Generalized Navier–Stokes equations for active turbulence

by

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Abstract

Recent experiments show that active fluids stirred by swimming bacteria or ATP-powered microtubule networks can exhibit complex flow dynamics and emergent pattern scale selection. Here, I will investigate a simplified phenomenological approach to ‘active turbulence’, a chaotic non-equilibrium steady-state in which the solvent flow develops a dominant vortex size. This approach generalizes the incompressible Navier–Stokes equations by accounting for active stresses through a linear instability mechanism, in contrast to externally driven classical turbulence. This minimal model can reproduce experimentally observed velocity statistics and is analytically tractable in planar and curved geometry. Exact stationary bulk solutions include Abrikosov-type vortex lattices in 2D and chiral Beltrami fields in 3D. Numerical simulations for a plane Couette shear geometry predict a low viscosity phase mediated by stress defects, in qualitative agreement with recent experiments on bacterial suspensions. Considering the active analog of Stokes’ second problem, our numerical analysis predicts that a periodically rotating ring will oscillate at a higher frequency in an active fluid than in a passive fluid, due to an activity-induced reduction of the fluid inertia. The model readily generalizes to curved geometries. On a two-sphere, we present exact stationary solutions and predict a new type of upward energy transfer mechanism realized through the formation of vortex chains, rather than the merging of vortices, as expected from classical 2D turbulence. In 3D simulations on periodic domains, we observe spontaneous mirror-symmetry breaking realized through Beltrami-like flows, which give rise to upward energy transfer, in contrast to the classical direct Richardson cascade. Our analysis of triadic interactions supports this numerical prediction by establishing an analogy with forced rigid body dynamics and reveals a previously unknown triad invariant for classical turbulence.

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Exact periodic solutions of Eqs. (3.1) include inviscid vortex lattices. (a) Linear stability analysis of Eqs. (3.1), for a square domain with periodic boundary conditions identifies three types of Fourier modes: dissipative (blue), active (red), and neutral (black circles). Neutral modes can be combined through the superposition (3.2) to form exact stationary stress-free solutions of the nonlinear Eqs. (3.1). (b) Square lattice solution, \( \psi = \cos(\frac{kx}{\sqrt{2}}) \cos(\frac{ky}{\sqrt{2}}) \), corresponding to the green squares in panel (a). (c) Hexagonal Kagome lattice solution, \( \psi = 2 \cos(\frac{ky}{2}) \cos(\frac{\sqrt{3}kx}{2}) - \cos(ky) \), corresponding to the orange hexagons in (a). This flow topology is very similar to Abrikosov lattices found in quantum superfluids, cf. Figures in [1, 109]. (d) Triangular lattice solution, \( \psi = -\cos(ky) - \cos(k\frac{\sqrt{3}x-y}{2}) - \sin(k\frac{\sqrt{3}x+y}{2}) \), also corresponding to the orange hexagons in (d), but with different mode amplitudes. In all three cases, \( k \) can be chosen as the radius of the inner or the outer inviscid circle (black) in panel (a).
Active shear flows exhibit qualitatively different dynamics, velocity statistics and symmetry-breaking behavior depending on confinement geometry \((L_x, L_y)\) and applied shear speed \(V\) (see also Fig. 3-3). (a) For wide channels of width \(L_y = 5\Lambda\) and weak shear \(V = 0.57U\), typical flow configurations realize advectively mixed vortex lattices (App. A). The characteristic circulation speed \(U\) and growth-time scale \(\tau\) of the bulk vortices are defined in Chapter 2. Scale bar shows bulk vortex size \(\Lambda\) and colormap encodes vorticity. (b-c) For narrow channels with \(L_y = 2.2\Lambda\) and strong shear \(V = 1.65U\), the active fluid exhibits spontaneous symmetry-breaking resulting in unidirectional transport of the fluid’s center of mass (App. A and Fig. A-2). Depending on initial conditions, qualitatively different low-energy (b; App. A) and high-energy (c; App. A) flow states can arise for identical system parameters. (d-e) The kinetic energy time series \(E(t)\) for the simulations in panels (a-c) illustrate relaxation to statistically stationary states. The center-of-mass velocity \(V_{CM}(t)\) indicates persistent macroscopic average flows through the channel (Fig. A-2). (f-g) The spatially averaged mean shear stresses \(\Sigma^\pm(t)\), rescaled by kinematic viscosity \(\Gamma_0\) and shear rate \(\gamma\), reveal top-bottom symmetry-breaking in narrow channels (b,c,g) and exhibit large temporal variability, resulting in a substantial variance of the effective shear viscosity (see also Fig. A-3). Vertical dotted lines indicate the start time \(T\) of the temporal averaging periods for the results depicted in Fig. 3-3(a-c). (h-i) Shear stress histograms constructed from the time series in (f-g) for \(t > T\) reflect the top-bottom flow asymmetry in narrow channels.
Numerical simulations of Eq. (3.1) identify the conditions for spontaneous left-right symmetry-breaking and low-viscosity states. (a) The relative mean kinetic energy of the fluid's center-of-mass signals spontaneous symmetry-breaking. Averages over 10 simulations for each of the 208 simulated parameter pairs (markers) were connected using spline interpolations. Black dots: simulations settle into a single class of statistically stationary kinetic energy ground-states. Red circles: In addition to the ground-state, long-lived excited states are observed for randomly sampled initial conditions (App. A.1). Gray triangles indicate the occurrence of dynamical symmetry-breaking characterized by uni-directional fluid transport with average speed $> 0.25U$ persisting over time-scales $> 100\tau$. $\Lambda$ and $\tau$ are the characteristic bulk vortex size and the characteristic timescale in an equivalent system with periodic boundary conditions. The black diagonal line marks the corresponding characteristic flow speed $U$, separating the low-shear from the high-shear regime. (b) The dark blue domains in the effective viscosity phase diagram correspond to quasi-inviscid parameter regimes. (c) Vertical cut through panel (b) at constant shear rate $\dot{\gamma} = 2.7\tau^{-1}$ showing oscillatory behavior of the shear viscosity with boundary separation $L_y$. Viscosity fluctuations are maximal when an integer number $n$ of vortices fits between the boundaries, $L_y \approx n\Lambda$. (d) Horizontal cut through panel (b) at constant separation $L_y = 3\Lambda$, illustrating the suppression of viscosity fluctuations at high shear (see also Fig. A-3). (e-f) Representative flow fields, with local speed (top) and vorticity (center) shown as background, and corresponding stress fields (bottom) in the low-viscosity regime (e; App. A) and the high-viscosity regime (f; App. A). These simulations were performed at the same shear rate but different boundary separations, as indicated in panels (b,c). The spectral norm $||\sigma||_2$, corresponding to the largest eigenvalue of the stress tensor $\sigma$, and the associated director field reveal the presence of zero-stress defects in the bulk as well as half-loops in the stress-field lines along the edges for the low-viscosity states (e, bottom; App. A).
3-4 Transition from a low-viscosity to a normal fluid by changing the activity parameter $\Gamma_2 = \Gamma_2^* + \delta \Gamma_2$, starting from the quasi-inviscid state with $\Gamma_2^* < 0$ in Fig. 3-3(e) and keeping $L_y = 3\Lambda$ and $\dot{\gamma} = 2.7\tau^{-1}$ fixed. (a) Increasing $\Gamma_2$ via $\delta \Gamma_2$ corresponds to an effective reduction in activity. As the activity is decreased, the effective shear viscosity first increases before dropping to the value $\nu/\Gamma_0 = 1$, expected for a passive fluid with kinematic viscosity $\Gamma_0$. (b) Shear-stress histograms for the colored points in (a) show the transition from a low viscosity flow (blue) to normal laminar flow (red) through a highly viscous state (green). In the vicinity of the critical point (orange) the fluid can fluctuate between low-stress and high-stress states. Histograms and mean values were sampled from 12 long runs for each value of $\delta \Gamma_2$.

3-5 Decreasing the aspect ratio $\alpha = L_x/L_y$ stabilizes flow states capable of performing mechanical work. (a-b) Steady-state flows for a narrow channel $L_y = 2\Lambda$ and moderate shear $\dot{\gamma} = 0.77\tau^{-1}$, shown for two different aspect ratios: (a) $\alpha = 5$ and (b) $\alpha = 3$. (c) The kinetic energy time series indicates that, for $\alpha = 3$, the flow locks into a time-independent steady state, in which fluctuations are completely suppressed by the no-slip shear boundary conditions. (d) Shear stresses $\Sigma^\pm(t)$ acting on the top (+) and bottom (−) boundary for the simulations in (a-b) yield a negative effective viscosity in both cases, implying that the fluid is pushing the boundaries. This negative-viscosity effect is enhanced for the stationary state observed at smaller aspect ratios (b).
4-1 Typical flow and stress fields for an active fluid with vortex size $\Lambda$ and wide vortex-size distribution $\kappa_w = 1.5/\Lambda$, confined to a planar disk geometry (radius $R = 2.67\Lambda$) with boundary held fixed. The presence of the stress-free defects allows the stress director field to develop complex configurations, enabling a nontrivial response to time-dependent boundary conditions, see Figs. 4-2 and 4-3.

4-2 Stokes' second problem for a ring-shaped container pendulum coupled to a torsional spring. Response of active fluids with large (a,b: $\kappa_w = 1.5/\Lambda$) and small (c,d: $\kappa_s = 0.63/\Lambda$) vortex-size distributions to oscillatory boundary conditions (App. B). The boundary speed is sinusoidal with amplitude $A$ and angular frequency $\Omega$. (b,d) The activity-induced relative change $\lambda$ in the effective inertia experienced by the ring pendulum. Negative values of $\lambda$ imply that the pendulum oscillates at a higher frequency in an active fluid than in a passive fluid.
Einstein–de Haas analogue effect for an active fluid with narrow vortex-size distribution $\kappa_s = 0.63/\Lambda$. When isolated, the fluid can significantly shake the enclosing container, a thin rigid ring of radius $R$. (a) The standard deviation $\sigma_L$ of the container angular momentum $L$ depends on the radius $R$ and the fluid-to-ring mass ratio $\alpha$. The fluctuations $\sigma_L$ are independent of $\alpha$ for heavy containers ($\alpha \ll 1$) but start to decrease monotonically with $\alpha$ when the containers become light ($\alpha \gg 1$). As $R$ varies, the fluctuations oscillate with the period set by the characteristic vortex scale $\Lambda$ (see also App. B). Black dots represent 323 simulated parameter pairs, the color code shows linear interpolation. (b) Representative time series of the container angular momentum for two different radii $R = 3.67\Lambda$ and $R = 3.33\Lambda$ and fixed mass ratio $\alpha = 1$. (c) The standard deviation $\sigma_\phi \sim \alpha \sigma_L$ of the container angular speed $\dot{\phi}$. (d) Horizontal cuts through (a) and (c) at a constant radius of $R = 4\Lambda$. In particular, to maximize both the angular momentum and velocity fluctuations, the fluid mass should match the container mass ($\alpha \sim 1$).

Stationary solutions of Eqs. (5.2) are superpositions of the form (5.3) with $f(-\ell(\ell + 1) + 4) = 0$. (a) An exact stationary solution with $\ell = 6$ which is also approximately realized as a transient state in the time-dependent burst solution of Fig. 5-2. (b) Complex symmetric solutions can be constructed by choosing the expansion coefficients $\psi_{ml}$ accordingly [159]. In both panels, the stream functions are normalized by their maxima; see App. C for coefficients $\psi_{ml}$.
Phase diagrams (a,b) and representative snapshots (c-e) from simulations showing quasi-stationary burst dynamics (B-phase), anomalous vortex-network turbulence (A-phase), and classical 2D turbulence (T-phase). (a,b) The A- and T-phase are approximately separated by the condition $\kappa\Lambda = 1$ (vertical dashed line) and differ by the average number of vortices (a), the geometry of the branches in the tension field (b), and the energy spectra (Fig. 5-3). The B-phase arises for narrowband energy injection $\kappa R \lesssim 1$ when only a single $\ell$-mode is active (region right below the dashed-dotted line); decreasing $\kappa$ further gives a passive fluid (white region). (c-e) Top: Instantaneous vorticity fields normalized by their maxima. Bottom: Surface tension fields normalized by the maximum deviation from the mean. (c) Quasi-stationary pre-burst state resembling the exact solution in Fig. 5-1(a). (d) For subcritical curvature and intermediate energy injection bandwidths, $R^{-1} < \kappa < \Lambda^{-1}$, the flows develop a percolating vortex-chain network structure, with an accumulation of tension and vorticity along the edges. (e) For broadband energy injection $\kappa\Lambda > 1$, smaller eddies merge to create larger vortices, as typical of classical 2D turbulence. Parameters: (a) $\alpha = 0.5$; (c) $R/\Lambda = 2$, $\tau = 4.9$ s, $\kappa\Lambda = 0.29$; (d) $R/\Lambda = 10$, $\tau = 14.9$ s, $\kappa\Lambda = 0.5$; (e) $R/\Lambda = 10$, $\tau = 11.7$ s, $\kappa\Lambda = 2.0$. Panels (a, b) show steady-state time averages over $[50\tau, 100\tau]$. Solid curves in (c-e) indicate streamlines of the velocity fields.
Time-averaged energy spectra and fluxes indicate two qualitatively different types of upward energy transport. (a) For narrowband energy injection \( \kappa \Lambda < 1 \), the energy spectrum exhibits a peak corresponding to the dominant vortex size \( \Lambda \) (red curve). For broadband injection \( \kappa \Lambda \sim 2 \), the spectra decay monotonically (blue and black curves). (b) In all four examples, the fluxes confirm inverse energy transport, albeit with different origins. For broadband energy injection (blue and black curves), the upward energy flux to larger scales is due to vortex mergers [Fig. 5-2(e)]. By contrast, for narrowband injection (red curve), a relatively stronger upward energy flux arises from the collective motion of vortex chains [Fig. 5-2(d)]. The shaded regions indicate the energy injection ranges with colors matching those of the corresponding curves, respectively. Parameters: \( R/\Lambda = 10 \) for a unit sphere, \( \tau = 11.7s \), time step \( 5 \cdot 10^{-4} \tau \), total simulation time \( 500\tau \). Spectra and fluxes were determined after relaxation by averaging over \([150\tau, 500\tau]\). For \( \kappa \Lambda \gg 1 \), energy steadily accumulates at larger scales and the absence of a large-scale dissipative mechanism leads to a divergent total enstrophy and kinetic energy on the sphere.
6-1 Exact Beltrami-flow solutions and spontaneous mirror symmetry breaking in 3D simulations. (a) Linear stability analysis of Eqs. (1.2) distinguishes three different regions in Fourier space: Domains I and III are dissipative, whereas II represents active modes. The radius of the active shell II corresponds approximately to the inverse of characteristic pattern formation scale $\Lambda$. The bandwidth $\kappa$ measures the ability of the active fluid component to concentrate power input in Fourier space. (b) Two examples of exact stationary bulk solutions of Eqs. (1.2) realizing Beltrami vector fields of opposite helicity, obtained from Eq. (6.4) by combining modes of the same helicity located on one of the marginally stable grey surfaces in (A). (c) Simulations with random initial condition spontaneously select one of two helicity branches. The histogram represents an average over 150 runs with random initial conditions, sampled over the statistically stationary state starting at time $t = 20\tau$ (dashed line). Simulation parameters: $\Lambda = 75\mu m$, $U = 72\mu m/s$, $\kappa_1 = 0.9/\Lambda$, $L = 8\Lambda$ (see also Fig. 6-2 and App. D for larger simulations).
Active fluids spontaneously break mirror symmetry by realizing Beltrami-type flows. (a) Snapshot of a representative vorticity component field \( \omega_x \) for an active fluid with small bandwidth \( \kappa_S = 0.63/\Lambda \), as defined in Fig. 6-1(a). (b) The corresponding helicity field signals parity-symmetry breaking leading to a positive-helicity flow in this example. (c) Histograms of the angles between velocity \( \mathbf{v} \) and vorticity \( \mathbf{\omega} \) quantify the alignment between the two fields for different active bandwidths \( \kappa_S < \kappa_1 < \kappa_W \): The smaller the bandwidth, the stronger the alignment between \( \mathbf{v} \) and \( \mathbf{\omega} \). (d) Numerically estimated distributions of the Beltrami measure, \( \beta = \mathbf{v} \cdot \mathbf{\omega} / (\lambda |\mathbf{v}|^2) \), shown on a log-scale. An ideal Beltrami flow with \( \omega = \lambda \mathbf{v} \) produces a delta-peak centered at \( \beta = 1 \). Identifying \( \lambda \) with the midpoint of the active shell (\( \lambda \approx \pi/\Lambda \)), which approximately corresponds to the most unstable wavenumber and the characteristic pattern formation scale, we observe that a smaller active bandwidth leads to a sharper peak and hence more Beltrami-like flows. Data were taken at a single representative time-point long after the characteristic relaxation time. Simulation parameters: \( \Lambda = 75 \mu m \), \( U = 72 \mu m/s \), \( L = 32\Lambda \).
Scale selection controls mirror symmetry breaking and induces an inverse energy cascade. We demonstrate these effects for active fluids with a (a-d) small active bandwidth $\kappa_S$ and (e-h) wide bandwidth $\kappa_W$ [Fig. 6-1(a)]. The intermediate case $\kappa_I$ is presented in Fig. S3. (a) Energy spectra $e^{\pm}(k)$ of the helical velocity-field modes show strong symmetry breaking for small bandwidth parameter $\kappa_S$. In this example, the system spontaneously selects positive helicity modes, such that $e^+(k) > e^-(k)$ at all dominant wavenumbers. Dashed vertical lines indicate the boundaries of the energy injection domain II. (b) The resulting energy fluxes $\Pi^{\pm}(k)$ combine into the total flux $\Pi(k)$, which is negative in region I and positive in III, signaling inverse and forward energy cascades, respectively. (c) Contributions to the energy flow $\langle T_{klpq}^{spq} \rangle$ between the three spectral domains I,II, and III (18 possibilities, columns) from the eight types of triad interactions (rows). In reflection-invariant turbulence, this table remains unchanged under upside-down flipping (+ ↔ -). Instead, we observe a strong asymmetry, with two cumulative triads (d) dominating the energy transfer. Red and blue arrows represent transfer towards large and small scales, respectively, and thickness represents the magnitude of energy flow. Green arrows represent transfer within the same spectral domain. The direction of the energy flow is in agreement with the instability assumption of Waleffe [216]. In this case, 18.2% of the injected energy is transferred from region II to region I and 81.8% is transferred from II to III. (e-h) Same plots for an active fluid with wide active bandwidth $\kappa_W$. (e and f) Energy spectra show weaker parity breaking (e) and suppression of the inverse energy cascade (f). (g) The energy flow table partially recovers the upside-down (+ ↔ -) symmetry. (h) The most active triads now favor the forward cascade, so that only 1.1% of the injected energy flows into region I, while 98.9% are transferred into region III. Data represent averages over single runs (Fig. S2). Simulation parameters are identical to those in Fig. 6-2.
7-1  (a) Dispersion relations $\xi(k)$ for the polynomial GNS model (7.2) and the Gaussian GNS model (7.29). Modes with $\xi(k) < 0$ define the energy injection and scale selection domain typical of active turbulence.
(b) Illustration of the key model parameters in 3D Fourier space. The spectral bandwidth $\kappa$ defines the width of the unstable domain II (red), which is localized around the characterized vortex scale $\Lambda$ and separates dissipative Fourier modes at large (region I) and small scales (region III).

7-2  Numerical simulations of (7.5) with polynomial dispersion (7.2) initiated with random complex initial conditions show that active triads $(p < k < q)$ are unstable when forced at large wavenumbers $q$. Energy and helicity increase exponentially (a), reflecting the exponential growth of the forced helical mode (d) and underdamped decay of the passive helical modes (b, c). Parameters: $\{k, p, q\} = \{(-5, 9, 0), (1, 2, 0), (4, -11, 0)\}$, box size $L = 24\Lambda$.

7-3  Numerical simulations of (7.5) with polynomial dispersion (7.2) initiated with random complex initial conditions show that active triads $(p < k < q)$ are unstable when forced at small wavenumbers $p$. Energy and helicity increase exponentially (a), reflecting the exponential growth of the forced helical mode (c) and overdamped decay of the passive helical modes (b, d). Parameters: $\{k, p, q\} = \{(-14, -13, 0), (4, -11, 0), (10, 24, 0)\}$, box size $L = 24\Lambda$.  

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Numerical simulations of (7.5) with polynomial dispersion (7.2) initiated with random complex initial conditions show that active triads \( p < k < q \) are stable when forced at intermediate scales \( k \). The energy and helicity \( a \) as well as the amplitudes of the helical modes \( b-d \) stay bounded and soon take the form of very rapid charge-discharge bursts, reflecting the collapse of the dynamics onto a limit cycle, see Fig. 7-5. Note the different \( y \)-scales in \( b-d \), which indicate that the energy produced by the intermediate scale is primarily sent to large scales. This is a manifestation of the upward transfer at the level of a single triad. Parameters: \( \{k,p,q\} = [(12,1,0),(3,7,0),(-15,-8,0)] \), box size \( L = 24\Lambda \).

Asymptotic analysis of the results in Fig. 7-4: the dynamics of stable active triads eventually collapses onto a limit cycle in a real three-dimensional subspace. (a) The cubic quantity \( \text{Im}(A_k A_p A_q) \) decays in accordance with (7.13) until the machine double-precision limit is reached. (b) Complex trajectories traced out by the modes \( A(t) = (A_k(t), A_p(t), A_q(t)) \) approach straight lines at an exponential rate. (c) Trajectories in (b) for \( t > 100\tau \). The lines are characterized by the angles \( \phi_k, \phi_p, \phi_q = (0.759, -0.185, -0.574) \), such that \( \phi_k + \phi_p + \phi_q = 0 \), as required by vanishing of \( \text{Im}(A_k A_p A_q) \). (d) The phases define the change of variables \( (A'_k, A'_p, A'_q) = (e^{-i\phi_k} A_k, e^{-i\phi_p} A_p, e^{-i\phi_q} A_q) \) and \( (B'_k, B'_p, B'_q) = (e^{-i\phi_k} B_k, e^{-i\phi_p} B_p, e^{-i\phi_q} B_q) \), which leaves the differential Eqs. (7.5) unchanged, but rotates the complex trajectories so that the variables \( (A'_k(t), A'_p(t), A'_q(t)) \) become real in the limit \( t \to \infty \). (e) In this three dimensional real subspace, \( A'(t) \) collapses onto a stable limit cycle. (f–h) Projections of the limit cycle of \( A'(t) \) onto the coordinate planes. (i) \( B'(t) \) also develops a limit cycle, shown is the real part. (j–k) The corresponding projections of \( \text{Re}B'(t) \) onto the coordinate planes.
7-6 Types of orbits \( A(t) \) in the complex plane for the classical system (7.5a) with \( D = 0 \) include: fixed points (not shown), circular orbits for initial conditions in the set \( Z_3 \) (a), orbits resulting from trajectories on a three-torus for generic initial conditions (b), straight lines for initial conditions with \( C = 0 \) in (7.26), in which case the system reduces to the classical Euler equations for a rigid body (c).

7-7 Simulation results for the Gaussian model (7.29). (a) Energy and helicity time series show the initial relaxation phase and the subsequent statistically stationary stage. Time instants and interval labels refer to Fig. 7-8. (b) Normalized histograms of the angles between velocity \( \mathbf{v} \) and vorticity \( \omega \) at three different time instants confirm that mirror-symmetry breaking is achieved by developing Beltrami-type flows, where velocity and vorticity are nearly aligned. (c) Snapshot of the helicity density field at \( t = 60\tau \) showing spontaneous symmetry breaking towards positive values.

7-8 Numerical results for the Gaussian activity model (7.29) based on the simulation in Fig. 7-7. Instantaneous (a-f) and average (g-i) energy spectra, fluxes and dominant integrated triads for time instants and intervals indicated in Fig. 7-7(a). Vertical dashed lines mark the energy injection range.
A-1 To estimate the effective viscosity at fixed separation $L_y$ and shear rate $\dot{\gamma}$ from an ensemble average, we generate $\geq 10$ simulations with initial data corresponding to a randomly perturbed linear shear profile (App. A.1). (a) Time series of the kinetic energy $E(t)$ for multiple runs. (b) Time series of the shear stress $\Sigma^+(t)$ on the upper boundary for $L_y = 5\Lambda$, $\dot{\gamma} = 1.4 T^{-1}$. The vertical dotted line indicates the relaxation time $T$. (c) Combined time series for $t > T$ from all runs of the shear stress $\Sigma(t) = \Sigma^+(t) + \Sigma^-(t)$ rescaled by the kinematic viscosity $\Gamma_0$ and shear rate $\dot{\gamma}$. (d) Histogram corresponding to the combined time series in (c) yields the estimates for the mean viscosity $\nu = \langle \Sigma \rangle / \dot{\gamma}$ and its variance. (e) Convergence of the mean viscosity estimates as a function of the averaging interval $\Delta$. (f) Relative magnitude of the Fourier-Chebyshev coefficients of the vorticity field at a random representative time of the simulation demonstrates geometric convergence to zero, confirming that the number of modes used in the simulations suffices to completely resolve the dynamics at double precision accuracy ($\epsilon \sim 10^{-16}$).

A-2 Additional flow examples for various channel widths $L_y$ and shear rates $\dot{\gamma}$. The shear stress histograms represent averages over $\geq 10$ runs.

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A-3 (a) Validation that the flow symmetry breaking occurs is observed with equal probability for both directions. Same parameters as Fig. 3-2(b-c) \( (L_y = 2.2\Lambda, V = 1.65U) \). For 300 runs, we obtained 46.3:53.7 for the relative proportions of left-right symmetry breaking. (b) Standard deviation of the effective viscosity shown in Fig. 3-3(b). We distinguish between two regimes whose boundary (black line) is defined by the shear speed \( V \) being equal to the characteristic vortex speed \( U \). At small shear, \( V < U \), the standard deviation is inversely proportional to the shear rate. That is, in the weak-shear regime, the fluctuations of the shear stress \( \Sigma \) depend only on the channel width \( L_y \). At large shear, \( V > U \), the flow becomes more stable and the standard deviation quickly becomes orders of magnitude smaller than \( \Gamma_0 \). Blue lines indicate horizontal and vertical cuts shown in Fig. 3-3(c,d).

A-4 Flow structures for four other choices of higher-order boundary conditions. The following quantities were set to zero pointwise on the boundary: (a) higher-order stress contributions, \( \sigma^{\Gamma_2}|_{\partial D} \) and \( \sigma^{\Gamma_4}|_{\partial D} \), (b) vorticity and its normal derivative, \( \omega \) and \( \partial_n\omega \), (c) normal components of Laplacian and bi-Laplacian of vorticity, \( \partial_n\nabla^2\omega|_{\partial D} \) and \( \partial_n\nabla^4\omega|_{\partial D} \), (d) normal components of vorticity and Laplacian of vorticity \( \partial_n\omega|_{\partial D} \) and \( \partial_n\nabla^2\omega|_{\partial D} \). Parameters same as in Fig. 3-3(e).

B-1 Stokes’ second problem: Driving protocol. (a) The driving amplitude, cf. Eq. (B.4), increases according to the prefactor \( f(t) \) defined in Eq. (B.5). (b) Mean kinetic energy time series for the simulation shown in Fig. 4-2(b) of Chapter 4 shows that the system relaxes well before the start time of the temporal averaging periods for the mean power \( \langle P \rangle \) (vertical dashed lines).
B-2 Stokes’ second problem: Additional examples. The middle row is the same as Figs. 2(b,d) in Chapter 4. For narrow vortex-size distribution \( \kappa_s \) (right column), it takes higher driving frequency \( \Omega \) to disrupt the structure of the stress field, a core filled with low-stress defects and half-loops of the stress director field near the boundary. As a result, the resonance peak in Fig. 4-2(c) of Chapter 4 appears at a higher frequency for \( \kappa_s \) than for \( \kappa_w \).

B-3 The linear relation between the torque \( T(t) \) and angular speed \( \dot{\phi}(t) \) (4.3) of Chapter 4 approximately holds for active fluids. The formula becomes very accurate as the driving frequency \( \Omega \) and amplitude \( A \) become larger than the corresponding active fluid characteristic pattern formation parameters \( 2\pi/\tau \) and \( U \), respectively. (a,d) Normalized power spectral density \( |T_\omega|^2/\sum_{\omega'}|T_{\omega'}|^2 \) of the (discrete) steady-state time series \( T_n \) for the two simulations shown in Figs. 2(b,d) of Chapter 4. The complex amplitudes \( T_\omega \) are obtained by applying the Discrete Fourier Transform to \( T_n \). The proportion of the energy concentrated at the driving frequency \( \Omega \) (b,c,e,f) as well as at the second most energetic frequency (insets) as a function of \( \Omega \) (b,e), the oscillation amplitude \( A \) (c,f) for active fluids with the wide (b,c) and small (e,f) active bandwidths \( \kappa_s \) and \( \kappa_w \).
B-4 Inertial (a,d) and dissipative (b,e) response parameters $f$ and $\gamma$ as a function of the oscillating frequency $\Omega$ and amplitude $A$ (insets) that appear in the relation (4.3) of Chapter 4 for active fluids with wide $\kappa_w = 1.5/\Lambda$ (a-c) and small $\kappa_s = 0.63/\Lambda$ (d-f) spectral bandwidths. The parameters were computed using Eq. (B.6). (c,f) Average power input per unit length $\langle P \rangle$ in the steady state normalized by the value $\langle P \rangle$ expected from the classical Stokes’ problem for a semi-infinite plate shows relative resonance at the characteristic frequency $2\pi/\tau$ of the active flow patterns. The markers indicate power input as computed from the full time series of the torque $T(t)$ while the lines indicate the contribution derived from the linear relation (4.3) of Chapter 4.

B-5 A passive fluid ($\Gamma_2 = \Gamma_4 = 0$) with viscosity $\Gamma_0 = 10^{-6} \text{m}^2/\text{s}$ confined to a disk domain of radius $R = 200\mu\text{m}$ subject to oscillatory boundary conditions in Eq. (B.4) with angular frequency $\Omega = 3.14\text{rad/s}$ and amplitude $A = 628\mu\text{m/s}$ responds effectively as a rigid body. This is because for such parameters, typical for the active Stokes’ second problem presented in Fig. 4-2 of Chapter 4, the penetration depth $\delta$ of the passive fluid is much bigger than the domain size $R$. (a) Representative snapshot of the vorticity and flow fields illustrates the rigid body-like response. (b) The corresponding power input (solid line) of the passive fluid driven according to the protocol described in Eq. (B.4) follows accurately the formula (B.9) for the power input of a rigid body rotating about the z-axis represented by a disk with a mass equal to that of the fluid (broken line).
B-6  Einstein–de Haas analogue effect for an active fluid with narrow vortex-size distribution \( \kappa_S = 0.63/\Lambda \). (a) Angular momentum fluctuations \( \sigma_L \) as a function of the domain size for heavy containers obtained from Fig. 4-3(a) of Chapter 4 by averaging over \( \alpha \in [0.01, 0.1] \). (b) Angular speed fluctuations \( \sigma_\alpha \) as a function of the domain size for light containers obtained from Fig. 4-3(c) of Chapter 4 by averaging over \( \alpha \geq 10 \). (c-d) Zoom-in of the time series of the container’s angular momentum (blue) calculated from Eq. (B.10) shown in Fig. 4-3(b) of Chapter 4 for domain radius \( R = 3.33 \) (c) and \( R = 3.67 \) (d). Additionally, to illustrate the angular momentum conservation in the fluid-container system, we show the time series of the fluid’s angular momentum (orange) calculated independently using the formula \( L_{\text{fluid}} = \rho \int_0^R \int_0^{2\pi} rv_\theta rd\theta \).

C-1  Vortex detection scheme. Miller cylindrical projection of the sphere showing the unprocessed normalized vorticity field (a), and the thresholded vorticity field with values \( \omega(x,t) \in [\alpha_\omega \min_{x \in S^2} \omega(x,t), \alpha_\omega \max_{x \in S^2}] \) removed for (b) \( \alpha_\omega = 0.25 \), and (c) \( \alpha_\omega = 0.5 \). The chain-like branched structures in the vorticity field remain preserved after thresholding.

C-2  Phase diagram for \( \alpha_\omega = 0.25 \) (a) and \( \alpha_\omega = 0.75 \) (b), showing that qualitative changes in the different turbulent phases are robust with regard to variations in \( \alpha_\omega \); cf. Fig. 5-2(a) in Chapter 5. Color scales show normalized Betti number defined in Eq. (C.94).

C-3  The ratio between the mean kinetic energy and mean enstrophy also differentiates between the A- and T- phases in Fig. 5-2(a) of Chapter 5, which are approximately separated by the dashed vertical line \( \kappa \Lambda = 1 \).
D-1 Numerical estimation of the stationary energy spectra for the narrow bandwidth in Fig. 6-3(a) of Chapter 6. (a and b) Kinetic energy (a) and helicity time series (b) are used to determine the relaxation time $\mathcal{R}$ to a stationary state (vertical broken line, $\mathcal{R} = 31.7\tau$ in this case). (c) Convergence of the energy spectra estimates using $|\langle \epsilon^\pm \rangle_{\mathcal{R},\Delta} - \langle \epsilon^\pm \rangle_{\mathcal{R},\Delta_{\text{max}}} |_2 / |\langle \epsilon^\pm \rangle_{\mathcal{R},\Delta_{\text{max}}} |_2$, see Eq. (D.31), as a function of the averaging interval $\Delta$. (d) Relative difference ($l^2$-norm) between the momentary energy spectra and their (stationary) mean as a function of time.

D-2 Mirror-symmetry breaking and inverse energy cascade for an active fluid with the intermediate bandwidth $\kappa_I$, showing same quantities as in Fig. 6-3 of Chapter 6. Overall, 15.4% of the injected energy flows into region I, while 84.6% flows into region III.

D-3 Characterization of the inverse energy cascade. (a and b) Two horizontal cuts through the 3D simulation domain for a small bandwidth $\kappa_S$, showing that the inverse cascade is not characterized by vortex mergers, but rather by chain-like complexes. (c and d) Same flow-field snapshots as in (a) and (b) but now represented through the local helicity field. The chain-like large-scale structures carry most of the helicity. They do not merge, but rather form extended filaments and clusters that move throughout the simulation domain. Domain size $L = 32\Lambda$. (e) The proportion of the energy injected by the active component that is transported to region I [corresponding to large scales, cf. Fig. 6-1(a) of Chapter 6] as a function of the active bandwidth $\kappa$. (f) The absolute value of the energy flux for an active fluid with small bandwidth $\kappa_S$ for different simulation domain sizes. In region I, corresponding to large scales, the upward transfer is non-inertial at intermediate wavenumbers with the flux exhibiting $k^3$ scaling. For $k \to 0$, however, the flux approaches a constant plateau value, indicating that inertial effects begin to dominate at very large scales $\gg \Lambda$. 

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D-4 (a) Relaxation time for spontaneous symmetry breaking depends on the domain size. (b and c) Kinetic energy (b) and helicity (c) as a function of time for a very large domain ($L = 48\Lambda$). The relaxation proceeds in two stages, the initial stage characterized by a rapid exponential growth rate ($t < 20\tau$), followed by a slower linear growth until full relaxation ($t \approx 100\tau$). (d-f) Energy spectra at various stages of the relaxation process [cf. broken lines in (b) and (c)] show how the system realizes a state with broken mirror symmetry.

E-1 Numerical simulations of the triad dynamics (7.5) initiated with generic complex initial conditions show that active triads ($p < k < q$) are unstable when forced at intermediate $k$ and small $q$ scales. Energy and helicity increase exponentially (a), reflecting the exponential growth of one of the forced modes (d) and underdamped decay of the remaining forced mode (b) and the passive mode (c). Parameters: $\{k, p, q\} = [(4, -11, 0), (-9, -1, 0), (5, 12, 0)]$, box size $L = 24\Lambda$.

E-2 Numerical simulations of the triad dynamics (7.5) initiated with generic complex initial conditions show that active triads ($p < k < q$) are unstable when forced at large $p$ and intermediate $k$ scales. Energy and helicity increase exponentially (a), reflecting the exponential growth of one of the forced modes (c) and overdamped decay of the remaining forced mode (b) and the passive mode (d). Parameters: $\{k, p, q\} = [(5, 11, 0), (8, 8, 0), (-13, -19, 0)]$, box size $L = 24\Lambda$.

E-3 Instantaneous (a,b) and average (c) helicity fluxes for the Gaussian activity model (7.29) for time instants and intervals indicated in Fig. 7-7(a). Vertical dashed lines mark the energy injection range.
Chapter 1

Introduction

Turbulence, the chaotic motion of liquids and gases, remains one of the most widely studied phenomena in classical physics [72, 141]. Turbulent flows determine energy transfer and material mixing over a vast range of scales, from the interstellar medium [92, 65] and solar winds [36] to the Earth’s atmosphere [148, 130], and ocean currents [206]. Of particular recent interest is the interplay of turbulence and active biological matter [61], owing to its relevance for carbon fixation and nutrient transport in marine ecosystems [203].

Classical turbulence concerns externally driven flows at high Reynolds number [72, 64]. By contrast, energy injection in suspensions of self-motile structures [149] is delocalized and inherently coupled to the flow field. For example, swimming microorganisms [142, 55, 156, 97, 59] stir the surrounding fluid, but also respond to the flow field and interact through the fluid. Similar flow-dependent forcing mechanisms are present in suspensions of artificial micro-swimmers [95, 220, 33] and ATP-driven microtubule networks [177]. When the concentration of such active objects is sufficiently high, self-sustained chaotic flow patterns emerge; this phenomenon is commonly referred to as active turbulence nowadays [228, 222, 77, 32, 213]. A striking difference between classical and active turbulence is that the latter often exhibits characteristic scales, leading to a preferred eddy size [194, 195, 222, 59, 177].

In this thesis, we study a class of phenomenological models of active turbulence that combine a generic linear instability mechanism with a conventional advective
nonlinearity. Specifically, we consider pattern-forming nonequilibrium fluids consisting of a passive solvent component, such as water, and a stress-generating active component, which could be bacteria [194], ATP-driven microtubules [177], or Janus particles [99, 38]. In contrast to earlier studies that analyzed the velocity field of the active matter component [222, 59, 32], we focus here on the incompressible solvent velocity field $v(t, x)$ described by

$$\nabla \cdot v = 0,$$  

(1.1a)

$$\partial_t v + v \cdot \nabla v = -\nabla p + \nabla \cdot \sigma,$$  

(1.1b)

where $p(t, x)$ is the local pressure. The effective stress tensor $\sigma(t, x)$ is composed of passive contributions from the intrinsic solvent fluid viscosity and active contributions representing the stresses exerted by the microswimmers on the fluid [181, 175, 139, 164]. Experiments [55, 45, 177, 194, 59, 227] show that active stresses typically lead to vortex scale selection in the ambient solvent fluid, in stark contrast to the scale-free vortex structures in externally driven classical fluid turbulence. We describe this mesoscale pattern-formation through the phenomenological stress tensor [184, 185]

$$\sigma = (\Gamma_0 - \Gamma_2 \nabla^2 + \Gamma_4 \nabla^4)[\nabla v + (\nabla v)^\top],$$  

(1.1c)

where the higher-order derivatives $\nabla^{2n} \equiv (\nabla^2)^n$, $n \geq 2$ account for non-Newtonian effects [18], and the parameters $\{\Gamma_i\}_{i=0,2,4}$ are constant. Inserting the ansatz (1.1c) into the Cauchy momentum equation (1.1b) yields the following higher-order incompressible Generalized Navier–Stokes (GNS) equations

$$\nabla \cdot v = 0,$$  

(1.2a)

$$\partial_t v + v \cdot \nabla v = -\nabla p + \Gamma_0 \nabla^2 v - \Gamma_2 \nabla^4 v + \Gamma_4 \nabla^6 v.$$

(1.2b)

The 1D version of (1.2) was first proposed by Nikolaevskiy to describe an active geophysical medium [151]. More generally, closely related GNS models have also been studied in the context of soft-mode turbulence and seismic waves [19, 212]. Numer-
ical solutions of the GNS Eqs. (1.2) in 2D show phenomenological similarities with planar magneto-hydrodynamic (MHD) flows driven by electromagnetic stresses [215], suggesting that the results below may also apply to astrophysical systems.

The crucial difference between the GNS equations (1.2) and models of classical turbulence is the nature of the forcing mechanism. In the latter, the fluid is typically forced by an external, random, time-uncorrelated, Gaussian body force [62, 31]. By contrast, the kinetic energy in (1.2) is injected into the solvent through a momentum-conserving instability obtained by setting $\Gamma_2 < 0$, while keeping $\Gamma_0$ and $\Gamma_4$ positive, which creates a bandwidth of linearly unstable modes. The goal of this thesis is to examine the consequences of such flow-dependent forcing schemes. We will study the resulting flow structures in 2D and 3D, as well as on curved surfaces. We will explore their bulk properties and interactions with boundaries. In particular, we will focus on symmetry-breaking phenomena and the resulting interscale energy transport. Importantly, we will show that the main predictions of this thesis derive from the presence of linearly unstable modes and are insensitive to the particular ansatz (1.1c) for the stress tensor. We therefore expect the results described herein to be valid for a broad class of passive fluids that are driven by a linear instability mechanism.

In Chapter 2, we develop a basic intuition for the GNS model (1.2). We perform a linear stability analysis and identify the relevant scales that characterize the resulting nonequilibrium flow structures. Specifically, we map the model parameters $\Gamma_i$'s to the most unstable vortex size $\Lambda$, its turnover time $\tau$ and the spectral bandwidth $\kappa$, which measures the size of the forcing range. We then fit these experimentally relevant parameters to available laboratory data on active fluids driven by bacterial suspensions and ATP-fueled microtubule networks and demonstrate that the model (1.2) reproduces the observed velocity statistics.

In Chapter 3, we study the response of (1.2) to shear boundary conditions in two dimensions. We first present exact bulk analytical solutions including stress-free vortex lattices and introduce a computational framework that allows the efficient treatment of higher-order shear boundary conditions. Through large-scale parameter scans, we identify the conditions for spontaneous flow symmetry breaking, geometry-

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In Chapter 4, we focus on an active analog of Stokes' second problem by studying the thin-film flows enclosed by a moving ring-shaped container. Our numerical analysis predicts that a periodically rotating ring will oscillate at a higher frequency in an active fluid than in a passive fluid, due to an activity-induced reduction of the fluid inertia. In the case of a freely suspended fluid-container system that is isolated from external forces or torques, we find that active fluid stresses can induce giant fluctuations in the container’s angular momentum if the confinement radius matches certain multiples of the intrinsic vortex size of the active suspension. Chapter 4 is currently under revision at *Phys. Rev. Lett.*: J. Slomka, A. Townsend, and J. Dunkel. Stokes' second problem in active fluids [188].

In Chapter 5, we generalize the model (1.2) to curved surfaces and derive exact stationary solutions for the case of a spherical bubble geometry. Numerical simulations reveal a curvature-induced transition from a burst phase to an anomalous turbulent phase that differs distinctly from externally forced, classical 2D Kolmogorov turbulence. This new type of active turbulence is characterized by the self-assembly of vortices into linked chains of anti-ferromagnetic order, which percolate through the entire fluid domain, forming a dynamic network. The coherent motion of the vortex chain network provides an efficient mechanism for upward energy transfer from smaller to larger scales, presenting an alternative to the conventional energy cascade in classical 2D turbulence. The results of Chapter 5 are published in: O. Mickelein, J. Slomka, K. J. Burns, D. Lecoanet, G. M. Vasil, L. M. Faria, and J. Dunkel. Anomalous chained turbulence in actively driven flows on spheres. *Phys. Rev. Lett.*, 120(16):164503, 2018 [143].

In Chapter 6, we investigate bulk flows and energy transport properties of (1.2) in 3D. We show that chiral Beltrami vector fields provide exact stationary solutions of (1.2) and use numerical simulations to show that active fluids spontaneously break
mirror-symmetry by developing helical flows. These flows are Beltrami-like: their velocity and vorticity fields are preferentially aligned. As a consequence of this parity violation, we predict that an inverse energy transport to larger scales develops in otherwise homogeneous and isotropic active turbulence, in contrast to classical turbulence, where only forward energy cascades to smaller scales occur in 3D. We further show that the upward energy transfer is sustained by a chiral subset of all triad interactions as predicted by Waleffe [216] about 25 years ago. The results of Chapter 6 are published in: J. Słomka and J. Dunkel. Spontaneous mirror-symmetry breaking induces inverse energy cascade in 3D active fluids. Proc. Natl. Acad. Sci. U.S.A., 114(9):2119-2124, 2017 [186].

In Chapter 7, we rationalize the numerical findings presented in Chapter 6 by studying the triad truncations of the polynomial model (1.2) and a Gaussian GNS model. Identifying a previously unknown cubic invariant for the triads, we show that their asymptotic dynamics reduces to that of a forced rigid body coupled to a particle moving in a magnetic field. This analogy allows us to classify triadic interactions by their asymptotic stability: unstable triads correspond to rigid-body forcing along the largest and smallest principal axes, whereas stable triads arise from forcing along the middle axis. Analysis of the polynomial GNS model reveals that unstable triads induce exponential growth of energy and helicity, and stable triads develop a limit cycle of bounded energy and helicity. This suggests that the unstable triads dominate the initial relaxation stage of the full hydrodynamic equations, whereas the stable triads determine the statistically stationary state. To test whether this hypothesis extends beyond polynomial dispersion relations, we introduce and investigate an alternative Gaussian active turbulence model. Similar to the polynomial case, the steady-state chaotic flows in the Gaussian model spontaneously accumulate non-zero mean helicity while exhibiting Beltrami statistics and upward energy transport. The results of Chapter 7 are published in: J. Słomka, P. Suwara, and J. Dunkel. The nature of triad interactions in active turbulence. J. Fluid Mech., 841:702-731, 2018 [187].

Finally, in Chapter 8, we conclude the study of the GNS model. We discuss strengths and weaknesses of the model and propose directions for future research.
Chapter 2

Basic properties of the Generalized Navier–Stokes equations

2.1 Introduction

In this short chapter, we study basic properties of the active turbulence model (1.2). We perform a linear stability analysis and identify the relevant scales that characterize the resulting nonequilibrium flow structures. Specifically, we map the model parameters $\Gamma_i$'s to the most unstable vortex size $\Lambda$, its turnover time $\tau$ and the spectral bandwidth $\kappa$, which determines the size of the spectral forcing range. All numerical simulations and analytical calculations throughout this thesis will be normalized in terms of these three scales as well as the associated circulation speed $U = 2\pi \Lambda / \tau$. We then fit these experimentally relevant parameters to available laboratory data on active fluids that are driven by bacterial suspensions [59] and ATP-fueled microtubule networks [177] and demonstrate that the model (1.2) reproduces the observed velocity statistics.

2.2 Linear stability analysis and characteristic scales

We consider the motion of an active fluid described by Eq. (1.2) on an unbounded 3D domain. Since the velocity field $\mathbf{v}$ is incompressible, we introduce a vector potential $\mathbf{\psi}$
to eliminate the divergence-free constraint (1.2a) by setting \( \mathbf{v} = \nabla \times \psi \). Introducing the vorticity field \( \omega = \nabla \times \mathbf{v} \) and taking the curl of the momentum Eq. (1.2b) yields the following equivalent vorticity-vector potential formulation of the equations of motion (1.2)

\[
\nabla^2 \psi = -\omega, \quad (2.1a)
\]
\[
\partial_t \omega + \nabla \times (\omega \times \mathbf{v}) = \mathcal{L} \omega, \quad (2.1b)
\]

where \( \mathcal{L} = \Gamma_0 \nabla^2 - \Gamma_2 \nabla^4 + \Gamma_4 \nabla^6 \). We now focus on the evolution of a single Fourier mode \( k \). Inserting \( \psi(t, \mathbf{x}) = \hat{\psi}(t, k)e^{ik \cdot \mathbf{x}} \) into (2.1) yields

\[
\partial_t \hat{\psi}(t, \mathbf{k}) = -\xi(k) \hat{\psi}(t, \mathbf{k}), \quad (2.2)
\]

where \( k = |\mathbf{k}| \) and the linear growth rate associated with the higher order operator \( \mathcal{L} \) is

\[
\xi(k) = \Gamma_0 k^2 + \Gamma_2 k^4 + \Gamma_4 k^6. \quad (2.3)
\]

Setting \( \Gamma_2 < 0 \) introduces a bandwidth of linearly unstable modes with \( \xi(k) < 0 \), see Fig. 2-1(a). The most unstable mode \( k_m = \arg \max \xi(k) \) is well approximated by the maximum \( k_p \) of the function \( f(k) = \xi(k)/k^2 = \Gamma_0 + \Gamma_2 k^2 + \Gamma_4 k^4 \), yielding

\[
k_p^2 = \frac{-\Gamma_2}{2 \Gamma_4}. \quad (2.4)
\]

We prefer to express characteristic scales in terms of \( k_p \) (instead of \( k_m \)) as this simplifies subsequent formulas, and \( k_p \) is generally close to \( k_m \) for experimentally relevant parameters. The associated wavelength is \( \lambda_p = 2\pi/k_p \). This wavelength represents two vortices, one with positive and one with negative vorticity, each of characteristic diameter

\[
\Lambda = \frac{\lambda_p}{2} = \pi \sqrt{\frac{2 \Gamma_4}{-\Gamma_2}}. \quad (2.5)
\]
Figure 2-1:  (a) Dispersion relations \( \xi(k) \) for the polynomial GNS model (2.3) and the Gaussian GNS model (7.29) considered in Chapter 7. Modes with \( \xi(k) < 0 \) define the energy injection and scale selection domain typical of active turbulence. (b) Illustration of the key model parameters in 3D Fourier space. The spectral bandwidth \( \kappa \) defines the width of the unstable domain II (red), which is localized around the characteristic vortex scale \( \Lambda \), and separates dissipative Fourier modes at large (region I) and small scales (region III). (c) Snapshot of a representative vorticity component field \( \omega_z \) for an active fluid with bandwidth \( \kappa = 0.63/\Lambda \). Domain size \( L = 32\Lambda \).

The corresponding growth rate is

\[
\xi(k_p) = \frac{\Gamma_2}{2\Gamma_4} \left( \frac{\Gamma_0}{\Gamma_2} - \frac{\Gamma_2^2}{4\Gamma_4} \right),
\]

which defines the turnover time scale

\[
\tau = -\frac{1}{\xi(k_p)}.
\]  

(2.6)

The parameters \( \Lambda \) and \( \tau \) can be used to define a characteristic circulation speed \( U = 2\pi\Lambda/\tau \). Finally, we denote the size of the interval of linearly unstable modes (the size of the forcing range) by \( \kappa = k_+ - k_- \), where \( k_\pm \) are the two nontrivial zeros of the dispersion relation \( \xi(k) \), see Fig. 2-1(b). We find that

\[
\kappa = \left( \frac{-\Gamma_2}{\Gamma_4} - 2\sqrt{\frac{\Gamma_0}{\Gamma_4}} \right)^{1/2}.
\]  

(2.7)

In summary, the relations (2.5, 2.6, 2.7) show that the three model parameters \( (\Gamma_0, \Gamma_2, \Gamma_4) \) uniquely determine the three scales \( (\Lambda, \tau, \kappa) \) that characterize the resulting
nonequilibrium flow structures, see Fig. 2-1(c). Similarly, given the characteristic scales, we can recover the model parameters according to

\[
\Gamma_0 = -\frac{(\kappa^2 \Lambda^2 - 2\pi^2)^2}{\pi^2 \kappa^2 \tau (\kappa^2 \Lambda^2 - 4\pi^2)},
\]

\[
\Gamma_2 = \frac{8\Lambda^2}{\kappa^2 \tau (\kappa^2 \Lambda^2 - 4\pi^2)},
\]

\[
\Gamma_4 = -\frac{4\Lambda^4}{\pi^2 \kappa^2 \tau (\kappa^2 \Lambda^2 - 4\pi^2)}.
\]

In the next section, we determine \((\Lambda, \tau, \kappa)\) and hence \((\Gamma_0, \Gamma_2, \Gamma_4)\) from available experimental data on two microscopically distinct active fluids: bacterial suspensions and ATP-driven microtubule networks.

2.3 Fit to existing experimental data

**Bacterial suspensions.** The experiments reported in Ref. [59] studied dense suspensions of rod-like *Bacillus subtilis* bacteria swimming in a quasi-3D microfluidic channel (height \(\sim 80\mu m\), radius \(\sim 750\mu m\)). The bacterial velocity field was reconstructed from bright-field microscopy videos using particle imaging velocimetry (PIV), and the solvent flow dynamics were measured by particle tracking velocimetry (PTV) using micron-sized fluorescent tracer beads. The experimental setup allowed the observation of 2D slices through the 3D velocity field, yielding data for the in-plane velocity components. From these 2D data, velocity distributions and correlation functions for bacteria and passive tracer particles were reconstructed, showing close correlations between bacterial dynamics and solvent flows. To compare the experimental measurements in Ref. [59] with our 3D simulations (see Chapter 6), we mimicked the experimental setup by selecting arbitrary 2D planes in our 3D simulation volume. We then measured the in-plane velocity components and compared the numerically calculated velocity statistics with the corresponding experimental data [Fig. 2-2(a-c)].

Figure 2-2(a) compares the experimentally measured velocity distribution for bacteria (open circles) and solvent tracer particles (filled circles), with the statistics of
the five-parameter model for the bacterial velocity field considered in Ref. [59] (black line labeled ‘theory’) and the GNS model (blue line). As discussed by the authors of Ref. [59], their model for the bacterial dynamics fails to capture the tails of the velocity distributions as it includes an effective fourth-order velocity potential (representing steric alignment interactions) that dominates the tails of velocity distributions in their simulations. By contrast, the GNS models accurately capture the experimentally measured Gaussian velocity probability distribution functions (PDFs) over the whole range of the available experimental data [see Fig. 2-2(c) legend for a summary of fit parameters].

Figure 2-2(b) compares the equal-time (in-plane) velocity correlation functions (VCFs) for the bacteria, tracer particles and theories. As mentioned in Ref. [59], the VCFs for tracer particles become unreliable at large distances $r > 50\mu m$, due to the deliberately low seeding densities of the tracer particles in these experiments. Low tracer densities were required to minimize feedback from passive tracers on the active suspension dynamics. This meant however that tracer particle pairs with large spatial separation $r$ are significantly less frequently observed. Notwithstanding such experimental limitations, we find that the two complementary continuum models for bacterial and solvent flow yield qualitatively and quantitively similar VCFs, correctly reflecting the typical vortex size $\sim 50 - 70\mu m$ in the negative part of the VCFs.

Figure 2-2(c) compares the simulation results for the velocity auto-correlation functions (VACFs) with the corresponding experimental results at different bacterial activities [59] due to oxygen depletion. PTV-based VACFs were not given in Ref. [59] as a specific tracer particle typically spends only a short time in the 2D field of view of the microscope before diffusing out of view. As evident from Fig. 2-2(c), the GNS model can correctly reproduce the functional form of the PIV-based VACFs at high (green), intermediate (blue) and low (magenta) activities. With regard to a future quantitative characterization and classification of active fluids, we find it encouraging that a three-parameter model can account for the key velocity statistics reported in Ref. [59].
ATP-driven microtubule networks. The GNS equations defined in Eqs. (1.2) merely assume that active stresses in an otherwise passive fluid lead to scale-selection. They should therefore also apply to other types of active fluids, including ATP-driven microtubule suspensions. To test this hypothesis, we performed additional simulations to compare the model (1.2) with experimental data published recently in Ref. [177]. The authors of this study report velocity correlation function (VCF) data for tracer-particles diffusing in fluid flows driven by predominantly extensile microtubule-kinesin bundles that form complex, approximately isotropic networks. The flows created by these active networks exhibit turbulent vortices on scales larger than the typical bundle-bending radii, suggesting that these flows are generated by the collective extensile dynamics of the bundles. Figure 2-2(d) shows the experimental VCF data reported in Ref. [177] (colored circles and lines), and a fit (black solid line) obtained from simulations of the GNS model using the parameters specified in the legend. Different ATP-controlled activity levels can be reproduced in the model (1.2) through a trivial adjustment of the velocity scale $U$. Strikingly, changing the activity does not significantly alter the shape of the VCF after rescaling by kinetic energy for both bacterial and active microtubule suspensions [Fig. 2-2(b,d)], corroborating the idea that active suspensions can be robustly described by the leading-order terms of stress tensor expansions. More generally, the good agreement between the GNS model and two microscopically distinct active fluids supports the view that the main results and predictions of our study apply to a broad range of pattern-forming nonequilibrium fluids.

1These parameters yield the typical velocity, length and time scales that are of the same order of magnitude as expected from microbial suspensions.
Figure 2-2: Fit results for the GNS model defined in (1.2) compared with recent experimental data for (a-c) bacterial suspensions [59] and (d) microtubule networks [177]. (a) PDFs of the Cartesian in-plane velocity components, normalized by their mean values and standard deviations. The black curve labeled ‘theory’ represents a five-parameter continuum model for the bacterial dynamics described in Ref. [59]. The blue curve is obtained for the three-parameter GNS model for the solvent flow defined in (1.2), using the fit parameters listed in the description of (c). Note that only the GNS model correctly captures the tails of the velocity distribution. (b) The equal-time velocity correlation functions (VCFs) indicate the characteristic pattern formation scale. The black curve labeled ‘theory’ again represents the continuum model for the bacterial dynamics described in Ref. [59]. The blue curve is obtained for the GNS model for the solvent flow defined in (1.2), using the fit parameters listed in the description of (c). (c) VACFs obtained for three different values of the fit parameters corresponding to three different activity levels of the bacteria: \( \Gamma_0 = 4.77 \, \mu m^2/s \), \( \Gamma_2/\Gamma_0 = -1.15 \times 10^3 \mu m^2 \), \( \Gamma_4/\Gamma_0 = 9.80 \times 10^4 \mu m^4 \) (magenta line), \( \Gamma_0 = 6.82 \mu m^2/s \), \( \Gamma_2/\Gamma_0 = -1.15 \times 10^3 \mu m^2 \), \( \Gamma_4/\Gamma_0 = 9.80 \times 10^4 \mu m^4 \) (blue lines) \( \Gamma_0 = 1.59 \times 10^1 \mu m^2/s \), \( \Gamma_2/\Gamma_0 = -1.15 \times 10^3 \mu m^2 \), \( \Gamma_4/\Gamma_0 = 9.80 \times 10^4 \mu m^4 \) (green line). In terms of the characteristic vortex size \( \Lambda \), growth time \( \tau \), speed \( U \) and bandwidth \( \kappa \), these parameters correspond to: \( \Lambda = 41 \mu m, \tau = 15s, U = 17.2\mu m/s \) and \( \kappa = 73 mm^{-1} \) (magenta), \( \Lambda = 41 \mu m, \tau = 10.5s, U = 24.5\mu m/s \) and \( \kappa = 73 mm^{-1} \) (blue), and \( \Lambda = 41 \mu m, \tau = 4.5s, U = 57.2\mu m/s \) and \( \kappa = 73 mm^{-1} \) (green). (d) Equal-time VCF for an active fluid driven by a microtubule network. The black line is obtained for the GNS model for the solvent flow defined in (1.2) using fit parameters \( \Gamma_0 = 9.05 \times \mu m^2/s \), \( \Gamma_2/\Gamma_0 = -8.61 \times 10^3 \mu m^2 \), \( \Gamma_4/\Gamma_0 = 7.37 \times 10^6 \mu m^4 \) corresponding to \( \Lambda = 130 \mu m, \tau = 125s, U = 6.5\mu m/s \) and \( \kappa = 21 mm^{-1} \). All simulations were performed on a large 3D domain of size \( L = 32\Lambda \) to exclude finite size effects.
Chapter 3

Shear boundary conditions

3.1 Introduction

In this chapter, we investigate flow pattern formation and viscosity reduction mechanisms in active fluids by studying the polynomial GNS model (1.2) that captures the experimentally observed bulk vortex dynamics in microbial suspensions. We present exact analytical solutions in free-space in 2D including stress-free vortex lattices and introduce a computational framework that allows the efficient treatment of higher-order shear boundary conditions in a plane Couette geometry. Large-scale parameter scans identify the conditions for spontaneous flow symmetry breaking, geometry-dependent viscosity reduction and negative-viscosity states amenable to energy harvesting in confined suspensions. The theory uses only generic assumptions about the symmetries and long-wavelength structure of active stress tensors, suggesting that inviscid phases may be achievable in a broad class of non-equilibrium fluids by tuning confinement geometry and pattern scale selection.

Previous theoretical work [88, 41, 87, 174, 78, 67, 172, 71] identified viscosity reduction mechanisms [193, 133] in dilute active suspensions, corresponding to the limit

case when steric and near-field interactions between active components and collective dynamical effects become negligible. Important analytical and numerical insights into the dynamics and rheology of dilute suspensions were obtained by considering how individual microswimmers and their force dipoles align with an externally imposed shear flow [88], and by studying kinetic models [174, 5] that couple effective one-particle Fokker-Planck equations for the particle dynamics with Stokes flows [175]. Considerably less is known about the viscous properties of non-dilute active suspensions since perturbative approaches become inaccurate when the bulk dynamics is dominated by the vortical flow patterns that are collectively generated by interacting bacteria or sperm cells [55, 166, 195, 222, 194]. To understand better the rheology of dense pattern-forming active fluids, we investigate here a generalization of the classical Navier-Stokes (NS) equations [184], based on a phenomenological description of non-Newtonian fluids through higher-order stress tensors [19, 212, 211]. As demonstrated in Chapter 2, a 3D version of this model captures essential aspects of the experimentally measured bulk fluid velocity statistics in bacterial and ATP-driven microtubule suspensions [59, 177]. Here, we focus on 2D shear geometries relevant to thin film experiments [195].

3.2 Analytical solutions and zero-viscosity modes

The GNS equations (1.2) and (1.1c) are valid in arbitrary dimensions. Here, we focus on the 2D case relevant to free-standing thin-film experiments [195]. In a planar 2D geometry $\mathcal{D}$ with boundary $\partial \mathcal{D}$, we may rewrite Eqs. (1.2) and (1.1c) in vorticity-stream function form (App. A.2)

\[
\begin{align*}
\partial_t \omega + \nabla \omega \times \nabla \psi &= -H \cdot \nabla \omega + \Gamma_0 \nabla^2 \omega - \Gamma_2 \nabla^4 \omega + \Gamma_4 \nabla^6 \omega, \\
\nabla^2 \psi &= -\omega,
\end{align*}
\]

\[(3.1a)\quad (3.1b)\]

\(^1\text{To describe thin-film experiments performed on a substrate, one could add a linear damping term } -\gamma_0 \psi \text{ in the NS equations to account phenomenologically for the substrate friction. However, such a modification would merely lead to a trivial shift of the dispersion relation. Therefore, if the damping is not supercritical and active vortical flows are not completely suppressed, then one can expect that the main results of this study remain valid qualitatively for films on substrates.}\]
Figure 3-1: Exact periodic solutions of Eqs. (3.1) include inviscid vortex lattices. (a) Linear stability analysis of Eqs. (3.1), for a square domain with periodic boundary conditions identifies three types of Fourier modes: dissipative (blue), active (red), and neutral (black circles). Neutral modes can be combined through the superposition (3.2) to form exact stationary stress-free solutions of the nonlinear Eqs. (3.1). (b) Square lattice solution, \( \psi = \cos(\frac{kr}{\sqrt{2}}) \cos(\frac{k\gamma}{\sqrt{2}}) \), corresponding to the green squares in panel (a). (c) Hexagonal Kagome lattice solution, \( \psi = 2 \cos(\frac{k\gamma}{2}) \cos(\frac{\sqrt{3}kr}{2}) - \cos(k\gamma) \), corresponding to the orange hexagons in (a). This flow topology is very similar to Abrikosov lattices found in quantum superfluids, cf. Figures in [1, 109]. (d) Triangular lattice solution, \( \psi = -\cos(k\gamma) - \cos(k\sqrt{3x-y}) - \sin(k\sqrt{3x+y}) \), also corresponding to the orange hexagons in (d), but with different mode amplitudes. In all three cases, \( k \) can be chosen as the radius of the inner or the outer inviscid circle (black) in panel (a).

where the vorticity \( \omega = \nabla \times \mathbf{v} = \epsilon_{ij} \partial_i v_j \) is defined in terms of the 2D Levi-Civita tensor \( \epsilon_{ij} = -\epsilon_{ji} \) such that \( \epsilon_{11} = 0, \epsilon_{12} = 1 \), \( \psi \) is the stream function and \( \mathbf{H} \) is a harmonic field related to the fluid’s center-of-mass (CM) motion. The components of the flow field \( \mathbf{v} = (v_1, v_2) \) are recovered from the Hodge decomposition [179] as \( v_i = \epsilon_{ij} \partial_j \psi + H_i \) (App. A.2).

We construct a family of exact nontrivial stationary solutions of the nonlinear partial differential equations (PDEs) (3.1) in free space. To this end, we focus on the center-of-mass frame with \( \mathbf{H} = 0 \) and consider the stream-function ansatz

\[
\psi(r, \theta) = \int_0^{2\pi} d\phi \hat{\psi}(\phi) e^{-ikr \cos(\phi - \theta)},
\]

where \( k = \sqrt{k_x^2 + k_y^2} \) is a fixed wavenumber radius, and \( (r, \theta) \) are polar position coordinates. The superposition (3.2) yields the vorticity

\[
\omega = -\nabla^2 \psi = k^2 \psi,
\]
and hence eliminates the nonlinear advection term in Eq. (3.1a), because

$$\nabla \omega \times \nabla \psi = k^2 \nabla \psi \times \nabla \psi = 0. \quad (3.4)$$

Thus, to obtain a stationary solution of Eqs. (3.1), we need to fix \( k \) such that the right hand side of Eq. (3.1a) vanishes. This criterion can be fulfilled if \( k \) satisfies the polynomial equation

$$k^2(\Gamma_0 + \Gamma_2 k^2 + \Gamma_4 k^4) = 0, \quad (3.5)$$

which has real roots if \( \Gamma_2 < 0 \) and \( \Gamma_2^2 > 4 \Gamma_0 \Gamma_4 \).

One can further show that the stress tensor defined in Eq. (1.1c) vanishes identically, \( \sigma \equiv 0 \), for stationary solutions of this type. Thus, these solutions are stress-free modes, describing effectively frictionless flow states [Fig. 3-1(a)]. An interesting subclass of exact stationary solutions included in Eq. (3.2) are vortex lattices [Fig. 3-1(b-d)]. By superimposing a small number of \( k \)-modes that lie on one of the two stress-free rings, with \( \psi \) being a sum of suitably weighted Dirac delta-functions, one can construct rectangular, hexagonal and triangular lattices [Fig. 3-1(b-d)], whereas oblique lattices are forbidden by rotational symmetry. The stress-free solutions lie at the interface of the stable and unstable modes [Fig. 3-1(a)]. We next demonstrate through simulations that effectively inviscid behavior remains observable in numerical shear experiments for optimized geometries.

### 3.3 Simulations of shear boundary conditions

**Numerical shear experiments.** To study the rheology of Eqs. (3.1), we simulate a typical shear experiment [133] in which two parallel boundaries are moved in opposite directions, both at a constant speed \( V \) [Fig. 3-2(a-d)]. Specifically, we consider a rectangular domain \((x, y) \in D = [-L_x/2, L_x/2] \times [-L_y/2, L_y/2]\) with periodic boundary conditions in the \( x \)-direction and non-periodic shear boundary conditions in the \( y \)-direction [Fig. 3-2(a)]. In this case, the harmonic field \( H(t, x) = V_{CM}(t) \hat{x} \) coincides
with the center-of-mass velocity and, hence, is governed by Newton's force-balance law, where the force acting on the fluid is obtained by integrating the stress tensor $\sigma$ over the boundary (App. A.1).

As common in the shear analysis of passive fluids [120], we assume no-slip boundary conditions for the flow field $\mathbf{v}(x, \pm L_y/2) = (\pm V, 0)$, which translate into an overdetermined system [161] for the stream function $\psi$ (App. A.1). In contrast to the classical second-order NS equations, the sixth-order PDE (3.1a) requires additional higher-order boundary conditions to specify solutions. Active components in a fluid can form complex boundary-layer structures [227, 136, 226], which are poorly understood experimentally and theoretically. To identify physically acceptable boundary conditions, we tested different types of higher-order conditions. These test simulations showed that imposing $\nabla^2 \omega = 0$ and $\nabla^4 \omega = 0$ at the boundaries reproduces the vortical bulk flow patterns observed in free-standing thin bacterial films [195], whereas stiffer boundary conditions generally do not produce the experimentally observed flow structures. We therefore fix $\nabla^2 \omega = 0$ and $\nabla^4 \omega = 0$ at the upper and lower boundaries in the following discussion. In a rectangular geometry, these soft higher-order boundary conditions mean that integrated force contributions coming from the higher-order stress terms vanish.$^{2}$

Numerical solution of the coupled nonlinear sixth-order PDEs (3.1) with non-periodic boundary conditions for experimentally relevant domain sizes [55, 195, 222, 59] is computationally challenging. We implemented an algorithm that achieves the required numerical accuracy by combining a well-conditioned Chebyshev-Fourier spectral method [153, 209] with a third-order semi-implicit time-stepping scheme [14] and integral conditions for the vorticity field [161] (App. A.1). This computationally efficient code, which runs in real-time on conventional CPUs, can be useful in simulations of a wide range of fluid-based pattern-formation processes, including Kolmogorov flows [155].

**Parameters and observables.** We performed systematic large-scale parameter scans of realistic bulk coefficients ($\Gamma_0, \Gamma_2, \Gamma_4$) and boundary conditions ($\dot{\gamma}, L_x, L_y$),

$^{2}$Other choices of boundary conditions are discussed and illustrated in App. A.4
Figure 3-2: Active shear flows exhibit qualitatively different dynamics, velocity statistics and symmetry-breaking behavior depending on confinement geometry \((L_x, L_y)\) and applied shear speed \(V\) (see also Fig. 3-3). (a) For wide channels of width \(L_y = 5\Lambda\) and weak shear \(V = 0.57U\), typical flow configurations realize advectively mixed vortex lattices (App. A). The characteristic circulation speed \(U\) and growth-time scale \(T\) of the bulk vortices are defined in Chapter 2. Scale bar shows bulk vortex size \(\Lambda\) and colormap encodes vorticity. (b-c) For narrow channels with \(L_y = 2.2\Lambda\) and strong shear \(V = 1.65U\), the active fluid exhibits spontaneous symmetry-breaking resulting in unidirectional transport of the fluid’s center of mass (App. A and Fig. A-2). Depending on initial conditions, qualitatively different low-energy (b; App. A) and high-energy (c; App. A) flow states can arise for identical system parameters. (d-e) The kinetic energy time series \(E(t)\) for the simulations in panels (a-c) illustrate relaxation to statistically stationary states. The center-of-mass velocity \(V_{CM}(t)\) indicates persistent macroscopic average flows through the channel (Fig. A-2). (f-g) The spatially averaged mean shear stresses \(\Sigma^\pm(t)\), rescaled by kinematic viscosity \(\Gamma_0\) and shear rate \(\dot{\gamma}\), reveal top-bottom symmetry-breaking in narrow channels (b,c,g) and exhibit large temporal variability, resulting in a substantial variance of the effective shear viscosity (see also Fig. A-3). Vertical dotted lines indicate the start time \(T\) of the temporal averaging periods for the results depicted in Fig. 3-3(a-c). (h-i) Shear stress histograms constructed from the time series in (f-g) for \(t > T\) reflect the top-bottom flow asymmetry in narrow channels.

where \(\dot{\gamma} = V/L_y\) is the shear rate [Fig. 3-2(a-c)]. Non-dimensionalization reduces the effective number of parameters to four, which we chose to be \((\Gamma_2, \dot{\gamma}, L_x, L_y)\). We explored >200 experimentally relevant parameter combinations in total. For a given parameter set, we repeated numerical shear experiments at least 10 times, initializing simulations with a randomly perturbed linear shear profile (App. A.1). For each
simulation, we recorded the spatial averages of the kinetic energy [Fig. 3-2(d,e)]

\[ E(t) = \frac{1}{L_x L_y} \int_D dx dy \left( \nu^2 / 2 \right) \]  

and the kinematic shear stresses

\[ \Sigma^\pm(t) = \frac{1}{L_x} \int_{\partial D^\pm} dx \sigma_{yx} \]  

acting on the top and bottom boundaries [Fig. 3-2(f-i)]. The statistics of these time series are analyzed for an interval \([T, T + \Delta]\), where \(T\) is chosen larger than the numerically determined flow relaxation time. The averaging interval \(\Delta\) is taken sufficiently long to ensure convergence of statistical observables [Fig. 3-2(f,g), Fig. A-1(a-e)]. For each time series \(O(t)\), we compute mean values

\[ \langle O \rangle = \lim_{T, \Delta \to \infty} \frac{1}{\Delta} \int_T^{T+\Delta} dt \, O(t) \]  

and histograms [Fig. 3-2(h,i)], by performing additional ensemble averaging over simulation runs with identical parameters but different initial conditions [Fig. A-1(a-d)].

Of particular interest for the subsequent analysis are measurements of the total shear stress on the two boundaries, \(\langle \Sigma \rangle = \langle \Sigma^+ \rangle + \langle \Sigma^- \rangle\), and the associated mean kinematic viscosity

\[ \nu = \langle \Sigma \rangle / \dot{\gamma} . \]  

3.4 Spontaneous circulation and low-viscosity phases

Dynamic symmetry breaking and directed transport. Recent experimental studies of bacterial suspensions [225] and ATP-driven active liquid crystals [53] in long narrow channels observed the spontaneous formation of persistent unidirectional macro-scale flows [164, 229]. The GNS model reproduces this dynamical symmetry-breaking effect (Fig. 3-2 and Fig. A-3) and predicts optimal geometries that maximize
directed transport [Fig. 3-3(a)]. Fixing $\Gamma_2 < 0$ to realize bacterial vortex structures as described above, we investigate how the boundary separation $L_y$ and the shear rate $\dot{\gamma}$ affect the mean velocity $V_{CM}$ of an active fluid modeled by Eqs. (1.2), which is governed by [see Eq. (A.7) in App. A.2]

$$\frac{dV_{CM}}{dt} = \frac{1}{L_y} (\Sigma^+ - \Sigma^-). \quad (3.10)$$

For wide channels with $L_y \gg \Lambda$, the flow structures found in the simulations typically resemble a mixed vortex lattice [Fig. 3-2(a)]. In this case, the mean flow can fluctuate but is typically undirected [Fig. 3-2(d); App. A]. By contrast, for narrow channels, the center-of-mass velocity $V_{CM}(t)$ can spontaneously select a persistent mean-flow direction [Fig. 3-2(b,c,e)]. Our parameter scans show that this broken-symmetry phase extends over a wide range of shear rates if approximately two ($L_y \sim 2\Lambda$) or four ($L_y \sim 4\Lambda$) rows of vortices fit between the boundaries [Fig. 3-3(a)]. These results are in good qualitative agreement with recent microfluidic measurements in linearly confined bacterial suspensions; cf. Fig. 4 in Ref. [225].

**Frustrated vortex packings.** In addition to unidirectional center-of-mass motions, our simulations predict another secondary top-bottom symmetry-breaking phenomenon. When the boundary separation is close to $2\Lambda$, the stress statistics for the two boundaries can be substantially different at high shear $V > U$ [Fig. 3-2(g,i)]. Intuitively, this statistical asymmetry can be explained by the fact that two counter-rotating vortices cannot simultaneously satisfy the externally imposed shear boundary conditions. Thus, one of the two vortices will be effectively pushed away from the boundary. The resulting asymmetric vortex alignment produces unequal shear forces on upper and lower boundaries even after long-time averaging [Fig. 3-2(i)], illustrating that the rheological analysis of active fluids requires more sensitive measures than in the case of passive fluids.

**Low-viscosity phases and edge-stresses.** Recent experiments [133] reported the observation of zero- and negative-viscosity states in concentrated *Escherichia coli* suspensions. Adopting typical bacterial parameters ($\Lambda, \tau, U$) as described above, we
Figure 3-3: Numerical simulations of Eq. (3.1) identify the conditions for spontaneous left-right symmetry-breaking and low-viscosity states. (a) The relative mean kinetic energy of the fluid’s center-of-mass signals spontaneous symmetry-breaking. Averages over 10 simulations for each of the 208 simulated parameter pairs (markers) were connected using spline interpolations. Black dots: simulations settle into a single class of statistically stationary kinetic energy ground-states. Red circles: In addition to the ground-state, long-lived excited states are observed for randomly sampled initial conditions (App. A.1). Gray triangles indicate the occurrence of dynamical symmetry-breaking characterized by uni-directional fluid transport with average speed > 0.25U persisting over time-scales > 100τ. A and τ are the characteristic bulk vortex size and the characteristic timescale in an equivalent system with periodic boundary conditions. The black diagonal line marks the corresponding characteristic flow speed U, separating the low-shear from the high-shear regime. (b) The dark blue domains in the effective viscosity phase diagram correspond to quasi-inviscid parameter regimes. (c) Vertical cut through panel (b) at constant shear rate \( \dot{\gamma} = 2.7\tau^{-1} \) showing oscillatory behavior of the shear viscosity with boundary separation \( L_y \). Viscosity fluctuations are maximal when an integer number \( n \) of vortices fits between the boundaries, \( L_y \approx nA \). (d) Horizontal cut through panel (b) at constant separation \( L_y = 3A \), illustrating the suppression of viscosity fluctuations at high shear (see also Fig. A-3). (e-f) Representative flow fields, with local speed (top) and vorticity (center) shown as background, and corresponding stress fields (bottom) in the low-viscosity regime (e; App. A) and the high-viscosity regime (f; App. A). These simulations were performed at the same shear rate but different boundary separations, as indicated in panels (b,c). The spectral norm \( ||\sigma||_2 \), corresponding to the largest eigenvalue of the stress tensor \( \sigma \), and the associated director field reveal the presence of zero-stress defects in the bulk as well as half-loops in the stress-field lines along the edges for the low-viscosity states (e, bottom; App. A).
investigate how the boundary separation $L_y$ and the shear rate $\dot{\gamma}$ affect the effective viscosity $\nu$ in the general NS model [Fig. 3-3(b-f)]. Consistent with the experimental observations [133], the numerically obtained $(\dot{\gamma}, L_y)$-phase diagram confirms the existence of an effectively inviscid phase with $\nu/\Gamma_0 \ll 1$ at low-to-intermediate values of the shear rate $\dot{\gamma}$, when the boundary separation is around $3\Lambda$ (blue domain in [Fig. 3-3(b)]. Varying the shear rate $\dot{\gamma}$ at constant separation $3\Lambda$, one observes a viscosity minimum when $\dot{\gamma}$ matches approximately the inverse vortex growth rate $1/\tau$ [Fig. 3-3(d)]. In this quasi-inviscid regime, three counterrotating vortices fit between the boundaries, so that the flow near the top and bottom aligns optimally with the boundary velocity [Fig. 3-3(e), top; App. A]. The nematic field lines of the associated stress field (1.1c) are defined by the eigenspace axis of the largest eigenvalue $||\sigma||_2$. In the low-viscosity state, these director field lines connect primarily to the same boundary, and they are separated by stress-free defects concentrated in the bulk region [Fig. 3-3(e), bottom; App. A]. Thus, only a few stress-carrying strings connect the two boundaries, resulting in a significantly reduced shear viscosity.

**Viscosity resonances.** In contrast to a passive Newtonian fluid, the effective viscosity $\nu$ of the active fluid generally depends nonlinearly on both the shear rate $\dot{\gamma}$ and boundary separation $L_y$ [Fig. 3-3(b-d)]. Qualitatively, we can distinguish between two characteristic regimes, corresponding to shear speeds $V = \dot{\gamma}L_y$ larger or smaller than the characteristic bulk vortex speed $U$ [(black lines in Fig. 3-3(a,b)]. At small shear speeds, $V < U$, the effective viscosity $\nu$ and its fluctuations depend primarily on the boundary separation $L_y$, exhibiting oscillatory behavior as $L_y$ increases [Fig. 3-3(a,b)]. Viscosity minima occur at selected integer multiples of the characteristic bulk vortex size $\Lambda$ and are separated by maxima that can exceed $\Gamma_0$ by more than a factor 2 [Fig. 3-3(c)]. In such high-viscosity states, the stress field is nearly defect-free and similar to that of a laminar Newtonian fluid, with most of the stress field lines connecting the two boundaries [Fig. 3-3(f), bottom; App. A]. At supercritical shear speeds, $V > U$, the viscosity $\nu$ depends on both $L_y$ and $\dot{\gamma}$, and viscosity fluctuations decrease strongly with $\dot{\gamma}$, signaling that the bulk dynamics becomes dominated by the no-slip boundary conditions at high shear [Fig. 3-3(b,d)].
3.5 Inviscid transition and power extraction

Inviscid transition. The \((\dot{\gamma}, L_y)\)-parameter scans confirm the existence of low-viscosity phases when confinement geometry and shear-rate resonate with the natural bulk vortex size and circulation time scale of an active fluid [Fig. 3-3(b)]. The presence of an active driving mechanism is essential for the emergence of intrinsic length- and time-scales in the statistically stationary non-equilibrium flow states [59]. It is therefore interesting to explore how a decrease in the activity, which can be realized experimentally through oxygen or nutrient depletion [59, 133], affects the quasi-inviscid behavior. We study this process numerically through a systematic change of \(\Gamma_2\), while keeping all other parameter fixed. Starting from the low-viscosity state with \(\Gamma_2^* < 0\) shown in Fig. 3-3(e), we increase \(\Gamma_2\) by adding an increment \(\delta \Gamma_2 > 0\) to \(\Gamma_2^*\), corresponding to a decrease in activity. As \(\delta \Gamma_2\) increases, the average viscosity undergoes a rapid increase before dropping to the value \(\nu/\Gamma_0 = 1\) expected for a passive fluid with kinematic viscosity \(\Gamma_0\) [Fig. 3-4(a)]. The viscosity peak separating the active from the passive phase can be explained by studying the stress distributions [Fig. 3-4(b)]: Away from the transition region, the system remains locked in
Figure 3-5: Decreasing the aspect ratio $\alpha = L_x/L_y$ stabilizes flow states capable of performing mechanical work. (a-b) Steady-state flows for a narrow channel $L_y = 2\Lambda$ and moderate shear $\dot{\gamma} = 0.77 \tau^{-1}$, shown for two different aspect ratios: (a) $\alpha = 5$ and (b) $\alpha = 3$. (c) The kinetic energy time series indicates that, for $\alpha = 3$, the flow locks into a time-independent steady state, in which fluctuations are completely suppressed by the no-slip shear boundary conditions. (d) Shear stresses $\Sigma(t)$ acting on the top (+) and bottom (−) boundary for the simulations in (a-b) yield a negative effective viscosity in both cases, implying that the fluid is pushing the boundaries. This negative-viscosity effect is enhanced for the stationary state observed at smaller aspect ratios (b).

the quasi-inviscid or the laminar ground-state [blue and red curves in Fig. 3-4(b)]. In the critical transition regime, large fluctuations can cause the dynamics to oscillate between a low-stress ground-state and excited higher-stress states, resulting in a bi-modal stress distribution and a higher average viscosity [green and orange curves in Fig. 3-4(b)].

Active fluids as motors. Work extraction from active suspensions has been investigated both theoretically [74, 102, 204] and experimentally [127, 192, 101] in recent years, resulting in a number of promising design proposals for bacteria-powered motors [94] and rectification devices [75, 103]. Moreover, recent experiments [133] report long-lived (> 25s) negative viscosity flows in bacterial suspensions, supporting theoretical predictions that suggested the possibility of extracting work from polar active fluids [74]. Equations (1.2) offer an alternative mechanism for constructing microbial ‘motors’ by exploiting long-lived turbulent states that perform work on the boundaries. Conditions for the existence of such states can be deduced analytically.
from energy balance considerations (App. A.3), which yield for the power input

\[ P = \sum_k k^2(\Gamma_0 + \Gamma_2 k^2 + \Gamma_4 k^4)\varepsilon(k), \]  

(3.11)

where \( \varepsilon(k) \) is the energy spectrum at wavenumber \( k \). For active fluids with \( \Gamma_2 < 0 \), the power input \( P \) can become negative if the boundary conditions are tuned such that the energy spectrum \( \varepsilon(k) \) favors modes that produce a negative rhs. in Eq. (3.11). Spectra of this type allow the extraction of mechanical work from the active fluid. We tested this idea by scanning different spectra \( \varepsilon(k) \) through variation of the aspect ratio \( \alpha = L_x/L_y \) of the simulation domain. Our numerical results confirm the existence of long-lived work-performing states in the low-shear regime \( V < U \) (Fig. 3-5). In particular, when the aspect ratio is not too large, \( \alpha \sim 3 \), and the boundary separation matches twice the bulk vortex scale, \( L_y \sim 2\Lambda \) [Fig. 3-5(b)], then the active flow is found to lock into a stationary state, in which the shear forces exerted on the boundaries remain constant and have a negative sign. In this case, a simple active fluid motor is obtained by connecting the ends of the domain in Fig. 3-5(b), to form a cylindrical film. Such a setup could, in principle, be realizable with bacterial soap films [193].

**Superfluid analogy.** The observation of frictionless flow states in *E. coli* suspensions [133], which have been termed ‘superfluid’ [133, 201], raises the question whether there might exist certain phenomenological similarities between the flow dynamics in quantum superfluids [125] and active suspensions. Effective hydrodynamic models as in Eq. (3.1) can provide a useful starting point for systematic future investigations that explore the parallels and differences at the mean-field level. Such a comparison is made possible by the fact that quantum fluids can also be effectively described in terms of hydrodynamic equations after applying a Madelung transformation [137, 157, 20] to the complex order-parameters in the Ginzburg-Landau [1, 109] and Gross-Pitaevskii [83, 158] equations. An important physical and mathematical difference between the incompressible active suspension model (3.1) and the quantum hydrodynamic equations is that the latter are compressible and feature
a quantum pressure that depends nonlinearly on the density [20]. It is interesting that, despite such differences, the periodic bulk solutions of Eqs. (3.1) include inviscid vortex lattices [Fig. 3-1(b-d)] reminiscent of those in Ginzburg-Landau quantum fluids [1, 109, 135, 122, 25]. In particular, the lattice shown in Fig. 3-1(c) is of Abrikosov-type (cf. Figures in [1, 109]), suggesting that frictionless flow states may share universal vortex signatures despite fundamental differences in the microscopic details and in the form of the governing equations. Similar to quantum vortex lattices, the marginally stable lattice solutions of the model (1.2) are exact only in the quasi-infinite fluid, and they become replaced by ‘cavity-modes’ in the presence of confinement. Yet, lattice remnants remain visible in simulations with shear boundaries (App. A).

Another interesting observation is that the half-loops in the stress-field lines which form along moving boundaries in the low-viscosity state [Fig. 3-3(e), bottom; App. A] bear a resemblance to the presumed edge-current structure in solid-state quantum Hall devices [cf. Fig. 1(c) in Ref. [107]]. The role played by the stress tensor for force transmission in an active fluid is comparable to that of the conductivity tensor for charge current transport in a quantum superfluid [126, 104, 4]. The ‘superfluid’ defects in the stress field of an active fluid reflect an interruption of force transmission lines between the boundaries giving rise to low-viscosity states [Fig. 3-3(e), bottom; App. A]. The suggested phenomenological similarities between active and quantum fluids can likely be traced back to the fact that these two distinct classes of systems share two key features: (i) the governing equations describe collective low-energy excitations in the form of coherent vortex structures, and (ii) unlike classical turbulence the emergent flow structures have a dominant length scale [16]. In the quantum case, vortices can be supported by an external magnetic field, whereas in active fluids vortices arise spontaneously from the microscopic and hydrodynamic interactions of bacteria [194, 59], ATP-driven microtubule bundles [177] or other active components. In the future, it will be interesting to investigate whether, in the quasi-incompressible limit, quantum hydrodynamic equations can be systematically approximated by equations similar to Eqs. (3.1) through a suitably truncated
Madelung transformation [137], or by eliminating one of the two velocity fields in two-fluid models [118]. Moreover, it would be interesting to explore both theoretically and experimentally whether biologically or chemically driven non-equilibrium flows described by Eqs. (1.2) and (1.1c) can mimic other defining characteristics of conventional quantum superfluids, such as wall-climbing Rollin films [167, 15] or the Hess-Fairbank effect [91, 126].
Chapter 4

Stokes’ second problem in active fluids

4.1 Introduction

In this Chapter, we study active flows as described by the polynomial GNS model (1.2) that are enclosed by a moving ring-shaped container. Considering Stokes’ second problem, which concerns the motion of an oscillating boundary, our numerical analysis predicts that a periodically rotating ring will oscillate at a higher frequency in an active fluid than in a passive fluid, due to an activity-induced reduction of the fluid inertia. In the case of a freely suspended fluid-container system that is isolated from external forces or torques, active fluid stresses can induce giant fluctuations in the container’s angular momentum if the confinement radius matches certain multiples of the intrinsic vortex size of the active suspension. This effect could be utilized to transform microscopic swimmer activity into macroscopic motion in optimally tuned geometries.

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Chapter 4 is currently under revision at *Phys. Rev. Lett.*: J. Slomka, A. Townsend, and J. Dunkel. Stokes’ second problem in active fluids [188].
Pendulums swinging in air or water exhibit periods longer than those predicted based on gravity and buoyancy [198, 150]. In his famous mid-19th century work [198], George G. Stokes resolved the discrepancy by postulating an additional parameter, the 'index of friction' (viscosity), in the hydrodynamic equations that now bear his name. Building on this insight, Stokes was able to calculate the terminal velocity of sedimenting globules set by the viscous drag, providing a partial explanation for the suspension of clouds [93]. Since then, the term Stokes’ problems (SPs) has become synonymous with the investigation of objects that move either uniformly or in an oscillatory manner through a liquid [223, 237]. Nowadays, the traditional SPs provide important reference points for the rheology of active fluids, such as water-based solutions driven by swimming bacteria [195, 59] or microtubule networks [177, 77]. Recent experiments show that sufficiently dense bacterial suspensions can significantly reduce the drag experienced by a moving sphere [193] or rotated cylindrical walls [133]. Several theories have been proposed to rationalize the observed decrease in shear viscosity, ranging from microscopic and Fokker–Planck-based approaches [87, 174, 172, 201] for dilute suspensions to active liquid crystal continuum models [88, 41, 78, 67, 71] and phenomenologically GNS equations for dense suspensions [185] (Chapter 3). By contrast, the effects of oscillatory boundary conditions – Stokes’ second problem – have thus far only been partially explored in dilute active fluids [85]. Therefore, it is currently unknown how the collective microbial swimming dynamics in dense suspensions, which typically exhibit active turbulence with characteristic vortex length scale $A$ and correlation time $\tau$ [55, 45, 177, 194, 59, 200, 227], interacts with oscillating boundaries. In particular, it is not known how the frequency of a pendulum is altered by the presence of an active fluid component. Here, we will show that activity effectively reduces the fluid inertia, thus increasing the frequency relative to that of an identical pendulum swinging in water.

Two recent experiments on bacterial [225] and microtubule [231] suspensions under channel confinement showed that active liquids can spontaneously achieve persistent circulation by exerting net forces on boundaries [229, 185, 205]. Such non-equilibrium force generation raises interesting questions as to the combined dynamics of isolated
active-fluid–container systems [74], implying a natural extension of the classic SPs. Whereas for passive fluids viscous friction eventually suppresses any container motion, active fluids can continually transform chemical into kinetic energy. This suggests that, under suitable conditions, mesoscopic bulk fluid vortices arising from collective microbial swimming could induce macroscopic fluctuations in the container’s angular momentum, realizing an approximate non-equilibrium analogue of the Einstein–de Haas (EdH) effect [60]. In this analogy, the angular momenta of the bacterial vortices assume the role of the magnetic EdH spin degrees of freedom, whose collective dynamics induces a measurable angular net motion of the macroscopic sample. An active EdH analog experiment could be performed by measuring the rotational fluctuations of a ring containing a thin active film in its interior, similar to the setup reported in Ref. [195].

To explore Stokes’ second problem and the possibility of EdH-type work extraction for dense microswimmer suspensions, we investigate a hydrodynamic model [185, 186] describing non-dilute active fluids subject to either oscillating boundary conditions or confined by a container that responds freely to the internal fluid stresses. Inspired by recent experiments [195, 6], we consider free-standing thin liquid films enclosed by a ring-shaped container of radius $R$. For containers periodically forced by a torsional spring, our simulations predict an activity-induced reduction of the fluid inertia and, hence, oscillation period. For freely suspended containers driven solely by active fluid stresses, we predict giant angular momentum fluctuations that are tunable through the container’s diameter and the fluid-container mass ratio.

Focusing on a planar disk domain of radius $R$, we can rewrite Eqs. (1.2) in the vorticity-stream function form

$$\partial_t \omega + \nabla \times \nabla \psi = \Gamma_0 \nabla^2 \omega - \Gamma_2 \nabla^4 \omega + \Gamma_4 \nabla^6 \omega,$$

$$\nabla^2 \psi = -\omega,$$

where the vorticity pseudo-scalar $\omega = \nabla \times \mathbf{v} = \epsilon_{ij} \partial_i v_j$ is defined in terms of the 2D Levi–Civita tensor $\epsilon_{ij}$, and $\psi$ is the stream function. In polar coordinates $(r, \theta)$,
one recovers the radial and azimuthal velocity components from $v_r = (1/r)\partial_\theta \psi$ and $v_\theta = -\partial_r \psi$. An impermeable container wall imposes the radial boundary condition $v_r(t, R, \theta) = 0$. The tangential component satisfies the no-slip condition $v_\theta(t, R, \theta) = V(t)$. We will consider three cases: a stationary boundary $V(t) = 0$, periodic forcing $V(t) \approx A \cos(\Omega t)$ and freely suspended boundaries, where the fluid stresses induce a rigid body motion $V(t)$ of the container. Additionally, we fix soft higher-order boundary conditions $\nabla^2 \omega(R, \theta) = \nabla^4 \omega(R, \theta) = 0$ throughout, which have been shown previously to reproduce the experimentally observed bulk flow dynamics and viscosity reduction in rectangular shear geometries [185].

4.2 Numerical method

To solve Eq. (4.1) numerically with spectral accuracy, we implemented a recently developed disk analogue of the double Fourier sphere method [224]. The underlying algorithm uses a polar coordinate representation while avoiding the introduction of an artificial boundary at the origin. We combined this method with a third-order IMEX time-stepping scheme, which decouples the system of PDEs (4.1) and treats the non-linear advection term explicitly. Spatial differential operators were discretized using the Fourier spectral method in $\theta$ and the ultraspherical spectral method in $r$ [153]. This procedure generates a sparse spectrally-accurate discretization that can be solved in a cost of $O(n^2 \log n)$ operations per time step, where $n$ is the number of Fourier-Chebyshev modes employed in $\theta$ and $r$. To avoid aliasing errors, the 3/2-rule [39] was used to evaluate the advection term. Additional mode-filtering prevents unphysical oscillations in the solution (see [208] for details). The no-slip boundary conditions were enforced via integral conditions on the vorticity field [161].
Figure 4-1: Typical flow and stress fields for an active fluid with vortex size $\Lambda$ and wide vortex-size distribution $\kappa_w = 1.5/\Lambda$, confined to a planar disk geometry (radius $R = 2.67\Lambda$) with boundary held fixed. The presence of the stress-free defects allows the stress director field to develop complex configurations, enabling a nontrivial response to time-dependent boundary conditions, see Figs. 4-2 and 4-3.

4.3 Stationary boundaries

We first solved Eqs. (4.1) for the fixed boundary conditions, $v_\theta(t, R, \theta) = 0$. In the case of a relatively wide bandwidth $\kappa_w = 1.5/\Lambda$, the active flow spontaneously forms vortices spanning a range of different diameters in the vicinity of the preferred value $\Lambda$ (Fig. 4-1), in agreement with recent simulations [205] of multi-field models. The traceless nematic stress tensor field $\sigma$ defined in Eq. (1.1c) is uniquely characterized by its largest eigenvalue $\|\sigma\|_2$ and the director field of the corresponding eigenvector. We generally find that the stress director fields develop locally ordered domains, which are punctured and separated by stress-free topological defects (Fig. 4-1 right). As we shall see below, the defects facilitate activity-induced reduction of the fluid inertia, when the container is periodically forced.
Figure 4-2: Stokes' second problem for a ring-shaped container pendulum coupled to a torsional spring. Response of active fluids with large (a,b: $\kappa_w = 1.5/\Lambda$) and small (c,d: $\kappa_w = 0.63/\Lambda$) vortex-size distributions to oscillatory boundary conditions (App. B). The boundary speed is sinusoidal with amplitude $A$ and angular frequency $\Omega$. (b,d) The activity-induced relative change $\lambda$ in the effective inertia experienced by the ring pendulum. Negative values of $\lambda$ imply that the pendulum oscillates at a higher frequency in an active fluid than in a passive fluid.

4.4 Stokes’ second problem in active fluids

To connect with Stokes' second problem, we next consider the motion of a ring pendulum consisting of a circular container coupled to a torsional spring. The torque exerted by an active fluid of mass $m_f$ on the ring is (App. B)

$$T = -\frac{m_f}{\pi} \int_0^{2\pi} d\theta \sigma_{r\theta}(t, R, \theta),$$  \hspace{1cm} (4.2)

with $\sigma_{r\theta}$ the normal-tangential component of the stress tensor (1.1c) in polar coordinates. Our simulations show that relation between $T$ and the angular speed of the ring, $\dot{\phi} = \frac{v_\theta}{R}$, is dominated by a linear response (App. B),

$$T = -I_R \ddot{\phi} - \gamma \dot{\phi},$$  \hspace{1cm} (4.3)
with the inertial and dissipative parameters $I_f$ and $\gamma$ depending on the driving frequency, geometry and fluid parameters. For passive fluids at low Reynolds number, Eqs. (4.3) holds exactly, and $I_f$ and $\gamma$ can be calculated for simple geometries, owing to the linearity of the Stokes' equations [119]. For our active fluid model, we can determine $I_f$ and $\gamma$ directly from the numerically measured power spectral densities (App. B). To find out how activity affects the pendulum frequency, we follow Stokes’ original argument [198] and balance $T$ with the torque exerted by the torsional spring of stiffness $k$, which yields

$$(I_c + I_f)\ddot{\phi} + \gamma \dot{\phi} + k\phi = 0,
$$

where $I_c = m_c R^2$ is moment of inertia of a ring of mass $m_c$. Since the effect of $\gamma$ is generally quite small (App. B), we find that to leading order $v_\phi(t, R, \theta) = A \cos (\Omega t)$, where $\Omega = [k/(I_c + I_f)]^{1/2}$. For passive fluids, this is exactly the result obtained by Stokes’, who concluded that the added fluid inertia $I_f$ reduces a pendulum’s frequency $\Omega$. Moreover, by expressing $I_f$ in terms of viscosity, he was then able to explain several puzzling experiments [198]. For parameters relevant to microbial experiments, a passive fluid essentially behaves as a rigid body since the penetration depth $\sqrt{2\Gamma_0/\Omega}$ is much larger than the container radius $R$ (App. B). In this case, the moment of inertia of the passive fluid equals that of a solid disk, $I_{fp} = \frac{1}{2} m_f R^2$. Using $I_{fp}$ as a natural reference point, we express the effective inertia of an active fluid as $I_{fa} = (1 + \lambda) I_{fp}$, where $\lambda$ is the relative added inertia due to activity. To explore how confinement geometry, driving protocol (App. B) and active fluid properties affect $\lambda$, we varied systematically the amplitude $A$, the oscillation frequency $\Omega$, and the container radius $R$ in our simulations, comparing active fluids with wide ($\kappa_w = 1.5/\Lambda$) and small ($\kappa_s = 0.63/\Lambda$) energy injection bandwidths, respectively (Fig. 4-2). Interestingly, we find that for both values of $\kappa$, the added inertia is negative, $\lambda < 0$, across a wide range of driving frequencies $\Omega$ and amplitudes $A$ [Figs. 4-2(b,d)]. This implies that the fluid activity effectively reduces the amount of inertia transferred to the pendulum, and hence increases the oscillation frequency compared with a passive
fluid. At high frequencies $\Omega \gg 2\pi/\tau$, which can be achieved by using sufficiently stiff springs, $\lambda \approx -1$ implying that the pendulum does not acquire additional inertia and oscillates as if placed in a vacuum. In this regime, the bulk flow effectively decouples from the boundary due to the presence of defects in the stress field.

4.5 Einstein–de Haas analogue effect

Last but not least, we consider an EdH-type container-fluid system isolated from external forces or torques ($\gamma, k \to 0$) so that the container responds solely to the stresses generated by the enclosed fluid. In passive fluids, viscosity dissipates energy and such a system will eventually converge to a state of rest or rigid rotation if it had nonzero initial angular momentum. By contrast, active fluids are continuously supplied with kinetic energy through conversion of chemical energy and may thus induce a permanent dynamic response of the container. Focusing as before on a thin rigid ring-shaped container governed by Newton’s second law, the angular dynamics of the ring is determined by (App. B)

$$\ddot{\phi} = -\frac{\alpha}{\pi} \int_0^{2\pi} d\theta \sigma_{\phi}(t, R, \theta), \quad (4.5)$$

where $\alpha = m_f/m_c$ is the ratio of total fluid mass and ring mass. We solve Eqs. (4.1) and (4.5) simultaneously using $V(t) = R\dot{\phi}$ as boundary condition for Eqs. (4.1).

To interpret the simulation results, we note that the characteristic length and time scales $\Lambda$ and $\tau$ of an active fluid give rise to a natural unit of angular momentum. Regarding a single vortex as a thin rigid disk of radius $\Lambda/2$ rotating at the constant angular speed $2\pi/\tau$, one finds the characteristic kinematic angular momentum $L_v = \pi^2 \Lambda^4/(16\tau)$. A planar disk of radius $R$ can carry about $N_v = (2R/\Lambda)^2$ vortices, so it is convenient to introduce the normalization factor $\ell = \sqrt{N_v L_v}$. Adopting $\ell$ as a basic unit, one would expect specific angular momentum fluctuations of order one if $N_v$ vortices contributed randomly in an uncorrelated manner. Larger fluctuations indicate correlated collective angular momentum transfer between vortices and the
Figure 4-3: Einstein–de Haas analogue effect for an active fluid with narrow vortex-size distribution $\kappa_s = 0.63/\Lambda$. When isolated, the fluid can significantly shake the enclosing container, a thin rigid ring of radius $R$. (a) The standard deviation $\sigma_L$ of the container angular momentum $L$ depends on the radius $R$ and the fluid-to-ring mass ratio $\alpha$. The fluctuations $\sigma_L$ are independent of $\alpha$ for heavy containers ($\alpha \ll 1$) but start to decrease monotonically with $\alpha$ when the containers become light ($\alpha \gg 1$). As $R$ varies, the fluctuations oscillate with the period set by the characteristic vortex scale $\Lambda$ (see also App. B). Black dots represent 323 simulated parameter pairs, the color code shows linear interpolation. (b) Representative time series of the container angular momentum for two different radii $R = 3.67\Lambda$ and $R = 3.33\Lambda$ and fixed mass ratio $\alpha = 1$. (c) The standard deviation $\sigma_\dot{\phi} \sim \alpha \sigma_L$ of the container angular speed $\dot{\phi}$. (d) Horizontal cuts through (a) and (c) at a constant radius of $R = 4\Lambda$. In particular, to maximize both the angular momentum and velocity fluctuations, the fluid mass should match the container mass ($\alpha \sim 1$).

Focusing on an active fluid with narrow vortex-size distribution ($\kappa_s = 0.63/\Lambda$), we performed parameter scans to determine how the standard deviations $\sigma_L$ and $\sigma_\dot{\phi}$ of the ring’s angular momentum $L$ and angular speed $\dot{\phi}$ depend on the ring radius $R$ and fluid-to-ring mass ratio $\alpha$. Our simulations show that for a heavy container ($\alpha \ll 1$), the fluctuations $\sigma_L$ are approximately independent of $\alpha$, in which case their magnitude is the same as if the boundary was held fixed (cf. Fig 4-1). Once the container becomes lighter ($\alpha \sim 1$), the fluctuations start to decrease, with the decay rate approaching $1/\alpha$ for very light containers ($\alpha \gg 1$) [Figs 4-3(a,d)]. Similarly, the angular velocity fluctuations $\sigma_\dot{\phi} \sim \alpha \sigma_L$ are independent of $\alpha$ for light containers,
but increase linearly for heavy containers. In particular, $\sigma_\phi$ vanishes as $\alpha \to 0$, implying that the container becomes stationary as its mass becomes very large, as expected [Figs 4-3(c,d)]. We also conclude that to maximize the angular velocity fluctuations $\sigma_\phi$ without significantly reducing the angular momentum transfer to the boundary the fluid mass should match the container mass ($\alpha \sim 1$). Strikingly, we find that the fluctuations oscillate as a function of $R$, with the period set by the vortex size $\Lambda$ [Figs 4-3(a,c); App. B]. This result corroborates the idea that non-monotonic energy spectra, which the system (1.2) develops [186], generically result in oscillatory forces on boundaries [123]. For optimal combinations of $(\alpha, R)$, $\sigma_L$ can be more than an order of magnitude larger than $\ell$, indicating that the bulk vortices transfer angular momentum to the container collectively [Fig. 4-3(b)]. Such giant non-equilibrium fluctuations offer a novel way of extracting work from active suspensions (e.g., with fluctuation-driven microelectrical alternators), complementing recently proposed ratchet-based devices [192, 127]. For example, at the peak of the bacterial activity we may take $\tau \sim 2s$ and $\Lambda \sim 50\mu m$ [59, 186]. At the mass ratio $\alpha \sim 1$ and the container radius $R = 200\mu m$, Fig. 4-3(d) gives $\sigma_\phi \sim 0.02(2\pi/\tau) \sim 0.06\, \text{rad}/\text{s}$, comparable with the rotation rates of bacteria-powered microscopic gears of similar size reported in [192].

In conclusion, recent experiments [225, 227, 226, 231] have successfully utilized the interplay between characteristic flow pattern scales in active turbulence and confinement geometry to rectify and stabilize collective dynamics in natural and synthetic microswimmer suspensions. The above analysis extends these ideas to the time domain to achieve dynamic control, similar in spirit to actuation-controlled classical turbulence [202]. Our two main predictions about activity-induced reduction of fluid inertia and geometrically quantized giant fluctuations for a freely suspended container-fluid system should be testable with recently developed experimental techniques.
Chapter 5

GNS equations on curved surfaces

5.1 Introduction

Recent experiments [105, 183, 43] demonstrate the importance of substrate curvature for actively forced fluid dynamics. Yet, the covariant formulation and analysis of continuum models for nonequilibrium flows on curved surfaces still poses theoretical challenges. Here, we introduce and study a covariant GNS model for fluid flows driven by active stresses in nonplanar geometries. The analytical tractability of the theory is demonstrated through exact stationary solutions for the case of a spherical bubble geometry. Numerical simulations reveal a curvature-induced transition from a burst phase to an anomalous turbulent phase that differs distinctly from externally forced classical 2D Kolmogorov turbulence. This new type of active turbulence is characterized by the self-assembly of finite-size vortices into linked chains of anti-ferromagnetic order, which percolate through the entire fluid domain, forming an active dynamic network. The coherent motion of the vortex chain network provides an efficient mechanism for upward energy transfer from smaller to larger scales, presenting an alternative to the conventional energy cascade in classical 2D turbulence.

Substrate geometry profoundly affects dynamics and energy transport in complex fluids flowing far from equilibrium [49, 105, 238]. Examples range from magneto-hydrodynamic turbulence on stellar surfaces [49] to the rich microscale dynamics of topological defects in active nematic vesicles [105, 238]. Studying the interplay between spatial curvature and actively driven fluid flows is also essential for understanding microbial locomotion [183], biofilm formation [43] and bioremediation [51] in soils [30], tissues [47] and water [169, 170, 168]. Over the past two decades, important breakthroughs have been made in characterizing active-stress driven matter flows in planar Euclidean geometries both theoretically [139, 228, 175, 35] and experimentally [195, 232, 59]. More recently, theoretical work has begun to focus on incorporating curvature effects into active matter models [98, 176, 90, 108, 2, 68, 69]. Despite some promising progress, the hydrodynamic description of pattern-forming nonequilibrium liquids in non-Euclidean spaces continues to pose conceptual challenges, attributable to the difficulty of formulating exactly solvable continuum models and devising efficient spectral methods in curved geometries.

Aiming to help improve upon these two issues, we introduce and investigate here the covariant extension of the GNS model (1.2) describing incompressible active fluid flow on an arbitrarily curved surface. Focusing on a spherical ‘bubble’ geometry, we derive exact stationary solutions and numerically explore the effects of curvature on the steady-state flow dynamics, using the open-source spectral code Dedalus [37]. The numerically obtained phase diagrams, energy spectra and flux curves predict an anomalous turbulent phase when the spectral bandwidth of the active stresses becomes sufficiently narrow. This novel type of 2D turbulence supports an unexpected upward energy transfer mechanism, mediated by the large-scale collective dynamics of self-organized vortex chains, akin to actively moving anti-ferromagnetic spin chains. At high curvature, the anomalous turbulence transforms into a quasi-stationary burst phase, whereas for broadband spectral forcing the flow dynamics transitions to classical 2D Kolmogorov turbulence, accumulating energy in a few large-scale vortices. We next motivate and define the covariant GNS model for an arbitrary 2D surface; analytical and numerical results for the sphere case will be discussed subsequently.
5.2 Covariant formulation of the polynomial GNS model

Recent experiments have investigated the collective dynamics of swimming bacteria [195] and algae [84] in thin quasi-2D soap films held by a coplanar wire frame. Generalizing to non-Euclidean geometries [105, 238], which can be realized with soap bubbles or curved wire frames [80], we consider here a free-standing nonplanar 2D film in which the fluid flow is driven by active stresses, as in suspensions of swimming bacteria [121, 57] or ATP-driven microtubule networks [162, 177]. On a curved manifold, the fluid velocity field components \( v^a \) satisfy incompressibility and Cauchy momentum conservation

\[
\nabla_a v^a = 0, \quad (5.1a)
\]

\[
\partial_t v^a + v^b \nabla_b v^a = \nabla^a \sigma + \nabla_b T^{ab}, \quad (5.1b)
\]

where \( \nabla_b v^a \) denotes the covariant derivative of \( v^a \), \( a, b = 1, 2 \) and \( \sigma \) is the (surface) tension. The stress tensor \( T^{ab} \) includes passive and active contributions from the solvent fluid viscosity and the stresses exerted by the microswimmers on the fluid. Below, we study the covariant version of the linear active-stress model [19, 212, 185, 186]

\[
T^{ab} = f(\nabla^2)(\nabla^a u^b + \nabla^b u^a), \quad (5.1c)
\]

\[
f(\nabla^2) = \Gamma_0 - \Gamma_2 \nabla^2 + \Gamma_4 \nabla^2 \nabla^2,
\]

where \( \nabla^2 = \nabla^a \nabla_a \) is the tensor Laplacian. In qualitative agreement with experimental observations for active suspensions [55, 195, 59, 177], the polynomial ansatz for \( f \) in Eq. (5.1c) generates vortices of characteristic size \( \Lambda \) and growth time \( \tau \), provided that \( \Gamma_2 < 0 \), which introduces a bandwidth \( \kappa \) of linearly unstable modes (Chapter 2). General mathematical stability considerations demand \( \Gamma_0, \Gamma_4 > 0 \). The phenomenological model (5.1) is minimal in the sense that it assumes the active stresses create
to leading order a linear instability, while neglecting energy transfer within the active component. As verified in Chapter 2, the linear active-stress model (5.1c) suffices to quantitatively reproduce the experimentally measured velocity distributions and flow correlations in 3D bacterial [59] and ATP-driven microtubule [177] suspensions.

5.3 Analytical solutions

Exact stationary solutions of Eqs. (5.1) for a sphere of radius \( R \) can be constructed from the vorticity-stream function formulation (App. C)

\[
\begin{align*}
\Delta \psi &= -\omega, \\
\partial_t \omega + \{\omega, \psi\} &= f(\Delta + 4K)(\Delta + 2K)\omega,
\end{align*}
\]  

(5.2a, 5.2b)

where \( \psi \) and \( \omega \) are the stream function and vorticity. The advection term in spherical coordinates \((\theta, \phi)\) reads \( \{\omega, \psi\} = (\partial_\theta \omega \partial_\phi \psi - \partial_\phi \omega \partial_\theta \psi) / (R^2 \sin \theta) \). \( K = R^{-2} \) is the Gaussian curvature and \( \Delta \) the standard spherical Laplacian. Since the spherical harmonics \( Y^m_\ell \) diagonalize the Laplacian, \( \Delta Y^m_\ell = -R^{-2} \ell(\ell + 1)Y^m_\ell \) for integers \( \ell, m \) such that \( \ell \geq 0 \) and \( |m| \leq \ell \), an arbitrary superposition

\[
\psi = \sum_{|m| \leq \ell} \psi_{m\ell} Y^m_\ell
\]

(5.3)

solves the system (5.2) exactly, provided that the eigenvalue \( \ell \) is an integer root of \( f(-\ell(\ell + 1) + 4) = 0 \) (App. C). As usual, the velocity field is tangent to the level sets of the stream function. Two particular exact solutions are shown in Fig. 5-1. The first example, Fig. 5-1(a), is reminiscent of the square lattice solutions found earlier in the flat 2D case [185]. The second example in Fig. 5-1(b) illustrates a flow field with five-fold symmetry, obtained by applying the superposition procedure of Ref. [159]. Although these exact solutions are not stable, they provide some useful intuition about the instantaneous flow patterns expected in dynamical simulations (Fig. 5-2), similar to exact coherent structures [218] in conventional turbulence [219].
Figure 5-1: Stationary solutions of Eqs. (5.2) are superpositions of the form (5.3) with \( f(-\ell(\ell+1)+4) = 0 \). (a) An exact stationary solution with \( \ell = 6 \) which is also approximately realized as a transient state in the time-dependent burst solution of Fig. 5-2. (b) Complex symmetric solutions can be constructed by choosing the expansion coefficients \( \psi_{mt} \) accordingly [159]. In both panels, the stream functions are normalized by their maxima; see App. C for coefficients \( \psi_{mt} \).

5.4 Numerical method

To find and analyze time-dependent solutions of Eqs. (5.1), we performed numerical simulations using Dedalus [37], an open-source framework for solving differential equations with spectral methods. The equations (5.1) were solved directly as a coupled partial differential-algebraic system for the scalar tension \( \sigma \) and vector velocity \( v^a \). To spatially discretize the system, we used spin-weighted spherical harmonics, which are a parameterized family of basis functions that correctly capture the analytical behavior of spin-weighted functions on the sphere (App. C). Under this spectral expansion, the system (5.1) is reduced to a set of coupled ordinary differential-algebraic equations for the time evolution of the expansion coefficients. We solve these equations using mixed implicit-explicit time stepping, in which the linear terms of the evolution equations are integrated implicitly, the linear constraints are enforced implicitly, and the nonlinear terms are integrated explicitly. This allows us to simultaneously evolve the velocity field while enforcing the incompressibility constraint, and with a time step that is limited by the advective Courant-Friedrichs-Lewy time condition rather than the diffusive time at any scale.
Figure 5-2: Phase diagrams (a,b) and representative snapshots (c-e) from simulations showing quasi-stationary burst dynamics (B-phase), anomalous vortex-network turbulence (A-phase), and classical 2D turbulence (T-phase). (a,b) The A- and T-phase are approximately separated by the condition $\kappa \Lambda = 1$ (vertical dashed line) and differ by the average number of vortices (a), the geometry of the branches in the tension field (b), and the energy spectra (Fig. 5-3). The B-phase arises for narrowband energy injection $\kappa R \lesssim 1$ when only a single $\ell$-mode is active (region right below the dashed-dotted line); decreasing $\kappa$ further gives a passive fluid (white region). (c-e) Top: Instantaneous vorticity fields normalized by their maxima. Bottom: Surface tension fields normalized by the maximum deviation from the mean. (c) Quasi-stationary pre-burst state resembling the exact solution in Fig. 5-1(a). (d) For subcritical curvature and intermediate energy injection bandwidths, $R^{-1} < \kappa < \Lambda^{-1}$, the flows develop a percolating vortex-chain network structure, with an accumulation of tension and vorticity along the edges. (e) For broadband energy injection $\kappa \Lambda > 1$, smaller eddies merge to create larger vortices, as typical of classical 2D turbulence. Parameters: (a) $\alpha_\omega = 0.5$; (c) $R/\Lambda = 2$, $\tau = 4.9$ s, $\kappa \Lambda = 0.29$; (d) $R/\Lambda = 10$, $\tau = 14.9$ s, $\kappa \Lambda = 0.5$; (e) $R/\Lambda = 10$, $\tau = 11.7$ s, $\kappa \Lambda = 2.0$. Panels (a, b) show steady-state time averages over $[50\tau, 100\tau]$. Solid curves in (c-e) indicate streamlines of the velocity fields.

5.5 Flow statistics and interscale energy transfer

The parameters $(\Gamma_0, \Gamma_2, \Gamma_4)$ in Eqs. (5.1) define a characteristic time scale $\tau$, a characteristic vortex diameter $\Lambda$, and a characteristic spectral bandwidth $\kappa$, which can be directly inferred from experimental data [186]; explicit expressions are derived in the App. C. Given a sphere of radius $R$, fixing $(\tau, \Lambda, \kappa)$ uniquely determines the parameters $(\Gamma_0, \Gamma_2, \Gamma_4)$. To explore the interplay between curvature and activity, we run 351 simulations, using $R/\Lambda \in [2, 10]$ and $\kappa \cdot \Lambda \in [0.1, 2.0]$. Typical vortex diameters for
bacterial and microtubule suspensions are $\Lambda \sim 50 - 100$ $\mu$m with $\tau$ of the order of seconds [55, 195, 177, 59]. Time steps were in the range $[5 \cdot 10^{-4} \tau, 5 \cdot 10^{-3} \tau]$ with a total simulation time $100\tau$, allowing the system to fully develop its dynamics after an initial relaxation phase during which active stresses inject energy until the viscous dissipation and activity balance on average. In the remainder, it will be convenient to regard $\Lambda$ as reference length and compare the flow topologies across the $(\kappa, R)$ parameter plane.

Our simulations reveal three qualitatively distinct flow regimes (Fig. 5-2): a quasi-stationary burst phase for $\kappa R \lesssim 1$ [domain B in Fig. 5-2(a)], an anomalous turbulence for $R^{-1} < \kappa < \Lambda^{-1}$ [domain A in Fig. 5-2(a)], and normal 2D turbulence for $\kappa \Lambda > 1$ [domain T in Fig. 5-2(a)]. Representative vorticity and tension fields from the corresponding steady-state dynamics are shown in Fig. 5-2(c-e).

In the B-phase, the energy injection bandwidth $\kappa$ is close to the wavenumber spacing set by the sphere curvature $R^{-1}$, leaving only a single active wavenumber $\ell$. Decreasing $\kappa$ further completely suppresses active modes resulting in globally damped fluid motion [white domain in Fig. 5-2(a)]. The B-phase is characterized by the formation of intermittent quasi-stationary flow patterns that lie in the vicinity of the exact stationary solutions (5.3), cf. Figs. 5-1(a) and 5-2(c). Once formed, the amplitude of these flow patterns grows exponentially (Fig. S3) until nonlinear advection becomes dominant and eventually causes energy to be released through a rapid burst. Afterwards, the dynamics become quasi-linear again with the flow settling into a new quasi-stationary pattern. These burst cycles are continuously repeated.

The two turbulent phases A and T in Fig. 5-2(a) can be distinguished through topological, geometric and spectral measures. We demonstrate this by determining the topology of the vorticity fields, the geometry of the high-tension domains and the energy spectra for each simulation after flows had reached the chaotic steady-state.

To study the vortex topology, we fix a threshold $\alpha_\omega \in [0, 1]$ and identify regions in which the vorticity is larger (or smaller) than $\alpha_\omega$ times the maximum (or minimum) vorticity (App. C). This thresholding divides the sphere into patches of high absolute vorticity (Fig. S1). The number of connected domains, given by the zeroth Betti
number, counts the vortices in the system. For a fixed pair \((\kappa, R)\), we denote the vortex number at time \(t\) by \(N_\omega(\kappa, R; t)\). Although more sophisticated methods for vortex detection exist [100], the thresholding criterion proved to be sufficient for our analysis (Fig. S2). To normalize vortex numbers across the parameter space, we fix a reference value \(\kappa_* = 0.3/\Lambda\). With this, we can define a normalized Betti number as

\[
\text{Betti}_\omega(\kappa, R) = \frac{\langle N_\omega(\kappa, R; t) - N_\omega(\kappa_*, R; t) \rangle}{\langle N_\omega(\kappa_*, R; t) \rangle},
\]

where the time average \(\langle \cdot \rangle\) is taken after the initial relaxation period. Intuitively, large values of \(\text{Betti}_\omega\) indicate many vortices of comparable circulation, whereas small values suggest the presence of a few dominant eddies. The variation of \(\text{Betti}_\omega\) in the \((\kappa, R)\)-parameter plane is color-coded in Fig. 5-2(a). In the anomalous turbulent A-phase, vortices of diameter \(\approx \Lambda\) eventually cover the surface of the sphere, with stronger vortices forming chains of anti-ferromagnetic order [Fig. 5-2(d) top]. By contrast, in the T-phase characterized by broadband energy injection \(\kappa > \Lambda^{-1}\), smaller eddies merge to create a small number of larger vortices, as typical of classical 2D turbulence [29] [Fig. 5-2(e) top]. Interestingly, the A-phase shares phenomenological similarities with the low-entropy states found in quasi-2D superfluid models [182], while the vortex-condensation in the T-phase corresponds approximately to the negative “temperature” regime in Onsager’s statistical hydrodynamics [154]. Moreover, the upper region of Fig. 5-2(a), which corresponds to the small-curvature limit \(R/\Lambda \gg 1\), suggests that the two phases extend to planar geometries, provided boundary effects remain negligible.

To obtain a more detailed geometric characterization of the turbulent A- and T-phases, we next consider the corresponding tension fields. Analogously to the case of vorticity above, we focus on regions where the local tension \(\sigma(t, x)\) is larger than the instantaneous global mean value. For each connected component of the identified high-tension regions, we denote by \(A\) its total area and by \(\partial A\) its total boundary area in pixels. The ratio \(\partial A/A\) is a measure of chain-like structures in the tension fields, a large value signaling a highly branched structure, whereas smaller values
Figure 5-3: Time-averaged energy spectra and fluxes indicate two qualitatively different types of upward energy transport. (a) For narrowband energy injection $\kappa \Lambda < 1$, the energy spectrum exhibits a peak corresponding to the dominant vortex size $\Lambda$ (red curve). For broadband injection $\kappa \Lambda \sim 2$, the spectra decay monotonically (blue and black curves). (b) In all four examples, the fluxes confirm inverse energy transport, albeit with different origins. For broadband energy injection (blue and black curves), the upward energy flux to larger scales is due to vortex mergers [Fig. 5-2(e)]. By contrast, for narrowband injection (red curve), a relatively stronger upward energy flux arises from the collective motion of vortex chains [Fig. 5-2(d)]. The shaded regions indicate the energy injection ranges with colors matching those of the corresponding curves, respectively. Parameters: $R/A = 10$ for a unit sphere, $\tau = 11.7s$, time step $5 \cdot 10^{-4} \tau$, total simulation time $500\tau$. Spectra and fluxes were determined after relaxation by averaging over $[150\tau, 500\tau]$. For $\kappa \Lambda > 1$, energy steadily accumulates at larger scales and the absence of a large-scale dissipative mechanism leads to a divergent total enstrophy and kinetic energy on the sphere.

Indicate less branching. Denoting the instantaneous sum of the ratios $\partial A/A$ over all connected high-tension domains by $A_\sigma(\kappa, R; t)$, a normalized branching index can then be defined by (App. C)

$$\text{Branch}_\sigma(\kappa, R) = \frac{\langle A_\sigma(\kappa, R; t) - A_\sigma(\kappa_\ast, R; t) \rangle}{\langle A_\sigma(\kappa_\ast, R; t) \rangle},$$

(5.5)

where the time average is again taken after the initial relaxation. As evident from the phase diagram in Fig. 5-2(b) and the corresponding tension fields in Fig. 5-2(d,e) the geometric characterization confirms the existence of an anomalous turbulent phase, in which vortices combine to form percolating dynamic networks with high-tension being localized along the edges [Fig. 5-2(d) bottom].
To compare the energy transport in the anomalous turbulent phase with classical 2D turbulence, we analyze the energy spectra and fluxes for the A- and T-phases. Expanding in spherical harmonics, $\psi = \sum_{m,\ell} \psi_{m\ell} Y_{\ell}^m$, the energy of mode $\ell$ is $E(\ell) = \sum_{|m| \leq \ell} \ell(\ell+1)|\psi_{m\ell}|^2$. The corresponding mean energy flux across $\ell$ in the statistically stationary state is obtained as (App. C)

$$\Pi(\ell) = -2 \sum_{\ell' \geq \ell} f[4 - \ell'(\ell' + 1)](2 - \ell'(\ell' + 1)](E\ell')$$

where $f$ is the polynomial defined in Eq. (5.1c). Figure 5-3 shows the numerically obtained energy spectra $E(\ell)$ and fluxes $\Pi(\ell)$ for four active bandwidths $\kappa$. In all four cases, the kinetic energy produced in the injection range ($\ell \sim \pi R/\Lambda$) propagates to both large ($\ell < \pi R/\Lambda$) and small ($\ell > \pi R/\Lambda$) scales, as indicated by negative and positive values of $\Pi(\ell)$, respectively. Energy transfer to large scales is a prominent feature of classical 2D turbulence [116, 29, 199] and our results show that it also occurs in active turbulence. However, the transfer mechanisms can be dramatically different, as already implied by the preceding analysis of the vorticity and tension fields. For broadband spectral forcing $\kappa \Lambda \gg 1$, the classical 2D turbulence picture of vortex mergers and energy condensation at large scales prevails [Fig. 5-2(e)]. For $\kappa \Lambda \lesssim 2$ the spectrum follows a $k^{-1}$-scaling, indicating the formation of a dilute-vortex system [117]. For even larger values of $\kappa$, additional large-scale dissipation is needed to bound the upward energy transfer, in which case the spectrum is expected to approach the Kolmogorov $k^{-5/3}$-scaling [29]. By contrast, for narrowband driving $\kappa \Lambda \lesssim 1$, the upward energy transfer is realized through the coherent motion of high-tension vortex chains. Interestingly, only this anomalous type of inverse energy cascade appears to persist in 3D active bulk fluids [186], where it is sustained by spontaneous chiral symmetry-breaking [27].

In summary, we have presented analytical and numerical solutions for the GNS equations describing actively driven nonequilibrium flows on a sphere. Our calculations predict that spectrally localized active stresses can induce a novel turbulent phase, in which finite-size vortices self-organize into chain complexes of anti-
ferromagnetic order that percolate through the surface [56]. The collective motion of these chain networks enables a significant upward energy transport and may thus provide a basis for efficient fluid mixing in quasi-2D active and magnetohydrodynamic flows. Future generalizations to rotating spheres could thus promise insights into pattern formation in planetary and stellar atmospheres [70].
Chapter 6

Spontaneous mirror-symmetry breaking in 3D

6.1 Introduction

Classical turbulence theory assumes that energy transport in a 3D turbulent flow proceeds through a Richardson cascade whereby larger vortices successively decay into smaller ones [72]. By contrast, an additional inverse cascade characterized by vortex growth exists in 2D fluids and gases, with profound implications for meteorological flows and fluid mixing [29]. The possibility of a helicity-driven inverse cascade in 3D fluids had been rejected in the 1970s based on equilibrium-thermodynamic arguments [115]. Recently, however, it was proposed that certain symmetry-breaking processes could potentially trigger a 3D inverse cascade [216, 27], but no physical system exhibiting this phenomenon has been identified to date. In this chapter, we present analytical and numerical evidence for the existence of an inverse energy cascade in the polynomial 3D active turbulence model (1.2), describing microbial suspension flows that spontaneously break mirror symmetry. We show analytically

that self-organized scale selection, a generic feature of many biological and engineered nonequilibrium fluids, can generate parity-violating Beltrami flows. Our simulations further demonstrate how active scale selection controls mirror-symmetry breaking and the emergence of a 3D inverse cascade.

Turbulence, the chaotic motion of liquids and gases, remains one of the most widely studied phenomena in classical physics [72, 141]. Turbulent flows determine energy transfer and material mixing over a vast range of scales, from the interstellar medium [92, 65] and solar winds [36] to the Earth’s atmosphere [148, 130], ocean currents [206], and our morning cup of coffee. Of particular recent interest is the interplay of turbulence and active biological matter [61], owing to its relevance for carbon fixation and nutrient transport in marine ecosystems [203]. Although much has been learned about the statistical and spectral properties of turbulent flows both experimentally [152, 129, 235] and theoretically [112, 113, 111, 114, 116, 216, 106, 160] over the last 75 years, several fundamental physical and mathematical [66] questions remain. One of the most important among them, with profound implications for the limits of hydrodynamic mixing, concerns whether 3D turbulent flows can develop an inverse cascade that transports energy from smaller to larger scales [115, 216, 27].

Kolmogorov’s 1941 theory of turbulence [112] assumes that turbulent energy transport in 3D proceeds primarily from larger to smaller scales through the decay of vortices. This forward (Richardson) cascade is a consequence of the fact that the 3D inviscid Euler equations conserve energy [72]. In 1967, Kraichnan [114] realized that the presence of a second conserved quantity, enstrophy, in 2D turbulent flows implies the existence of two dual cascades [29]: a vorticity-induced cascade to smaller scales and an inverse energy cascade to larger scales [50, 106]. Two years later, Moffatt [145] discovered a new invariant of the 3D Euler equations, which he termed helicity. Could helicity conservation generate an inverse turbulent cascade in 3D? Building on thermodynamic considerations, Kraichnan [115] argued in 1973 that this should not be possible, but he also conceded that turbulent flows do not necessarily follow equilibrium statistics. Since then, insightful theoretical studies [216, 27] have elucidated other important conditions for the emergence of helicity-driven inverse cascades in 3D.
Parity

\[ P = 000 \]

represents active modes. The radius of the active shell II corresponds approximately to the inverse of characteristic pattern formation scale \( \Lambda \). The bandwidth \( \kappa \) measures the ability of the active fluid component to concentrate power input in Fourier space.

(b) Two examples of exact stationary bulk solutions of Eqs. (1.2) realizing Beltrami vector fields of opposite helicity, obtained from Eq. (6.4) by combining modes of the same helicity located on one of the marginally stable grey surfaces in (A). (c) Simulations with random initial condition spontaneously select one of two helicity branches. The histogram represents an average over 150 runs with random initial conditions, sampled over the statistically stationary state starting at time \( t = 20\tau \) (dashed line). Simulation parameters: \( \Lambda = 75 \mu m, U = 72 \mu m/s, \kappa_1 = 0.9/\Lambda, L = 8\Lambda \) (see also Fig. 6-2 and App. D for larger simulations).

fluids, in particular identifying mirror-symmetry breaking as a key mechanism [27]. However, no natural or artificially engineered fluid system exhibiting this phenomenon has been identified to date.

Our analysis in this Chapter predicts that fluid flows in active nonequilibrium liquids, such as bacterial suspensions, can spontaneously break mirror symmetry, resulting in a 3D inverse cascade. Broken mirror symmetry plays an important role in nature, exemplified by the parity-violating weak interactions [230] in the standard model of particle physics, by the helical structure of DNA [221] or, at the macroscale, by chiral seed pods [8]. Another, fluid-based realization [229] of a spontaneously broken chiral symmetry was recently observed in confined bacterial suspensions [227, 226], which form stable vortices of well-defined circulation when the container dimensions match the correlation scale \( \sim 70 \mu m \) of the collective cell motion in bulk [194, 59]. Motivated by these observations, we investigate the polynomial GNS model [184, 185] for pattern-forming nonequilibrium fluids that are driven by an active component,
which could be swimming bacteria [194, 59] or ATP-driven microtubules [177, 77] or artificial microswimmers [33, 220, 207].

To demonstrate the existence of a helicity-driven inverse cascade in 3D active bulk fluids, we first verify analytically the existence of exact parity-violating Beltrami-flow [9, 54, 63] solutions. We then confirm numerically that active bulk flows starting from random initial conditions approach attractors that spontaneously break mirror symmetry and are statistically close to Beltrami-vector fields. Finally, we demonstrate that the broken mirror symmetry leads to an inverse cascade with triad interactions as predicted by Waleffe [216] about 25 years ago.

6.2 Analytical solutions and parity violation

Exact Beltrami-flow solutions and broken mirror symmetry. The higher- order Navier-Stokes equations defined by Eqs. (1.2) are invariant under the parity transformation $x \rightarrow -x$. The solutions however, can spontaneously break this mirror symmetry. To demonstrate this explicitly, we construct a family of exact nontrivial stationary solutions in free space by decomposing the Fourier series $v(t, k)$ of the divergence-free velocity field $v(t, x)$ into helical modes [216, 27]

$$v(t, k) = u^+(t, k) h^+(k) + u^-(t, k) h^-(k), \quad (6.1)$$

where $h^\pm$ are the eigenvectors of the curl operator, $i k \times h^\pm = \pm k h^\pm$ with $k = |k|$. Projecting Eq. (1.2b) onto helicity eigenstates [216] yields the evolution equation for the mode amplitudes $u^\pm$

$$[\partial_t + \xi(k)] u^\pm(t, k) = \sum_{(p, q) : k + p + q = 0} f^\pm(t; k, p, q), \quad (6.2)$$
where $\xi(k) = \Gamma_0 k^2 + \Gamma_2 k^4 + \Gamma_4 k^6$ is the active stress contribution, and the nonlinear advection is represented by all triadic interactions [216, 27]

$$f_{sk}(t; k, p, q) = -\frac{1}{4} \sum_{s_p, s_q} (s_p p - s_q q) \times \left[ \overline{(h_{sp} \times \overline{h_{sq}})} \cdot \overline{h_{sk}} \right] \overline{u^{s_p}} \overline{u^{s_q}}$$

between helical $\mathbf{k}$-modes and $\mathbf{p}, \mathbf{q}$-modes, where $s_k, s_p, s_q \in \{ \pm \}$ are the corresponding helicity indices (overbars denote complex conjugates of $h_{sp} = h_{sp}(p)$, etc. [216]).

There are two degrees of freedom per wavevector, and hence eight types of interactions for every triple $(k, p, q)$. As evident from Eq. (6.3), arbitrary superpositions of modes with identical wavenumber $p = q = k$, and same helicity index annihilate the advection term, because $s_p p - s_q q = 0$ in this case. Therefore, by choosing $k_*$ to be a root of the polynomial $\xi(k)$, corresponding to the grey surfaces in Fig. 6-1(a), we obtain exact stationary solutions

$$v^\pm(x) = \sum_{k, k=k_*} u^\pm(k) h^\pm(k) e^{ikx}, \quad \text{(6.4)}$$

where $u^\pm(-k) = \overline{u^\pm(k)}$ ensures real-valued flow fields. In particular, these solutions (6.4) correspond to Beltrami flows [9, 54, 63], obeying $\nabla \times v^\pm = \pm k_* v^\pm$. Applying the parity operator to any right-handed solution $v^+(x)$ generates the corresponding left-handed solution $v^-(x)$ and vice versa [Fig. 6-1(b)].

Although the exact solutions $v^\pm(x)$ describe stationary Beltrami fields [9, 54, 63] of fixed total helicity $H^\pm = \int d^3 x \mathbf{v}^\pm \cdot \mathbf{w}^\pm$, where $\mathbf{w} = \nabla \times \mathbf{v}$ is the vorticity, it is not yet clear whether parity violation is a generic feature of arbitrary time-dependent solutions of Eqs. (1.2). However, as we will demonstrate next, simulations with random initial conditions do indeed converge to statistically stationary flow states that spontaneously break mirror symmetry and are close to Beltrami flows.

**Spontaneous mirror symmetry breaking in time-dependent solutions.**

We simulate the full nonlinear Eqs. (1.2) on a periodic cubic domain (size $L$) using a spectral algorithm (App. D.1). Simulations are performed for typical bac-
Figure 6-2: Active fluids spontaneously break mirror symmetry by realizing Beltrami-type flows. (a) Snapshot of a representative vorticity component field $\omega_x$ for an active fluid with small bandwidth $\kappa_S = 0.63/\Lambda$, as defined in Fig. 6-1(a). (b) The corresponding helicity field signals parity-symmetry breaking leading to a positive-helicity flow in this example. (c) Histograms of the angles between velocity $\mathbf{v}$ and vorticity $\mathbf{\omega}$ quantify the alignment between the two fields for different active bandwidths $\kappa_S < \kappa_1 < \kappa_W$: The smaller the bandwidth, the stronger the alignment between $\mathbf{v}$ and $\mathbf{\omega}$. (d) Numerically estimated distributions of the Beltrami measure, $\beta = \mathbf{v} \cdot \mathbf{\omega} / (|\lambda| |\mathbf{v}|^2)$, shown on a log-scale. An ideal Beltrami flow with $\mathbf{\omega} = \lambda \mathbf{v}$ produces a delta-peak centered at $\beta = 1$. Identifying $\lambda$ with the midpoint of the active shell ($\lambda \approx \pi/\Lambda$), which approximately corresponds to the most unstable wavenumber and the characteristic pattern formation scale, we observe that a smaller active bandwidth leads to a sharper peak and hence more Beltrami-like flows. Data were taken at a single representative time-point long after the characteristic relaxation time. Simulation parameters: $\Lambda = 75\mu m$, $U = 72 \mu m/s$, $L = 32\Lambda$. 

Material parameters ($\Gamma_0, \Gamma_2, \Gamma_4$), keeping the vortex scale $\Lambda = 75\mu m$ and circulation speed $U = 72 \mu m/s$ fixed [194, 59] and comparing three different spectral bandwidths $\kappa_S = 0.63/\Lambda = 8.4 \text{mm}^{-1}$, $\kappa_1 = 0.90/\Lambda = 12 \text{mm}^{-1}$ and $\kappa_W = 2.11/\Lambda = 28.1 \text{mm}^{-1}$, corresponding to active fluids with a small (S), intermediate (I) and wide (W) range of energy injection scales. A small bandwidth means that the active stresses inject energy into a narrow shell in Fourier space, whereas a wide bandwidth means energy is pumped into a wide range of Fourier modes [Fig. 6-1(a)]. All simulations are initiated with weak incompressible random flow fields. For all three values of $\kappa$, we observe spontaneous mirror-symmetry breaking indicated by the time evolution of the mean helicity $H = (1/L^3) \int d^3x \, h$, where $h = \mathbf{v} \cdot \mathbf{\omega}$ is the local helicity. During the initial relaxation phase, the flow dynamics is attracted to states of well-defined total helicity and remains in such a statistically stationary configuration for the rest of the simulation. As an illustration, Fig. 6-1(c) shows results from 150 runs for $\kappa = \kappa_1$ and $L = 8\Lambda$, with flow settling into a positive (negative) mean helicity state 72 (78).
times. This spontaneous mirror-symmetry breaking is robust against variations of the bandwidth and simulation box size, as evident from the local vorticity and helicity fields for $\kappa = \kappa_S$ and $L = 32\lambda$ in Fig. 6-2(a,b).

**Beltrami-flow attractors.** Having confirmed spontaneous parity violation for the time-dependent solutions of Eqs. (1.2), we next characterize the chaotic flow attractors. To this end, we measure and compare the histograms of the angles between the local velocity field $v(t, x)$ and vorticity field $\omega(t, x)$ for the three bandwidths $\kappa_S < \kappa_1 < \kappa_W$. Our numerical results reveal that a smaller active bandwidth, corresponding to a more sharply defined scale selection, causes a stronger alignment of the two fields [Fig. 6-2(c)]. Recalling that perfect alignment, described by $\omega = \lambda v$ with eigenvalue $\lambda$, is the defining feature of Beltrami flows [9, 54, 63], we introduce the Beltrami measure $\beta = v \cdot \omega / (\lambda |v|^2)$. For ideal Beltrami fields, the distribution of $\beta$ becomes a delta peak centered at $\beta = 1$. Identifying $\lambda$ with the midpoint of the active shell ($\lambda \approx \pi / \Lambda$), which approximately corresponds to the most dominant pattern formation scale in Eqs. (1.2), we indeed find that the numerically computed flow fields exhibit $\beta$-distributions that are sharply peaked at $\beta = 1$ [Fig. 6-2(d)]. Keeping $\Lambda$ and $U$ constant, the sharpness of the peak increases with decreasing active bandwidth $\kappa$. These results imply that active fluids with well-defined intrinsic scale selection realize flow structures that are statistically close to Beltrami fields, as suggested by the particular analytical solutions derived earlier.

### 6.3 Inverse cascade, triad interactions and mixing

**Spontaneous parity breaking vs. surgical mode removal.** Important previous studies identified bifurcation mechanisms [138, 48, 110] leading to parity-violation in 1D and 2D [73] continuum models of pattern-forming nonequilibrium systems [79, 210]. The above analytical and numerical results generalize these ideas to 3D fluid flows, by showing that an active scale selection mechanism can induce spontaneous helical mirror-symmetry breaking. Such self-organized parity violation can profoundly affect energy transport and mixing in 3D active fluids, which do not satisfy the
Figure 6-3: Scale selection controls mirror symmetry breaking and induces an inverse energy cascade. We demonstrate these effects for active fluids with a (a-d) small active bandwidth $\kappa_S$ and (e-h) wide bandwidth $\kappa_W$ [Fig. 6-1(a)]. The intermediate case $\kappa_I$ is presented in Fig. S3. (a) Energy spectra $e^{\pm}(k)$ of the helical velocity-field modes show strong symmetry breaking for small bandwidth parameter $\kappa_S$. In this example, the system spontaneously selects positive helicity modes, such that $e^+(k) > e^-(k)$ at all dominant wavenumbers. Dashed vertical lines indicate the boundaries of the energy injection domain II. (b) The resulting energy fluxes $H'(k)$ combine into the total flux $H(k)$, which is negative in region I and positive in III, signaling inverse and forward energy cascades, respectively. (c) Contributions to the energy flow ($T^{KLPQ}_{KLPQ}$) between the three spectral domains I, II, and III (18 possibilities, columns) from the eight types of triad interactions (rows). In reflection-invariant turbulence, this table remains unchanged under upside-down flipping (+ ↔ −). Instead, we observe a strong asymmetry, with two cumulative triads (d) dominating the energy transfer. Red and blue arrows represent transfer towards large and small scales, respectively, and thickness represents the magnitude of energy flow. Green arrows represent transfer within the same spectral domain. The direction of the energy flow is in agreement with the instability assumption of Waleffe [216]. In this case, 18.2% of the injected energy is transferred from region II to region I and 81.8% is transferred from II to III. (e-h) Same plots for an active fluid with wide active bandwidth $\kappa_W$. (e and f) Energy spectra show weaker parity breaking (e) and suppression of the inverse energy cascade (f). (g) The energy flow table partially recovers the upside-down (+ ↔ −) symmetry. (h) The most active triads now favor the forward cascade, so that only 1.1% of the injected energy flows into region I, while 98.9% are transferred into region III. Data represent averages over single runs (Fig. S2). Simulation parameters are identical to those in Fig. 6-2.
premises of Kraichnan's thermodynamic no-go argument [115]. An insightful recent study [27], based on the classical Navier-Stokes equation, found that an ad hoc projection of solutions to positive or negative helicity subspaces can result in an inverse cascade but it has remained an open question whether or not such a surgical mode removal can be realized experimentally in passive fluids. By contrast, active fluids spontaneously achieve helical parity-breaking [Fig. 6-1(c)] by approaching Beltrami flow states [Fig. 6-2(c,d)], suggesting the possibility of a self-organized inverse energy cascade even in 3D. Before testing this hypothesis in detail, we recall that the generic minimal model defined by Eqs. (1.2) merely assumes the existence of linear active stresses to account for pattern scale selection as observed in a wide range of microbial suspensions [55, 195, 59, 177], but does not introduce nonlinearities beyond those already present in the classical Navier-Stokes equations. That is, energy redistribution in the solvent fluid is governed by the advective nonlinearities as in conventional passive liquids.

**Inverse cascade in 3D active fluids.** To quantify how pattern scale selection controls parity breaking and energy transport in active fluids, we analyzed large-scale simulations \([L = 32\Lambda; \text{Fig. 6-2(a,b)}]\) for different values of the activity bandwidth \(\kappa\) [Fig. 6-1(a)] while keeping the pattern scale \(\Lambda\) and the circulation speed \(U\) fixed. The active shell [red domain II in Fig. 6-1(a)] corresponds to the energy injection range in Fourier space and provides a natural separation between large flow scales (blue domain I) and small flow scales (blue domain III). Consequently, the forward cascade corresponds to a net energy flux from domain II to domain III, whereas an inverse cascade transports energy from II to I. We calculate energy spectra \(e(k) = e^+(k) + e^-(k)\) and energy fluxes \(\Pi(k) = \Pi^+(k) + \Pi^-(k)\) directly from our simulation data, by decomposing the velocity field into helical modes as in Eq. (6.1), which yields a natural splitting into cumulative energy and flux contributions \(e^\pm(k)\) and \(\Pi^\pm(k)\) from helical modes \(u^\pm(k)\) lying on the wavenumber shell \(|k| = k\) (App. D.1). Time-averaged spectra and fluxes are computed for each simulation run after the system has relaxed to a statistically stationary state (Fig. D-1). For a small injection bandwidth \(\kappa_s\), the energy spectra \(e^\pm(k)\) reflect the broken mirror-symmetry, with
most of the energy being stored in either the positive-helicity or the negative-helicity modes [Fig. 6-3(a)], depending on the initial conditions. Moreover, in addition to the expected 3D forward transfer, the simulation data for $\kappa_S$ also show a significant inverse transfer, signaled by the negative values of the total flux $\Pi(k)$ [yellow curve in Fig. 6-3(b)] in domain I. As evident from the blue curves in Fig. 6-3(a) and (b), this inverse cascade is facilitated by the helical modes that carry most of the energy. For a large injection bandwidth $\kappa_W \gg \kappa_S$, the energy spectra continue to show signatures of helical symmetry breaking [Fig. 6-3(e)], but the energy transported to larger scales becomes negligible relative to the forward cascade, as contributions from opposite-helicity modes approximately cancel in the long-wavelength domain I [Fig. 6-3(f)]. Results for the intermediate case $\kappa_M$ still show a significant inverse transfer [Fig. S3, S4(e)], demonstrating how the activity bandwidth – or, equivalently, the pattern selection range – controls both parity violation and inverse cascade formation in an active fluid. The upward transfer is non-inertial at intermediate scales, as indicated by the wavenumber dependence of the energy flux [Fig. 6-3(b)]. At very large scales $\gg \Lambda$, however, the flux approaches an inertial plateau (App. D.2). In contrast to the energy-mediated 2D inverse cascade in passive fluids, which is characterized by vortex mergers, the helicity-driven 3D inverse cascade in active fluids is linked to the formation of extended vortex chain complexes that move collectively through the fluid (App. D.2).

**Triad interactions.** Our numerical flux measurements confirm directly the existence of a self-sustained 3D inverse cascade induced by spontaneous parity violation, consistent with earlier projection-based arguments for the classical Navier-Stokes equations [27]. An inverse energy cascade can exist in 3D active fluids because mirror symmetry breaking favors only a subclass of all possible triad interactions, which describe advective energy transfer in Fourier space between velocity modes \{\(v(k), v(p), v(q)\)\} with \(k + p + q = 0\), cf. Eq. (6.3). To analyze in detail which triads are spontaneously activated in a pattern-forming nonequilibrium fluid, we consider combinations \(K, P, Q \in \{I, II, III\}\) of the spectral domains in Fig. 6-1(a) and distinguish modes by their helicity index \(s_K, s_P, s_Q \in \{\pm\}\). The helicity-resolved integrated
energy flow into the region \((K, s_K)\) due to interaction with regions \((P, s_P)\) and \((Q, s_Q)\) is given by (App. D.1)

\[
\mathcal{J}_{KPQ}^{s_Ks_Ps_Q} = \frac{1}{2} (\mathcal{J}_{KPQ}^{s_Ks_Ps_Q} + \mathcal{J}_{KQP}^{s_Ks_Qs_P}),
\]

(6.5)

where the non-symmetric flows are defined by

\[
\mathcal{J}_{KPQ}^{s_Ks_Ps_Q} = -\int d^3 x \mathbf{v}_K^{s_K} \cdot [(\mathbf{v}_P^{s_P} \cdot \nabla)\mathbf{v}_Q^{s_Q}],
\]

(6.6)

with \(\mathbf{v}_K^{s_K}(t, \mathbf{x})\) denoting the helical Littlewood-Paley velocity components, obtained by projecting on modes of a given helicity index \(s_K \in \{-, +\}\) restricted to the Fourier space domain \(K\). Intuitively, entries of \(\mathcal{T}\) are large when the corresponding triads are dominant.

For active fluids, Fourier space is naturally partitioned into three regions [Fig. 6-1(a)] and there are \(2^3 = 8\) helicity index combinations. The triad tensor \(\mathcal{T}\) is symmetric in the last two indices, so that \(\mathcal{T}\) has \(8 \times 18\) independent components encoding the fine-structure of the advective energy transport. Stationary time-averages for \(\langle \mathcal{T} \rangle\), measured directly from our simulations (App. D.1) for small \((\kappa_S)\) and wide \((\kappa_W)\) energy injection bandwidths are shown in Fig. 6-3(c) and (g). For reflection-symmetric turbulent flows, these two tables would remain unchanged under an upside-down flip \((+ \leftrightarrow -)\). By contrast, we find a strong asymmetry for a narrow bandwidth \(\kappa_S\) [Fig. 6-3(c)], which persists in weakened form for \(\kappa_W\) [Fig. 6-3(g)]. Specifically, we observe for \(\kappa_S\) two dominant cumulative triads with energy flowing out of the active spectral range II into the two passive domains I and III [Fig. 6-3(d)]. These cumulative triads visualize dominant entries of the tables in Figs. 6-3(c,g) and represent the total contributions from all triadic interactions between modes with given helicity indexes and with ‘legs’ lying in the specified spectral domain. The observed energy transfer directions, with energy flowing out of the intermediate domain II when the small-scale modes carry the same helicity index, are in agreement with a turbulent instability mechanism proposed by Waleffe [216]. Interestingly, however, our numerical results show that both ‘R’-interaction channels \(+++\) and \(+--\) contribute substan-
tially even in the case of strong parity-breaking \( \kappa_S \); when one surgically projects the full dynamics onto states with fixed parity, only the \( +++ \) channel remains [27]. By contrast, for a wide bandwidth \( \kappa_w \), the dominating triad interactions [Fig. 6-3(h)] favor the forward cascade. Hence, the inverse energy cascade in a 3D active fluids is possible because only a subset of triadic interactions is active in the presence of strong mirror-symmetry breaking. This phenomenon is controlled by the spectral bandwidth of the scale selection mechanism.

**Enhanced mixing.** Eqs. (1.2) describe a 3D isotropic fluid capable of transporting energy from smaller to larger scales. Previously, self-organized inverse cascades were demonstrated only in effectively 2D flows [116, 106, 29, 148, 189, 190, 191, 234, 144, 42, 233]. The 2D inverse cascade has been intensely studied in meteorology [148, 130], a prominent example being Jovian atmospheric dynamics [214], because of its importance for the mixing of thin fluid layers [171, 22, 21]. Analogously, the 3D inverse cascade and the underlying Beltrami-flow structure is expected to enhance mixing and transport in active fluids. Seminal work by Arnold [9] showed that steady solutions of the incompressible Euler equations include Beltrami-type ABC flows [54] characterized by chaotic streamlines. Similarly, the Beltrami structure of the active-flow attractors of Eqs. (1.2) implies enhanced local mixing. Combined with the presence of an inverse cascade, which facilitates additional large scale mixing through the excitation of long-wavelength modes, these results suggest that active biological fluids, such as microbial suspensions [55, 195, 59], can be more efficient at stirring fluids and transporting nutrients than previously thought.

### 6.4 Conclusions

To detect Beltrami flows in biological or engineered active fluids, one has to construct histograms and spectra as shown in Figs. 6-2(c,d) and 6-3(a,e) from experimental fluid velocity and helicity data, which is in principle possible with currently available fluorescence imaging techniques [235, 59]. Moreover, helical tracer particles [86] can help distinguish left-handed and right-handed flows. The above analysis predicts
that Beltrami flow structures, mirror symmetry breaking and the inverse 3D cascade appear more pronounced when the pattern selection is focused in a narrow spectral range. Our simulations further suggest that the relaxation time required for completion of the mirror-symmetry breaking process depends on the domain size (Fig. D-4). For small systems, the relaxation is exponentially fast, whereas for large domains relaxation proceeds in two stages, first exponentially and then linearly. In practice, it may therefore be advisable to accelerate relaxation by starting experiments from rotating initial conditions.
Chapter 7

Triad analysis

7.1 Introduction

In Chapter 6, we have shown that the polynomial GNS equations describing 3D active fluids with flow-dependent narrow spectral forcing possess numerical solutions that can sustain significant energy transfer to larger scales by realizing chiral Beltrami-type chaotic flows. In this Chapter, we rationalize these findings by studying the triad truncations of polynomial and Gaussian GNS models focusing on modes lying in the energy injection range. Identifying a previously unknown cubic invariant for the triads, we show that their asymptotic dynamics reduces to that of a forced rigid body coupled to a particle moving in a magnetic field. This analogy allows us to classify triadic interactions by their asymptotic stability: unstable triads correspond to rigid-body forcing along the largest and smallest principal axes, whereas stable triads arise from forcing along the middle axis. Analysis of the polynomial GNS model reveals that unstable triads induce exponential growth of energy and helicity, whereas stable triads develop a limit cycle of bounded energy and helicity. This suggests that the unstable triads dominate the initial relaxation stage of the full hydrodynamic

equations, whereas the stable triads determine the statistically stationary state. To
test whether this hypothesis extends beyond polynomial dispersion relations, we in-
troduce and investigate an alternative Gaussian active turbulence model. Similar to
the polynomial case, the steady-state chaotic flows in the Gaussian model sponta-
neously accumulate non-zero mean helicity while exhibiting Beltrami statistics and
upward energy transport. Our results suggest that self-sustained Beltrami-type flows
and an inverse energy cascade may arise generically in the presence of flow-dependent
narrow spectral forcing.

Originally introduced by Kraichnan [115] to study energy transfer in inertial tur-
bulence, the triad truncation projects the fluid dynamics onto three Fourier modes
with wavevectors \( \{ k, p, q \} \) such that \( k + p + q = 0 \). The truncated dynamics of isolated
triads differs from the exact fluid flow, failing for example to conserve the topology
of the vorticity field [147]. Notwithstanding, the analysis of triadic interactions has
yielded important qualitative insights about the direction of energy transfer in exter-
nally forced [216, 217] and magnetohydrodynamic [128, 131, 132] turbulence. Kraich-
nan [115] combined the triad truncation with absolute equilibrium considerations to
argue against the possibility of an inverse inertial energy cascade in 3D helical turbu-
lence [34]. Numerical simulations of the Navier–Stokes equations (NS) verified later
that such turbulence indeed produces only direct energy and helicity cascades [31].
In the meantime, Waleffe [216, 217] formulated his instability assumption, suggesting
that there exists a subclass of triads capable of transferring energy to larger scales,
but that this subclass is not dominant in isotropic and reflection-invariant turbulence.
To amplify the impact of such upward-cascading triads, Biferale et al. [27, 28] studied
a projection of the NS equations onto positive helicity states, which breaks reflection-
invariance and eliminates triads promoting forward energy transfer, and found that
inverse energy transfer can develop in such a reduced system. Similar conclusions ap-
ply to NS-like equations where the nonlinear term is modified to weight various types
of triads differently [173]. New analytical properties of the triadic system continue
to be discovered, including pseudo-invariants for a subclass of the interactions [163],
with direct implications for externally driven turbulence in passive fluids.
On a periodic cubic domain, the Fourier representation of (1.2) reads

$$\left[ \frac{\partial}{\partial t} + \xi(k) \right] \hat{v}_a(k, t) = -i \sum_{k+p+q=0} P_{ab}(k) q_c \hat{v}_c^*(p, t) \hat{v}_b^*(q, t),$$

(7.1)

where $k = |k|$, the projector $P_{ab} = \delta_{ab} - k_3 k_b / k^2$ enforces incompressibility, and the dispersion relation is given by the polynomial

$$\xi(k) = \Gamma_0 k^2 + \Gamma_2 k^4 + \Gamma_4 k^6,$$

(7.2)

see Fig. 7-1(a). As discussed in Chapter 2, microswimmer activity is modeled by letting $\Gamma_2 < 0$, which introduces a band of linearly unstable modes with $\xi(k) < 0$, while $\Gamma_0 > 0$ and $\Gamma_4 > 0$ represent damping at large and small scales with $\xi(k) > 0$. The most unstable wavenumber $k_A$ determines the typical eddy size $\Lambda = \pi / k_A$, the corresponding growth rate sets the timescale $\tau = -\xi(k_A)^{-1}$, and we denote by $\kappa$ the bandwidth of the unstable modes, see Fig. 7-1(b). The parameters $(\Lambda, \tau, \kappa)$, uniquely determined by $(\Gamma_0, \Gamma_2, \Gamma_4)$, characterize the resulting flow structures and can be inferred from experimental data (Chapter 2).
In this Chapter, we investigate analytically and numerically the dynamical system arising from the triad truncation of (7.1). In contrast to the approach typically adopted when studying the inertial energy transfer in classical turbulence, our analysis does not neglect the linear term $\xi(k)$, although we will later discuss the implications for the classical case $\xi(k) \equiv 0$ as well. Specifically, we focus on the subclass of all possible triad interactions in which one or two ‘legs’ lie in the energy injection range, while the remaining legs are dissipative. We refer to such triads as ‘active triads’, to distinguish them from the ‘classical triads’ for which $\xi(k) \equiv 0$. Utilizing a previously unrecognized cubic invariant, we show that the resulting triad dynamics is asymptotically equivalent to a coupled system of a rigid body and a particle moving in a magnetic field. This analogy allows us to classify the active triads by their asymptotic stability: Triads forced at the small or large scale are unstable and increase energy and helicity exponentially, whereas triads forced at the intermediate scale are stable and develop a limit cycle. This asymptotic behavior of the active triads is in stark contrast to the classical triadic dynamics, for which the rigid body analogy does not hold in general but whose solutions one can classify using the cubic invariant. For the untruncated system (7.1), it is plausible that unstable active triads dominate the initial relaxation characterized by helicity growth, whereas stable active triads determine the subsequent statistically stationary stage. To support this hypothesis, and to demonstrate that the predictions of the polynomial model generalize to a broader class of flow-dependent forcing schemes, we will also consider a non-polynomial active turbulence model (7.29) which combines the usual viscous dissipation $\sim \Gamma_0 k^2$ with a Gaussian forcing term, see blue solid curve in Fig. 7-1(a). We will use direct numerical simulations to show that the Gaussian activity model develops strongly aligned steady-state velocity and vorticity fields and an inverse energy cascade. The upward energy transfer is non-inertial, yet the weak dependence of the energy flux on the wavenumber generates energy spectra that approximately follow $-5/3$ scaling [112] at large wavelengths. These results suggest that Beltrami-type flows and an inverse energy cascade are generic features of 3D active turbulence models with flow-dependent narrow spectral forcing.
7.2 Triad truncation and its asymptotic dynamics

We introduce the triad truncation of (7.1) for $\xi(k) \neq 0$, extending the approach of Kraichnan [115] who considered the case $\xi(k) \equiv 0$ corresponding to the inertial range approximation. We adopt the notation and build on the results of Moffatt [147].

7.2.1 Truncation

Triad truncation is the projection of the dynamics (7.1) onto three Fourier modes $\{\hat{v}(k, t), \hat{v}(p, t), \hat{v}(q, t)\}$ such that $k + p + q = 0$. The truncation is a first step beyond full linearization (which decouples the Fourier modes), to keep the smallest non-trivial portion of the quadratic nonlinearity. The velocity field reduces to

$$v(x, t) = \hat{v}(k, t)e^{ikx} + \hat{v}(p, t)e^{ipx} + \hat{v}(q, t)e^{iqx} + \text{c.c.,} \quad (7.3)$$

where c.c. denotes complex conjugate terms which ensure that $v(x, t)$ is real. Since the triad $\{k, p, q\}$ forms a triangle, it may be taken to lie in the $(x, y)$-plane by rotating the coordinate system, implying that the velocity field is independent of the spatial variable $z$. This allows one to introduce a stream function $\psi$ and write the velocity field as $v = (\partial\psi/\partial y, -\partial\psi/\partial x, v_z)$. Thus, rather than working with the representation (7.3), it is more convenient to introduce the triadic expansions of the scalars $\psi$ and $v_z$ [147]

$$\psi(x, y, t) = A_k(t)e^{ikx} + A_p(t)e^{ipx} + A_q(t)e^{iqx} + \text{c.c.,} \quad (7.4a)$$
$$v_z(x, y, t) = B_k(t)e^{ikx} + B_p(t)e^{ipx} + B_q(t)e^{iqx} + \text{c.c..} \quad (7.4b)$$

Following step by step the derivation in [147], the triad truncation of (7.1) in terms of the complex vectors $A = (A_k, A_p, A_q)$ and $B = (B_k, B_p, B_q)$ results in the following system of coupled differential equations

$$IA + DIA = 2\Delta(IA^* \times A^*), \quad (7.5a)$$
$$B + DB = 2\Delta(B^* \times A^*), \quad (7.5b)$$
where $\Delta = (k_y p_y - k_y p_x)/2$ is the area of the triangle formed by $\{k, p, q\}$ and

$$I = \text{diag}(k^2, p^2, q^2), \quad D = \text{diag}(\xi(k), \xi(p), \xi(q)). \quad (7.6)$$

The positive and negative entries of $D$ represent dissipation and forcing of the three modes, respectively. The key difference between the system (7.5) and the classical triad truncation is the matrix $D$, which vanishes in the latter case. The typically studied case $D = 0$ is suitable for the inertial range considerations in classical turbulence and arises formally from the truncation of the inviscid Euler equation. In the context of active turbulence, we are interested in the case $D \neq 0$.

Energy $E$ and helicity $H$ of the triad are given by [147]

$$2E = k^2 |A_k|^2 + p^2 |A_p|^2 + q^2 |A_q|^2 + |B|^2, \quad (7.7a)$$

$$H = IA \cdot B^* + IA^* \cdot B. \quad (7.7b)$$

In the remainder, we restrict our analysis to the triads obeying

$$\text{tr}(D) = \xi(k) + \xi(p) + \xi(q) > 0. \quad (7.8)$$

Since in a finite spatial domain the number of active modes with $\xi(k) < 0$ is finite, this condition is always satisfied for triads with at most two active legs, say $\xi(p) < 0$ and $\xi(k) < 0$ but $\xi(q) > 0$, provided the forcing is sufficiently weak.

Finally, we express the helical decomposition [46, 216, 3, 26] in terms of $A$ and $B$. Since the triad lies in the $(x,y)$-plane, the curl eigenmodes can be taken as

$$h_{\pm}(k) = \hat{z} \times \hat{k} \pm i \hat{z} = (-k_y, k_x, \pm ik)/k. \quad (7.9)$$

Projecting $\hat{v}(k)$ onto these eigenmodes gives the helical decomposition

$$a_{\pm}(k) = \frac{1}{2} h_{\pm}(k)^* \cdot \hat{v}(k) = -\frac{i}{2} (kA_k \pm B_k). \quad (7.10)$$
Analogous expressions hold for $p$ and $q$.

### 7.2.2 Asymptotic rigid body dynamics: A cubic invariant

Since, according to (7.5), the dynamics of $A$ affects $B$, but not vice versa, we study Eqs. (7.5a) first. In components, (7.5a) reads

\[
\begin{align*}
    k^2 \dot{A}_k + \xi(k) k^2 A_k &= 2\Delta(p^2 - q^2) A_p^* A_q^* \\
p^2 \dot{A}_p + \xi(p) p^2 A_p &= 2\Delta(q^2 - k^2) A_q^* A_k^* \\
q^2 \dot{A}_q + \xi(q) q^2 A_q &= 2\Delta(k^2 - p^2) A_k^* A_p^*
\end{align*}
\]

This system has the following three properties:

(i) If the initial conditions are real, then $A(t)$ is real for all $t$. In this case, Eqs. (7.11) reduce to the Euler equations for the rotation of a rigid body.

(ii) The change of variables given by the constant phase shifts $(\phi_k, \phi_p, \phi_q)$

\[
(A'_k, A'_p, A'_q) = (A_k e^{-i\phi_k}, A_p e^{-i\phi_p}, A_q e^{-i\phi_q}) \text{ where } \phi_k + \phi_p + \phi_q = 0
\]

leaves the Eqs. (7.11) unchanged.

(iii) The following identity holds

\[
\text{Im}(A_k A_p A_q) = |A_k||A_p||A_q| \sin(\phi_k + \phi_p + \phi_q) = C \exp[-\text{tr}(D)t],
\]

where $C = \text{Im}[A_k(0)A_p(0)A_q(0)]$ and we introduced polar representations $A_k = |A_k|e^{i\phi_k}$, etc. Eq. (7.13) also implies that

\[
\text{Im}(k^2 A'_k A_k) = k^2 \det \begin{bmatrix} \text{Re}\dot{A}_k & \text{Re}A_k \\
\text{Im}\dot{A}_k & \text{Im}A_k \end{bmatrix} = 2\Delta(p^2 - q^2) C \exp[-\text{tr}(D)t],
\]

where we introduced the real and imaginary components, $A_k = \text{Re}A_k + i\text{Im}A_k$. Analogous expressions hold for $A_p$ and $A_q$. Eq. (7.14) has a useful geometrical
interpretation: It gives the areal velocity (rate at which area is swept out) as a function of time of the complex trajectory traced out by the mode $A_k(t)$. Since we focus on triads with $\text{tr}(D) > 0$, this immediately implies that the mode eventually vanishes, becomes stationary, or its trajectory approaches a straight line through the origin.

The property (i) was pointed out in [216, 147]. The second property is easily verified by direct substitution. To derive the last property, multiply the first equation in (7.11) by $A_p A_q$, etc., to obtain

$$
\begin{align*}
  k^2 \dot{A}_k A_p A_q &+ \xi(k) k^2 A_k A_p A_q = 2\Delta(p^2 - q^2)|A_p|^2 |A_q|^2 \\
p^2 A_k \dot{A}_p A_q &+ \xi(p) p^2 A_k A_p A_q = 2\Delta(q^2 - k^2)|A_q|^2 |A_k|^2 \\
q^2 A_k A_p \dot{A}_q &+ \xi(q) q^2 A_k A_p A_q = 2\Delta(k^2 - p^2)|A_k|^2 |A_p|^2
\end{align*}
$$

Subtract from each equation its complex conjugate and add the resulting expressions

$$
\dot{A}_k A_p A_q + A_k \dot{A}_p A_q + A_k A_p \dot{A}_q + [\xi(k) + \xi(p) + \xi(q)] A_k A_p A_q - \text{c.c} = 0. \tag{7.16}
$$

Now use the chain rule and substitute $\xi(k) + \xi(p) + \xi(q) = \text{tr}(D)$

$$
\frac{d}{dt} (A_k A_p A_q - A_k^* A_p^* A_q^*) = -\text{tr}(D) (A_k A_p A_q - A_k^* A_p^* A_q^*). \tag{7.17}
$$

Property (iii) then follows from integrating this first order equation. To derive (7.14), multiply the first equation in (7.11) by $A_k^*$, etc., subtract from each such obtained equation its complex conjugate and then use (7.13).

We note that (iii) also implies that $\text{Im}(A_k A_p A_q)$ is conserved in the inertial range of classical turbulence, where $D = 0$ holds. This adds a cubic invariant to a list of quadratic invariants of the classical triadic system [216, 147, 163]. In section 7.4 we combine the cubic invariant with the conservation of in-plane energy and enstrophy [147] to obtain a detailed geometric classification of the solutions of the system (7.5a) when $D = 0$. 

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7.2.3 Asymptotic dynamics: rigid body and particle in a magnetic field

We use the properties (i-iii) to argue that the dynamics (7.11) is asymptotically equivalent to that of a forced rigid body with principal moments of inertia \((k^2, p^2, q^2)\). Since we consider triads for which \(\text{tr}(D) > 0\), Eq. (7.13) suggests that the phase curves of (7.11) approach the following algebraic subset \(S\) at an exponential rate

\[
\text{Im}(A_k A_p A_q) = |A_k| |A_p| |A_q| \sin(\phi_k + \phi_p + \phi_q) = 0. \tag{7.18}
\]

For the purposes of asymptotic analysis, we assume it is sufficient to consider initial conditions \(A(0)\) lying on the attractor \(S\). There are two possibilities:

\[
|A_i| = 0 \text{ for some } i \in \{k, p, q\} \quad \text{or} \quad \phi_k + \phi_p + \phi_q = n\pi. \tag{7.19}
\]

Regardless which of the three conditions \(A(0)\) satisfies, the property (ii) implies it is always possible to perform a change of variables that makes \(A(0)\) a real vector without altering the dynamics (7.11). But then it follows from property (i) that \(A(t)\) is real for all \(t\). The asymptotic dynamics of the complex system (7.11) therefore becomes effectively equivalent to that of the real system

\[
I \dot{\omega} + DI \omega = I \omega \times \omega, \tag{7.20}
\]

where \(\omega = (\omega_k, \omega_p, \omega_q)\) is a real vector. Eq. (7.20) has the structure of the Euler equations for a forced rigid body with inertia tensor \(I\) and angular velocity \(\omega\). When a triadic leg lies in the active or passive range, the rigid body is either forced or damped along the corresponding axis of inertia. Importantly, the forcing/damping is proportional to the component of angular momentum \(I \omega\) along that axis. The system (7.20) admits exact solutions corresponding to exponential growth or decay of rotations about one principal axis only, for example \(\omega = c(e^{-D_k t}, 0, 0)\).

We now focus on the asymptotic dynamics of the system (7.5b) for \(B\). Since by
the above analysis $A$ can be eventually taken to be the real vector $\omega$, the real and imaginary parts of $B$ asymptotically decouple into two equations

\begin{align}
\text{Re}\, \dot{B} + D\text{Re} B &= \text{Re} B \times \omega, \quad (7.21a) \\
\text{Im}\, \dot{B} + D\text{Im} B &= -\text{Im} B \times \omega. \quad (7.21b)
\end{align}

The first equation has the structure of Newton's equations for a forced particle with velocity $u = \text{Re} B$ and charge $+1$ moving in a magnetic field $\omega$. The second equation describes an analogous dynamics with velocity $\text{Im} B$ and charge $-1$. Since for real-valued $\omega$ the helicity (7.7b) is determined by the real part of $B$, we may conclude that the triadic system (7.5), in the long-time limit, becomes equivalent to the following equations for the real vectors $\omega$ and $u$

\begin{align}
I\dot{\omega} + D I \omega &= I\omega \times \omega, \quad (7.22a) \\
\dot{u} + D u &= u \times \omega. \quad (7.22b)
\end{align}

Eq. (7.22b) means that the angular velocity $\omega$ of the rigid body acts as a magnetic field for a forced particle moving with velocity $u$. In this notation, the triad helicity is the dot product between the rigid body angular momentum and the particle velocity $H = 2I\omega \cdot u$. (7.23)

Thus, the helicity is positive when the particle moves in the direction of the angular momentum and negative when it moves in the opposite direction.

### 7.3 Triad classification

We would like to classify active triads according to their long-time behavior; we call triads stable or unstable if they tend to produce bounded or unbounded energy, respectively. It is useful to develop first an intuitive understanding based on the asymptotic correspondence with the 'rigid body and a particle in a magnetic field'
system (7.22). Subsequently, we will confirm the intuitive picture through explicit numerical simulations.

Without forcing, $D = 0$ in (7.22a), the rigid body dynamics admits three fixed points, which correspond to constant angular velocity rotation about one of the three principal axes. The rotations about the small ($p^2$) or large ($q^2$) axis are stable fixed points, whereas rotation about the middle axis ($k^2$) is unstable [10]. With forcing, $D \neq 0$, the linear part of (7.22a) promotes exponential growth of the mode for which $D_{ii} < 0$ and damping of the remaining modes. It is conceivable that, when combined with the Eulerian nonlinearity, the coupled dynamical system (7.22), and hence the system (7.5), becomes unstable (asymptotically produces infinite energy) when the rigid body is forced at the small or large principal axis, for in this case the nonlinearity does not counteract the exponential growth. However, when forced at the middle principal axis, the nonlinearity should induce motion about the remaining axes. Since these axes are dissipative, the system should soon realign with the middle principal axis, until the nonlinearity becomes dominant again, and so on. Numerical investigations presented in section 7.3.2 suggest that the dynamics (7.5a) indeed approaches a limit cycle, although we do not rule out the possibility of more complicated attractors for some particular triads and forcing parameters.

In all numerical simulations of (7.5) we use the polynomial dispersion relation $\xi(k)$ given by (7.2) with parameters $(\Gamma_0, \Gamma_2, \Gamma_4)$ corresponding to the characteristic triple $(\Lambda = 75 \mu m, \tau = 6.4 s, \kappa = 8.4 \text{ mm}^{-1})$, as studied in [186]. For time-stepping, we use the classical Runge–Kutta method (RK4).

7.3.1 Unstable triads: rigid body forced at the small or large principal axis

Suppose the triadic system is forced at the small scale $q$, implying that $D_{qq} < 0$ but $D_{kk} > 0$ and $D_{pp} > 0$ in (7.5). The rigid body correspondence suggests the $q$-mode

---

1 Indeed, as shown by Chen and Lee [44] the dynamics described by Eq. (7.22a) can develop strange attractors. Interestingly, the famous Lorenz system [134] can also be cast into the form of an axisymmetric rigid body forced through a general linear instability.
Figure 7-2: Numerical simulations of (7.5) with polynomial dispersion (7.2) initiated with random complex initial conditions show that active triads \((p < k < q)\) are unstable when forced at large wavenumbers \(q\). Energy and helicity increase exponentially (a), reflecting the exponential growth of the forced helical mode (d) and underdamped decay of the passive helical modes (b, c). Parameters: \(\{k, p, q\} = \{(-5, 9, 0), (1, 2, 0), (4, -11, 0)\}\), box size \(L = 24A\).

should become unstable as the exponential growth and the nonlinearity reinforce each other. Indeed, \(A = c(0, 0, e^{-Dqt})\) is an exact unstable solution of (7.5a). The remaining part of the triadic system (7.5) is the Eq. (7.5b) for \(B\). In the long-time limit, when \(A \rightarrow c(0, 0, e^{-Dqt})\), we find the exact solution \(B = c'(0, 0, e^{-Dqt})\). Our simulations suggest that this solution is an attracting phase curve for generic initial conditions, confirming the rigid body correspondence in this case, see Fig. 7-2.

The asymptotic growth of the forced modes \(A_q\) and \(B_q\) implies that both energy and helicity increase exponentially, as confirmed in Fig. 7-2(a). Thus, at the level of a single triad, the mirror symmetry breaking may be generated by the following process in the full model (7.1): the rigid body quickly approaches a state in which it is rotating about the \(q\)-axis, with the angular speed growing exponentially, while the particle accelerates in the direction of \(q\) or in the direction directly opposite,
producing positive or negative helicity, respectively, depending on initial conditions. Spontaneous generation of non-zero helicity thus is a direct consequence of forcing through a linear instability.

A similar description characterized by exponential growth of energy and helicity applies when active triads are forced at the large scale \( p \), see Fig. 7-3. What distinguishes the two types of forcing is the nature of the damping of the dissipative modes. When forced at large wavenumbers \( q \), the decay is underdamped exhibiting oscillations, Figs. 7-2(b,c), whereas forcing at the small wavenumbers \( p \) results in overdamped dynamics, as shown in Figs. 7-3(b,d), a direct consequence of the dependence of the damping force on the wavenumber magnitude. The asymptotic response of the system (7.5) when two modes are forced is identical to the above scenarios when one mode is forced, as discussed in the App. E.1.
Figure 7-4: Numerical simulations of (7.5) with polynomial dispersion (7.2) initiated with random complex initial conditions show that active triads \((p < k < q)\) are stable when forced at intermediate scales \(k\). The energy and helicity (a) as well as the amplitudes of the helical modes (b–d) stay bounded and soon take the form of very rapid charge-discharge bursts, reflecting the collapse of the dynamics onto a limit cycle, see Fig. 7-5. Note the different \(y\)-scales in (b–d), which indicate that the energy produced by the intermediate scale is primarily sent to large scales. This is a manifestation of the upward transfer at the level of a single triad. Parameters: \(\{k, p, q\} = [(12, 1, 0), (3, 7, 0), (-15, -8, 0)]\), box size \(L = 24\Lambda\).
7.3.2 Stable triads: rigid body forced at the middle principal axis

For a rigid body forced at the middle principal axis we expect periodic behavior since the nonlinearity destabilizes the action of the linear forcing in this case. Numerical simulations of (7.5) with $D_{kk} < 0$ but $D_{pp} > 0$ and $D_{qq} > 0$ show that the system equilibrates by developing periodic bursts characterized by alternating exponential growth and decay of energy, helicity and the helical modes, suggesting the existence of a stable limit cycle, see Fig. 7-4.

To numerically verify the existence of a limit cycle in the system (7.5a) initiated with generic complex initial conditions, we now illustrate how to determine the three-dimensional real subspace onto which the system converges. We first note that the numerical solutions obey the property (iii) until the machine precision is reached; see Fig. 7-5(a). As a consequence, Eqs. (7.14) imply that each mode either vanishes, stops moving, or its trajectory in the complex plane approaches a straight line through the origin. In the present case of forcing the intermediate wavenumber $k$ all three modes follow the last scenario: the complex trajectories $(A_k(t), A_p(t), A_q(t))$ become straight lines, see Fig. 7-5(b), with well-defined phase angles $(\phi_k, \phi_p, \phi_q)$, that satisfy $\phi_k + \phi_p + \phi_q = 0$, see Fig. 7-5(c). We use these angles to define the change of variables $(A'_k, A'_p, A'_q) = (e^{-i\phi_k} A_k, e^{-i\phi_p} A_p, e^{-i\phi_q} A_q)$ and $(B'_k, B'_p, B'_q) = (e^{-i\phi_k} B_k, e^{-i\phi_p} B_p, e^{-i\phi_q} B_q)$. This change of variables merely rotates the complex trajectories so that the three modes $(A'_k, A'_p, A'_q)$ approach a real three-dimensional subspace at an exponential rate, Fig. 7-5(d). The asymptotic trajectory in that subspace reveals a limit cycle, Figs. 7-5(e–h), as expected from the rigid body correspondence. The limit cycle represents exponential growth of the rotation rate about the $k$-axis until the nonlinear effects destabilize it, followed by a rapid discharge along the two dissipative axes. The discharge along the $q$-axis represents energy transfer to small scales, while the discharge along the $p$-axis represents energy transfer to large scales. This behavior likely explains, at the level of individual triads, the origin of the steady-state upscale energy transfer in the full system (7.1).
Figure 7-5: Asymptotic analysis of the results in Fig. 7-4: the dynamics of stable active triads eventually collapses onto a limit cycle in a real three-dimensional subspace. (a) The cubic quantity $\text{Im}(A_k A_p A_q)$ decays in accordance with (7.13) until the machine double-precision limit is reached. (b) Complex trajectories traced out by the modes $\mathbf{A}(t) = (A_k(t), A_p(t), A_q(t))$ approach straight lines at an exponential rate. (c) Trajectories in (b) for $t > 100r$. The lines are characterized by the angles $(\phi_k, \phi_p, \phi_q) = (0.759, -0.185, -0.574)$, such that $\phi_k + \phi_p + \phi_q = 0$, as required by vanishing of $\text{Im}(A_k A_p A_q)$. (d) The phases define the change of variables $(A'_k, A'_p, A'_q) = (e^{-i\phi_k} A_k, e^{-i\phi_p} A_p, e^{-i\phi_q} A_q)$ and $(B'_k, B'_p, B'_q) = (e^{-i\phi_k} B_k, e^{-i\phi_p} B_p, e^{-i\phi_q} B_q)$, which leaves the differential Eqs. (7.5) unchanged, but rotates the complex trajectories so that the variables $(A'_k(t), A'_p(t), A'_q(t))$ become real in the limit $t \to \infty$. (e) In this three dimensional real subspace, $\mathbf{A}'(t)$ collapses onto a stable limit cycle. (f–h) Projections of the limit cycle of $\mathbf{A}'(t)$ onto the coordinate planes. (i) $\mathbf{B}'(t)$ also develops a limit cycle, shown is the real part. (j–k) The corresponding projections of $\text{Re}\mathbf{B}'(t)$ onto the coordinate planes.
7.3.3 Only stable triads admit a fixed point

We still mention that the triadic system (7.5) forced at the intermediate wavenumber (and only in that case) exhibits a family of fixed points (see App. E.2 for details)

\[
\begin{bmatrix}
A_k \\
A_p \\
A_q \\
B_k \\
B_p \\
B_q
\end{bmatrix} = \sqrt{\alpha} \begin{bmatrix}
\sqrt{|p^2 - q^2|/|D_{kk}|}\frac{1}{k} \\
\sqrt{|q^2 - k^2|/|D_{pp}|}\frac{1}{p} \\
\sqrt{|k^2 - p^2|/|D_{qq}|}\frac{1}{q}
\end{bmatrix}
\]

(7.24a)

\[
\begin{bmatrix}
B_k \\
B_p \\
B_q
\end{bmatrix} = \begin{bmatrix}
k\sqrt{|p^2 - q^2|/|D_{kk}|}[c_1 + ic_2(-k^2 + p^2 + q^2)] \\
p\sqrt{|q^2 - k^2|/|D_{pp}|}[c_1 + ic_2(k^2 - p^2 + q^2)] \\
q\sqrt{|k^2 - p^2|/|D_{qq}|}[c_1 + ic_2(k^2 + p^2 - q^2)]
\end{bmatrix}
\]

(7.24b)

where

\[
\alpha = -\det(ID)/(4\Delta^2|p^2 - q^2||q^2 - k^2||k^2 - p^2|).
\]

(7.24c)

The arbitrary real constants \(c_1\) and \(c_2\) determine energy and helicity. The property (ii) in section 7.2.2 also implies that we can rotate the solution in the complex plane provided the three phases sum to zero. The fixed points are unstable to linear perturbations.

7.4 Implications for classical triads

In this section, we classify the geometry of the solutions of the Eq. (7.5a) for the case \(D = 0\) corresponding to the triad truncation of the Euler equations

\[
I\dot{A} = 2\Delta(IA^* \times A^*).
\]

(7.25)
The system (7.25) exhibits three constants of motion:

\[
\begin{align*}
    k^2|A_k|^2 + p^2|A_p|^2 + q^2|A_q|^2 &= E \\
    k^4|A_k|^2 + p^4|A_p|^2 + q^4|A_q|^2 &= \Omega \\
    A_k A_p A_q - A_k^* A_p^* A_q^* &= C
\end{align*}
\] (7.26)

The quadratic constants \( E \) and \( \Omega \) were found by [147], the new cubic constant \( C \) was derived in section 7.2.2 above. The triple (7.26) suggests that the system (7.25) is confined to a three-dimensional surface in a six-dimensional phase space. We next summarize a series of results classifying the solutions to (7.25), which are rigorously proven in the App. E.3.

In the six-dimensional phase space for the system for \( A(t) \), we consider separately the following subsets of \( \mathbb{R}^6 \):

\[
Z_1 = \{ A_p = 0, A_q = 0 \} \cup \{ A_q = 0, A_k = 0 \} \cup \{ A_k = 0, A_p = 0 \}, \\
Z_3 = \{ \text{Re}(A_k A_p A_q) = 0 \} \cap \\
\{ |A_q|^2|A_k|^2 k^2 q^2 (k^2 - q^2) + |A_p|^2|A_k|^2 p^2 k^2 (p^2 - k^2) + |A_p|^2|A_q|^2 q^2 p^2 (q^2 - p^2) = 0 \}
\] (7.27)

Initial conditions in \( Z_1 \) correspond to fixed points of (7.25). For initial conditions in \( Z_3 \), the system (7.25) is solved exactly by a quasi-periodic motion with constant amplitudes and phases evolving linearly in time according to

\[
\begin{align*}
    \phi_k &= \pm (p^2 - q^2)|A_p||A_q|/(k^2|A_k|)t + c_k \\
    \phi_p &= \pm (q^2 - k^2)|A_k||A_p|/(p^2|A_p|)t + c_p \\
    \phi_q &= \pm (k^2 - p^2)|A_k||A_p|/(q^2|A_q|)t + c_q
\end{align*}
\] (7.28)

where the equalities hold modulo \( 2\pi \) and the constants \( c_i \) are chosen so that \( \phi_k + \phi_p + \phi_q = \pi/2 \) or \( \phi_k + \phi_p + \phi_q = 3/2\pi \) holds, as required by the definition of \( Z_3 \). Importantly, for initial conditions in \( Z_3 \) the sum of phases is conserved, so the system (7.25) stays in \( Z_3 \) and the phase space in fact can be reduced to a torus \( \mathbb{T}^2 \). A typical trajectory for initial conditions on \( Z_3 \) is shown in Fig. 7-6(a). In summary, fixed points and quasi-
periodic motion completely characterize the solutions of (7.25) for initial conditions in \( Z_1 \) and \( Z_3 \), respectively.

We now consider the most important generic case of initial conditions in the complement \( N = \mathbb{R}^6 \setminus (Z_1 \cap Z_3) \). In \( N \), the differential of (7.26) has full rank, implying that (7.26) defines a three-dimensional manifold in \( N \), that is, the solutions of (7.25) are confined to a smooth three-dimensional surface. For generic values of the triple \((E, \Omega, C)\), this surface is in fact a three-torus \( \mathbb{T}^3 \) (or several copies of such tori). A typical trajectory in such a generic case is shown in Fig. 7-6(b). There are also special cases of \((E, \Omega, C)\) for which the manifold looks like (copies of) a product of a line and a torus \( \mathbb{R} \times \mathbb{T}^2 \) and/or (copies of) \( \mathbb{T}^3 \). The reader is referred to App. E.3 for more details and rigorous proofs.

Finally, we still note that, since the solutions \( A(t) \) remain continuous and bounded for all \( t \), the linear system for \( B(t) \) can be solved exactly, at least formally, in terms of time-ordered matrix exponentials [76]. In the future, it would be interesting to explore in detail potential relations between the above results and recent studies of turbulence in two-dimensional three-component flows [26].
7.5 Gaussian active turbulence model

The behavior of individual active triads suggests that the mirror-symmetry breaking and upward energy transfer observed in the GNS system (7.1) is first triggered by unstable active triads and then sustained by stable active triads. Moreover, these effects likely hold for a broader class of forcing schemes provided the dispersion relation $\xi(k)$ in (7.1) introduces a narrow band of unstable modes. To test this hypothesis, we numerically study an alternative GNS model where the dispersion relation has the form

$$\xi(k) = \Gamma_0 k^2 - \alpha \exp[-(k - k_0)^2/(2\sigma^2)].$$

(7.29)

The main difference between (7.29) and the polynomial model (1.2) is that Gaussian activity model (7.29) behaves like a Newtonian fluid with viscosity $\Gamma_0$ at both large and small scales, see Fig. 7-1(a). Eq. (7.29) leads to an integro-partial differential equation in position space. In our simulations, we always fix $\Gamma_0 = 10^{-6}$ m$^2$s$^{-1}$, corresponding to the kinematic viscosity of water. To relate the parameters $(\alpha, k_0, \sigma)$ to the characteristic triple $(\Lambda, \tau, \kappa)$, we must solve

$$\xi'(k_\Lambda) = 0, \quad \tau = -\xi^{-1}(k_\Lambda), \quad \xi(k_\pm) = 0, \quad \kappa = k_+ - k_-,$$

(7.30)

where $k_\Lambda = \pi/\Lambda$ is the most unstable wavenumber and $k_\pm$ are the non-trivial zeros of the dispersion relation $\xi(k)$. Since no closed-form solutions exist, we solve the system (7.30) numerically. We set $(\alpha, k_0, \sigma) = (2.544165\text{ms}^{-1}, 52.36\text{mm}^{-1}, 10\text{mm}^{-1})$, yielding $(\Lambda, \tau, \kappa) = (65.14\mu\text{m}, 0.1\text{s}, 1.94\text{mm}^{-1})$, which is in the range of typical bacterial suspension values [186]. Non-dimensionalising according to

$$x = \frac{L}{2\pi} \tilde{x}, \quad t = T \tilde{t}, \quad v = \frac{L/(2\pi)}{T} \tilde{v}, \quad k = \frac{2\pi}{L} \tilde{k},$$

(7.31)

gives, after dropping the tildes and setting $T = (L/2\pi)^2/\Gamma_0$,\

$$\xi(k) = k^2 - T \alpha \exp\left\{ -[(2\pi/L)k - k_0]^2/(2\sigma^2) \right\}.$$\

(7.32)
We simulate the dimensionless system in the vorticity-vector potential formulation as described in [186] using the Fourier pseudo-spectral method with the '3/2'-rule [39], discretization size $N = 243^3$ and time step $dt = 5 \times 10^{-4} \tau / T$. We set the domain size $L = 42\Lambda$, which corresponds to the most unstable wavenumber at $k_\Lambda = 21$. For time-stepping, we use a third-order semi-implicit backward differentiation scheme [14]. For this choice of parameters, the resulting flow patterns arise from the balance between the active and passive stresses mediated by the nonlinear advection. Hence, advection and stresses are equally important, suggesting an effective Reynolds number of the order of unity. Indeed, adopting the standard definition of the integral-scale Reynolds number $Re$ [72], we find that $Re \sim 4.7$ for the simulations in Fig.

To discuss the results of numerical simulations, we use the helical decomposition [46, 216] to expand the velocity field in an orthogonal basis of curl operator eigenvectors $h^\pm$

$$v(t, \mathbf{k}) = u^+(t, \mathbf{k}) h^+(\mathbf{k}) + u^-(t, \mathbf{k}) h^-(\mathbf{k}), \quad (7.33)$$

where $h^\pm$ satisfy $i \mathbf{k} \times h^\pm = \pm k h^\pm$ with $k = |\mathbf{k}|$. The decomposition (7.33) yields a splitting into cumulative energy and flux contributions $e^\pm(k)$ and $\Pi^\pm(k)$ from helical
Figure 7-8: Numerical results for the Gaussian activity model (7.29) based on the simulation in Fig. 7-7. Instantaneous (a-f) and average (g-i) energy spectra, fluxes and dominant integrated triads for time instants and intervals indicated in Fig. 7-7(a). Vertical dashed lines mark the energy injection range.
modes $u^\pm(k)$ lying on the wavenumber shell $k$. Specifically, $\Pi^+(k) = \sum_{i=1}^{4} \Pi^i(k)$ and $\Pi^-(k) = \sum_{i=5}^{8} \Pi^i(k)$, where $\Pi^i(k)$ is one of the eight types of helicity-resolved fluxes and the summation follows the binary ordering of [216]. To analyze which triads are spontaneously activated at various time instants, we consider combinations $K, P, Q \in \{ I, II, III \}$ of the three spectral domains in Fig. 7-1(b), with region I corresponding to large scales, II to the energy injection range and III to small scales, and distinguish modes by their helicity index $s_K, s_P, s_Q \in \{ \pm \}$. The helicity-resolved integrated energy flow into the region $(K, s_K)$ due to interaction with regions $(P, s_P)$ and $(Q, s_Q)$ is given by

$$\mathcal{T}^{s_K s_P s_Q}_{K P Q} = \frac{1}{2} (\tilde{\mathcal{T}}^{s_K s_P s_Q}_{K P Q} + \tilde{\mathcal{T}}^{s_K s_Q s_P}_{K Q P}), \quad (7.34)$$

where the non-symmetric flows are defined by

$$\tilde{\mathcal{T}}^{s_K s_P s_Q}_{K P Q} = -\int d^3x \, v^s_K \cdot [(v^s_P \cdot \nabla) v^s_Q], \quad (7.35)$$

with $v^s_K(t, x)$ denoting the helical Littlewood-Paley velocity components, obtained by projecting on modes of a given helicity index $s_K \in \{ \pm \}$ restricted to the Fourier space domain $K$. Entries of the tensor $\mathcal{T}$ are large when the corresponding triads are dominant. For example, a positive (negative) value of $\mathcal{T}^{+ + +}_{I, III, III}$ indicates that energy flows into (out of) large scale (I) positive helicity modes due to interactions of these modes with positive helical modes corresponding to energy injection range (II) and small scales (III).

Our numerical simulations show that the Gaussian-forcing model (7.29) and the polynomial model (7.2) exhibit qualitatively similar behavior, cf. Figs. 7-7, 7-8 and corresponding plots in [186]. The Gaussian activity model also undergoes mirror symmetry breaking and spontaneously develops a non-zero net helicity, by realizing chaotic Beltrami-type flow states in which velocity $v$ and vorticity $\omega$ are almost aligned, see Fig. 7-7. Fig. 7-8 shows instantaneous and time-averaged energy spectra, energy fluxes and the dominant entries of the integrated triadic energy flows (7.34) for the time instants and intervals marked in Fig. 7-7(a). The energy spectra in Figs. 7-
8(a,d,g) indicate that the system spontaneously selects positive helicity modes at all relevant wavenumbers in this particular realization, while the energy fluxes in Figs. 7-8(b,e,h) are always negative at scales larger than the energy injection range (vertical dashed lines), demonstrating the inverse energy cascade. The corresponding helicity fluxes are shown in Fig. E-3 of the App. E. Similarly to the polynomial case, the upward energy transfer in the Gaussian model is not inertial, as indicated by the wavenumber dependence of the energy fluxes. In particular, it is not necessary to introduce hyper-viscosity at large scales to arrest the upward transfer, and the instantaneous and averaged energy fluxes can be computed directly from the balance between the nonlinear and dissipative and active terms according to Eqs. (S23) and (S33) in [186]. Unlike the polynomial model, however, the long-time spectra of the Gaussian model develop an approximate Kolmogorov $-5/3$ scaling at large wavelengths, see Fig. 7-8(d). We attribute this effect to the weak dependence of the energy flux on the wavenumber. More importantly, the dominant integrated energy flows shown in Figs. 7-8(c,h,i), where broken arrows indicate the direction of the inter-scale energy transfer and their thickness the relative magnitude of the transfer, are in agreement with the hypothesis that unstable triads drive the initial relaxation until stable triads become dominant and sustain the statistically stationary chaotic flow states.

7.6 Conclusions

We derived a previously unknown cubic invariant for the triad dynamics and used it to analyze and compare the triad truncations of two GNS models and the classical Euler equations. In the GNS case, we focused on active triads with one or two modes in the energy injection range and found that their dynamics is asymptotically equivalent to a coupled system consisting of a forced rigid body and a forced particle in a magnetic field. This analogy allows one to distinguish unstable and stable active triads, based on whether the rigid body is forced along the small/large principal axes (large/small scales) or the middle principal axis (intermediate scales), respectively. The dynamics
of the active GNS triads differs strongly from those of the classical Euler triads, for which the rigid body analogy does not hold in general and solutions are confined to a three-torus for generic initial conditions (section 7.4).

The existence of unstable and stable triads explains recent numerical results in Chapter 6, which suggested that the polynomial 3D GNS models can spontaneously break mirror symmetry by developing Beltrami-like flow states and upward energy transfer: Unstable triads induce exponential helicity growth from small perturbations and dominate the initial relaxation. Because of the nonlinear coupling between the triads, the stable triads eventually become dominant and the system settles into a statistically stationary chaotic flow state. In the stationary regime, energy is transferred from the spectral injection range to both large and small scales. This is consistent with the behavior of stable triads, which develop a limit cycle. In the rigid body analogy, this limit cycle represents a periodic two-phase process. During the first phase, the rigid body accumulates energy by increasing its spinning rate along the middle principal axis; during the second phase, the accumulated energy is released along the small and large principal axes. This release of the energy corresponds to energy transfer to large and small scales in the untruncated hydrodynamic equations. We confirmed the above picture for an alternative GNS model (7.29), which combines viscous dissipation and active Gaussian forcing, by computing the integrated energy flow between the three spectral domains (large scales, energy injection range and small scales). Unlike the previously studied polynomial model, the Gaussian active turbulence model develops energy spectra that approximately follow Kolmogorov's $-5/3$ scaling at large wavelengths, which may be desirable in applications to microbial suspensions.

More broadly, the above results suggest that parity violation and an inverse energy cascade may be generic features of turbulence models where the forcing term depends on the velocity field. The degree to which the mirror-symmetry is broken or the proportion of energy that is transferred to small and large scales should depend on the particular forcing considered. The two GNS models (7.2) and (7.29) analyzed here are basic examples that introduce a bandwidth of linearly unstable modes. These models
can help guide theoretical efforts to find other forcing schemes that realize specific desired features, such as the magnitude of the upward transfer or its inertial character. Biological and engineered active fluids are promising candidates for the experimental implementation, as GNS models can be fitted to reproduce experimentally observed velocity correlation functions (Chapter 2). However, the general nature of the triad-based arguments presented here suggests that other non-equilibrium fluids might also be capable of breaking mirror-symmetry and developing upward energy transfer. Last but not least, our results indicate that helical flows [146] and the Beltrami-type flows in particular, which have been primarily studied as exact stationary solutions of the Euler equations [13] and in the context of magnetodynamics [140, 236, 96], could be more ubiquitous than previously thought.
Chapter 8

Concluding remarks

In this thesis, we modeled self-sustained active fluids, such as water driven by bacteria or ATP-fueled microtubule networks, as incompressible and momentum-conserving non-Newtonian fluids that obey the usual Cauchy momentum equation

\[ \nabla \cdot \mathbf{v} = 0, \quad (8.1a) \]
\[ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \mathbf{\sigma}. \quad (8.1b) \]

We proposed to describe the inherent activity in such systems through phenomenological constitutive relations between the stress tensor \( \mathbf{\sigma} \) and the rate of strain \( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \) that are linear and isotropic, as in the case of Newtonian fluids, but are otherwise nonlocal. The non-locality assumption is expressed by promoting the kinematic viscosity to a differential operator given by

\[ \mathbf{\sigma} = (\Gamma_0 - \Gamma_2 \nabla^2 + \Gamma_4 \nabla^4)[\nabla \mathbf{v} + (\nabla \mathbf{v})^T], \quad (8.1c) \]

where the \( \Gamma_i \)'s are constant. Substituting (8.1c) in (8.1) yields the generalized Navier–Stokes equations (GNS)

\[ \nabla \cdot \mathbf{v} = 0, \quad (8.2a) \]
\[ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \Gamma_0 \nabla^2 \mathbf{v} - \Gamma_2 \nabla^4 \mathbf{v} + \Gamma_4 \nabla^6 \mathbf{v}. \quad (8.2b) \]
Activity is modeled by setting $\Gamma_2 < 0$, which introduces a bandwidth of linearly unstable modes; stability at large and small scales is retained by keeping $\Gamma_0, \Gamma_4 > 0$. As a result of the advection-mediated balance between the energy injection and dissipation, a chaotic nonequilibrium flow structures emerge. This so called 'active turbulence' is characterized by the typical vortex size $\Lambda$, its turnover time $\tau$ and the active bandwidth $\kappa$ that measures the size of the forcing range (Chapter 2), in contrast to scale-free classical turbulence.

The GNS model (8.2) is a turbulence model in which the fluid is driven by a linear instability rather than an external random body force characteristic of classical turbulence\(^1\). In this thesis, we studied the consequences of such flow-dependent forcing schemes. We showed that the model (8.2) reproduces recent experimental results focusing on rheology of bacterial suspensions and the motion of active fluids in confined geometries (Chapter 3). We also made a number of predictions regarding the nature of fluid-structure interactions, symmetry breaking phenomena and energy transport properties in 2D and 3D.

In Chapter 3, we studied the response of the GNS model to shear boundary conditions in a plane Couette geometry. We found that the model (8.2) admits a low viscosity phase, in agreement with recent experiments on bacterial suspensions [193, 133]. We rationalized this effective viscosity reduction through the presence of point stress-free defects in the bulk of the fluid that allow the stress director field to develop a half-loop topology near the boundaries [Fig. 3-3(e)]. Such a half-loop topology implies that the shear stress oscillates between positive and negative values as one moves along the boundary. Summing up these stress contributions leads to substantial cancellations, reducing the overall force on the boundary. We also found that the model (8.2) can generate spontaneous flow of the fluid's center of mass, a phenomenon that has been observed in active fluids confined to 'racetracks' [225, 231].

In Chapter 4, we studied the active analog of Stokes' second problem, which concerns the oscillatory motion of boundaries. We investigated an oscillating rigid

\(^1\)To the best of the authors' knowledge, the key distinction between conservative systems forced through an external force field and a linear instability was first emphasized in the context of turbulence by V. I. Arnold [12].
circular ring encapsulating an active fluid obeying (8.2). By analyzing the linear response of the torque exerted by the fluid on the container, we predicted that a periodically rotating ring will oscillate at a higher frequency in an active fluid than in a passive fluid, due to an activity-induced reduction of the fluid inertia (Fig. 4-2). In the case of a freely suspended fluid-container system that is isolated from external forces or torques, we predicted that active fluid stresses can induce giant fluctuations in the container’s angular momentum if the confinement radius matches certain multiples of the intrinsic vortex size of the active suspension (Fig. 4-3).

In Chapter 5, we generalized the GNS model (8.2) to curved surfaces and derived exact stationary solutions for a two-sphere. Through numerical simulations, we found a curvature-induced transition from a burst phase to an anomalous turbulent phase that differs distinctly from externally forced classical 2D Kolmogorov turbulence. This new type of active turbulence is characterized by the self-assembly of finite-size vortices into linked chains (Fig. 5-2). The coherent motion of the vortex chain network provides an efficient mechanism for upward energy transfer from smaller to larger scales, presenting an alternative to the conventional energy cascade in classical 2D turbulence characterized by vortex mergers.

In Chapter 6, we investigated bulk flows and energy transport properties of (8.2) in 3D. We first showed that chiral Beltrami vector fields provide exact stationary solutions of (8.2) and then used numerical simulations to demonstrate that active fluids spontaneously develop helical Beltrami-like flows with preferentially aligned velocity and vorticity fields. As a consequence of this parity violation, we predicted that an inverse energy transport to larger scales develops in an otherwise homogeneous and isotropic active turbulence, in stark contrast to classical turbulence, where only the forward Richardson cascade to smaller scales occurs in 3D.

In Chapter 7, we rationalized the numerical findings presented in Chapter 6 by studying the triad truncations of the polynomial (8.2) and Gaussian GNS models. We identified a previously unknown cubic invariant for the triads, and showed that their asymptotic dynamics reduce to that of a forced rigid body coupled to a particle moving in a magnetic field. We used this analogy to classify triadic interactions by
their asymptotic stability: unstable triads correspond to rigid-body forcing along the largest and smallest principal axes, whereas stable triads arise from forcing along the middle axis. Analysis of the polynomial GNS model revealed that unstable triads induce exponential growth of energy and helicity, whereas stable triads develop a limit cycle of bounded energy and helicity. This suggests that the unstable triads dominate the initial relaxation stage of the full hydrodynamic equations, whereas the stable triads determine the statistically stationary state. To test whether this hypothesis extends beyond polynomial dispersion relations, we introduced and investigated an alternative Gaussian active turbulence model. Similar to the polynomial case, the steady-state chaotic flows in the Gaussian model spontaneously accumulate nonzero mean helicity while exhibiting Beltrami statistics and upward energy transport.

Last but not least, we stress that the qualitatively similar behavior of the polynomial and Gaussian GNS models (Chapter 7) strongly suggests that the presence of linearly unstable modes is the key feature of our phenomenological approach to active turbulence. The main predictions discussed above are insensitive to the particular ansatz (8.1c) for the effective stress tensor, as long as it introduces such unstable modes. We therefore expect the results described in this thesis to be valid for a broad class of passive fluids driven through a linear instability.

8.1 Limitations of the GNS model

"All models are wrong but some are useful." (George E. P. Box)

The GNS model (8.2) successfully reproduces key aspects of the experimentally observed velocity statistics [186] in bacterial and ATP-driven microtubule suspensions. Being a phenomenological model, it does so without explicitly accounting for the microscopic details of the active forcing mechanisms. For example, although the model parameters can be estimated straightforwardly from experimental flow data (Chapter 2), it is unknown how the effective parameters $(\Gamma_0, \Gamma_2, \Gamma_4)$ depend on bacterial properties, such as size, swimming speed, and concentration. To establish this dependence, one would have to derive the GNS equations from a microscopic model such
as a coupled system of the classical Navier–Stokes equations forced by a suspension of self-propelled particles.

A possible derivation of the GNS equations from microscopic models could follow kinetic theory approaches that have been used to derive mean field models of active suspensions [89, 165, 205]. Starting from overdamped Langevin equations for the swimmers, one can obtain the corresponding Fokker-Planck equations for the one-particle probability density function for the swimmer position and orientation. When supplemented with closure schemes, the Fokker-Planck equations yield equations for the moments of the one-particle distribution. The three lowest-order moments yield continuum equations for the concentration $c$, the polar $p$ and the nematic $Q$ order parameters for the swimmers. For sufficiently dense suspensions, the concentration can be taken as constant. Furthermore, the equations for $p$ and $Q$ couple to the Navier–Stokes equations for the solvent velocity $v$ through the addition of active stresses. Importantly, to reduce the coupled system for $\{v, p, Q\}$ to the GNS equations, one has to assume a linear and isotropic response between the solvent flow and the order parameters of the form

$$p = \sum_{n=1}^{\infty} p_n \nabla^{2n} v, \quad Q = \sum_{n=1}^{\infty} q_n \nabla^{2n} (\nabla v)^+ \tag{8.3}$$

where the superscript '+' denotes the traceless symmetric tensor part. Further assuming that the conventional viscous stresses and the active stresses are additive, that the leading active stresses are linear in the order parameters $p$ and $Q$, and by truncating at order $n = 3$, one is led to a sixth-order stress tensor as in Eq. (8.1c).

As is common in such hydrodynamic coarse-graining schemes, several challenges arise when trying to connect the effective parameters $(\Gamma_0, \Gamma_2, \Gamma_4)$ to bacterial properties by carrying out the program outlined above. The derivations based on kinetic theory typically rely on several simplifying assumptions such as neglecting fluid inertia through the use of the Stokes’ equation or factorization of the many-particle distribution function into one-particle distribution function. The GNS equations describe the solvent flow in the intermediate-to-high concentration regime, where col-
lective swimming persists. In this regime, both aforementioned approximations break down: the Reynolds number is no longer small and the correlations between the positions and orientations of different swimmers cannot be neglected. This implies that, for example, two-particle distribution function does not factorize into a product of one-particle distribution functions, at least for separations smaller than the coherence length. Even with these restrictions in mind, one still has to identify a set of bacterial parameters that lead to a dynamical state characterized by the linear response (8.3). This last step is challenging as it underpins the emergence of collective dynamics. Yet, such slaving between the order parameters and the flow field has been confirmed experimentally, see Fig. 3 (a) in [59], which demonstrates that the solvent and bacterial flow statistics become tightly linked.

Last but not least, we note that, by construction, the GNS model neglects non-advective nonlinearities such as nematic alignment interactions that become important at very high swimmer concentrations. It also neglects the conversion of kinetic to elastic energy that might be stored in the active component. Such additional energy transfer mechanisms play a key role in the dynamics of a polymer solution even in a very dilute regime [81, 82], and could also be important in active fluids.

8.2 Future directions

The results of this work suggest two interesting directions for future work on the GNS model. The first of these concerns topological aspects of the upward energy transfer mechanisms in 3D active turbulence. In classical turbulence, such transfer only exists in 2D and is often linked to the merging of vortices [29]. By contrast, in 3D active turbulence the flow phenomenology underlying the inverse cascade appears considerably more complex. For example, spontaneous generation and fluctuations of helicity (Chapter 6) reflect complex interactions and reconnections of tangled vortex lines. One can therefore expect that topological properties of the flow field play a key role in sustaining 3D inverse transfer. Building on a correspondence between helicity and vortex line-linking discovered by Vladimir Arnold [11], efforts in this direction will
require developing an efficient numerical framework to investigate these topological properties. This is a computationally challenging problem, since Arnold’s correspondence holds only for simply connected domains, such as the interior of a sphere, ruling out fast standard Fourier spectral method for periodic domains. As an intermediate step, one can focus on developing a fast spectral solver for the vorticity-vector potential formulation of the GNS model in bounded 3D domains, which requires constructing a consistent integral formulation of the no-slip boundary conditions [161].

The second intriguing direction for future research concerns possible connections between pattern formation in the 2D GNS model and spatial statistics of random sums of monochromatic plane waves, a special case of a Gaussian random field, in which the energy spectrum is concentrated around a single wavenumber. The results of Chapter 5 suggest that the GNS model samples dynamically from such a random field ensemble. It would therefore be interesting to characterize this behavior in detail, by studying the zero sets [40, 178] and exploring analogies with previous investigations of quantum chaos [23, 197, 17].
Appendix A

Appendix to Chapter 3

A.1 Numerical Methods

We simulate typical shear experiments [133] in which two parallel boundaries move in opposite directions, both at a constant speed $V$ [Fig. 3-2(a)]. After rescaling by $L_x/(2\pi)$ and $L_y/2$, the simulation domain is a rectangle $(x, y) \in \mathcal{D} = [-\pi, \pi] \times [-1, 1]$ with periodic boundary conditions in the $x$-direction and non-periodic conditions in the $y$-direction. The usual no-slip boundary conditions for the velocity field $\mathbf{v}$ translate into $\partial_y \psi(x, \pm 1) = \pm V - V_{CM}$ and $\psi(x, \pm 1) = 0$.

A well-known challenge when working in the vorticity-stream function formulation is that the Poisson equation (3.1b) is overdetermined by the combined Dirichlet $\psi(x, \pm 1) = 0$ and Neumann $[\partial_y \psi(x, \pm 1) = \pm V - V_{CM}]$ boundary conditions for $\psi$. For the standard incompressible NS equations with no-slip boundary conditions this issue was resolved by Quartpelle and Valz-Gris [161], who proposed to reinterpret the Neumann data for $\psi$ as a set of integral conditions for the vorticity $\omega$. In practice, the implementation of these integral conditions involves computing all the harmonic functions on a given domain.

To solve Eqs. (3.1) numerically, we translate the integral conditions from the corresponding classical Navier–Stokes problem, which specifies two boundary conditions. Because Eq. (3.1a) is a sixth-order PDE, we need four more constraints to determine the solution. We therefore additionally impose $\nabla^2 \omega = 0$ and $\nabla^4 \omega = 0$ at $y = \pm 1$. This
phenomenological choice corresponds to the assumption that the total force on the boundary coming from the higher-order terms (proportional to $\Gamma_2$ and $\Gamma_4$) vanishes in a rectangular geometry. Combined with the no-slip assumption, these higher-order conditions suffice to close the system (3.1).

To evolve Eqs. (3.1) in time, we use a third-order semi-implicit backward differentiation formula time-stepping scheme introduced by Ascher et al. [14], calculating the nonlinear advection term explicitly, while inverting the linear part implicitly. The instantaneous center-of-mass velocity is computed by integrating Eq. (A.7) with the forward Euler method. For the spatial discretization, we adopt a spectral method, expanding functions in a basis composed of Fourier modes and Chebyshev polynomials of the first kind. The implicit inversion is discretized using the well-conditioned scheme introduced by Olver and Townsend [153, 209]. Since the system is periodic in the horizontal direction, the linear operator separates into one-dimensional operators, one for each Fourier mode. The resulting one-dimensional discretized linear operators augmented with integral and boundary conditions are sparse and almost banded, and therefore can be efficiently inverted. The explicit calculation of advection is done by collocation, that is, the relevant derivatives of $\omega$ and $\psi$ are evaluated on the Fourier–Chebyshev grid using the discrete Fourier transform (DFT) and the discrete cosine transform (DCT), then multiplied, and subsequently converted back to the expansion coefficients using the inverse DFT and the inverse DCT. Furthermore, the 3/2-zero-padding rule [39] is applied during the explicit step, to ensure that no spurious terms arising from the finite discretization affect the collocation calculation. Advection is the most expensive part with a complexity of $O(n \log n)$ when using the computationally optimal fast Fourier transform (FFT) for a discretization with $n = n_C \times n_F$, where $n_C$ and $n_F$ is the number of Chebyshev and Fourier modes, respectively. In our simulations, a discretization size of $n_C, n_F \sim 10^2$ suffices to obtain geometric convergence to double-precision accuracy [Fig. A-1(f)].

Simulation runs are initiated as follows. For fixed shear rate $\dot{\gamma}$, a linear shear profile corresponds to a constant vorticity field $\omega_0 = -2\dot{\gamma}$. We set $\omega = \omega_0 + \text{small noise}$, and then correct $\omega$ by projection so that it obeys the integral and boundary conditions.
We then solve the Poisson Eq. (3.1b) for $\psi$. The such generated pair $(\omega, \psi)$ is then used to start the time-stepping scheme.

Prior to scanning the parameter space relevant to the shear experiments, we validated our algorithm against results obtained earlier [184] for the periodic case. When the separation between the boundaries is large compared to the vortex size, the effect of the boundaries becomes negligible, and we recover energy spectra consistent with those obtained for periodic boundary conditions as well as with corresponding analytical results. After this cross-validation, we applied the Chebyshev–Fourier spectral method to simulate shear experiments in active fluids.

### A.2 Hodge decomposition

In a two-dimensional planar region $\mathcal{D}$ with boundary $\partial \mathcal{D}$, the Hodge decomposition for a vector field $\mathbf{v}$ reduces to

$$\mathbf{v} = \nabla \phi + \nabla \times \psi + \nabla g + \mathbf{H}, \quad (A.1)$$

where $\phi$ and $\psi$ are scalar functions satisfying the boundary conditions $\phi|_{\partial \mathcal{D}} = \psi|_{\partial \mathcal{D}} = 0$, $g$ is a harmonic function $\nabla^2 g = 0$ with arbitrary boundary data, and $\mathbf{H}$ is a harmonic vector field ($\nabla \cdot \mathbf{H} = 0, \nabla \times \mathbf{H} = 0$) that is tangential to the boundary ($\mathbf{H}^\perp = 0$).

For divergence-free flow fields, Eq. (A.1) simplifies to

$$\mathbf{v} = \nabla \times \psi + \nabla g + \mathbf{H}, \quad (A.2)$$

because $\nabla \cdot \mathbf{v} = 0$ makes $\phi$ harmonic with zero boundary data, implying $\phi = 0$ throughout $\mathcal{D}$. Moreover, imposing no penetration through the boundary ($\mathbf{v}^\perp = 0$) fixes Neumann data for $g$ as $\mathbf{n} \cdot \nabla g = 0$ on $\partial \mathcal{D}$, and therefore $g = \text{constant}$ throughout $\mathcal{D}$. We are then left with

$$\mathbf{v} = \nabla \times \psi + \mathbf{H}. \quad (A.3)$$
Figure A-1: To estimate the effective viscosity at fixed separation $L_y$ and shear rate $\dot{\gamma}$ from an ensemble average, we generate $\geq 10$ simulations with initial data corresponding to a randomly perturbed linear shear profile (App. A.1). (a) Time series of the kinetic energy $E(t)$ for multiple runs. (b) Time series of the shear stress $\Sigma^+(t)$ on the upper boundary for $L_y = 5\Delta, \dot{\gamma} = 1.4\tau^{-1}$. The vertical dotted line indicates the relaxation time $T$. (c) Combined time series for $t > T$ from all runs of the shear stress $\Sigma(t) = \Sigma^+(t) + \Sigma^-(t)$ rescaled by the kinematic viscosity $\Gamma_0$ and shear rate $\dot{\gamma}$. (d) Histogram corresponding to the combined time series in (c) yields the estimates for the mean viscosity $\nu = \langle \Sigma \rangle/\dot{\gamma}$ and its variance. (e) Convergence of the mean viscosity estimates as a function of the averaging interval $\Delta$. (f) Relative magnitude of the Fourier-Chebyshev coefficients of the vorticity field at a random representative time of the simulation demonstrates geometric convergence to zero, confirming that the number of modes used in the simulations suffices to completely resolve the dynamics at double precision accuracy ($\epsilon \sim 10^{-16}$).

Given that $\psi$ vanishes on the boundary, the physical interpretation of the harmonic field $H$ is that it accounts for the center-of-mass motion of the fluid. This follows from

$$\int_{\mathcal{D}} \mathbf{v} = \int_{\mathcal{D}} H,$$

(A.4)

since $\int_{\mathcal{D}} \nabla \times \psi$ vanishes because of $\psi|_{\partial \mathcal{D}} = 0$.

Now consider a rectangle with periodic boundary conditions in the $x$-direction. We write the harmonic field as $H = H_x \hat{x} + H_y \hat{y}$. Since $H$ is harmonic, both $H_x$ and
Figure A-2: Additional flow examples for various channel widths $L_y$ and shear rates $\dot{\gamma}$. The shear stress histograms represent averages over $\geq 10$ runs.

$H_y$ satisfy Laplace's equation. Additionally, $H^\perp = 0$ requires that $H_y = 0$ on the boundary, and hence $H_y = 0$ throughout the domain. The divergence-free condition, $\nabla \cdot H = 0$, requires that $H_x$ is a function of $t$ and $y$ only, $H_x(t,y)$. The curl-free condition, $\nabla \times H = 0$, further reduces $H_x$ to be solely a function of time. From Eq. (A.4) we see that $H_x$ represents the center-of-mass speed,

$$H = V_{CM}(t)\hat{x},$$

(A.5)

where $\hat{x}$ is the unit vector along $x$-axis. The dynamical equation for $V_{CM}$ follows from Newton's Second Law

$$M \frac{dV_{CM}}{dt} = F^+ - F^-, \quad (A.6)$$

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where $M$ is the total fluid mass, and $\mathbf{F}^+ = F^+ \hat{x}$ and $\mathbf{F}^- = -F^- \hat{x}$ are the forces on the upper and lower boundary (i.e., $F^+ = F^-$ if the boundaries are pulled in opposite direction with equal force). Since $M = \rho L_x L_y$, where $\rho$ is the constant two-dimensional fluid density, we obtain

$$
\frac{dV_{CM}}{dt} = \frac{1}{L_y} (\Sigma^+ - \Sigma^-),
$$

(A.7)

where $\Sigma^\pm(t) = \frac{1}{L_x} \int dx \sigma_{yx}(x, y = \pm 1)$ are the mean kinematic stresses as defined in Chapter 3.

The Hodge decomposition is also quite natural from an energetic perspective, for it provides an orthogonal splitting of the kinetic energy. In the present case, we have for the total kinetic energy,

$$
\mathcal{E}(t) = \frac{1}{2} \int_D dxdy \psi^2 = \frac{1}{2} \int_D dxdy \left[ (\partial_y \psi + V_{CM})^2 + (\partial_x \psi)^2 \right] = \mathcal{E}_\psi + \mathcal{E}_{CM},
$$

(A.8)

where the cross-term $\int_D dxdy \partial_y \psi V_{CM}$ vanishes by virtue of the boundary conditions imposed on $\psi$. Thus, the total kinetic energy splits into the vortical kinetic energy

$$
\mathcal{E}_\psi = \frac{1}{2} \int_D dxdy \left[ (\partial_y \psi)^2 + (\partial_x \psi)^2 \right]
$$

and the center-of-mass kinetic energy

$$
\mathcal{E}_{CM} = \frac{1}{2} \int_D dxdy V_{CM}^2.
$$

Figure 3-3(a) shows the proportions of how the total kinetic energy splits between the two components.
Figure A-3: (a) Validation that the flow symmetry breaking occurs is observed with equal probability for both directions. Same parameters as Fig. 3-2(b-c) \((L_y = 2.2\Lambda, V = 1.65U)\). For 300 runs, we obtained 46.3:53.7 for the relative proportions of left-right symmetry breaking. (b) Standard deviation of the effective viscosity shown in Fig. 3-3(b). We distinguish between two regimes whose boundary (black line) is defined by the shear speed \(V\) being equal to the characteristic vortex speed \(U\). At small shear, \(V < U\), the standard deviation is inversely proportional to the shear rate. That is, in the weak-shear regime, the fluctuations of the shear stress \(\Sigma\) depend only on the channel width \(L_y\). At large shear, \(V > U\), the flow becomes more stable and the standard deviation quickly becomes orders of magnitude smaller than \(\Gamma_0\). Blue lines indicate horizontal and vertical cuts shown in Fig. 3-3(c,d).

### A.3 Energy balance

We derive the energy balance, Eq. (8) of Chapter 3, by considering how the total kinetic energy, \(E(t) = \frac{1}{2} \int_D dx\,dy \, v^2\), changes with time (using an Einstein summation convention),

\[
\frac{dE}{dt} = \int_D dx\,dy \, v_i \partial_i v_i \\
= \int_D dx\,dy \, v_i (-v_j \partial_j v_i - \partial_i p + \partial_j \sigma_{ji}) \\
= \int_D dx\,dy \left\{ -\partial_i [v_i (\frac{1}{2}v^2 + p)] + v_i \partial_j \sigma_{ji} \right\} \\
= -\int_{\partial D} dx \left[ v_\perp (\frac{1}{2}v^2 + p) \right] + \int_D dx\,dy \, v_i \partial_j \sigma_{ji} \\
= \int_D dx\,dy \, v_i \partial_j \sigma_{ji}. \tag{A.9}
\]

In the second line, we used the equation of motion [Eq. (1b)], in the third the incompressibility condition [Eq. (1a)], in the fourth the divergence theorem (\(v_\perp\) is the
normal component to the boundary $\partial \mathcal{D}$), and in the last the fact that there is no penetration of the fluid through the walls ($v_\perp = v_y = 0$ at $y = \pm 1$). Integration by parts further gives

$$
\frac{d\mathcal{E}}{dt} = \int_\mathcal{D} dxdy [\partial_j (v_i \sigma_{ji}) - (\partial_j v_i) \sigma_{ji}]
$$

$$
= \int_{\partial \mathcal{D}} dx \sigma_{yi} v_i - \int_\mathcal{D} dxdy (\partial_j v_i) \sigma_{ji}
$$

$$
= V(F^+ + F^-) - \int_\mathcal{D} dxdy (\partial_j v_i) \sigma_{ji}. \quad (A.10)
$$

In the second line, we used the divergence theorem and in the last line the no slip boundary condition. $F^+$ and $F^-$ are the magnitudes of the force acting on the upper and lower boundaries, as defined above, and $V$ is the speed of the boundaries. We recognize $V(F^+ + F^-)$ as the power input, $P$, and therefore, for steady states with $d\mathcal{E}/dt = 0$, we have

$$
P = \int_\mathcal{D} dxdy (\partial_j v_i) \sigma_{ji}. \quad (A.11)
$$

Using the explicit form of the stress tensor [Eq. (2) of Chapter 3], we obtain

$$
P = \int_\mathcal{D} dxdy (\partial_j v_i)(\Gamma_0 - \Gamma_2 \partial_{nn} + \Gamma_4 \partial_{nn}^2) \partial_j v_i.
$$

In terms of Fourier modes, $v = \sum_k \hat{\mathbf{v}}(k)e^{ik \cdot x}$, the balance reads

$$
P = \sum_k k^2(\Gamma_0 + \Gamma_2 k^2 + \Gamma_4 k^4)|\hat{\mathbf{v}}(k)|^2, \quad (A.12)
$$

where $k = |\mathbf{k}|$. We now introduce the energy spectrum $\varepsilon(k) = \sum_{k',|k'|=k}|\hat{\mathbf{v}}(k')|^2$ to recover Eq. (7) of Chapter 3

$$
P = \sum_\mathbf{k} k^2(\Gamma_0 + \Gamma_2 k^2 + \Gamma_4 k^4)\varepsilon(k). \quad (A.13)
$$

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Figure A-4: Flow structures for four other choices of higher-order boundary conditions. The following quantities were set to zero pointwise on the boundary: (a) higher-order stress contributions, \( \sigma^{T^2} \big|_{\partial D} \) and \( \sigma^{T^4} \big|_{\partial D} \), (b) vorticity and its normal derivative, \( \omega \) and \( \partial_n \omega \), (c) normal components of Laplacian and bi-Laplacian of vorticity, \( \partial_n \nabla^2 \omega \big|_{\partial D} \) and \( \partial_n \nabla^4 \omega \big|_{\partial D} \), (d) normal components of vorticity and Laplacian of vorticity \( \partial_n \omega \big|_{\partial D} \) and \( \partial_n \nabla^2 \omega \big|_{\partial D} \). Parameters same as in Fig. 3-3(e).

### A.4 Boundary conditions

The dynamical system described by Eqs. (3.1) is of sixth order in the spatial derivatives. One therefore needs to specify two more boundary conditions in addition to the usual no-slip conditions. In contrast to passive flows [120] or passive liquid crystal models, the physically correct boundary conditions for continuum models describing active polar and/or active nematic suspensions are generally not well-understood, as they may depend on swimmer type [58, 196], details of cell-cell and cell-surface interactions [136], boundary geometry [227], etc. The boundary conditions considered in Chapter 3 (no slip plus \( \nabla^2 \omega = 0 \) and \( \nabla^4 \omega = 0 \) at the upper and lower boundaries)
were selected because they produce a bulk flow dynamics similar to those observed in recent bacteria experiments [225]. In this section, we illustrate how changing the higher-boundary conditions affects the bulk flow solutions. Examples are shown in Fig. A-4.

Keeping no-slip boundary conditions throughout, we consider separately the different higher-order contributions to the stress tensor as well as the behavior of the Laplacian and bi-Laplacian of the vorticity and its normal derivative on the boundary. In the vorticity-stream function formulation, the shear component of the stress tensor given in Eq. (1.1c) reads

\[
\sigma_{xy} = (\Gamma_0 - \Gamma_2 \nabla^2 + \Gamma_4 \nabla^4)(-\partial_{xx} + \partial_{yy})\psi \\
\equiv \sigma_{xy}^{\Gamma_0} + \sigma_{xy}^{\Gamma_2} + \sigma_{xy}^{\Gamma_4},
\]

where \(\sigma_{xy}^{\Gamma_i}\) represents the contribution to stress proportional to \(\Gamma_i\). Thus, one way of generating higher-order boundary conditions is to fix the various stress contributions separately or combination of them. Figure A-4(a) shows the flow structures obtained by setting \(\sigma_{xy}^{\Gamma_2} = 0\) and \(\sigma_{xy}^{\Gamma_4} = 0\) on the boundary \(\partial D\) with the same bulk flow parameters as in Fig. 3-3(e). The half-loop topology of the stress field lines is still present in these case, although it appears less regular than in Fig. 3-3(e) of Chapter 3.

An alternative way of specifying boundary conditions is to control the Laplacian and bi-Laplacian of the vorticity, \(\nabla^2 \omega\) and \(\nabla^4 \omega\), and/or their normal derivatives, \(\partial_n \nabla^2 \omega\) and \(\partial_n \nabla^4 \omega\). The boundary conditions adopted in Chapter 3, \(\nabla^2 \omega = 0\) and \(\nabla^4 \omega = 0\), fall into this category. This choice implies that the integrated higher-order stress contributions vanish, that is \(\int_{\partial D} \sigma_{xy}^{\Gamma_2} = 0\) and \(\int_{\partial D} \sigma_{xy}^{\Gamma_4} = 0\). Figures A-4(b-d) show flow structures for three other possibilities, again using the same parameters \((L_y, \gamma)\) as in Fig. 3-3(e). The ‘stiff’ combination \(\omega = 0\) and \(\partial_n \omega = 0\) enforces an essentially linear shear profile [Fig. A-4(b)]. By contrast, the softer choice \(\partial_n \nabla^2 \omega = 0\) and \(\partial_n \nabla^4 \omega = 0\) yields vortical structures with half-loops in the stress lines [Fig. A-4(c)]. Finally, the semi-stiff condition \(\partial_n \omega = 0\) and \(\partial_n \nabla^2 \omega = 0\) [Fig. A-4(d)] produces a more linear stress field topology without the half-loops but still allows a directed
motion of the fluids center of mass. For bacterial suspensions, the stiffer boundary conditions appear to be ruled out by the experimental results in Ref. [225].
Appendix B

Appendix to Chapter 4

B.1 Nondimensionalization

For numerical simulations, we non-dimensionalize the equations of motion (4.1) of Chapter 4 by rescaling according to

\[ t' = T_0 t, \quad x'_i = R x_i, \quad \omega' = \omega_0 \omega, \quad \psi' = \psi_0 \psi, \]

(B.1)

which gives, after dropping the primes,

\[ \partial_t \omega + T_0 \frac{\psi_0}{R^2} (\partial_y \psi) \partial_x \omega - T_0 \frac{\psi_0}{R^2} (\partial_z \psi) \partial_y \omega = \frac{T_0 \Gamma_0}{R^2} (\nabla^2 \omega - \frac{\Gamma_2}{\Gamma_0 R^2} \nabla^4 \omega + \frac{\Gamma_4}{\Gamma_0 R^4} \nabla^6 \psi) \]

\[ \frac{\psi_0}{R^2} \nabla^2 \psi = -\omega_0 \omega. \]

We set \( \psi_0 = \frac{R^2}{T_0} \) and \( \omega_0 = \frac{1}{T_0} \) and \( T_0 = \frac{R^2}{\Gamma_0} \), which leads to

\[ \partial_t \omega + (\partial_y \psi) \partial_x \omega - (\partial_z \psi) \partial_y \omega = \nabla^2 \omega - \gamma_2 \nabla^4 \omega + \gamma_4 \nabla^6 \omega, \]

(B.3)

\[ \nabla^2 \psi = -\omega, \]

where \( \gamma_2 = \frac{\Gamma_2}{\Gamma_0 R^2} \) and \( \gamma_4 = \frac{\Gamma_4}{\Gamma_0 R^4} \).
B.2 Stokes' second problem: driving protocol

We describe in detail the driving protocol for the active Stokes' second problem. At \( t = 0 \), we initiate the simulations with both the boundary and fluid at rest plus a small random perturbing flow \( \delta \mathbf{v} \left( \| \delta \mathbf{v} \|_1 / U \ll 1 \right) \), where \( U \) is the characteristic speed of the turbulent patterns). We then turn on the periodic driving by applying the no-slip boundary condition

\[
v_\theta(t, R, \theta) = f(t) A \cos(\Omega t),
\]

where

\[
f(t) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{t - 30\tau}{2.5\tau} \right) \right],
\]

and \( \tau \) is the characteristic time scale of the active flow patterns as defined in Chapter 4. Thus, for the time interval \( \sim 30\tau \) the boundary remains stationary. During that time, the bulk flow relaxes and active turbulence develops. At about \( \sim 30\tau \), the periodic driving sets in and the boundary condition (B.4) quickly approaches \( v_\theta(t, R, \theta) = A \cos(\Omega t) \). Calculation of the mean power \( \langle P \rangle \) presented in Fig. 4-2 of Chapter 4 occurs during the time interval \([60\tau, 200\tau]\), long after the relaxation, see Fig. B-1.

Figure B-1: Stokes’ second problem: Driving protocol. (a) The driving amplitude, cf. Eq. (B.4), increases according to the prefactor \( f(t) \) defined in Eq. (B.5). (b) Mean kinetic energy time series for the simulation shown in Fig. 4-2(b) of Chapter 4 shows that the system relaxes well before the start time of the temporal averaging periods for the mean power \( \langle P \rangle \) (vertical dashed lines).
Figure B-2: Stokes' second problem: Additional examples. The middle row is the same as Figs. 2(b,d) in Chapter 4. For narrow vortex-size distribution $\kappa_s$ (right column), it takes higher driving frequency $\Omega$ to disrupt the structure of the stress field, a core filled with low-stress defects and half-loops of the stress director field near the boundary. As a result, the resonance peak in Fig. 4-2(c) of Chapter 4 appears at a higher frequency for $\kappa_s$ than for $\kappa_w$.

**B.3 Stokes' second problem: Linear relation between torque and angular speed**

In this section we quantify how accurately the formula (4.3) of Chapter 4, which relates the fluid-induced torque $\mathcal{J}(t)$ on the container to the container angular speed $\dot{\phi}(t)$, describes the response of an active fluid subject to oscillatory boundary conditions. This formula approximately holds if the power spectral density (PSD) $|\mathcal{J}(\omega)|^2$ of the time-series $\mathcal{J}(t)$ is concentrated at the driving frequency $\Omega$. To see this, we follow the usual argument, see for example [119]. We write the container angular speed as $\dot{\phi} = \phi_0 R(e^{i\Omega t})$, where $\phi_0 = A/R$. If the PSD $|\mathcal{J}(\omega)|^2$ is concentrated at $\Omega$,
then

\[ \mathcal{T}(t) \approx \Re \{ \mathcal{T}(t)e^{i\Omega t} \} = \Re \{ [\mathcal{T}_r(\Omega) + i\mathcal{T}_i(\Omega)]e^{i\Omega t} \} \]

\[ = \Re \left\{ \mathcal{T}_r(\Omega) e^{i\Omega t} + \frac{\mathcal{T}_i(\Omega)}{\Omega} \frac{d}{dt} e^{i\Omega t} \right\} = \frac{\mathcal{T}_r(\Omega)}{\phi_0} \dot{\phi}(t) + \frac{\mathcal{T}_i(\Omega)}{\phi_0} \ddot{\phi}(t). \]  

(B.6)

Setting \( I_t = -[\mathcal{T}_r(\Omega)]/\phi_0 \) and \( \gamma = -[\mathcal{T}_i(\Omega)]/[\phi_0 \Omega] \), we obtain Eq. (4.3) of Chapter 4.

To verify that the PSD \( |\mathcal{T}(\omega)|^2 \) of the time-series \( \mathcal{T}(t) \) is concentrated at the driving frequency \( \Omega \), we performed spectral analysis of the steady-state part of \( \mathcal{T}(t) \). Let \( \mathcal{T}_n \) be the discrete time series obtained in simulations, where \( n \) denotes the time step. The number of time steps is always taken large enough to ensure that the physical time interval is at least two orders of magnitudes greater than the larger of the two quantities: the characteristic pattern formation scale \( \tau \) or the driving period \( T = 2\pi/\Omega \). The time series itself is obtained by integrating the stress-tensor over the container according to the Eq. (B.11). We apply the Discrete Fourier Transform to \( \mathcal{T}_n \) to obtain the discrete PSD \( |\mathcal{T}_\omega|^2 \).

Fig. B-3 quantifies the shape of the power spectral density by displaying the proportion of the PSD concentrated at \( \Omega \) as well as at the second most energetic frequency. Figs. SB-3(a,d) show the full PSD \( |\mathcal{T}_\omega|^2 \) normalized by the total energy \( \sum_\omega |\mathcal{T}_\omega|^2 \) for the two simulations shown in Figs. 2(b,d) of Chapter 4. Two strong peaks at the driving frequency \( \Omega = 2\pi/\tau \) confirm that the formula (4.3) of Chapter 4 holds in these two cases. In general, we measured the proportion of the PSD stored in the driven mode \( |\mathcal{T}_\Omega|^2 / \sum_\omega |\mathcal{T}_\omega|^2 \) [Figs. SB-3(b,c,e,f)] as well as in the second most energetic mode [insets in Figs. SB-3(b,c,e,f)] for different container radii \( R \), driving frequencies [Figs. SB-3(b,e)], driving amplitudes [Figs. SB-3(c,f)] and the wide [Figs. SB-3(b,c)] and small [Figs. SB-3(e,f)] active bandwidths \( \kappa_s \) and \( \kappa_w \) defined in Chapter 4. We found that at least about half of the energy is always concentrated at the driven mode and that this proportion quickly becomes larger than 90% once the driving frequency \( \Omega \) and amplitude \( A \) become larger than the corresponding active fluid characteristic pattern formation parameters \( 2\pi/\tau \) and \( U \), respectively. The second most energetic mode typically contains an order or two orders of magnitude.
Figure B-3: The linear relation between the torque $\mathcal{T}(t)$ and angular speed $\dot{\phi}(t)$ (4.3) of Chapter 4 approximately holds for active fluids. The formula becomes very accurate as the driving frequency $\Omega$ and amplitude $A$ become larger than the corresponding active fluid characteristic pattern formation parameters $2\pi/\tau$ and $U$, respectively.

(a,d) Normalized power spectral density $|\mathcal{T}_\omega|^2/\sum_{\omega'} |\mathcal{T}_{\omega'}|^2$ of the (discrete) steady-state time series $\mathcal{T}_n$ for the two simulations shown in Figs. 2(b,d) of Chapter 4. The complex amplitudes $\mathcal{T}_\omega$ are obtained by applying the Discrete Fourier Transform to $\mathcal{T}_n$. The proportion of the energy concentrated at the driving frequency $\Omega$ (b,c,e,f) as well as at the second most energetic frequency (insets) as a function of $\Omega$ (b,e), the oscillation amplitude $A$ (c,f) for active fluids with the wide (b,c) and small (e,f) active bandwidths $\kappa_s$ and $\kappa_w$.

less energy than the driven mode. Overall, Fig. B-3 confirms that the response of the torque $\mathcal{T}(t)$ is typically concentrated around the driving frequency $\Omega$, validating the relation (4.3) of Chapter 4 in the case of active fluids.

B.4 Stokes’ second problem: dissipative response

The discussion of the active Stokes’ second problem in Chapter 4 focused on the inertial response characterized by the parameter $I_I$ in Eq. (4.3) of Chapter 4. In this section, we focus on the dissipative response described by the parameter $\gamma$ in that equation. Both, $I_I$ and $\gamma$ are displayed in Fig. B-4.

Specifically, we are interested in the energy transfer between container and active
fluid, reflected in the average power input per unit length $\langle P \rangle$ needed to sustain
the oscillations. As will be shown in the next section, a passive Newtonian fluid
$(\Gamma_2 = \Gamma_4 = 0)$ such as water confined to a circular container responds effectively
as a rigid body under the conditions typical for active fluids experiments. Since
an ideal rigid body is a conservative system, we instead benchmark the active fluid
dissipative response against the response of a passive Newtonian fluid filling the upper
half-plane and driven horizontally along the $x$-axis. In this classical setting, Stokes’
second problem can be solved analytically yielding the power input per unit area
$\langle P \rangle = \rho A^2 \sqrt{\Omega \Gamma_0 / 8}$, where $\rho$ and $\Gamma_0$ are the density and kinematic viscosity of the
fluid [119]. Adapting this classical result to thin-films by interpreting $\langle P \rangle$ as power
per unit length and $\rho$ as area density, $\langle P \rangle = \rho A^2 \sqrt{\Omega \Gamma_0 / 8}$ defines a reference for the
dissipative response of the active fluid.

We computed the power input $\langle P \rangle$ in two different ways: using the full time series
for the torque $T(t)$, which gives $\langle P \rangle = \langle T \dot{\phi} \rangle$, or approximately, using the relation (4.3)
of Chapter 4, for which $\langle P \rangle \approx \gamma (A/R)^2 / 2$. To explore how the confinement geometry,
driving protocol and active fluid properties affect $\langle P \rangle$ computed in these two ways, we
varied systematically the amplitude $A$, the oscillation frequency $\Omega$, and the container
radius $R$ in our simulations, comparing active fluids with wide $(\kappa_w = 1.5/\Lambda)$ and small
$(\kappa_s = 0.63/\Lambda)$ spectral bandwidths, respectively. The results of these parameter scans
are summarized in Figs. SB-4(c,f). The two ways of computing $\langle P \rangle$, through the exact
[markers in Figs. SB-4(c,f)] and approximate [lines in Figs. SB-4(c,f)] formulae, yield
almost identical results, further verifying the validity of Eq. (4.3) of Chapter 4.

Changing the driving amplitude $A$ while keeping the other parameters fixed, we
find that the classical power-amplitude scaling $\langle P \rangle \sim A^2$ remains preserved in active
fluids to within a good approximation [insets in Figs. SB-4(c,f)]. Our simulations
predict, however, that passive and active fluids exhibit a fundamentally different
response to frequency variations. For both $\kappa \Lambda > 1$ and $\kappa \Lambda < 1$, we observe deviations
from the $1/2$ exponent characterized by a relative resonance when the external driving
period $T = 2\pi / \Omega$ becomes of the order of the intrinsic vortex growth time scale $\tau$
[Figs. SB-4(c,f)]. Away from the resonance, the growth is faster than predicted by
Figure B-4: Inertial (a,d) and dissipative (b,e) response parameters $I_f$ and $\gamma$ as a function of the oscillating frequency $\Omega$ and amplitude $A$ (insets) that appear in the relation (4.3) of Chapter 4 for active fluids with wide $\kappa_w = 1.5/A$ (a-c) and small $\kappa_w = 0.63/A$ (d-f) spectral bandwidths. The parameters were computed using Eq. (B.6).

(c,f) Average power input per unit length $\langle P \rangle$ in the steady state normalized by the value $\langle P \rangle$ expected from the classical Stokes' problem for a semi-infinite plate shows relative resonance at the characteristic frequency $2\pi/\tau$ of the active flow patterns. The markers indicate power input as computed from the full time series of the torque $T(t)$ while the lines indicate the contribution derived from the linear relation (4.3) of Chapter 4.

The 1/2 exponent at small frequencies and slower than the 1/2 exponent at large frequencies; the precise growth rates depend on the domain size $R$, a signature of the interplay between activity and confinement. However, the relative resonance itself is robust against variations in $R$. 
In this section, we analyze the response of a passive fluid \((\Gamma_2 = \Gamma_4 = 0)\) with water viscosity \(\Gamma_0 = 10^{-6} \text{m}^2/\text{s}\) to the oscillatory boundary conditions presented in Fig. 4-2 of Chapter 4. We first compare the penetration depth \(\delta\) [119]

\[
\delta = \sqrt{2\Gamma_0/\Omega},
\]

with the domain size \(R\). Typical values of the characteristic time scale \(\tau\) and vortex size \(\Lambda\) at the peak of bacterial activity are \((\tau, \Lambda) = (2\text{s}, 50\mu\text{m})\) [59]. Therefore, in a potential experiment realizing the set-up in Fig. 4-2 of Chapter 4, one expects frequencies and domain sizes of the order \(\Omega \sim 2\pi/\tau \sim 3.14 \text{ rad/s}, R \sim 4\Lambda \sim 200 \mu\text{m}\).

For such frequencies, a passive fluid with water viscosity has the penetration depth

\[
\delta \sim 1\text{mm} \gg R.
\]
We see that the penetration depth is much bigger than the domain size, which implies that, for the range of domain sizes and driving frequencies relevant to the active Stokes’ second problem, the passive fluid effectively responds as a rigid body. Since a rigid body performing harmonic oscillations behaves like a conservative system, one expects a negligible power input in that case. Specifically, a flat disk of radius $R$ and thickness $z$ filled with water with density $\rho = 10^3 \text{kg/m}^3$ has mass $m_f = \rho \pi R^2 z$. The corresponding moment of inertia is

$$I = m_f R^2 / 2 = \rho \pi R^4 z / 2.$$ 

The angular speed of the disk is $\dot{\phi}(t) = (A/R) \cos(\Omega t)$, see Eq. (B.4). Then the energy of the rigid disk is $E = I \dot{\phi}^2 / 2$. Differentiating with respect to time yields the power of the disk undergoing sinusoidal oscillations about the $z$-axis

$$P = \dot{E} = I \ddot{\phi} = -I (A/R)^2 \Omega \cos(\Omega t) \sin(\Omega t). \quad (B.9)$$

Averaging the above expression over a period yields zero power input, as expected. We compared this exact expression with the power input for a passive fluid subject to the oscillatory boundary conditions in the disk geometry presented in Fig. 4-2 of Chapter 4 with driving parameters $(R, A, \Omega) = (200 \mu\text{m}, 628 \mu\text{m/s}, 3.14 \text{rad/s})$, typical for the active problem. Fig. B-5 shows the vorticity profile and time series for the power input in a representative simulation. The evolution of the power input is sinusoidal and follows the exact expression (B.9) very closely, implying that the fluid indeed behaves like a rigid body, as expected from the above penetration depth estimates.

The above analysis confirms that a passive fluid responds to the oscillatory boundary conditions in the disk geometry presented in Fig. 4-2 of Chapter 4 effectively as a rigid body. In the notation given by Eq. (4.3) of Chapter 4, the passive response is characterized by $I_f = I_{f, p} = m_f R^2 / 2$ and $\gamma_{\text{passive}} \approx 0$, justifying the definition of the activity-induced relative added mass $\lambda$ given by $I_{f, a} = (1 + \lambda) I_{f, p}$. 

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B.6 Container angular momentum equation

When a container encapsulating an active fluid is isolated from external forces and torques, it is subject solely to the fluid stresses. The container is taken to be a uniform rigid ring of radius $R$ and mass $m_c$. The fluid is assumed to form a planar free standing thin film supported on the ring. Since the fluid is incompressible, the center of mass of the ring is stationary. However, the ring can acquire angular momentum, because the fluid can exert nonzero torque on the container. The ring's angular momentum is

$$L_c = I \dot{\phi},$$
\[(B.10)\]

where $I = m_c R^2$ is the moment of inertia and $\dot{\phi}$ is the angular speed. Assuming the ring lies in the $(x, y)$-plane and its center is at the origin, working in polar coordinates $(r, \theta)$, we find that the torque due the fluid stress on a small segment $Rd\theta$ of the ring is

$$\text{torque on a small segment} = R \hat{r} \times [\rho \sigma \cdot (-\hat{r}) Rd\theta] = -\rho R^2 \sigma_{r\theta} d\theta,$$
\[(B.11)\]

where $\sigma_{r\theta} = \hat{r} \cdot \sigma \cdot \hat{\theta}$. The two-dimensional fluid density $\rho$ appears explicitly, since in Chapter 4 it is our convention that in the stress tensor

$$\sigma = (\Gamma_0 - \Gamma_2 \nabla^2 + \Gamma_4 \nabla^4)[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$$
\[(B.12)\]

the parameters $\Gamma_i$ are kinematic quantities. Integrating over the entire ring gives the total torque, and thus the evolution of the ring angular momentum obeys

$$\frac{d}{dt} L_c = -\rho R^2 \int_0^{2\pi} d\theta \sigma_{r\theta}.$$  
\[(B.13)\]

In terms of the ring angular acceleration, we have

$$m_c \ddot{\phi} = -\rho \int_0^{2\pi} d\theta \sigma_{r\theta}.$$  
\[(B.14)\]
Figure B-6: Einstein–de Haas analogue effect for an active fluid with narrow vortex-size distribution $\kappa_S = 0.63/A$. (a) Angular momentum fluctuations $\sigma_L$ as a function of the domain size for heavy containers obtained from Fig. 4-3(a) of Chapter 4 by averaging over $\alpha \in [0.01, 0.1]$. (b) Angular speed fluctuations $\sigma_\omega$ as a function of the domain size for light containers obtained from Fig. 4-3(c) of Chapter 4 by averaging over $\alpha \geq 10$. (c-d) Zoom-in of the time series of the container’s angular momentum (blue) calculated from Eq. (B.10) shown in Fig. 4-3(b) of Chapter 4 for domain radius $R = 3.33$ (c) and $R = 3.67$ (d). Additionally, to illustrate the angular momentum conservation in the fluid-container system, we show the time series of the fluid’s angular momentum (orange) calculated independently using the formula $L_{\text{fluid}} = \rho \int_0^R \int_0^{2\pi} r v_\theta r dr d\theta$.

Non-dimensionalizing as in the first section, we obtain

$$ m_c \ddot{\phi} = -\rho R^2 \int_0^{2\pi} d\theta \sigma_{r\theta}, \quad \text{(B.15)} $$

where $\sigma = (1 - \gamma_2 \nabla^2 + \gamma_4 \nabla^4)[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$. The fluid mass is $m_f = \rho \pi R^2$. We introduce a dimensionless parameter $\alpha$ representing the ratio of the fluid mass to the ring mass,

$$ \alpha = \frac{m_f}{m_c}. \quad \text{(B.16)} $$

We then find that the angular speed of the ring obeys

$$ \ddot{\phi} = -\frac{\alpha}{\pi} \int_0^{2\pi} d\theta \sigma_{r\theta}. \quad \text{(B.17)} $$
Appendix C

Appendix to Chapter 5

C.1 Vorticity-stream formulation

Here, we derive the vorticity-stream function formulation (2) of Chapter 5 from the momentum conservation (1b) of Chapter 5 for surfaces of constant Gaussian curvature $K$. We proceed by first rewriting the momentum equation explicitly in terms of the velocity vector field $v^a$. This involves the standard computation of the divergence of the rate of strain on curved surfaces as well as commuting covariant derivatives acting on rank-2 symmetric traceless tensors, which generates additional curvature terms. Once the equation for the contravariant vector $v^a$ is known, we lower the indices to find the corresponding equation satisfied by the covariant vector $v_a$. We then apply the Hodge decomposition [179] to $v_a$, which determines the unique (up to a constant) stream function $\psi$, and take the (surface) curl of the equation of motion for $v_a$ to find the equation for the vorticity function $\omega$. Finally, we consider the case of a sphere of radius $R$ and introduce spherical coordinates. The final system takes the simple form of a higher order partial differential equation for two scalar functions $\psi$ and $\omega$ on a sphere, where the differential operators are the familiar Jacobian and spherical Laplacian.

**Divergence of rate of strain tensor.** To express the momentum conservation (1b) of Chapter 5 in terms of the velocity field $v^a$, we must compute the diver-
gence $\nabla_b T^{ab}$ of the stress tensor $T^{ab}$

\begin{align}
T^{ab} &= f(\nabla^2)2S^{ab}, \\
f(\nabla^2) &= \Gamma_0 - \Gamma_2 \nabla^2 + \Gamma_2 \nabla^2 \nabla^2,
\end{align}

where $\nabla^2 = \nabla^c \nabla_c$ is the tensor Laplacian defined in terms of the covariant (Levi-Civita) derivative $\nabla_c$ and

$$S^{ab} = \frac{1}{2}(\nabla^a v^b + \nabla^b v^a),$$

is the rate of strain tensor for a stationary surface [180]. To this end, we first recall the standard calculation of the divergence of the rate of strain tensor $\nabla_a S^{ab}$, see for example [7]. By definition, the divergence is

$$2\nabla_a S^{ab} = \nabla_a \nabla^a v^b + \nabla_a \nabla^b v^a.$$  \hfill (C.3)

Before we simplify the last term by using the incompressibility condition $\nabla_a v^a = 0$, we must compute the commutator $[\nabla_a, \nabla^b] v^a$. From the definition of the Riemann curvature tensor $R_{edac}$, we have

$$\nabla_a \nabla_c v_d - \nabla_c \nabla_a v_d = -v^e R_{edac}.  \hfill (C.4)$$

For a two-dimensional surface with Gaussian curvature $K$, the Riemann tensor is

$$R_{edac} = K(g_{ea} g_{cd} - g_{ec} g_{ad}), \hfill (C.5)$$

and the commutator becomes

$$\nabla_a \nabla_c v_d - \nabla_c \nabla_a v_d = -v^e K(g_{ea} g_{cd} - g_{ec} g_{ad}) = -K(v_a g_{cd} - v_c g_{ad}). \hfill (C.6)$$
Contracting $a$ and $d$, and using the incompressibility $\nabla_v v^a = 0$ condition we obtain
\begin{equation}
\nabla^a \nabla_c v_a = K v_c. \tag{C.7}
\end{equation}

Substituting this for the last term in Eq. (C.3), we recover the known result [7] that the divergence of the rate of strain tensor has the form
\begin{equation}
2\nabla_a S^{ab} = \nabla_a \nabla^a v^b + K v^b = (\nabla^2 + K) v^b, \tag{C.8}
\end{equation}
which implies that, on curved spaces, additional forces arise due to the curvature, beyond the viscous term proportional to the Laplacian.

**Computing $[\nabla_a, \nabla^2]$ on rank-two symmetric traceless tensors.** Now that we have the divergence of the rate of strain tensor (C.8), we can express the divergence of the stress tensor $\nabla_b T^{ab}$ in terms of $v^a$. To this end, we must know how to commute the tensor Laplacian $\nabla^2 = \nabla^c \nabla_c$ with the operator $\nabla_a$. Since the velocity field is incompressible, the rate of strain tensor is traceless $S^{aa} = 0$ and so is any tensor Laplacian of it, $\nabla^2 S^a_a = 0$ and $\nabla^2 \nabla^2 S^a_a = 0$. We thus have to compute the commutator $[\nabla_a, \nabla^2]$ on a symmetric and traceless rank-2 tensor. The following calculation is very similar to that presented in Section 3 in [52]. For a rank-three covariant tensor $H_{cde}$ we have
\begin{equation}
[\nabla_a, \nabla_b] H_{cde} = - R^f_{\ cab} H_{fde} - R^f_{\ dab} H_{cfe} - R^f_{\ eab} H_{cdf}, \tag{C.9}
\end{equation}
Contracting $a$ with $d$ and $b$ with $c$ yields
\begin{equation}
[\nabla^a, \nabla^b] H_{bae} = - R^{fa}_{\ ba} H_{fae} - R^{fa}_{\ a} H_{bfe} - R^{fa}_{\ ab} H_{baf}. \tag{C.10}
\end{equation}
The first two terms on the right hand side cancel out
\begin{equation}
R^{fa}_{\ ba} H_{fae} + R^{fa}_{\ a} H_{bfe} = (R^{fa}_{\ ba} + R^{fa}_{\ ab}) H_{fae} = (R_f^{\ \ b} R_{a b f}) H^{fa} e = (R_{fhab} + R_{abhf}) g_{\ \ b} H^{fa} e = 0 \tag{C.11}
\end{equation}
since \( R_{fhab} = R_{hfab} = -R_{abhf} \), which reduces (C.10) to

\[
[\nabla^a, \nabla^b] H_{bae} = R^e_{
abla b} H_{abf}. \tag{C.12}
\]

Similarly, for rank-two covariant tensors, we have

\[
[\nabla_a, \nabla_b] H_{ed} = -R^e_{cob} H_{ed} - R^e_{dab} H_{ce}. \tag{C.13}
\]

Contracting \( a \) and \( c \) gives

\[
(\nabla^a \nabla_b - \nabla_b \nabla^a) H_{ad} = R^e_{b} H_{ed} - R^e_{d} b H_{ae}, \tag{C.14}
\]

where \( R^e = K\delta^e_b \) is the Ricci tensor, and further contraction with \( \nabla^b \) yields

\[
(\nabla^b \nabla^a \nabla_b - \nabla^b \nabla_b \nabla^a) H_{ad} = \nabla^b (R^e_{b} H_{ed}) - \nabla^b (R^e_{d} a b H_{ae}). \tag{C.15}
\]

We now restrict to the specific case of constant Gaussian curvature \( K \). Since \( K \) is constant and we are working with the Levi-Civita connection, we have

\[
(\nabla^b \nabla^a \nabla_b - \nabla^b \nabla_b \nabla^a) H_{ad} = R^e_{b} \nabla^b H_{cd} - R^e_{d} a b \nabla^b H_{ae} \\
= K \nabla^e H_{ed} - R^e_{d} a b \nabla^b H_{ae}. \tag{C.16}
\]

The last term is

\[
R^e_{d} a b H_{ae} = K \nabla_d H^a_{e} - K \nabla^e H_{de}. \tag{C.17}
\]

Since our symmetric tensor is traceless, we have

\[
(\nabla^b \nabla^a \nabla_b - \nabla^b \nabla_b \nabla^a) H_{ad} = 2K \nabla^e H_{de}. \tag{C.18}
\]

We now combine Eq. (C.12) with Eq. (C.18) by setting \( H_{bae} = \nabla_b H_{ae} \). For this choice,
Eq. (C.12) gives

\[ \nabla^a \nabla^b \nabla_b H_{ae} = \nabla^b \nabla^a \nabla_b H_{ae} + R^f_{\ e \ ab} \nabla_a H_{bf}. \]  

(C.19)

Using (C.18) to replace the first term on the right hand side gives

\[ \nabla^a \nabla^b \nabla_b H_{ae} = \nabla^b \nabla_b \nabla^a H_{ae} + 2 K \nabla^f H_{ef} + R^f_{\ e \ ab} \nabla_a H_{bf}. \]  

(C.20)

We rewrite the last term above as

\[ R^f_{\ e \ ab} \nabla_a H_{bf} = K (g f^a \delta_e^b - g f^b \delta_e^a) \nabla_b H_{ef} \]

\[ = K \nabla^f H_{ef} - K \nabla_e H_f = K \nabla^f H_{ef}. \]  

(C.21)

Finally, we obtain the following expression

\[ \nabla_a \nabla^b \nabla_b H_{ae} = \nabla^b \nabla_b \nabla^a H_{ae} + 3 K \nabla_a H_{ae} = (\nabla^2 + 3 K) \nabla_a H_{ae}, \]  

(C.22)

which, on surfaces of constant Gaussian curvature \( K \), allows us to commute the tensor Laplacian with the divergence operator acting on any symmetric and traceless tensor \( H_{ae} \).

**Divergence of stress tensor and equation for \( v^a \).** To calculate the divergence of the stress tensor, we combine the results of the two previous subsections. We obtain

\[ \nabla_b T^{ab} = \nabla_b f (\nabla^2)2 S^{ab} = f (\nabla^2 + 3 K)2 \nabla_b S^{ab} = f (\nabla^2 + 3 K)(\nabla^2 + K)v^b, \]  

(C.23)

where we used Eq. (C.22) in the first line, remembering that \( S^{a}_a = \nabla^2 S^{a}_a = \nabla^2 \nabla^2 S^{a}_a = 0 \), and Eq. (C.8) in the second line. We can now express the momentum conservation equation (1b) of Chapter 5 solely in terms of the velocity vector field \( v^a \)

\[ \partial_t v^a + v^b \nabla_b v^a = f (\nabla^2 + 3 K)(\nabla^2 + K)v^a + \nabla^a \sigma. \]  

(C.24)

**Coordinate-free equation for \( v_a \).** We lower the indices in (C.24) to obtain the
corresponding equation for the covariant vector (one-form) $v_a$

\[
\partial_t v_a + v^b \nabla_b v_a = f(\nabla^2 + 3K)(\nabla^2 + K)v_a + \nabla_a \sigma. \tag{C.25}
\]

Since we are ultimately interested in applying the Hodge decomposition to the one-form $v_a$, we first replace the tensor Laplacian $\nabla^2 = \nabla_c \nabla_c$ with the Hodge Laplacian $\Delta_H = \delta d + d \delta$, where $d$ and $\delta$ are the differential and codifferential operators. The Weitzenböck identity [24] for one-forms reads

\[
\Delta_H v_a = -\nabla^c \nabla_c v_a + R_{ab} v^b. \tag{C.26}
\]

As before, for two dimensional surfaces $R_{ab} = K g_{ab}$, and the above identity reduces to

\[
\Delta_H v_a = -\nabla^c \nabla_c v_a + K v_a. \tag{C.27}
\]

In terms of the Hodge Laplacian, the equations of motion read

\[
\nabla^a v_a = 0, \tag{C.28}
\]

\[
\partial_t v_a + v^b \nabla_b v_a = f(-\Delta_H + 4K)(-\Delta_H + 2K)v_a + \nabla_a \sigma
\]

To simplify the subsequent calculations, we rewrite the above equations in the coordinate-free form. Denote by $v$ the contravariant field $v^a$ and by $v$ the corresponding covariant one-form $v_a$. In the new notation (C.28) reads

\[
\delta v = 0, \tag{C.29}
\]

\[
\partial_t v + \nabla v = d\sigma + f(-\Delta_H + 4K)(-\Delta_H + 2K)v,
\]

where again, $d$ and $\delta$ denote the differential and codifferential operators. It is useful to apply the following identity relating the Lie and covariant derivatives of one-forms [13]

\[
L_v v = \nabla_v v + \frac{1}{2} d v^2, \tag{C.30}
\]
where \( v^2/2 = \nu_\alpha \nu_\alpha/2 \) is the kinetic energy density. In terms of the Lie derivative, the equation of motion reads

\[
\delta v = 0, \quad (C.31a)
\]
\[
\partial_t v + L_v v = f(-\Delta_H + 4K)(-\Delta_H + 2K)v + d(\sigma + \frac{1}{2}v^2). \quad (C.31b)
\]

**Hodge decomposition and vorticity-stream function formulation.** Since \( v \) is co-exact (\( v \) is divergence-free), we can use the Hodge decomposition [179] to write \( v = \delta \tilde{\psi} \), for some two-form (pseudoscalar) \( \tilde{\psi} \) that we will later identify with the stream function. Introducing \( \tilde{\psi} \) automatically satisfies Eq. (C.31a). We take the differential \( d \) of Eq. (C.31b) to derive the vorticity equation. The great advantage of the above coordinate-free representation is that the differential commutes both with the Lie derivative and Hodge Laplacian. We get

\[
\partial_t \tilde{\omega} + L_v \tilde{\omega} = f(-\Delta_H + 4K)(-\Delta_H + 2K)\tilde{\omega}, \quad (C.32)
\]

where we introduced the vorticity pseudoscalar \( \tilde{\omega} = dv = d\delta \tilde{\psi} = \Delta_H \tilde{\psi} \). Thus, the equations of motion become

\[
\Delta_H \psi = \tilde{\omega}, \quad (C.33)
\]
\[
\partial_t \tilde{\omega} + L_v \tilde{\omega} = f(-\Delta_H + 4K)(-\Delta_H + 2K)\tilde{\omega}.
\]

Above, \( \tilde{\psi} \) and \( \tilde{\omega} \) are both pseudoscalars (two-forms in 2D). We now apply the Hodge star \( * \), which commutes with the Hodge Laplacian, to find equations for the scalars \( \omega = *\tilde{\omega} \) and \( \psi = *\tilde{\psi} \)

\[
\Delta_H \psi = \omega, \quad (C.34)
\]
\[
\partial_t \omega + * L_v \tilde{\omega} = f(-\Delta_H + 4K)(-\Delta_H + 2K)\omega.
\]

Since on scalars the Hodge Laplacian equals the negative of the familiar Laplace-
Beltrami operator, \( \Delta_H = -\Delta \), we finally arrive at

\[
\Delta \psi = -\omega, \quad \text{(C.35)}
\]

\[
\partial_t \omega + *L_\nu \tilde{\omega} = f(\Delta + 4K)(\Delta + 2K)\omega.
\]

**Spherical case.** In this section, we specialize to the case of a sphere of radius \( R \) and explicitly write Eqs. (C.35) in spherical coordinates \((\theta, \phi)\). The metric and its inverse are \( g_{ij} = R^2 \text{diag}(1, \sin^2 \theta) \) and \( g^{ij} = R^{-2} \text{diag}(1, 1/\sin^2 \theta) \) and the determinant volume prefactor is \( \sqrt{|g|} = R^2 \sin \theta \).

Let the pseudoscalar stream function be

\[
\tilde{\psi} = \psi(\theta, \phi) \sqrt{|g|} d\theta \wedge d\phi = \psi(\theta, \phi) R^2 \sin \theta d\theta \wedge d\phi, \quad \text{(C.36)}
\]

where \( \psi = *\tilde{\psi} \) is the stream function on the sphere. We now compute the velocity field \( \nu = \delta \tilde{\psi} \). The codifferential for a 2D Riemannian manifold is \( \delta = -*d* \), and since \( \psi = *\tilde{\psi} \), we obtain

\[
d * \tilde{\psi} = \partial_\theta \psi d\theta + \partial_\phi \psi d\phi. \quad \text{(C.37)}
\]

Applying the Hodge star yields

\[
* d * \tilde{\psi} = (d * \tilde{\psi})^t \sqrt{|g|} \varepsilon_{ij} dx^j = -\frac{1}{\sin \theta} \partial_\phi \psi d\theta + \partial_\theta \psi \sin \theta d\phi, \quad \text{(C.38)}
\]

and the velocity one-form becomes

\[
\nu = \delta \tilde{\psi} = \frac{1}{\sin \theta} \partial_\phi \psi d\theta - \partial_\theta \psi \sin \theta d\phi. \quad \text{(C.39)}
\]

By raising the indices, we obtain the corresponding velocity vector field

\[
\nu = \frac{1}{R^2 \sin \theta} \frac{1}{R} \partial_\phi \psi \partial_\theta - \frac{1}{R^2 \sin \theta} \frac{1}{R} \partial_\theta \psi \partial_\phi. \quad \text{(C.40)}
\]

In terms of the usual unit vectors \( \hat{\theta} \) and \( \hat{\phi} \), this is \( \nu = \frac{1}{R \sin \theta} \partial_\phi \psi \hat{\theta} - \frac{1}{R} \partial_\theta \psi \hat{\phi} \). We now
compute the vorticity pseudoscalar
\[
\tilde{\omega} = dv = -\left[ \frac{1}{\sin \theta} \partial_{\phi} \psi + \partial_{\theta} (\sin \theta \partial_{\phi} \psi) \right] d\theta \wedge d\phi
\]
\[= \omega(\theta, \phi) R^2 \sin \theta d\theta \wedge d\phi,
\]
where
\[
\omega(\theta, \phi) = -\frac{1}{R^2} \left[ \frac{1}{\sin \theta} \partial_{\theta} (\sin \theta \partial_{\phi} \psi) + \frac{1}{\sin^2 \theta} \partial_{\phi} \psi \right] \psi(\theta, \phi),
\]
and we recover \( \Delta \psi = -\omega \) as required, since \( \omega = *\tilde{\omega} = \omega(\theta, \phi) \). We finally compute the Lie derivative
\[
L_v \tilde{\omega} = di_v \tilde{\omega}.
\]
We start with
\[
i_v \tilde{\omega} = \omega(\theta, \phi) R^2 \sin \theta (v^\theta d\phi - v^\phi d\theta) = \omega(\theta, \phi) \partial_{\theta} \psi d\theta + \omega(\theta, \phi) \partial_{\phi} \psi d\phi,
\]
then
\[
L_v \tilde{\omega} = \left[ \partial_{\theta} (\omega \partial_{\phi} \psi) - \partial_{\phi} (\omega \partial_{\theta} \psi) \right] d\theta \wedge d\phi = \left( \partial_{\theta} \omega \partial_{\phi} \psi - \partial_{\phi} \omega \partial_{\theta} \psi \right) d\theta \wedge d\phi.
\]
Finally,
\[
* L_v \tilde{\omega} = \frac{1}{R^2 \sin \theta} \frac{1}{R^2} (\partial_{\theta} \omega \partial_{\phi} \psi - \partial_{\phi} \omega \partial_{\theta} \psi).
\]
To sum up, in spherical coordinates, the equations of motion [Eq. (2) in Chapter 5] read
\[
\Delta \psi = -\omega, 
\]
\[
\partial_t \omega = \frac{1}{R^2 \sin \theta} (\partial_{\theta} \omega \partial_{\phi} \psi - \partial_{\phi} \omega \partial_{\theta} \psi) + f(\Delta + 4K)(\Delta + 2K) \omega,
\]
\[167\]
where $\omega$ and $\psi$ are scalars (not pseudoscalars) on a sphere and $\Delta$ is the usual Laplace-Beltrami operator for a sphere of radius $R$.

**Expansion in spin-weighted spherical harmonics.** For numerical integration within Dedalus [37], each vector and tensor is expanded in terms of a basis of coherent spin weight: $e_{\pm} = (e_\theta \mp ie_\phi)/\sqrt{2}$. A unitary matrix transforms between the spin basis and the coordinate basis. The tensor product of unit vectors of coherent spin weight adds their individual spin, i.e. the rank-4 tensor basis element $e_+ e_- e_+ e_+$ carries spin-weight $+1 - 1 + 1 + 1 = 2$. We expand the components of a tensor in terms of spin-weighted spherical harmonics depending on the spin-weight of their basis vectors. The surface tension, $\sigma$, is a pure spin-0 field. The velocity, $v$, is a sum of ±1 components. Higher-order tensors comprise a range of spin-weights; e.g., $v \otimes v$ contains components with spin±2 and spin-0.

For example, with an rank-$r$ tensor,

$$T = \sum_{\sigma_i = \pm 1} \sum_{m = -L}^{L} \sum_{\ell_0} T_{\ell,m}^{\sigma_1 \ldots \sigma_r} Y^{m,s}_\ell(\theta, \phi)e_{\sigma_1} \ldots e_{\sigma_r} \quad (C.48)$$

where $s = \sigma_1 + \ldots + \sigma_r$, $\ell_0 = \max(|m|, |s|)$, and $T_{\ell,m}^{\sigma_1 \ldots \sigma_r}$ represent an array of spectral coefficients. The intrinsic gradient operator on the two-sphere acts in a coherent way with regard to spin.

$$\nabla (Y^{m,s}_\ell e_{\sigma_1} \ldots e_{\sigma_r}) = (k_{\ell,s}^+ Y^{m,s+1}_\ell e_+ + k_{\ell,s}^- Y^{m,s-1}_\ell e_-) e_{\sigma_1} \ldots e_{\sigma_r} \quad (C.49)$$

where

$$k_{\ell,s}^\mu = -\mu \sqrt{\frac{(\ell - \mu s)(\ell + \mu s + 1)}{2}}. \quad (C.50)$$

This means that the spectral coefficients act in a particularly simple way under differentiation.

$$\nabla T \longleftrightarrow k_{\ell,s}^\pm T_{\ell,m}^{\sigma_1 \ldots \sigma_r} \quad (C.51)$$
The spin-weighed basis renders computations in the sphere almost identical to Fourier series from an algorithmic perspective. Said another way: the gradient of a traditional spherical harmonic function is not a series of traditional spherical harmonic functions. But it is a very small number of other kinds of functions. This is philosophically the same as saying that the derivative of a cosine function is not naturally a series of cosines functions. But it is a very simple expression in terms of sine functions, and vice versa. In fact, sine and cosine functions are the spherical harmonic basis for the one-dimensional sphere (also known as the circle); so it's more than just a convenient analogy, the same underlying structure is at play in both cases.

The spin-weighted spherical harmonic function are each orthonormal under integration on the unit sphere. We therefore use Gauss quadrature to transform from field on a Legendre quadrature grid to the spectral coefficients. Linear operations happen in spectral space, nonlinear multiplications happen locally on the grid. Since the GNS equations are linearly decoupled for different values of \( m \), the scheme can be easily parallelized. This is done automatically via MPI in Dedalus, allowing the simulations to be run on up to \( \ell_{\text{max}} \) cores simultaneously, where \( \ell_{\text{max}} \) is predetermined cut-off of the spectral expansion.

With the above definitions, we can compute the intrinsic Laplacian in two-dimensions. Acting on an individual spin component,

\[
\nabla \cdot \nabla T \leftrightarrow (k_{\ell,s+1}^- k_{\ell,s}^+ + k_{\ell,s-1}^+ k_{\ell,s}^-) T_{\ell,m}^{\sigma_1 \ldots \sigma_r}, \tag{C.52}
\]

where

\[
k_{\ell,s+1}^- k_{\ell,s}^+ + k_{\ell,s-1}^+ k_{\ell,s}^- = -\ell(\ell+1) + s^2. \tag{C.53}
\]

Equation (C.52) gives a slightly different formula than would result from taking the three-dimensional Laplacian and restricting the result to the surface of the unit 2-sphere. In this case, additional terms result from contracting in the third dimension. Equation (C.52) defines what is often called the rough Laplacian, connection Laplacian, or intrinsic Laplacian. In curved geometry, it is possible to define several linear,
second-order, elliptic differential operators with a reasonable claim to the title of Laplacian. The Weitzenböck identity ensures that any two such Laplacians differ by a scalar curvature term at most; i.e., a term with no derivatives. In the case of the restricted three-dimensional Laplacian,

\[
\nabla_{3D} \cdot \nabla_{3D} T\bigg|_{s^2} \longleftrightarrow -\left(\ell(\ell + 1) - s^2 + r\right) T_{\ell,m}^{\sigma_1\ldots\sigma_r},
\]

where \( r \) gives the tensor rank of \( T \). In the case of simple vectors, \( e_\pm, s^2 = r = 1 \). In applications, the most appropriate Laplacian depends on the details of the underlying physics.

C.2 Nondimensionalization and exact solutions

Nondimensionalization. Before we construct exact stationary solutions of (C.47), we first nondimensionalize the equation by introducing the length scale \( R \) and the time scale \( T \), and set

\[
\psi \rightarrow \frac{R^2}{T^2} \psi, \quad \omega \rightarrow \frac{1}{T} \omega. \tag{C.54}
\]

For a sphere of radius \( R \), the Gaussian curvature is \( K = R^{-2} \). The equations of motion (C.47) become

\[
\Delta \psi = -\omega,
\]

\[
\partial_\theta \omega = f(\Delta + 4)(\Delta + 2)\omega - \frac{1}{\sin \theta} (\partial_\theta \omega \partial_\phi \psi - \partial_\phi \omega \partial_\theta \psi), \tag{C.55}
\]

where \( \Delta = (\sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta) + (\sin \theta)^{-2} \partial_\phi \partial_\phi \) is the usual Laplacian on the unit sphere and

\[
f(\Delta + 4)(\Delta + 2)\omega = [1 - \gamma_2(\Delta + 4) + \gamma_4(\Delta + 4)^2]\Gamma_0 \frac{T}{R^2}(\Delta + 2)\omega, \tag{C.56}
\]
where \( \gamma_2 = \Gamma_2/(\Gamma_0 R^2) \) and \( \gamma_4 = \Gamma_4/(\Gamma_0 R^4) \). To summarize, the nondimensionalized equations are

\[
\begin{align*}
\Delta \psi &= -\omega, \\
\partial_t \omega &= -\frac{1}{\sin \theta} \left( \partial_\theta \omega \partial_\theta \psi - \partial_\phi \omega \partial_\phi \psi \right) + \left[ 1 - \gamma_2 (\Delta + 4) + \gamma_4 (\Delta + 4)^2 \right] (\Delta + 2) \omega,
\end{align*}
\]

where we set the time scale to \( T = R^2/\Gamma_0 \).

**Exact stationary solutions.** We start the construction of exact stationary solutions of (C.57) by first noting that the spherical harmonics \( Y^m_\ell(\theta, \phi) \) are eigenstates of the Laplace operator \( \Delta = (\sin \theta)^{-1} \partial_\theta (\sin \theta \partial_\theta) + (\sin \theta)^{-2} \partial_\phi \phi \)

\[
\Delta Y^m_\ell = -\ell (\ell + 1) Y^m_\ell.
\]

Taking the linear combination \( \psi = \sum_m a_m Y^m_\ell \), where \( -\ell \leq m \leq \ell \) and \( a_m \) are arbitrary real numbers, annihilates the nonlinear term in Eq. (C.47b) because \( \omega = \ell (\ell + 1) \psi \) by Eq. (C.47a). This reduces (C.47b) to the polynomial equation

\[
0 = \left\{ 1 + \gamma_2 [\ell (\ell + 1) - 4] + \gamma_4 [\ell (\ell + 1) - 4]^2 \right\} [\ell (\ell + 1) - 2].
\]

The index \( \ell \) is a non-negative integer; if it coincides with a positive root of the above polynomial, we obtain an exact stationary solution. There are two possibilities: either

\[
1 + \gamma_2 [\ell (\ell + 1) - 4] + \gamma_4 [\ell (\ell + 1) - 4]^2 = 0
\]

or

\[
\ell (\ell + 1) - 2 = 0.
\]

The first possibility is a direct consequence of the higher-order nature of Eqs (C.57). In this case, non-trivial roots exist when \( \gamma_2 < 0 \), which introduces linearly unstable modes. Example of solutions of this type are shown in Fig. 5-1 of Chapter 5. The second possibility gives \( \ell = 1 \) as the only admissible solution and arises even for
the classical Navier-Stokes equations. Superposition of $\ell = 1$ spherical harmonics
corresponds to flow patterns representing rigid rotation of the whole sphere with
rotation rate and rotation axis specified by the three constants $\{a_{-1}, a_0, a_1\}$. We stress
that this second possibility arises only when one derives the equations of motion from
the Cauchy equations (Eq. 1 in Chapter 5) instead of starting with the equations
for the velocity field in the flat space and promoting the corresponding differential
operators to covariant ones, implying that the latter approach is incorrect.

C.3 Energy spectrum and energy flux

Energy spectrum. For flows on periodic domains, the energy spectrum is typically
defined by expanding energy using the Fourier series. For flows on a sphere, the most
natural analogue is obtained by the spherical harmonics basis. The kinetic energy
density function is $e = v_a v^a/2$. Integrating over the sphere surface gives the total
kinetic energy

$$E = \int_S e d\Omega = \frac{1}{2} \int_S v_i v^i d\Omega,$$

where $d\Omega$ is the area element. In coordinate free notation

$$\int_S v_i v^i d\Omega = \int_S v \wedge *v = \langle v, v \rangle,$$

where $\langle v, v \rangle$ is the inner product of one-forms. We use the Green’s formula [179]
(integration by parts) to get

$$2E = \langle v, v \rangle = \langle \delta \tilde{\psi}, \delta \tilde{\psi} \rangle = \langle \tilde{\psi}, d \delta \tilde{\psi} \rangle = \langle \tilde{\psi}, \Delta_H \tilde{\psi} \rangle$$

$$= \langle \tilde{\psi}, \tilde{\omega} \rangle = \int_S \tilde{\psi} \wedge *\tilde{\omega} = \int_S \tilde{\psi} \omega = \int_S \psi \omega d\Omega.$$ (C.64)

Since $\Delta \psi = -\omega$, we obtain for the total energy

$$E = \frac{1}{2} \int_S \psi \omega d\Omega = \frac{1}{2} \int_S \psi \omega d\Omega - \frac{1}{2} \int_S \psi \Delta \psi d\Omega.$$ (C.65)
Expanding $\psi$ in the spherical harmonics basis

$$\psi = \sum_{m, \ell} \psi_m \ell Y^m_\ell,$$  \hfill (C.66)

and applying the orthogonality condition

$$\int_S Y^m_\ell Y^m_{\ell'} d\Omega = \delta_{\ell\ell'} \delta_{mm'},$$  \hfill (C.67)

yields

$$E = \frac{1}{2} \sum_{m, \ell} \ell (\ell + 1) |\psi_m |^2 = \frac{1}{2} \sum_{\ell} E_\ell,$$  \hfill (C.68)

where the energy in mode $\ell$ is given by

$$E_\ell = \sum_{|m| \leq \ell} \ell (\ell + 1) |\psi_m |^2.$$  \hfill (C.69)

**Energy flux.** We derive the expression for the energy flux $\Pi(\ell)$ across the wavenumber $\ell$ shown in Fig. 5-3 of Chapter 5.

Denote by $S$ the spherical harmonics transform, that maps a function defined on a sphere to its coefficients in the spherical harmonics basis. Apply $S$ to the equations of motion to get

$$\omega_m \ell = \ell (\ell + 1) \psi_m \ell,$$  \hfill (C.70a)

$$\partial_t \omega_m \ell = -8 m \ell \left\{ \frac{1}{\sin \theta} (\partial_{\theta} \omega \partial_{\theta} \psi - \partial_{\phi} \omega \partial_{\phi} \psi) \right\} + \mathcal{F}(\ell) \omega_m \ell,$$

where

$$\mathcal{F}(\ell) = \left\{ 1 - \gamma_2 [4 - \ell (\ell + 1)] + \gamma_4 [4 - \ell (\ell + 1)]^2 \right\} \left[ 2 - \ell (\ell + 1) \right].$$  \hfill (C.71)

Since the energy spectrum is given by $E_\ell = \sum_{|m| \leq \ell} \ell (\ell + 1) |\psi_m |^2$, we use (C.70a) to
substitute for $\omega_{m\ell}$ in (C.70b) to find the evolution equation for $\psi_{m\ell}$

$$\ell(\ell + 1)\partial_{\ell}\psi_{m\ell} = -S_{m\ell}\left\{\frac{1}{\sin \theta} (\partial_{\phi}\omega \partial_{\phi}\psi - \partial_{\phi}\omega \partial_{\theta}\psi)\right\} + F(\ell)\ell(\ell + 1)\psi_{m\ell}. \quad (C.72)$$

We multiply both sides by $\psi^*_{m\ell}$ and add the resulting equation to its complex conjugate to get

$$\ell(\ell + 1)\partial_{\ell}|\psi_{m\ell}|^2 = 2F(\ell)\ell(\ell + 1)|\psi_{m\ell}|^2$$

$$-\left(\psi^*_{m\ell}S_{m\ell}\left\{\frac{1}{\sin \theta} (\partial_{\phi}\omega \partial_{\phi}\psi - \partial_{\phi}\omega \partial_{\theta}\psi)\right\} + \text{c.c.}\right). \quad (C.73)$$

Suming over $|m| \leq \ell$ and rearranging yields the evolution equation for $E_{\ell}$

$$\partial_{\ell}E_{\ell} - 2F(\ell)E_{\ell} = -\sum_{|m| \leq \ell} \left(\psi^*_{m\ell}S_{m\ell}\left\{\frac{1}{\sin \theta} (\partial_{\phi}\omega \partial_{\phi}\psi - \partial_{\phi}\omega \partial_{\theta}\psi)\right\} + \text{c.c.}\right). \quad (C.74)$$

Since for the Euler equations on a sphere $F(\ell) = 0$ holds, we identify the right hand side above as the nonlinear energy transfer into mode $\ell$. In the statistically stationary state and after time averaging we expect $\langle \partial_{\ell}E_{\ell} \rangle = 0$ and hence

$$-2\langle F(\ell)E_{\ell} \rangle = -\left\langle \sum_{|m| \leq \ell} \left(\psi^*_{m\ell}S_{m\ell}\left\{\frac{1}{\sin \theta} (\partial_{\phi}\omega \partial_{\phi}\psi - \partial_{\phi}\omega \partial_{\theta}\psi)\right\} + \text{c.c.}\right) \right\rangle. \quad (C.75)$$

The energy flux across $\ell$ is then finally given by

$$\Pi(\ell) = -2\sum_{\ell' \geq \ell} \langle F(\ell')E_{\ell'} \rangle. \quad (C.76)$$

In Eq. (6) of Chapter 5, we express the prefactor $F(\ell)$ in terms of the polynomial $f$ defined in Eq. 1(c) of Chapter 5 as

$$F(\ell) = f\left(4 - \ell(\ell + 1)\right) [2 - \ell(\ell + 1)]. \quad (C.77)$$
Figure C-1: Vortex detection scheme. Miller cylindrical projection of the sphere showing the unprocessed normalized vorticity field (a), and the thresholded vorticity field with values $\omega(x, t) \in [\alpha_\omega \min_{x \in S^2} \omega(x, t), \alpha_\omega \max_{x \in S^2} \omega(x, t)]$ removed for (b) $\alpha_\omega = 0.25$, and (c) $\alpha_\omega = 0.5$. The chain-like branched structures in the vorticity field remain preserved after thresholding.

C.4 Expansion coefficients for exact solutions

The spectrum of the instantaneous snapshot in Fig 2(c) shows a peak at $\ell = 6$, with values three orders of magnitude larger than other values of $\ell$. Extracting the values $\psi_{m6}$ from this snapshot, we constructed the exact solution in Fig 1(a) in Chapter 5 by letting it have only these non-vanishing expansion coefficients. This results in

$$
\psi_{06} = -4.4, \psi_{16} = -2.4-1.8i, \psi_{26} = 0.4+1.3i, \psi_{36} = 143.9+40.8i, \psi_{46} = -15.2+2.5i, \\
\psi_{56} = -2.9-20.0i, \psi_{66} = 40.8+14.5i, \text{and } \psi_{-m\ell} = (-1)^m \psi_{m\ell}.
$$

The exact solution in Fig 1(b) in Chapter 5 has $\ell = 30$ and non-vanishing expansion coefficients $\psi_{0\ell} = 8\sqrt{95993978542907}$, $\psi_{\pm 5\ell} = \pm 6\sqrt{2266150070307981}$, $\psi_{10\ell} = 369\sqrt{6048837670715}$, $\psi_{\pm 15\ell} = \pm 496\sqrt{5224419474285}$, $\psi_{20\ell} = 6483\sqrt{5330890838}$, $\psi_{\pm 25\ell} = \pm 30502\sqrt{8224777}$, $\psi_{30\ell} = 79290599\sqrt{77}$.

C.5 Derivation of characteristic parameters

Before non-dimensionalizing the time scale, the linearized equation for the stream function on a sphere of radius $R$ has the form

$$
\partial_t \omega = f(\Delta + 4K)(\Delta + 2K)\omega.
$$
Writing the shorthand \( \delta = (\ell(\ell + 1) - 4)/R^2 \) and moving to the spherical harmonics basis, this becomes

\[
\partial_t \omega_{m\ell} = - \left( \delta + \frac{2}{R^2} \right) \left( \Gamma_0 + \Gamma_2 \delta + \Gamma_4 \delta^2 \right) \omega_{m\ell} .
\] (C.79)

The time-dependent solution of this is

\[
\omega_{m\ell}(t) = \omega_{m\ell}(0)e^{\sigma_{m\ell} t},
\] (C.80)

where

\[
\sigma_{m\ell} = - \left( \delta + \frac{2}{R^2} \right) \left( \Gamma_0 + \Gamma_2 \delta + \Gamma_4 \delta^2 \right).
\] (C.81)

The flow of Eq. (C.80) exhibits a characteristic wave number \( \ell_c \) given by the maximum of \( \sigma_{m\ell} \). We approximate this maximum by the maximum of the function

\[- (\Gamma_0 + \Gamma_2 \delta + \Gamma_4 \delta^2). \] (C.82)

This results in \( \delta_c = - \frac{\Gamma_2}{2\Gamma_4} \) and

\[
\ell_c = \frac{1}{2} \left( 1 + 2 \sqrt{\frac{17}{4} - \frac{\Gamma_2}{2\Gamma_4} R^2} \right).
\] (C.83)

There is then an associated wavelength \( \lambda_c = 2\pi R/\ell_c \), corresponding to two vortices - each of characteristic diameter

\[ \Lambda = \frac{2\pi R}{2\sqrt{\frac{17}{4} - \frac{\Gamma_2}{2\Gamma_4} R^2} - 1} . \] (C.84)

Next, at the characteristic wave number \( \ell_c \), the flow of Eq. (C.80) has the characteristic time-scale

\[ \tau = \sigma_{m\ell_c}^{-1} = \left[ \left( \frac{\Gamma_2}{2\Gamma_4} - \frac{2}{R^2} \right) \cdot \left( \Gamma_0 - \frac{\Gamma_2}{4\Gamma_4} \right) \right]^{-1} , \] (C.85)
and a characteristic spectral bandwidth $\kappa$, defined by

$$\kappa = \frac{\ell_+ - \ell_-}{R},$$  \hspace{1cm} (C.86)

where $\ell_\pm$ are the $\ell$-values corresponding to the positive roots $\delta_\pm$ of

$$\Gamma_0 + \Gamma_2 \delta + \Gamma_4 \delta^2 = 0. \hspace{1cm} (C.87)$$

We have (remember that $-\Gamma_2 > 0$)

$$\delta_\pm = \frac{1}{2\Gamma_4}(-\Gamma_2 \pm \sqrt{\Gamma_4^2 - 4\Gamma_0 \Gamma_4}). \hspace{1cm} (C.88)$$

This results in

$$\ell_\pm = \frac{1}{2}(-1 + \sqrt{17 + 4\delta_\pm R^2}), \hspace{1cm} (C.89)$$

so the bandwidth is

$$\kappa = \frac{1}{2R}\left\{\sqrt{17 + 4\delta_+ R^2} - \sqrt{17 + 4\delta_- R^2}\right\}. \hspace{1cm} (C.90)$$

We further manipulate

$$\kappa = \left\{\frac{17}{2R^2} + \delta_+ + \delta_- - \frac{1}{2R^2}\left[(17 + 4\delta_+ R^2)(17 + 4\delta_- R^2)\right]\right\}^{\frac{1}{2}} \hspace{1cm} (C.91)$$

$$\kappa = \left\{\frac{17}{2R^2} + \delta_+ + \delta_- - \left[\frac{17^2}{4R^4} + \frac{17}{R^2}(\delta_+ + \delta_-) + 4\delta_+ \delta_-\right]\right\}^{\frac{1}{2}}$$

$$\kappa = \sqrt{\frac{17}{2R^2} - \frac{\Gamma_2}{\Gamma_4} - 2\sqrt{\frac{17^2}{16R^4} - \frac{17}{4R^2} \frac{\Gamma_2}{\Gamma_4} + \frac{\Gamma_0}{\Gamma_4}}}.$$ 

In the limit $R \to \infty$, we recover $\kappa \to \sqrt{-\frac{\Gamma_2}{\Gamma_4} - 2\sqrt{\frac{\Gamma_0}{\Gamma_4}}}$ which is the expression for the bandwidth in the flat case [185, 186].
C.6 Vortex detection scheme

We fix a threshold $\alpha_\omega \in [0, 1]$ and define

$$\tilde{\omega}(x, t) = \begin{cases} 0, & \text{if } \min_{x \in S^2} \omega(x, t) < \frac{\omega(x, t)}{\alpha_\omega} < \max_{x \in S^2} \omega(x, t), \\ \omega(x, t), & \text{otherwise.} \end{cases}$$

(C.92)

The number of vortices present on the sphere at time $t$, $N_\omega(\kappa, R; t)$, is then defined to be the number of connected components of the region $\{x : \tilde{\omega}(x, t) \neq 0\}$. Fig. C-1 demonstrates this procedure. The large-scale branched structure of the vorticity field is captured well after thresholding, justifying this simple vortex detection scheme.

Next, we characterize the geometrical difference in the behavior of the surface tension chains. Calculating the average surface tension $\bar{\sigma}(t)$ on the sphere, we define a thresholded surface tension by

$$\bar{\sigma}(x, t) = \begin{cases} \bar{\sigma}(t), & \text{if } \min_{x \in S^2} \sigma(x, t) < \bar{\sigma}(t), \\ \sigma(x, t), & \text{otherwise.} \end{cases}$$

(C.93)

For each connected component of the region where $\bar{\sigma}(x, t) > \bar{\sigma}(t)$, we measure its area $A$, together with the area of its boundary pixels $\partial A$. The ratio $\partial A/A$ is then a measure of the chain-like structure in the tension fields, with a large value signaling a highly branched structure, whereas smaller values indicate less branching.

We denote the Betti number of vortices for a parameter pair $(\kappa, R)$ at time $t$ as $N_\omega(\kappa, R; t)$, and the sum of the ratios $\partial A/A$ for every connected component in the region where $\bar{\sigma}(x, t) \neq 0$ by $A_\sigma(\kappa, R; t)$. To normalize these quantities, we define a reference value $\kappa_* = 0.3/A$, corresponding to a flow pattern exhibiting the anomalous turbulent phase, for all measured values of $A$. With this, we can define a normalized Betti number of vortices as

$$\text{Betti}_\omega = \frac{\langle N_\omega(\kappa, R; t) - N_\omega(\kappa_*, R; t) \rangle}{\langle N_\omega(\kappa_*, R; t) \rangle}$$

(C.94)
and a relative branching index for the high-tension areas

\[
\text{Branch}_{\sigma} = \frac{\langle A_{\sigma}(\kappa, R; t) - A_{\sigma}(\kappa, R; t) \rangle}{\langle A_{\sigma}(\kappa, R; t) \rangle},
\]

where the averages are taken over time after the initial relaxation period.

Figure C-2: Phase diagram for \(\alpha_\omega = 0.25\) (a) and \(\alpha_\omega = 0.75\) (b), showing that qualitative changes in the different turbulent phases are robust with regard to variations in \(\alpha_\omega\); cf. Fig. 5-2(a) in Chapter 5. Color scales show normalized Betti number defined in Eq. (C.94).

**Robustness of phase transition to thresholding.** Fig. C-2 shows phase diagrams using the thresholding parameters \(\alpha_\omega = 0.25\) (a) and \(\alpha_\omega = 0.75\) (b). The phase transition exhibits the same qualitative behavior for these parameter value as compared to Fig. 5-2(a) in Chapter 5, indicating robustness to the method used.

### C.7 Energy–enstrophy characterization

The stationary mean of the enstrophy, together with the mean kinetic energy, provides another useful characterization of the A- and T-phases in Fig. 5-2(a) of Chapter 5. Figure C-3 shows the ratio of the mean kinetic energy and mean enstrophy normalized by the characteristic pattern scale \(\Lambda^2\), which gives an estimate of an average kinetic energy per vortex. For \(\kappa \Lambda < 1\) (A-phase), the energy spectra tend to be concentrated around the forcing scale \(\pi / \Lambda\), see Fig. 5-3 of Chapter 5, in which case we expect the ratio to be approximately \(\pi^{-2} \sim 0.1\) which is close to the measured value in Fig. C-3. For \(\kappa \Lambda > 1\) (T-phase), the spectra are broadband and kinetic energy concentrates
Figure C-3: The ratio between the mean kinetic energy and mean enstrophy also differentiates between the A- and T- phases in Fig. 5-2(a) of Chapter 5, which are approximately separated by the dashed vertical line $\kappa \Lambda = 1$.

around few large vortices; this is reflected by the increase of the energy-enstrophy ratio.
Appendix D

Appendix to Chapter 6

D.1 Numerical Methods

Numerical simulations were performed using a Fourier spectral method with 3/2-rule to avoid aliasing when calculating the advection term through collocation [39]. We typically used grids of size $243^3$. Larger resolutions are not necessary, because the highest order term in Eq. (1) of Chapter 6 provides strong damping $\sim k^6$ at large wavenumbers $k$. We find solutions to Eq. (1) of Chapter 6 by using the Hodge decomposition [179] and solving the corresponding vorticity-vector potential problem

\[
\begin{align*}
\partial_t \omega + \nabla \times (\omega \times v) &= \Gamma_0 \nabla^2 \omega - \Gamma_2 \nabla^4 \omega + \Gamma_4 \nabla^6 \omega, \\
\nabla^2 \psi &= -\omega,
\end{align*}
\]  

(D.1a)  
(D.1b)

where $\omega = \nabla \times v$ is the vorticity and $\psi$ is the divergence-free vector potential related to the velocity field through $v = \nabla \times \psi$. Equations (D.1) are evolved in time using a third-order semi-implicit backward differentiation time-stepping scheme [14], calculating the nonlinear advection term explicitly while inverting the linear part implicitly. The discretized Eqs. (D.1) maintain $\omega$ and $\psi$ divergence-free in exact arithmetic. To avoid slow build up of non-zero divergence when working in double-precision arithmetic, numerical solutions are projected onto the divergence-free manifold during the
time-stepping. To calculate the energy transfer tables in Figs. 6-3 (c) and (g) of Chapter 6 efficiently, we decompose the velocity field into Littlewood-Paley components and use collocation.

**Vorticity-vector potential formulation.** We find the vorticity-vector potential formulation of the system (1) of Chapter 6 on the three-torus, $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. This is a manifold without boundary, so the usual Hodge Decomposition applies [179]. For a vector field $\mathbf{v}$, the decomposition takes the form

$$\mathbf{v} = \nabla \phi + \nabla \times \psi + \mathbf{H}, \quad (D.2)$$

where $\mathbf{H}$ is an element of the three-dimensional space of harmonic vector fields, which implies on a torus that $\mathbf{H} = H_x \hat{x} + H_y \hat{y} + H_z \hat{z}$ for some constants $H_x, H_y, H_z$. For divergence-free flows, we have $\nabla^2 \phi = 0$ and hence $\phi = 0$ since $\mathbb{T}^3$ is compact and without boundary. In this case, we interpret $\mathbf{H}$ as the fluid center of mass motion. By working in the center of mass frame, we are left with

$$\mathbf{v} = \nabla \times \psi. \quad (D.3)$$

Taking the curl of Eqs. (1) of Chapter 6 gives

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{v} = \mathcal{L} \omega, \quad (D.4)$$

where $\omega = \nabla \times \mathbf{v}$ and

$$\mathcal{L} = \Gamma_0 \nabla^2 - \Gamma_2 \nabla^4 + \Gamma_4 \nabla^6 \quad (D.5)$$

We can simplify the advection term by using the following standard identity

$$\nabla \times (\omega \times \mathbf{v}) = \omega (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \omega) + \mathbf{v} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{v} \quad (D.6)$$

$$= \mathbf{v} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{v},$$
since both fields are divergence-free. This identity speeds up the computational cost of evaluating the advection term, as it requires fewer applications of the Fast Fourier Transform. Since the divergence of $\psi$ does not affect the decomposition (D.2), we fix the gauge and work with a divergence-free vector potential. In this case, taking the curl of Eq. (D.3) gives $\nabla^2 \psi = -\omega$. In summary, in the vorticity-vector potential formulation, the equations of motion read

$$\begin{align*}
\partial_t \omega + \nabla \times (\omega \times v) &= \mathcal{L} \omega, \\
\nabla^2 \psi &= -\omega.
\end{align*}$$

(D.7a) (D.7b)

**Non-dimensionalization in numerical simulations.** For simulation purposes, we rescale time and space as $t = T\tilde{t}$ and $x = L\tilde{x}$, where $L$ is the domain size. We further introduce $\psi = \psi_0\tilde{\psi}$ and $\omega = \omega_0\tilde{\omega}$. Equations (D.7) then become (after dropping the tildes)

$$\begin{align*}
\frac{1}{T} \partial_t \omega + \frac{\psi_0}{L^2} \nabla \times (\omega \times v) &= \frac{\Gamma_0}{L^2} \left[ \nabla^2 \omega - \frac{\Gamma_2}{\Gamma_0 L^2} \nabla^4 \omega + \frac{\Gamma_4}{\Gamma_0 L^4} \nabla^6 \omega \right], \\
\frac{\psi_0}{L^2} \nabla^2 \psi &= -\omega_0\omega.
\end{align*}$$

(D.8a) (D.8b)

Setting $\omega_0 = \frac{1}{T}$, $\psi_0 = \frac{L^2}{T}$, and $T = \frac{L^2}{\Gamma_0}$, and defining $\gamma_2 = \frac{\Gamma_2}{\Gamma_0 L^2}$ and $\gamma_4 = \frac{\Gamma_4}{\Gamma_0 L^4}$, we obtain the non-dimensionalized equations

$$\begin{align*}
\partial_t \omega + \nabla \times (\omega \times v) &= \nabla^2 \omega - \gamma_2 \nabla^4 \omega + \gamma_4 \nabla^6 \omega, \\
\nabla^2 \psi &= -\omega.
\end{align*}$$

(D.9a) (D.9b)

**Time discretization.** For the time stepping, we use the third-order semi-implicit backward differentiation scheme introduced by Ascher et al. [14]

$$\left( \frac{11}{6} - \Delta t \mathcal{L} \right) \omega^{n+1} = 3 \omega^n - \frac{3}{2} \omega^{n-1} + \frac{1}{3} \omega^{n-2} - \Delta t \left( 3 \mathcal{N}^n - 3 \mathcal{N}^{n-1} + \mathcal{N}(D^2) \right).$$

(D.10a)
where

\[ \mathcal{L} = \nabla^2 - \gamma_2 \nabla^4 + \gamma_4 \nabla^6, \quad \text{(D.10b)} \]

\[ \mathbf{N}(\psi, \omega) = \nabla \times (\omega \times \mathbf{v}), \quad \text{(D.10c)} \]

recalling that \( \mathbf{v} = \nabla \times \psi \). We then solve for the vector potential

\[ \nabla^2 \psi^{n+1} = -\omega^{n+1}. \quad \text{(D.11)} \]

**Space discretization.** We work with a Fourier spectral method. If we denote the rhs. of Eq. (D.10a) by \( b^{n+1} \), then the update formula for the Fourier coefficients reads

\[ \omega^{n+1}(k) = \frac{1}{p(k)} b^{n+1}(k), \quad \text{(D.12a)} \]

\[ \psi^{n+1}(k) = \frac{1}{k^2} \omega^{n+1}(k), \quad \text{(D.12b)} \]

where \( p(k) = 11/6 + \Delta t(k^2 + \gamma_2 k^4 + \gamma_4 k^6) \). In addition, we always have \( \omega(t, k = 0) = 0 \), because the vorticity is defined by taking the curl of \( \mathbf{v} \), and we can set \( \psi(t, k = 0) = 0 \) by gauge freedom. Because both \( \omega \) and \( \psi \) are divergence-free, we have to impose

\[ k \cdot \omega = 0, \quad k \cdot \psi = 0. \quad \text{(D.13)} \]

If we initiate the simulations with divergence free fields, then the update rule (D.12) preserves this property in exact arithmetic. Nevertheless, numerical errors will always build up after several iterations in double-precision arithmetic. We project back onto the divergence-free manifold every several steps by mimicking gauge transformation. Suppose \( \omega \) has small divergence which we want to remove. We set \( f = \nabla \cdot \omega \). We then solve the Poisson equation

\[ \nabla^2 \lambda = f, \quad \text{(D.14)} \]
and subsequently remove the divergence from $\omega$ according to

$$\omega \rightarrow \omega - \nabla \lambda.$$  \hspace{1cm} (D.15)

**Calculation of shell interactions.** We next explain how the energy spectra, fluxes and energy flow tables are calculated numerically (Fig. 6-3 of Chapter 6 and Fig. D-2). To establish notation, we first recall the derivation of the energy balance equation as given in Waleffe [216]. Expanding the velocity and pressure fields in Fourier series, Eqs. (1) of Chapter 6 give

$$k_i \cdot v_i(t, k) = 0,$$

$$[\partial_t + \xi(k)]v_i(t, k) = -ik_i p(t, k) - i \sum_{q+p=k} v_j(t, p) q_j v_i(t, q),$$

where $\xi(k) = \Gamma_0 k^2 + \Gamma_2 k^4 + \Gamma_4 k^6$. By projecting on helical modes one finds Eq. (3) of Chapter 6. To find the equation for the energy in mode $k$ we relabel $k \rightarrow -k$ in the second equation, multiply by $v_i(t, k)$, sum over $i$ and use the incompressibility condition to get

$$v_i(k)[\partial_t + \xi(k)]v_i(-k) = -i \sum_{k+p+q=0} v_j(p) q_j v_i(q) v_i(k),$$

where we dropped the explicit time dependence for ease of notation. We now add the above equation to its complex conjugate and use $v_i(-k) = \overline{v_i(k)}$

$$[\partial_t + 2\xi(k)]|v(k)|^2 = -i \sum_{k+p+q=0} v_j(p) q_j v_i(q) v_i(k) + c.c.$$

The energy in shell $|k| = k$ is defined as

$$\epsilon(t, k) = \frac{1}{2} \sum_{|k'|=k} |v(t, k')|^2.$$  \hspace{1cm} (D.16)
The corresponding evolution equation is

$$[\partial_t + 2\xi(k)]\epsilon(t, k) = \sum_p \sum_q \tilde{t}(t; k, p, q),$$

where

$$\tilde{t}(t; k, p, q) = \frac{i}{2} \sum_{\text{shells}} \delta_{k+p+q,0} v_j(t, p) q_j v_i(t, q) v_i(t, k) + \text{c.c.}$$

$$= -i \sum_{\text{shells}} \delta_{k+p+q,0} v_j(t, p) q_j v_i(t, q) v_i(t, k). \quad (D.17)$$

We used the fact that the sum over all modes can be split into radial and shell parts

$$\sum_k f(k) = \sum_k \sum_{|k'|=k} f(k')$$

and we defined

$$\sum_{\text{shells}} f(k, q, p) \equiv \sum_{|k'|=k} \sum_{|p'|=p} \sum_{|q'|=q} f(k', q', p').$$

Symmetrizing as $t(t; k, p, q) = \tilde{t}(t; k, p, q) + \tilde{t}(t; k, q, p)$ gives the usual energy balance equation [72, 216]

$$[\partial_t + 2\xi(k)]\epsilon(t, k) = T(t, k), \quad (D.18a)$$

where

$$T(t, k) = \frac{1}{2} \sum_p \sum_q t(t; k, p, q). \quad (D.18b)$$

The quantity $t(t; p, k, q)$ is the energy transfer into the shell $k$ due to all triad interactions with shells $p$ and $q$ at time $t$, and $T(t, k)$ is the energy transfer into the shell $k$ due to all triad interactions. The energy flux across $k$ is defined as

$$\Pi(t, k) = \sum_{k'>k} T(t, k'), \quad (D.19)$$

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Figure D-1: Numerical estimation of the stationary energy spectra for the narrow bandwidth in Fig. 6-3(a) of Chapter 6. (a and b) Kinetic energy (a) and helicity time series (b) are used to determine the relaxation time $\mathcal{R}$ to a stationary state (vertical broken line, $\mathcal{R} = 31.7\tau$ in this case). (c) Convergence of the energy spectra estimates using $|\langle \epsilon^{\pm} \rangle_{\mathcal{R},\Delta} - \langle \epsilon^{\pm} \rangle_{\mathcal{R},\Delta_{\text{max}}}|^2/|\langle \epsilon^{\pm} \rangle_{\mathcal{R},\Delta_{\text{max}}}|_2^2$, see Eq. (D.31), as a function of the averaging interval $\Delta$. (d) Relative difference ($l^2$-norm) between the momentary energy spectra and their (stationary) mean as a function of time.

and represents energy flow from wavenumbers below $k$ to those above it at time $t$.

Projecting the velocity field onto the helical modes reveals additional substructure [216]. The energy spectrum splits into two helical components

$$\epsilon(t, k) = \epsilon^+(t, k) + \epsilon^-(t, k), \quad \epsilon^{\pm}(t, k) = \sum_{|k'|=k} |u^{\pm}(t, k')|^2.$$  \hspace{1cm} (D.20)

The energy flow and energy flux split into eight components, one for each possible assignment of the helicity index over the triads

$$t(t; k, p, q) = \sum_{i=1}^{8} t^{(i)}(t; k, p, q), \quad T(t; k) = \sum_{i=1}^{8} T^{(i)}(t; k), \quad \Pi(t, k) = \sum_{i=1}^{8} \Pi^{(i)}(t, k).$$  \hspace{1cm} (D.21)
where we follow the binary ordering as in [216] \((i = 1 \text{ corresponds to } + + +, i = 2 \text{ to } + + -, \text{ etc.})\). The energy conservation for helical shells becomes

\[
[\partial_t + 2\xi(k)]\epsilon^{\pm}(t, k) = T^{\pm}(t, k), \tag{D.22}
\]

where \(T^{+}(t, k) = \sum_{i=1}^{4} T^{(i)}(t, k)\) and \(T^{-}(t, k) = \sum_{i=5}^{8} T^{(i)}(t, k)\).

We now consider time averages. For a quantity \(O\) we define

\[
\langle O \rangle = \lim_{\mathcal{R},\Delta \to \infty} \frac{1}{\Delta} \int_{\mathcal{R}}^{\mathcal{R}+\Delta} dt \, O(t). \tag{D.23}
\]

In practice, \(\mathcal{R}\) is the relaxation time for the system, and \(\Delta\) is the length of the averaging interval. In the stationary regime, the averages become time-independent,

---

**Figure D-2:** Mirror-symmetry breaking and inverse energy cascade for an active fluid with the intermediate bandwidth \(\kappa_1\), showing same quantities as in Fig. 6-3 of Chapter 6. Overall, 15.4\% of the injected energy flows into region I, while 84.6\% flows into region III.
which we denote by

\[
\langle \epsilon(t, k) \rangle = \langle \epsilon(k) \rangle, \quad \langle t(t; k, p, q) \rangle = \langle t(k, p, q) \rangle
\]  
(D.24)

and

\[
\langle \partial_t \epsilon(t, k) \rangle = \partial_t \langle \epsilon(t, k) \rangle = 0.
\]  
(D.25)

Taking averages, reduces Eq. (D.18a) to

\[
2\xi(k)\langle \epsilon(k) \rangle = \langle T(k) \rangle.
\]  
(D.26)

Thus, in the stationary regime, the energy flux can be derived from the spectrum according to

\[
\langle \Pi(k) \rangle = \sum_{k' > k} 2\xi(k')\langle \epsilon(k') \rangle.
\]  
(D.27)

Similarly, for the helical projections we get

\[
2\xi(k)\langle \epsilon^\pm(k) \rangle = \langle T^\pm(k) \rangle,
\]  
(D.28)

and

\[
\langle \Pi^\pm(k) \rangle = \sum_{k' > k} 2\xi(k')\langle \epsilon^\pm(k') \rangle.
\]  
(D.29)

We numerically estimate the discrete stationary spectra \(\langle \epsilon^\pm(k) \rangle\) as follows. At each time step \(n\), we calculate

\[
\epsilon^\pm_n(k) = \sum_{|k'|=k} |u^\pm_n(k')|^2.
\]  
(D.30)
We then apply the discrete version of the formula (D.23),

\[ \langle \epsilon^\pm(k) \rangle_{\mathcal{R}, \Delta} = \frac{\mathcal{R} + \Delta}{\Delta} \sum_{n=\mathcal{R}}^{\mathcal{R} + \Delta} \epsilon^\pm_n(k), \]  

(D.31)

where we choose \( \mathcal{R} \) to be the relaxation time of the energy and helicity time series and the averaging interval \( \Delta \) is taken long enough to ensure convergence of statistical observables (Fig. D-1). We recover the helical flux contributions using formula (D.29), and the total flux from \( \langle \Pi(k) \rangle = \langle \Pi^+(k) \rangle + \langle \Pi^-(k) \rangle \).

For plotting purposes, we connect the discrete energy spectra to their continuous definitions. The mean kinetic energy in the system of size \( L \) is

\[ E = \sum_k \epsilon(k) = \sum_k \frac{\sum_{k' \in [k, k+\Delta k)} \epsilon(k')}{\Delta k} \Delta k, \]  

(D.32)

where \( \tilde{k} = n \frac{2\pi}{L} \), \( n = 1, 2, 3, \ldots \) and \( \Delta k = \frac{2\pi}{L} \). In the limit \( L \to \infty \), we recover the continuous definition of the energy spectrum

\[ E = \int_0^{\infty} e(\tilde{k}) d\tilde{k}, \]  

(D.33)

\[ e(\tilde{k}) = \lim_{L \to \infty} \sum_{k' \in [k, k+\Delta k)} \frac{\epsilon(k')}{\Delta k}. \]  

(D.34)

When plotting energy spectra, we thus use the discrete (finite box size) approximation of the continuous definition

\[ e^\pm(\tilde{k}) = \frac{\sum_{k' \in [k, k+\Delta k)} (\epsilon^\pm(k'))_{\mathcal{R}, \Delta}}{\Delta k}. \]  

(D.35)

The spectral domains I, II and III in Fig. 6-1(a) of Chapter 6 have finite thickness. To calculate the energy flow between the regions, we have to sum over shells contained in a given region. For example,

\[ \mathcal{I}_{I, II, III} = \frac{1}{2} \sum_{k \in \text{region I}} \sum_{p \in \text{region II}} \sum_{q \in \text{region III}} t(k, p, q) \]  

(D.36)
is the energy flow into the region I due to all triad interactions with modes supported on regions II and III. To calculate $\mathcal{J}_{KPQ}$, where $K, P, Q \in \{I, II, III\}$ consider the following integral

$$\tilde{\mathcal{J}}_{KPQ} = -\int d^3x \, \mathbf{v}_K \cdot [(\mathbf{v}_P \cdot \nabla)\mathbf{v}_Q],$$

(D.37)

where $\mathbf{v}_K$ is the Littlewood-Paley component corresponding to region $K$, similarly for $\mathbf{v}_P$ and $\mathbf{v}_Q$. Specifically, $\mathbf{v}_K$ is obtained from $\mathbf{v}$ by keeping only the Fourier amplitudes supported on the region $K$, etc. In terms of Fourier series, we find that

$$\tilde{\mathcal{J}}_{KPQ} = \sum_{k \in \text{region } K} \sum_{p \in \text{region } P} \sum_{q \in \text{region } Q} \tilde{t}(k,p,q),$$

(D.38)

where $\tilde{t}(k,p,q)$ is given by Eq. (D.17). We symmetrize in the last to indices, by defining

$$\mathcal{J}_{KPQ} = \frac{1}{2} (\tilde{\mathcal{J}}_{KPQ} + \tilde{\mathcal{J}}_{KQP}).$$

(D.39)

To split $\mathcal{J}_{KPQ} = \sum_{i=1}^{8} \mathcal{J}_{KPQ}^{(i)}$ into the contributions from the eight types of helical triad interactions, it is convenient to consider equivalent integral representations of the form

$$\tilde{\mathcal{J}}_{KPQ}^{s_Ks_Ps_Q} = -\int d^3x \, \mathbf{v}_K^{s_K} \cdot [(\mathbf{v}_P^{s_P} \cdot \nabla)\mathbf{v}_Q^{s_Q}],$$

(D.40)

where $\mathbf{v}_K^{s_K}$ is constructed from $\mathbf{v}_K$ by projection onto modes with helicity index $s_K \in \{\pm\}$, etc. The symmetrization

$$\mathcal{J}_{KPQ}^{s_Ks_Ps_Q} = \frac{1}{2} (\tilde{\mathcal{J}}_{KPQ}^{s_Ks_Ps_Q} + \tilde{\mathcal{J}}_{KQP}^{s_Ks_Qs_P}),$$

(D.41)

represents the energy flow into modes with helicity index $s_K$ lying in region $K$, due to interactions with modes with helicity indices $s_P$ and $s_Q$ in regions $P$ and $Q$, respectively. Expressions of the form (D.40) are calculated, at a given time step, by
collocation: Evaluate the three projections in the physical domain on equally spaced grid, perform the point-wise multiplication, go back to Fourier space and integrate by reading off the value of the zeroth Fourier mode. All these operations are done efficiently using the Fast Fourier Transform. To calculate the stationary energy flows \( \mathcal{J}_{K_PQ} \) shown in the energy flow tables [Figs. 6-3(c) and (g) in Chapter 6 and Fig. D-2(c)], we adopt a procedure analogous to that used to estimate the energy spectra.

Figure D-3: Characterization of the inverse energy cascade. (a and b) Two horizontal cuts through the 3D simulation domain for a small bandwidth \( \kappa_S \), showing that the inverse cascade is not characterized by vortex mergers, but rather by chain-like complexes. (c and d) Same flow-field snapshots as in (a) and (b) but now represented through the local helicity field. The chain-like large-scale structures carry most of the helicity. They do not merge, but rather form extended filaments and clusters that move throughout the simulation domain. Domain size \( L = 32A \). (e) The proportion of the energy injected by the active component that is transported to region I [corresponding to large scales, cf. Fig. 6-1(a) of Chapter 6] as a function of the active bandwidth \( \kappa \). (f) The absolute value of the energy flux for an active fluid with small bandwidth \( \kappa_S \) for different simulation domain sizes. In region I, corresponding to large scales, the upward transfer is non-inertial at intermediate wavenumbers with the flux exhibiting \( k^3 \) scaling. For \( k \to 0 \), however, the flux approaches a constant plateau value, indicating that inertial effects begin to dominate at very large scales \( \gg \Lambda \).
D.2 Cascade characteristics

The phenomenology of the inverse cascade in passive 2D turbulent flows is often characterized in terms of vortex mergers. By contrast, in active fluids with a well-defined vortex scale $\Lambda$ and a small injection bandwidth $\kappa_S$, vortex mergers are suppressed by the dominant pattern-scale selection processes. This raises the question how the inverse cascade, which can transport a considerable fraction of energy to larger scales [Fig. D-3(e)], manifests itself in the flow field structure of a 3D active fluid. Our simulations demonstrate that pattern-forming nonequilibrium fluids can achieve energy transport to larger scales by forming chain-like vortex complexes that propagate through the fluid. To illustrate this phenomenon in more detail, Figs. D-3(a-d) show two horizontal 2D $(x, y)$-slices of a large 3D simulation domain (size $L = 32\Lambda$) at a fixed representative time for an active fluid with small active bandwidth $\kappa_S$ (using the same parameters as in Chapter 6). In Figs. D-3(a) and (b), the flow field is visualized through the perpendicular $z$ component of the vorticity, $\omega_z$, and in Figs. D-3(c) and (d) through the local helicity $h$. The thin black lines in Figs. D-3(a) and (b) indicate in-plane portions of filaments consisting of alternating vortices that correspond to 3D filamentous clusters of high helicity in Figs. D-3(c) and (d). The kinetic energy transported to large scales manifests itself as the formation and motion of such vortex chains. These results illustrate that the helicity-driven 3D inverse cascade in active fluids is distinctly different from the energy-driven 2D inverse cascade in passive fluids.

A detailed spectral characterization of this helicity-driven 3D active turbulence can be obtained by analyzing the upward energy transfer into region I in Fourier space [defined in Fig. 6-1(a) of Chapter 6]. Figure D-3(f) shows the absolute value of the energy flux for an active fluid with small bandwidth $\kappa_S$ for three different simulation domain sizes. In the case of an inertial energy cascade, one expects the flux to be independent of the wavenumber $k$, at least over some range. In such inertial ranges there is no dissipation of energy, just purely nonlinear redistribution. In our simulations, we see that the energy flux, upon entering the spectral region I from above (i.e., coming from region II), is at first non-inertial with an approximate $k^3$
scaling, implying that the transfer is assisted by strong dissipation effects. At very large scales $\gg \Lambda$, however, the flux develops a plateau, indicating that the transfer becomes mostly inertial. Increasing the simulation domain size broadens the plateau and increases the magnitude of the flux that reaches the plateau. Interestingly, at these very large scales, the model defined in Eqs. (1) of Chapter 6 effectively reduces to the classical Navier-Stokes equations.

Figure D-4: (a) Relaxation time for spontaneous symmetry breaking depends on the domain size. (b and c) Kinetic energy (b) and helicity (c) as a function of time for a very large domain ($L = 48\Lambda$). The relaxation proceeds in two stages, the initial stage characterized by a rapid exponential growth rate ($t < 20\tau$), followed by a slower linear growth until full relaxation ($t \approx 100\tau$). (d-f) Energy spectra at various stages of the relaxation process [cf. broken lines in (b) and (c)] show how the system realizes a state with broken mirror symmetry.
Appendix E

Appendix to Chapter 7

E.1 Triads forced at two legs

Figs. E-1 and E-2 show the results of numerical simulations of the system (7.5) when it is forced at intermediate and small scales (Fig. E-1) and at large and intermediate scales (Fig. E-2). In both cases, even though the intermediate scale is forced, it is eventually suppressed and the asymptotic behavior becomes identical to the single-mode forcing case, as described in section 7.3.1.

E.2 Fixed points of the active triadic system and their linear stability

We show that the triadic system (7.5) forced at the intermediate wavenumber exhibits a linearly unstable fixed point. To this end, we first look for time-independent solutions of (7.5a), satisfying

\[ DIA = 2\Delta (IA^* \times A^*). \]  

(E.1)
Figure E-1: Numerical simulations of the triad dynamics (7.5) initiated with generic complex initial conditions show that active triads ($p < k < q$) are unstable when forced at intermediate $k$ and small $q$ scales. Energy and helicity increase exponentially (a), reflecting the exponential growth of one of the forced modes (d) and underdamped decay of the remaining forced mode (b) and the passive mode (c). Parameters: $\{k, p, q\} = [(4, -11, 0), (-9, -1, 0), (5, 12, 0)]$, box size $L = 24\Lambda$. 
Figure E-2: Numerical simulations of the triad dynamics (7.5) initiated with generic complex initial conditions show that active triads \((p < k < q)\) are unstable when forced at large \(p\) and intermediate \(k\) scales. Energy and helicity increase exponentially (a), reflecting the exponential growth of one of the forced modes (c) and overdamped decay of the remaining forced mode (b) and the passive mode (d). Parameters: \(\{k, p, q\} = [(5, 11, 0), (8, 8, 0), (-13, -19, 0)]\), box size \(L = 24\Delta\).
Remembering the convention $p < k < q$ and using the polar representation $A_k = |A_k|e^{i\phi_k}$ we demand that

\[ -s_k|D_{kk}|k^2|A_k|e^{i\phi} = 2\Delta|p^2 - q^2||A_p||A_q|, \]  
(E.2)

\[ s_p|D_{pp}|p^2|A_p|e^{i\phi} = 2\Delta|q^2 - k^2||A_q||A_k|, \]  
(E.3)

\[ s_q|D_{qq}|q^2|A_q|e^{i\phi} = 2\Delta|k^2 - p^2||A_k||A_p|, \]  
(E.4)

where $s_k = 1$ if $D_{kk} > 0$ and $s_k = -1$ if $D_{kk} < 0$ and $\phi = \phi_k + \phi_p + \phi_q$. Matching the phases, requires that

\[ \phi + \phi_{-s_k} = \phi + \phi_{s_p} = \phi + \phi_{s_q} = 0, \]  
(E.5)

where the equalities hold modulo $2\pi$. The only way to satisfy the above restrictions is to choose $s_k = -1$ and $s_p = s_q = 1$, that is, a fixed point can exist only when the intermediate wavenumber is forced. Of course, we must then have $\phi = 0$, which leaves a two-parameter family of fixed points. Without loss of generality, we can set all phases to zero $\phi_k = \phi_p = \phi_q = 0$. Matching the amplitudes gives

\[ |D_{kk}|k^2|A_k| = 2\Delta|p^2 - q^2||A_p||A_q|, \]  
(E.6)

\[ |D_{pp}|p^2|A_p| = 2\Delta|q^2 - k^2||A_q||A_k|, \]  
(E.7)

\[ |D_{qq}|q^2|A_q| = 2\Delta|k^2 - p^2||A_k||A_p|. \]  
(E.8)

Furthermore, we still have the following two identities

\[ -|D_{kk}|k^2|A_k|^2 + |D_{pp}|p^2|A_p|^2 + |D_{qq}|q^2|A_q|^2 = 0, \]  
(E.9)

\[ -|D_{kk}|k^4|A_k|^2 + |D_{pp}|p^4|A_p|^2 + |D_{qq}|q^4|A_q|^2 = 0, \]  
(E.10)

which represent energy and in-plane enstrophy balance: energy and enstrophy produced at the wavenumber $k$ are dissipated at wavenumbers $p$ and $q$. The two con-
strain leave one degree of freedom represented by the line

\[ \begin{bmatrix} |A_k|^2 \\ |A_p|^2 \\ |A_q|^2 \end{bmatrix} = \alpha \begin{bmatrix} \frac{|p^2 - q^2|}{|D_{kk}|k^2} \\ \frac{|q^2 - k^2|}{|D_{pp}|p^2} \\ \frac{|k^2 - p^2|}{|D_{qq}|q^2} \end{bmatrix} \]  \hspace{1cm} (E.11)

The positive constant \( \alpha \) is fixed by inserting the above expression into (E.6), which then yields for \( A \) the fixed point

\[ \begin{bmatrix} A_k \\ A_p \\ A_q \end{bmatrix} = \alpha^{1/2} \begin{bmatrix} \sqrt{\frac{|p^2 - q^2|}{|D_{kk}|k^2}} \\ \sqrt{\frac{|q^2 - k^2|}{|D_{pp}|p^2}} \\ \sqrt{\frac{|k^2 - p^2|}{|D_{qq}|q^2}} \end{bmatrix} \]  \hspace{1cm} (E.12)

where

\[ \alpha = -\det(ID)/(4\Delta^2|p^2 - q^2||q^2 - k^2||k^2 - p^2|). \]  \hspace{1cm} (E.13)

All other fixed points are obtained by the transformation

\[ \begin{bmatrix} A_k \\ A_p \\ A_q \end{bmatrix} \rightarrow \begin{bmatrix} A_k e^{i\phi_k} \\ A_p e^{i\phi_p} \\ A_q e^{i\phi_q} \end{bmatrix} \]  \hspace{1cm} (E.14)

where \( \phi_k + \phi_p + \phi_q = 0 \).

We now turn to the fixed points of the system for \( B(t) = \Re B(t) + i\Im B(t) \), that is, we look for time-independent solutions of (7.5b) with \( A \) given by (E.12). In this case, the system decouples into two linear equations for the real and imaginary parts

\[ D\Re B = 2\Delta \Re B \times A, \]  \hspace{1cm} (E.15)

\[ D\Im B = -2\Delta \Im B \times A. \]  \hspace{1cm} (E.16)
In both cases the null-space is one dimensional, generated by the vectors

\[
\begin{bmatrix}
\Re B_k \\
\Re B_p \\
\Re B_q
\end{bmatrix} = \begin{bmatrix}
k\sqrt{\frac{p^2 - q^2}{|D_{kk}|}} \\
p\sqrt{\frac{|q^2 - k^2|}{|D_{pp}|}} \\
q\sqrt{\frac{|k^2 - p^2|}{|D_{qq}|}}
\end{bmatrix}, \quad \begin{bmatrix}
\Im B_k \\
\Im B_p \\
\Im B_q
\end{bmatrix} = \begin{bmatrix}
k\sqrt{\frac{p^2 - q^2}{|D_{kk}|}}(-k^2 + p^2 + q^2) \\
p\sqrt{\frac{|q^2 - k^2|}{|D_{pp}|}}(k^2 - p^2 + q^2) \\
q\sqrt{\frac{|k^2 - p^2|}{|D_{qq}|}}(k^2 + p^2 - q^2)
\end{bmatrix}
\] (E.17)

The fixed point for \( B \) is obtained by combining the real and imaginary parts,

\[
\begin{bmatrix}
B_k \\
B_p \\
B_q
\end{bmatrix} = \alpha^{1/2} \begin{bmatrix}
k\sqrt{\frac{p^2 - q^2}{|D_{kk}|}}(c_1 + ic_2(-k^2 + p^2 + q^2)) \\
p\sqrt{\frac{|q^2 - k^2|}{|D_{pp}|}}(c_1 + ic_2(k^2 - p^2 + q^2)) \\
q\sqrt{\frac{|k^2 - p^2|}{|D_{qq}|}}(c_1 + ic_2(k^2 + p^2 - q^2))
\end{bmatrix}, \quad (E.18)
\]

where \( c_1 \) and \( c_2 \) are some arbitrary real constants and the prefactor \( \alpha^{1/2} \) has been factored out for convenience. Note that if we started with any other fixed point for \( A \) obtained by the transformation (E.14), then the above argument still applies, provided we apply the same phase transformation to the vector \( B \). The real constants \( c_1 \) and \( c_2 \) set the helicity and energy of the fixed point. Indeed

\[
H = 2I A \cdot \Re B = 2\alpha c_1 \left( k^2 \frac{|p^2 - q^2|}{|D_{kk}|} + p^2 \frac{|q^2 - k^2|}{|D_{pp}|} + q^2 \frac{|k^2 - p^2|}{|D_{qq}|} \right), \quad (E.19)
\]

and

\[
\frac{2E}{\alpha} = \frac{|p^2 - q^2|}{|D_{kk}|} + \frac{|q^2 - k^2|}{|D_{pp}|} + \frac{|k^2 - p^2|}{|D_{qq}|} + \\
c_1^2 \left( k^2 \frac{|p^2 - q^2|}{|D_{kk}|} + p^2 \frac{|q^2 - k^2|}{|D_{pp}|} + q^2 \frac{|k^2 - p^2|}{|D_{qq}|} \right) + \\
c_2^2 \left( (p^2 + q^2 - k^2)^2 + p^2 \frac{|q^2 - k^2|}{|D_{pp}|} (q^2 + k^2 - p^2)^2 + q^2 \frac{|k^2 - p^2|}{|D_{qq}|} (k^2 + p^2 - q^2)^2 \right). \quad (E.20)
\]

We now show that the fixed point for the triadic system (7.5) is linearly unstable by studying the perturbation \( A = \bar{A} + \delta A \) around the fixed point \( \bar{A} \) given by (E.12). The real and imaginary parts of the linearized dynamical Eq. (7.5a) for \( \delta A = \Re \delta A + i \Im \delta A \)
\[ \Re \delta \dot{A} + D \Re \delta A = 2 \Delta I^{-1} (I \Re \delta A \times \dot{A}) + 2 \Delta I^{-1} (I \dot{A} \times \Re \delta A), \]  \hspace{1cm} (E.21) \\
\Im \delta \dot{A} + D \Im \delta A = -2 \Delta I^{-1} (I \Im \delta A \times \dot{A}) - 2 \Delta I^{-1} (I \dot{A} \times \Im \delta A). \]  \hspace{1cm} (E.22)

Since these two equations are decoupled, it suffices to show linear instability of the first equation. The corresponding Jacobian \( J \) reads

\[ J = -D - 2 \Delta I^{-1} M_A I + 2 \Delta I^{-1} M_I A, \]  \hspace{1cm} (E.23)

where \( M_w \) denotes the antisymmetric matrix with components \( M_{ab} = \epsilon_{abc} w_c \), corresponding to the cross product with \( w \). Direct computation reveals that the Jacobian has the following properties

\[ \text{tr}(J) = -\text{tr}(D), \quad \text{tr}(J^2) - \text{tr}^2(J) = 0, \quad \text{det}(J) = 4 \text{det}(D). \]  \hspace{1cm} (E.24)

We recall the Routh-Hurwitz stability criteria for the eigenvalues of a \( 3 \times 3 \) matrix \( M \) to have negative real parts [76]

\[ \text{tr}(M) < 0, \quad \text{det}(M) < 0, \quad \text{tr}(M)\{\text{tr}(M^2) - \text{tr}^2(M)\} > -2 \text{det}(M). \]  \hspace{1cm} (E.25)

The Jacobian \( J \) satisfies the first condition because of our restriction (7.8), it also satisfies the second condition because the fixed point only exists for \( \text{det} D < 0 \). But it violates the last one, since for the fixed point one always has \( \text{det} D < 0 \). Thus \( J \) has an eigenvalue with positive or vanishing real part. We now show that the real part is always positive, implying that the fixed point is linearly unstable. To this end, note that the properties (E.24) imply that the characteristic equation of \( J \) has the form

\[ \text{det}(\lambda I - J) = \lambda^3 + \lambda^2 \text{tr}(D) + 4|\text{det}(D)| = 0. \]  \hspace{1cm} (E.26)
Since we assume that $\text{tr}(D) > 0$, this cubic equation has negative discriminant

\[-16\text{tr}^3(D)|\text{det}(D)| - 432|\text{det}(D)|^2 < 0,\]  

(E.27)

implying that (E.26) has one real root and two non-real complex conjugate roots. Equivalently, (E.26) must have the form

\[
\text{det}(\lambda I - J) = (\lambda - r_1)(\lambda - r_2)(\lambda - r_2^*),
\]  

(E.28)

where $r_1$ is real and $r_2$ is complex. Thus, we want to eliminate the possibility that $\Re(r_1) = r_1 = 0$ or $\Re(r_2) = 0$. If $r_1 = 0$, then (E.28) reduces to

\[
\text{det}(\lambda I - J) = \lambda(\lambda - r_2)(\lambda - r_2^*) = \lambda^3 - \lambda^2(r_2 + r_2^*) + \lambda|r_2|^2,
\]  

(E.29)

which is incompatible with (E.26), since $|\text{det}(D)| \neq 0$ for the active triads considered here. If $\Re(r_2) = 0$, then (E.28) reduces to, for some real $r$,

\[
\text{det}(\lambda I - J) = (\lambda - r_1)(\lambda - ir)(\lambda + ir) = \lambda^3 - \lambda^2r_1 + \lambda r^2 - r_1r^2,
\]  

(E.30)

which is also incompatible with (E.26), since imposing that $r = 0$ to eliminate the term proportional to $\lambda$, also eliminates the constant term. Thus, $J$ has at least one eigenvalue with positive real part, implying that the fixed point (7.5) is linearly unstable.
E.3 The phase space of the system for $A(t)$ when $D = 0$

E.3.1 Geometry of the solutions

Consider the system (7.5a) when $D = 0$

\[
\begin{align*}
  k^2 \dot{A}_k &= 2\Delta(p^2 - q^2)A_p^*A_q^* \\
  p^2 \dot{A}_p &= 2\Delta(q^2 - k^2)A_q^*A_k^* \\
  q^2 \dot{A}_q &= 2\Delta(k^2 - p^2)A_k^*A_p^* 
\end{align*}
\]  

(E.31)

which has the three constants of motion

\[
\begin{align*}
  k^2 |A_k|^2 + p^2 |A_p|^2 + q^2 |A_q|^2 &= E \\
  k^4 |A_k|^2 + p^4 |A_p|^2 + q^4 |A_q|^2 &= \Omega \\
  A_k^*A_p - A_k^*A_p^* &= C 
\end{align*}
\]  

(E.32)

The quadratic invariants $E$ and $\Omega$ were found by [147], and the cubic invariant $C$ was derived in section 7.2.2.

Eqs. (E.32) provide three constraints for $(A_k, A_p, A_q) \in \mathbb{C}^3 \simeq \mathbb{R}^6$ depending on $(E, \Omega, C) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times i\mathbb{R}$. Denote by $M(E, \Omega, C)$ the set defined by these equations. We will show that for generic values of $(E, \Omega, C)$ the set $M(E, \Omega, C)$ is a compact three-dimensional manifold (possibly empty) and that each of its connected components is a three-torus.

To show that the system (E.32) defines a manifold in an appropriate subset of $\mathbb{R}^6$, it is enough to show that its differential $J$ has full rank on that subset. Differentiating (E.32) with respect to $\partial A_i$ and $\partial A_i^*$ yields

\[
J = \begin{bmatrix}
  k^2 A_k^* & k^2 A_k & p^2 A_p^* & p^2 A_p & q^2 A_q^* & q^2 A_q \\
  k^4 A_k^* & k^4 A_k & p^4 A_p^* & p^4 A_p & q^4 A_q^* & q^4 A_q \\
  A_p A_q & -A_p A_q^* & A_k A_q & -A_k A_q^* & A_k A_p & -A_k A_p^*
\end{bmatrix}
\]  

(E.33)
Note that the matrix above is in fact the complexification of \( J \), which has the same rank. First, consider the minor \( J_{123} \):

\[
J_{123} = \det \begin{bmatrix}
k^2A_k^* & k^2A_k & p^2A_p^* \\
k^4A_k^* & k^4A_k & p^4A_p^* \\
A_p A_q & -A_p^* A_q & A_k A_q
\end{bmatrix} = 2p^2k^2(p^2 - k^2)A_p^* \Re(A_k A_p A_q). \tag{E.34}
\]

We see that \( \Re(A_k A_p A_q) \neq 0 \) implies that \( J \) has full rank. We now consider the various cases when \( \Re(A_k A_p A_q) = 0 \).

**Case 1.** Two (or more) modes vanish, say \( A_p = A_q = 0 \). Then the last row of \( J \) is zero and thus \( J \) can have rank at most 2. Therefore, we will consider the subset

\[
Z_1 = \{ A_p = 0, A_q = 0 \} \cup \{ A_q = 0, A_k = 0 \} \cup \{ A_k = 0, A_p = 0 \} \tag{E.35}
\]

of \( \mathbb{R}^6 \) separately.

**Case 2.** One mode vanishes, say \( A_k = 0 \) but \( A_p \neq 0 \) and \( A_q \neq 0 \). The differential \( J \) takes the form

\[
J|_{A_k = 0} = \begin{bmatrix}
0 & 0 & p^2A_p^* & p^2A_p & q^2A_q^* & q^2A_q \\
0 & 0 & p^4A_p^* & p^4A_p & q^4A_q^* & q^4A_q \\
A_p A_q & -A_p^* A_q & 0 & 0 & 0 & 0
\end{bmatrix} \tag{E.36}
\]

Taking linear combination of the first two rows gives

\[
\tilde{J}|_{A_k = 0} = \begin{bmatrix}
0 & 0 & 0 & 0 & q^2A_q^*(p^2 - q^2) & q^2A_q(p^2 - q^2) \\
0 & 0 & p^4A_p^* & p^4A_p & q^4A_q^* & q^4A_q \\
A_p A_q & -A_p^* A_q & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{E.37}
\]

which has full rank, since \( A_p \neq 0 \) and \( A_q \neq 0 \).

**Case 3.** None of the modes vanish, i.e. \( |A_k||A_p||A_q| \neq 0 \), but \( \Re(A_k A_p A_q) = 0 \). To simplify the analysis, note that the system (E.32) has the property (ii) of section 7.2.2,
that is, it is invariant under the change of variables

$$(A'_k, A'_p, A'_q) = (A_k e^{i\psi_k}, A_p e^{i\psi_p}, A_q e^{i\psi_q}) \quad \text{where } \psi_k + \psi_p + \psi_q = 0. \quad (E.38)$$

Therefore, without loss of generality we can assume $A_p \in \mathbb{R}, A_q \in \mathbb{R}$, and then $\text{Re}(A_k A_p A_q) = 0$ together with $|A_k A_p A_q| \neq 0$ implies $A_k \in i\mathbb{R}$. The differential becomes

$$J = \begin{bmatrix}
-k^2 A_k & k^2 A_k & p^2 A_p & p^2 A_p & q^2 A_q & q^2 A_q \\
-k^4 A_k & k^4 A_k & p^4 A_p & p^4 A_p & q^4 A_q & q^4 A_q \\
A_p A_q & -A_p A_q & A_k A_q & A_k A_q & A_k A_p & A_k A_p 
\end{bmatrix}. \quad (E.39)$$

The second, fourth and sixth columns are, up to a sign, the same as the first, third and fifth columns, respectively. Thus $J$ has full rank if and only if the minor $J_{135}$ is nonzero. We have:

$$J_{135} = \det \begin{bmatrix}
-k^2 A_k & p^2 A_p & q^2 A_q \\
-k^4 A_k & p^4 A_p & q^4 A_q \\
A_p A_q & A_k A_q & A_k A_p 
\end{bmatrix}
= -A_p^2 A_q^2 k^2 q^2 (k^2 - q^2) - A_p^2 A_q^2 k^2 (p^2 - k^2) + A_p^2 A_q^2 q^2 p^2 (q^2 - p^2)
= |A_q|^2 |A_k|^2 k^2 q^2 (k^2 - q^2) + |A_p|^2 |A_k|^2 p^2 k^2 (p^2 - k^2) +
|A_p|^2 |A_q|^2 q^2 p^2 (q^2 - p^2).
\quad (E.40)$$

Therefore, we must treat the following subset separately:

$$Z_3 = \{ |A_k| |A_p| |A_q| \neq 0, \text{Re}(A_k A_p A_q) = 0 \} \cap
\{ |A_q|^2 |A_k|^2 k^2 q^2 (k^2 - q^2) + |A_p|^2 |A_k|^2 p^2 k^2 (p^2 - k^2) +
|A_p|^2 |A_q|^2 q^2 p^2 (q^2 - p^2) = 0 \}, \quad (E.41)$$

which will be analysed in section E.3.2.

We conclude that the system $(E.32)$ defines a foliation of $N = \mathbb{R}^6 \setminus (Z_1 \cup Z_3)$ by three-dimensional manifolds since the differential $J$ has full rank on $N$. Precisely, $N$
is foliated by the manifolds \( \tilde{M}_{(E,\Omega,C)} = M_{(E,\Omega,C)} \cap N \). We call the closed set

\[
Z = \{ \text{rk} J < 3 \} = Z_1 \cup Z_3 = \{ \Re(A_k A_p A_q) = 0 \} \cap (E.42)
\]

\[
\{|A_q|^2|A_k|^2k^2q^2(k^2 - q^2) + |A_p|^2|A_k|^2p^2k^2(p^2 - k^2) + |A_p|^2|A_q|^2q^2p^2(q^2 - p^2) = 0\},
\]

the singular locus (of \( J \)). Its complement, \( N = \mathbb{R}^6 \setminus Z \), is called the regular locus (of \( J \)).

The considerations above prove that \( \tilde{M}_{(E,\Omega,C)} \) is a three-dimensional smooth submanifold of \( \mathbb{R}^6 \). We now prove that for generic values of \( (E,\Omega,C) \) the set \( M_{(E,\Omega,C)} \) does not intersect \( Z \) and thus is equal to \( \tilde{M}_{(E,\Omega,C)} \), and therefore is a compact three-dimensional submanifold. Moreover, we prove that it is in fact a sum of disjoint copies of the three-torus \( T^3 \).

First, note that \( M_{(E,\Omega,C)} \), as well as the sets \( Z_1 \) and \( Z_3 \) are invariant under the change of variables

\[
(A_k, A_p, A_q) \mapsto (A_k e^{i\psi_k}, A_p e^{i\psi_p}, A_q e^{i\psi_q}) \quad \text{where} \quad \psi_k + \psi_p + \psi_q = 0 \quad (E.43)
\]

which defines a group action of the two-torus \( T^2 = S^1 \times S^1 \) on \( M_{(E,\Omega,C)} \). For \( g \in T^2 \), denote by \( g \cdot x \) the action of the group element \( g \) on \( x \). Moreover, this action is free on \( \mathbb{R}^6 \setminus Z_1 \), and in particular on every \( \tilde{M}_{(E,\Omega,C)} \). By Corollary 21.6 and Theorem 21.10 in [124] the orbit space \( \tilde{O}_{(E,\Omega,C)} = \tilde{M}_{(E,\Omega,C)}/T^2 \) is a smooth manifold of dimension 1. Thus, the manifold \( \tilde{M}_{(E,\Omega,C)} \) is a fiber bundle over \( \tilde{O}(E,\Omega,C) \) with fiber \( T^2 \) (in fact, it is a principal \( T^2 \)-bundle). We denote the quotient map by \( \Pi : \tilde{M}_{(E,\Omega,C)} \to \tilde{O}(E,\Omega,C) \).

Since \( \tilde{O}(E,\Omega,C) \) is 1-dimensional, it is a union of circles \( S^1 \) and lines \( \mathbb{R} \). Consider any component \( \tilde{O} \) of \( \tilde{O}(E,\Omega,C) \) and the component \( \tilde{M} \) of \( \tilde{M}_{(E,\Omega,C)} \) projecting to \( \tilde{O} \), i.e. \( \tilde{M} = \Pi^{-1}(\tilde{O}) \). Suppose \( \tilde{O} \) is diffeomorphic to \( \mathbb{R} \). Since \( \mathbb{R} \) is contractible, every fiber bundle over it is trivial, so \( \tilde{M} \) is diffeomorphic to \( \mathbb{R} \times T^2 \).

Suppose now \( \tilde{O} \) is diffeomorphic to \( S^1 \). Consider the map \( \gamma : [0, 1] \to \tilde{O} \simeq S^1 \) given by \( \gamma(t) = e^{2\pi it} \). Lift this map to a map \( \tilde{\gamma} : [0, 1] \to \tilde{M} \), that is, take any map such that \( \Pi(\tilde{\gamma}(t)) = \gamma(t) \). Note that \( \gamma(0) = \gamma(1) = 1 \), thus \( \tilde{\gamma}(0), \tilde{\gamma}(1) \) belong to the same fiber.

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of \( \Pi, \Pi^{-1}(1) \). But \( T^2 \) acts transitively on the fibers of \( \Pi \), thus there is an element \( g \in T^2 \) such that \( \tilde{\gamma}(0) = g \cdot \tilde{\gamma}(1) \). Now take a path \( s : [0, 1] \to T^2 \) such that \( s(1) = g \) is as above and \( s(0) \) is the identity element. Then the map \( \eta(t) = s(t) \tilde{\gamma}(t) \) has the property that \( \eta(0) = \eta(1) \), and thus it descends to a map \( \eta : S^1 \to T^2 \) such which lifts \( \gamma \), i.e. \( \Pi \circ \eta = \gamma \). Finally, after smoothing \( \eta \), the map \( F : T^3 = S^1 \times T^2 \to \tilde{M} \) given by \( F(t, g) = g \cdot \eta(t) \) gives the desired diffeomorphism of \( T^3 \) and \( \tilde{M} \).

In particular, what follows is that whenever \( M_{(E, \Omega, C)} \) does not intersect \( Z \), it is a disjoint union of a finite number of three-tori. This may be empty when \( M_{(E, \Omega, C)} \) is empty, for instance if \( \Omega/E > \max(k^2, p^2, q^2) \), or \( \Omega/E < \min(k^2, p^2, q^2) \) etc. Now we determine a residual subset of triples \((E, \Omega, C) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times i\mathbb{R}\) for which \( M_{(E, \Omega, C)} \cap Z = \emptyset \).

Consider \((A_k, A_p, A_q) \in M_{(E, \Omega, C)} \cap Z\). Since \( \text{Re}(A_k A_p A_q) = 0 \), we have \( C = A_k A_p A_q - A_k^* A_k A_k^* A_q = 2i\text{Im}(A_k A_p A_q) = 2A_k A_p A_q \), thus \( |C|^2 = 4|A_k|^2 |A_p|^2 |A_q|^2 \). Denote \( x = |A_k|^2, y = |A_p|^2, z = |A_q|^2 \). The system (E.32) together with the equations defining \( Z \) thus implies

\[
\begin{align*}
  k^2 x + p^2 y + q^2 z &= E \\
  k^4 x + p^4 y + q^4 z &= \Omega \\
  k^2 q^2 (k^2 - q^2) xz + p^2 k^2 (p^2 - k^2) xy + q^2 p^2 (q^2 - p^2) yz &= 0 \\
  4xyz &= |C|^2
\end{align*}
\]

(E.44)

The first two equations express \( y, z \) as linear functions of \( x \). Inserting these into the third equation one obtains a quadratic equation for \( x \) with a non-zero leading term, which has at most 2 solutions. These solutions give at most 2 possible values of \( |C| \) using the last equation. Denote the set of triples \((E, \Omega, C) \) obtained this way by \( S \). This is a codimension 1 subset, thus a generic \((E, \Omega, C) \) does not belong to \( S \), and for such a triple \((E, \Omega, C) \) outside of \( S \) the set \( M_{(E, \Omega, C)} \) is deemed to be a sum of three-tori as explained earlier.

Since the differential \( J \) is of full rank on \( N = \mathbb{R}^6 \setminus Z \), the set \( F^{-1}(S) \cap N \) is of codimension 1, where \( F : \mathbb{R}^6 \to \mathbb{R}^3 \) is the map determined by (E.32). However, the set \( Z \) is of codimension 1, too, and since \( F^{-1}(S) = (F^{-1}(S) \cap N) \cup (F^{-1}(S) \cap Z) \), we
conclude that $F^{-1}(S)$ is of codimension 1. The complement of this set is foliated by three-tori, so taking all things together it follows that a generic point in $\mathbb{R}^6$ lies on one of these smooth three-tori.

**E.3.2 Exact solutions for initial conditions on $Z_1$ and $Z_3$**

To finish this section, we comment on the nature of the dynamics (E.31) when the initial conditions are taken from the subsets $Z_1$ and $Z_3$. It is easy to see that points on $Z_1$ are simply fixed points. Taking initial conditions on $Z_3$ results in evolution with constant amplitudes $|A_k|$, $|A_p|$ and $|A_q|$ and phases exhibiting periodic motion on two-torus. Indeed, consider the following ansatz $(A_k, A_p, A_q) = (|A_k|e^{i\phi_k(t)}, |A_p|e^{i\phi_p(t)}, |A_q|e^{i\phi_q(t)})$, where the amplitudes are independent of time. The system (E.31) gives

\[
\begin{align*}
\dot{k}^2 |A_k| \phi_k &= (p^2 - q^2) |A_p| |A_q| e^{-i(\phi_k + \phi_p + \phi_q + \pi/2)} \\
\dot{p}^2 |A_p| \phi_p &= (q^2 - k^2) |A_q| |A_k| e^{-i(\phi_k + \phi_p + \phi_q + \pi/2)} \\
\dot{q}^2 |A_q| \phi_q &= (k^2 - p^2) |A_k| |A_p| e^{-i(\phi_k + \phi_p + \phi_q + \pi/2)}
\end{align*}
\]

(E.45)

On $Z_3$, $e^{-i(\phi_k + \phi_p + \phi_q + \pi/2)} = \pm 1$. If we assume that this holds for any time $t$, we easily find solutions to these equations:

\[
\begin{align*}
\phi_k &= \pm [(p^2 - q^2) |A_p| |A_q| / (k^2 |A_k|)] t + c_k \\
\phi_p &= \pm [(q^2 - k^2) |A_q| |A_k| / (p^2 |A_p|)] t + c_p \\
\phi_q &= \pm [(k^2 - p^2) |A_k| |A_p| / (q^2 |A_q|)] t + c_q
\end{align*}
\]

(E.46)
Figure E-3: Instantaneous (a,b) and average (c) helicity fluxes for the Gaussian activity model (7.29) for time instants and intervals indicated in Fig. 7-7(a). Vertical dashed lines mark the energy injection range.

Moreover, multiplying the first equation of (E.45) by \( p^2q^2|A_p||A_q| \), the second by \( k^2q^2|A_k||A_q| \), the third by \( k^2p^2|A_k||A_p| \) and adding them together gives

\[
\pm k^2 p^2 q^2 |A_k||A_p||A_q| (\dot{\phi}_k + \dot{\phi}_p + \dot{\phi}_q) = (p^2 - q^2)p^2q^2 |A_p|^2 |A_q|^2 + (q^2 - k^2)q^2k^2 |A_q|^2 |A_k|^2 + (k^2 - p^2)k^2p^2 |A_k|^2 |A_p|^2 = 0. \tag{E.47}
\]

The right-hand side is zero by the definition of \( Z_3 \), implying that the sum of phases \( \phi_k + \phi_p + \phi_q \) is indeed constant and equal to \( \pi / 2 \) or \( 3\pi / 2 \) also by the definition of \( Z_3 \). Therefore (E.46) gives the solutions to the system (E.31) on \( Z_3 \) and these exhibit quasi-periodic motion.
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