Causal and Statistical Analyses of Dithered Systems Containing Three-Level Quantizers

by

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ABSTRACT

A dithered system is one which has an external signal introduced to improve the performance of the original system. Dither will eliminate limit cycle phenomena and will also cause the system to behave as if the quantizer was replaced by a linear gain. The input-output characteristic of a relay may be simulated by a three - level quantizer.

Two analyses are here used to explain the behavior of a dithered system. The first analysis under consideration makes use of the causal or deterministic properties of the dither signal. The dither signal is a sine wave. The amplitude of the sine wave necessary to control limit cycles is obtained using a describing function technique. The modified describing function of a dithered relay is derived and used to analyse a third order system. The results of the analysis are verified by a simulation of the system on a digital computer.

The statistical properties of the dither are employed to explain how the system is linearized by the dither. Both sinusoidal and first order Gaussian dither are considered. The statistical approach predicts the linearization properties of the dither and gives the same overall results as those given by the causal analysis.

Thesis Supervisor: Bernard Widrow
Title: Assistant Professor of Electrical Engineering

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CHAPTER 1

INTRODUCTION AND BACKGROUND

One of the methods used to smooth out discontinuous nonlinearities in control systems is called dithering. Some of the first applications of dither signals were with valves and bearings. These systems were vibrated or dithered to make them perform more linearly and to prevent them from exhibiting discontinuous behavior caused by the static friction. In about 1945 dither signals or independent sine waves were introducted into the loops of control systems to make them finer and to prevent low frequency, high amplitude, limit cycles.

1.1 HISTORY OF PROBLEM

One of the first analyses of a system with a dither source was presented by MacColl.[1] He considered the nonlinearity of the type of limiter or, a nonlinearity that has a constant output with a polarity determined by the input signal. By calculating the equivalent gain of the limiter to the input signal as a function of dither signal, he was able to determine the amplitude of dither necessary to "linearize" the system. The input to the limiter was assumed to be made up of control signal plus a dither signal. Both were sine waves but the dither frequency was much higher than the frequency of the control signal. As the two signals passed through the limiter, they were mixed and the resulting output was expressed in terms of the sums of harmonics and cross-products of the two frequencies. MacColl considered only the control signal frequency and thereby was able to derive an equivalent gain for this frequency. He assumed that the limiter was followed by linear elements that have low pass characteristics, as these elements will attenuate all the higher harmonics and cross product frequencies. Thus it can be seen that the limiter behaves as a gain or linear element when it is dithered.

The way the limiter is linearized by the dither can be seen physically if the output of the limiter is considered. If the input is only a dither, the output will be a series of square waves with a zero average. If a positive signal is introduced with the dither, the square wave will be positive longer and therefore the average of the output will be positive. This indicates that the signal has been transmitted through the nonlinearity as though it were a linear gain.

Using the above methods of analysis, systems were designed and built[2,3] using relays as the nonlinear elements. By sine wave dithering the relay, these systems were found to be comparable in performance to their linear equivalents with savings in weight and gain in reliability of operation.

Now the question arises: what are the limitations of the above analyses? First, only simple types of nonlinearities have been considered. Secondly, they do not predict how limit cycles are formed due to insufficient amplitude of dither signal.

To understand how limit cycles are formed in on-off type systems, (systems with a discontinuous nonlinearity), the nonlinearity may be thought of as a variable gain that is infinite at switching times. Due to this infinite gain, the system is made unstable and it tries to go to another state. However, the gain soon becomes zero and the system coasts to the next infinite gain point. The system will limit cycle at fixed amplitude and the oscillation will not grow unbounded because of the discrete levels of the nonlinearity.

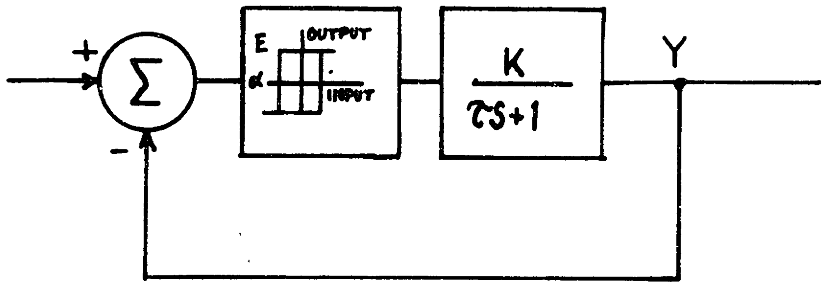
1.2 EXAMPLE OF DITHERED NONLINEARITY

The existance of limit cycles can be seen by examining simple systems. For example, in Figure 1.1, we assume that the output is of the form of equation 1.1.

$$KE\,(1 - e^{-\alpha\gamma}) \qquad (1.1)$$

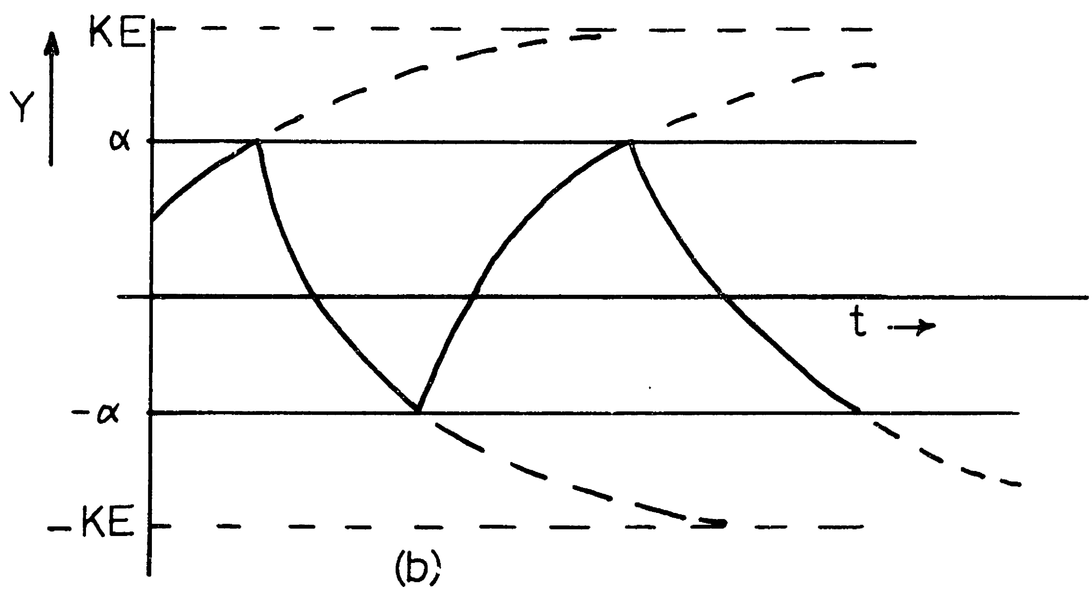
If the output reaches α , the non linear element switches and the system moves in the other direction. There will be movement only in the direction of switching because the output is first order and a function of initial position only. By solving equation 1.2 for T, the frequency

$$KE\,(1 - e^{-\alpha T}) = 2\alpha \qquad (1.2)$$



(a)

1st Order System



(b)

Output of System
fig. 1.1

of oscillation can be determined. Its magnitude, of course, will be α. As the value of α is decreased, the frequency increases and the amplitude goes down, and in the limiting case the oscillation occurs with infinite frequency and zero amplitude.

The condition for no oscillation will be where KE is less than α. For linear elements with a high order transfer function, the derivatives of the output at switching time must be considered in order to obtain a complete picture of the system's behavior and determine the frequency and amplitude of the limit cycle.

If one were to add a dither signal to this system, the piecewise solution of the equation would become cumbersome. However, it is not difficult to understand qualitavely how the dither produces low amplitude outputs of the same frequency as the dither signal. If the relay were turned on and off quickly, the exponential output would not have time to reach any large value before it was switched to the opposite polarity. As we can see, the output will be of the same form as the self limit cycle but instead of switching at its own natural rate, (which is low), it is forced to switch at the dither rate.

If a signal were introduced, the dither would just average out to zero causing the system to behave "linearily".

Two approaches will be used to study the effects of dither: Causal and Statistical. Causally, the dither changes the effective gain of the nonlinearity for sine wave inputs. Statistically, the same effect is accounted for by a gain and an independent noise source replacing the nonlinearity. This replacement can only be made if certain conditions are met by the input's signal statistics.

1.3 DESCRIBING FUNCTION

In the causal method, the input function to the nonlinearity is thought of as a deterministic function of time. One way of finding the frequency and amplitude of a limit cycle in a nonlinear control system is the describing function. This method is based on the steady state sinusoidal response of the nonlinear element.

The describing function method is an approximate one which assumes that the wave shape at the input to the nonlinearity is known. This wave shape is usually assumed to be sinusoidal. When the sine wave passes through the nonlinearity, a fundamental and higher harmonics are generated. They now pass through the linear elements and the higher frequencies are attenuated by the low pass action of these "filters". The linear elements effectively act as a time integrator and filter out all the higher harmonics. When the output of the nonlinearity is now fed back, it is a sinusoid of the fundamental frequency. Since this is the same wave shape as originally postulated for the input to the nonlinearity, the system will oscillate with that wave shape.

1.4 STABILITY AND LIMIT CYCLES

By ploting the locus of the gain and phase of the linear elements as a function of frequency, an intersection can be found with the locus of the nonlinear elements. This locus is the negative reciprocal of the gain and phase of the nonlinear elements as a function of sine wave input amplitude. Stated in mathematical terms this is equation 1.3:

$$-\frac{1}{N(\alpha)} = K(j\omega) \tag{1.3}$$

When the locus $-\frac{1}{N(\alpha)}$ of $K(j\omega)$ is drawn on the gain-phase plane, there can be four cases resulting of interest to our application: see Figure 1.2 ,

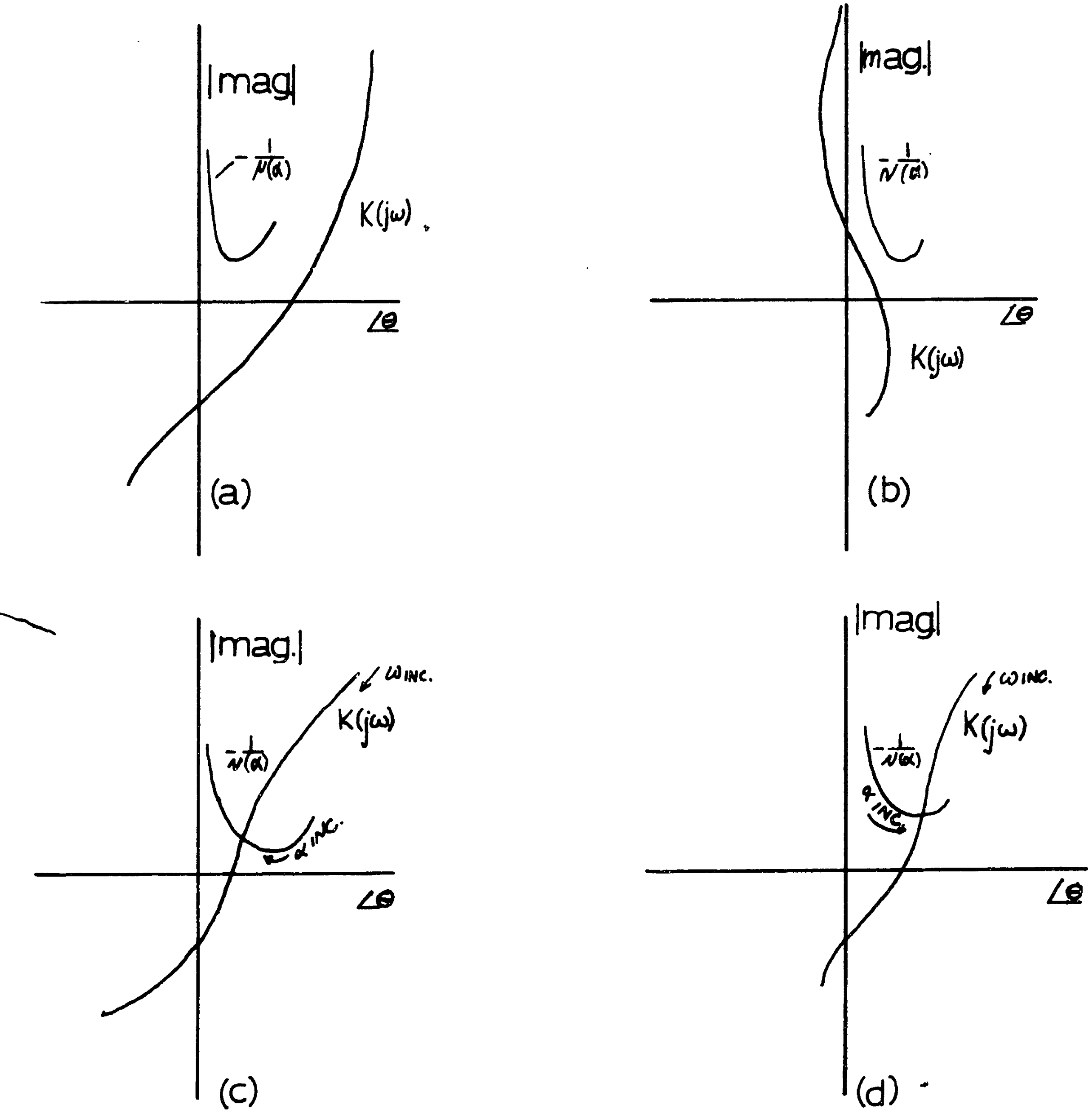
a) No intersection - No limit cycle

b) No intersection - No limit cycle. System is unstable. Oscillation grows without bounds.

c) Intersection - Stable, limit cycles for all excitations.

d) Intersection - Unstable limit cycle, cannot exist in that mode.

At any intersection point the exact nature of the intersection can be determined by a modified Nyquist criteria. The points on the $-\frac{1}{N(\alpha)}$ locus can be thought of as gain points. If, as the oscillation tends to increase in amplitude,(by the Nyquist curve having net circulation of the point in the $-\frac{1}{N(\alpha)}$ locus), the system brings the point on the $-\frac{1}{N(\alpha)}$ locus out of the Nyquist region, then the oscillation will be stable at the intersection's amplitude and frequency.



Behavior of Nonlinear System

fig. 1.2

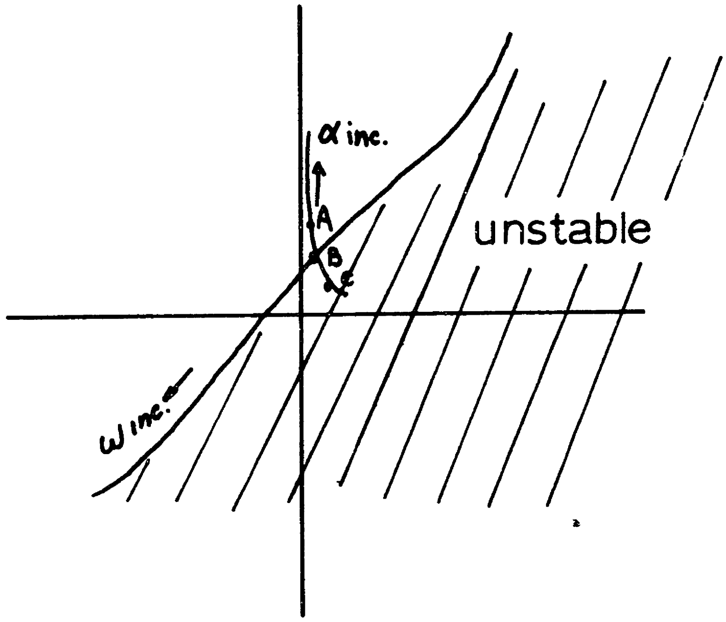
A study of Figure 1.3 shows a stable equilibrium point. Consider point C first. This point is an unstable gain point. The linear element will tend to make the oscillation grow to point B. As soon as it goes into region A B, where this system causes disturbances to die down, the decreasing amplitude brings its operation point back to point B.

Throughout this analysis one assumption has been made: the wave shape at the input to the nonlinearity has been known and by its propagation around the loop this wave shape was supported. This assumption infers that the nonlinear element must be followed by low pass elements to attenuate higher harmonics. Morover no subharmonics can be generated. It must be realized that if the harmonics and subharmonics are taken into account at the input to the nonlinearity, this analysis would give correct results.

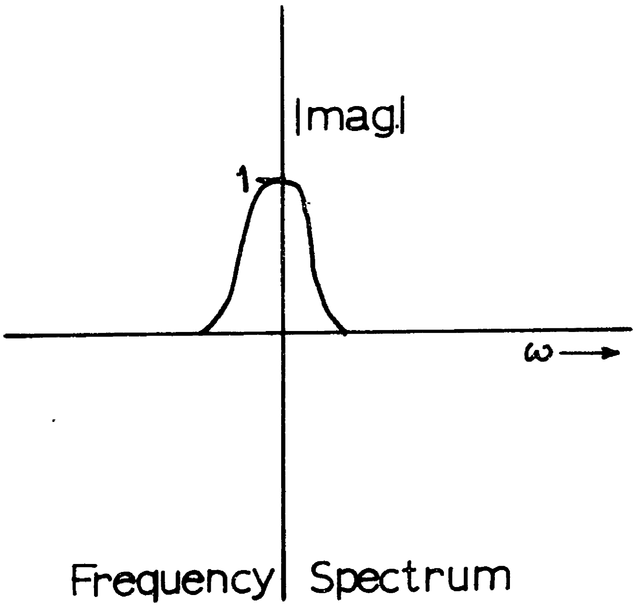
For dithered systems this analysis may also be used because the assumptions stated above are met. The dither signal must also be attenuated before it reaches the ouput. In further chapters the describing function will be extended to systems with dither.

1.5 INTRODUCTION TO STATISTICAL METHODS OF ANALYSIS

In order to understand how dithered systems are analysed by statistical means, it is necessary to review certain aspects of sample data, probability and the statistical theory of quantization.



Stable Limit Cycle
fig. 1.3



Frequency Spectrum
fig. 1.4

1.6 IMPULSE MODULATION AND SAMPLING THEORY

The action of a sampler in a continuous system is similar to that of a conventional AM modulator. In AM broadcast the modulator produces sidebands around the carrier frequency. The impulse modulator, however, produces an infinite amount of sidebands about a zero frequency carrier. The action of the impulse modulator in the time domain produces an impulse of area equal to the height of the time wave form at each sampling instance.

The action of the modulator can easily be seen by examining the frequency spectrum of an input wave shape. Assume that the input wave shape has an amplitude distribution as shown in Figure 1.4. The modulator will produce a frequency spectrum as in Figure 1.5 i.e., Figure 1.4 repeated at the sampling frequency. Due to this modulator each lump (or individual spectra) is attenuated by $\frac{1}{T}$ where T is the sampling period.

A few consequences of the modulation process can immediately be seen should one want to recover the original information from the modulated signal. This can be done exactly by using a theoretical low pass filter if there is no overlap of the lump centered at the origin. Therefore, the three conditions for exact recovery are:

1) The signal must be band limited.
2) The sampling frequency must be high enough to prevent overlap.
3) It must be possible to ideally filter the signal.

In practice, the signals are never band limited and ideal filter cannot be realized. However, by properly controlling the sampling frequency the physical limitations imposed by the signal and filter can be made negligible and a good engineering device may be developed.

1.7 DISTRIBUTION DENSITIES AND CHARACTERISTIC FUNCTIONS.

A distribution density may be derived from a strip chart of a time wave form by first constructing a bargraph. The width of the bar is made equal to a range of amplitude. The height of each bar is made proportional to the total time that the wave shape is in this band of amplitudes. This graph will indicate for any Δx what fraction of time the signal spends in that interval. By permitting the band of amplitudes to approach zero, a continuous curve will be formed in the limit. This is called the first order probability density of the signal. The moments or weighted averages of this function about the origin are the first order moments of the signal. These can be expressed mathematically as equation 1.4.

$$\overline{x^n} = \int_{-\infty}^{+\infty} x^n \; w(x) \; dx \qquad (1.4)$$

$\overline{x}$ indicates the above averaging process

$\overline{x^0} = 1$ or the probability of finding all values of the signal at a given time is 1.

$\bar{x}$ is called the average of the waveshape.

$\bar{x}^2$ is called the second moment of the signal, etc.

A useful parameter that indicates the spread of amplitudes about the average value is the variance

$$\sigma^2 = \overline{(x(t) - \bar{x}(t))^2} \quad \text{or} \quad \bar{x}^2 - (\bar{x})^2 \tag{1.5}$$

and σ is called the standard deviation. If one now takes a different type of average or as equation 1.6

$$p(u) = \int_{-\infty}^{\infty} w(x) \, e^{+jux} \, dx \tag{1.6}$$

where e^{+jux} replaces x^2 in the previous average (equation 1.4) the result is called the characteristic function or moment generation function. The function looks exactly like the Fourier Transform of a signal $W(X)$. It can be used to generate moments by the following formula:

$$\bar{x}^n = (-j)^n \left. \frac{\partial^n P_x(u)}{\partial u^n} \right|_{n=0} \tag{1.7}$$

This formula can be shown by examining the definition of $p(u)$. By differentiating as indicated in equation 1.7 we obtain equation 1.8

$$\frac{\partial P(u)}{\partial u} = \frac{\partial}{\partial u} \int_{-\infty}^{\infty} e^{jux} w(x) \, dx \tag{1.8}$$

By differentiating under the integral sign and comparing the results to the definition of the moments, we obtain

$$\frac{\partial p(u)}{\partial u} = \int_{-\infty}^{\infty} jx\, W(x)\, e^{jux}\, dx \tag{1.9}$$

(for reference)

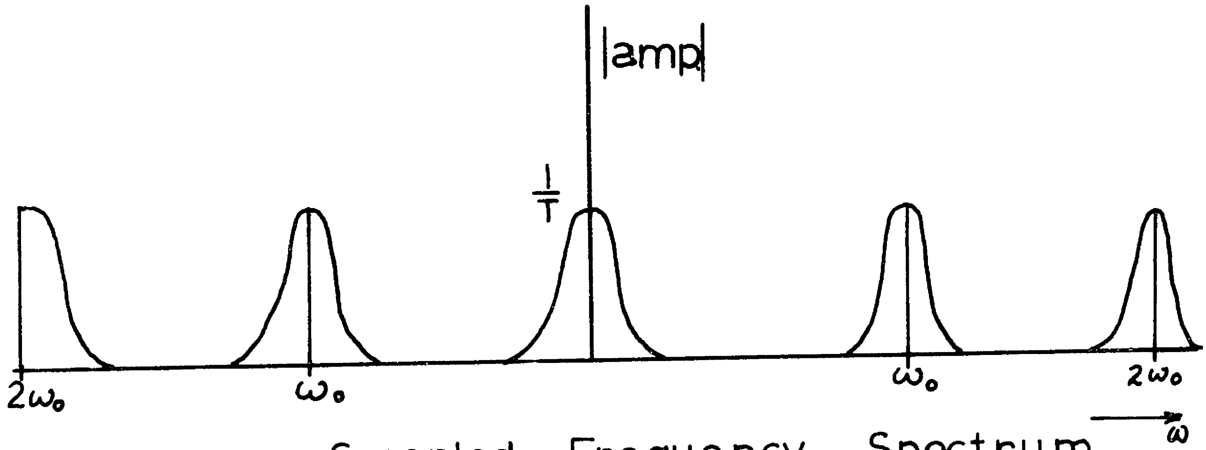
$$\bar{x} = \int_{-\infty}^{\infty} xW(x)\, dx \tag{1.4a}$$

If we now set $u = 0$ the equations (1.9) and (1.4a) become identical except for the j. Continuing with this process, it is easily seen that the equation is true.

1.8 SECOND ORDER DISTRIBUTIONS

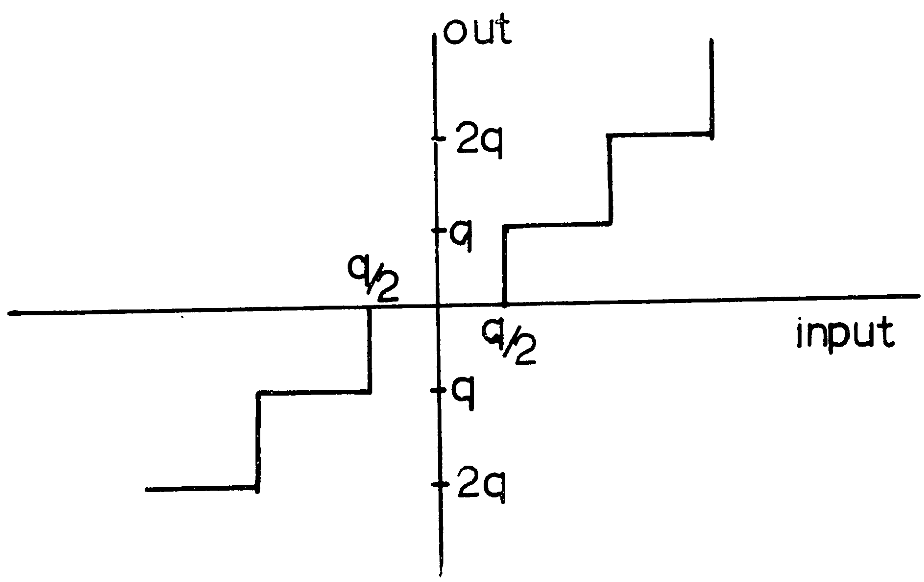
These one dimensional concepts of averages can now be extended to multidimensional densities. A multidimensional process is one that gives, after n samples of the process are taken, $x_1 \ldots\ldots x_n$, the probability that x_{n+1} is a certain value. This, of course, must be plotted in multidimensional space.

The second order distribution has three variables: x_1 and x_2 and a value τ which is the time between samples. The distribution density for



Sampled Frequency Spectrum

fig. 1.5



Typical Quantizer

fig. 1.6

this second order process is now written as equation 1.10,

$$W_{x_1 x_2}(x_1, x_2)_\Upsilon \tag{1.10}$$

and averages now can be expressed:

$$\overline{x^k x^l} = \int_{-\infty}^{\infty} x^k x^l \, W(x_1, x_2)_\Upsilon dx_1 \, dx_2 \tag{1.11}$$

The characteristic function of Fourier Transforms of the multidimentional distributions can also be used to generate the moments as equation 1.12

$$\overline{x^k x^l} = (-j)^{k+l} \frac{\partial^{k+l} P(u_1, u_2)}{\partial u_1^k \partial u_2^l} \Bigg|_{u_1=u_2=0} \tag{1.12}$$

and are called the k, lth moment.

1.9 POWER DENSITY SPCTRA AND AUTOCORRELATION

Of particular interest is the 1,1 moment or the autocorrelation. The autocorrelation function is obtained by allowing Υ to be a variable. The autocorrelation coefficient is obtained from the autocorrelation function by dividing it by the mean square value.

If a process has the same first order distribution for all time it is a called stationary one. Since the processes considered are stationary, it does not make a difference if the normalizing factor for the autocorrelaion coeffecient is $\overline{x_1}^2$ or $\overline{x_2}^2$.

The Fourier Transform of the autocorrelation function is called the power density spectrum of the waveshape. This function is a measure of the power contained in the frequency components of the function. The total area under the power density spectrum is the average power. A Fourier Transform pair can now be written between the autocorrelation function and power density spectrum.

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{11}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad (1.13)$$

$$\varphi_{11}(\tau) = \int_{-\infty}^{\infty} \Phi_{11}(\omega)\, e^{j\omega\tau}\, d\omega \qquad (1.14)$$

It may be noted from equation (1.14) that $\varphi_{11}(0)$ is the average power of the wave. From equation 1.11 it can be seen that $\varphi_{11}(0)$ is the mean square of the signal.

A useful relationship for the autocorrelation, of the sum of two statistically independent signals x and y where one has zero mean, is equation 1.15

$$\varphi_{oo} = \varphi_{xx} + \varphi_{yy} \qquad (1.15)$$

1.10 STATISTICAL THEORY OF QUANTIZATION

A quantizer is a nonlinear device that has a constant output for inputs in a certain band. A graphical representation of a quantizer is shown in Figure 1.6. Using the ideas of statistics and sampling it is possible to obtain a description of the way quantizers operate in a system.

The input signal to the quantizer will be described in terms of its statistical properties instead of its deterministic ones. The quantizer will now operate in the following manner. It will sample all the amplitude in a certain band (area under w(x) in a certain band) and call it one value, the value at the center of the box. This is analogous to impulse sampling but here the area of the function is sampled. The output characteristic function of the quantizer can now be obtained from the input density by multipling it by a $\frac{\sin \frac{\pi u}{\phi}}{\frac{\pi u}{\phi}}$ function and the resulting product sampled by an impulse modulator. The area sampling gives rise to a new term " $\frac{\sin \frac{\pi u}{\phi}}{\frac{\pi u}{\phi}}$ " but in other respects it is analogous to impulse sampling. ϕ is analogous to Ω and q, the grain size of the quantizer, is analogous to T, the sampling period. The output characteristic function will now have the shape of the input characteristic function multiplied by $\frac{\sin \frac{\pi u}{\phi}}{\frac{\pi u}{\phi}}$ and will be repeated in characteristic function space. Since the moments are derivitives of the characteristic function evaluated at the origin, see equation 1.12, the low frequency behavior of the characteristic function is important if the moments are to be preserved in the quantization process. It is also of interest to note that multiplication of two characteristic functions is the same as addition of two statistically independent functions. Since the output of the quantizer is a product, the individual terms may be identified with a gain and a noise source. The noise source corresponds to the characteristic function on $\frac{\sin x}{x}$ and the gain to the input c.f. A corresponding distribution of "$\frac{\sin x}{x}$" density (d.d) is flat topped between $\pm \frac{q}{2}$ and has a height of $\frac{1}{q}$. This replacement is exact if there is no overlap in characteristic function space. This is completely analogous to the recovery of time function in impulse sampling.

These first order concepts must now be extended to second order distributions in order to determine the effect of the correlation of the input on the correlation of the noise source. This was shown by Widrow[4] to be none, if the second order quantizing theorem is satisfied, i.e., there is no overlap in two dimensional characteristic function space at the origin. The noise source that replaces the quantizer is now not only flat topped but first order also.

It is necessary to know when the quantizing theorems are applied to system analysis how well the quantizing theorems are satisfied.

1.11 ANALYSIS OF GAUSSIAN SIGNALS

Widrow found for Gaussian distributions that there are small errors in the mean square and mean fourth of the output of the quantizer, for quantization levels up to 2 & 3σ. He also found that the quantization noise was practically uncorrelated and first order for the input correlations of about 80%. The error in the moments of the output was evaluated by determining the overlap of the characteristic function at the origin.

1.12 ANALYSIS OF SINE WAVES

The next distribution studied was that of the sine wave. Furman[5] found that there is small overlap in first order characteristic function space for quantization levels to about 1/4 of the amplitude of the sine wave. The output noise of the quantizer was found to be practically uncorrelated for input correlation to about 80% for quantization levels up to half the amplitude of the sine wave.

1.13 QUANTIZERS AND DITHER

The usefulness in analysing dithered quantizating systems is now apparent. It is now known what amplitude of sine waves are needed to approximitely satisfy the quantizating theorem. Any other independent input can only make the characteristic function of the sum narrower, because the characteristic function of independent sources multiply. If the theorem had been satisfied by the sine wave dither acting alone, it would surely be satisfied by the combination of the two signals. The quantizer may now be replaced by an equivalent noise source and linear gain. The system can now be analysed as though it were a linear one.

1.14 PROPAGATION OF STATISTICS IN SYSTEMS

In order to determine if the quantizing theorem is satisfied, it is necessary to know how statistics propagate through the system. If the d.d. is known at one point in the system,it can be found for all points. A method for accomplishing this was worked out by Widrow. The propagation of the varience and the autocorrelation are of particular importance in this problem.

Since the mean square is the value of the autocorrelation function at the origin, it is necessary to show only how the autocorrelation or its power density spectrum propagates through linear elements. It is known that

$$\Phi_{oo}(\omega) = |H(\omega)|^2 \, \Phi_{ii}(\omega) \qquad (1.16)$$

By inverse transforming the above equation we obtain:

$$\varphi_{oo}(\tau) = \int_{-\infty}^{\infty} dt_1 \; H_1^2(t_1) \; \varphi_{ii}(\tau) \qquad (1.17)$$

Now by setting ($\tau = 0$) to find the mean square value of the signal at the origin, we obtain:

$$\overline{x_o^2} = \int_{-\infty}^{\infty} dt_1 \; H^2(t_1) \, \phi_{ii}(0) \quad = \overline{x_i^2} \int_{-\infty}^{\infty} dt_1 \; H^2(t_1) \qquad (1.18)$$

Equation 1.18 now relates the output mean square to the input mean square or, in other words, the output mean square is found by multipling the input by the area squared of the impulse response.

If the system has quantizers as well as linear elements, the transmission of the autocorrelation through these elements must also be determined. If the quantizing theorem is satisfied, (multidimensional one), Widrow[5] showed that the autocorrelation is unchanged except for its behavior at the origin. There it picks up an impulse of height $q^2/12$ due to the quantization noise (independent noise source). As the theorem is violated by the input distribution, this noise becomes correlated and distortion is introduced in the output autocorrelation.

With this understanding of the causal and statistical methods, an analysis of a specific system will be made to see how useful these two methods will be for examining the same phenonema.

CHAPTER 11

CAUSAL ANALYSIS

In the present chapter a single system will be analysed. It is believed that this study will entail most of the ideas necessary for the analysis of dithered systems.

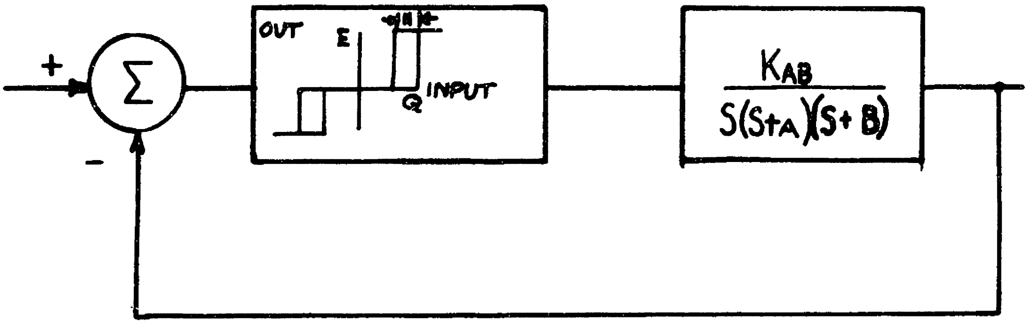
2.1 DESCRIPTION OF SYSTEM

The system to be studied is one that contains a nonlinear element of the type relay in the feed forward path. See Figure 2.1. As the input signal is increased from zero, the output signal remains zero until the input reaches a level Q. The output then has a constant value. As the input is now decreased from level Q, the output remains constant until it reaches a level Q-H. The output then becomes zero.

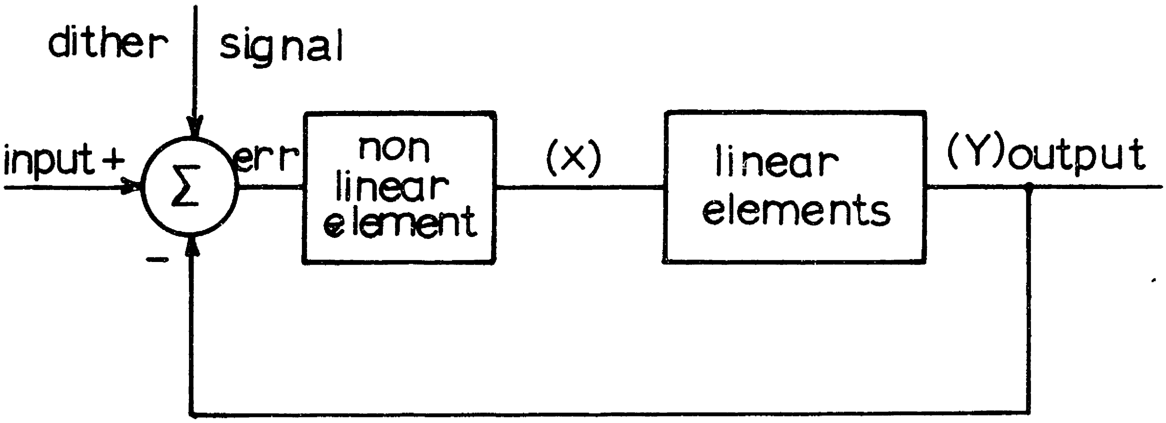
The linear part of the system is a pure integration followed by two first order lags. This may be thought of as a mathematical model of a motor.

A third order system was chosen because it is of high enough order to filter the dither signal before it reaches the output. This condition is necessary if the system is to be improved by dithering. A second order system will also provide filtering; however, due to its low order, the usefulness of this analysis will not be exhibited to its fullest extent.

This system is a practical case where an on-off type element can be used as a power amplifier. To make the system finer, the relay is dithered



Nonlinear System
fig. 2.1



Block Diagram of Dithered System
fig. 2.2

externally. For a block diagram, see Figure 2.2.

2.2 PIECEWISE LINEAR ANALYSIS

It will be instructive to first consider the system before any dither is applied. If a disturbance is applied, the system can exist in two states after the transients have died down. It can come to rest or can continue to limit cycle for all time.

Since the ouput of the nonlinearity is always a constant or zero, it is possible to find the time response of the system for any initial disturbing conditions. The output of the relay can be thought of as being made up of a series of steps. See Figure 2.3. The transfer function of the linear element is equation 2.1

$$\frac{Y(s)}{X(s)} = \frac{Kab}{s\,(s+a)\,(s+b)} \tag{2.1}$$

where the input x(s) is $\frac{+E}{s}$ with zero initial conditions. The output, Y(s), is equation

$$Y(s) = \frac{KEab}{s^2(s+a)(s+b)} = KE\left\{\frac{1}{s^2} - \frac{a+b}{ab} - \frac{ab}{a^2(a-b)} + \frac{ab}{b^2(a-b)}\right\} \tag{2.2}$$

By calling a=kb and bt=t' inverse transforming equation 2.2 to find the time response, we obtain

$$Y(t) = \frac{KE}{kb}\left\{kt' - \frac{1}{k-1}\left[k^2(1-e^{-t'}) - (1-e^{-kt'})\right]\right\} \tag{2.3}$$

A plot of this function appears in Figure 2.4 for values of K=3 KE=b=1

Equation 2.3 represents the time response to a step at $t=0$. Because the output of the relay is made up of a step of +E at $t=0$ and a step of -E at $t=t_1$ etc., it may be expressed as equation 2.4

$$Y(t) = y_o + \sum_{k=1}^{N} y_s(t-t_k)\ u(t-t_k) \tag{2.4}$$

where t_1, t_2, etc., are the pull in and dropout times of the relay. y o is the initial value of y with $\dot{y}_o$ and $\ddot{y}_o$ equal to zero

To calculate the y(t), it is necessary to perform a summation. For the case considered here, it is easier to affect this summation graphically if we break the $y_s(t)$ into two parts:

$$y_{s_1}(t) = \frac{KE}{b}\ t \tag{2.5}$$

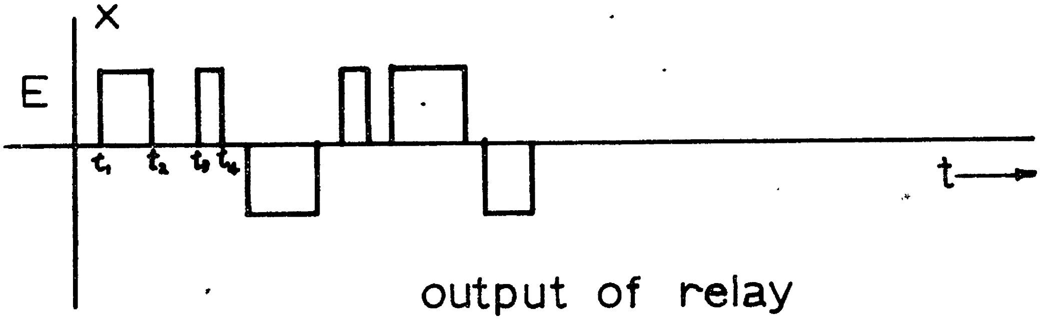
and

$$y_{s_2}(t) = \frac{-KE}{kb}\left\{\frac{1}{k-1}\left[k^2(1-e^{-t'}) - (1-e^{-kt'})\right]\right\} \tag{2.6}$$

where

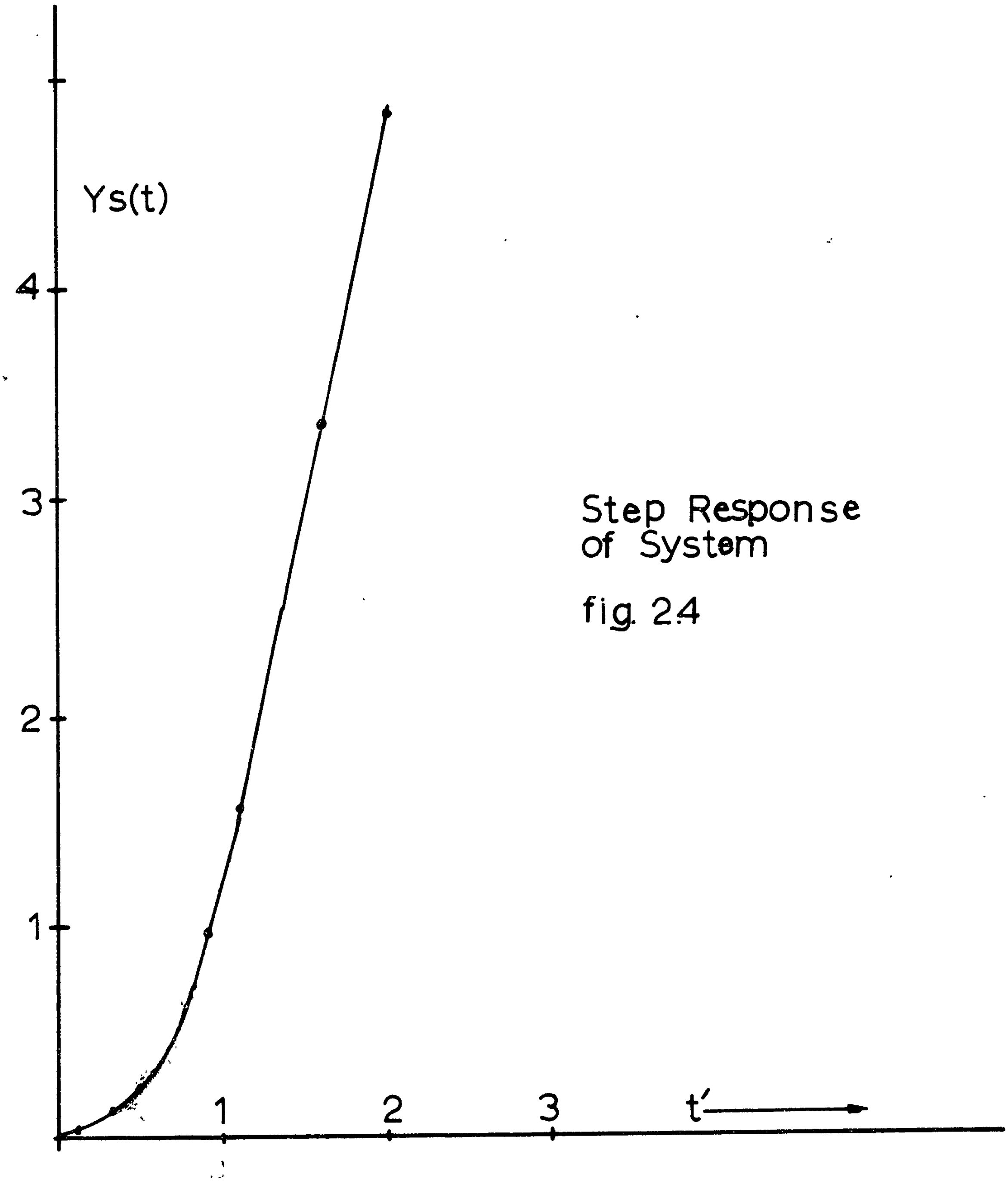
$$y_s(t) = y_{s_1}(t) + y_{s_2}(t) \tag{2.7}$$

The sum of the $ys_1(t)$ will always be either a ramp with a positive or negative slope or a constant value. The partial sum of $ys_2(t)$ will reach a constant value for $t \gg 3k$.



output of relay

fig. 23



Step Response of System

fig. 2.4

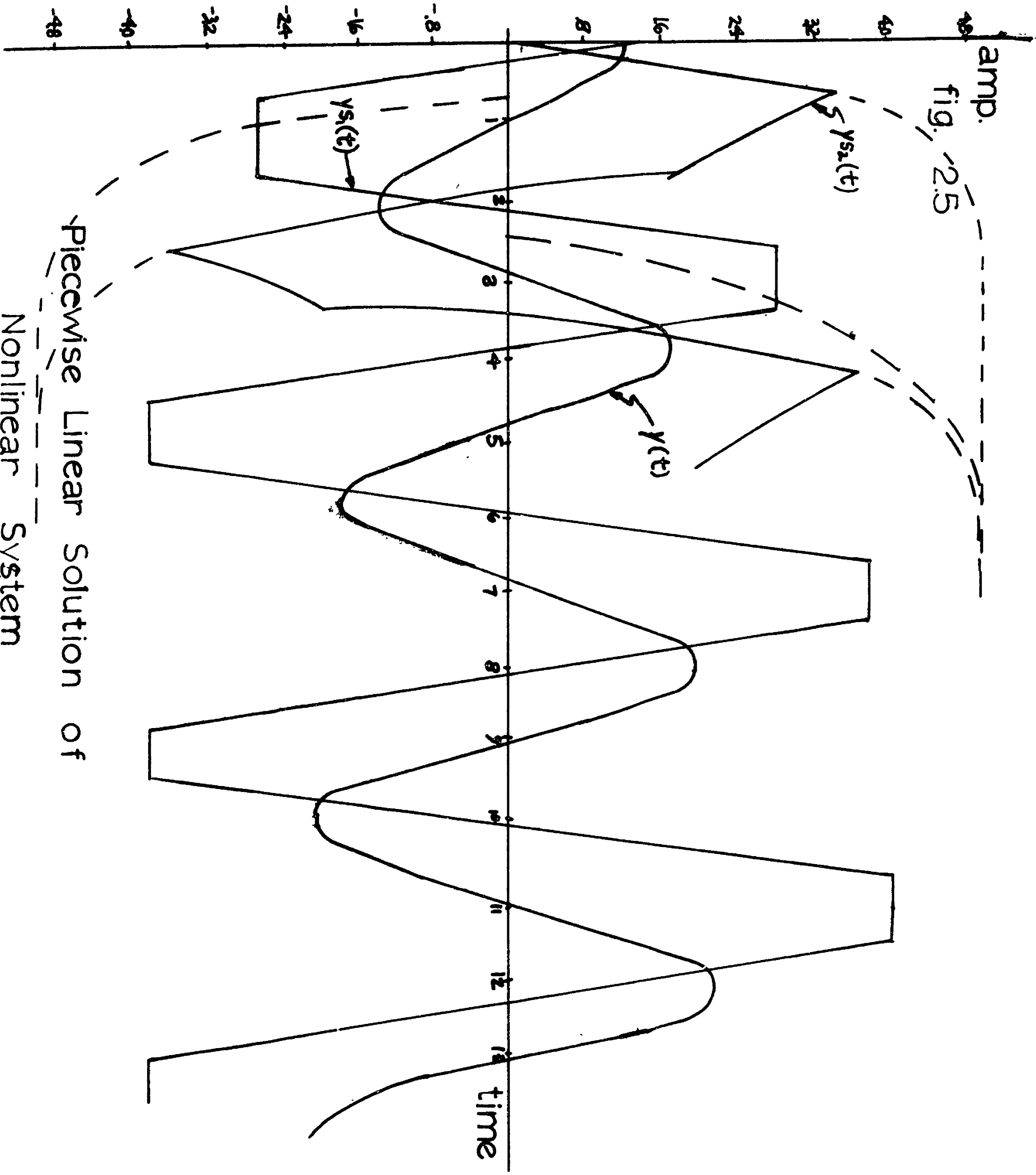


fig. 2.5

Piecewise Linear Solution of Nonlinear System

2.3 COMPARISON WITH COMPUTER RESULTS

This sum was carried out for yo=1.2, k=3, b=1 and KE=4.6 See Figure 2.5. The results show that a limit cycle of amplitude equal to 2 units and a period of (T=4.01 sec) exists.

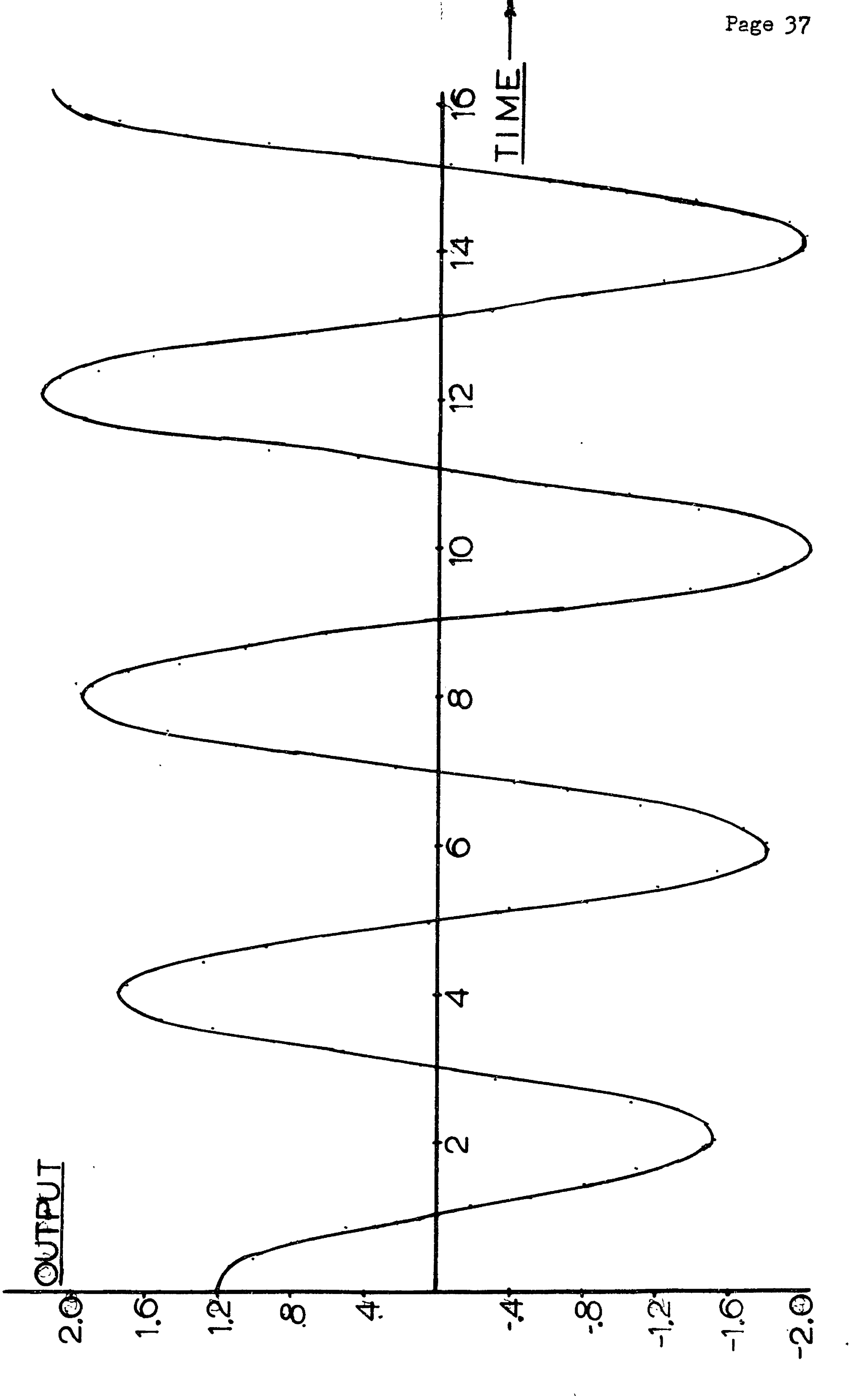
As a further check on these results the system was simulated on a digital computer. (See appendix 1). The program used the differential equation resulting from the linear parts of the system and logic to simulate the switching of the relay. A limit cycle of the same amplitude and frequency resulted. See Figure 2.5. Since the ouput was nearly sinusoidal, the assumption that the linear elements provide effective low pass filtering is verified. This analysis is necessary and serves as a check that the assumptions underlying the describing function analysis are met. Because of this low pass filtering action, it is also possible to derive improvements in systems performance by dithering.

2.4 DESCRIBING FUNCTION

The describing function for a relay can be derived in the following manner. Let the input to the relay be a sine wave of amplitude $\alpha > Q$.

The output of the relay is now a series of pulses. To find the sine and cosine components of the output waveshape we take

$$A_1 = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x \sin \omega t \, dt \qquad (2.8)$$



LIMIT CYCLE - NO INPUT

FIGURE-2.6

and

$$B_1 = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x \cos\omega t \; dt \qquad (2.9)$$

where

$$X(t) = A_1 \sin\omega t + B_1 \cos\omega t + \cdots \qquad (2.10)$$

A_1 and B_1 can now be evaluated for the ouput wave shape

$$A_1 = \frac{2E}{\pi}\int_{\theta_1}^{\theta_2} \sin\theta \, d\theta = -\frac{2E}{\pi} \cos\theta \Big|_{\theta_1}^{\theta_2} \qquad (2.11)$$

$$= \frac{2E}{\pi} (\cos\theta_2 - \cos\theta_1) \qquad (2.11a)$$

Similarly

$$B_1 = \frac{2E}{\pi} (\sin\theta_1 - \sin\theta_2) \qquad (2.12)$$

X (t) can now be expressed in terms of a magnitude and phase.

$$\sqrt{A_1^2 + B_1^2} \quad \sin(\omega t + \phi) \qquad (2.13)$$

By changing angle θ_2 from θ_2 to $\pi - \theta_2'$, a simplification of the results can be affected.

Rememberihg that

$$\cos(\pi - \theta_2') = -\cos\theta_2' \qquad (2.14)$$

$$\sin(\pi - \theta_2') = \sin\theta_2'$$

the magnitude of the describing function can now be put in form of equation

$$\sqrt{A_1^2 + B_1^2} = \frac{4E}{\pi}\left(\cos^2\theta_1' + \cos^2\theta_2' + 2\cos\theta_1'\cos\theta_2' + \sin^2\theta_1' + \sin^2\theta_2' + 2\sin\theta_1'\sin\theta_2' \right)^{\frac{1}{2}} \quad (2.16)$$

Using

$$\cos\theta_1'\cos\theta_2' - \sin\theta_1'\sin\theta_2' = \cos(\theta_1 + \theta_2') \quad (2.17)$$

and

$$2\cos\frac{\theta_1 + \theta_2'}{2} = \sqrt{2\left(1 + \cos(\theta_1 + \theta_2')\right)}$$

we find as equation

$$\sqrt{A_1 + B_1} = \frac{4}{\pi} E \cos\left(\frac{\theta_1 + \theta_2'}{2}\right)$$

The phase can similarly be obtained as

$$\tan\phi = \frac{A_1}{B_1} = -\frac{\sin\theta_1 - \sin\theta_2'}{\cos\theta_1 + \cos\theta_2'} \quad (2.20)$$

$$\phi = -\frac{\theta_1 - \theta_2'}{2} \quad (2.21)$$

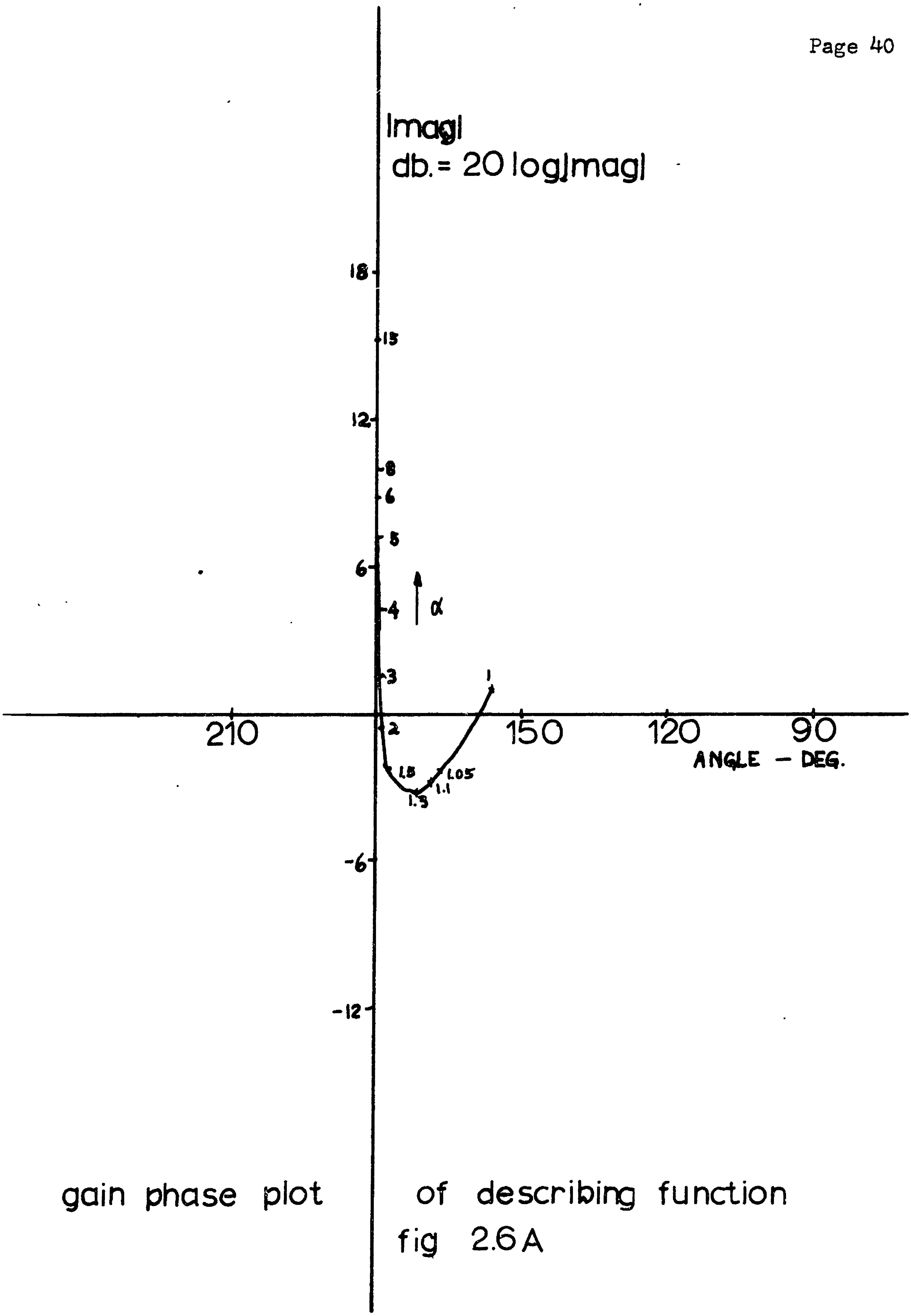
or expressed as the describing function which is

$$N(\alpha) = \begin{cases} 0 & \alpha < Q \\ \sqrt{A_1^2 + B_1^2}/\alpha \, e^{-j\phi} & \alpha > Q \end{cases} \quad (2.22)$$

Now the values of θ_1 and θ_2 are calculated when

$$x \sin\theta_1 = Q \quad (2.23) \qquad \text{and} \qquad x \sin\theta_2 = Q - H \quad (2.24)$$

For system analysis the $\frac{1}{N(\alpha)}$ locus is plotted on the gain phase plane as explained in Chapter 1. The plot of $\frac{-1}{N(\alpha)}$ appears in Figure 2.6a for values of Q=1, H=.25, E=2. Using the same constants as those in the case under study



gain phase plot of describing function
fig 2.6A

in the proceeding section, a stable intersection was found to be at a frequency of 1.6 or perioa of 4.06.seconds and amplitude of 1.8. units. This compares favorably with the previous results.

2.5 MODIFIED DESCRIBING FUNCTION FOR DITHERED NONLINEARITIES

As explained in Chapter 1, the describing function analysis is only successful when the waveshape is known at the input to the nonlinearity. It was shown by the linear piecewise analysis of the system that any signal that is fedback tends to be sinusoidal due to the filtering action of the linear elements. When dither is applied to the system under consideration, the input to the nonlinearity will consist of a signal that is a sine wave of a frequency in the pass band of the linear elements, plus a high frequency sine wave dither. The dither that is fedback around the loop has been attenuated by the low pass elements just as the higher harmonics of the signal frequency have been.

In order to derive a describing function for the dithered relay, we proceed in a manner analogous to the derivation of the regular describing function. We assume the input to consist of $X = S \sin \omega t + d \sin (n\omega t + \alpha)$ where the dither signal can have any phase with respect to the signal frequency.

The output will now be expressed as a Fourier series. Actually the output will consist of a series of pulses with the width determined by the average amplitude of the signal at the point. The Fourier series is expressed as equation 2.23 as with the previous describing function.

$$X(t) = A_1 \sin t + B_1 \sin t + \ldots \qquad (2.25)$$

However, now there is an additional difficulty encountered in determining the pull in and drop out angles θ_1 and θ_2 etc. The equation to be solved for the pull in values is

$$S \sin(\theta_1) + D \sin(n\theta_2 + \alpha) = Q \qquad (2.26)$$

and for the drop out values is

$$S \sin(\theta_2) + D \sin(n\,\theta_2 + \alpha) = Q - H \qquad (2.27)$$

When values of θ_1 and θ_2 are subsituted in the integrals (see equation 2.8, 2.9) $N(\alpha)$ is now just $\sqrt{A^2+B^2}/\alpha\, e^{-j\phi}$ where $A_1 B_1$ and ϕ have the same meaning as before.

In general, A_1 can be expressed as

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} \sin\theta \, d\theta \qquad (2.28a)$$

$$= \frac{1}{\pi}\sum_{n-1}^{N} - \cos\theta \Big|_{\theta_n}^{\theta_{n+1}} \qquad (2.28b)$$

$$= \frac{1}{\pi} - \sum_{n=1}^{N} \cos\theta_{n+1} - \cos\theta_n \qquad (2.28c)$$

where the upper index N is the number of square waves per period of signal frequency.

Similarly,

$$B_1 = \frac{1}{\pi}\sum_{n=1}^{N} \sin\theta_{n+1} - \sin\theta_n \qquad (2.29)$$

The actual values of $\theta_1 \ldots \theta_n$ were found by a trial and error process. Values of θ were substituted in equation 2.26 until a value that satisfied the equation was found. Since there can be as many as 4N trial and error solutions for each group of values of signal, dither and alpha, this computation was performed by a digital computer. For the actual details of the program see appendix 11.

The results of the computer runs show, as expected, that the Describing Function does approach a linear gain. This is seen from the resulting curves because they become independent of the input amplitude.

The overall gain of a dithered relay does become smaller as the dither amplitude is increased due to saturation. However, this gain can be increased by increasing the value of E until the overall gain of the relay is unity for all values of input amplitude.

There are actually three parameters: amplitude, frequency and phase of dither signal (alpha), that can be expected to influence the magnitude and phase of the describing function. The program showed that the effect of frequency ratio (ratio of dither signal frequency) was small. The slight variation in Describing Function was primarily due to limitations of the accuracies of the program. The program used increments of 0.9 degrees. It was found that increasing the increment to .225 degrees resulted in a variation in the third significant figure only.

The variation of describing function amplitude with alpha was found to be approximately 5%. However, as alpha was changed, the describing function picked up about three degrees when the phasing of the dither was 180 degrees out of phase with the signal or alpha was 18 degrees. If the frequency ratio is increased, this effect can be made completely negligible.

The only variation that was found, as noted above, was as the dither amplitude was increased. The effect of dither for small amplitude, about .25, is that it takes out the phase shift introduced by the hysteresis of the nonlinearity. The shape of the magnitude curve is practically the same as that for dither = 0 (Figure 2.7) except for the increased sensitivity in the range 0-1.

The results of the program are plotted in Figures 2.7, 2.8 and 2.9. The first two are plotted as magnitude of describing function vs. input amplitude. The phase was not plotted because the shift is of the order of one degree. The variation that occured was due to inaccuracies in calculating the pull in and drop out values of theta.

The plot of Figure 2.8 and 2.9 is used with a plot of K $(j\omega)$ in the gain phase plane. This serves as a calibration of the gain axis because there is no phase shift for any value of input amplitude.

The modified describing function will be used to analyse the system. A plot of the linear elements transfer function appears in Figure 2.10. By properly adjusting the gain of the linear elements, the system can be made to have the desired time response. The additional parameter that may be adjusted

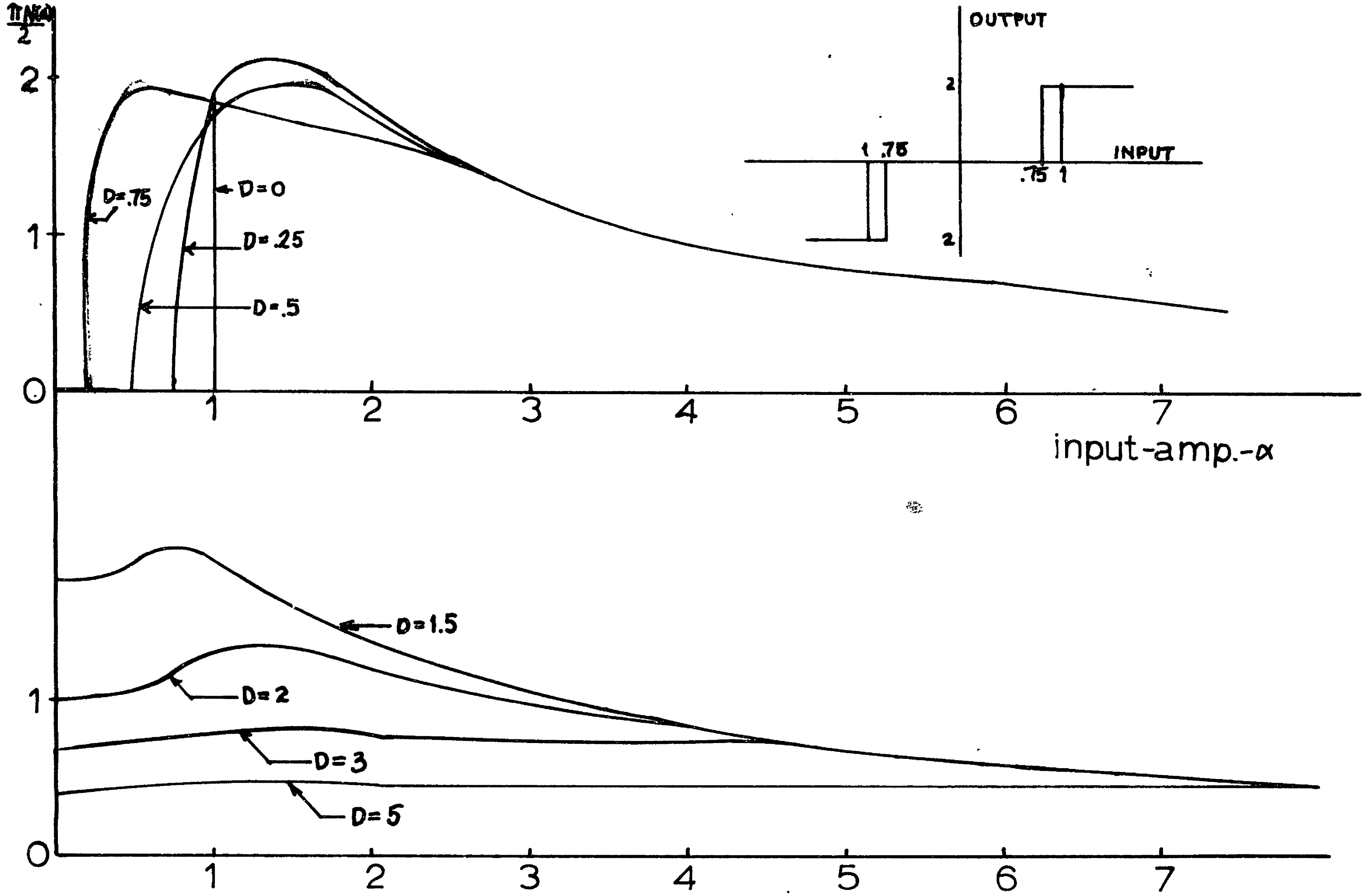
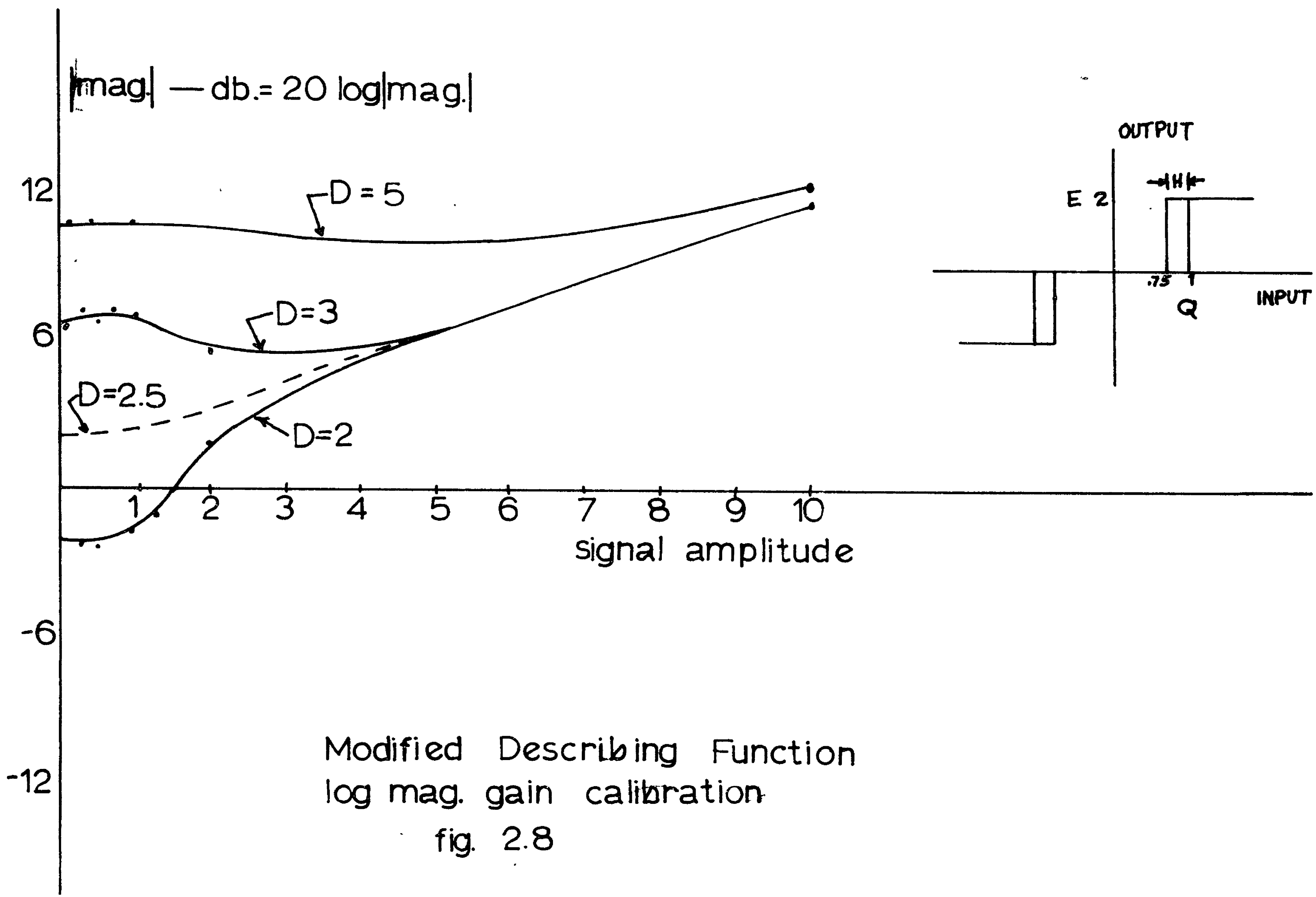
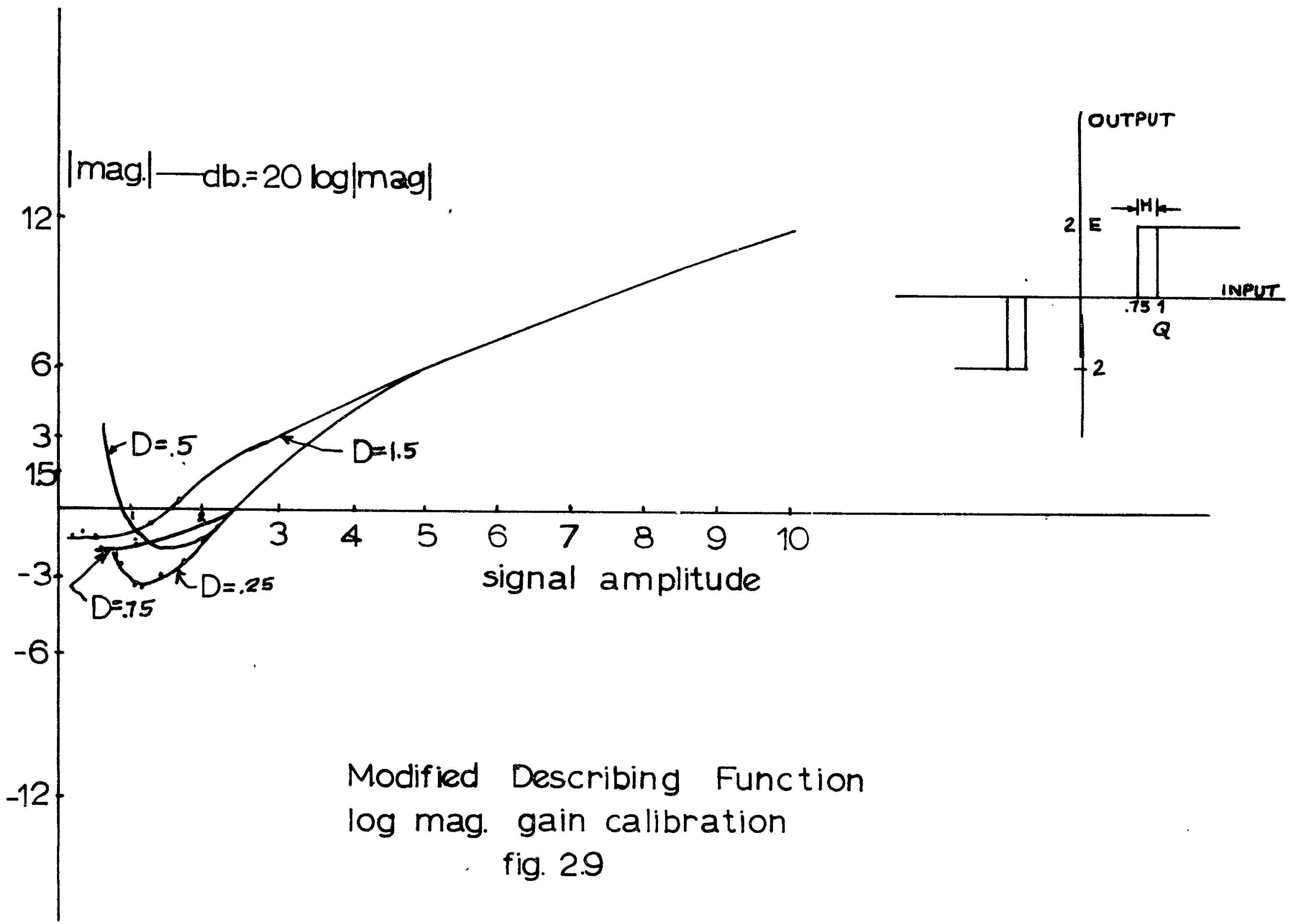


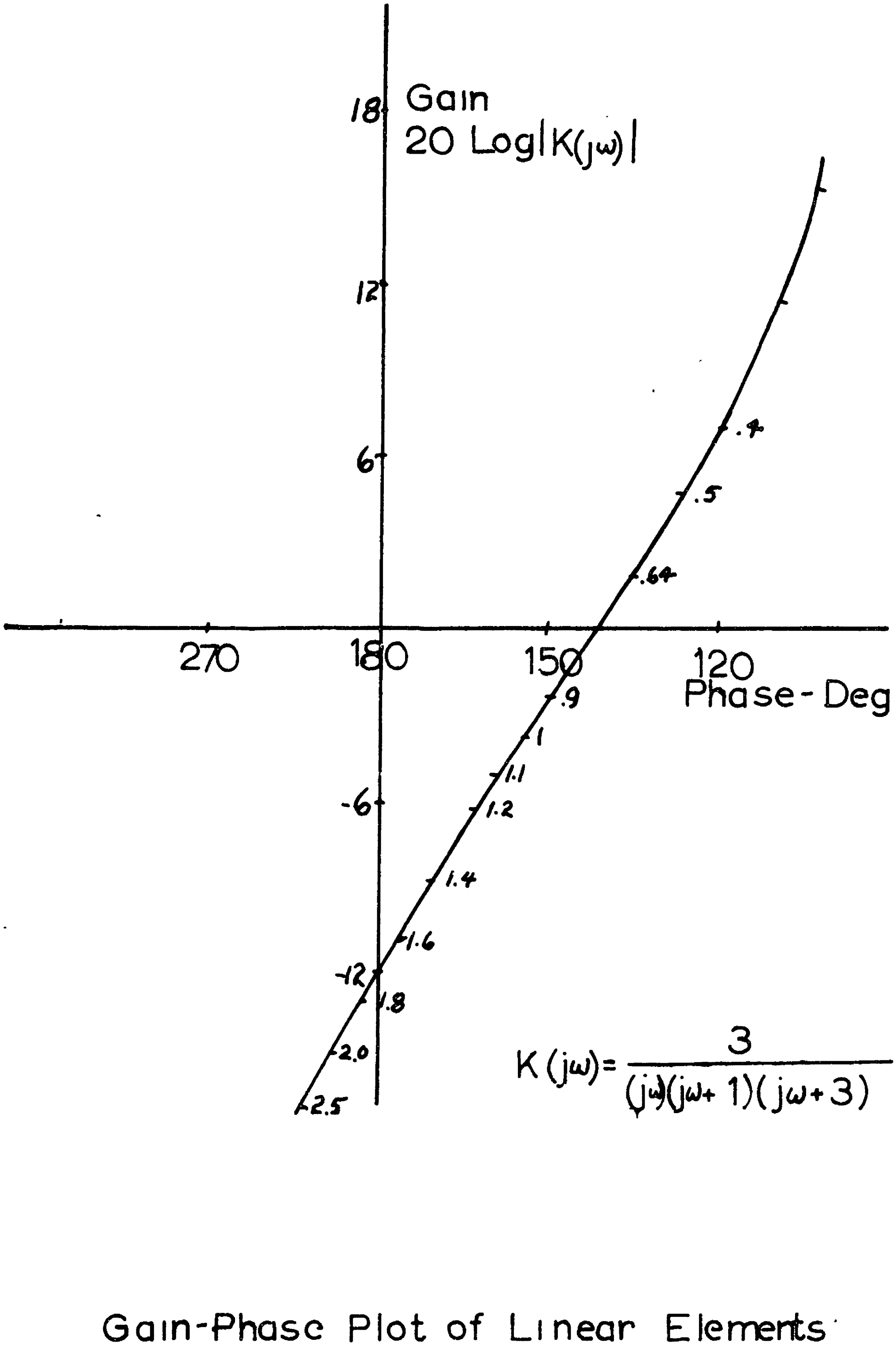
fig. 2.7 Modified Describing Function



Modified Describing Function
log mag. gain calibration
fig. 2.8



Modified Describing Function
log mag. gain calibration
fig. 2.9



Gain-Phase Plot of Linear Elements

FIG-2.10

is the dither amplitude. This may be adjusted so that the system does not limit cycle by moving the $-\frac{1}{N}(\alpha)$ locus back to the gain axis or it may be adjusted so that it behaves completely linearly with only a slight ripple appearing in the output due to the dither.

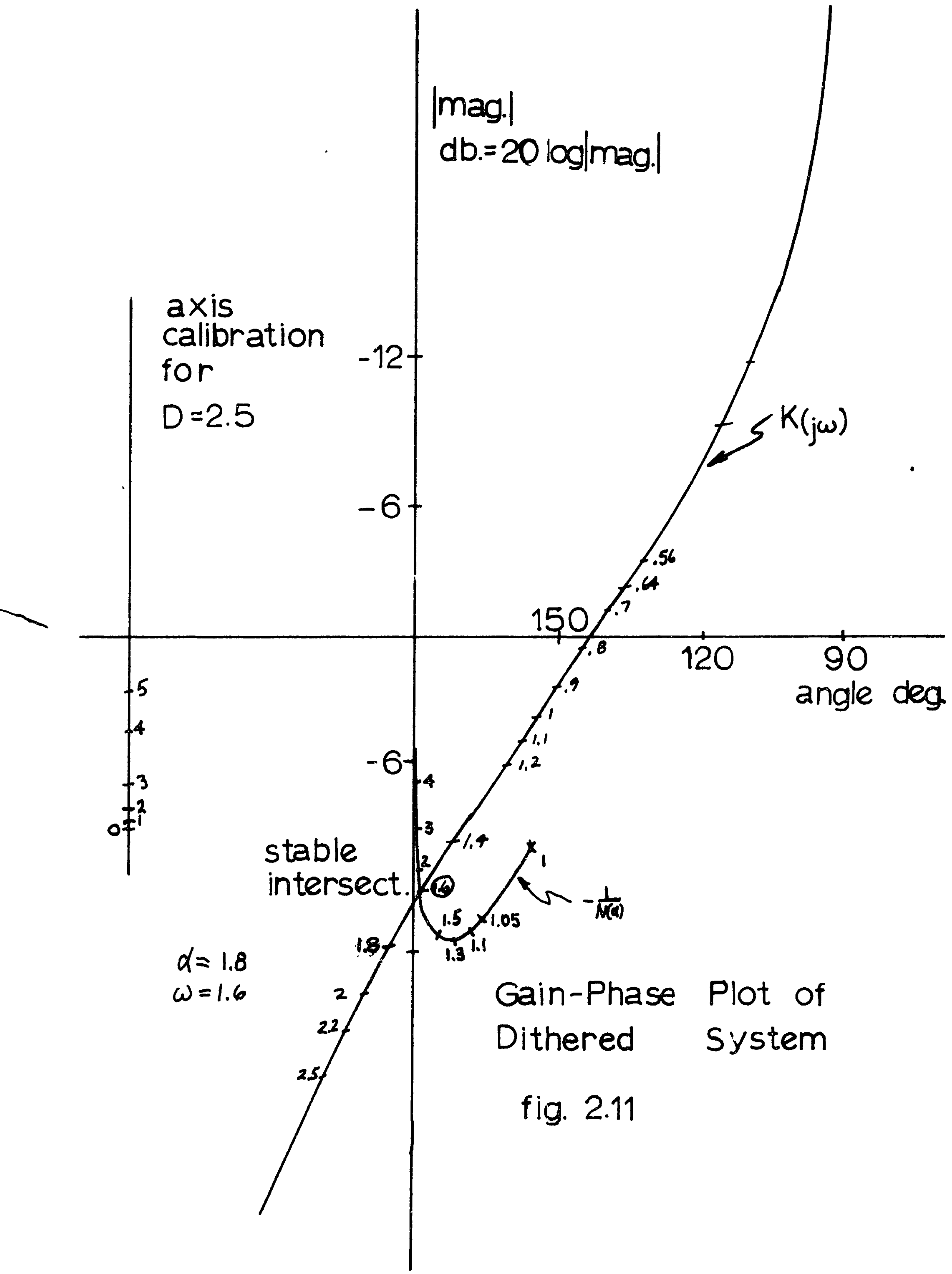
The above methods can now be applied to the analysis of the system described in section 2.4. For the same gain setting as in section 2.4, the system exhibits a stable limit cycle.

This operation is shown in figure 2.11. When the system is dithered, the $-\frac{1}{N}(\alpha)$ curves for the dithered nonlinearity must be used in the analysis. Using the appropriate $-\frac{1}{N}(\alpha)$ curve, (figure 2.8), no intersection was found. This indicates that for a dither of 2.5 the system will be stable. (Figure 2.11 shows the dithered $-\frac{1}{N}(\alpha)$ curve) This curve was plotted as a gain axis calibration to avoid confusing the various curves.

From these curves it is also possible to estimate the range of input signals that will linearize the system. Figure 2.11 shows that for signal values up to 1.5 the variation in the equivalent gain of the system is negligible. The nonlinearity may be thought of as a constant gain for this range of inputs.

2.6 COMPARISON OF RESULTS WITH COMPUTER SIMULATION

The above system was also simulated on the digital computer. (See Appendix I.) The output of the system appears in figure 2.12. The signal is .2 the amplitude and 0.1 the frequency of the dither. The system shows



Gain-Phase Plot of Dithered System

fig. 2.11

that the output has a ratio of signal to dither of 7. This means that the dither has been attenuated 25 times while the signal frequency has been transmitted as though the system were a linear one. As a further check of the modified describing function analysis, the gain and phase shift of the system were calculated at the signal frequency. The transmittion from input to output is equation 2.30

$$T_{io} = \frac{K'}{-j\omega^3 \quad -(A+B)\,\omega^2+(AB)j\omega+K'} \qquad (2.30)$$

where K' is the combined gain of the linear element and the equivalent gain of the nonlinear element.

Substituting the values of frequency $\omega = .7$, the equivalent gain of the nonlinearity/Δ1.1 $\frac{2}{\pi}$ = .7, and the gain K = 8.4 K' becomes (8.4) (.7) = 5.87. T_{io} reduces to equation 2.31

$$T_{io} = \frac{5.87}{3.07 + j1.757} = 1.66 \;\angle -33^\circ \qquad (2.31)$$

Comparing this to the values taken from Figure 2.12 T_{io} is found to be:

$$T_{io} = \frac{.71}{.5} \;\angle -33^\circ = 1.41 \;\angle -33^\circ \qquad (2.32)$$

This compares favorably with the values previously found using the equivalent gain of the nonlinear element.

The computer simulation was also taken for different values of input signal using the same values of dither signal. The purpose of these runs was to verify the linear behavior of the system.

The describing function analysis showed that the gain of the nonlinearity is approximately the same for input amplitudes of 1.5. The computer results

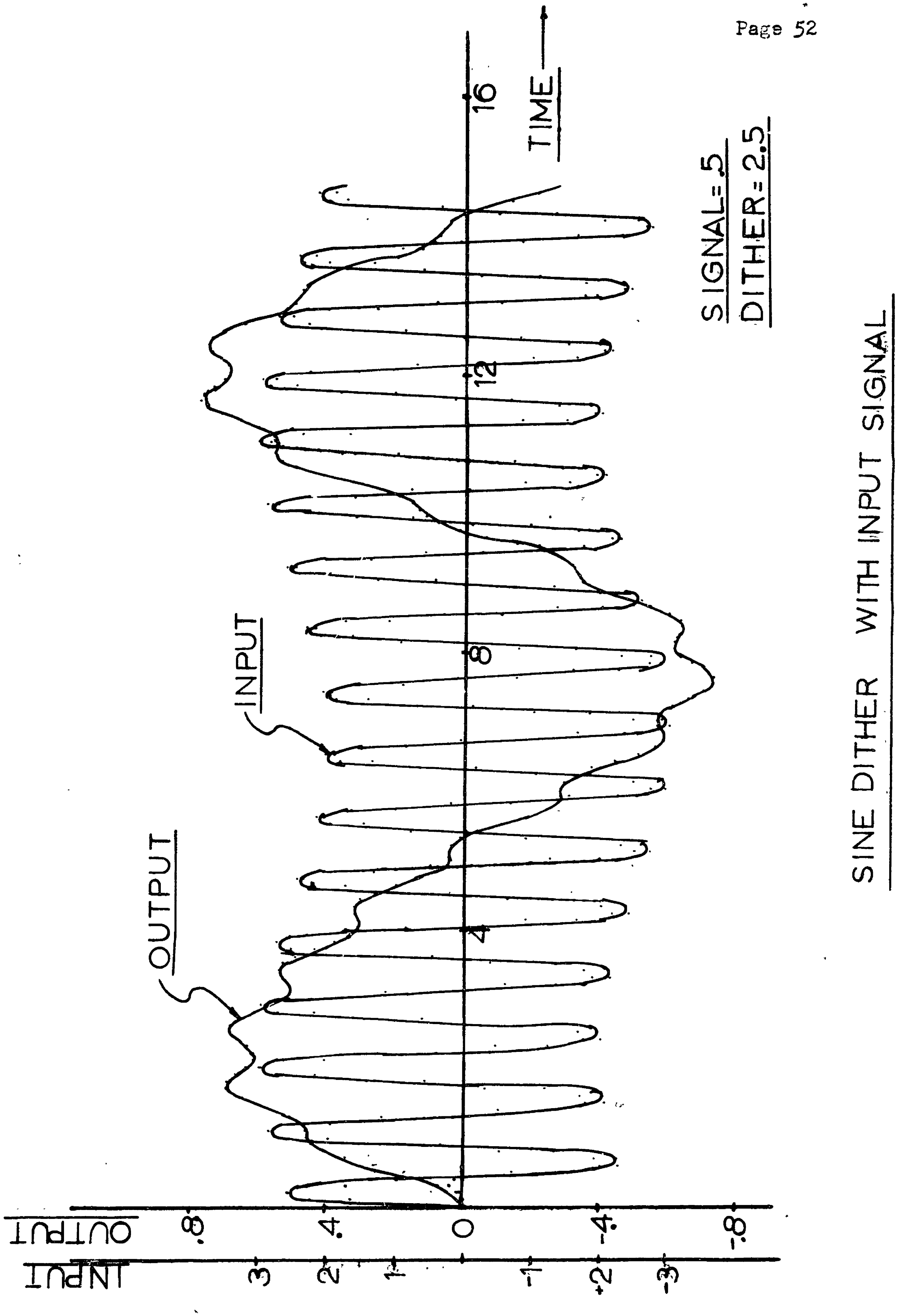
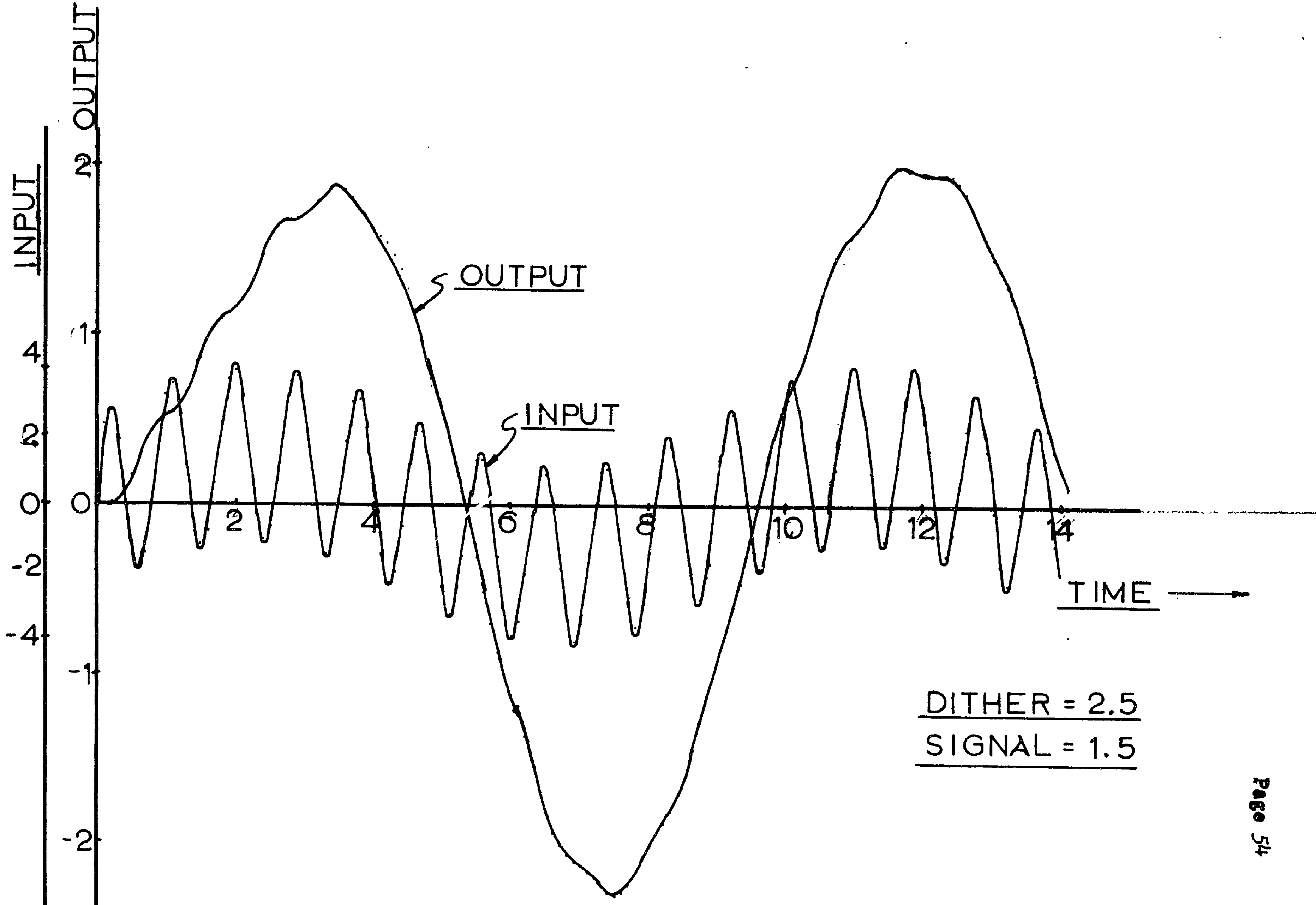

SINE DITHER WITH INPUT SIGNAL
SIGNAL=.5
DITHER=2.5
OUTPUT
INPUT
TIME
4
8
12
16
OUTPUT
.8
.4
0
-.4
-.8
INPUT
3
2
1
-1
-2
-3

FIGURE 2.12

(see Figure 2.13) indicated that the gain of the total system is 2/1.5=1.33. The variation in gain is due to the inaccuracies which arise when measuring the amplitude of the output wave shape.

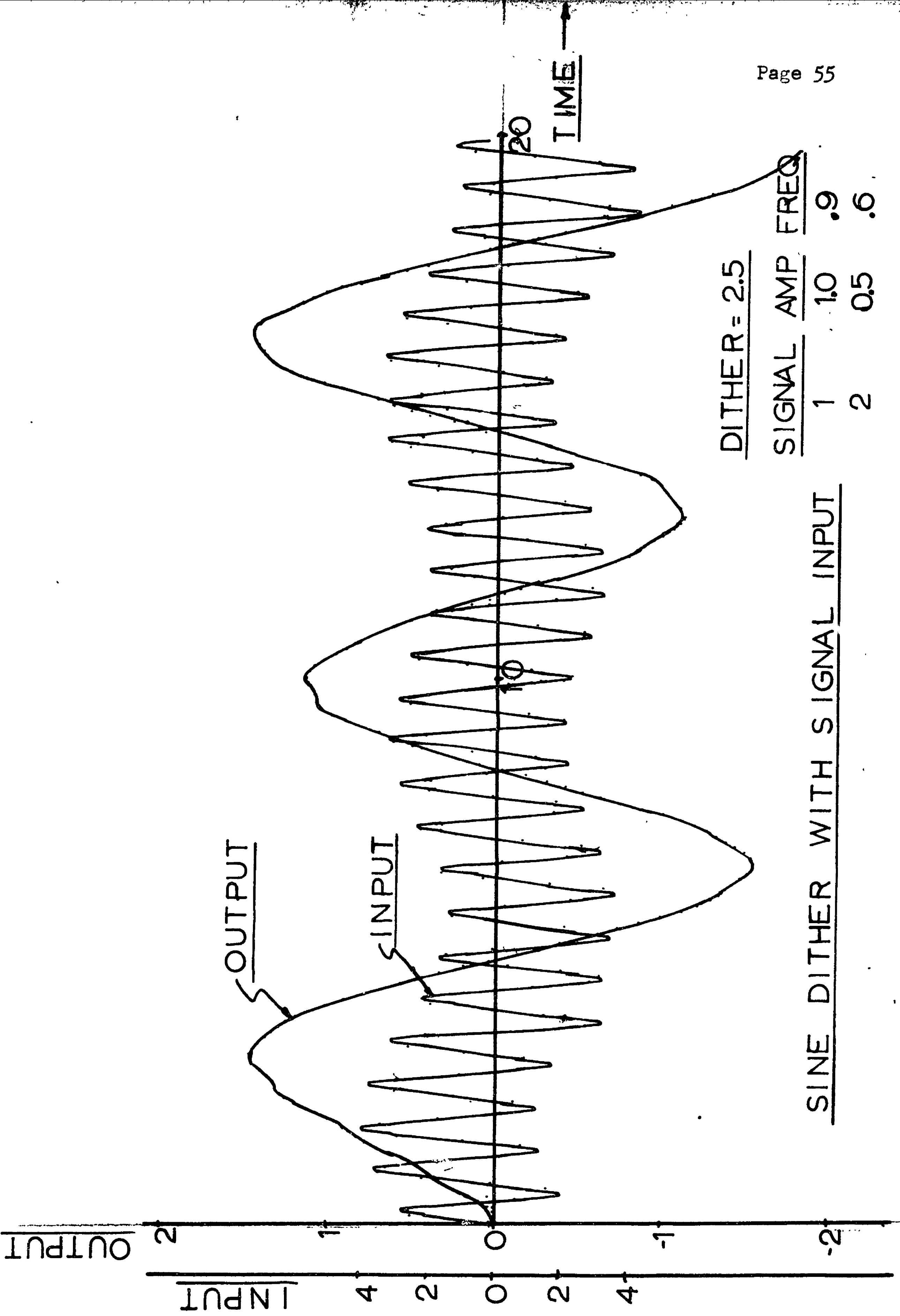
The next computer run (See Figure 2.14) of the system was taken with two sine waves of different frequencies and amplitudes applied simultaneously. The system was also dithered with a sine wave of 2.5. The resultant output was the sum of the two input sine waves.

If the nonlinear element was not linearized by the dither, the output would exhibit components due to the sum and difference frequencies of the two input signals. The frequency of these signals would be $\omega = 1.5$ and $\omega = .3$, respectively. However, examination of Figure 2.15 shows that there is practically no components in the output due to the sum and difference frequencies.



SINE DITHER WITH SIGNAL INPUT

FIGURE 2.13



SINE DITHER WITH SIGNAL INPUT

FIGURE 2.14

CHAPTER 111

STATISTICAL ANALYSIS

The effects of the relay in a control system can be explained by the use of the statistical theory of quantization.

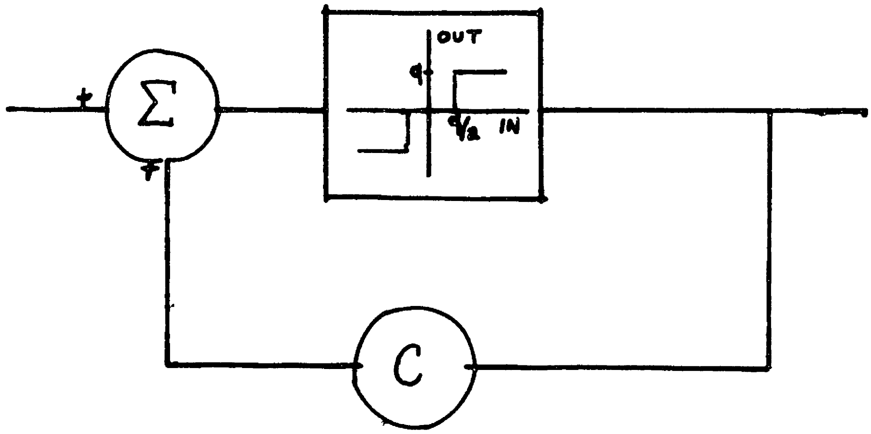
The statistical approach will give a different picture of the same phenomena previously discussed and will also be useful in explaining the effects of dithers other than sine waves.

3.1 INTRODUCTION AND ASSUMPTIONS

The relay can be represented by a quantizer with a positive feedback path around it.[5] The operation of the equivalent system can be seen by examining Figure 3.1.

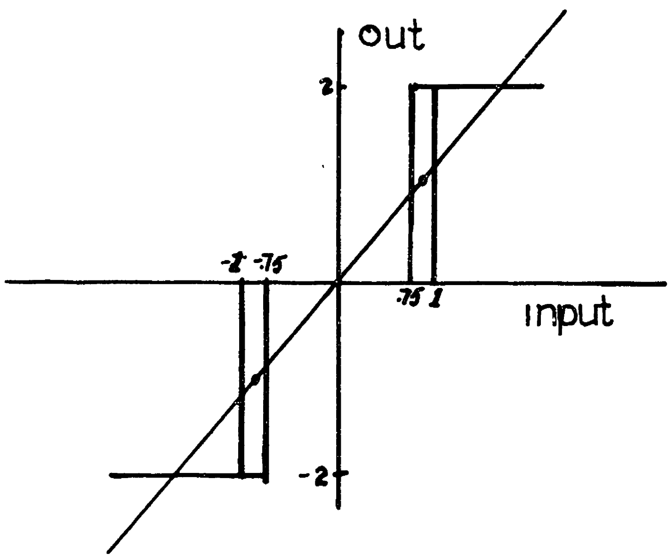
If the input signal is of a level higher than $q/2$, the output will be equal to q. As the level of the input is decreased from $q/2$, the positive feedback will maintain the output. The output will remain at this value until the input is decreased to a point $q/2 - cq$. Then it will become zero. The input - output characteristic is thus seen to be the same as that of a relay.

In order to analyse the dithered system, it is necessary only to know how well the quantizing theorem is satisfied by the dither. If the theorem is satisfied, the quantizer may be replaced by a gain and a noise



Equivalent Representation for a Relay

FIG-31



Equivalent Gain of the Quantizer

FIG-3.2

source and the system can be analysed as a linear one.

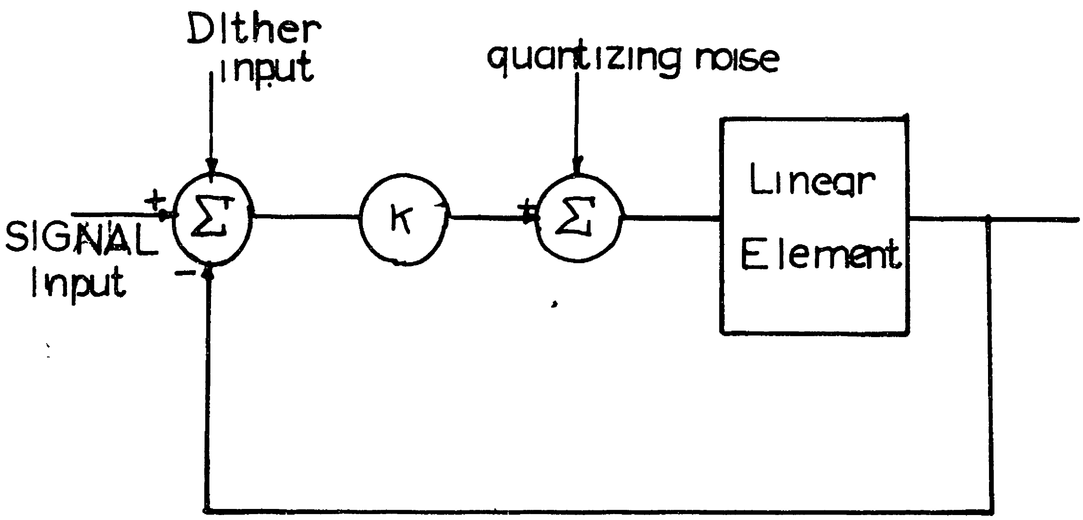
In the statistical description of the quantizer, the inputs and outputs are described in terms of their distribution densities rather than their time wave forms.

A causal signal is a signal that is known for all times; therefore, its statistical description must include all the multidimensional distribution densities. The first and second order distributions are the ones of primary concern because most design information can be obtained from these two densities.

The analysis may begin with the assumption that the quantizing theorem is satisfied, at least in the first order sense. The feedback path around the quantizer will be accounted for by modifying the quantization levels and eliminating this path. Since the statistical analysis will give estimates of the system's behavior, this replacement will simplify the analysis and introduce little error in our estimate of dither amplitude. The equivalent gain of the quantizer is normally taken as the slope of the line that is passed through the midpoints of each of the quantization levels. See Figure 3.2. If the box size is slightly modified to have a value of .875 instead of 1, the gain is increased to 1.14.

The system can now be represented as in Figure 3.3.

This system shows that a signal will propagate as though the system was a linear one. This replacement is only valid if the quantizing theorems are satisfied. In chapter one it was noted that the satisfaction of the



Equivalent System for Statistical Analysis

FIG - 3.3

first order quantizing theorem by a signal; i.e., there was no overlap in characteristic function space, assured us that the moments of the signal were preserved as they passed through the quantizer. The moments describe only the "low frequency" properties of the distribution density. As the distribution density or characteristic function, propagates through the linear elements, the low pass filtering effectively filters the characteristic function and removes the "high frequency" parts. The moments now can describe the distribution density at the output adequately. Thus it is seen that as far as the transmission of the signal through the system is concerned, the moments of the distribution density must be preserved by the quantization process in order to preserve the distribution density of the output.

There is practically no dither signal fedback as the dither has been chosen high enough in frequency to be attenuated by the linear elements. This further simplifies the analysis as the system behaves as though it were a straight cascaded one, with respect to the dither.

It remains, now, to see whether a dither signal can be supplied that satisfies the quantizing theorems.

3.2 SATISFACTION OF QUANTIZING THEOREM BY SINE WAVES

A sine wave dither will now be applied to the system. Since the system has no elements beforethe quantizer, the same amplitude that is applied appears at the input to the quantizer.

Furman showed that in order to satisfy the quantizing theorem, to a good approximation in the first order sense, the ratio of input amplitude to quantizer box size should be about 3. However, it must be remembered that the quantizer representing the relay in this paper is actually a saturating quantizer, i.e., levels larger than $\frac{3}{2}$ q all give the same output. If the quantizer was dithered with an amplitude larger than $\frac{3}{2}$ q, a signal of any amplitude would bring the total signal into the saturation level. Therefore the quantizing theorem will never be able to be satisfied to a very high degree.

The partial satisfaction of the quantizing theorem will usually be enough to insure that the system will approximately linearize for sine wave dithers of about 1 to 2 times the box size of -q.

Furman showed that the second moment is in error by about 30% due to overlap at the origin in characteristic function space. This gives a usable range of dither of about 2 to 4 for the system discussed in section 2.4.

3.3 AUTOCORRELATION OF THE QUANTIZER NOISE

The second order quantizating theorem must also be examined to determine if the equivalent noise is correlated or not. The characteristics of the noise source are easily understood by examining its autocorrelation. The dither input has a periodic autocorrelation that is $\frac{E^2}{2} \cos \omega\tau$ for a sine wave. The noise source will also be periodic with $\omega\tau$. However, as the second order quantizing theorem is more closely met, the noise autocorrelation will become more peaked at the origin and at $\omega\tau = \pm n\pi$ where n is any even integer. Since the input is a sine wave, there are two values per period that have a correlation of 1, i.e., the values for $\omega\tau = \pm n\pi$. The noise, therefore, also is correlated with a value of $q^2/12$ at these points.

Furman's curves, which relate the input correlation to the noise correlation for sine waves, were used to determine the theoretical auto-correlation of the noise. This was plotted (See Figure 3.4) for different values of m, the ratio of input amplitude to grain size. Figure 3.4 shows that, as the quantizing theorem is more closely satisfied by increasing m, the noise becomes more peaked at the origin. It should be noted that the assumption made when plotting these curves is that the first order quantizing theorem is satisfied by the dither signal. However, for the values of m used as a parameter, this is not true, as noted above, and the mean square must be corrected due to overlap at the origin in characteristic function space.

In order to check this result, the noise of the quantization was calculated for the dither of 2.5 and its autocorrelation found. (See Appendix 111) Figure 3.5 shows that the autocorrelation is very close in form to the one calculated for m = 1.5. However, Furman's curves were drawn conservatively to exaggerate the noise correlation and therefore there are differences between the theoretical and experimental curves.

The value of the mean square was found by the experiment to be .565. The value of $q^2/12$ is .333 and the correction to the mean square due to the first order quantiting theorem not being satisfied by the dither is the factor 1.5.

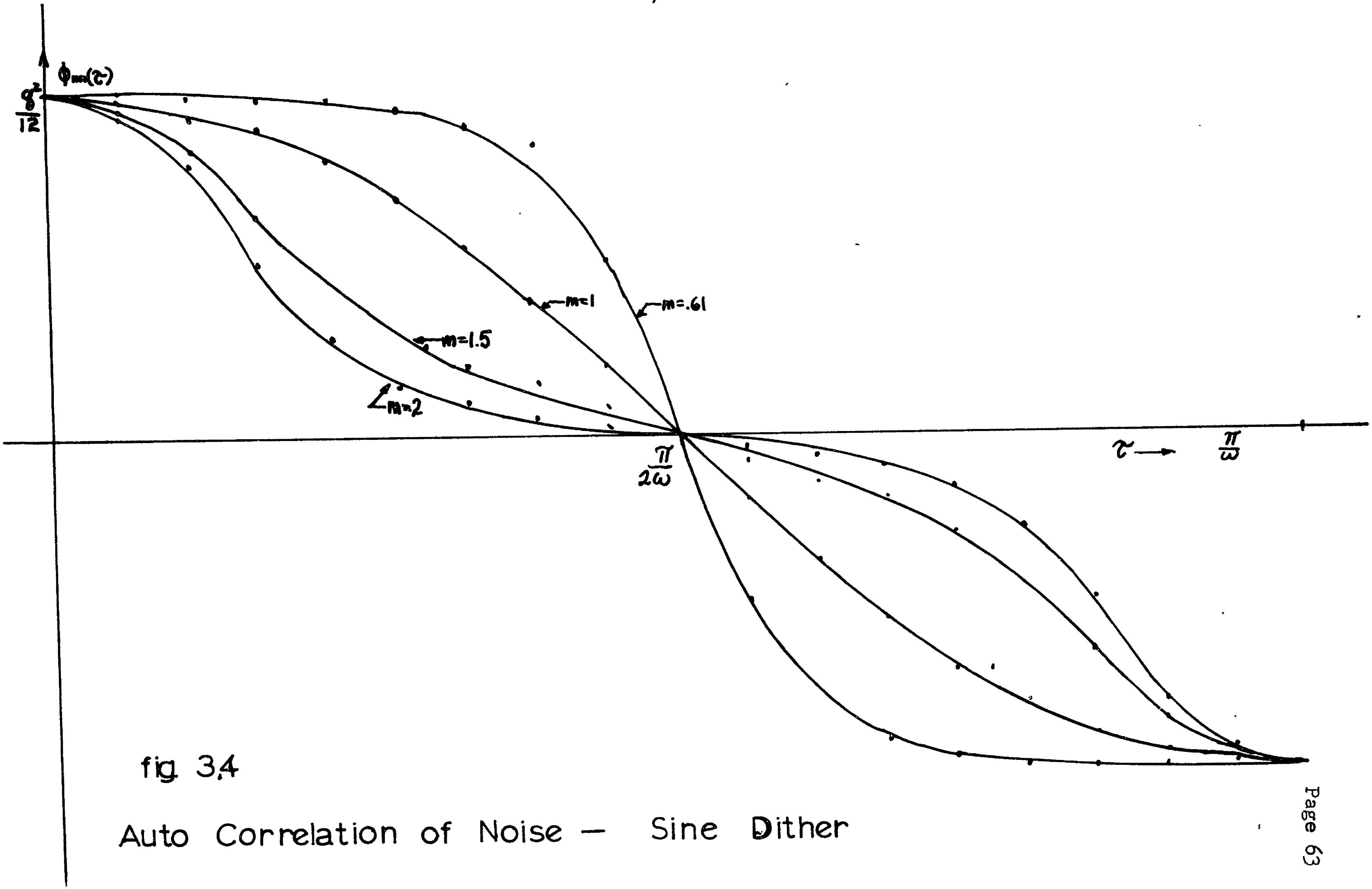


fig. 3.4

Auto Correlation of Noise — Sine Dither

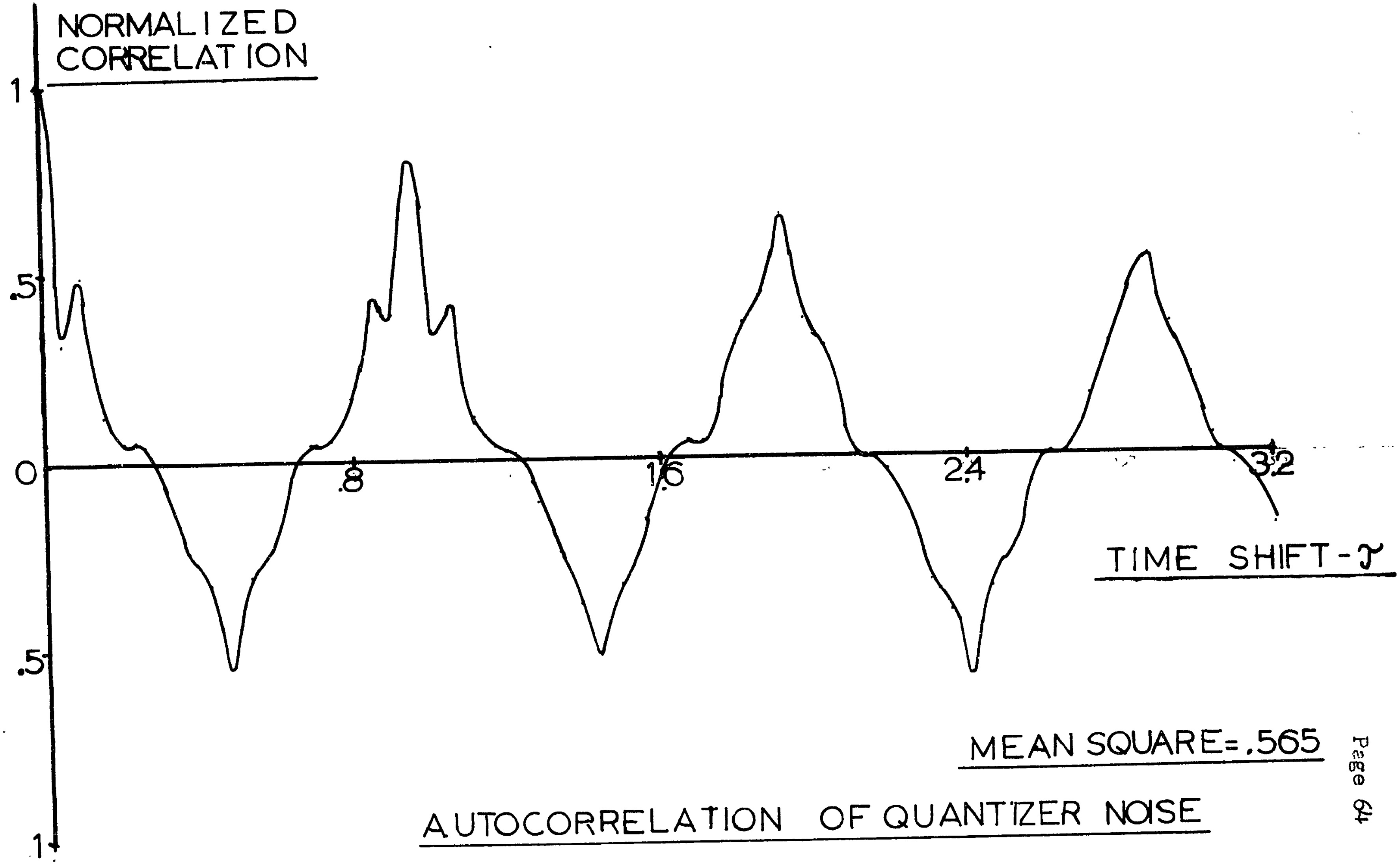

NORMALIZED CORRELATION
1
.5
0
.5
1
.8
1.6
2.4
3.2
TIME SHIFT-τ
MEAN SQUARE=.565
AUTOCORRELATION OF QUANTIZER NOISE

FIGURE 3.5

The theoretical mean square, therefore, becomes 1.5 (.333) = .5. This agrees with the value given by the experiment.

The experimental autocorrelation shows that the noise is of the same fundamental frequency as the dither. This should be expected because the input correlation becomes unity twice each period. The noise correlation, therefore, must also reach a maximum at these points.

Although the autocorrelagram shows that the quantizer noise is correlated, its frequency is either the same or higher than the dither itself. This means that the noise will be attenuated by the linear elements, as will the dither.

If the noise is truly periodic, as stated above, the autocorrelation function should reach the same peak at all time shifts equal to $\pm\frac{n\pi}{\omega}$ where n is an integer. However, attenuation for large time shifts was found. (See Figure 3.5) This was due to the way the autocorrelation was computed. Since a limited amount of data was used to calculate the autocorrelation, the summation of equation was divided by N+1 instead of N-K. For large time shifts this will make the variation of the autocorrelation small. This is done to prevent erratic fluctuations of the autocorrelation due to small sample size, for large time shifts.

The result of the statistical analysis may now be summarized. By using sine wave dither of amplitudes 2 - 4 units, the system may be linearized. The noise source which replaces the quantizer will be approximately flat topped (quantizing Theorem then is satisfied to about 30% in second moments). However, as seen by the autocorrelagram, it will not be first order. The noise, nevertheless, will be filtered by the system as it is of the same frequency as the dither, which is itself filtered out by the linear elements.

The values of dither found agree with the experimental runs reported in chapter II. See Figures 2.12, 2.13 and 2.14. Figure 2.12 and 2.13 show that with a dither of 2.5 the system behaves linearly, i.e., an increase in the input produces a proportional increase in the output. The equivalent gain of the system, when the first order quantizing theorem is satisfied, is 1.14, as shown above. This is a higher value of gain than the one given by the causal analysis. The corresponding overall gain is 1.21. See equation 2.30. If this value is compared with equation 2.31, it is seen that the statistical analysis gives higher values of gain than those found causally. These two analyses give the upper and lower bound on the range of equivalent gain. The actual gain found by experiment was midway between the two.

Figure 2.14 is easily explained by a statistical arguement. When the quantizing theorems (first and second order) are approximately satisfied, the system behaves as if a linear gain replaced the quantizer. The quantizer noise source should be uncorrelated or at least contain predominately periodic components of the dither frequency. When other signals are added, the statistics of the quantization noise and the equivalent of the quantizer should remain substantially unchanged. This is demonstrated by the output plotted in Figure 2.14. This shows that the output contains only the input frequencies plus the dither.

The next dither to be considered is Gaussian.

3.4 FIRST ORDER GAUSSIAN DITHER

Widrow has shown[5] that for quantization levels as rough as $\sigma = q/3$ the error in the output mean square is 13 per cent. Using sine wave dither, the same error in mean square occurs when the amplitude is equal to 2q.

One of the features of Gaussian dither is that it is first order and therefore produces uncorrelated noise. The Gaussian dither will be attenuated by the system because its power spectrum is very broad.

Two experiments were performed to find the minimum values of Gaussian dither that could be used to eliminate limit cycles and linearize the system. (See Appendix I.) First order Gaussian dither was generated by summing 12 random flat topped dictributed noises. The output noise approached a Gaussian to a very good approximation. A small sampling time was chosen (one time increment). The Gaussian noises, therefore, were uncorrelated after one time increment.

Figure 3.6 shows that with a sigma of 1, the limit cycles are suppressed. It should be noted that there is negligible amount of

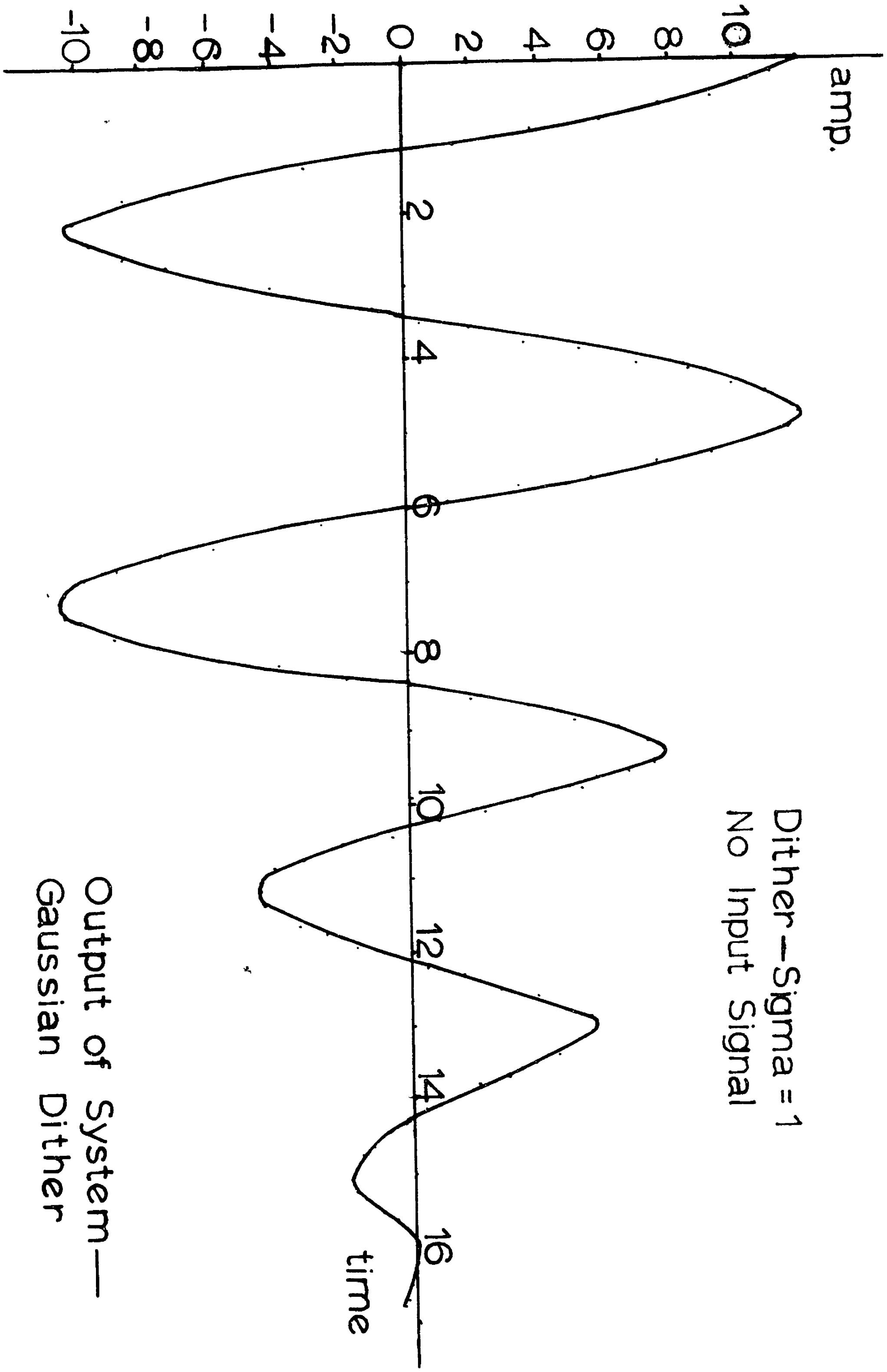
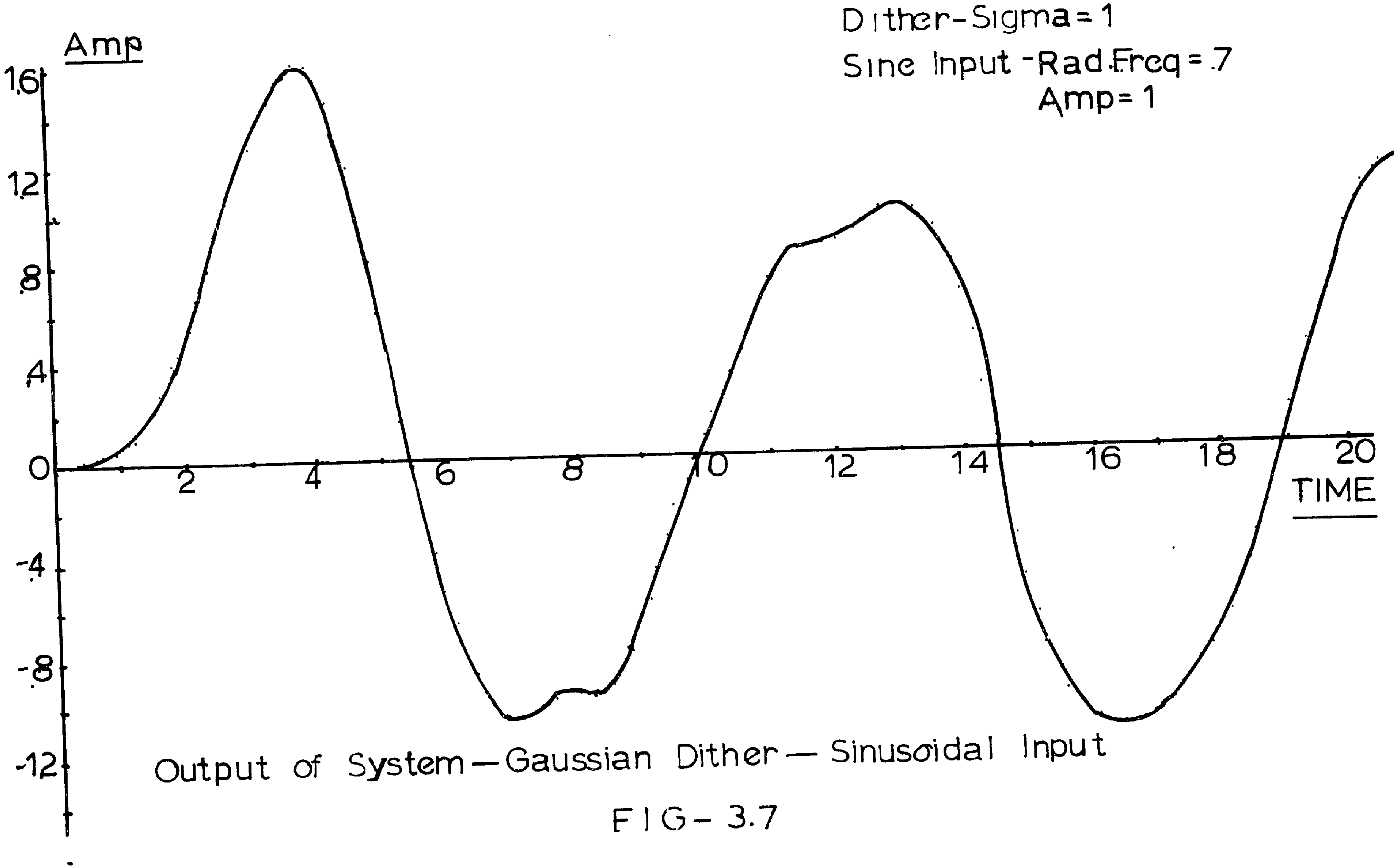


fig.36

Output of System—
Gaussian Dither



Output of System — Gaussian Dither — Sinusoidal Input

FIG- 3.7

dither component in the output.

Although the first order quantizing theorem is not fully satisfied, (mean square is in error by 13%), the system nevertheless will follow the input. This is demonstrated in Figure 3.7 where the variance of the dither is 3/8 of a quantum level.

In summary: the first order quantizing theorem has to be satisfied by at least 20 or 30% in the mean square by either sinusoidal or Gaussian dither to prevent limit cycle phenomena and to linearize the system.

CHAPTER IV

SUMMARY AND CONCLUSIONS

In the preceeding pages two approaches were used to describe the same phenomena. The values of dither amplitude necessary to linearize the system and prevent limit cycle were determined.

The causal approach dealt with the deterministic properties of the sine wave dither. It was shown that by deriving a modified describing function it is possible to determine the gain of the system needed to eliminate limit cycles. A digital computer simulation of the system verified these results.

In order to explain how the system was linearized, a statistical description of the dither was employed. It was found that the system behaved approximately linearly when the first order quantizing theorem was satisfied to about 30% in the mean square. By determining that the dither quantizer could be replaced by a linear gain, with respect to the propagation of the moments, this behavior was able to be predicted. Due to the filtering of the linear elements, the moments described the distribution density of the signal at the output of the system. The dither and the noise generated by the quantizer were filtered by the linear elements. Therefore, the output contained predominately input signal components. This, too, was verified by a computer simulation.

Gaussian dither was considered and the results of the analysis verified by a dither computer simulation. It was found that for sigmas as low as 3/8 of a quantum level, limit cycles are prevented.

APPENDIX 1

PROGRAM TO SIMULATE A NONLINEAR SYSTEM

The system of figure 2.1 was simulated on the IBM 650 using the following program. The conversion program that was used is called MAC (Machine Algebra Conversion).

MAC allows the programer to write a program using several statements. Some of the statements and format are explained below.[8]

1) An algebraic statement is written with the variable to be computed on the left hand side of the equal sign and the other variables on the right. The format follows the rules of algebra. All the variables on the right hand of the equal sign side must be known at the time the equation is reached in the program.

2) The Read and Punch statements are self explanatory.

3) The "Go to " statement can be of two forms:

   a) Go to N indicates that the next step of the program is equation N and the program will continue from there on.

   b) A "Computed go to" has an expression after the "go to". If the expression is equal to zero, it goes to the first equation on the list, one, the second equation, etc.

4) The "If" statement is a conditional statement. If the condition of the statement is met, it executes the order after the comma. If not, the program ignores the statement.

5) Differential equations may also be written as statements. The

highest derivative appears to the left as the algebraic statements. (For exact operation of this statement see appendix B of "Short Guide to MAC".) It is sufficient for the purposes of this discussion to state that with proper increments of the independent variable, accuracies of seven places may be obtained. The "Difiq" controls the numerical approximation process used in the solution of the differential equation.

For further information regarding this, the reader is referred to "Short Guide to MAC".

A block diagram of the system appears in Figure A.1, and the MAC program appears in Figure A.2.

The input can be selected by setting "IMP" on the DATA card to values found in table A.1 and Figure A.3. Similarly, a dither signal may be chosen. The output will appear in tabulated form. For a sample output see Figure A.4.

Initial conditions are set in by using table A.2. A sample input-DATA card print out appears in Figure A.4.

The actual working of the program can be understood by following Figure A.1.

The program begins by reading in the data and setting up output format. After initial conditions are set up, the program finds the input and sums it with the previous value of the output to obtain an error signal. This error signal then is used to select a value for X by a series of "if" statements.

The relay is simulated by a series of logical statements that determine the relative value of the error and choose appropriate outputs. These outputs

Block Diagram for Nonlinear System Simulation

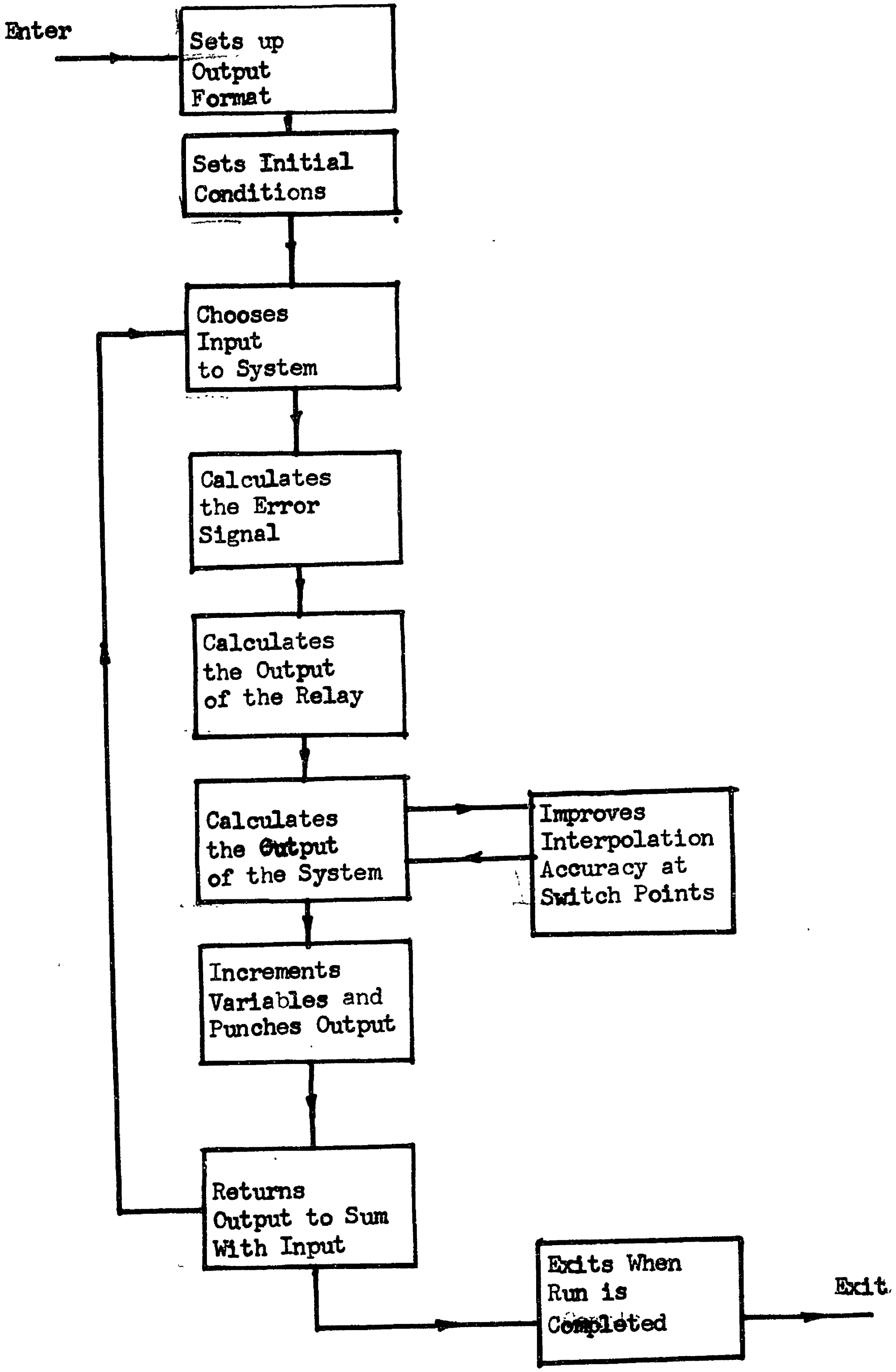


FIGURE A.1

```
10000 SYSTEM WITH NONLINEARITY          0000 08-24-59
M0010 READ E,H,G,K,A,B,IMP,AMP,DIT,          1
M0020 VAL,N,ALPHA,SIGMA,INCT,DOS,
M0030 STOP,INCY,INCDY,OMEGA,FRES
M0040 PUNCH MSG,SP2
M0050 SYSTEM WITH NONLINEARITY
M0070 PUNCH E,H,O,K,A,B,IMP,AMP,DIT
M0080 ,VAL,N,ALPHA,SIGMA,INCT,DOS,
M0090 STOP.INCY,INCDY,OMEGA,FRES
M0100 PUNCH HDG,SP2
E0110                          .
M0120   T     I    ERR    Y    Y    /
M0130 X=0
M0140 Y=INCY
M0150 DY/DT=INCDY
E0160  2    2
M0170 D Y/DT =0
M0180 T=0
M0190 DT=INCT
M0200 I=0
M0210 COUNTER =0
M0220 INTP=0                               2
M0230 GO TO (IMP-1),3,4,5,6,555
M0240 I=AMP                                3
```

Figure A2a

```
10000 SYSTEM WITH NONLINEARITY          0000 08-24-59
M0250 GO TO 6
M0260 I=AMP T                                      4
M0270 GO TO 6
M0280 I= AMP SIN(OMEGA T )                         5
M0285 GO TO 6
M0286 I=AMP SIN(OMEGA T) + POS SIN(               555
M0287 FREQ T)
M0290 GO TO (DIT-1),7,9,8                          6
M0300 I=I+VAL SIN( T+ALPHA)                        7
M0310 GO TO 9
M0320 I=I+VAL RNDM (SIGMA)                         8
M0330 ERR= I - Y                                   9
M0340 IF ABS(ERR)-C POS, GO TO 14
M0350 IF ABS(ERR)-C+H POS, GO TO 12               10
M0360 GO TO 13                                    11
M0370 IF X  NZ, GO TO 14                          12
S0380       B
M0390 X=0                                         13
M0400 GO TO 125
M0410 X = SGN(ERR) F                              14
M0415 IF T ZERO,GO TO 15                         125
M0420 IF X -X ZERO,GO TO 15
S0430       B
```

Figure A.2b

```
10000 SYSTEM WITH NONLINEARITY          0000 08-24-59
M0432 IF ABS(ERR)-ERR  ZERO,GO TO
S0433                B
M0434 15
M0440 INTP=1
M0450 X=X
S0460    B
M0470 IF X ZERO,GO TO 135
S0480      b
M0490 DT=-INCT(Q-H-ABS(ERR)/ERR -
S0500                             B
M0510 ABS(ERR))
M0520 GO TO 115
M0530 DT=-INCT(ABS(ERR)-Q/ABS(ERR)-   135
M0540 ERR )
S0550    B
M0560 INTP=INTP+1
M0570 GO TO 115
M0580 X =X                             15
S0590  B
E0600  3    3          2    2
M0610 D Y/DT =-(A+B)D Y/DT -ABDY/DT
M0620 +KX                              115
M0630 GO TO (1-INTP),155,145
M0640 ERR =ABS(ERR)                    145
S0650    B
M0660 PUNCH T/53,I/53,ERR/53,Y/53,
```

Figure A.2c

```
10000 SYSTEM WITH NONLINEARITY        0000 08-24-59
M0670 DY/DT  53,X/50
M0680 IF COUNTER NZ, GO TO 215        155
M0690 IF T -STOP POS, GO TO 16
M0700 IF COUNTER ZERO,COUNTER = 4     215
M0710 COUNTER =COUNTER -  1
M0720 DIFEQ T,DT
M0730 IF COUNTER NZ,GO TO 115
M0740 GO TO (4-INTP),195,195,165,
M0750  175,2
M0760 X=0                             165
M0770 GO TO 185
M0780 X=SGN(ERR) E                    175
M0790 DT=-DT                          185
M0800 INTP=INTP+2
M0810 GO TO 15
M0820 DT=INCT                         195
M0830 GO TO 2
M0840 GO TO 1                          16
M0850 START AT 1
```

Figure A.2d

VALUE OF IMP	SELECTS INPUT	EQUATION	FIGURE
1	step	AMP u(T)	A.3a
2	ramp	AMP T	A.3a
3	sine	AMP Sin (Omega T)	A.3a
4	no input		
5	two sine	AMP Sin (Omega T) + Sin (Fres T)	A.3a

TABLE A.1

VALUE OF DIT	SELECT DITHER SIGNAL	EQUATION	FIGURE
1	sin wave		A.3b
2	no dither		
3	gaussian 1st order		A.3c

TABLE A.2

INTITAL CONDITION	VARIABLE	EQUATION	FIGURE
INCY	Y		
INCDY	DY/DT		
INCT	Time increments		
Stop	Lenght of time run		

TABLE A.3

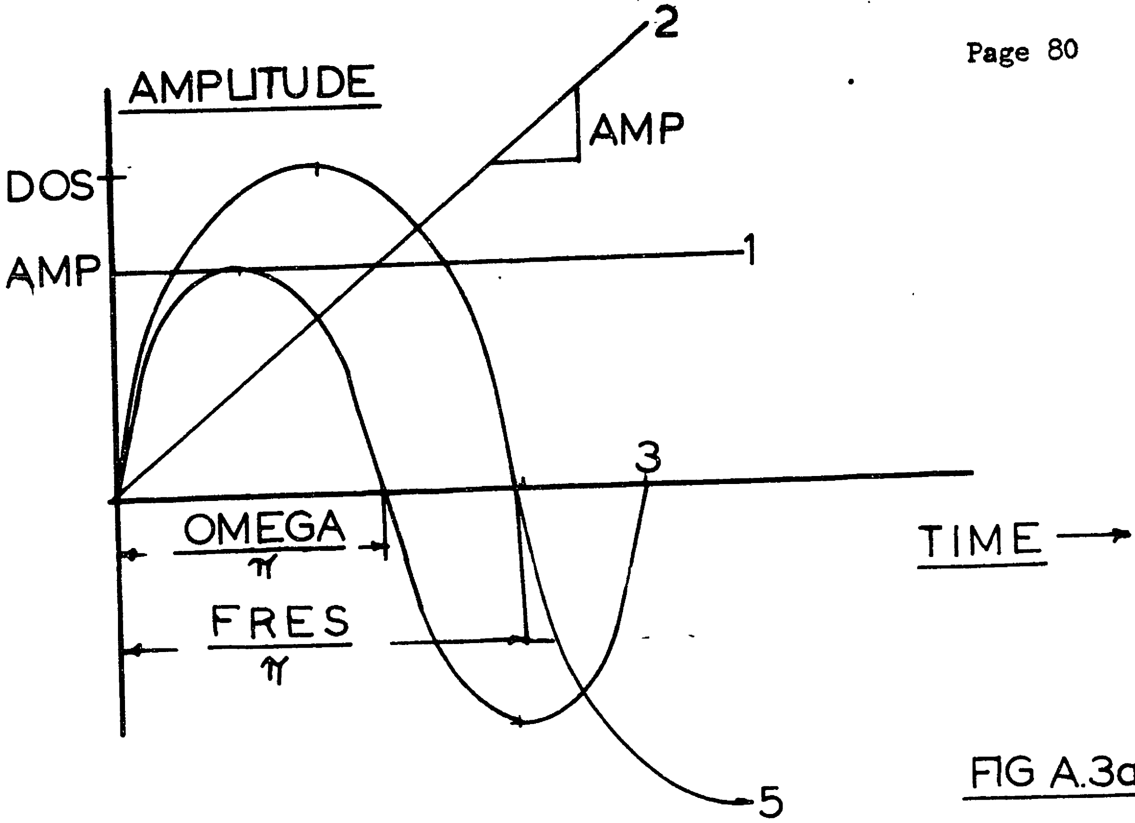


FIG A.3a

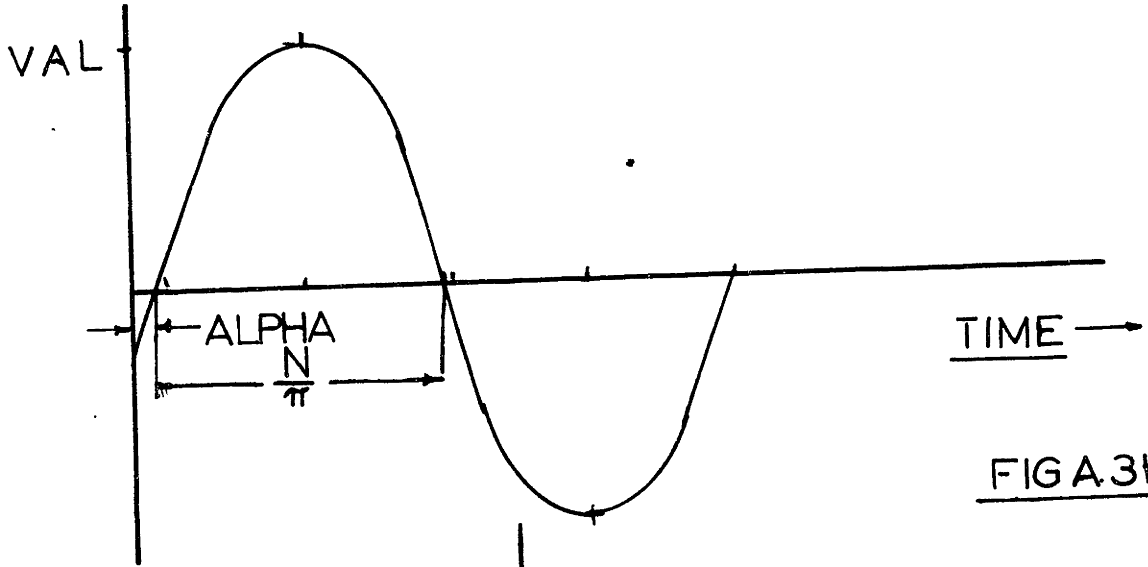


FIG A.3b

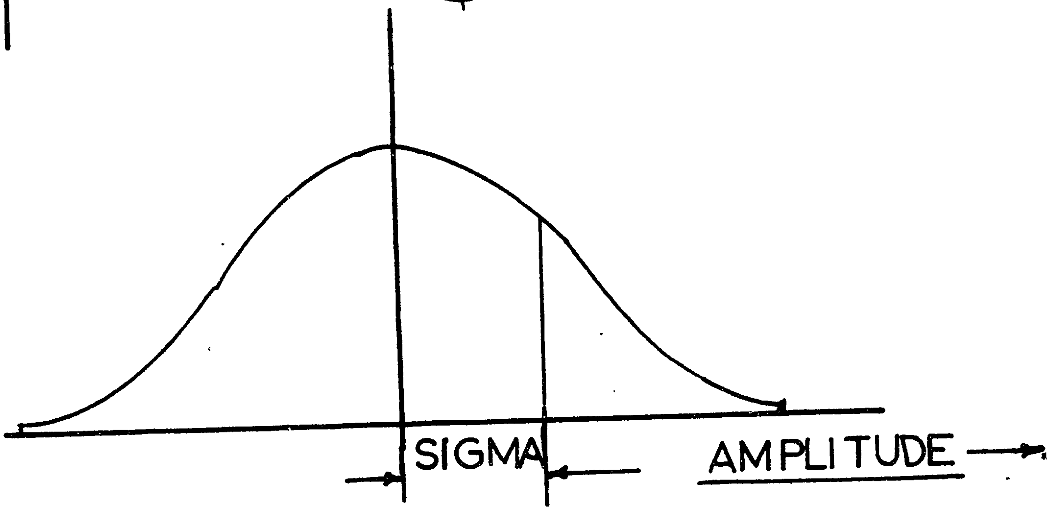


FIG A.3c

INPUTS AVAILABLE FOR SYSTEM

10000 SYSTEM WITH NONLINEARITY   0000 08-24-59

SAMPLE INPUT

	20000000 51	25000000 50	10000000 51	84000000 51	30000000 51	10000000 51
1	30000000 51	15000000 51	10000000 51	25000000 51	70000000 51	
2		50000000 49		15000000 52		
3	70000000 50					

SAMPLE OUTPUT

SYSTEM WITH NONLINEARITY

1	20000000 51	25000000 50	10000000 51	84000000 51	10000000 51	30000000 51
2	40000000 51	20000000 51	10000000 51	10000000 51		10000000 51
3	80000000 49	16000000 52	12000000 51		10000000 51	

	T	I	ERR	Y	$\dot{Y}$	X
4			-00120000 53	00120000 53		20000000 51
5	00008000 53		-00119868 53	00119868 53	-00004840 53	20000000 51
6	00016000 53		-00119020 53	00119020 53	-00017461 53	20000000 51
7	00024000 53		-00116931 53	00116931 53	-00035526 53	20000000 51
8	00032000 53		-00113239 53	00113239 53	-00057248 53	20000000 51
9	00040000 53		-00107710 53	00107710 53	-00081269 53	20000000 51
10	00048000 53		-00100203 53	00100203 53	-00106565 53	20000000 51
11	00056000 53		-00090647 53	00090647 53	-00132372 53	20000000 51
12	00064000 53		-00079025 53	00079025 53	-00158127 53	20000000 51
13	00072000 53		-00065359 53	00065359 53	-00183421 53	
14	00080000 53		-00049830 53	00049830 53	-00203127 53	
15	00088000 53		-00033090 53	00033090 53	-00214104 53	

Figure A.4

are then used to solve equation A.1

$$\frac{d^3y}{dt^3} = -(A + B)\frac{d^2y}{dt^2} - AB\frac{dy}{dt} + K x \qquad (A.1)$$

by an approximate numerical technique.

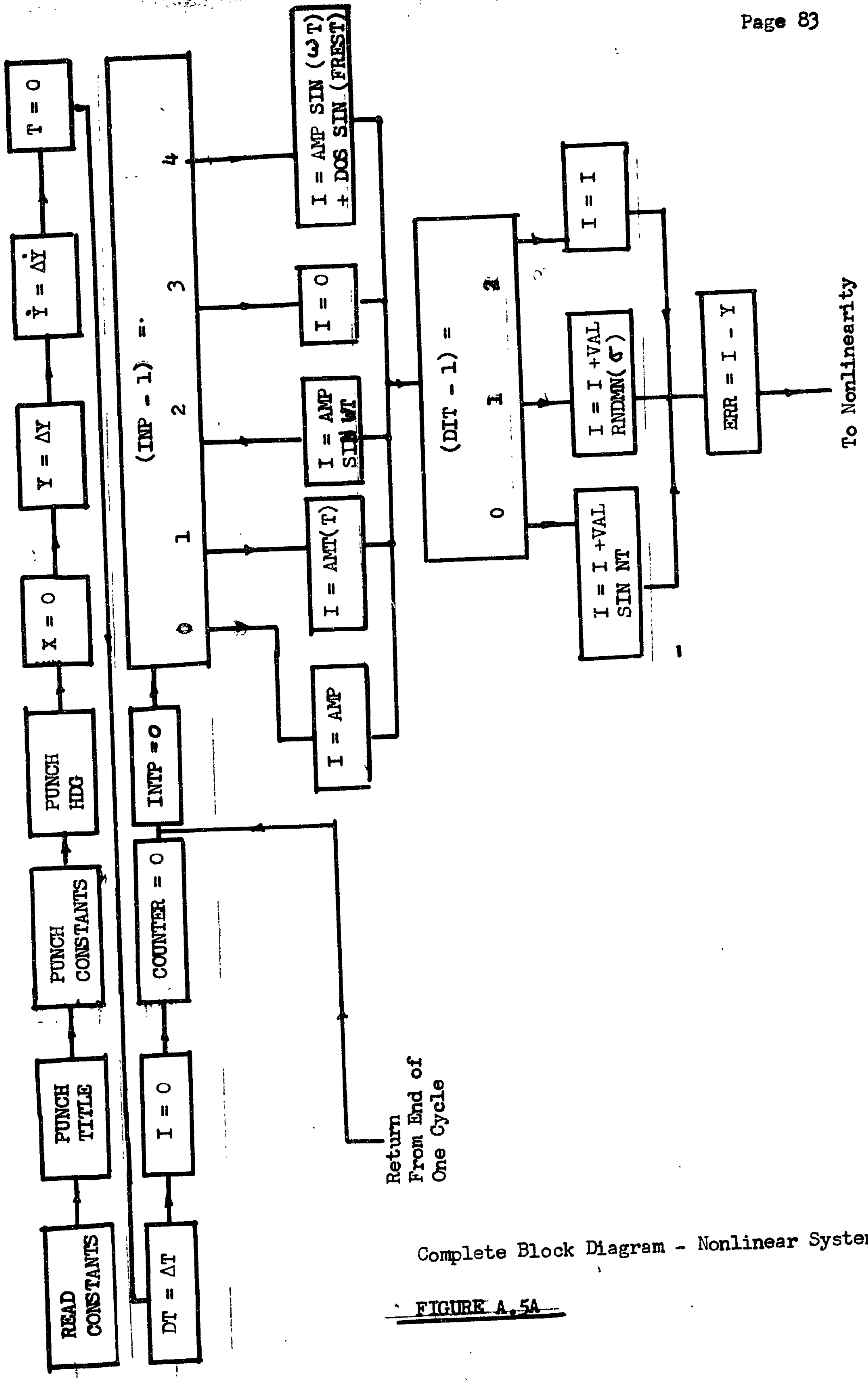
To improve the accuracy of the solution an intpolation routine was added.

Since time is incremented in uniform steps, a value of the error may change by an amount that exceeds the switching value. However, the program did not know this until the next increment and for the latter part of this increment the equation was actually incorrect. This was rectified by performing a linear interpolation of the error when a switching occurred.

The equation relating X and y therefore is now incremented backward in time by an amount determined by interpolation. The value of X is changed to the correct one and the equation is incremented foward by the same amount to place the solution in sychronism with the previous solution times. During the interpolation routine, the output is suppressed and only values of the variable at the sychronous times are recorded.

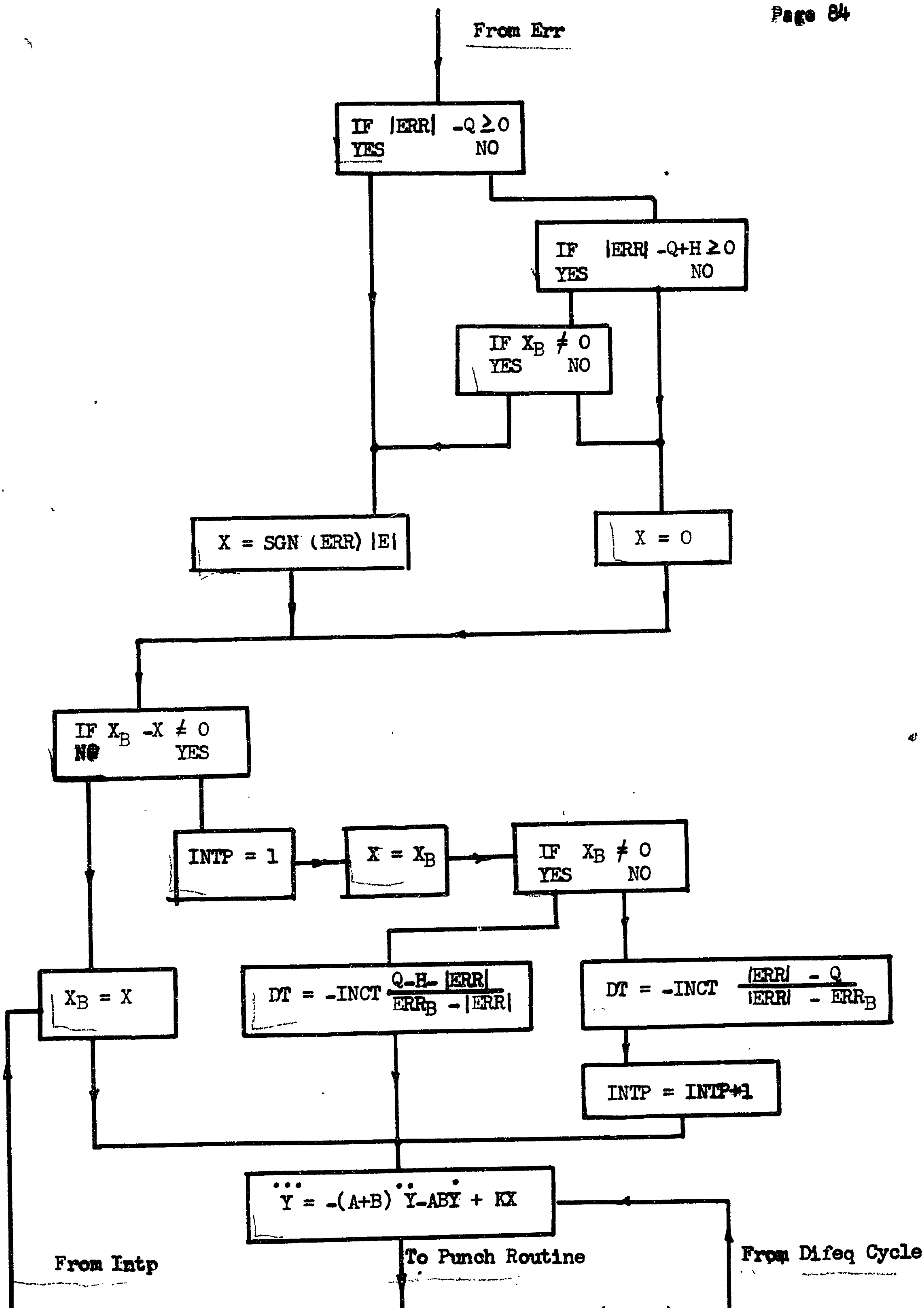
A more detailed block diagram of the operation of the program appears in Figure A5.

The running time for the program is 3.5 seconds per time increment. This is running time on the IBM 650.



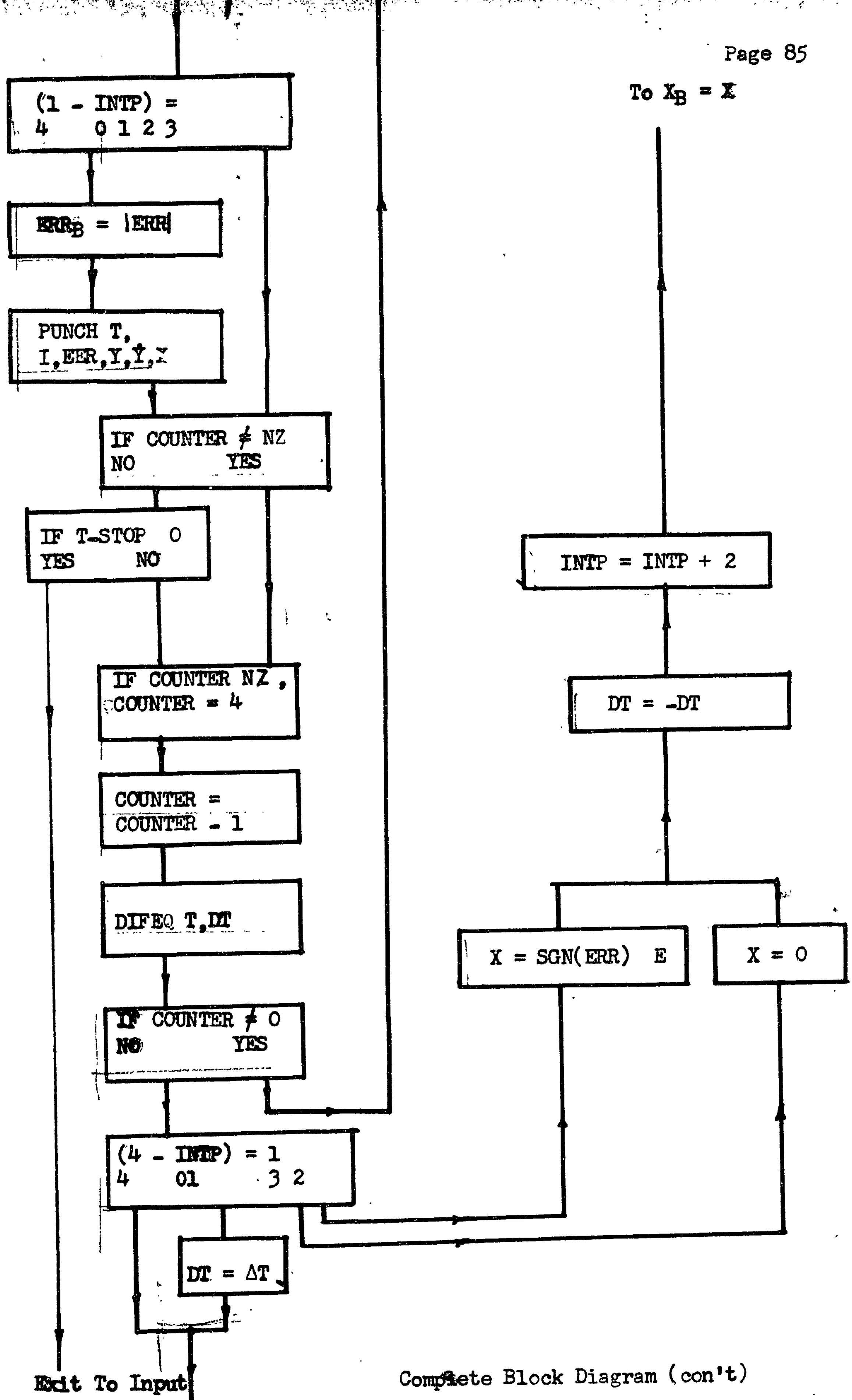
Complete Block Diagram - Nonlinear System

FIGURE A.5A



Complete Block Diagram (con't)

FIGURE A5B



Complete Block Diagram (con't)

FIGURE A.5C

APPENDIX 11

PROGRAM FOR CALCULATION OF DESCRIBING FUNCTION

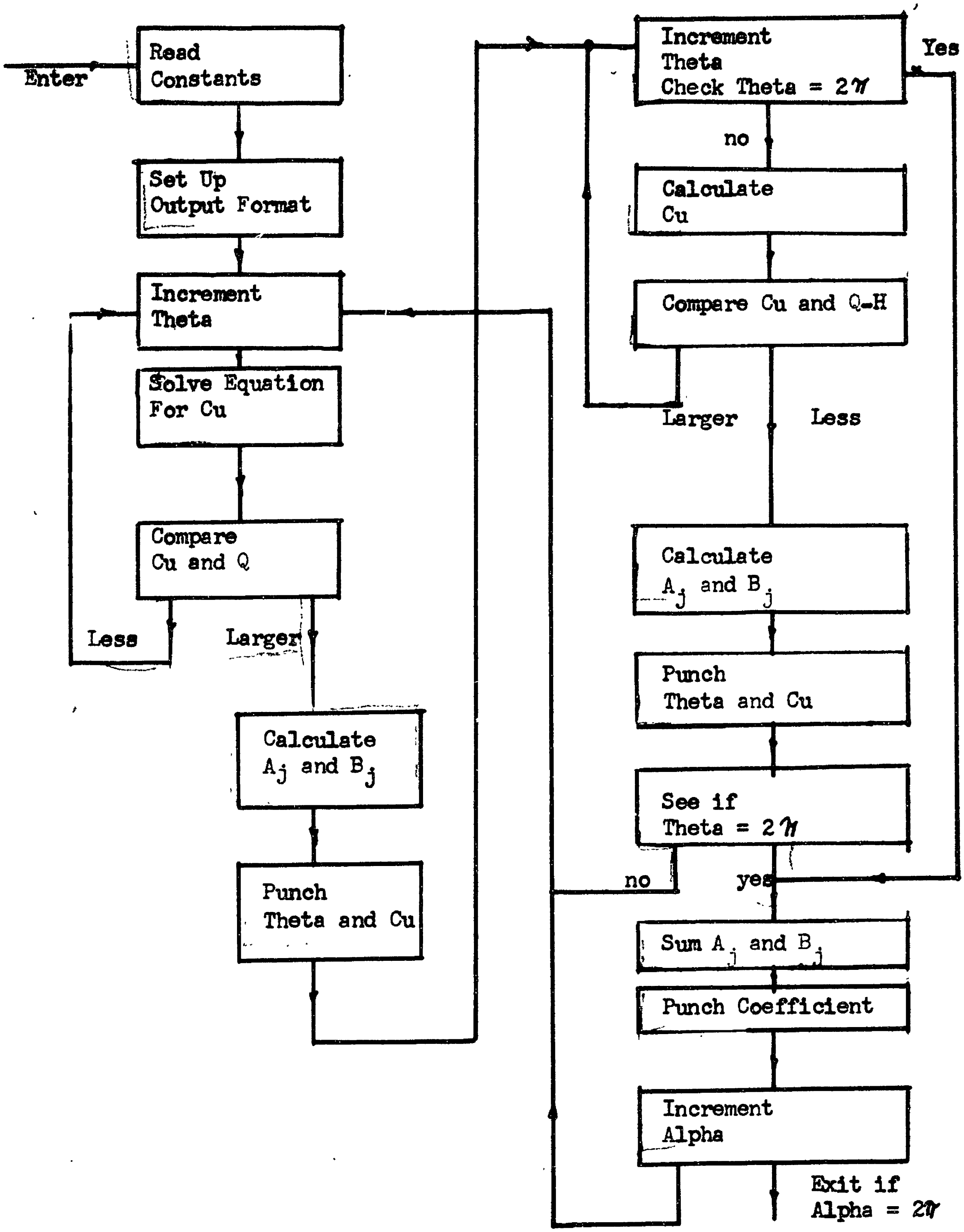
In order to calculate the output of the nonlinearity and the coefficients of the fundamental term of the Fourier series of this output, the equation to be solved is:

Sig Sin (Theta) + Dit Sin (N Theta - Alpha) = CU B.1

This equation was solved using a digital computer by substituting values of Theta in increments and observing when the equation was satisfied. It also calculated the coefficients of the Fourier series for the signal frequency.

The operation of the program can be understood by examining Figure B1. The program was written using MAC. See Figure B.1. For an explanation of the statements see Appendix 1.

The first statements set up the output and read in data. The program then sets Theta equal to Alpha and increments Theta by steps until it has incremented it through one period or 2π radians. At each increment it calculates CU and compares it with the pull in and drop out values of the relay, respectively. If the value is greater than the pull in value, it will punch Theta and calculate the individual A_j and B_j for each Theta. (The program operates similarly on drop out values.) Because the inital value of Theta chosen may be larger than the pull in value of the relay by more than one increment, a check was included in the program to reset A_1 and B_1 to their



Block Diagram for Calculation of Modified Describing Function

FIGURE B.1

correct valves. A_1 and B_1 are correctly calculated at the end of the program.

After the program has incremented Theta through the range of 2π radians it sums all the A_j's and B_j's. The individual A_j's and B_j's are the individual terms of A as seen by equation.

$$A = \frac{E}{\pi} \int_{0\ N}^{2\pi} \sin\theta \, d\theta \qquad \text{(B.2)}$$

$$= \frac{E}{\pi} \sum_{n-1} - \cos\theta_n + \cos\theta n{+}1 \qquad \text{(B.2a)}$$

$$A_j \stackrel{\Delta}{=} - \cos\theta_j \quad j \text{ odd for drop out values} \qquad \text{(B.3)}$$

$$A_j \stackrel{\Delta}{=} \cos\theta_j \quad j \text{ even for pull in values} \qquad \text{(B.4)}$$

and similarly for B;

When the program has calculated A and B it returns to calculate A and B again for different values of Alpha.

For a sample of the program output see Figure B.3

Because the program must calculate the sine and cosine of the agreement twice for each value of Theta and four times for each pull in and drop out valves of Theta, the running time on the IBM 650 is about 400 ms for each increment of Theta. The running time on the IBM 704 is about 10 ms per increment of Theta.

```
10000 DESCRIBING FUNCTION                         0000 08-24-59
M0010 READ SIG,DIT,N,U,D,INCT                       1
M0011 READ INCA
M0020 PUNCH MSG
M0030 DESC FUNCTION FOR
M0040 PUNCH SIG,DIT,N,U,D,INCT
M0041 PUNCH INCA,SP2
M0060 PUNCH HDG,SP2
M0070   A   + JB ALPHATHETA CU
M0080 INDEX J
M0100 RESERVE A     ,B
S0110              200   200
M0120 ALPHA=0
M0125 J=1                                           2
M0126 ASET=0
M0130 THETA=ALPHA-INCT
M0140 THETA=THETA+INCT                              3
M0150 IF(THETA-6.28-ALPHA)POS,GO TO
M0160 8                                             4
M0170 CU=SIG SIN(THETA)+DIT SIN(
M0180 N THETA-N ALPHA)                              5
M0190 IF(ABS(CU)-U)NEG, GO TO 3
M0192 Z = 57.3THETA
```

Figure B.2a

```
10000 DESCRIBING FUNCTION                           0000 08-24-59
M0195 PUNCH 0,0,0,Z/53,CU
M0196 IF(THETA-ALPHA-INCT/2) NEG
M0197 ,GO TO 15
M0198 GO TO 25
M0199 ASET=1                                         15
M0200 A =-SGN(CU)SIN(THETA)                          25
S0210  J
M0220 B =SGN(CU)COS(THETA)
S0230  J
M0240 J=J+1                                           6
M0250 THETA=THETA+INCT                                7
M0260 IF(THETA-6.28-ALPHA)POS,GO TO
M0270 9
M0280 CU=SIG SIN(THETA)+DIT SIN(
M0290 N THETA-N ALPHA)
M0300 IF(ABS(CU)-D) PNZ, GO TO 7
M0302  Z = 57.3THETA
M0305 PUNCH 0,0,0,Z/53,CU
M0310 A = SGN(CU)SIN(THETA)
S0320  J
M0330 B =-SGN(CU)COS(THETA)
S0340  J
M0350 J=J+1
M0360 GO TO 3
```

Figure B.2b

```
10000 DESCRIBING FUNCTION                              0000 08-24-59

M0364 IF(ASET-1)ZERO,A =A                                   8
S0365                  1   J-1

M0366 IF(ASET-1)ZERO,B =B
S0367                  1   J-1

M0369 J=J-1

M0371 J=J-1                                                 9

M0390 K=J

M0400 A=0

M0410 B=0

M0419 J=0

M0420 J=J+1                                                10

M0430 A =A + A
S0440          J

M0450 B =B   + B
S0460            J

M0462 IF(J-K)NEG, GO TO 10

M0470 PUNCH (A/SIG),(B/SIG),ALPHA

M0480 ALPHA=ALPHA +INCA

M0490 IF(ALPHA - 6.28/N)POS,GO TO 11

M0500 GO TO 2

M0510 GO TO 1                                              11

M0520 START AT 1
```

Figure B.2c

```
10000 DESCRIBING FUNCTION              0000 08-24-59

      SAMPLE INPUT

         1    00755000 52  25000000 50  10000000 52  10000000 51  75000000 50  15700000 49
         2    15700000 50
         3    01100000 52  25000000 50  10000000 52  10000000 51  75000000 50  15700000 49
         4    15700000 50

      SAMPLE OUTPUT

      DESC FUNCTION FOR
         1    75499999 50  25000000 50  10000000 52  10000000 51  75000000 50  15700000 49

         2    15700000 50

                   A           + JB         ALPHA         THETA          CU

         3
         4                                              08906138 53  10015359 51
         5                                              09895709 53  74779930 50
         6    13083775 51  22790039 50  15700000 50
         7                              31400000 50
         8                                              26898320 53 -10009027 51
         9                                              27977651 53 -71051175 50
        10    13053413 51  24844597 50  47099999 50
```

Figure B.3

APPENDIX III

PROGRAM FOR CALCULATION OF AUTOCORRELATION FUNCTION

This program calculates the autocorrelation function for the value of ERR - X_B or the quantization noise. The summation

$$\phi(t) \approx \frac{1}{N+1} \sum_{j=1}^{N-T} X_j X_{j+t} \qquad (C.1)$$

that is used to approximate the auto-correlation function is summed using one statement of MAC. This statement is called the "Do to" statement and it controls the incrementing of a variable in an equation. The numbers after the equal sign are the initial value, increment and final value of the variable to be incremented. The operation of the program is similar to that of the one in appendix II.

The running time of the program on the IBM 650 for each autocorrelation point is 30 seconts.

For a printup of the program see Figure. C.1.

```
10000 AUTO CORRELATION OF NOISE        0000 08-24-59

M0010 READ  LIMT,NDATA,H                  1

M0020 PUNCH HDG,SP2

M0030   T   PHI INPUT

M0040 INDEX J,T

M0050 RESERVE   X
S0060            400

M0070 B =0
S0080  P

M0090 DO TO 2 FOR J=1(1)NDATA

M0100 READ N,N,X ,N,N,B
S0110            J

M0120 X =B-B H-X
S0130  J    P   J

M0140 B =B
S0150  P

M0155 PUNCH 0,0,X /51
S0156             J

M0160 DO TO 4 FOR T=0(1)LIMT

M0170 PHI=0

M0180 DO TO 3 FOR J=1(1)(NDATA-T)

M0190 PHI =PHI + X X                      3
S0200             J J+T

M0210 PUNCH  T/53,(PHI/NDATA+1)           4

M0220 GO TO 1

M0230 START AT 1
```

Figure C.1

APPENDIX 1V

GLOSSARY OF TERMS

SYMBOL	MEANING
A	Time Constant of System
A_1	Fourier Coefficient
Alpha	Phase of Dither
Amp	Amplitude of Input
B	Time Constant of System
B_1	Fourier Coefficient
D	Amplitude of Dither
Dit	Dither Selector
Dos	Amplitude of Input
E	Output of Relay
e	Base of Natural Log Arithm
Err	Error Signal
Fres	Frequency of Input
H	Hysteresis
H(ω)	General Transiter Function
Imp	Input Selector
Incdy	Initial Valve of Velocity
Inct	Increment of Time
Incy	Initial Value of Y

SYMBOL	MEANING
j	$\sqrt{-1}$
j	Index of Summation
K	Gain of Linear Elements
k	Ratio of Time Constants
$K(j\omega)$	Transfer Function for Linear Elements
N	Frequency of Dither
$N(\alpha)$	Describing Function
Omega	Frequency of Input
P(u)	Characteristic Function
Q	Pull in Voltage of Relay
q	Grain Size of Quantizer
Signa	Standard Deviation
Stop	Limit of Time
T	Period of Wave Shape
u	Variable in C.F, Space
$u(t-t_1)$	Step Function at t_1
Val	Amplitude of Dither
W(x)	Distribution Density
x	Output of nonlinearity
x	A variable
Y	Output of System
Y s{t}	Output due to Step Input

SYMBOL	MEANING
α	Input Amplitude to Nonlinearity
α	Time Shift of Dither
α_H	Hysteresis of Limiter
θ_1	Pull in or Drop Out Angle
σ	Standard Deviation
σ^2	Variance
τ	Time Constant, or Time Shift
$\Phi(\omega)$	Power Density Spectrum
Φ	Quantization Frequency
$\Phi_{11}(\tau)$	Auto Correlation Function
Ω	Sampling Frequency
ω	Radian Frequency

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