}}\right] . \qquad (6)$$

(Eq. (6) should not be accepted too readily, in view of the fact that elongation was defined in terms of an "angle", but we write it down as a first approximation to the true relation for T_e .)

Eqs. (4), (5), and (6) may be solved to yield a, c, and μ only as functions of T_s , T_λ , and "T_e" . e.g.

$$a = \frac{T_e^2 - T_\lambda^2}{T_e^2 + T_\lambda^2} - \left(\frac{2\pi}{T_s}\right)^2 \frac{(T_e^2 - T_\lambda^2)^3}{\left[(T_e^2 + T_\lambda^2)^{\frac{3}{2}} - (T_e^2 - T_\lambda^2)^{\frac{3}{2}}\right]^2} \times \frac{(T_e^2 + T_\lambda^2)^2}{16\, T_\lambda^2} \{ \ldots\ldots \}$$

$$c = \ldots\ldots$$

$$\mu = \ldots\ldots$$

We see here that the "r = a coordinate" for the test particle P can be given a real meaning, in terms of T_s , T_λ , and T_e .

We now have all the parameters necessary in Shapiro's simple case of his particular problem in terms of observables at E. If we wish, we can substitute the parameters back into Eq. (3) and get a relation for the proper-time delay T , which does not depend on an observer's interpreting the coordinates (in the Euclidean sense) that he used to write down his metric and equations of motion.

Although Shapiro's example does not completely give

an interpretation to the coordinates - it does not give an operational meaning to ϕ (which he found he did not need in this simple case) - it does illustrate the general scheme proposed. Our discussion of the rotating coordinate system will attempt to interpret the coordinates fully.

The Rotating Coordinate System

We now turn to an analysis of the interpretation of coordinates in the rotating coordinate system. There are many reasons why the rotating coordinate system is important:

1. A discussion of the rotating coordinate system is probably the most widely-used introduction to General Relativity, yet its real meaning has not been fully investigated. We will exhibit an interpretation of the coordinates in this system.

2. The rotating coordinate system has cylindrical symmetry, as opposed to the Schwarzschild solution, which is spherically symmetric. Analysis of the problem may indicate general rules (or possibly prohibitions) for the interpretation of coordinate systems with different symmetries.

3. Free particle motion in the rotating coordinate system is always unbounded, unlike the case of periodic motion usually investigated in the Schwarzschild solution. This will introduce additional parameters of motion, and will require a somewhat different approach.

4. The rotating coordinate system does not have too many parameters in the metric, nor is it too difficult

conceptually. It will serve as a good case to analyze completely.

5. The problem is not a moot one; in the near future, it may serve as another test of the validity of Einstein's equations, if such a system is set up in a region of space where the sun's gravitational influence is "negligible".

6. The understanding of coordinates in this system may be very useful in the larger problem of cosmological space.

Analysis

The line element in the rotating coordinate system is

$$ds^2 = (c^2 - r^2\omega^2)dt^2 - dr^2 - r^2 d\theta^2 - dz^2 - 2\omega r^2 d\theta\, dt. \quad (1)$$

Using the convention

$$x^0 \equiv ct$$
$$x^1 \equiv r$$
$$x^2 \equiv \theta$$
$$x^3 \equiv z \quad ,$$

we can write the metric as

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & -\frac{\omega r^2}{c} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{\omega r^2}{c} & 0 & \left(1 - \frac{r^2\omega^2}{c^2}\right) \end{pmatrix} \quad (2)$$

where ω and c are constants.

Before going further, there is an important characteristic of the rotating coordinate system that should be discussed. (We note that this point is not always made in the standard textbooks of General Relativity.) The line element (1) can be obtained from the line element for the inertial system, written in cylindrical coordinates as

$$ds^2 = c^2dt'^2 - dr'^2 - r'^2d\theta'^2 - dz'^2 \quad , \qquad (3)$$

by the transformation

$$\begin{aligned} t &= t' \\ r &= r' \\ \theta &= \theta' - \omega t' \\ z &= z' \end{aligned} \qquad (4)$$

We know from our previous discussion that the intrinsic curvature of the four space does not change when we transform from one coordinate system to another. Thus, in our rotating coordinate system the four space is still flat ! i.e. all the components of $R^{i}_{klm} = 0$ for the four-space described by (1). Note that this does not mean that we have no gravitational effects present in this system, for by the Principle of Equivalence, a non-inertial system always corresponds to a gravitational field. What it does

mean is that the gravitational field is not a permanent one - a transformation of coordinates can be made to eliminate the field everywhere (in this case we can substitute (4) into (1) with ω replaced by $-\omega$ and get (3)).

Again recalling our previous discussion, it is clear that the three dimensional spatial metric is curved (non-Euclidean), so that we cannot interpret r, θ, z (and t) as we normally would. (i.e. There exist components of the fourth-rank curvature tensor $R^{\cdot}_{\cdot\cdot\cdot}$ of the three-dimensional space that are not equal to zero.)

We proceed with our analysis. The non-zero contravariant components of our metric tensor, Eq. (2), given by

$$g^{ik} = \frac{A_{ik}}{g} = g^{ki}$$

where g is the determinant of g_{ij} and A_{ik} is the associated co-factor of g_{ik}, are found to be (Appendix A)

$$g^{11} = -1 \qquad g^{22} = \frac{-\left(1-\frac{r^2\omega^2}{c^2}\right)}{r^2} \qquad g^{33} = -1 \qquad g^{00} = 1$$

$$g^{02} = g^{20} = -\frac{\omega}{c}$$

Consequently, the non-zero elements of Γ^{i}_{kl} are (Appendix A)

$$\Gamma^{1}_{22} = -r \qquad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r}$$

$$\Gamma^{1}_{20} = \Gamma^{1}_{02} = -\frac{\omega r}{c} \qquad \Gamma^{2}_{10} = \Gamma^{2}_{01} = \frac{\omega}{rc}$$

$$\Gamma^{1}_{00} = -\frac{r\omega^2}{c^2} \qquad (5)$$

Our ultimate aim is to enable an observer at rest at a point in the rotating coordinate system to give a complete and meaningful interpretation to his coordinate system.

A.

Following our procedure, our first goal is to solve the equations of motion for a light ray in the rotating coordinate system. The geodesic equations of motion for a light ray are known to be [9]

$$\frac{d^2x^i}{dq^2} + \Gamma^{i}_{kl}\frac{dx^k}{dq}\frac{dx^l}{dq} = 0 \qquad (6)$$

where q is an arbitrary parameter. The light-ray trajectories are null-geodesic lines, $ds^2 = 0$. With the aid of Eqs. (5), we get

$$\frac{d^2r}{dq^2} - r\left(\frac{d\theta}{dq}\right)^2 - 2\omega r\frac{d\theta}{dq}\frac{dt}{dq} - r\omega^2\left(\frac{dt}{dq}\right)^2 = 0 \qquad (7)$$

$$\frac{d^2\theta}{dq^2} + \frac{2}{r}\frac{dr}{dq}\frac{d\theta}{dq} + \frac{2\omega}{r}\frac{dr}{dq}\frac{dt}{dq} = 0 \qquad (8)$$

$$\frac{d^2z}{dq^2} = 0 \qquad (9)$$

$$\frac{d^2t}{dq^2} = 0 \qquad (10)$$

Equation (9) yields immediately

$$\frac{dz}{dq} = \text{constant},$$

which may be taken to be equal to zero for convenience, without losing any physical generality.

Equation (10) yields

$$\frac{dt}{dq} = \alpha = \text{constant} \quad . \qquad (11)$$

These equations can now be handled if we include our equation for the line element of a light ray, $ds^2 = 0$. By (1), with a dot denoting $\frac{d}{dq}$, we have

$$0 = \dot{r}^2 + r^2\dot{\theta}^2 + 2\omega\alpha r^2\dot{\theta} - c^2\alpha^2 + r^2\omega^2\alpha^2 \qquad (12)$$

(8) and (11) can be written as

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \frac{2\omega}{r}\dot{r}\alpha = 0 \qquad (8)$$

$$\dot{t} = \alpha \qquad (11)$$

Multiplying (8) by r^2 gives

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + 2\omega r\dot{r}\alpha = 0$$

$$= \frac{d}{ds}(r^2\dot{\theta} + \omega r^2\alpha)$$

$$\therefore \quad r^2\dot{\theta} + \omega\alpha r^2 = \beta = \text{constant} \qquad (13)$$

By (13),

$$r^2\dot{\theta} = \beta - \omega\alpha r^2$$

and

$$r^2\dot{\theta}^2 = \frac{\beta^2}{r^2} - 2\omega\alpha\beta + \omega^2\alpha^2 r^2 .$$

Substituting these relations into (12) yields

$$0 = \dot{r}^2 + \frac{\beta^2}{r^2} - c^2\alpha^2$$

$$\therefore \quad \dot{r}^2 = c^2\alpha^2 - \frac{\beta^2}{r^2} \qquad (14)$$

To find r as a function of t, we have from (11) and (14);

$$\left(\frac{dr}{dt}\right)^2 = \frac{\dot{r}^2}{\dot{t}^2} = \frac{\dot{r}^2}{\alpha^2}$$

$$= c^2 - \frac{\beta^2}{\alpha^2 r^2}$$

$$= c^2\left(1 - \frac{\beta^2}{\alpha^2 c^2 r^2}\right)$$

Defining a constant

$$d \equiv \frac{\beta}{\alpha c} \quad , \tag{15}$$

we get

$$\left(\frac{dr}{dt}\right)^2 = c^2\left(1 - \frac{d^2}{r^2}\right) \quad . \tag{16}$$

Eqs. (14) and (15) give us

$$\dot{r}^2 = c^2\alpha^2 - \frac{\beta^2}{r^2}$$

$$= \alpha^2 c^2\left(1 - \frac{\beta^2}{\alpha^2 c^2 r^2}\right)$$

$$= \alpha^2 c^2\left(1 - \frac{d^2}{r^2}\right)$$

$$\therefore \quad \dot{r} = \pm \alpha c \sqrt{1 - \frac{d^2}{r^2}} \tag{17}$$

Using Eq. (13), we get

$$\dot{\theta} = \frac{\beta}{r^2} - \omega\alpha \quad , \tag{18}$$

and combining (17) and (18) yields

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{\pm \alpha c \sqrt{1 - \frac{d^2}{r^2}}}{\frac{\beta}{r^2} - \omega \alpha}$$

$$\rightarrow \quad \frac{dr}{d\theta} = \frac{\pm r^2 \sqrt{1 - \frac{d^2}{r^2}}}{d - \frac{\omega}{c} r^2} \qquad (19)$$

B.

We now proceed to obtain the time delay between two points in our space - r_1 and r_2 . Using Eq. (16) ,

$$\frac{dr}{dt} = \pm c \sqrt{1 - \frac{d^2}{r^2}}$$

we get

$$c\,dt = \frac{\pm\, dr}{\sqrt{1 - \frac{d^2}{r^2}}} = \frac{\pm\, r\, dr}{\sqrt{r^2 - d^2}} \,.$$

For r = d lying between r_1 and r_2 , which is the case with which we will be working in this analysis, integration yields for the coordinate time taken by light to travel from r_1 to r_2

$$t_{12} = \frac{1}{c}\left[\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right]$$

Since the roundtrip is symmetric for an observer at rest in the rotating coordinate system, the round-trip coordinate-

time-delay is

$$t = \frac{2}{c}\left[\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right] \qquad (20)$$

We are really interested in the proper time delay as measured by the observer at rest in the system (sitting at r_1, say). We use Eq. (1) to convert our results to proper time. As is well known [7]

$$ds^2 = c^2 d\tau^2$$

$$= (c^2 - r_1^2\omega^2)\, dt^2$$

$$= c^2\left(1 - \frac{\omega^2}{c^2} r_1^2\right) dt^2$$

$$\therefore \quad \tau = \left(1 - \frac{\omega^2}{c^2} r_1^2\right)^{\frac{1}{2}} t \qquad , \qquad (21)$$

and the proper time delay between an observer fixed at r_1, and r_2 is

$$\tau = \frac{2}{c}\left(1 - \frac{\omega^2}{c^2} r_1^2\right)^{\frac{1}{2}}\left[\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right] . \qquad (22)$$

We wish to eliminate d. From (19)

$$\frac{dr}{d\theta} = \frac{\pm r^2\sqrt{1 - \frac{d^2}{r^2}}}{d - \frac{\omega}{c} r^2} \qquad , \qquad (19)$$

we get

$$d\theta = \frac{\pm\, dr\left(d - \frac{\omega}{c} r^2\right)}{r^2\sqrt{1 - \frac{d^2}{r^2}}}$$

$$= \frac{\pm\, dr\left(\beta - \frac{\omega}{c} r^2\right)}{r\sqrt{r^2 - \beta^2}}$$

$$= \pm\left[\frac{\beta\, dr}{r\sqrt{r^2-\beta^2}} - \frac{\frac{\omega}{c} r\, dr}{\sqrt{r^2-\beta^2}}\right] .$$

Integrating between (r_1, θ_1) and (r_2, θ_2) yields

$$\theta_1 - \theta_d = \cos^{-1}\left(\frac{d}{r_1}\right) - \frac{\omega}{c}\sqrt{r_1^2 - d^2}$$

$$\theta_d - \theta_2 = \cos^{-1}\left(\frac{d}{r_2}\right) - \frac{\omega}{c}\sqrt{r_2^2 - d^2}$$

$$\therefore\ (\theta_1 - \theta_d) + (\theta_d - \theta_2) =$$

$$\theta_1 - \theta_2 = \cos^{-1}\left(\frac{d}{r_1}\right) + \cos^{-1}\left(\frac{d}{r_2}\right) - \frac{\omega}{c}\left(\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right)$$

(23)

Taking the cosine of both sides of (23) gives

$$\cos(\theta_1 - \theta_2) = \cos\left[(A+B) - C\right]$$

$$= \cos(A+B)\cos C + \sin(A+B)\sin C$$

$$= (\cos A \cos B - \sin A \sin B) \cos C$$

$$+ (\sin A \cos B + \cos A \sin B) \sin C$$

where

$$A \equiv \cos^{-1}\left(\frac{d}{r_1}\right) \qquad B \equiv \cos^{-1}\left(\frac{d}{r_2}\right)$$

$$C \equiv \frac{\omega}{c}\left(\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right)$$

$$\therefore \cos(\theta_1 - \theta_2) =$$

$$\left(d^2 - \sqrt{r_1^2 - d^2}\,\sqrt{r_2^2 - d^2}\right)\frac{\cos C}{r_1 r_2} +$$

$$\left(\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right)\frac{d \sin C}{r_1 r_2} \qquad (24)$$

Eq. (24) is the relation that determines d in terms of the boundary conditions. It is a transcendental equation, whose solution requires a great deal of numerical analysis, but which yields little physical insight.

We can obtain an analytic solution for d , by considering the case where ω is small, and solving to first order in ω .

Rearranging Eq. (23),

$$\cos^{-1}\left(\frac{d}{r_1}\right) + \cos^{-1}\left(\frac{d}{r_2}\right) = (\theta_1 - \theta_2) + \frac{\omega}{c}\left(\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right)$$

and taking the cosine of both sides

$$\cos\left[\cos^{-1}\left(\frac{d}{r_1}\right) + \cos^{-1}\left(\frac{d}{r_2}\right)\right] =$$

$$\cos\Delta\theta - \frac{\omega}{c}\sin\Delta\theta\left[\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right]$$

to first order in ω , with

$$\Delta\theta \equiv \theta_1 - \theta_2$$

We finally get

$$\frac{1}{r_1 r_2}\left[d^2 - \sqrt{r_1^2 - d^2}\sqrt{r_2^2 - d^2}\right] = \cos\Delta\theta -$$

$$\frac{\omega}{c}\sin\Delta\theta\left[\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right] \quad (25)$$

We now want to solve (25) for d. It will be seen later that, for the situation we are concerned with, r_1 , $r_2 \gg$ d. Therefore

$$\frac{1}{r_1 r_2}\left[d^2 - r_1\left(1 - \frac{d^2}{2r_1^2}\right) r_2\left(1 - \frac{d^2}{2r_2^2}\right)\right] =$$

$$\cos\Delta\theta - \frac{\omega}{c}\sin\Delta\theta\left[r_1\left(1 - \frac{d^2}{2r_1^2}\right) + r_2\left(1 - \frac{d^2}{2r_2^2}\right)\right]$$

$$\frac{d^2}{r_1 r_2} - 1 + \frac{d^2}{2r_1^2} + \frac{d^2}{2r_2^2} - \frac{d^4}{4(r_1 r_2)^2} =$$

$$\cos\Delta\theta - \frac{\omega}{c}\sin\Delta\theta\,(r_1+r_2) + \frac{\omega}{2c}\sin\Delta\theta\, d^2\,\frac{(r_1+r_2)}{(r_1 r_2)}$$

Multiplying both sides by $4(r_1 r_2)^2$ and combining terms, we get

$$0 = d^4 + \left[\frac{2\omega}{c}\sin\Delta\theta\,(r_1 r_2)(r_1+r_2) - 4(r_1 r_2) - 2(r_1^2+r_2^2)\right] d^2$$

$$+ 4(r_1 r_2)^2\left[1 + \cos\Delta\theta - \frac{\omega}{c}\sin\Delta\theta\,(r_1+r_2)\right]$$

or

$$d^4 + B d^2 + C = 0 \qquad (26)$$

where

$$B \equiv \frac{2\omega}{c}\sin\Delta\theta\,(r_1 r_2)(r_1+r_2) - 4(r_1 r_2) - 2(r_1^2+r_2^2)$$

$$C \equiv 4(r_1 r_2)^2\left[1 + \cos\Delta\theta - \frac{\omega}{c}\sin\Delta\theta\,(r_1+r_2)\right]$$

$$\therefore\quad d^2 = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$

$$= \frac{-B \pm |B|\sqrt{1 - \frac{4C}{B^2}}}{2}$$

$$= \frac{-B \pm |B| \left(1 - \frac{2C}{B^2}\right)}{2}$$

since $\frac{4C}{B^2} << 1$.

Since $B < 0$, we may take our solution as

$$d^2 \simeq -\frac{C}{B}$$

$$= \frac{2(r_1 r_2)^2 \left[1 + \cos\Delta\theta - \frac{\omega}{c} \sin\Delta\theta \, (r_1 + r_2)\right]}{(r_1 + r_2)\left[(r_1 + r_2) - \frac{\omega}{c} \sin\Delta\theta \, (r_1 r_2)\right]} \qquad (27)$$

(the other possible solution to (26) being only a consequence of the method of approximation used). We note that for $\omega = 0$, i.e. for flat space, Eq.(27) becomes the relation for $\underline{d}$ = the radial distance of closest approach to the origin by the light ray, as we would expect.

Turning back to our equation for time delay, Eq.(22) can be written to first order in ω as

$$\tau = \frac{2}{c}\left[\sqrt{r_1^2 - d^2} + \sqrt{r_2^2 - d^2}\right]$$

$$= \frac{2}{c}\left[r_1\left(1 - \frac{d^2}{2r_1^2}\right) + r_2\left(1 - \frac{d^2}{2r_2^2}\right)\right]$$

$$= \frac{2}{c}\left[(r_1 + r_2) - \frac{d^2}{2}\frac{(r_1 + r_2)}{(r_1 r_2)}\right] .$$

Substituting-in Eq.(27) for d^2, we get

$$\tau = \frac{2}{c}\left\{(r_1+r_2) - (r_1 r_2)\,\frac{\left[1+\cos\Delta\theta - \frac{\omega}{c}\sin\Delta\theta\,(r_1+r_2)\right]}{\left[(r_1+r_2) - \frac{\omega}{c}\sin\Delta\theta\,(r_1 r_2)\right]}\right\}$$

$$= \frac{2}{c}\left\{(r_1+r_2) - \frac{(r_1 r_2)}{(r_1+r_2)}\left[1+\cos\Delta\theta - \frac{\omega}{c}\sin\Delta\theta\,(r_1+r_2)\right]\times\right.$$

$$\left.\left[1+\frac{\omega}{c}\sin\Delta\theta\,\frac{(r_1 r_2)}{(r_1+r_2)}\right]\right\}$$

which finally yields

$$\tau = \frac{2}{c}\left\{(r_1+r_2) - \frac{r_1 r_2}{(r_1+r_2)}\left[1+\cos\Delta\theta\right] + \right.$$

$$\left.\frac{\omega}{c}\sin\Delta\theta\,\frac{(r_1 r_2)}{(r_1+r_2)}\left[(r_1+r_2) - \frac{(r_1 r_2)}{(r_1+r_2)} - \frac{\cos\Delta\theta\,(r_1 r_2)}{(r_1+r_2)}\right]\right\} \qquad (28)$$

We now have an expression for the time delay of a light signal - sent by an observer at (r_1, θ_1) and bounced back by a test particle at some (r_2, θ_2), as measured by the observer's clock - solely in terms of the coordinates of the observer and test particle and the metric parameters (ω and c).

C.

We now wish to solve the geodesic equations of motion of a free particle in our space. The geodesic equations are

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0 \quad .$$

Using Eqs.(5) gives us, with a dot denoting $\frac{d}{ds}$:

$$\ddot{r} - r\dot{\theta}^2 - 2\omega r\dot{\theta}\dot{t} - r\omega^2\dot{t}^2 = 0 \qquad (29)$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} + \frac{2\omega}{r}\dot{r}\dot{t} = 0 \qquad (30)$$

$$\ddot{z} = 0 \qquad (31)$$

$$\ddot{t} = 0 \qquad (32)$$

again, we find

$$\dot{z} = \text{constant} \quad ,$$

which may be taken = 0. And

$$\dot{t} = \alpha = \text{constant} \qquad (33)$$

(of course, α is different from before).

Multiplying (30) by r^2 and substituting in (33) gives

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + 2\omega r\dot{r}\alpha = 0$$

$$= \frac{d}{ds}\left(r^2\dot{\theta} + \omega r^2\alpha\right)$$

$$\therefore\ r^2\dot{\theta} + \omega\alpha r^2 = \beta = \text{constant} \qquad (34)$$

(different β than before).

By Eq.(1)

$$1 = c^2\alpha^2 - r^2\omega^2\alpha^2 - \dot{r}^2 - r^2\dot{\theta}^2 - 2\omega r^2\alpha\dot{\theta} \qquad (35)$$

Combining (34) and (35) yields

$$\dot{r}^2 = c^2\alpha^2 - 1 - \frac{\beta^2}{r^2}$$

$$\dot{r} = \pm\sqrt{(c^2\alpha^2 - 1) - \frac{\beta^2}{r^2}}\ , \qquad (36)$$

We wish to find r as a function of coordinate time t. Therefore, by (36) and (33),

$$\frac{dr}{dt} = \left(\frac{dr}{ds}\right)\left(\frac{ds}{dt}\right)$$

$$= \dot{r}\,\frac{1}{\alpha} = \pm\frac{1}{\alpha}\sqrt{c^2\alpha^2 - 1 - \frac{\beta^2}{r^2}}$$

Let

$$\alpha \equiv \frac{b}{c}\ , \qquad (37)$$

where δ is some constant.

$$\therefore \quad \frac{dr}{dt} = \pm \frac{c}{\delta} \sqrt{(\delta^2-1) - \frac{\beta^2}{r^2}} \qquad (38)$$

Note that $\frac{dr}{dt} = 0$ at

$$\bar{r} = \frac{\beta}{\sqrt{\delta^2-1}} \qquad (38')$$

From (38),

$$\int \frac{dr}{\sqrt{(\delta^2-1) - \frac{\beta^2}{r^2}}} = \pm \frac{c}{\delta} \int dt$$

Solving for the case where $r = \bar{r}$ at some time $\bar{t}$, $0 < \bar{t} < t$, we get

$$(\delta^2-1)\, r^2 = \left[\frac{c(\delta^2-1)}{\delta} t - A \right]^2 + \beta^2 \qquad (39)$$

with

$$A \equiv \sqrt{(\delta^2-1)\, r_0^2 - \beta^2} \qquad (39')$$

Let us look more closely at the constant δ introduced in Eq.(37). From (33), we know

$$\frac{dt}{ds} = \alpha = \frac{\delta}{c} .$$

We know also that

$$ds = c\,d\tau \quad ,$$

where τ is the proper time of a clock moving with the particle. Therefore

$$\frac{dt}{ds} = \frac{dt}{c\,d\tau} = \frac{b}{c}$$

$$\frac{dt}{d\tau} = b$$

$$dt = b\,d\tau$$

Without making any approximations, Eq.(21) shows that t is the proper time measured by an observer situated at $r = 0$. But at that exact point, if we can imagine it, the observer is not "rotating" and his space is a flat Euclidean space; thus Special Relativity holds for him. If a particle is moving close to where he is located, he is approximately observing a particle moving freely in a flat space, and by Special Relativity, the relation between his time and the particle's proper time is given by

$$dt_o = \frac{d\tau_p}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where v is the velocity of the particle in a flat space. Since r = 0 is an allowable point in our rotating coordinate system, we can correctly interpret $\mathcal{L}$ as having the magnitude of the Lorentz factor. (Note that in the Schwarzschild case, one of the equations of motion of a free particle is [9]

$$\left(1 - \frac{2m}{r}\right) \frac{dt}{ds} = \mathcal{L} = \text{constant},$$

and as we go to a region where both particle and observer are at r very large, we see that $\mathcal{L}$ also unambiguously shows itself to be the Lorentz factor. We may state that, for a space whose metric approaches the flat space case in some region R, and in which the free particle geodesic equation of motion has the form

$$\frac{dt}{ds} = \text{constant} \times f(x^1, x^2, x^3) ,$$

where $f(x^1, x^2, x^3)$ is a function of the spatial coordinates whose value approaches 1 in R, the constant may unambiguously be taken to be the Lorentz factor.)

Without attaching any meaning to r, except that the coordinate does not close-in on itself, Eq.(39) tells us that a free particle's motion in this space is unbound.

From Eqs.(34), (33), and (37) we get

$$\frac{d\theta}{dt} = \dot{\theta}\frac{1}{\alpha} = \frac{\beta}{r^2\alpha} - \omega = \frac{c\beta}{r^2 b} - \omega \quad .$$

This can be solved (Appendix B) to yield

$$\theta = \theta_0 + \tan^{-1}\frac{c\beta t}{b r_0^2 - ctA} - \omega t \qquad (40)$$

Eqs.(39) and (40) give us r and θ as a function of coordinate time for the motion of a free particle in the space given by (1).

We now want to get all the parameters in terms of the observer's proper time. We take the unit of time to be a specified number of oscillations of the observer's atomic clock. We set the observer's position as

$$\begin{aligned} r_1 &= 1 \\ \theta_1 &= 0 \end{aligned} \quad , \qquad (41)$$

thereby setting our spatial scale.

In order to get all the remaining parameters in our equations - c, ω, r, θ_0, β, and b - in terms of proper time, we have to make a number of measurements. We are going to bounce signals off a free particle moving in our space. We can consider a particle moving such that $r_0 >> \beta$, and $\theta_0 = \frac{\pi}{2}$ (whatever that pictorially means - which is unimportant!) with $\theta(t)$ changing very little from θ_0 as

t increases. (We see here our justification for considering r_1 , r_2 >> d before.)

Our time-delay equation, with (41),

$$\tau = \frac{2}{c}\left\{ (1+r) - \frac{r}{(1+r)}\left[1+\cos\theta\right] + \right.$$

$$\left. \frac{\omega}{c}\sin\theta\,\frac{r}{(1+r)}\left[(1+r) - \frac{r}{(1+r)} - \frac{\cos\theta\, r}{(1+r)}\right]\right\} \quad (28)$$

where r and θ refer to the particle, can now be written as

$$\tau = \frac{2}{c}\left[1 + r + \frac{\omega}{c} r\theta\right] \quad . \quad (42)$$

D. & E.

Observables

The first measurement is by the observer at t = 0. By (42)

$$\tau_0 = \frac{2}{c}\left[1 + r_0 + \frac{\omega}{c} r_0 \theta_0\right]$$

$$\therefore \quad \theta_0 = \frac{c}{\omega r_0}\left[\frac{c\tau_0}{2} - 1 - r_0\right] \quad (43)$$

To avoid unwieldy mathematics, we place another observer at $r_{observer} << 1$ ($r_{observer} << 1$ means the observer is near the origin of the rotating coordinate

system. This position in space could be determined independently of our analysis - although its exact value could not be so determined (it is not needed) - for example, by initially picking a point as the origin and rotating our system with respect to it, etc.), and allow him too to make measurements. (Note that our progress up to this point indicates that all measurements could indeed be made by the single observer at $r = 1$, although the mathematical analysis would be much more complicated.) Eq.(21) shows us that

$$\tau = t$$

to first order in ω, so that both observer's clocks are "ticking" at the same rate - thus they can immediately relate their measurements to each other. Eq.(28) gives for the time delay as measured by an observer at $r_{observer} << 1$, $\bar{\tau}$,

$$\bar{\tau} = \frac{2}{c} r \qquad (44)$$

where r is the coordinate of the particle. (Note that this is by no means the Euclidean r.) We are perfectly justified in using measurements made at a point $r_{observer} << 1$, as long as we do not rely solely on $\bar{\tau}$ measurements to determine ω to sufficient accuracy.

At t = 0, the second observer $\bar{O}$, also makes a time delay measurement off the particle. By (44), at t = 0

$$\bar{\tau}_0 = \frac{2}{c} r_0$$

$$\therefore \quad r_0 = \frac{c}{2} \bar{\tau}_0 \equiv B c \qquad (45)$$

where $B \equiv \frac{c}{2}$.

By Eq.(44), $\bar{\tau}$ is a minimum when

$$\frac{dr}{dt} = \frac{d\bar{\tau}}{dt} = 0 \quad .$$

Eq.(38') tells us this occurs at a time t = T , where

$$\bar{r} = \frac{\beta}{\sqrt{b^2-1}}$$

Using Eq.(39) gives

$$(b^2-1)\,\bar{r}^2 = \left[\frac{c(b^2-1)}{b} T - A\right]^2 + \beta^2 = \beta^2$$

$$\therefore \quad \frac{c(b^2-1)}{b} T = A$$

$$\frac{c^2(b^2-1)^2}{b^2} T^2 = (b^2-1) r_0^2 - \beta^2$$

$$= (b^2-1)\frac{c^2\bar{\tau}_0^2}{4} - \beta^2$$

by (45).

$$\therefore \beta^2 = (\sigma^2-1)\frac{c^2\bar{r}_0^2}{4} - \frac{c^2(\sigma^2-1)^2}{\sigma^2}T^2$$

$$\beta^2 = c^2(\sigma^2-1)\left[\frac{\bar{r}_0^2}{4} - \frac{(\sigma^2-1)}{\sigma^2}T^2\right] \qquad (46)$$

Let $\sigma \equiv 1+\delta$

For particles not moving too fast, δ will be very small.

Therefore

$$\sigma^2-1 = 2\delta = \frac{\sigma^2-1}{\sigma} = \frac{\sigma^2-1}{\sigma^2}$$

At t = 1 unit of time, Eq.(39) gives

$$(\sigma^2-1)r_1^2 = \left[\frac{c(\sigma^2-1)}{\sigma} - A\right]^2 + \beta^2$$

where $r_1 = r(t=1)$

$$\therefore \quad 2\delta r_1^2 = [c2\delta - A]^2 + \beta^2$$

$$= c^2 4\delta^2 - 4c\delta A + 2\delta r_0^2$$

$$r_1^2 = c^2 2\delta - 2cA + r_0^2$$

$$r_1^2 = c^2 2\delta - 2^{\frac{3}{2}} c\sqrt{\delta}\, r_0 + r_0^2 \qquad (47)$$

By Eq.(44),

$$r_1 = \frac{c}{2}\,\bar{\tau}_1$$

where $\bar{\tau}_1$ is the proper time delay measured by $\bar{O}$ at t = 1. Substituting (45) and (48) into (47) yields

$$D = 2\delta - \sqrt{2\delta}\;\bar{\tau}_0 \qquad (49)$$

where

$$D \equiv \frac{\bar{\tau}_1^{\,2}}{4} - \frac{\bar{\tau}_0^{\,2}}{4} \,.$$

Rearranging (49) and squaring both sides gives

$$4\delta^2 - 2(2D + \bar{\tau}_0^{\,2})\,\delta + D^2 = 0$$

$$\therefore\quad \delta = \frac{2(2D+\bar{\tau}_0^{\,2}) \pm \sqrt{4(2D+\bar{\tau}_0^{\,2}) - 16D^2}}{8}$$

$$= \frac{1}{4}\left[(2D + \bar{\tau}_0^{\,2}) \pm \bar{\tau}_0\sqrt{4D + \bar{\tau}_0^{\,2}}\,\right]$$

$$= \frac{1}{8}\,(\bar{\tau}_1 \pm \bar{\tau}_0)^2 \,.$$

Reflection on our previous discussion of δ will show that the correct choice of sign is - . Therefore

$$\delta = \frac{1}{8}\,(\bar{\tau}_1 - \bar{\tau}_0)^2 \qquad (50)$$

Substituting (50) and (46) yields

$$\beta^2 = c^2(k^2-1)\left[\frac{\bar{\tau}_0^2}{4} - \frac{(k^2-1)}{k^2}T^2\right] \qquad (46)$$

$$= c^2 2\delta\left[\frac{\bar{\tau}_0^2}{4} - 2\delta T^2\right]$$

$$= \frac{c^2(\bar{\tau}_1-\bar{\tau}_0)^2}{4}\left[\frac{\bar{\tau}_0^2}{4} - \frac{(\bar{\tau}_1-\bar{\tau}_0)^2}{4}T^2\right]$$

$$= \frac{c^2(\bar{\tau}_1-\bar{\tau}_0)^2}{16}\left[\bar{\tau}_0^2 - (\bar{\tau}_1-\bar{\tau}_0)^2 T^2\right]$$

$$\therefore \beta = \left\{\frac{(\bar{\tau}_1-\bar{\tau}_0)^2}{16}\left[\bar{\tau}_0^2 - (\bar{\tau}_1-\bar{\tau}_0)^2 T^2\right]\right\}^{\frac{1}{2}} c$$

$$\equiv D\,c \qquad (51)$$

where

$$D \equiv \left\{\frac{(\bar{\tau}_1-\bar{\tau}_0)^2}{16}\left[\bar{\tau}_0^2 - (\bar{\tau}_1-\bar{\tau}_0)T^2\right]\right\}^{\frac{1}{2}} \qquad (51')$$

Again, at t = 1 (47) and (45) give

$$r_1^2 = c^2 2\delta - 2^{\frac{3}{2}} c\sqrt{\delta}\, r_0 + r_0^2 \qquad (47)$$

$$= c^2 2\delta - \sqrt{2}\, c^2\sqrt{\delta}\,\bar{\tau}_0 + \frac{c^2\bar{\tau}_0^2}{4}$$

$$= \left(2\delta - \sqrt{2}\,\delta\,\bar{\tau}_0 + \frac{\bar{\tau}_0^2}{4}\right) c^2$$

$$\therefore r_1 = \left(2\delta - \sqrt{2\delta}\,\bar{\tau}_0 + \frac{\bar{\tau}_0^2}{4}\right)^{\frac{1}{2}} c \equiv E\,c \qquad (52)$$

where

$$E \equiv \left(2\delta - \sqrt{2\delta}\,\bar{\tau}_0 + \frac{\bar{\tau}_0^2}{4}\right)^{\frac{1}{2}} \qquad (52')$$

Eq.(40) becomes, for t = 1

$$\theta_1 = \theta_0 + \frac{c\beta}{br_0^2 - ctA} - \omega \quad .$$

Using (51) and (45) yields

$$\theta_1 = \theta_0 + \frac{Dc^2}{(1+\delta)\frac{c^2\bar{\tau}_0^2}{4} - c^2\sqrt{\frac{\delta}{2}}\,\bar{\tau}_0} - \omega$$

$$\theta_1 = \theta_0 - \omega + \frac{D}{(1+\delta)\frac{\bar{\tau}_0^2}{4} - \sqrt{\frac{\delta}{2}}\,\bar{\tau}_0} = \theta_0 - \omega + F \qquad (53)$$

where

$$F \equiv \frac{D}{(1+\delta)\frac{\bar{\tau}_0^2}{4} - \sqrt{\frac{\delta}{2}}\,\bar{\tau}_0} \quad . \qquad (53')$$

Therefore the proper time delay at $r_1 = 1$ at $t = 1$ is given by (42) as

$$\tau_1 = \frac{2}{c}\left[1 + r_1 + \frac{\omega}{c} r_1 \theta_1\right]$$

which, with the aid of (52), (53), (45), and (43), becomes

$$\frac{c\tau_1}{2} = 1 + r_1 + \frac{\omega}{c} r_1 \theta_1 = 1 + Ec + \omega E \theta_1$$

$$= 1 + Ec + \omega E\left(\theta_0 - \omega + F\right)$$

$$= 1 + Ec + \omega EF + \frac{Ec}{r_0}\left[\frac{c\tau_0}{2} - 1 - r_0\right]$$

$$\therefore \quad \left[\frac{\tau_1}{2E} - 1 - \frac{(\tau_0 - \bar{\tau}_0)}{\bar{\tau}_0}\right] c = \left(\frac{1}{E} - \frac{2}{\bar{\tau}_0}\right) + F\omega$$

or

$$G\,c = H + F\omega \tag{54}$$

with

$$G \equiv \frac{\tau_1}{2E} - 1 - \frac{(\tau_0 - \bar{\tau}_0)}{\bar{\tau}_0}$$

$$H \equiv \frac{1}{E} - \frac{2}{\bar{\tau}_0} \tag{54'}$$

We now consider another measurement at $r_1 = 1$, at $t = \delta t \equiv t'$. At $t = t'$, r and θ can be taken as

$$r = r_0 - \delta r$$

$$\theta = \theta_0 + \delta\theta \tag{55}$$

Eq.(39) becomes

$$2\delta r^2 = [2\delta c t' - A]^2 + \beta^2$$

giving

$$r^2 = (r_0 - \delta r)^2 = 2\delta c^2 t'^2 - 2ct'A + r_0^2 \,,$$

which finally yields

$$\delta r = \frac{2}{c\bar{\tau}_0}\left[c^2 t' \sqrt{\frac{\delta}{2}}\,\bar{\tau}_0 - \delta c^2 t'^2\right]$$

$$\delta r = \frac{2}{\bar{\tau}_0}\left[t'\sqrt{\frac{\delta}{2}}\,\bar{\tau}_0 - \delta t'^2\right] c \equiv I c \qquad (56)$$

where

$$I \equiv \frac{2}{\bar{\tau}_0}\left[t'\sqrt{\frac{\delta}{2}}\,\bar{\tau}_0 - \delta t'^2\right] \qquad (56')$$

Eq. (40) gives

$$\delta\theta = \frac{c\beta t'}{\delta r_0^2 - c t' A} - \omega t'$$

Analagous to Eq.(53), we get

$$\delta\theta = \frac{D t'}{(1+\delta)\frac{\bar{\tau}_0^2}{4} - t'\sqrt{\frac{\delta}{2}}\,\bar{\tau}_0} - \omega t' \equiv J - \omega t' \qquad (57)$$

where

$$J \equiv \frac{D t'}{(1+\delta)\frac{\bar{\tau}_0^2}{4} - t'\sqrt{\frac{\delta}{2}}\,\bar{\tau}_0} \,. \qquad (57')$$

Combining Eqs.(42), (43), (55), (56), and (57), the proper time delay τ' at t = t' can be written as

$$\tau' = \tau_0 - \delta\tau = \frac{2}{c}\left[1 + r_0 - \delta r + \frac{\omega}{c}(r_0 - \delta r)(\theta_0 + \delta\theta)\right]$$

$$\therefore \quad \frac{c\,\delta\tau}{2} = \delta r - \frac{\omega}{c} r_0\,\delta\theta + \frac{\omega}{c}\delta r\,\theta_0$$

$$= I c - \omega \left(\frac{\bar{\tau}_0}{2} J\right) + \frac{2I}{\bar{\tau}_0}\left[\frac{c}{2}(\tau_0 - \bar{\tau}_0) - 1\right]$$

which gives

$$\left[\frac{\delta\tau}{2} - I - \frac{I}{\bar{\tau}_0}(\tau_0 - \bar{\tau}_0)\right] c + \left(\frac{\bar{\tau}_0}{2} J\right)\omega + \frac{2I}{\bar{\tau}_0} = 0$$

or

$$K c - L \omega + N = 0 \qquad (58)$$

where

$$K \equiv \frac{\delta\tau}{2} - I - \frac{I}{\bar{\tau}_0}(\tau_0 - \bar{\tau}_0)$$

$$L \equiv -\frac{\bar{\tau}_0}{2} J \qquad N \equiv \frac{2I}{\bar{\tau}_0}$$

Combining Eqs.(54) and (58),

$$G c = H + F \omega \qquad (54)$$

$$K c - L \omega + N = 0 \qquad (58)$$

yields

$$\omega = \frac{\frac{KH}{G} + N}{L - \frac{KF}{G}} \qquad (59)$$

and

$$C = \frac{H}{G} + \frac{F}{G}\,\frac{\left(\frac{KH}{G} + N\right)}{\left(L - \frac{KF}{G}\right)} \qquad (60)$$

Recalling Eqs.(51), (50), (45), and (43)

$$\beta = D\,C \qquad (51)$$

$$\delta = \frac{1}{8}\left(\bar{\tau}_1 - \bar{\tau}_0\right)^2 \qquad (\delta = 1+\delta) \qquad (50)$$

$$r_0 = B\,C \qquad (45)$$

$$\theta_0 = \frac{C}{\omega r_0}\left[\frac{C\tau_0}{2} - 1 - r_0\right] , \qquad (43)$$

we see that we have successfully expressed all our parameters in terms of proper time; for K, H, G, N, L, F, D, and B, are functions only of τ_0 , $\bar{\tau}_0$, T , $\bar{\tau}_1$, τ_1 , t', and δT , where:

τ_0 = time delay measured by observer 0 (r_1=1, θ_1=0) at t = 0.

$\bar{\tau}_0$ = time delay measured by observer $\bar{0}$ ($r_{obs.} \ll 1$) at t = 0.

T = time that minimum time delay occurs, as measured by $\bar{0}$.

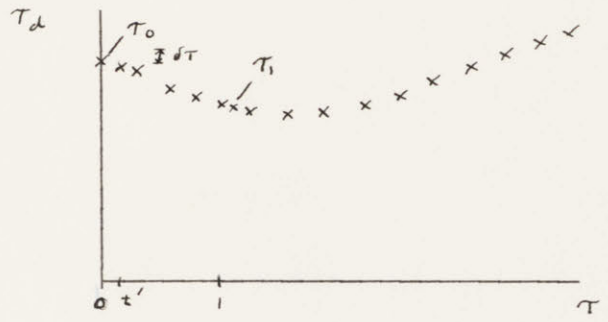
$\bar{\tau}_1$ = time delay measured by $\bar{0}$ at t = 1.

τ_1 = time delay measured by O at t = 1.

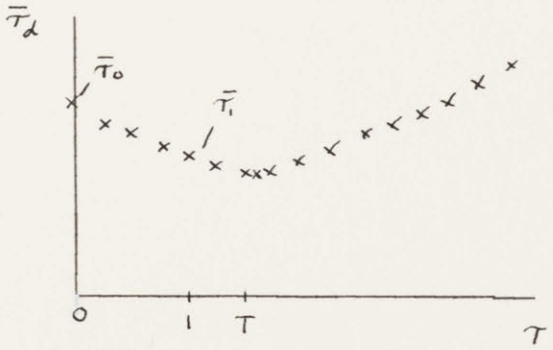
$\delta\tau$ = change in time delay from t = 0 to $\underline{t = \delta t = t'}$ as measured by O.

(We label this set of measurements $\{T\}$.)

We can show our measurements pictorially. We plot the proper time delay τ_d as a function of the clock's proper time τ . For O, a typical plot might look like (We can ignore the particle's motion in the time span of a single time delay.) -



and for $\bar{O}$ -



We started with a space labelled by a certain set of coordinates, and by writing our equations in terms of these coordinates we were able to completely "eliminate" the coordinates in this space in terms of proper-time observables $\{T\}$: Substituting Eqs.(59), (60), (51), (50), (45), and (43) into Eqs.(39) and (40) gives a complete determination and unambiguous interpretation and meaning to r and θ , in terms of $\{T\}$. (Our analysis was restricted to motion on a z plane, for convenience. For more general motions, z is also given in terms of observables.)

Conclusion

We have shown the role played by coordinates in General Relativity, and their lack of reality except in terms of observers' proper time measurements. We have established a criterion for interpreting the coordinate system in a space. And we have shown that we can successfully analyze and thereby interpret the rotating coordinate system.

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Appendix A

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & \frac{-\omega r^2}{c} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{-\omega r^2}{c} & 0 & \left(1 - \frac{r^2\omega^2}{c^2}\right) \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 0 \end{matrix}$$

$$g^{ik} = \frac{A_{ik}}{g} = g^{ki}$$

$$g = -r^2\left(1 - \frac{r^2\omega^2}{c^2}\right) - \left(\frac{\omega r^2}{c}\right)^2 = -r^2$$

$\therefore$ Non zero elements of g^{ik} :

$$g^{11} = -1 \qquad g^{22} = \frac{-\left(1 - \frac{r^2\omega^2}{c^2}\right)}{r^2} \qquad g^{33} = -1 \qquad g^{00} = 1$$

$$g^{02} = g^{20} = -\frac{\omega}{c}$$

$$\Gamma^{i}_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right) = \Gamma^{i}_{lk}$$

The non zero elements of $\frac{\partial g_{ij}}{\partial x^k}$ are, with $g'_{ij} = \frac{\partial g_{ij}}{\partial r}$

$$g_{22}' = -2r \qquad g_{00}' = -\frac{2rw^2}{c^2} \qquad g_{20}' = g_{02}' = -\frac{2wr}{c}$$

We have to compute the following elements:

Γ^{i}_{11} Γ^{i}_{22} Γ^{i}_{33} Γ^{i}_{00}

Γ^{i}_{12} Γ^{i}_{23} Γ^{i}_{30}

Γ^{i}_{13} Γ^{i}_{20}

Γ^{i}_{10}

$$\Gamma^{i}_{kl} = \frac{1}{2} g^{i1}\left(\frac{\partial g_{1k}}{\partial x^l} + \frac{\partial g_{1l}}{\partial x^k} - \frac{\partial g_{kl}}{\partial r}\right) + \frac{1}{2} g^{i2}\left(\frac{\partial g_{2k}}{\partial x^l} + \frac{\partial g_{2l}}{\partial x^k}\right) + \frac{1}{2} g^{i3}\left(\frac{\partial g_{3k}}{\partial x^l} + \frac{\partial g_{3l}}{\partial x^k}\right) + \frac{1}{2} g^{i0}\left(\frac{\partial g_{0k}}{\partial x^l} + \frac{\partial g_{0l}}{\partial x^k}\right)$$

$$i = 1 : \qquad \Gamma^{1}_{kl} = -\frac{1}{2}(1)\left(-\frac{\partial g_{kl}}{\partial r}\right)$$

$$\Gamma^{1}_{11} = -\tfrac{1}{2}(1)(0) = 0 \qquad \Gamma^{1}_{22} = \tfrac{1}{2}(1)(-2r) = -r$$

$$\Gamma^{1}_{12} = -\tfrac{1}{2}(1)(0) = 0 \qquad \Gamma^{1}_{23} = -\tfrac{1}{2}(1)(0) = 0$$

$$\Gamma^{1}_{13} = -\tfrac{1}{2}(1)(0) = 0 \qquad \Gamma^{1}_{20} = \tfrac{1}{2}(1)\left(-\frac{2wr}{c}\right) = -\frac{wr}{c}$$

$$\Gamma^{1}_{10} = -\tfrac{1}{2}(1)(0) = 0 \qquad \Gamma^{1}_{33} = -\tfrac{1}{2}(1)(0) = 0$$

$$\Gamma^1_{30} = -\tfrac{1}{2}(1)(0) = 0$$

$$\Gamma^1_{00} = \tfrac{1}{2}(1)\left(-\frac{2r\omega^2}{c^2}\right) = -\frac{r\omega^2}{c^2}$$

$i = 2$: $$\Gamma^2_{kl} = -\frac{\left(1-\frac{r^2\omega^2}{c^2}\right)}{2r^2}\left(\frac{\partial g_{2k}}{\partial x^l} + \frac{\partial g_{2l}}{\partial x^k}\right) - \frac{\omega}{2c}\left(\frac{\partial g_{0k}}{\partial x^l} + \frac{\partial g_{0l}}{\partial x^k}\right)$$

$$\Gamma^2_{11} = 0$$

$$\Gamma^2_{12} = \frac{\left(1-\frac{r^2\omega^2}{c^2}\right)}{2r^2}(0+2r) + \frac{\omega}{2c}\left(0+\frac{2\omega r}{c}\right) = \frac{1}{r}$$

$$\Gamma^2_{13} = 0$$

$$\Gamma^2_{10} = \frac{\left(1-\frac{r^2\omega^2}{c^2}\right)}{2r^2}\left(0+2\frac{\omega r}{c}\right) + \frac{\omega}{2c}\left(0+\frac{2r\omega^2}{c^2}\right) = \frac{\omega}{rc}$$

$$\Gamma^2_{22} = 0 \qquad \Gamma^2_{33} = 0$$

$$\Gamma^2_{23} = 0 \qquad \Gamma^2_{30} = 0$$

$$\Gamma^2_{20} = 0 \qquad \Gamma^2_{00} = 0$$

$i = 3$: $$\Gamma^3_{kl} = -\frac{1}{2}\left(\frac{\partial g_{3k}}{\partial x^l} + \frac{\partial g_{3l}}{\partial x^k}\right)$$

all $\Gamma^3_{kl} = 0$

$i = 0$: $\Gamma^{0}_{kl} = -\frac{\omega}{2c}\left(\frac{\partial g_{2k}}{\partial x^{l}} + \frac{\partial g_{2l}}{\partial x^{k}}\right) + \frac{1}{2}\left(\frac{\partial g_{0k}}{\partial x^{l}} + \frac{\partial g_{0l}}{\partial x^{k}}\right)$

$\Gamma^{0}_{11} = 0$

$\Gamma^{0}_{12} = \frac{\omega}{2c}(0+2r) - \frac{1}{2}(0+\frac{2\omega r}{c}) = 0$

$\Gamma^{0}_{13} = 0$

$\Gamma^{0}_{11} = \frac{\omega}{2c}(0+\frac{2\omega r}{c}) - \frac{1}{2}(0+2r\frac{\omega^{2}}{c^{2}}) = 0$

$\Gamma^{0}_{22} = 0$ $\qquad$ $\Gamma^{0}_{33} = 0$

$\Gamma^{0}_{23} = 0$ $\qquad$ $\Gamma^{0}_{30} = 0$

$\Gamma^{0}_{20} = 0$ $\qquad$ $\Gamma^{0}_{00} = 0$

$\therefore$ Non zero elements are:

$\Gamma^{1}_{22} = -r$ $\qquad$ $\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r}$

$\Gamma^{1}_{20} = \Gamma^{1}_{02} = -\frac{\omega r}{c}$ $\qquad$ $\Gamma^{2}_{10} = \Gamma^{2}_{01} = \frac{\omega}{rc}$

$\Gamma^{1}_{00} = -\frac{r\omega^{2}}{c^{2}}$

Appendix B

$$\frac{d\theta}{dt} = \frac{c\beta}{r^2 b} - \omega$$

$$\int_{\theta_0}^{\theta} d\theta = \frac{c\beta}{b} \int_0^t \frac{dr}{r^2} - \omega \int_0^t dt$$

$$= \frac{c\beta}{b} (b^2-1) \int_0^t \frac{dt}{(xt-A)^2+\beta^2} - \omega t$$

by Eq.(39), with

$$x \equiv \frac{c(b^2-1)}{b} \qquad A \equiv \sqrt{(b^2-1) r_0^2 - \beta^2}$$

$$\theta - \theta_0 = \left[\tan^{-1} \frac{xt-A}{\beta} \right]_0^t - \omega t$$

$$= \tan^{-1} \frac{xt-A}{\beta} + \tan^{-1} \frac{A}{\beta} - \omega t$$

$$\tan(\theta - \theta_0 + \omega t) = \tan(P+Q)$$

$$= \frac{\tan P + \tan Q}{1 - \tan P \tan Q}$$

where

$$P \equiv \tan^{-1} \frac{xt-A}{\beta} \qquad Q \equiv \tan^{-1} \frac{A}{\beta}$$

$$\therefore \tan(\theta - \theta_0 + \omega t) = \frac{\frac{xt-A}{\beta} + \frac{A}{\beta}}{1 - \frac{(xt-A)A}{\beta^2}}$$

$$= \frac{xt\beta}{\beta^2 + A^2 - xtA} = \frac{ct\beta}{b r_0^2 - ctA}$$

$$\therefore \quad \theta = \theta_0 + \tan^{-1} \frac{c\beta t}{b r_0^2 - c t A} - \omega t$$

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