# Scaling limits of random plane partitions and six-vertex models

by

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## Abstract

We present a collection of results about the scaling limits of several models from integrable probability.

Our first result concerns the asymptotic behavior of the bottom slice of a Hall-Littlewood random plane partition. We show the latter concentrates around a limit shape and in two different scaling regimes identify the fluctuations around this shape with the GUE Tracy-Widom distribution and the narrow wedge initial data solution to the Kardar-Parisi-Zhang (KPZ) equation. The second result concerns the limiting behavior of a class of six-vertex models in the quadrant, and we obtain the GUE-corners process as a scaling limit for this class near the boundary. Our final result, joint with Ivan Corwin, demonstrates the (long predicted) transversal 2/3 critical exponent for the height functions of the stochastic six-vertex model and asymmetric simple exclusion process (ASEP).

The algebraic parts of our arguments involve the construction and use of degenerations and modifications of the Macdonald difference operators to obtain rich families of observables for the models we consider. These formulas are in terms of multiple contour integrals and provide a direct access to quantities of interest. The analytic parts of our arguments include the detailed asymptotic analysis of Fredholm determinants and contour integrals through steepest descent methods. An important aspect of our approach, is the combination of exact formulas with more probabilistic arguments, based on various Gibbs properties enjoyed by the models we study.


Thesis Supervisor: Alexei Borodin
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# Contents

# Chapter 1

# Introduction

This thesis presents a collection of results about the scaling limits of large stochastic systems, which have been the object of immense study in a relatively young area of mathematics called integrable probability. The majority of the models we investigate can be formulated as random plane partitions or random six-vertex models, which in turn can be viewed as random two-dimensional surfaces. The probability distribution of any of the models we consider depends on several parameters and when these parameters converge to their critical values the system size stochastically increases. Our main goal is to describe the behavior of these systems as they approach criticality. The general flavor of the results we present is that if one appropriately shifts and rescales these random surfaces, they will converge to limiting probabilistic objects.

It is believed that many of the limits of random surfaces are *universal*, in the sense that large classes of models should converge to the same objects regardless of the specifics of their distributions. Proving such a statement remains out of reach in the general case; however, there are now several integrable models (including the ones presented in this thesis) for which there are tools to partially verify this belief. In the context of this thesis, the integrability of the distributions we consider comes from their connection to special classes of symmetric functions – Hall-Littlewood functions [64] and their rational generalizations [20]. The structural dependence of our models on symmetric functions allows the use of purely algebraic tools, which provide exact formulas for rich families of observables of the systems. Once these formulas are available, one can study them asymptotically and combine them with additional combinatorial and probabilistic arguments to derive precise statements about the limits of the systems.

The results of this thesis are split into three chapters, which are for the most part self-contained and may be read in any order. We summarize each chapter below.

In Chapter 2 we consider a probability distribution $\mathbb{P}^{q,t}_{HL}$ on plane partitions, which arises as a one parameter generalization of the standard $q^{volume}$ measure. This generalization is closely related to the classical multivariate Hall-Littlewood polynomials, and it was first introduced by Vuletić in [81]. We prove that as the plane partitions become large ($q$ goes to 1, while the Hall-Littlewood parameter $t$ is fixed), the scaled bottom slice of the random plane partition converges to a deterministic limit shape, and that one-point fluctuations around the limit shape are asymptotically given by the GUE Tracy-Widom distribution. On the other hand, if $t$ simultaneously converges to its own critical value of 1, the fluctuations instead converge to the one-dimensional Kardar-Parisi-Zhang (KPZ) equation with the so-called narrow wedge initial data.

The connection of $\mathbb{P}_{HL}^{q,t}$ to Hall-Littlewood functions, allows us to apply the (more general) formalism of Macdonald difference operators from [24] to our problem. In the Hall-Littlewood setting the operators approach gives access to a single observable and we find a (general) Fredholm determinant formula for its $t$-Laplace transform. In order to prove our main results we specialize the general formula for the $t$-Laplace transform to the particular measure we consider. Subsequently, we find two different representations of this formula that are suitable for the two limiting regimes. When $t \in (0,1)$ is fixed and $q \to 1^-$ the $t$-Laplace transform converges to an indicator function and our Fredholm determinant formula converges to the CDF of the Tracy-Widom GUE distribution. When both $q, t \to 1^-$ the $t$-Laplace transform converges to the usual Laplace transform and our Fredholm determinant formula converges to the Laplace transform of the partition function of the continuous directed random polymer [5,35]. The main difficulties in establishing the above convergence results are finding suitable contours for our Fredholm determinants and representations for the integrands. We reduce the convergence results to verifying certain exponential bounds for the integrands, which are obtained through a careful analysis on the (specially) constructed contours. This detailed asymptotic analysis of the arising Fredholm determinants forms the analytic part of our arguments. Chapter 2 is based on the paper

[47] E. Dimitrov, KPZ and Airy limits of Hall-Littlewood random plane partitions, *Ann. Inst. Henri Poincaré Probab. Stat.*, to appear, 2016. Preprint, arXiv:1602.00727

In Chapter 3 we consider a class of probability distributions on the vertically inhomogeneous six-vertex model, which originates from the higher spin vertex models of [33]. These distributions are closely related to a remarkable family of symmetric rational functions $\mathsf{F}_\lambda$, parametrized by non-negative signatures $\lambda = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0$. These functions form a one-parameter generalization of the classical Hall-Littlewood polynomials [64] and enjoy many of the same structural properties [20]. Our approach to studying the vertically inhomogeneous six-vertex model is based on a new class of operators $D_N^k$, inspired by the Macdonald difference operators. These operators act diagonally on the functions $\mathsf{F}_\lambda$, whenever $\lambda$ has distinct parts and can be used to derive formulas for the probability of observing certain arrow configurations in different locations of the model.

The main goal of Chapter 3 is to use the correlation functions obtained from our operators to analyze a particular class of homogeneous six-vertex models as the system size becomes large. For the class of models we consider, the correlation functions can be expressed in terms of multiple contour integrals, which are suitable for asymptotic analysis. For a particular choice of parameters we analyze the limit of the correlation functions through a steepest descent method. Combining this asymptotic statement with some new results about Gibbs measures on Gelfand-Tsetlin cones and patterns, we show that certain configurations of holes (absence of arrows or empty edges) weakly converge to the GUE-corners process as the size of the system tends to infinity. An important ingredient in the proof is a classification result, which identifies the GUE-corners process as the unique probability measure that satisfies the continuous Gibbs property and has the correct marginal distribution on the right edge.

Chapter 3 is based on the paper

[48] E. Dimitrov, Six-vertex models and the GUE-corners process, *Int. Math. Res. Notices*, to appear, 2016. Preprint, arXiv:1610.06893

In Chapter 4 we prove the long predicted transversal 2/3 exponent for the asymmetric simple exclusion process (ASEP) [63,76] and the stochastic six vertex model [53] – two closely related $1+1$ dimensional random interface growth models in the Kardar-Parisi-Zhang (KPZ) universality class. We work with step initial data for both models and demonstrate that their height functions, scaled in space by $T^{2/3}$ and in fluctuation size by $T^{1/3}$, are tight as spatial processes as time $T$ goes to infinity. We also show that all subsequential limits of the scaled height function (shifted by a parabola) have increments, which are absolutely continuous with respect to a Brownian bridge measure. Conjecturally, the limit process should be the $\text{Airy}_2$ process and we provide further evidence for this conjecture by uncovering a Gibbsian line ensemble structure behind these models, which formally limits to that of the Airy line ensemble [42].

Our approach is based on the study of a class of measures on discrete line ensembles that satisfy what we call the 'Hall-Littlewood Gibbs' resampling property. This Gibbs property implies that conditional on the second curve in the line ensemble, the top curve has a law expressible in terms of an explicit Radon-Nikodym derivative with respect to the trajectory of a random walk. By controlling this Radon-Nikodym derivative as $T$ goes to infinity, we are able to control quantities like the maximum, minimum and modulus of continuity of the prelimit continuous curves, which translates into a tightness statement in the space of continuous curves. By exploiting a strong coupling of random walk and Brownian bridges we can further deduce the absolute continuity of subsequential limits with respect to Brownian bridges of appropriate variance. The results we establish for line ensembles are quite general and can be applied to the six-vertex model using the recent developments in [23] and to the ASEP using the results in [3, 27]. Chapter 4 is based on the joint paper with Ivan Corwin

[41] I. Corwin and E. Dimitrov, Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood Gibbsian line ensembles, *Comm. Math. Phys.*, to appear, 2017. Preprint, arXiv:1703.07180

# Chapter 2

# KPZ and Airy limits of Hall-Littlewood random plane partitions

## 2.1 Introduction

The purpose of this chapter is to use the Macdonald difference operators [64] to study the $q = 0$ degeneration of the Macdonald process [24], called the Hall-Littlewood process. Our motivation for studying the Hall-Littlewood process is that it arises naturally in a problem of random plane partitions. The distribution on plane partitions we consider, called $\mathbb{P}_{HL}^{r,t}$ in the text and defined in the next section, was first considered by Vuletić in [81], where she discovered a generalization of the famous MacMahon formula and identified an important geometric structure of the measure. The measure $\mathbb{P}_{HL}^{r,t}$ is a one-parameter generalization of the usual $r^{vol}$ measure on plane partitions, which is recovered if one sets $t = 0$ (the volume parameter is usually denoted by $q$ in the literature, but we reserve this letter for the $q$ in the Macdonald polynomials and use $r$ instead for the remainder of the text).

The algebraic part of our arguments consists of developing a framework for the Macdonald difference operators in the Hall-Littlewood case. Although our discussion is parallel to the one for the $q$-Whittaker case in [24], we remark that there are several technical modifications that need to be made. In the Hall-Littlewood setting the operators approach gives access to a single observable and we find a Fredholm determinant formula for its $t$-Laplace transform. This result is given in Proposition 2.3.10 and we believe it to be of separate interest as it can be applied to generic Hall-Littlewood measures and its Fredholm determinant form makes it suitable for asymptotic analysis. For the particular model we consider, the observable is insufficient to study the 3-dimensional diagram; however, we are able to use it to analyze the one-point marginal distribution of the bottom part of the diagram.

The main results of the chapter (Theorems 2.1.2 and 2.1.3 below) describe the asymptotic distribution of the bottom slice of a plane partition, distributed according to $\mathbb{P}_{HL}^{r,t}$, in two limiting regimes: when $r \to 1^-$, $t \in (0, 1)$ - fixed and when $r, t \to 1^-$ in some critical fashion. In both cases one observes the same limit shape, while the fluctuations in the first limiting regime converge to the Tracy-Widom GUE distribution [78], and to the distribution of the Hopf-Cole solution to the KPZ equation with narrow wedge initial data [6, 10] in the second one. The latter results suggest that our model belongs to the KPZ universality class [40], although some care needs to be taken. Typically, models belonging to the KPZ universality class are characterized by some dynamics (interacting particle systems, growing interfaces, random polymers etc.), so that the system evolves with time. In sharp contrast, the model we consider is *stationary*, i.e. there is no notion of time.

We now turn to carefully describing the measure $\mathbb{P}^{r,t}_{HL}$ and explaining our results in detail.

### 2.1.1 The measure $\mathbb{P}^{r,t}_{HL}$

We recommend Section 2.2.1 for a brief overview of some concepts related to partitions and Young diagrams. A plane partition is a Young diagram filled with positive integers that form non-increasing rows and columns. A *connected component* of a plane partition is the set of all connected boxes of its Young diagram that are filled with the same number. The number of connected components in a plane partition $\pi$ is denoted by $k(\pi)$. Figure 2-1 shows an example of a plane partition and the 3-d Young diagram representing it. The connected components, which are separated in the Young diagram with bold lines, naturally correspond to the grey terraces in the 3-d diagram.

| | | | | | |
|---|---|---|---|---|---|
| 5 | 5 | 4 | 3 | 3 | 3 |
| 4 | 4 | 3 | 3 | 3 | 3 |
| 4 | 3 | 3 | 3 | 3 | 3 |
| 3 | 3 | 3 | 2 | 2 | 1 |
| 3 | 3 | 3 | 2 | 2 | |
| 1 | 1 | 1 | | | |

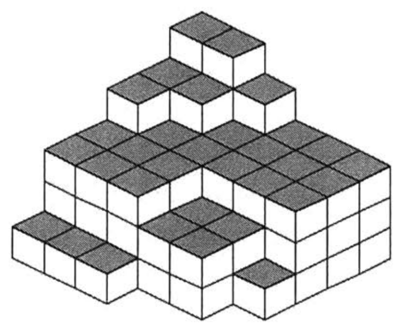

Figure 2-1: A plane partition and its 3-d Young diagram. In this example $k(\pi) = 7$.

If a box $(i, j)$ belongs to a connected component $C$, we define its *level* $h(i, j)$ as the smallest $h \in \mathbb{N}$ such that $(i + h, j + h) \notin C$. A *border component* is a connected subset of a connected component where all boxes have the same level. We also say that the border component is of this level. For the example above, the border components and their levels are illustrated in Figure 2-2.

For each connected component $C$ we define a sequence $(n_1, n_2, ...)$ where $n_i$ is the number of $i$-level border components of $C$. We set

$$P_C(t) := \prod_{i \geq 1} (1 - t^i)^{n_i}.$$

Let $C_1, C_2, ...C_{k(\pi)}$ be the connected components of $\pi$. We define

$$A_\pi(t) := \prod_{i=1}^{k(\pi)} P_{C_i}(t). \tag{2.1.1}$$

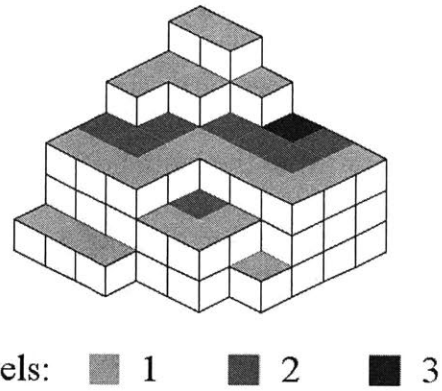


Figure 2-2: Border components and their levels.

For the example above $A_\pi(t) = (1-t)^7(1-t^2)^3(1-t^3)$.

Given two parameters $r, t \in (0,1)$ we define $\mathbb{P}_{HL}^{r,t}$ to be the probability distribution on plane partitions such that

$$\mathbb{P}_{HL}^{r,t}(\pi) \propto r^{|\pi|} A_\pi(t),$$

where $|\pi|$ denotes the volume of $\pi$, i.e. the number of boxes in its 3-d Young diagram. In [81] it was shown that

$$\sum_{\pi} r^{|\pi|} A_\pi(t) = \prod_{n=1}^{\infty} \left( \frac{1 - tr^n}{1 - r^n} \right)^n =: Z(r,t). \tag{2.1.2}$$

The above explicitly determines $\mathbb{P}_{HL}^{r,t}$ as

$$\mathbb{P}_{HL}^{r,t}(\pi) := Z(r,t)^{-1} r^{|\pi|} A_\pi(t), \tag{2.1.3}$$

with $Z(r,t)$ as in (2.1.2).

*Remark* 2.1.1. In Section 2.2.4 it will be shown that $\mathbb{P}_{HL}^{r,t}$ arises as a limit of certain Macdonald processes. These processes are defined in terms of Hall-Littlewood symmetric functions, which explains the "HL" in our notation.

The distribution $\mathbb{P}_{HL}^{r,t}$ has been studied in the cases $t = 0$ and $t = -1$. When $t = 0$ we have $\mathbb{P}_{HL}^{r,0}(\pi) = Z(r,0)^{-1} r^{|\pi|}$, where $Z(r,0)$ is given by the famous MacMahon formula

$$Z(r,0) = \sum_{\pi} r^{|\pi|} = \prod_{n=1}^{\infty} \left( \frac{1}{1 - r^n} \right)^n. \tag{2.1.4}$$

We summarize a few of the known results when $t = 0$. In [36] it was shown that under suitable scaling a partition $\pi$, distributed according to $\mathbb{P}_{HL}^{r,0}$, converges to a particular limit shape as $r \to 1^-$ (see also [57]). In [69] it was shown that $\mathbb{P}_{HL}^{r,0}$ is described by a Schur process and has the structure of a determinantal point process with an explicit correlation kernel, suitable for asymptotic analysis. In [51] it was shown that under suitable scaling the edge of the limit shape converges to the Airy process.

When $t = -1$ the measure $\mathbb{P}_{HL}^{r,-1}$ concentrates on strict plane partitions (these are plane partitions such that all border components have level 1) and is described by a shifted Schur process as discussed in [80]. The shifted Schur process is shown to have the structure of a Pfaffian point process with an explicit correlation kernel, which can be analyzed as $r \to 1^-$. A limiting point density can be derived, which suggests a limit-shape phenomenon similar to the $t = 0$ case. To the author's knowledge there are no results regarding the edge asymptotics in this case.

In this chapter we study the distribution $\mathbb{P}^{r,t}_{HL}$ for $t \in (0,1)$. In particular, we will be interested in the behavior of a plane partition, distributed according to $\mathbb{P}^{r,t}_{HL}$, as the parameter $r$ goes to $1^-$. Part of the difficulty in dealing with the case $t \in (0,1)$ comes from the fact that a determinantal or Pfaffian point process structure is no longer availbable. Instead, we will use the formalism of Macdonald difference operators (see [24] and [28]) to obtain formulas for a certain class of observables for a plane partition $\pi$, distributed accodrding to $\mathbb{P}^{r,t}_{HL}$. These formulas can be asymptotically analyzed and imply one-point convergence results for the bottom slice of $\pi$.

### 2.1.2 Main results

For a partition $\lambda$, we let $\lambda_1'$ denote its largest column (i.e. the number of non-zero parts). Given a plane partition $\pi$, we consider its diagonal slices $\lambda^t$ (alternatively $\lambda(t)$) for $t \in \mathbb{Z}$, i.e. the sequences

$$\lambda^k = \lambda(k) = (\pi_{i,i+k}) \quad \text{for } i \geq \max(0, -k).$$

For $r \in (0,1)$, $\tau \in \mathbb{R}$ we define

$$N(r) := \frac{1}{1-r} \quad \text{and } \chi := \left[\frac{e^{-|\tau|/2}}{(1+e^{-|\tau|/2})^2}\right]^{-1/3} = \left[\frac{4}{\cosh^2(\tau/4)}\right]^{-1/3}. \tag{2.1.5}$$

Below we analyze the large $N$ asymptotics of $\lambda_1'(\lfloor \tau N(r) \rfloor)$ of a random plane partition, distributed according to $\mathbb{P}^{r,t}_{HL}$.

**Theorem 2.1.2.** *Consider the measure $\mathbb{P}^{r,t}_{HL}$ on plane partitions, given in (2.1.3), with $t \in (0,1)$ fixed. Then for all $\tau \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}$ we have*

$$\lim_{r \to 1^-} \mathbb{P}^{r,t}_{HL}\left(\frac{\lambda_1'(\lfloor \tau N(r) \rfloor) - 2N(r)\log(1+e^{-|\tau|/2})}{\chi^{-1}N(r)^{1/3}} \leq x\right) = F_{GUE}(x),$$

*where $F_{GUE}$ is the GUE Tracy-Widom distribution [78] and $N(r), \chi$ are as in (2.1.5).*

**Theorem 2.1.3.** *Consider the measure $\mathbb{P}^{r,t}_{HL}$ on plane partitions, given in (2.1.3). Suppose $T > 0$ is fixed and $\frac{-\log t}{(1-r)^{1/3}} = \chi(T/2)^{1/3}$. Then for all $\tau \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}$ we have*

$$\lim_{r \to 1^-} \mathbb{P}^{r,t}_{HL}\left(\frac{\lambda_1'(\lfloor \tau N(r) \rfloor) - 2N(r)\log(1+e^{-|\tau|/2})}{\chi^{-1}N(r)^{1/3}(T/2)^{-1/3}} + \log(N(r)^{1/3}\chi^{-1}(T/2)^{-1/3}) \leq x\right) = F_{CDRP}(x)$$

*where $F_{CDRP}(x) = \mathbb{P}(\mathcal{F}(T,0) + T/24 \leq x)$ and $\mathcal{F}(T,X)$ is the Hopf-Cole solution to the Kardar-Parisi-Zhang equation with narrow wedge initial data [6, 10]. The coefficients $N(r)$ and $\chi$ are as in (2.1.5).*

The definitions of $F_{GUE}(x)$ and $F_{CDRP}(x)$ are provided below in Definition 2.1.7. In Sections 2.4 and 2.5 we will reduce the proofs of the above results to claims on certain asymptotics of Fredholm determinant formulas. Throughout the chapter, we will, rather informally, refer to the limiting regime in Theorem 2.1.2 as "the GUE case" and to the one in Theorem 2.1.3 as "the CDRP case".

*Remark* 2.1.4. The exclusion of the case $\tau = 0$ appears to be a technical assumption, necessary for our proofs to work. It is possible that the arguments in this chapter can be modified to include this case, but we will not pursue this goal.

Before we record the limiting distributions that appear in our results, we briefly discuss the definition of $\mathcal{F}(X,T)$. The *continuous directed random polymer* (CDRP) is a universal scaling limit for $1+1$ dimensional directed random polymers [5, 35]. Its partition function with respect to general boundary perturbations is given as follows (cf. [26, Definition 1.7]).

**Definition 2.1.5.** The partition function for the continuum directed random polymer with boundary perturbation $\ln \mathcal{Z}_0(X)$ is given by the solution to the stochastic heat equation (SHE) with multiplicative Gaussian space-time white noise and $\mathcal{Z}_0(X)$ initial data:

$$\partial_T \mathcal{Z} = \frac{1}{2}\partial_X^2 \mathcal{Z} + \mathcal{Z}\dot{\mathcal{W}}, \qquad \mathcal{Z}(0,X) = \mathcal{Z}_0(X). \tag{2.1.6}$$

The initial data $\mathcal{Z}_0(X)$ may be random but is assumed to be independent of the Gaussian space-time white noise $\dot{\mathcal{W}}$ and is assumed to be almost surely a sigma-finite positive measure. Observe that even if $\mathcal{Z}_0(X)$ is zero in some regions, the stochastic PDE makes sense and hence the partition function is well-defined.

A detailed description of the SHE and the class of initial data for which it is well-posed can be found in [6, 10]. Provided, $\mathcal{Z}_0$ is an almost surely sigma-finite positive measure, it follows from the work of Mueller [66] that, almost surely, $\mathcal{Z}(T,X)$ is positive for all $T > 0$ and $X \in \mathbb{R}$ and hence its logarithm is a well-defined random space-time function. The following is Definition 1.8 in [26].

**Definition 2.1.6.** For $\mathcal{Z}_0$ an almost surely sigma-finite positive measure define the free energy for the continuous directed random polymer with boundary perturbation $\ln \mathcal{Z}_0(X)$ as

$$\mathcal{F}(T,X) = \ln \mathcal{Z}(T,X).$$

The random space-time function $\mathcal{F}$ is also the Hopf-Cole solution to the Kardar-Parisi-Zhang equation with initial data $\mathcal{F}_0(X) = \ln \mathcal{Z}_0(X)$ [6, 10]. In this chapter, we will focus on the case when $\mathcal{Z}_0(X) = \mathbf{1}_{\{X=0\}}$, which is known as the *narrow wedge* or 0-*spiked* initial data [6, 26]. In [26, Theorem 1.10] it was shown that when $\mathcal{Z}_0(X) = \mathbf{1}_{\{X=0\}}$, one has the following formula for the Laplace tansform of $\exp(\mathcal{F}(T,0) + T/24)$.

$$\mathbb{E}\left[e^{-e^x \exp(\mathcal{F}(T,0)+T/24)}\right] = \det(I - K_{CDRP})_{L^2(\mathbb{R}_+)}, \tag{2.1.7}$$

where the right-hand-side (RHS) denotes the Fredholm determinant (see Section 2.2.5) of the operator $K_{CDRP}$, given in terms of its integral kernel

$$K_{CDRP}(\eta, \eta') := \int_{\mathbb{R}} dt \frac{e^x}{e^x + e^{-t/\sigma}} Ai(t+\eta) Ai(t+\eta'). \tag{2.1.8}$$

In the above formula $\sigma = (2/T)^{1/3}$, $x \in \mathbb{R}$ and $Ai(\cdot)$ is the Airy function.

We now record the definitions of the limiting distributions that appear in Theorems 2.1.2 and 2.1.3. The first part of the following definition is [26, Definition 1.6].

**Definition 2.1.7.** The GUE Tracy-Widom distribution [78] is defined as

$$F_{GUE}(x) := \det(I - K_{Ai})_{L^2(x,\infty)},$$

where $K_{Ai}$ is the Airy kernel, that has the integral representation

$$K_{Ai}(\eta, \eta') = \frac{1}{(2\pi\iota)^2} \int_{e^{-2\pi\iota/3}\infty}^{e^{2\pi\iota/3}\infty} dw \int_{e^{-\pi\iota/3}\infty}^{e^{\pi\iota/3}\infty} dz \frac{1}{z-w} \frac{e^{z^3/3 - z\eta'}}{e^{w^3/3 - w\eta}},$$

where the contours $z$ and $w$ do not intersect.

Suppose $\mathcal{F}(T,X)$ is the free energy for the CDRP with boundary perturbation $\ln \mathcal{Z}_0(X)$ and $\mathcal{Z}_0(X) = \mathbf{1}_{\{X=0\}}$ as in Definition 2.1.6. Then we define

$$F_{CDRP}(x) := \mathbb{P}(\mathcal{F}(T,0) + T/24 \leq x).$$

### 2.1.3 Outline

The introductory section above formulated the problem statement and gave the main results of the chapter. In Section 2.2 we present some background on partitions, symmetric functions, Macdonald processes and Fredholm determinants. In Section 2.3 we derive a formula for the $t$-Laplace transform of a certain random variable in terms of a Fredholm determinant using the approach of Macdonald difference operators. In Sections 2.4 and 2.5 we extend the results of Section 2.3 to a setting suitable for asymptotic analysis in the GUE and CDRP cases respectively and prove Theorems 2.1.2 and 2.1.3. Section 2.6 summarizes various technical results used in the proofs of Theorems 2.4.7 and 2.5.3. Section 2.7 presents a sampling algorithm for random plane partitions, formuates conjectural extensions of the results of this chapter and provides some empirical evidence supporting them.

## 2.2 General definitions

In this section we summarize some facts about symmetric functions and Macdonald processes. Macdonald processes were defined and studied in [24], which is the main reference for what follows together with the book of Macdonald [64]. We explain how the measure $\mathbb{P}_{HL}^{r,t}$ arises as a limit of a certain sequence of Macdonald processes and end with some background on Fredholm determinants, used in the text.

### 2.2.1 Partitions and Young diagrams

We start by fixing terminology and notation. A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and all but finitely many elements are zero.

We denote the set of all partitions by $\mathbb{Y}$. The *length* $\ell(\lambda)$ is the number of non-zero $\lambda_i$ and the *weight* is given by $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ . If $|\lambda| = n$ we say that $\lambda$ *partitions* $n$, also denoted by $\lambda \vdash n$. There is a single partition of 0, which we denote by $\varnothing$. An alternative representation is given by $\lambda = 1^{m_1}2^{m_2}\cdots$, where $m_j(\lambda) = |\{i \in \mathbb{N} : \lambda_i = j\}|$ is called the *multiplicity* of $j$ in the partition $\lambda$. There is a natural ordering on the space of partitions, called the *reverse lexicographic order*, which is given by

$$\lambda > \mu \iff \exists k \in \mathbb{N} \text{ such that } \lambda_i = \mu_i, \text{ whenever } i < k \text{ and } \lambda_k > \mu_k.$$

A Young diagram is a graphical representation of a partition $\lambda$, with $\lambda_1$ left justified boxes in the top row, $\lambda_2$ in the second row and so on. In general, we do not distinguish between a partition $\lambda$ and the Young diagram representing it. The *conjugate* of a partition $\lambda$ is the partition $\lambda'$ whose Young diagram is the transpose of the diagram $\lambda$. In particular, we have the formula $\lambda'_i = |\{j \in \mathbb{N} : \lambda_j \geq i\}|$.

Given two diagrams $\lambda$ and $\mu$ such that $\mu \subset \lambda$ (as a collection of boxes), we call the difference $\theta = \lambda - \mu$ a *skew Young diagram.* A skew Young diagram $\theta$ is a *horizontal m-strip* if $\theta$ contains $m$ boxes and no two lie in the same column. If $\lambda - \mu$ is a horizontal strip we write $\lambda \succ \mu$. Some of these concepts are illustrated in Figure 2-3.

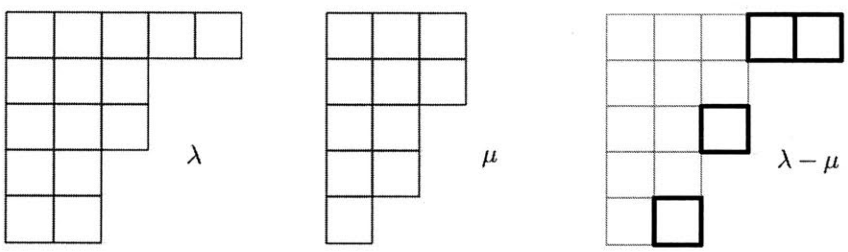


Figure 2-3: The Young diagram $\lambda = (5,3,3,2,2)$ and its transpose (not shown) $\lambda' = (5,5,3,1,1)$. The length $\ell(\lambda) = 5$ and weight $|\lambda| = 15$. The Young diagram $\mu = (3,3,2,2,1)$ is such that $\mu \subset \lambda$. The skew Young diagram $\lambda - \mu$ is shown in *black bold lines* and is a horizontal 4-strip.

A *plane partition* is a two-dimensional array of nonnegative integers

$$\pi = (\pi_{i,j}), \quad i,j = 0,1,2,...,$$

such that $\pi_{i,j} \geq \max(\pi_{i,j+1}, \pi_{i+1,j})$ for all $i,j \geq 0$ and the *volume* $|\pi| = \sum_{i,j\geq 0} \pi_{i,j}$ is finite. Alternatively, a plane partition is a Young diagram filled with positive integers that form non-increasing rows and columns. A graphical representation of a plane partition $\pi$ is given by a 3-*dimensional Young diagram*, which can be viewed as the plot of the function

$$(x,y) \to \pi_{\lfloor x \rfloor, \lfloor y \rfloor} \quad x,y > 0.$$

Given a plane partition $\pi$ we consider its diagonal slices $\lambda^t$ for $t \in \mathbb{Z}$, i.e. the sequences

$$\lambda^t = (\pi_{i,i+t}) \quad \text{for } i \geq \max(0,-t).$$

One readily observes that $\lambda^t$ are partitions and satisfy the following interlacing property

$$\cdots \prec \lambda^{-2} \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \lambda^2 \succ \cdots .$$

Conversely, any (finite) sequence of partitions $\lambda^t$, satisfying the interlacing property, defines a partition $\pi$ in the obvious way. Concepts related to plane partitions are illustrated in Figure 2-4.

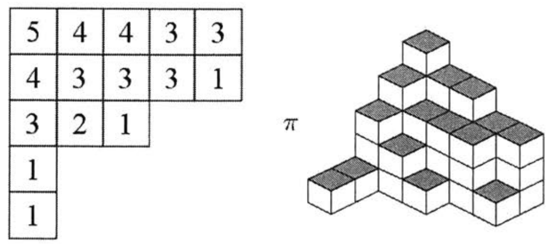


Figure 2-4: The plane partition $\pi = \varnothing \prec (1) \prec (1) \prec (3) \prec (4,2) \prec (5,3,1) \succ (4,3) \succ (4,3) \succ (3,1) \succ (3) \succ \varnothing$ . The volume $|\pi| = 41$.

### 2.2.2 Macdonald symmetric functions

We let $\Lambda_X$ denote the $\mathbb{Z}_{\geq 0}$ graded algebra over $\mathbb{C}$ of symmetric functions in variables $X = (x_1, x_2, ...)$, which can be viewed as the algebra of symmetric polynomials in infinitely many variables with bounded degree, see e.g. Chapter I of [64] for general information on $\Lambda_X$. One way to view $\Lambda_X$ is as an algebra of polynomials in Newton power sums

$$p_k(X) = \sum_{i=1}^{\infty} x_i^k, \quad \text{for } k \geq 1.$$

For any partition $\lambda$ we define

$$p_\lambda(X) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(X),$$

and note that $p_\lambda(X)$, $\lambda \in \mathbb{Y}$ form a linear basis in $\Lambda_X$.

An alternative set of algebraically independent generators of $\Lambda_X$ is given by the elementary symmetric functions

$$e_k(X) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad \text{for } k \geq 1.$$

In what follows we fix two parameters $q, t$ and assume that they are real numbers with $q, t \in (0,1)$. Unless the dependence on $q, t$ is important we will suppress them from our notation, similarly for the variable set $X$.

The Macdonald scalar product $\langle \cdot, \cdot \rangle$ on $\Lambda$ is defined via

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \left( \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \right) \left( \prod_{i=1}^{\lambda_1} i^{m_i(\lambda)} m_i(\lambda)! \right). \tag{2.2.1}$$

The following definition can be found in Chapter VI of [64].

**Definition 2.2.1.** Macdonald symmetric functions $P_\lambda$, $\lambda \in \mathbb{Y}$, are the unique linear basis of $\Lambda$ such that

1. $\langle P_\lambda, P_\mu \rangle = 0$ unless $\lambda = \mu$.

2. The leading (with respect to reverse lexicographic order) monomial in $P_\lambda$ is $\prod_{i=1}^{\ell(\lambda)} x_i^{\lambda_i}$.

*Remark* 2.2.2. The Macdonald symmetric function $P_\lambda$ is a homogeneous symmetric function of degree $|\lambda|$.

*Remark* 2.2.3. If we set $x_{N+1} = x_{N+2} = \cdots = 0$ in $P_\lambda(X)$, then we obtain the symmetric polynomials $P_\lambda(x_1, ..., x_N)$ in $N$ variables, which are called the Macdonald polynomials.

There is a second family of Macdonald symmetric functions $Q_\lambda$, $\lambda \in \mathbb{Y}$, which are dual to $P_\lambda$ with respect to the Macdonald scalar product:

$$Q_\lambda = \langle P_\lambda, P_\lambda \rangle^{-1} P_\lambda, \quad \langle P_\lambda, Q_\mu \rangle = \delta_{\lambda,\mu}, \ \ \lambda, \mu \in \mathbb{Y}.$$

For two sets of variables $X = (x_1, x_2, ...)$ and $Y = (y_1, y_2, ...)$ define

$$\Pi(X;Y) = \sum_{\lambda \in \mathbb{Y}} P_\lambda(X) Q_\lambda(Y).$$

Then from Chapter VI (2.5) in [64] we have

$$\Pi(X;Y) = \prod_{i,j=1}^{\infty} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}, \tag{2.2.2}$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$ is the $q$-Pochhammer symbol. The above equality holds when both sides are viewed as formal power series in the variables $X$, $Y$ and it is known as the Cauchy identity.

We next proceed to define the skew Macdonald symmetric functions (see Chapter VI in [64] for details). Take two sets of variables $X = (x_1, x_2, ...)$ and $Y = (y_1, y_2, ...)$ and a symmetric function $f \in \Lambda$. Let $(X, Y)$ denote the union of sets of variables $X$ and $Y$. Then we can view $f(X, Y) \in \Lambda_{(X,Y)}$ as a symmetric function in $x_i$ and $y_i$ together. More precisely, let

$$f = \sum_{\lambda \in \mathbb{Y}} C_\lambda p_\lambda = \sum_{\lambda \in \mathbb{Y}} C_\lambda \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i},$$

be the expansion of $f$ into the basis $p_\lambda$ of power symmetric functions (in the above sum

$C_\lambda = 0$ for all but finitely many $\lambda$). Then we have

$$f(X,Y) = \sum_{\lambda \in \mathbb{Y}} C_\lambda \prod_{i=1}^{\ell(\lambda)} (p_{\lambda_i}(X) + p_{\lambda_i}(Y)).$$

In particular, we see that $f(X,Y)$ is the sum of products of symmetric functions of $x_i$ and symmetric functions of $y_i$.

Skew Macdonald symmetric functions $P_{\lambda/\mu}$, $Q_{\lambda/\mu}$ are defined as the coefficients in the expansion

$$P_\lambda(X,Y) = \sum_{\mu \in \mathbb{Y}} P_\mu(X) P_{\lambda/\mu}(Y) \quad \text{and} \quad Q_\lambda(X,Y) = \sum_{\mu \in \mathbb{Y}} Q_\mu(X) Q_{\lambda/\mu}(Y) \tag{2.2.3}$$

*Remark* 2.2.4. The skew Macdonald symmetric function $P_{\lambda/\mu}$ is 0 unless $\mu \subset \lambda$, in which case it is homogeneous of degree $|\lambda| - |\mu|$.

*Remark* 2.2.5. When $\lambda = \mu$, $P_{\lambda/\mu} = 1$ and if $\mu = \varnothing$ (the unique partition of 0), then $P_{\lambda/\mu} = P_\lambda$.

We mention here two important special cases for the skew Macdonald symmetric functions. Suppose $x_2 = x_3 = \cdots = 0$. Then we have

$$P_{\lambda/\mu}(x_1) = \psi_{\lambda/\mu} x_1^{|\lambda|-|\mu|} \quad \text{and} \quad Q_{\lambda/\mu}(x_1) = \phi_{\lambda/\mu} x_1^{|\lambda|-|\mu|},$$

whenever $\lambda \succ \mu$ and zero otherwise. The coefficients $\phi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ have exact formulas as is shown in Chapter VI (6.24) of [64], and we write them below. Let $f(u) = (tu;q)_\infty/(qu;q)_\infty$. If $\lambda \succ \mu$ then

$$\phi_{\lambda/\mu}(q,t) = \prod_{1 \le i \le j \le \ell(\lambda)} \frac{f(q^{\lambda_i - \lambda_j} t^{j-i}) f(q^{\mu_i - \mu_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})}, \tag{2.2.4}$$

$$\psi_{\lambda/\mu}(q,t) = \prod_{1 \le i \le j \le \ell(\mu)} \frac{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})}, \tag{2.2.5}$$

otherwise the coefficients are zero.

### 2.2.3 The Macdonald process

A *specialization* $\rho$ of $\Lambda$ is a unital algebra homomorphism of $\Lambda$ to $\mathbb{C}$. We denote the application of $\rho$ to $f \in \Lambda$ as $f(\rho)$. One example of a specialization is the *trivial* specialization $\varnothing$, which takes the value 1 at the constant function $1 \in \Lambda$ and the value 0 at any homogeneous $f \in \Lambda$ of degree $\ge 1$. Since the power sums $p_n$ are algebraically independent generators of $\Lambda$, a specialization $\rho$ is uniquely defined by the numbers $p_n(\rho)$. Conversely, given any sequence $\alpha = a_1, a_2, \ldots$ of complex numbers, we can define a specialization $\rho_\alpha$ by setting $p_n(\rho_\alpha) = a_n$ and linearly extending to the rest of $\Lambda$.

Given two specializations $\rho_1$ and $\rho_2$ we define their union $\rho = (\rho_1, \rho_2)$ as the specialization defined on power sum symmetric functions via

$$p_n(\rho_1, \rho_2) = p_n(\rho_1) + p_n(\rho_2), \quad n \ge 1.$$

One specialization that we will consider frequently is of the form $x_1 = a_1, ..., x_N = a_N$ and $x_k = 0$ for $k > N$, where $a_1, ..., a_N$ are given complex numbers. That is, we set

$$p_n = \sum_{i=1}^{N} a_i^n \text{ for all } n \in \mathbb{N}.$$

Notice that the above is well defined even if $N = \infty$, provided that $\sum_i |a_i|^n < \infty$ for each $n \geq 1$, which is ensured if $\sum_i |a_i| < \infty$. If $N < \infty$ we call the above a *finite length* specialization.

**Definition 2.2.6.** We say that a specialization $\rho$ of $\Lambda$ is Macdonald nonnegative (or just 'nonnegative') if it takes nonnegative values on the skew Macdonald symmetric functions: $P_{\lambda/\mu}(\rho) \geq 0$ for any partitions $\lambda$ and $\mu$.

One can show (see e.g. Section 2.2 in [24]) that if we have $a_i \geq 0$ and $\sum_i a_i < \infty$ in the specialization we considered before, then it is nonnegative. Such a specialization is called *Pure alpha*. We remark that finite unions of nonnegative specializations are nonnegative (see Section 2.2 in [24]).

Let $\rho_1$ and $\rho_2$ be two non-negative specializations, then one defines

$$\Pi(\rho_1, \rho_2) = \sum_{\lambda \in \mathbb{Y}} P_\lambda(\rho_1) Q_\lambda(\rho_2),$$

the latter being well-defined in $[1, \infty]$ (observe that $P_\varnothing(\rho_1) = 1 = Q_\varnothing(\rho_2)$, so that $\Pi(\rho_1, \rho_2) \geq 1$).

We now formulate the definition of the *Macdonald process*. Let $N$ be a natural number and fix nonnegative specializations $\rho_0^+, ..., \rho_{N-1}^+, \rho_1^-, ..., \rho_N^-$, such that $\Pi(\rho_i^+, \rho_j^-) < \infty$ for all $i, j$. Consider two sequences of partitions $\lambda = (\lambda^1, ..., \lambda^N)$ and $\mu = (\mu^1, ..., \mu^{N-1})$. We define their weight as

$$\mathcal{W}(\lambda, \mu) = P_{\lambda^1}(\rho_0^+) Q_{\lambda^1/\mu^1}(\rho_1^-) P_{\lambda^2/\mu^1}(\rho_1^+) \cdots P_{\lambda^N/\mu^{N-1}}(\rho_{N-1}^+) Q_{\lambda^N}(\rho_N^-). \tag{2.2.6}$$

**Definition 2.2.7.** With the above notation, the Macdonald process $\mathbf{M}(\rho_0^+, ..., \rho_{N-1}^+; \rho_1^-, ..., \rho_N^-)$ is the probability measure on sequences $(\lambda, \mu)$, given by

$$\mathbf{M}(\rho_0^+, ..., \rho_{N-1}^+; \rho_1^-, ..., \rho_N^-)(\lambda, \mu) = \frac{\mathcal{W}(\lambda, \mu)}{\prod_{0 \leq i < j \leq N} \Pi(\rho_i^+; \rho_j^-)}.$$

Using properties of Macdonald symmetric functions one can show (see e.g. Proposition 2.4 in Section 2 of [24]) that the above definition indeed produces a probability measure, that is

$$\sum_{\lambda, \mu} \mathcal{W}(\lambda, \mu) = \prod_{0 \leq i < j \leq N} \Pi(\rho_i^+; \rho_j^-).$$

The Macdonald process with $N = 1$ is called the *Macdonald measure* and is written as $\mathbf{MM}(\rho^+; \rho^-)$.

One important feature of Macdonald processes is that if we pick out subsequences of $(\lambda, \mu)$, then their distribution is also a Macdonald process (with possibly different specializations). One special case that is important for us is the distribution of $\lambda^k$ under projection of the above law. As shown in Section 2 of [24], $\lambda^k$ is distributed according to the Macdonald measure $\mathbf{MM}(\rho^+_{[0,k-1]}; \rho^-_{[k,N]})$, where $\rho^\pm_{[a,b]}$ denotes the union of specializations $\rho^\pm_m$, $m = a, ..., b$.

### 2.2.4 The measure $\mathbb{P}^{r,t}_{HL}$ as a limit of Macdonald processes.

The main object of interest in this chapter is a distribution $\mathbb{P}^{r,t}_{HL}$ on plane partitions, depending on two parameters $r, t \in (0,1)$, which satisfies $\mathbb{P}_{HL}(\pi) \propto r^{|\pi|}A_\pi(t)$ for a certain explicit polynomial $A_\pi$, depending on the geometry of $\pi$ (see Section 2.1.1 for the details). We explain how this measure arises as a limit of Macdonald processes with $q = 0$.

Start by fixing a natural number $N$ and consider sequences of partitions $\lambda^{-N+1}, \dots, \lambda^{N-1}$

$$\varnothing \prec \lambda^{-N+1} \prec \dots \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \dots \succ \lambda^{N-1} \succ \varnothing.$$

The latter sequences exactly represent the set of plane partitions, whose support lies in a square of size $N$, i.e. the set $\{\pi : \pi_{i,j} = 0 \text{ if } i > N \text{ or } j > N\}$ (see Section 2.2.1). We next consider the collection of finite length specializations $\rho^+_n$, $\rho^-_n$ given by

$$\begin{aligned}
&\rho^+_n : x_1 = r^{-n-1/2}, x_2 = x_3 = \dots = 0 && -N \le n \le -1,\\
&\rho^-_n : x_1 = x_2 = x_3 = \dots = 0 && -N+1 \le n \le -1,\\
&\rho^-_n : x_1 = r^{n+1/2}, x_2 = x_3 = \dots = 0 && 0 \le n \le N-1,\\
&\rho^+_n : x_1 = x_2 = x_3 = \dots = 0 && 0 \le n \le N-2.
\end{aligned}$$

Consider the Macdonald process $\mathbf{M}(\rho^+_{-N}, ...\rho^+_{N-2}; \rho^-_{-N+1}, ...\rho^-_{N-1})$ and recall that the probability of a pair of sequences $(\lambda, \mu)$ with $\lambda = (\lambda^{-N+1}, \cdots, \lambda^{N-1})$ and $\mu = (\mu^{-N+1}, \cdots, \mu^{N-2})$ is given by

$$\mathbf{M}(\rho^+_{-N}, ...\rho^+_{N-2}; \rho^-_{-N+1}, ...\rho^-_{N-1}) = \frac{\prod_{n=-N+1}^{N-1} P_{\lambda^n/\mu^{n-1}}(\rho^+_{n-1}) Q_{\lambda^n/\mu^n}(\rho^-_n)}{\prod_{-N+1 \le i < j \le N-2} \Pi(\rho^+_i; \rho^-_j)}, \tag{2.2.7}$$

where we set $\mu^{-N} = \mu^{N-1} = \varnothing$. Using properties of skew Macdonald polynomials we see that the above product is zero unless

- $\mu^n = \lambda^n$ for $n < 0$ and $\mu^n = \lambda^{n+1}$ for $n \ge 0$,
- $\varnothing \prec \lambda^{-N+1} \prec \dots \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \dots \succ \lambda^{N-1} \succ \varnothing$.

Under the two conditions above the numerator in (2.2.7) equals (see Section 2.2.2)

$$\prod_{n=-N+1}^{0} \psi_{\lambda^n/\mu^{n-1}}(q,t) r^{(-2n+1)(|\lambda^n| - |\mu^{n-1}|)/2} \times \prod_{n=1}^{N} \phi_{\lambda^{n-1}/\mu^{n-1}}(q,t) r^{(2n+1)(|\lambda^{n-1}| - |\mu^{n-1}|)/2}.$$

Using that $\mu^n = \lambda^n$ for $n < 0$ and $\mu^n = \lambda^{n+1}$ for $n \ge 0$ we get

$$\sum_{n=-N+1}^{0} \frac{-2n+1}{2}(|\lambda^n| - |\mu^{n-1}|) + \sum_{n=1}^{N} \frac{2n+1}{2}(|\lambda^{n-1}| - |\mu^{n-1}|) = \sum_{n=-N+1}^{0} \frac{-2n+1}{2}(|\lambda^n| - |\lambda^{n-1}|)$$

$$+\sum_{n=1}^{N}\frac{2n+1}{2}(|\lambda^{n-1}|-|\lambda^n|)=\sum_{n=-N+1}^{-1}|\lambda^n|+\frac{1}{2}|\lambda^0|+\frac{1}{2}|\lambda^0|+\sum_{n=1}^{N-1}|\lambda^n|=\sum_{n=-N+1}^{N-1}|\lambda^n|=|\pi|$$

where we set $\lambda^{-N}=\lambda^N=\varnothing$ and $\pi$ is the plane partition corresponding to the diagonal slices $\lambda^n$ (see Section 2.2.1).

Letting $q\to 0$ in equations (2.2.4) and (2.2.5) we get (see (5.8) and (5.8′) in Chapter 3 of [64]):

$$\phi_{\lambda/\mu}(0,t)=\prod_{i\in I}(1-t^{m_i(\lambda)})\quad\text{and}\quad\psi_{\lambda/\mu}(0,t)=\prod_{j\in J}(1-t^{m_j(\mu)}).$$

In the above formula we assume $\lambda\succ\mu$ otherwise both expressions equal 0. The sets $I,J$ are:

$$I=\{i\in\mathbb{N}:\lambda'_{i+1}=\mu'_{i+1}\text{ and }\lambda'_i>\mu'_i\}\text{ and }J=\{j\in\mathbb{N}:\lambda'_{j+1}>\mu'_{j+1}\text{ and }\lambda'_j=\mu'_j\}.$$

Summarizing the above work, we see that $\mathbf{M}(\rho^+_{-N},...\rho^+_{N-2};\rho^-_{-N+1},...\rho^-_{N-1})$ induces a probability measure on sequences $\varnothing\prec\lambda^{-N+1}\prec\cdots\prec\lambda^{-1}\prec\lambda^0\succ\lambda^1\succ\cdots\succ\lambda^{N-1}\succ\varnothing$ and hence on plane partitions $\pi$, whose support lies in the square of size $N$. Call the latter measure $\mathbb{P}^{r,t,N}_{HL}$ and observe that

$$\mathbb{P}^{r,t,N}_{HL}(\pi)=Z_N^{-1}r^{|\pi|}\prod_{n=-N+1}^{0}\psi_{\lambda^n/\lambda^{n-1}}(0,t)\times\prod_{n=1}^{N}\phi_{\lambda^{n-1}/\lambda^n}(0,t)=Z_N^{-1}r^{|\pi|}B_\pi(t),$$

where $B_\pi(t)$ is an integer polynomial in $t$ and $Z_N$ is a normalizing constant. In [81] it was shown that $B_\pi(t)=A_\pi(t)$ and the normalizing constant was evaluated to equal

$$Z_N(r,t)=\prod_{i=1}^{N}\prod_{j=1}^{N}\frac{1-tr^{i+j-1}}{1-r^{i+j-1}}.$$

*Remark* 2.2.8. The "HL" in our notation stands for Hall-Littlewood, since in the limit $q\to 0$ the Macdonald symmetric functions $P_\lambda(X;q,t)$ and $Q_\lambda(X;q,t)$ degenerate to the Hall-Littlewood symmetric functions $P_\lambda(X;t)$ and $Q_\lambda(X;t)$.

As $N\to\infty$ the measures $\mathbb{P}^{r,t,N}_{HL}$ converge to the measure $\mathbb{P}^{r,t}_{HL}$ since $\lim_{N\to\infty}Z_N(r,t)=Z(r,t)$ - the normalizing constant in the definition of $\mathbb{P}^{r,t}_{HL}$ (see (2.1.2)). Thus, we indeed see that $\mathbb{P}^{r,t}_{HL}$ arises as a limit of Macdonald processes, in which the parameter $q$ is set to 0.

Our approach of studying $\mathbb{P}^{r,t}_{HL}$ goes through understanding the distribution of the diagonal slices $\lambda^k$. For $N>|k|$ we have that

$$\mathbb{P}^{r,t,N}_{HL}(\lambda^k=\lambda)=Z_N^{-1}P_\lambda(r^{1/2},\cdots,r^{(2N-1)/2};t)Q_\lambda(r^{1/2+|k|},\cdots,r^{(2N-1)/2},\underbrace{0,0,...,0}_{|k|};t),$$

where we used results in Section 2.2.3 and the proportionality of $P_\lambda$ and $Q_\lambda$ to combine the cases $k\geq 0$ and $k<0$. Letting $N\to\infty$ we conclude that

$$\mathbb{P}^{r,t}_{HL}(\lambda^k=\lambda)=Z(r,t)^{-1}P_\lambda(r^{1/2},r^{3/2},\cdots;t)Q_\lambda(r^{1/2+|k|},r^{3/2+|k|},\cdots;t).$$

Finally, using the homogeneity of $P_\lambda$ and $Q_\lambda$, we see that

$$\mathbb{P}^{r,t}_{HL}(\lambda^k = \lambda) = Z(r,t)^{-1} P_\lambda(a, ar, ar^2, \cdots; t) Q_\lambda(a, ar, ar^2, \cdots; t),$$

where $a(k) = r^{(1+|k|)/2}$. It is this distribution, which we call the *Hall-Littlewood measure* with parameters $a, r, t \in (0,1)$, that we will analyze in subsequent sections.

### 2.2.5 Background on Fredholm determinants

We present a brief background on Fredholm determinants. For a general overview of the theory of Fredholm determinants, the reader is referred to [75] and [61]. For our purposes the definition below is sufficient and we will not require additional properties.

**Definition 2.2.9.** Fix a Hilbert space $L^2(X, \mu)$, where $X$ is a measure space and $\mu$ is a measure on $X$. When $X = \Gamma$, a simple (anticlockwise oriented) smooth contour in $\mathbb{C}$ we write $L^2(\Gamma)$ where for $z \in \Gamma$, $d\mu(z)$ is understood to be $\frac{dz}{2\pi\iota}$.

Let $K$ be an *integral operator* acting on $f(\cdot) \in L^2(X, \mu)$ by $Kf(x) = \int_X K(x,y) f(y) d\mu(y)$. $K(x,y)$ is called the *kernel* of $K$ and we assume throughout $K(x,y)$ is continuous in both $x$ and $y$. If $K$ is a *trace-class* operator then one defines the Fredholm determinant of $I + K$, where $I$ is the identity operator, via

$$\det(I+K)_{L^2(X)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \cdots \int_X \det\left[K(x_i, x_j)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i), \tag{2.2.8}$$

where the latter sum can be shown to be absolutely convergent (see [75]).

A sufficient condition for the operator $K(x,y)$ to be trace-class is the following (see [61] page 345).

**Lemma 2.2.10.** *An operator $K$ acting on $L^2(\Gamma)$ for a simple smooth contour $\Gamma$ in $\mathbb{C}$ with integral kernel $K(x,y)$ is trace-class if $K(x,y) : \Gamma^2 \to \mathbb{R}$ is continuous as well as $K_2(x,y)$ is continuous in $y$. Here $K_2(x,y)$ is the derivative of $K(x,y)$ along the contour $\Gamma$ in the second entry.*

The expression appearing on the RHS of (2.2.8) can be absolutely convergent even if $K$ is not trace-class. In particular, this is so if $X = \Gamma$ is a piecewise smooth, oriented compact contour and $K(x,y)$ is continuous on $X \times X$. Let us check the latter briefly.

Since $K(x,y)$ is continuous on $X \times X$, which is compact, we have $|K(x,y)| \leq A$ for some constant $A > 0$, independent of $x, y \in X$. Then by Hadamard's inequality[1] we have

$$\left| \det\left[K(x_i, x_j)\right]_{i,j=1}^n \right| \leq n^{n/2} A^n.$$

This implies that

$$\left| \frac{1}{n!} \int_X \cdots \int_X \det\left[K(x_i, x_j)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i) \right| \leq \frac{n^{n/2} B^n}{n!},$$

[1] Hadamard's inequality: the absolute value of the determinant of an $n \times n$ matrix is at most the product of the lengths of the column vectors.

where $B = A|\mu|(X)$. The latter is absolutely summable because of the $n!$ in the denominator.

Whenever $X$ and $K$ are such that the RHS in (2.2.8) is absolutely convergent, we will still call it $\det(I+K)_{L^2(X)}$. The latter is no longer a Fredholm determinant, but some numeric quantity we attach to the kernel $K$. Of course, if $K$ is the kernel of a trace-class operator on $L^2(X)$ this numeric quantity agrees with the Fredholm determinant. Doing this allows us to work on the level of numbers throughout most of the text, and avoid constantly checking if the kernels we use represent a trace-class operator.

The following lemmas provide a framework for proving convergence of Fredholm determinants, based on pointwise convergence and estimates of their defining kernels.

**Lemma 2.2.11.** *Suppose that $\Gamma$ is a piecewise smooth contour in $\mathbb{C}$ and $K^N(x,y)$, $N \in \mathbb{N}$ or $N = \infty$, are measurable kernels on $\Gamma \times \Gamma$ such that $\lim_{N\to\infty} K^N(x,y) = K^\infty(x,y)$ for all $x, y \in \Gamma$. In addition, suppose there is a non-negative, measurable function $F(x)$ on $\Gamma$ with*

$$\sup_{N\in\mathbb{N}} \sup_{y\in\Gamma} |K^N(x,y)| \leq F(x) \text{ and } \int_\Gamma F(x)|d\mu(x)| = M < \infty.$$

*Then for each $n \geq 1$ and $N$ one has that $\det\left[K^N(x_i,x_j)\right]_{i,j=1}^n$ is integrable on $\Gamma^n$, so that in particular $\int_\Gamma \cdots \int_\Gamma \det\left[K^N(x_i,x_j)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)$ is well defined. Moreover, for each $N$*

$$\det(I+K^N)_{L^2(\Gamma)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_\Gamma \cdots \int_\Gamma \det\left[K^N(x_i,x_j)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)$$

*is absolutely convergent and $\lim_{N\to\infty} \det(I+K^N)_{L^2(\Gamma)} = \det(I+K^\infty)_{L^2(\Gamma)}$.*

*Proof.* The following is similar to Lemma 8.5 in [26]; however, it allows for infinite contours $\Gamma$ and assumes a weaker pointwise convergence of the kernels, while requiring a dominating function $F$. The idea is to use the Dominated Convergence Theorem multiple times.

Since $\lim_{N\to\infty} K^N(x,y) = K^\infty(x,y)$ we know that $\sup_{y\in\Gamma} |K^\infty(x,y)| \leq F(x)$ and also

$$\lim_{N\to\infty} \det\left[K^N(x_i,x_j)\right]_{i,j=1}^n = \det\left[K^\infty(x_i,x_j)\right]_{i,j=1}^n \text{ for all } x_1, \dots, x_n \in \Gamma.$$

By Hadamard's inequality we have $\left|\det\left[K^N(x_i,x_j)\right]_{i,j=1}^n\right| \leq n^{n/2}\prod_{i=1}^n F(x_i)$, which is integrable by assumption. It follows from the Dominated Convergence Theorem with dominating function $n^{n/2}\prod_{i=1}^n F(x_i)$ that for each $n \geq 1$ one has

$$\lim_{N\to\infty} \int_\Gamma \cdots \int_\Gamma \det\left[K^N(x_i,x_j)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i) = \int_\Gamma \cdots \int_\Gamma \det\left[K^\infty(x_i,x_j)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i).$$

Next observe that

$$\left|\int_\Gamma \cdots \int_\Gamma \det\left[K^N(x_i,x_j)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(x_i)\right| \leq \int_\Gamma \cdots \int_\Gamma \left|\det\left[K^N(x_i,x_j)\right]_{i,j=1}^n\right| \prod_{i=1}^n |d\mu(x_i)| \leq n^{n/2}M^n.$$

The latter shows the absolute convergence of the series, defining $\det(I + K^N)_{L^2(\Gamma)}$ for each $N$. A second application of the Dominated Convergence Theorem with dominating series $1 + \sum_{n \geq 1} \frac{n^{n/2} M^n}{n!}$ now shows the last statement of the lemma. □

**Lemma 2.2.12.** *Suppose that $\Gamma_1, \Gamma_2$ are piecewise smooth contours and $g^N_{x,y}(z)$ are measurable on $\Gamma_1^2 \times \Gamma_2$ for $N \in \mathbb{N}$ or $N = \infty$ and satisfy $\lim_{N\to\infty} g^N_{x,y}(z) = g^\infty_{x,y}(z)$ for all $x, y \in \Gamma_1$, $z \in \Gamma_2$. In addition, suppose that there exist bounded non-negative measurable functions $F_1$ and $F_2$ on $\Gamma_1$ and $\Gamma_2$ respectively such that*

$$\sup_{N \in \mathbb{N}} \sup_{y \in \Gamma_1} |g^N_{x,y}(z)| \leq F_1(x) F_2(z), \text{ and } \int_{\Gamma_i} F_i(u) |d\mu(u)| = M_i < \infty.$$

*Then $\int_{\Gamma_2} |g^N_{x,y}(z)||d\mu(z)| < \infty$ for each $N$ and $K^N(x,y) := \int_{\Gamma_2} g^N_{x,y}(z) d\mu(z)$ are well-defined and satisfy the conditions of Lemma 2.2.11 with $\Gamma = \Gamma_1$ and $F = M_2 F_1$.*

*Proof.* Since $\lim_{N\to\infty} g^N_{x,y}(z) = g^\infty_{x,y}(z)$ for all $x, y \in \Gamma_1$, $z \in \Gamma_2$ we know that $|g^\infty_{x,y}(z)| \leq F_1(x) F_2(z)$ as well. Observe that for each $x, y \in \Gamma_1$ and $N$ one has that

$$\int_{\Gamma_2} |g^N_{x,y}(z)||d\mu(z)| \leq \int_{\Gamma_2} F_1(x) F_2(z) |d\mu(z)| \leq M_2 F_1(x) < \infty.$$

Setting $K^N(x,y) = \int_{\Gamma_2} g^N_{x,y}(z) d\mu(z)$, we see that $|K^N(x,y)| \leq M_2 F_1(x)$ for each $x, y \in \Gamma_1$ and $N$. As an easy consequence of Fubini's Theorem one has that $K^N(x,y)$ is measurable on $\Gamma_1^2$ (the case of real functions and measures $\mu$ can be found in Corollary 3.4.6 of [15], from which the complex extension is immediate). Using the Dominated Convergence Theorem with dominating function $F_1(x) F_2(z)$ we see that $\lim_{N\to\infty} K^N(x,y) = K^\infty(x,y)$.

□

## 2.3 Finite length formulas

In this section, we derive formulas for the $t$-Laplace transform of the random variable $(1 - t)t^{-\lambda_1'}$, where $\lambda$ is distributed according to the finite length Hall-Littlewood measure $\mathbb{P}_{X,Y}$ (see Section 2.2.4). The main result in this section is Proposition 2.3.10, which expresses the $t$-Laplace transform as a Fredholm determinant. We believe that such a formula is of separate interest as it can be applied to generic Hall-Littlewood measures and its Fredholm determinant form makes it suitable for asymptotic analysis. The derivation of Proposition 2.3.10 goes through a sequence of steps that is very similar to the work in Sections 2.2.3, 3.1 and 3.2 of [24]. There are, however, several technical modifications that need to be made, which require us to redo most of the work there. In particular, the statements below do not follow from some simple limit transition from those in [24].

In all statements in the remainder of this chapter we will be working with the principal branch of the logarithm.

### 2.3.1 Observables of Hall-Littlewood measures

In this section we describe a framework for obtaining certain observables of Macdonald measures. Our discussion will be very much in the spirit of section 2.2.3 in [24]; however, the results we need do not directly follow from that work and so we derive them explicitly. In this chapter we will be primarily working with finite length specializations, which greatly simplifies the discussion; however, we mention that the results below can be derived in a much more general setting as is done in [28]. Finally, our focus will be on the case when $q = 0$ in the Macdonald measure and we call this degeneration a *Hall-Littlewood measure*.

In what follows we fix a natural number $N$ and consider the space of functions in $N$ variables. Inside this space lies the space of symmetric polynomials $\Lambda_X$ in $N$ variables $X = (x_1, ..., x_N)$.

**Definition 2.3.1.** For any $u \in \mathbb{R}$ and $1 \leq i \leq n$ define the shift operator $T_{u,x_i}$ by

$$(T_{u,x_i} F)(x_1, ..., x_N) := F(x_1, ..., ux_i, ..., x_N).$$

For any subset $I \subset \{1, ..., N\}$ of size $r$ define

$$A_I(X; t) := t^{\frac{r(r-1)}{2}} \prod_{i \in I, j \notin I} \frac{tx_i - x_j}{x_i - x_j}.$$

Finally, for any $r = 1, 2, ..., N$ define the Macdonald difference operator

$$D_N^r := \sum_{\substack{I \subset \{1,...,N\} \\ |I| = r}} A_I(X; t) \prod_{i \in I} T_{q,x_i}.$$

A key property of the Macdonald difference operators is that they are diagonalized by the Macdonald polynomials $P_\lambda$. Specifically, as shown in Chapter VI (4.15) of [64], we have

**Proposition 2.3.2.** *For any partition $\lambda$ with $\ell(\lambda) \leq N$*

$$D_N^r P_\lambda(x_1, ..., x_N; q, t) = e_r(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, ..., q^{\lambda_N}) P_\lambda(x_1, ..., x_N; q, t),$$

*where $e_r$ denote the elementary symmetric functions (see Section 2.2.2).*

In particular, we see that

$$D_N^1 P_\lambda(x_1, ..., x_N; q, t) = \left( \sum_{i=1}^N q^{\lambda_i} t^{N-i} \right) P_\lambda(x_1, ..., x_N; q, t).$$

We now let $q \to 0$, while $t \in (0, 1)$ is still fixed. In this limiting regime the Macdonald polynomials $P_\lambda(X; q, t)$ degenerate to the Hall-Littlewood polynomials $P_\lambda(X; t)$. In addition, the Macdonald difference operator $D_N^1$ degenerates to (we use the same notation)

$$D_N^1 = \sum_{i=1}^N \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{0,x_i}, \text{ and also } \sum_{i=1}^N q^{\lambda_i} t^{N-i} \to t^{N-\lambda_1'-1} + \cdots + t^0 = \frac{1 - t^{N-\lambda_1'}}{1-t}.$$

$D_N^1$ is still an operator on the space of functions in $N$ variables and we summarize the properties that we will need:

1. $D_N^1$ is linear.
2. If $F_n$ converge pointwise to a function $F$ in $N$ variables, then $D_N^1 F_n$ converge pointwise to $D_N^1 F$ away from the set $\{(x_1, ..., x_N) : x_i = x_j \text{ for some } i \neq j\}$ .
3. $D_N^1 P_\lambda(x_1, ..., x_N; t) = \frac{1 - t^{N-\lambda_1'}}{1-t} P_\lambda(x_1, ..., x_N; t)$.

**Proposition 2.3.3.** *Assume that* $F(u_1, ..., u_N) = f(u_1) \cdots f(u_N)$ *with* $f(0) = 1$. *Take* $x_1, ..., x_N > 0$ *and assume that* $f(u)$ *is holomorphic and non-zero in a complex neighborhood of an interval in* $\mathbb{R}$ *that contains* $x_1, ..., x_N$. *Then we have*

$$(D_N^1 F)(x_1, ..., x_N) = \frac{F(x_1, ..., x_N)}{2\pi\iota} \int_C \prod_{j=1}^{N} \frac{tz - x_j}{z - x_j} \frac{1}{f(z)} \frac{dz}{(t-1)z}, \tag{2.3.1}$$

*where* $C$ *is a positively oriented contour encircling* $\{x_1, ..., x_N\}$ *and no other singularities of the integrand.*

*Proof.* The following proof is very similar to the proof of Proposition 2.11 in [24]. First observe that from $t \in (0,1)$ and our assumptions on $f$ a contour $C$ will always exist. Using continuity of both sides in the variables $x_1, ..., x_N$ it suffices to prove the above when the $x_i$ are pairwise distinct. The contour encircles the simple poles at $x_1, ..., x_N$ and the residue at $x_i$ equals

$$\prod_{j \neq i}^{N} \frac{tx_i - x_j}{x_i - x_j} \frac{1}{f(x_i)}.$$

Using the Residue Theorem we conclude that the RHS of (2.3.1) equals

$$\sum_{i=1}^{N} F(x_1, ..., x_N) \prod_{j \neq i}^{N} \frac{tx_i - x_j}{x_i - x_j} \frac{1}{f(x_i)} = \sum_{i=1}^{N} \prod_{j \neq i}^{N} \frac{tx_i - x_j}{x_i - x_j} f(x_j) = (D_N^1 F)(x_1, ..., x_N).$$

□

We next consider the operator $\mathcal{D}_N = \left[\frac{(t-1)D_N^1 + 1}{t^N}\right]$. It satisfies Properties 1. and 2. above and Property 3. is replaced by

3.′ $\mathcal{D}_N P_\lambda(x_1, ..., x_N; t) = t^{-\lambda_1'} P_\lambda(x_1, ..., x_N; t)$.

**Proposition 2.3.4.** *Assume that* $F(u_1, ..., u_N) = f(u_1) \cdots f(u_N)$ *with* $f(0) = 1$. *Take* $x_1, ..., x_N > 0$ *and assume that* $f(u)$ *is holomorphic and non-zero in a complex neighborhood* $D$ *of an interval in* $\mathbb{R}$ *that contans* $x_1, ..., x_N$ *and* $0$. *Then for any* $k \geq 1$ *we have*

$$(\mathcal{D}_N^k F)(x_1, ..., x_N) = \frac{F(x_1, ..., x_N)}{(2\pi\iota)^k} \int_{C_{0,1}} \cdots \int_{C_{0,k}} \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^{k} \left[\prod_{j=1}^{N} \frac{z_i - x_j t^{-1}}{z_i - x_j}\right] \frac{1}{f(z_i)} \frac{dz_i}{z_i}, \tag{2.3.2}$$

*where $C_{0,a}$ are positively oriented simple contours encircling $x_1, ..., x_N$ and 0 and no zeros of $f(z)$. In addition, $C_{0,a}$ contains $t^{-1}C_{0,b}$ for $a < b$ and $C_{0,1} \subset D$.*

*Proof.* The proof is similar to the proof of Proposition 2.14 in [24]. In this proposition the existence of the contours $C_{0,a}$ depends on the properties of the function $f$. In what follows we will assume that they exist and whenever we use this result in the future with a particular function $f$ we will provide explicit contours satisfying the conditions in the proposition.

Using the continuity of both sides in $x_1, ..., x_N$ it suffices to show the result when the $x_i$ are pairwise distinct. We now proceed by induction on $k \in \mathbb{N}$.

*Base case:* $k = 1$. The RHS of (2.3.2) equals

$$\frac{F(x_1, ..., x_N)}{2\pi \iota} \int_{C_{0,1}} \left[\prod_{j=1}^{N} \frac{z_1 - x_j t^{-1}}{z_1 - x_j}\right] \frac{1}{f(z_1)} \frac{dz_1}{z_1}.$$

The contour $C_{0,1}$ encircles the simple poles of the integrand at $x_1, ..., x_N$ and 0 and the residue at 0 equals $t^{-N}$ (using $f(0) = 1$). If we now deform $C_{0,1}$ to a contour $C$, which no longer encircles 0 but does encirlce $x_1, ..., x_N$ we see, using the Residue Theorem, that the RHS of (2.3.2) equals

$$t^{-N} F(x_1, ..., x_N) + \frac{F(x_1, ..., x_N)}{2\pi \iota} \int_{C} \left[\prod_{j=1}^{N} \frac{z_1 - x_j t^{-1}}{z_1 - x_j}\right] \frac{1}{f(z_1)} \frac{dz_1}{z_1} = t^{-N} F(x_1, ..., x_N) +$$

$$(t-1)t^{-N} \frac{F(x_1, ..., x_N)}{2\pi \iota} \int_{C} \left[\prod_{j=1}^{N} \frac{t z_1 - x_j}{z_1 - x_j}\right] \frac{1}{f(z_1)} \frac{dz_1}{z_1(t-1)} = (\mathcal{D}_N F)(x_1, ..., x_N).$$

In the last equality we used Proposition 2.3.4 and the definition of $\mathcal{D}_N$. This proves the base case.

We next suppose that the result holds for $k \geq 1$ and wish to prove it for $k + 1$. In particular, we have

$$(\mathcal{D}_N^k F)(x_1, ..., x_N) = \frac{1}{(2\pi \iota)^k} \int_{C_{0,1}} \cdots \int_{C_{0,k}} \prod_{j=1}^{N} g(x_j; z_1, ..., z_k) \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^{k} \frac{1}{f(z_i)} \frac{dz_i}{z_i},$$

where $g(u; z_1, ..., z_k) = f(u) \prod_{i=1}^{k} \frac{z_i - u t^{-1}}{z_i - u}$.

We apply $\mathcal{D}_N$ to both sides in the above expression and observe we may switch the order of $\mathcal{D}_N$ and the integrals on the RHS. To see the latter, one may approximate the integrals by Riemann sums and use Property 1. of $\mathcal{D}_N$ to switch the order of the sums and the operator. Subsequently, one may use Property 2. to show that the change of the order also holds in the limit. We thus obtain

$$(\mathcal{D}_N^{k+1} F)(x_1, ..., x_N) = \frac{1}{(2\pi \iota)^k} \int_{C_{0,1}} \cdots \int_{C_{0,k}} (\mathcal{D}_N G)(x_1, ..., x_N; z_1, ..., z_k) \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^{k} \frac{dz_i}{z_i f(z_i)},$$

where $G(x_1,...,x_N;z_1,...,z_k) = \prod_{j=1}^N g(x_j;z_1,...,z_k)$. We now wish to apply the base case to the function $G$. Notice that $g(0)=1$ and the zeros of $g(u)$ coincide with those of $f(u)$ except that it has additional zeros at $tz_i$ for $i=1,...,k$. By assumption $tC_{0,i}$ contain $C_{0,k+1}$ for all $i=1,...,k$ so the additional zeros of $g(u)$ are not contained in $C_{0,k+1}$, while $x_1,...,x_N$ and 0 are. Thus the Base case is applicable and we conclude that

$$(\mathcal{D}_N^{k+1}F)(x_1,...,x_N) = \frac{1}{(2\pi\iota)^{k+1}}\int_{C_{0,1}}\cdots\int_{C_{0,k}}\int_{C_{0,k+1}} G(x_1,...,x_N;z_1,...,z_k)\left[\prod_{j=1}^N \frac{z_{k+1}-x_jt^{-1}}{z_{k+1}-x_j}\right]\times$$
$$\frac{1}{g(z_{k+1};z_1,...,z_k)}\prod_{1\leq a<b\leq k}\frac{z_a-z_b}{z_a-z_bt^{-1}}\frac{dz_{k+1}}{z_{k+1}}\prod_{i=1}^k\frac{1}{f(z_i)}\frac{dz_i}{z_i},$$

Expressing $g(z_{k+1};z_1,...,z_k)$ and $G(x_1,...,x_N;z_1,...,z_k)$ in terms of $f(z_i)$ and $F(x_1,...,x_N)$ we arrive at

$$(\mathcal{D}_N^{k+1}F)(x_1,...,x_N) = \frac{F(x_1,...,x_N)}{(2\pi\iota)^{k+1}}\int_{C_{0,1}}\cdots\int_{C_{0,k+1}}\prod_{1\leq a<b\leq k+1}\frac{z_a-z_b}{z_a-z_bt^{-1}}\prod_{i=1}^{k+1}\left[\prod_{j=1}^N\frac{z_i-x_jt^{-1}}{z_i-x_j}\right]\frac{dz_i}{f(z_i)z_i}.$$

This concludes the proof of the case $k+1$. The general result now proceeds by induction. □

Let $\rho_X$ and $\rho_Y$ be the nonnegative finite length specializations in $N$ variables $X=(x_1,...,x_N)$ and $Y=(y_1,...,y_N)$ respectively, with $x_i,y_i\in(0,1)$ for $i=1,...,N$. We consider the Macdonald measure $\mathbf{MM}(\rho_X;\rho_Y)$ with parameter $q=0$ and denote the probability distribution and expectation with respect to this measure by $\mathbb{P}_{X,Y}$ and $\mathbb{E}_{X,Y}$. Using the Cauchy identity (see equation (2.2.2)) with $q=0$ we get

$$\sum_{\lambda\in\mathbb{Y}}P_\lambda(x_1,...,x_N;t)Q_\lambda(y_1,...,y_N;t) = \prod_{i,j=1}^N\frac{1-tx_iy_j}{1-x_iy_j} = \prod_{i=1}^N f_Y(x_i) \text{ with } f_Y(u)=\textstyle\prod_{j=1}^N\frac{1-tuy_j}{1-uy_j}. \tag{2.3.3}$$

We want to apply $\mathcal{D}_N^k$ in the $X$ variable to both sides of (2.3.3). We observe that the sum on the LHS is absolutely convergent so from Properties 1. and 2. we see that

$$\mathcal{D}_N^k\sum_{\lambda\in\mathbb{Y}}P_\lambda(X;t)Q_\lambda(Y;t) = \sum_{\lambda\in\mathbb{Y}}\mathcal{D}_N^kP_\lambda(X;t)Q_\lambda(Y;t) = \sum_{\lambda\in\mathbb{Y}}t^{-k\lambda_1'}P_\lambda(X;t)Q_\lambda(Y;t), \tag{2.3.4}$$

where in the last equality we used Property 3.′ $k$ times. We remark that the latter sum is absolutely convergent as well, since $\lambda_1'\leq N$ on the support of $\mathbb{P}_{X,Y}$.

On the other hand, the RHS of (2.3.3) satisfies the conditions of Proposition 2.3.4 and in order to apply it we need to find suitable contours. The contours will exist provided $y_i$ are sufficiently small. So suppose $y_i<\epsilon\leq t^k$ for all $i$ and observe that the zeros of $f_Y(u)$, which are at $t^{-1}y_i^{-1}$, lie outside the circle of radius $\epsilon^{-1}t^{-1}$ around the origin. Let $C_{0,k}$ be the positively oriented circle around the origin of radius 1 and let $C_{0,a}$ be positively oriented circles of radius slightly bigger than $t^{a-k}$, so that $C_{0,a}$ contains $t^{-1}C_{0,b}$ for all $a<b$ and $C_{0,1}$ has radius less than $\epsilon^{-1}$. Clearly such contours exist and satisfy the conditions of Proposition

2.3.4. Consequently, we obtain

$$\mathcal{D}^k \prod_{i=1}^N f_Y(x_i) = \frac{\prod_{i=1}^N f_Y(x_i)}{(2\pi\iota)^k} \int_{C_{0,1}} \cdots \int_{C_{0,k}} \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^k \left[ \prod_{j=1}^N \frac{z_i - x_j t^{-1}}{z_i - x_j} \right] \frac{dz_i}{f_Y(z_i) z_i} \tag{2.3.5}$$

Equating the expressions in (2.3.4) and (2.3.5) and dividing by $\prod_{i=1}^N f_Y(x_i)$ we arrive at

$$\sum_{\lambda \in \mathbb{Y}} t^{-k\lambda_1'} \frac{P_\lambda(X;t) Q_\lambda(Y;t)}{\Pi(X;Y)} = \frac{1}{(2\pi\iota)^k} \int_{C_{0,1}} \cdots \int_{C_{0,k}} \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^k \left[ \prod_{j=1}^N \frac{z_i - x_j t^{-1}}{z_i - x_j} \right] \frac{dz_i}{f_Y(z_i) z_i},$$

in which we recognize the LHS as $\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right]$. We isolate the above result in a proposition.

**Proposition 2.3.5.** *Fix positive integers $k$ and $N$ and a parameter $t \in (0,1)$. Let $\rho_X$ and $\rho_Y$ be the nonnegative finite length specializations in $N$ variables $X = (x_1, ..., x_N)$ and $Y = (y_1, ..., y_N)$ respectively, with $x_i, y_i \in (0,1)$ for $i = 1, ..., N$. In addition, suppose $y_i < \epsilon$ for all $i$. Then we have*

$$\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right] = \frac{1}{(2\pi\iota)^k} \int_{C_{0,1}} \cdots \int_{C_{0,k}} \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^k \left[ \prod_{j=1}^N \frac{(z_i - x_j t^{-1})(1 - z_i y_j)}{(z_i - x_j)(1 - t z_i y_j)} \right] \frac{dz_i}{z_i},$$

*where $C_{0,a}$ are positively oriented simple contours encircling $x_1, ..., x_N$ and $0$ and contained in a disk of radius $\epsilon^{-1}$ around $0$. In addition, $C_{0,a}$ contains $t^{-1}C_{0,b}$ for $a < b$. Such contours will exist provided $\epsilon \leq t^k$.*

Proposition 2.3.5 is an important milestone in our discussion as it provides an integral representation for a class of observables for $\mathbb{P}_{X,Y}$. In subsequent sections, we will combine the above formulas for different values of $k$, similarly to the moment problem for random variables, in order to better understand the distribution $\mathbb{P}_{X,Y}$.

### 2.3.2 An alternative formula for $\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right]$

There are two difficulties in using Proposition 2.3.5. The first is that the contours that we use are all different and depend implicitly on the value $k$. The second issue is that the formula for $\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right]$ that we obtain holds only when $y_i$ are sufficiently small (again depending on $k$). We would like to get rid of this restriction by finding an alternative formula for $\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right]$. This is achieved in Proposition 2.3.7, whose proof relies on the following technical lemma. The following result is very similar to Proposition 7.2 in [22].

**Lemma 2.3.6.** *Fix $k \geq 1$ and $q \in (1, \infty)$. Assume that we are given a set of positively oriented closed contours $\gamma_1, ..., \gamma_k$, containing $0$, and a function $F(z_1, ..., z_k)$, satisfying the following properties:*

1. *$F(z_1, ..., z_k) = \prod_{i=1}^k f(z_i)$;*

2. *For all $1 \leq A < B \leq k$, the interior of $\gamma_A$ contains the image of $\gamma_B$ multiplied by $q$;*

3. *For all $1 \leq j \leq k$ there exists a deformation $D_j$ of $\gamma_j$ to $\gamma_k$ so that for all $z_1, ..., z_{j-1}, z_j, ..., z_k$ with $z_i \in \gamma_i$ for $1 \leq i < j$ and $z_i \in \gamma_k$ for $j < i \leq k$, the function $z_j \to F(z_1, ..., z_j, ..., z_k)$ is analytic in a neighborhood of the area swept out by the deformation $D_j$.*

*Then we have the following residue expansion identity:*

$$\int_{\gamma_1} \cdots \int_{\gamma_k} \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - qz_B} F(z_1, \cdots, z_k) \prod_{i=1}^{k} \frac{dz_i}{2\pi \iota z_i} = \sum_{\lambda \vdash k} \frac{(1-q)^k (-1)^k q^{\frac{-k(k-1)}{2}} k_q!}{m_1(\lambda)! m_2(\lambda)! \cdots} \int_{\gamma_k} \cdots \int_{\gamma_k} \det \left[ \frac{1}{w_i q^{\lambda_i} - w_j} \right]_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} f(w_j) f(w_j q) \cdots f(w_j q^{\lambda_j - 1}) \frac{dw_j}{2\pi \iota}, \tag{2.3.6}$$

*where $k_t! = \frac{(1-t)(1-t^2)\cdots(1-t^k)}{(1-t)^k}$.*

*Proof.* The proof of the lemma closely follows the proof of Proposition 7.2 in [22], and we will thus only sketch the main idea. We remark that in [22] the considered contours do not contain 0 and $q \in (0,1)$. Nevertheless, all the arguments remain the same and the result of that proposition hold in the setting of the lemma.

The strategy is to sequentially deform each of the contours $\gamma_{k-1}, \gamma_{k-2}, ..., \gamma_1$ to $\gamma_k$ through the deformations $D_i$ afforded from the hypothesis of the lemma. During the deformations one passes through simple poles, coming from $z_A - qz_B$ in the denominator of (2.3.6), which by the Residue Theorem produce additional integrals of possibly fewer variables. Once all the contours are expanded to $\gamma_k$ one obtains a big sum of multivariate integrals over various residue subspaces, which can be recombined into the following form (see equation (38) in [22]):

$$\sum_{\lambda \vdash k} \frac{(1-q)^k (-1)^k q^{\frac{-k(k-1)}{2}}}{m_1(\lambda)! m_2(\lambda)! \cdots} \int_{\gamma_k} \cdots \int_{\gamma_k} \det \left[ \frac{1}{w_i q^{\lambda_i} - w_j} \right]_{i,j=1}^{\ell(\lambda)} \times$$

$$E^q \left( w_1, qw_1, ..., q^{\lambda_1 - 1} w_1, ..., w_{\ell(\lambda)}, qw_{\ell(\lambda)}, ..., q^{\lambda_{\ell(\lambda)} - 1} w_{\ell(\lambda)} \right) \prod_{j=1}^{\ell(\lambda)} w_j^{\lambda_j} q^{\frac{\lambda_j(\lambda_j - 1)}{2}} \frac{dw_j}{2\pi \iota},$$

where

$$E^q(z_1, ..., z_k) = \sum_{\sigma \in S_k} \prod_{1 \leq B < A \leq k} \frac{z_{\sigma(A)} - qz_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \frac{F(z_{\sigma(1)}, ..., z_{\sigma(n)})}{\prod_{i=1}^{k} z_{\sigma(i)}}.$$

By assumption $\frac{F(z_1, ..., z_n)}{\prod_{i=1}^{k} z_i}$ is a symmetric function of $z_1, ..., z_k$ and thus can be taken out of the sum, while the remaining expression evaluates to $k_q!$ as is shown in equation (1.4) in Chapter III of [64]. Substituting this back and performing some cancellation we arrive at (2.3.6).

□

**Proposition 2.3.7.** *Fix positive integers $k$ and $N$ and a parameter $t \in (0,1)$. Let $\rho_X$ and $\rho_Y$ be the nonnegative finite length specializations in $N$ variables $X = (x_1, ..., x_N)$ and*

$Y = (y_1, ..., y_N)$ *respectively, with* $x_i, y_i \in (0,1)$ *for* $i = 1, ..., N$. *Let* $C_0$ *be a simple positively oriented contour, which is contained in the closed disk of radius* $t^{-1}$ *around the origin, such that* $C_0$ *encircles* $x_1, ..., x_N$ *and* $0$. *Then we have*

$$\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right] = \sum_{\lambda \vdash k} \frac{(t^{-1}-1)^k k_t!}{m_1(\lambda)! m_2(\lambda)! \cdots} \int_{C_0} \cdots \int_{C_0} \det\left[\frac{1}{w_i t^{-\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} \times$$
$$\prod_{j=1}^{\ell(\lambda)} \prod_{i=1}^{N} \frac{1 - x_i(w_j t)^{-1}}{1 - x_i(w_j t)^{-1} t^{\lambda_j}} \frac{1 - y_i(w_j t) t^{-\lambda_j}}{1 - y_i(w_j t)} \frac{dw_j}{2\pi\iota}, \text{ where } k_t! = \frac{(1-t)(1-t^2)\cdots(1-t^k)}{(1-t)^k}. \tag{2.3.7}$$

*Proof.* Let $C_{0,k} = C_0$ and let $C_{0,a}$ be such that $C_{0,a}$ contains $t^{-1}C_{0,b}$ for all $a < b$, $a, b \in \{1, ..., k\}$. Suppose $0 < \epsilon < t^k$ is sufficiently small so that $C_{0,1}$ is contained in the disk of radius $\epsilon^{-1}$ and suppose $y_i < \epsilon$ for $i = 1, ..., N$. Then we may apply Proposition 2.3.5 to get

$$\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right] = \frac{1}{(2\pi\iota)^k} \int_{C_{0,1}} \cdots \int_{C_{0,k}} \prod_{1 \le a < b \le k} \frac{z_a - z_b}{z_a - z_b t^{-1}} \prod_{i=1}^{k} \left[\prod_{j=1}^{N} \frac{(z_i - x_j t^{-1})(1 - z_i y_j)}{(z_i - x_j)(1 - t z_i y_j)}\right] \frac{dz_i}{z_i}.$$

We may now apply Lemma 2.3.6 (with $q = t^{-1}$) to the RHS of the above and get

$$\mathbb{E}_{X,Y}\left[t^{-k\lambda_1'}\right] = \sum_{\lambda \vdash k} \frac{(1-t^{-1})^k (-1)^k t^{\frac{k(k-1)}{2}} k_{t^{-1}}!}{m_1(\lambda)! m_2(\lambda)! \cdots} \int_{C_{0,k}} \cdots \int_{C_{0,k}} \det\left[\frac{1}{w_i t^{-\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} \times$$
$$\prod_{j=1}^{\ell(\lambda)} G(w_j) G(w_j t^{-1}) \cdots G(w_j t^{1-\lambda_j}) \frac{dw_j}{2\pi\iota}, \text{ where } G(w) = \prod_{j=1}^{N} \frac{w - x_j t^{-1}}{w - x_j} \frac{1 - y_j w}{1 - t y_j w}. \tag{2.3.8}$$

Observe that $(-1)^k t^{\frac{k(k-1)}{2}} k_{t^{-1}}!(1-t^{-1})^k = (t^{-1}-1)^k k_t!$ and also

$$\prod_{j=1}^{\ell(\lambda)} G(w_j) G(w_j t^{-1}) \cdots G(w_j t^{1-\lambda_j}) = \prod_{i=1}^{N} \prod_{j=1}^{\ell(\lambda)} \frac{1 - x_i(t w_j)^{-1}}{1 - x_i(t w_j)^{-1} t^{\lambda_j}} \frac{1 - y_i(w_j t) t^{-\lambda_j}}{1 - y_i(w_j t)}.$$

Substituting these expressions into (2.3.8) and recalling that $C_{0,k} = C_0$ we arrive at (2.3.7). What remains is to extend the result to arbitrary $y_1, ..., y_N \in (0,1)$ by analyticity. In particular, if we can show that both sides of (2.3.7) define analytic functions on $\mathbb{D}^N$ ($\mathbb{D}$ is the unit complex disk), then because they are equal on $(0,\epsilon)^N$ it would follow they are equal on $\mathbb{D}^N$. This would imply the full statement of the proposition.

We start with the RHS of (2.3.7). Observe that it is a finite sum of integrals over compact contours. Thus it suffices to show analyticity of the integrands in $y_i \in \mathbb{D}$. The integrand's dependence on $y_i$ is through $\prod_{j=1}^{\ell(\lambda)} \prod_{i=1}^{N} \frac{1 - y_i(w_j t) t^{-\lambda_j}}{1 - y_i(w_j t)}$, which is clearly analytic on $\mathbb{D}^N$ as $|w_j| \le t^{-1}$.

For the LHS of (2.3.7) we have:

$$\mathbb{E}_{x,y}\left[t^{-k\lambda_1'}\right] = \Pi(X;Y)^{-1}\sum_{\lambda\in\mathbb{Y}} P_\lambda(X)Q_\lambda(y_1,...,y_N),$$

where $\Pi(X;Y) = \prod_{i,j=1}^N \frac{1-tx_iy_j}{1-x_iy_j}$. Clearly $\Pi(X;Y)$ is analytic and non-zero on $\mathbb{D}^N$ (as $x_i \in (0,1)$) and then so is $\Pi(X;Y)^{-1}$. In addition, the sum is absolutely convergent on $\mathbb{D}^N$, since by the Cauchy identity

$$\sum_{\lambda\mathbb{Y}} |P_\lambda(X)Q_\lambda(y_1,...,y_N)| \leq \sum_{\lambda\in\mathbb{Y}} P_\lambda(X)Q_\lambda(|y_1|,...,|y_N|) = \prod_{i,j=1}^N \frac{1-tx_i|y_j|}{1-x_i|y_j|} < \infty.$$

As the absolutely converging sum of analytic functions is analytic and the product of two analytic functions is analytic we conclude that the LHS of (2.3.7) is analytic on $\mathbb{D}^N$. □

### 2.3.3 Fredholm determinant formula for $\mathbb{E}_{X,Y}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right]$

In this section we will combine Proposition 2.3.7 with different values of $k$ to obtain a formula for the *t-Laplace transform* of $(1-t)t^{-\lambda_1'}$, which is defined by $\mathbb{E}_{X,Y}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right]$. We recall that $(a;t)_\infty = (1-a)(1-at)(1-at^2)\cdots$ is the $t$-Pochhammer symbol.

**Proposition 2.3.8.** *Fix $N \in \mathbb{N}$ and $t \in (0,1)$. Let $\rho_X$ and $\rho_Y$ be the nonnegative finite length specializations in $N$ variables $X = (x_1,...,x_N)$ and $Y = (y_1,...,y_N)$ respectively, with $x_i, y_i \in (0,1)$ for $i = 1,...,N$. Suppose $|u| < t^{N+1}$ is a complex number. Then we have*

$$\lim_{M\to\infty}\sum_{k=0}^M \frac{u^k\mathbb{E}_{X,Y}[t^{-\lambda_1'k}]}{k_t!} = \mathbb{E}_{X,Y}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right]. \tag{2.3.9}$$

*Proof.* We have that

$$\sum_{k=0}^M \frac{u^k\mathbb{E}_{X,Y}[t^{-\lambda_1'k}]}{k_t!} = \sum_{c=0}^N \mathbb{P}_{X,Y}(\lambda_1'=c)\sum_{k=0}^M \frac{u^kt^{-ck}}{k_t!}$$

By our assumption on $u$ and Corollary 10.2.2a in [7] we have that the inner sum over $k$ converges to $\frac{1}{((1-t)ut^{-c};t)_\infty}$, as $M \to \infty$. Thus

$$\lim_{M\to\infty}\sum_{c=0}^N \mathbb{P}_{X,Y}(\lambda_1'=c)\sum_{k=0}^M \frac{u^kt^{-ck}}{k_t!} = \sum_{c=0}^N \frac{\mathbb{P}_{X,Y}(\lambda_1'=c)}{((1-t)ut^{-c};t)_\infty} = \mathbb{E}_{X,Y}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right].$$

□

**Proposition 2.3.9.** *Fix $N \in \mathbb{N}$, $t \in (0,1)$ and $x_i, y_i \in (0,1)$ for $i = 1,...,N$. Then there* *exists $\epsilon > 0$ such that for $|u| < \epsilon$ and $u \notin \mathbb{R}^+$ we have*

$$1 + \lim_{M\to\infty} \sum_{k=1}^{M} (t^{-1}-1)^k u^k \sum_{\lambda\vdash k} \frac{1}{m_1(\lambda)!m_2(\lambda)!\cdots} \int_{C_0}\cdots\int_{C_0} \det\left[\frac{1}{w_i t^{-\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)} \times$$
$$\prod_{j=1}^{\ell(\lambda)}\prod_{i=1}^{N} \frac{1-x_i(w_jt)^{-1}}{1-x_i(w_jt)^{-1}t^{\lambda_j}} \frac{1-y_i(w_jt)t^{-\lambda_j}}{1-y_i(w_jt)} \frac{dw_j}{2\pi\iota} = \det(I+K_u^N)_{L^2(C_0)}. \tag{2.3.10}$$

*In the above $C_0$ is the positively oriented circle of radius $t^{-1}$ around 0. $K_u^N$ is defined in terms of its integral kernel*

$$K_u^N(w;w') = \frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty} ds\Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g^N_{w,w'}(t^s),$$

*where*

$$g^N_{w,w'}(t^s) = \frac{1}{wt^{-s}-w'}\prod_{j=1}^{N}\frac{(1-x_j(wt)^{-1})(1-y_j(wt)t^{-s})}{(1-x_j(wt)^{-1}t^s)(1-y_j(wt))}.$$

The proof of Proposition 2.3.9 depends on two lemmas: Lemma 2.3.11 and Lemma 2.3.12, whose proof is postponed to Section 2.3.4. Our choice for $C_0$ is made in order to simplify the proof.

*Proof.* From Lemma 2.3.12 we know that $K_u^N$ is trace-class for $u \notin \mathbb{R}^+$. Consequently we have that

$$\det(I+K_u^N)_{L^2(C_0)} = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{C_0}\cdots\int_{C_0}\det\left[K_u^N(w_i,w_j)\right]_{i,j=1}^{n}\prod_{i=1}^{n}\frac{dw_i}{2\pi\iota} =$$

$$1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{C_0}\cdots\int_{C_0}\sum_{\sigma\in S_n} sign(\sigma)\prod_{i=1}^{n}\left[\frac{1}{2\pi i}\int_{1/2-\iota\infty}^{1/2+\iota\infty}\Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g_{w_i,w_{\sigma(i)}}(t^s)ds\right]\prod_{i=1}^{n}\frac{dw_i}{2\pi\iota}.$$

Using Lemma 2.3.11 and the above formula we can find an $\epsilon > 0$ such that for $|u| < \epsilon$ and $u \notin \mathbb{R}^+$ one has

$$\det(I+K_u^N) = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{C_0}\cdots\int_{C_0}\sum_{\sigma\in S_n} sign(\sigma)\prod_{i=1}^{n}\left[\sum_{j=1}^{\infty}u^j(t^{-1}-1)^j g_{w_i,w_{\sigma(i)}}(t^j)\right]\prod_{i=1}^{n}\frac{dw_i}{2\pi\iota}. \tag{2.3.11}$$

Let us introduce the following short-hand notation

$$B(c_1,...,c_n) := \int_{C_0}\cdots\int_{C_0}\det\left[\frac{1}{w_it^{-c_i}-w_j}\right]_{i,j=1}^{n}\prod_{j=1}^{n}\prod_{i=1}^{N}\frac{1-x_i(w_jt)^{-1}}{1-x_i(w_jt)^{-1}t^{c_j}}\frac{1-y_i(w_jt)t^{-c_j}}{1-y_i(w_jt)}\frac{dw_j}{2\pi\iota}.$$

Notice that $B(c_1,...,c_n)$ is invariant under permutation of its arguments and that $\frac{(m_1(\lambda)+m_2(\lambda)+\cdots)!}{m_1(\lambda)!m_2(\lambda)!\cdots}$

is the number of distinct permutations of the parts of $\lambda$. The latter suggests that

$$\sum_{\lambda \vdash k} \frac{(t^{-1}-1)^k u^k}{m_1(\lambda)! m_2(\lambda)! \cdots} B(\lambda_1, ..., \lambda_{\ell(\lambda)}) = \sum_{n \geq 1} \sum_{\substack{c_1, c_2, ..., c_n \geq 1 \\ \sum c_i = k}} \frac{(t^{-1}-1)^k u^k}{n!} B(c_1, ..., c_n).$$

Observe that for some positive constant $C$ we have

$$\left| \prod_{j=1}^{n} \prod_{i=1}^{N} \frac{1 - x_i (w_j t)^{-1}}{1 - x_i (w_j t)^{-1} t^{c_j}} \frac{1 - y_i (w_j t) t^{-c_j}}{1 - y_i (w_j t)} \right| \leq C^{Nn} t^{-Nk} \prod_{i=1}^{N} \frac{1}{(1-x_i)^n (1-y_i)^n}.$$

The above together with Hadamard's inequality and the compactness of $C_0$ implies that for some positive constants $P, Q$ (independent of $k$ and $n$) we have $|B(c_1, ..., c_n)| \leq n^{n/2} P^n Q^k$. The latter implies that for $|u| < \epsilon$ and $\epsilon$ sufficiently small the sum

$$\sum_{k=1}^{\infty} \sum_{n \geq 1} \sum_{\substack{c_1, c_2, ..., c_n \geq 1 \\ \sum c_i = k}} \frac{(t^{-1}-1)^k u^k}{n!} B(c_1, ..., c_n)$$

is absolutely convergent. In particular, the limit on the LHS of equation (2.3.10) exists and equals

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{c_1, c_2, ..., c_n \geq 1} [(t^{-1}-1)u]^{c_1 + \cdots + c_n} B(c_1, ..., c_n).$$

Expanding the determinant inside the integral in the definition of $B(c_1, ..., c_n)$ we see that the integrand equals $\sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} g_{w_i, w_{\sigma(i)}}(t^{c_i})$. Consequently the LHS of equation (2.3.10) equals

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{c_1, c_2, ..., c_n \geq 1} [(t^{-1}-1)u]^{c_1 + \cdots + c_n} \int_{C_0} \cdots \int_{C_0} \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} g_{w_i, w_{\sigma(i)}}(t^{c_i}) \frac{dw_i}{2\pi\iota}. \quad (2.3.12)$$

What remains is to check that the two expressions in (2.3.12) and (2.3.11) agree. Since both are absolutely converging sums over $n$, it suffices to show equality of the corresponding summands. I.e. we wish to show that

$$\sum_{c_1, c_2, ..., c_n \geq 1} [(t^{-1}-1)u]^{c_1 + \cdots + c_n} \int_{C_0} \cdots \int_{C_0} \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} g_{w_i, w_{\sigma(i)}}(t^{c_i}) \frac{dw_i}{2\pi\iota} =$$

$$= \int_{C_0} \cdots \int_{C_0} \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} \left[ \sum_{j=1}^{\infty} u^j (t^{-1}-1)^j g_{w_i, w_{\sigma(i)}}(t^j) \right] \frac{dw_i}{2\pi\iota}. \quad (2.3.13)$$

By Fubini's Theorem (provided $|u|$ is sufficiently small) we may interchange the order of the sum and the integrals and the LHS of equation (2.3.13) becomes

$$\int_{C_0}\cdots\int_{C_0}\sum_{c_1,c_2,\ldots,c_n\geq 1}[(t^{-1}-1)u]^{c_1+\ldots+c_n}\sum_{\sigma\in S_n}sign(\sigma)\prod_{i=1}^{n}g_{w_i,w_{\sigma(i)}}(t^{c_i})\frac{dw_i}{2\pi\iota}=$$

$$\int_{C_0}\cdots\int_{C_0}\sum_{\sigma\in S_n}sign(\sigma)\prod_{i=1}^{n}\left[\sum_{c_i\geq 1}[(t^{-1}-1)u]^{c_i}g_{w_i,w_{\sigma(i)}}(t^{c_i})\right]\frac{dw_i}{2\pi\iota}.$$

From the above equation (2.3.13) is obvious. This concludes the proof. □

**Proposition 2.3.10.** *Fix $N\in\mathbb{N}$ and a parameter $t\in(0,1)$. Let $\rho_X$ and $\rho_Y$ be the non-negative finite length specializations in $N$ variables $X=(x_1,\ldots,x_N)$ and $Y=(y_1,\ldots,y_N)$ respectively, with $x_i,y_i\in(0,1)$ for $i=1,\ldots,N$. Then for $u\notin\mathbb{R}^+$ one has that*

$$\mathbb{E}_{X,Y}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right]=\det(I+K_u^N)_{L^2(C_0)}. \tag{2.3.14}$$

*The contour $C_0$ is the positively oriented circle of radius $t^{-1}$, centered at 0, and the operator $K_u^N$ is defined in terms of its integral kernel*

$$K_u^N(w,w')=\frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty}ds\Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g_{w,w'}^N(t^s),$$

*where*

$$g_{w,w'}^N(t^s)=\frac{1}{wt^{-s}-w'}\prod_{j=1}^{N}\frac{(1-x_j(wt)^{-1})(1-y_j(wt)t^{-s})}{(1-x_j(wt)^{-1}t^s)(1-y_j(wt))}.$$

*Proof.* Using Propositions 2.3.7, 2.3.8 and 2.3.9 we have the statement of the proposition for $|u|<\epsilon$ and $u\notin\mathbb{R}^+$ for some sufficiently small $\epsilon>0$. To conclude the proof it suffices to show that both sides of (2.3.14) are analytic functions of $u$ in $\mathbb{C}\backslash\mathbb{R}^+$.

The RHS is analytic by Lemma 2.3.12, while the LHS of (2.3.14) equals $\sum_{n=0}^{N}\mathbb{P}_{x,y}(\lambda_1'=n)\frac{1}{(ut^{-n};t)_\infty}$, and is thus a finite sum of analytic functions and so also analytic on $\mathbb{C}\backslash\mathbb{R}^+$. □

### 2.3.4 Proof of Lemmas 2.3.11 and 2.3.12

Versions of the following two lemmas appear in Section 3.2 of [24].

**Lemma 2.3.11.** *Fix $N\in\mathbb{N}$, $t\in(0,1)$ and $x_i,y_i\in(0,1)$ for $i=1,\ldots,N$. Let $w,w'\in\mathbb{C}$ be such that $|w|=|w'|=t^{-1}$ and let*

$$g_{w,w'}^N(t^s)=\frac{1}{wt^{-s}-w'}\prod_{j=1}^{N}\frac{(1-x_j(wt)^{-1})(1-y_j(wt)t^{-s})}{(1-x_j(wt)^{-1}t^s)(1-y_j(wt))}.$$

*Then there exists $\epsilon>0$ such that if $\zeta\in\{\zeta:|\zeta|<\epsilon,\zeta\notin\mathbb{R}^+\}$, we have*

$$\sum_{n=1}^{\infty}g_{w,w'}^N(t^n)\zeta^n=\frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty}\Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}^N(t^s)ds. \tag{2.3.15}$$

*Proof.* For simplicity we suppress $N$ from our notation. Let $R_M = M + 1/2$ ($M \in \mathbb{N}$) and set $A_M^1 = 1/2 - \iota R_M$, $A_M^2 = 1/2 + \iota R_M$, $A_M^3 = R_M + \iota R_M$ and $A_M^4 = R_M - \iota R_M$. Denote by $\gamma_M^1$ the contour, which goes from $A_M^1$ vertically up to $A_M^2$, by $\gamma_M^2$ the contour, which goes from $A_M^2$ horizontally to $A_M^3$, by $\gamma_M^3$ the contour, which goes from $A_M^3$ vertically down to $A_M^4$, and by $\gamma_M^4$ the contour, which goes from $A_M^4$ horizontally to $A_M^1$. Also let $\gamma_M = \cup_i \gamma_M^i$ traversed in order (see Figure 2-5).

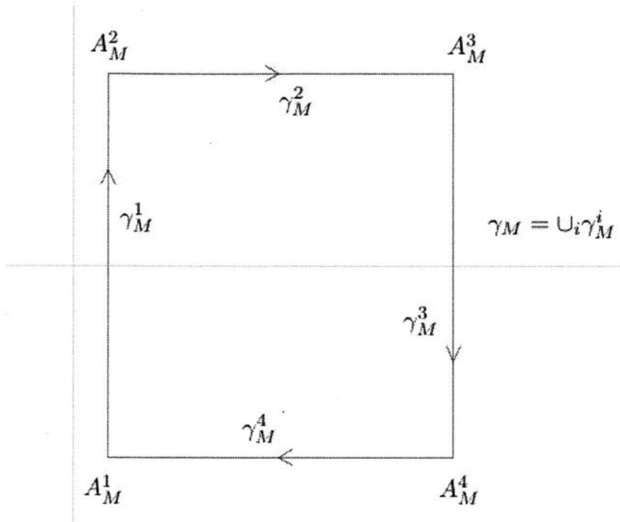


Figure 2-5: The contours $\gamma_M^i$ for $i = 1, ..., 4$.

We make the following observations:

1. $\gamma_M$ is negatively oriented.

2. The function $g_{w,w'}(t^s)$ is well-defined and analytic in a neighborhood of the closure of the region enclosed by $\gamma_M$. This follows from $|t^s| < 1$ for $Re(s) > 0$, which prevents any of the poles of $g_{w,w'}(t^s)$ from entering the region $Re(s) > 0$.

3. If $\text{dist}(s, \mathbb{Z}) > c$ for some fixed constant $c > 0$, then $\left|\frac{\pi}{\sin(\pi s)}\right| \leq c' e^{-\pi |Im(s)|}$ for some fixed constant $c'$, depending on $c$. In particular, this estimate holds for all $s \in \gamma_M$ since $\text{dist}(\gamma_M, \mathbb{Z}) = 1/2$ for all $M$ by construction.

4. If $-\zeta = re^{\iota\theta}$ with $|\theta| < \pi$ and $s = x + \iota y$ then

$$(-\zeta)^s = \exp\left((\log(r) + \iota\theta)(x + \iota y)\right) = \exp\left(\log(r)x - y\theta + \iota(\log(r)y + x\theta)\right),$$

since we took the principal branch. In particular, $|(-\zeta)^s| = r^x e^{-y\theta}$.

We also recall Euler's Gamma reflection formula

$$\Gamma(-s)\Gamma(1+s) = \frac{\pi}{\sin(-\pi s)}. \tag{2.3.16}$$

We observe for $s = x + \iota y$, with $x \geq 1/2$ that

$$|g_{w,w'}(t^s)| \leq \frac{\prod_{j=1}^N |1 - y_j(wt)t^{-s}|}{t^{-3/2} - t^{-1}} \prod_{i=1}^N \frac{2}{(1-y_i)(1-x_i)}.$$

In addition, we have $\prod_{j=1}^N |1 - y_j(wt)t^{-s}| \leq Ce^{cx}$ for some positive constants $C, c > 0$, depending on $N$, $t$ and $y_i$. Consequently, we see that if $\epsilon$ is chosen sufficiently small and $\zeta = re^{\iota\theta}$ with $r < \epsilon$ then

$$|g_{w,w'}(t^s)(-\zeta)^s| \leq Ce^{cx}\epsilon^x e^{|y\theta|} \leq Ce^{-cx}e^{|y\theta|},$$

with some new constant $C > 0$. In particular, the LHS in (2.3.15) is absolutely convergent, and we have

$$\sum_{n=1}^{\infty} g_{w,w'}(t^n)\zeta^n = \lim_{M\to\infty} \sum_{n=1}^{M} g_{w,w'}(t^n)\zeta^n.$$

From the Residue Theorem we have

$$\sum_{n=1}^{M} g_{w,w'}(t^n)\zeta^n = \frac{1}{2\pi\iota}\int_{\gamma_M} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)ds.$$

The last formula used $Res_{s=k}\Gamma(-s)\Gamma(1+s) = (-1)^{k+1}$ and observations 1. and 2. above. What remains to be shown is that

$$\lim_{M\to\infty} \frac{1}{2\pi\iota}\int_{\gamma_M} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)ds = \frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)ds. \tag{2.3.17}$$

Observe that on $Re(s) = 1/2$ we have that $|g_{w,w'}(t^s)|$ is bounded, while from (2.3.16) and observations 3. and 4. we have

$$|\Gamma(-s)\Gamma(1+s)(-\zeta)^s| = \left|\frac{\pi}{\sin(-\pi s)}(-\zeta)^s\right| \leq c' \exp((|\theta| - \pi)|Im(s)|)r^{1/2}, \tag{2.3.18}$$

which decays exponentially in $|Im(s)|$ since $|\theta| < \pi$. Thus the integrand on the RHS of (2.3.17) is exponentially decaying near $\pm\iota\infty$ and so the integral is well-defined. Moreover, from the Dominated Convergence Theorem we have that

$$\lim_{M\to\infty} \frac{1}{2\pi\iota}\int_{\gamma_M^1} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)ds = \frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)ds.$$

We now consider the integrals

$$\frac{1}{2\pi\iota}\int_{\gamma_M^i} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s),$$

when $i \neq 1$ and show they go to 0 in the limit. If true, (2.3.17) will follow.

Suppose that $i=2$ or $i=4$. Let $s=x+\iota y\in\gamma_M^i$, so $|y|=R_M$ and we get

$$|\Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)|\le Ce^{-cx}e^{|\theta y|}c'e^{-\pi|y|}\le Ce^{(|\theta|-\pi)R_M},$$

for some new constant $C>0$. Since $|\theta|-\pi<0$ we see that

$$\left|\frac{1}{2\pi\iota}\int_{\gamma_M^i}\Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)\right|\le CR_Me^{(|\theta|-\pi)R_M}\to 0\text{ as }M\to\infty.$$

Finally, let $i=3$. Let $s=x+\iota y\in\gamma_M^3$, so $x=R_M$ and we get

$$|\Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)|\le Ce^{-cx}e^{|\theta y|}c'e^{-\pi|y|}\le Cc'e^{-cR_M}.$$

Consequently, we obtain

$$\left|\frac{1}{2\pi\iota}\int_{\gamma_M^3}\Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(t^s)\right|\le 2R_MCc'e^{-cR_M}\to 0\text{ as }M\to\infty.$$

This concludes the proof of (2.3.17) and hence the lemma. □

**Lemma 2.3.12.** *Fix $N\in\mathbb{N}$ or $N=\infty$, $t\in(0,1)$ and $x_i,y_i\in(0,1)$ for $i=1,\dots,N$ such that $\sum_i x_i<\infty$, $\sum_i y_i<\infty$. Suppose $u\in\mathbb{C}\backslash\mathbb{R}^+$. Consider the operator $K_u^N$ on $L^2(C_0)$ (here $C_0$ is the positive circle of radius $t^{-1}$), which is defined in terms of its integral kernel*

$$K_u^N(w,w')=\frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty}ds\Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g_{w,w'}^N(t^s),$$

*where*

$$g_{w,w'}^N(t^s)=\frac{1}{wt^{-s}-w'}\prod_{j=1}^N\frac{(1-x_j(wt)^{-1})(1-y_j(wt)t^{-s})}{(1-x_j(wt)^{-1}t^s)(1-y_j(wt))}.$$

*Then $K_u^N$ is trace-class. Moreover, as a function of $u$ we have that $\det(I+K_u^N)$ is an analytic function on $\mathbb{C}\backslash\mathbb{R}^+$.*

*Proof.* We begin with the first statement of the lemma and suppress the dependence on $N$ and $u$ from the notation. From Lemma 2.2.10 it suffices to show that $K(w,w')$ is continuous on $C_0\times C_0$ and that $K_2(w,w')$ is continuous as well, where we recall that $K_2(w,w')$ is the derivative of $K(x,y)$ along the contour $C_0$ in the second entry.

In equation (2.3.18) we showed that if $-u(t^{-1}-1)=re^{\iota\theta}$ with $|\theta|<\pi$ and $s=1/2+\iota y$, then

$$|\Gamma(-s)\Gamma(1+s)(-\zeta)^s|\le C\exp((|\theta|-\pi)|y|)r^{1/2}$$

We observe that $g_{w,w'}(t^s)$ is continuous in $w,w'$ and moreover on $Re(s)=1/2$ we have

$$|g_{w,w'}(t^s)|\le M=\frac{1}{t^{-3/2}-t^{-1}}\prod_{j=1}^N\frac{(1+x_j)(1+y_jt^{-1/2})}{(1-x_jt^{1/2})(1-y_j)}<\infty$$

independently of $w, w'$. So if $(w_n, w_n') \to (w, w')$ we have that $g_{w_n,w_n'}(t^s) \to g_{w.w'}(t^s)$ and by the Dominated Convergence Theorem, we conclude that $K(w_n, w_n') \to K(w, w')$ so that $K(w, w')$ is continuous on $C_0 \times C_0$.

We next observe that

$$K_2(w, w') = \iota w' \frac{d}{dw'} K(w, w') = \iota w' \frac{1}{2\pi\iota} \int_{1/2-\iota\infty}^{1/2+\iota\infty} ds \Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s \frac{d}{dw'} g_{w,w'}(t^s),$$

where the change of the order of integration and differentiation is allowed by the exponential decay of the integrand. We have that $\frac{d}{dw'} g_{w,w'}(t^s) = -\frac{1}{wt^{-s}-w'} g_{w,w'}(t^s)$ so a similar argument as above now shows that $K_2(w, w')$ is continuous on $C_0 \times C_0$. We conclude that $K_u^N$ is indeed trace-class.

Since $K_u^N$ is trace-class we know that

$$\det(I + K_u^N) = 1 + \sum_{n \geq 1} \frac{1}{n!} \int_{C_0} \cdots \int_{C_0} \det \left[ K_u^N(w_i, w_j) \right]_{i,j=1}^n \prod_{i=1}^n \frac{dw_i}{2\pi\iota}.$$

We wish to show that the above sum is analytic in $u \in \mathbb{C} \backslash \mathbb{R}^+$.

We begin by showing that $K_u^N(w, w')$ is analytic in $u$ for each $(w, w') \in C_0 \times C_0$. Observe that on $(\mathbb{C} \backslash \mathbb{R}^+) \times (1/2 + \iota\mathbb{R})$, $\Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g_{w,w'}^N(t^s)$ is jointly continuous in $(u, s)$ and analytic in $u$ for each $s$. From Theorem 5.4 in Chapter 2 of [77] we know that for any $A \geq 0$

$$h_A(u) := \int_{1/2-\iota A}^{1/2+\iota A} \Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g_{w,w'}^N(t^s) ds$$

is an analytic function of $u \in \mathbb{C} \backslash \mathbb{R}^+$. In addition, using our earlier estimates we see that

$$|h_A(u) - K_u^N(w, w')| \leq 2|u|^{1/2} MC \int_A^\infty \exp((|\theta| - \pi)y) dy = \frac{2|u|^{1/2} MC}{\pi - |\theta|} \exp((|\theta| - \pi)A).$$

The latter shows that $h_A(u)$ converges uniformly on compact subsets of $\mathbb{C} \backslash \mathbb{R}^+$ to $K_u^N(w, w')$ as $A \to \infty$, which implies that $K_u^N(w, w')$ is analytic in $u$. Notice that when $A = 0$ the above shows that if $K'$ is a compact subset of $\mathbb{C} \backslash \mathbb{R}^+$ and $u \in K'$, we have $|K_u^N(w, w')| \leq C(K')$ for some contant $C > 0$ independent of $w, w'$.

We next observe that $K_u^N(w, w')$ is jointly continuous in $u$ and $(w, w')$ and analytic in $u$ for each $w, w'$ from our proof above. The latter implies that $\det \left[ K_u^N(w_i, w_j) \right]_{i,j=1}^n$ is continuous on $C_0^n \times \mathbb{C} \backslash \mathbb{R}^+$ and analytic in $u$ for each $(w_1, ..., w_n) \in C_0^n$. It follows from Theorem 5.4 in Chapter 2 of [77] that

$$H_n(u) = \frac{1}{n!} \int_{C_0} \cdots \int_{C_0} \det \left[ K_u^N(w_i, w_j) \right]_{i,j=1}^n \prod_{i=1}^n \frac{dw_i}{2\pi\iota},$$

is analytic in $u$.

Finally, suppose $K' \subset \mathbb{C} \backslash \mathbb{R}^+$ is compact and $u \in K'$. Then from Hadamard's inequality and our earlier estimate on $|K_u^N(w,w')|$ we know that

$$|H_n(u)| = \frac{1}{n!}\left|\int_{C_0}\cdots\int_{C_0}\det\left[K_u^N(w_i,w_j)\right]_{i,j=1}^n \frac{dw_i}{2\pi\iota}\right| \leq \frac{1}{n!}(t^{-1})^n n^{n/2} C(K')^n = B^n\frac{n^{n/2}}{n!}.$$

The latter is absolutely summable, and since the absolutely convergent sum of analytic functions is analytic and $K'$ was arbitrary, we conclude that $1+\sum_{n=1}^{\infty}H_n(u) = \det(I+K_u^N)_{L^2(C^0)}$ is analytic in $u$ on $\mathbb{C}\backslash\mathbb{R}^+$. This suffices for the proof. □

## 2.4 GUE asymptotics

In this section, we use the results from Section 2.3 to get formulas for the $t$-Laplace transform of $t^{1-\lambda_1'}$, with $\lambda$ distributed according to the Hall-Littlewood measure with parameters $a,r,t\in(0,1)$ (see Section 2.2.4). Subsequently, we analyze the formulas that we get in the limiting regime $r\to 1^-$, $t\in(0,1)$ - fixed and obtain convergence to the Tracy-Widom GUE distribution. In what follows, we will denote by $\mathbb{P}_{a,r,t}$ and $\mathbb{E}_{a,r,t}$ the probability distribution and expectation with respect to the Hall-Littlewood measure with parameters $a,r,t\in(0,1)$.

### 2.4.1 Fredholm determinant formula for $\mathbb{E}_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right]$

In the following results, unless otherwise specified, $\det(I+K)_{L^2(C)}$ dentotes the absolutely convergent sum on the RHS of (2.2.8) - see the discussion in Section 2.2.5.

**Proposition 2.4.1.** *Suppose $a,r,t\in(0,1)$ and let $\delta>0$ be such that $a<(1-\delta)$. Then for $u\in\mathbb{C}\backslash\mathbb{R}^+$ one has that*

$$\mathbb{E}_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right] = \det(I+K_u)_{L^2(C_0)}. \tag{2.4.1}$$

*The contour $C_0$ is a positively oriented piecewise smooth simple curve, contained in the closed annulus $A_{\delta,t}$ between the 0-centered circles of radius $t^{-1}$ and $\max\left(t^{-1}(1-\delta/2),t^{-3/4}\right)$. The kernel $K_u(w,w')$ is defined as*

$$K_u(w,w') = \frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty} ds\Gamma(-s)\Gamma(1+s)(-u(t^{-1}-1))^s g_{w,w'}(t^s), \tag{2.4.2}$$

*where*

$$g_{w,w'}(t^s) = \frac{1}{wt^{-s}-w'}\prod_{j=0}^{\infty}\frac{(1-ar^j(wt)^{-1})(1-ar^j(wt)t^{-s})}{(1-ar^j(wt)^{-1}t^s)(1-ar^j(wt))}.$$

*Remark* 2.4.2. Proposition 2.4.1 will be the starting point for our asymptotic analysis in both the GUE and CDRP cases. In the different limiting regimes, we will encounter different contours, which will be suitably picked contours contained in $A_{\delta,t}$.

*Proof.* We first prove the proposition when $C_0$ is the positively oriented circle of radius $t^{-1}$. The starting point is Proposition 2.3.10, from which we see that whenever $u\notin\mathbb{R}^+$ one has for every $N \in \mathbb{N}$

$$\mathbb{E}^N_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda'_1};t)_\infty}\right] = \det(I + K^N_u)_{L^2(C_0)}.$$

Here $\mathbb{E}^N_{a,r,t}$ stands for the expectation with respect to the Macdonald measure on partitions, corresponding to $q = 0$ and $x_i = y_i = ar^{i-1}$ for $i = 1, ..., N$ and $x_i = y_i = 0$ for $i > N$. The result would thus follow once we show that

1. $\lim_{N\to\infty} \mathbb{E}^N_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda'_1};t)}\right] = \mathbb{E}_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda'_1};t)}\right]$

2. $\lim_{N\to\infty} \det(I + K^N_u)_{L^2(C_0)} = \det(I + K_u)_{L^2(C_0)}$.

Before we prove the above two statements let us remark that the two limiting quantities are indeed well-defined. The fact that $K_u$ is a trace-class operator on $L^2(C_0)$ follows from Lemma 2.3.12. Next, we observe that if $u \not\in \mathbb{R}^+$ then for any $n$ we have that $\frac{1}{(ut^{-n};t)_\infty}$ is well defined and moreover there exists a constant $M(u)$ such that $\left|\frac{1}{(ut^{-n};t)_\infty}\right| \leq M$, for all $n$. Consequently, we can define unambiguously the expectation $\mathbb{E}_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda'_1};t)}\right]$ and it is a finite quantity.

We start with 1. Denote by $P^N_\lambda$ and $Q^N_\lambda$ the $N$-length specialization of the the Hall-Littlewood symmetric functions with $x_i = y_i = ar^{i-1}$ for $i = 1, ..., N$ and $x_i = y_i = 0$ for $i > N$ (here $N$ is a positive integer or $\infty$). Also let $Z^N$ be the normalization constant, which in the above case equals

$$Z^N = \prod_{i,j=1}^{N} \frac{1 - tar^{i-1}ar^{j-1}}{1 - ar^{i-1}ar^{j-1}} \text{ - this is the Cauchy identity in (2.2.2).}$$

We obtain

$$\mathbb{E}^N_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda'_1};t)_\infty}\right] = \frac{1}{Z^N}\sum_{\lambda\in\mathbb{Y}} P^N_\lambda Q^N_\lambda \frac{1}{((1-t)ut^{-\lambda'_1};t)_\infty}.$$

One readily verifies that $Z^N \nearrow Z^\infty$, $P^N_\lambda \nearrow P^\infty_\lambda$ and $Q^N_\lambda \nearrow Q^\infty_\lambda$ as $N \to \infty$. Thus from the Dominated Convergence Theorem (with dominating function $MP^\infty_\lambda Q^\infty_\lambda$) we get

$$\lim_{N\to\infty}\sum_{\lambda\in\mathbb{Y}} P^N_\lambda Q^N_\lambda \frac{1}{((1-t)ut^{-\lambda'_1};t)_\infty} = \sum_{\lambda\in\mathbb{Y}} P^\infty_\lambda Q^\infty_\lambda \frac{1}{((1-t)ut^{-\lambda'_1};t)_\infty}.$$

The latter implies that

$$\lim_{N\to\infty}\frac{1}{Z^N}\sum_{\lambda\in\mathbb{Y}} P^N_\lambda Q^N_\lambda \frac{1}{((1-t)ut^{-\lambda'_1};t)_\infty} = \frac{1}{Z^\infty}\sum_{\lambda\in\mathbb{Y}} P^\infty_\lambda Q^\infty_\lambda \frac{1}{((1-t)ut^{-\lambda'_1};t)_\infty},$$

which concludes the proof of 1.

Next we turn to 2. Firstly, we one readily observes that

$$g^N_{w,w'}(t^s) \to g_{w,w'}(t^s), \text{ as } N \to \infty$$

and moreover we have

$$|g^N_{w,w'}(t^s)| \leq \frac{1}{t^{-3/2}-t^{-1}} \prod_{j=0}^{\infty} \frac{(1+ar^j)(1+ar^jt^{-1/2})}{(1-ar^jt^{1/2})(1-ar^j)} = M < \infty,$$

independently of $N, w, w'$. Recall from (2.3.18) that

$$\left|\Gamma(-s)\Gamma(1+s)(-(t^{-1}-1)u)^s\right| \leq C \exp((|\theta|-\pi)|y|)r^{1/2},$$

where $-(t^{-1}-1)u = re^{\iota\theta}$ and $s = 1/2 + \iota y$. It follows by the Dominated Convergence Theorem (with dominating function $MC\exp((|\theta|-\pi)|y|)r^{1/2}$) that

$$\lim_{N\to\infty} K^N_u(w,w') = K_u(w,w'),$$

and moreover there exists a finite constant $M_2$ (depending on $u$) such that $|K^N_u(w,w')| \leq M_2$ for all $N, w, w'$. Next we have from the Bounded Convergence Theorem that for every $n$

$$\lim_{N\to\infty} \frac{1}{n!}\int_{C_0}\cdots\int_{C_0} \det\left[K^N_u(w_i,w_j)\right]^n_{i,j=1} \prod_{i=1}^n \frac{dw_i}{2\pi\iota} = \frac{1}{n!}\int_{C_0}\cdots\int_{C_0} \det\left[K_u(w_i,w_j)\right]^n_{i,j=1} \prod_{i=1}^n \frac{dw_i}{2\pi\iota}.$$

By Hadamard's inequality we have that for each $n$ the above is bounded (in absolute value) by $\frac{n^{n/2}t^{-n}M_2^n}{n!}$. Consequently, by the Dominated Convergence Theorem we have that

$$\lim_{N\to\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{C_0}\cdots\int_{C_0}\det\left[K^N_u(w_i,w_j)\right]^n_{i,j=1}\prod_{i=1}^n\frac{dw_i}{2\pi\iota} = \sum_{n=1}^{\infty}\frac{1}{n!}\int_{C_0}\cdots\int_{C_0}\det\left[K_u(w_i,w_j)\right]^n_{i,j=1}\prod_{i=1}^n\frac{dw_i}{2\pi\iota}.$$

This concludes the proof of 2.

We next wish to extend the result to a more general class of contours. Let $C$ be a positively oriented piecewise smooth simple contour contained in the annulus, described in the statement of the proposition. What we have proved so far is that

$$\mathbb{E}_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right] = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{C_0}\cdots\int_{C_0}\det\left[K_u(w_i,w_j)\right]^n_{i,j=1}\prod_{i=1}^n\frac{dw_i}{2\pi\iota}, \qquad (2.4.3)$$

where the latter sum is absolutely convergent. One readily verifies that $g_{w,w'}(t^s)$ is analytic in $w, w'$ on a neighborhood of $A_{\delta,t} \times A_{\delta,t}$ and by the exponential decay of $\Gamma(-s)\Gamma(1+s)(-(t^{-1}-1)u)^s$ near $1/2 \pm \iota\infty$ the same is true for $K_u(w,w')$. It follows that $\det\left[K_u(w_i,w_j)\right]^n_{i,j=1}$ is analytic on a neighborhood of $A^n_{\delta,t}$ and by Cauchy's theorem we may deform the contours $C_0$ in (2.4.3) to $C$, without changing the value of the integrals. This is the result we wanted. □

### 2.4.2 A formula suitable for asymptotics: GUE case

In this section we use Proposition 2.4.1 to derive an alternative $t$-Laplace transform, which is more suitable for asymptotic analysis in the GUE case. The following result makes references to two contours $\gamma_W(A)$ and $\gamma_Z(A)$, which depend on a real parameter $A \geq 0$, as well as a function $S_{a,r}(\cdot)$, which we define below.

**Definition 2.4.3.** For a parameter $A \geq 0$ define

$$\gamma_W(A) = \{-A|y| + \iota y : y \in I\} \text{ and } \gamma_Z(A) = \{A|y| + \iota y : y \in I\}, \text{ where } I = [-\pi, \pi].$$

The orientation is determined from $y$ increasing in $I$.

**Definition 2.4.4.** For $a, r \in (0,1)$ define

$$S_{a,r}(z) := \sum_{j=0}^{\infty} \log(1 + ar^j e^z) - \sum_{j=0}^{\infty} \log(1 + ar^j e^{-z}).$$

The function $S_{a,r}$ plays a central role in our arguments and the properties that we will need are summarized in Section 2.6. We isolate the most basic facts about $S_{a,r}$ in a lemma below. The lemma appears again in Section 2.6 as Lemma 2.6.1, where it is proved.

**Lemma 2.4.5.** *Suppose that $\delta \in (0,1)$. Consider $r \in (0,1)$ and $a \in (0, 1-\delta]$. Then there exists $\Delta'(\delta) > 0$ such that $S_{a,r}(z)$ is well-defined and analytic on $D_\delta = \{z \in \mathbb{C} : |Re(z)| < \Delta'\}$ and satisfies*

$$\exp(S_{a,r}(z)) = \prod_{j=0}^{\infty} \frac{1 + ar^j e^z}{1 + ar^j e^{-z}}. \tag{2.4.4}$$

**Proposition 2.4.6.** *Suppose $a, r, t \in (0,1)$ and let $\delta > 0$ be such that $a < (1-\delta)$. If $A > 0$ is sufficiently small (depending on $\delta$ and $t$) and $\gamma_W(A)$ and $\gamma_Z(A)$ are as in Definition 2.4.3, then for $\zeta \in \mathbb{C} \backslash \mathbb{R}^+$ one has*

$$\mathbb{E}_{a,r,t}\left[\frac{1}{(\zeta t^{1-\lambda_1'}; t)_\infty}\right] = \det(I - \tilde{K}_\zeta)_{L^2(\gamma_W)}.$$

*The kernel $\tilde{K}(W, W')$ has the integral representation*

$$\tilde{K}_\zeta(W, W') = \frac{e^W}{2\pi\iota} \int_{\gamma_Z(A)} \frac{dZ(-\zeta)^{f_t(Z,W)}}{e^{W'} - e^Z} G_{\zeta,t}(W, Z) \exp\left(S_{a,r}(Z) - S_{a,r}(W)\right). \tag{2.4.5}$$

*In the above formula, $S_{a,r}$ is as in Definition 2.4.4 and we have*

$$G_{\zeta,t}(W, Z) := \sum_{k \in \mathbb{Z}} \frac{\pi(-\log t)^{-1}(-\zeta)^{2\pi k\iota/(-\log t)}}{\sin(-\pi f_t(Z + 2\pi k\iota, W))} \text{ and } f_t(Z, W) := \frac{Z - W}{-\log t}. \tag{2.4.6}$$

*Proof.* We consider the contour $C_A := \{-t^{-1}e^{\iota\theta - A|\theta|} : \theta \in [-\pi, \pi]\}$, which is a positively oriented piecewise smooth contour. For $A > 0$ sufficiently small we know that $C_A$ is contained in the annulus $A_{\delta,t}$ in the statement of Proposition 2.4.1. Consequently, from (2.4.1) we know that

$$\mathbb{E}_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda_1'}; t)_\infty}\right] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{C_A} \cdots \int_{C_A} \det\left[K_u(w_i, w_j)\right]_{i,j=1}^n \prod_{i=1}^n \frac{dw_i}{2\pi\iota},$$

where $K_u(w,w')$ is as in (2.4.2) and the above sum is absolutely convergent. The $n$-th summand equals

$$\frac{1}{n!}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\det\left[K_u\left(-t^{-1}e^{\iota\theta_i-A|\theta_i|},-t^{-1}e^{\iota\theta_j-A|\theta_j|}\right)\right]_{i,j=1}^{n}\prod_{i=1}^{n}\frac{-t^{-1}e^{\iota\theta_i-A|\theta_i|}(\iota-Asign(\theta_i))d\theta_i}{2\pi\iota}.$$

Setting $y_i=\iota\theta_i-A|\theta_i|$ the above becomes

$$\frac{(-1)^n}{n!}\int_{\gamma_W(A)}\cdots\int_{\gamma_W(A)}\det\left[t^{-1}e^{y_i}K_u\left(-t^{-1}e^{y_i},-t^{-1}e^{y_i}\right)\right]_{i,j=1}^{n}\prod_{i=1}^{n}\frac{dy_i}{2\pi\iota}.$$

To conclude the proof it suffices to show that for $W,W'\in\gamma_W(A)$ and $\zeta=(t^{-1}-1)u$ one has

$$t^{-1}e^{W}K_u\left(-t^{-1}e^{W},-t^{-1}e^{W'}\right)=\tilde{K}_\zeta(W,W'). \tag{2.4.7}$$

Setting $Z=(-\log t)s+W$, using the Euler Gamma reflection formula from (2.3.16) and recalling $f_t(Z,W)=\frac{Z-W}{-\log t}$, we see that the LHS of (2.4.7) equals

$$\frac{e^W}{2\pi\iota}\int_{\frac{-\log t}{2}+W-\iota\infty}^{\frac{-\log t}{2}+W+\iota\infty}\frac{(-\log t)^{-1}\pi dZ}{\sin(-\pi f_t(Z,W))}(-\zeta)^{f_t(Z,W)}\frac{1}{e^{W'}-e^{Z}}\prod_{j=0}^{\infty}\frac{(1+ar^je^{-W})(1+ar^je^{Z})}{(1+ar^je^{-Z})(1+ar^je^{W})}.$$

If $W\in\gamma_W(A)$ we know that $Re\left[\frac{-\log t}{2}+W\right]\in\left[\frac{-\log t}{2}-\pi A,\frac{-\log t}{2}\right]$. In addition, the only poles of the integrand for $Re(Z)>0$ come from $\frac{1}{\sin(-\pi f_t(Z,W))}$ and are located at $W+(-\log t)\mathbb{Z}$. This implies that if $A$ is sufficiently small we may shift the $Z$- contour so that it passes through the point $A\pi$, without crossing any poles of the integrand (see Figure 2-6). The shift does not change the value of the integral by Cauchy's Theorem and the exponential decay of the integrand near $\pm\iota\infty$. Thus we get that the LHS of (2.4.7) equals

$$\frac{e^W}{2\pi\iota}\int_{A\pi-\iota\infty}^{A\pi+\iota\infty}\frac{(-\log t)^{-1}\pi dZ}{\sin(-\pi f_t(Z,W))}(-\zeta)^{f_t(Z,W)}\frac{1}{e^{W'}-e^{Z}}\prod_{j=0}^{\infty}\frac{(1+ar^je^{-W})(1+ar^je^{Z})}{(1+ar^je^{-Z})(1+ar^je^{W})}.$$

The next observation is that $e^{A\pi+\iota y}$ is periodic in $y$ with period $T=2\pi$. Using this we see that the LHS of (2.4.7) equals

$$\frac{e^W}{2\pi\iota}\sum_{k\in\mathbb{Z}}\int_{A\pi-\iota T/2+\iota kT}^{A\pi+\iota T/2+\iota kT}\frac{(-\log t)^{-1}\pi dZ}{\sin(-\pi f_t(Z,W))}(-\zeta)^{f_t(Z,W)}\frac{1}{e^{W'}-e^{Z}}\prod_{j=0}^{\infty}\frac{(1+ar^je^{-W})(1+ar^je^{Z})}{(1+ar^je^{-Z})(1+ar^je^{W})}=$$

$$=\frac{e^W}{2\pi\iota}\sum_{k\in\mathbb{Z}}\int_{A\pi-\iota T/2}^{A\pi+\iota T/2}dZ\frac{(-\zeta)^{\iota kT/(-\log t)}(-\log t)^{-1}\pi}{\sin(-\pi f_t(Z+\iota kT,W))}\frac{(-\zeta)^{f_t(Z,W)}}{e^{W'}-e^{Z}}\prod_{j=0}^{\infty}\frac{(1+ar^je^{-W})(1+ar^je^{Z})}{(1+ar^je^{-Z})(1+ar^je^{W})}.$$

Let $(-\zeta)=re^{\iota\theta}$ with $|\theta|<\pi$. Using a similar argument to (2.3.18), we have for $|k|\geq 1$

$$\left|\frac{(-\zeta)^{\iota kT/(-\log t)}}{\sin(-\pi f_t(Z+\iota kT,W))}\right|=\left|\frac{e^{-\theta kT/(-\log t)}}{\sin(-\pi f_t(Z+\iota kT,W))}\right|\leq Ce^{|k|T(|\theta|-\pi)/(-\log t)}, \tag{2.4.8}$$

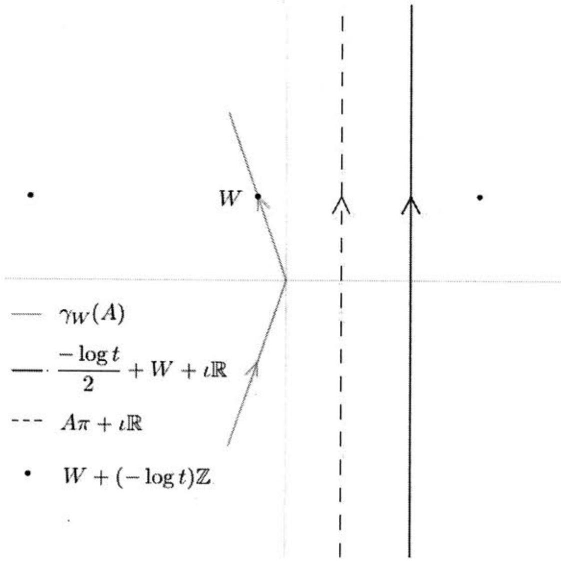


Figure 2-6: If $A$ is very small, no points of $W + (-\log t)\mathbb{Z}$ fall between $A\pi + \iota\mathbb{R}$ and $\frac{-\log t}{2} + W + \iota\mathbb{R}$, when $W \in \gamma_W(A)$.

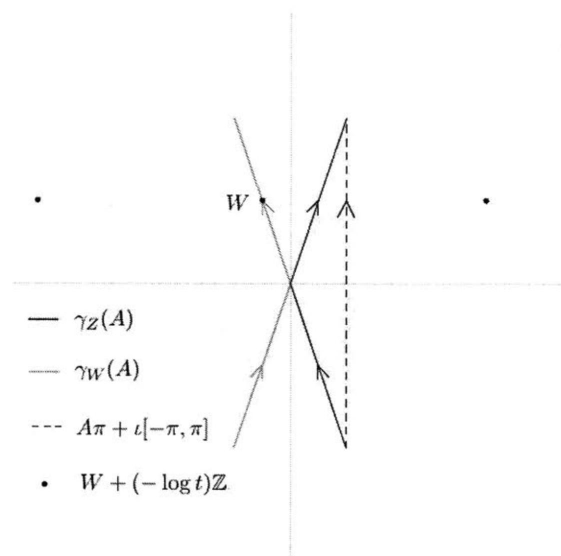


Figure 2-7: If $A$ is very small, no points of $W + (-\log t)\mathbb{Z}$ fall between $A\pi + \iota[-\pi, \pi]$ and $\gamma_Z(A)$, when $W \in \gamma_W(A)$.

where $C$ is some positive constant, independent of $Z$ and $W$, provided $W \in \gamma_W(A)$, $|Im(Z)| \leq \pi$ and $Re(Z) = A\pi$. We observe the latter is summable over $k$. Additionally,

$$\left| \frac{(-\zeta)^{f_t(Z,W)}}{e^{W'} - e^Z} \prod_{j=0}^{\infty} \frac{(1 + ar^j e^{-W})(1 + ar^j e^Z)}{(1 + ar^j e^{-Z})(1 + ar^j e^W)} \right| \leq \frac{1}{e^{A\pi} - 1} \left| (-\zeta)^{f_t(Z,W)} \prod_{j=0}^{\infty} \frac{(1 + ar^j e^{-W})(1 + ar^j e^Z)}{(1 + ar^j e^{-Z})(1 + ar^j e^W)} \right|,$$

and the latter is bounded by some constant $M(\zeta, B)$, provided $Re(Z) = A\pi$ and $W \in \gamma_W(A)$. By Fubini's theorem, we may change the order of the sum and the integral and get that LHS of (2.4.7) equals

$$\frac{e^W}{2\pi\iota} \int_{A\pi - \iota T/2}^{A\pi + \iota T/2} \frac{dZ(-\zeta)^{f_t(Z,W)}}{e^{W'} - e^Z} \left[ \sum_{k \in \mathbb{Z}} \frac{\pi(-\log t)^{-1}(-\zeta)^{\iota kT/(-\log t)}}{\sin(-\pi f_t(Z + \iota kT, W))} \right] \prod_{j=0}^{\infty} \frac{(1 + ar^j e^{-W})(1 + ar^j e^Z)}{(1 + ar^j e^{-Z})(1 + ar^j e^W)}.$$

From (2.4.8) we see that $G_{\zeta,t}(W, Z)$, which is given by

$$\frac{\pi(-\log t)^{-1}}{\sin(-\pi f_t(Z, W))} + \sum_{|k| \geq 1} \frac{\pi(-\log t)^{-1}(-\zeta)^{\iota kT/(-\log t)}}{\sin(-\pi f_t(Z + \iota kT, W))},$$

is the sum of $\frac{\pi(-\log t)^{-1}}{\sin(-\pi f_t(Z,W))}$ and an analytic function in $Z$ in the region $D = \{Z \in \mathbb{C} : |Im(Z)| \leq \pi \text{ and } Re(Z) \geq 0\}$. In particular, the poles of $G_{\zeta,t}(W, Z)$ in $D$ are exactly at $W + (-\log t)\mathbb{N}$. If we now deform the contour $[A\pi - \iota\pi, A\pi + \iota\pi]$ to $\gamma_Z(A)$ (see Figure 2-7) we will not cross any poles and from Cauchy's Theorem we obtain that the LHS of (2.4.7) is

$$\frac{e^W}{2\pi\iota}\int_{\gamma_Z(A)}\frac{dZ(-\zeta)^{f_t(Z,W)}}{e^{W'}-e^Z}G_{\zeta,t}(W,Z)\prod_{j=0}^{\infty}\frac{(1+ar^je^{-W})(1+ar^je^Z)}{(1+ar^je^{-Z})(1+ar^je^W)}.$$

From Lemma 2.4.5 (provided $A$ is sufficiently small so that $\gamma_Z(A),\gamma_W(A)\subset D_\delta$), we have

$$\prod_{j=0}^{\infty}\frac{(1+ar^je^{-W})(1+ar^je^Z)}{(1+ar^je^{-Z})(1+ar^je^W)}=\exp\left(S_{a,r}(Z)-S_{a,r}(W)\right).$$

Substituting this above we recognize the RHS of (2.4.7). □

### 2.4.3 Convergence of the $t$-Laplace transform (GUE case) and proof of Theorem 2.1.2

Here we state the regime, in which we scale parameters and obtain an asymptotic formula for $\mathbb{E}_{a,r,t}\left[\frac{1}{(\zeta t^{1-\lambda_1'};t)_\infty}\right]$. The formula is analyzed below and used to prove Theorem 2.1.2. One key reason we are considering the $t$-Laplace transform is that it asymptotically behaves like the expectation of an indicator function. The latter (as will be shown carefully below) allows one to obtain the limiting CDF of the properly scaled first column of a partition distributed according to the Hall-Littlewood measure with parameters $a,r,t$ and match it with $F_{GUE}$ (see Definition 2.1.7).

We summarize the limiting regime and some relevant expressions.

1. We will let $r\to 1^-$ and keep $t\in(0,1)$ fixed.
2. We assume that $a$ depends on $r$ and for some $\delta>0$ we have $\lim_{r\to 1^-}a(r)=a(1)\in(0,1-\delta]$.
3. We denote by $N(r)=\frac{1}{1-r}$, $M(r)=2\sum_{k=1}^{\infty}a(r)^k\frac{(-1)^{k+1}}{1-r^k}$ and $\alpha=\left[\frac{a(1)}{(1+a(1))^2}\right]^{-1/3}$.

$$\text{For a given } x\in\mathbb{R} \text{ set } \zeta_x=-t^{M(r)+x\alpha^{-1}N(r)^{1/3}}. \tag{2.4.9}$$

The following result is the key fact for the Tracy-Widom limit of the fluctuations of the first column of a partition distributed according to $\mathbb{P}_{a,r,t}$ in the GUE case. It shows that under the scaling regime described above the Fredholm determinant (and hence the $t$-Laplace transform) appearing in Proposition 2.4.6 converges to $F_{GUE}$.

**Theorem 2.4.7.** *Let $x\in\mathbb{R}$ be given and let $\zeta_x$ be given as in (2.4.9). If $A>0$ is sufficiently small (depending on $\delta$ and $t$) then*

$$\lim_{r\to 1^-}\det(I-\tilde{K}_{\zeta_x})_{L^2(\gamma_W(A))}=F_{GUE}(x), \tag{2.4.10}$$

*where $F_{GUE}$ is the GUE Tracy-Widom distribution (see Definition 2.1.7), $\gamma_W(A)$ is defined in Definition 2.4.3 and $\tilde{K}_{\zeta_x}$ is as in (2.4.5).*

In what follows we prove Theorem 2.1.2, assuming the validity of Theorem 2.4.7, whose proof is postponed until the next section.

We begin by summarizing the key results from our previous work as well as recalling a couple of lemmas from the literature. From Proposition 2.4.6 and Theorem 2.4.7 we have that under the scaling described in the beginning of the section and any $x \in \mathbb{R}$

$$\lim_{r \to 1^-} \mathbb{E}_{a,r,t} \left[ \frac{1}{(-t^{M(r)+\alpha^{-1}xN(r)^{1/3}} t^{1-\lambda_1'}; t)_\infty} \right] = F_{GUE}(x). \tag{2.4.11}$$

Set $\xi_r := \alpha N(r)^{-1/3} (\lambda_1' - M(r))$ and observe that (2.4.11) is equivalent to

$$\lim_{r \to 1^-} \mathbb{E}_{a,r,t} \left[ \frac{1}{((-t) \cdot t^{-[N(r)^{1/3}\alpha^{-1}(\xi_r - x)]}; t)_\infty} \right] = F_{GUE}(x). \tag{2.4.12}$$

The function that appears on the LHS under the expectation in (2.4.12) has the following asymptotic property.

**Lemma 2.4.8.** *Fix a parameter $t \in (0,1)$. Then*

$$f_q(y) := \frac{1}{((-t) \cdot t^{qy}; t)_\infty} = \prod_{k=1}^{\infty} \frac{1}{1 + t^{qy+k}} \tag{2.4.13}$$

*is increasing for all $q > 0$ and decreasing for all $q < 0$. For each $\delta > 0$ one has $f_q(y) \to 1_{\{y>0\}}$ uniformly on $\mathbb{R} \backslash [-\delta, \delta]$ as $q \to \infty$.*

*Proof.* This is essentially Lemma 5.1 in [50], but we present the proof for completeness. Each factor in the $t$-Pochhammer symbol $\frac{1}{1+t^{qy+k}}$ is positive, increases in $y$ when $q > 0$ and decreases in $y$ when $q < 0$. This proves monotonicity.

Let $\delta > 0$ be given. If $y < -\delta$ we have

$$0 \leq f_q(y) \leq \frac{1}{1 + t^{1+qy}} \leq \frac{1}{1 + t^{1-q\delta}} \to 0 \text{ as } q \to \infty. \tag{2.4.14}$$

If $y > \delta$ we have

$$0 \geq \log f_q(y) \geq -\sum_{k=1}^{\infty} \log\left[1 + t^{q\delta+k}\right] \to 0 \text{ as } q \to \infty, \tag{2.4.15}$$

where the latter statement follows from the Dominated Convergence Theorem with dominating function $\log\left[1 + t^k\right]$. Exponentiating (2.4.15) and combining it with (2.4.14) proves the second part of the lemma.

□

We will use the following elementary probability lemma (Lemma 4.1.39 of [24]).

**Lemma 2.4.9.** *Suppose that $f_n$ is a sequence of functions $f_n : \mathbb{R} \to [0,1]$, such that for each $n$, $f_n(y)$ is strictly decreasing in $y$ with a limit of 1 at $y = -\infty$ and 0 at $y = \infty$. Assume* *that for each $\delta > 0$ one has on $\mathbb{R}\backslash[-\delta, \delta]$, $f_n \to \mathbf{1}_{\{y<0\}}$ uniformly. Let $X_n$ be a sequence of random variables such that for each $x \in \mathbb{R}$*

$$\mathbb{E}[f_n(X_n - x)] \to p(x),$$

*and assume that $p(x)$ is a continuous probability distribution function. Then $X_n$ converges in distribution to a random variable $X$, such that $\mathbb{P}(X < x) = p(x)$.*

*Proof.* (Theorem 2.1.2) Let $r_n$ be a sequence converging to $1^-$ and set

$$f_n(y) = \frac{1}{((-t) \cdot t^{-\lfloor N(r_n)^{1/3}\alpha^{-1}y \rfloor}; t)_\infty} \text{ and } X_n = \xi_{r_n}.$$

Lemma 2.4.8 shows that $f_n$ satisfy the conditions of Lemma 2.4.9. Consequently, Lemma 2.4.9 and (2.4.12) show that $\xi_{r_n}$ converges weakly to the Tracy-Widom distribution. In particular, for each $x \in \mathbb{R}$ we have

$$\lim_{r \to 1^-} \mathbb{P}_{a,r,t}(\xi_r \leq x) = F_{GUE}(x). \tag{2.4.16}$$

Consider $a(r) = r^{(1+|\lfloor \tau N(r) \rfloor|)/2}$. Since, $\lim_{r\to 1^-} r^N = e^{-1}$, we see that $\lim_{r \to 1^-} a(r) = a(1) = e^{-|\tau|/2} < 1$ ( whenever $\tau \neq 0$). This means that $\alpha^{-1} := \left[\frac{a(1)}{(1+a(1))^2}\right]^{1/3} = \left[\frac{e^{-|\tau|/2}}{(1+e^{-|\tau|/2})^2}\right]^{1/3} =: \chi^{-1}$. From Section 2.2.4 we conclude that

$$\mathbb{P}^{r,t}_{HL}\left(\frac{\lambda_1'(\lfloor \tau N(r) \rfloor) - M(r)}{\chi^{-1}N^{1/3}} \leq x\right) = \mathbb{P}_{a,r,t}\left(\frac{\lambda_1' - M(r)}{\alpha^{-1}N^{1/3}} \leq x\right) = \mathbb{P}_{a,r,t}(\xi_r \leq x), \tag{2.4.17}$$

Combining (2.4.16) and (2.4.17) shows that if $\tau \neq 0$ one has

$$\lim_{r\to 1^-} \mathbb{P}^{r,t}_{HL}\left(\frac{\lambda_1'(\lfloor \tau N(r) \rfloor) - M(r)}{\chi^{-1}N^{1/3}} \leq x\right) = F_{GUE}(x).$$

In (2.6.8) we will show that $M(r) = 2N(r)\log(1+a(1)) + O(1) = 2N(r)\log(1+e^{-|\tau|/2}) + O(1)$. Substituting this above concludes the proof of the theorem. ☐

### 2.4.4 Proof of Theorem 2.4.7

We split the proof of Theorem 2.4.7 into four steps. In the first step we rewrite the LHS of (2.4.10) in a suitable form for the application of Lemmas 2.2.11 and 2.2.12. In the second step we verify the pointwise convergence and in the third step we provide dominating functions, which are necessary to apply the lemmas. In the fourth step we obtain a limit for the LHS of (2.4.10), subsequently we use a result from [26], to show that the limit we obtained is in fact $F_{GUE}$.

In Steps 2 and 3 we will require some estimates, which we summarize in Lemmas 2.4.10 and 2.4.11 below. The proofs are postponed until Section 2.6.

**Lemma 2.4.10.** *Let $A > 0$ be sufficiently small. Then for all large $N$ we have*

$$Re(S_{a,r}(z) - M(r)z) \leq -cN|z|^3 \text{ for all } z \in \gamma_Z(A) \text{ and} \tag{2.4.18}$$

$$Re(S_{a,r}(z) - M(r)z) \geq cN|z|^3 \text{ for all } z \in \gamma_W(A). \tag{2.4.19}$$

*In the above $c > 0$ depends on $A$ and $\delta$. In addition, we have*

$$Re(S_{a,r}(z) - M(r)z) = O(1) \text{ if } |z| = O(N^{-1/3}) \text{ and} \tag{2.4.20}$$

$$\lim_{N\to\infty} S_{a,r}(N^{-1/3}u) - M(r)N^{-1/3}u = u^3\alpha^{-3}/3 \text{ for all } u \in \mathbb{C}. \tag{2.4.21}$$

**Lemma 2.4.11.** *Let $t, u, U \in (0,1)$ be given such that $0 < u < U < \min(1, -\log t/10)$. Suppose that $z, w \in \mathbb{C}$ are such that $Re(w) \in [-U, 0]$, $Re(z) \in [u, U]$. Then there exists a constant $C > 0$, depending on $t$ such that the following hold*

$$\left|\frac{1}{e^z - e^w}\right| \leq Cu^{-1} \text{ and } \sum_{k\in\mathbb{Z}} \left|\frac{1}{\sin(-\pi f_t(z + 2\pi\iota k, w))}\right| \leq Cu^{-1}, \text{ where } f_t(z,w) = \tfrac{z-w}{-\log t}. \tag{2.4.22}$$

**Step 1.** For $A > 0$ define $\gamma_W'(A) = \{-A|y| + \iota y : y \in \mathbb{R}\}$ and $\gamma_Z'(A) = \{A|y| + \iota y : y \in \mathbb{R}\}$. Suppose $A > 0$ is sufficiently small, so that Proposition 2.4.6 holds. We consider the change of variables $z_i = N^{1/3}Z_i$ and $w_i = N^{1/3}W_i$ and observe that the LHS of (2.4.10) can be rewritten as $\det(I - \tilde{K}_x^N)_{L^2(\gamma_W'(A))}$, where

$$\tilde{K}_x^N(w,w') = \int_{\gamma_Z'(A)} g_{w,w'}^{N,x}(z)\frac{dz}{2\pi\iota}, \text{ and } g_{w,w'}^{N,x}(z) = \mathbf{1}_{\{\max(|Im(w)|,|Im(w')|,|Im(z)|) \leq N^{1/3}\pi\}} \times$$
$$\frac{e^{N^{-1/3}w}N^{-2/3}}{e^{N^{-1/3}w'} - e^{N^{-1/3}z}} G_{\zeta_x,t}(N^{-1/3}w, N^{-1/3}z)\frac{\exp(S_{a,r}(N^{-1/3}z) - MN^{-1/3}z - x\alpha^{-1}z)}{\exp(S_{a,r}(N^{-1/3}w) - MN^{-1/3}w - x\alpha^{-1}w)}. \tag{2.4.23}$$

We deform the contour $\gamma_Z'(A)$ inside the disc of radius $A^{-1}$ so that it is still piecewise smooth and contained in $\{z \in \mathbb{C} : Re(z) \geq 1/2\}$. Observe that the poles of $g_{w,w'}^{N,x}(z)$ in the right complex half-plane come from $G_{\zeta_x,t}$ and are thus located at least a distance of order $N^{1/3}$ from the imaginary axis. The later implies that if we perform, a deformation inside a disc of radius $O(1)$ we will not cross any poles provided $N$ is sufficiently large. In particular, our deformation does not change the value of $g_{w,w'}^{N,x}$ for all large $N$ by Cauchy's Theorem. We will continue to call the new contour by $\gamma_Z'(A)$. Deforming the contour has the advantage of shifting integration away from the singularity point 0.

**Step 2.** Let us now fix $w, w' \in \gamma_W'(A)$ and $z \in \gamma_Z'(A)$ and show that

$$\lim_{N\to\infty} g_{w,w'}^{N,x}(z) = g_{w,w'}^{\infty,x}(z), \text{ where } g_{w,w'}^{\infty,x}(z) := \frac{\exp(\alpha^{-3}z^3/3 - \alpha^{-3}w^3/3 - x\alpha^{-1}z + x\alpha^{-1}w)}{(w-z)(w'-z)}. \tag{2.4.24}$$

One readily observes that

$$\lim_{N\to\infty} e^{N^{-1/3}w}\frac{\mathbf{1}_{\{\max(|Im(w)|,|Im(w')|,|Im(z)|)\le N^{1/3}\pi\}}}{N^{1/3}\left(e^{N^{-1/3}w'}-e^{N^{-1/3}z}\right)}=\frac{1}{w'-z} \tag{2.4.25}$$

Using (2.4.21) we get

$$\lim_{N\to\infty}\frac{\exp(S_{a,r}(N^{-1/3}z)-MN^{-1/3}z-x\alpha^{-1}z)}{\exp(S_{a,r}(N^{-1/3}w)-MN^{-1/3}w-x\alpha^{-1}w)}=\exp(\alpha^{-3}(z^3/3-w^3/3)-x\alpha^{-1}z+x\alpha^{-1}w). \tag{2.4.26}$$

From (2.4.6) we have

$$N^{-1/3}G_{\zeta_x,t}(N^{-1/3}w,N^{-1/3}z)=N^{-1/3}\sum_{k\in\mathbb{Z}}\frac{\pi(-\log t)^{-1}(-\zeta_x)^{2\pi k\iota/(-\log t)}}{\sin(-\pi f_t(N^{-1/3}z+2\pi k\iota,N^{-1/3}w))}. \tag{2.4.27}$$

Using a similar argument as in (2.3.18) we see that for $|k|\ge 1$ and all large $N$ one has

$$\left|\frac{\pi(-\log t)^{-1}(-\zeta_x)^{2\pi k\iota/(-\log t)}}{\sin(-\pi f_t(N^{-1/3}z+2\pi k\iota,N^{-1/3}w))}\right|\le Ce^{-2|k|\pi/(-\log(t))}.$$

The latter is summable over $|k|\ge 1$ and killed by $N^{-1/3}$ in (2.4.27). We see that the only non-trivial contribution in (2.4.27) comes from $k=0$ and so

$$\lim_{N\to\infty}N^{-1/3}G_{\zeta_x,t}(N^{-1/3}w,N^{-1/3}z)=\lim_{N\to\infty}N^{-1/3}\frac{\pi(-\log t)^{-1}}{\sin\left(\frac{\pi N^{-1/3}}{-\log t}(w-z)\right)}=\frac{1}{w-z}. \tag{2.4.28}$$

Equations (2.4.25), (2.4.26) and (2.4.28) imply (2.4.24).

**Step 3.** We now proceed to find estimates of the type necessary in Lemma 2.2.12 for the functions $g^{N,x}_{w,w'}(z)$. If $z$ is outside of the disc of radius $A^{-1}$ (so lies on the undeformed portion of $\gamma'_Z(A)$) and $|Im(z)|\le\pi N^{1/3}$ the estimates of (2.4.18) are applicable (provided $A$ is small enough) and so we obtain

$$|\exp(S_{a,r}(N^{-1/3}z)-MN^{-1/3}z-x\alpha^{-1}z)|\le C\exp(-c|z|^3+|x\alpha^{-1}z|), \tag{2.4.29}$$

where $C,c$ are positive constants. Next suppose $z$ is contained the disc of radius $A^{-1}$ around the origin (i.e. lies on the portion of $\gamma'_Z(A)$ we deformed). From (2.4.21) we know that $S_{a,r}(N^{-1/3}z)-MN^{-1/3}z$ is $O(1)$. This implies that $|\exp(S_{a,r}(N^{-1/3}z)-MN^{-1/3}z-x\alpha^{-1}z)|$ is bounded and the estimate (2.4.29) continues to hold with possibly a bigger $C$.

If $w\in\gamma'_W(A)$ and $|Im(w)|\le\pi N^{1/3}$ the estimates of (2.4.19) are applicable (provided $A$ is small enough) and we obtain

$$|\exp(-S_{a,r}(N^{-1/3}w)+MN^{-1/3}w+x\alpha^{-1}w)|\le C\exp(-c|w|^3+|x\alpha^{-1}w|), \tag{2.4.30}$$

for some $C,c>0$.

If $A$ is sufficiently small so that $A\pi<\min(1,-\log t/10)$, then the estimates in Lemma

2.4.11 hold (with $u = (1/2)N^{-1/3}$ and $U = A\pi$), provided $\max(|Im(w)|, |Im(w')|, |Im(z)|) \leq N^{1/3}\pi$, $z \in \gamma_Z'(A)$ and $w', w \in \gamma_W'(A)$. Consequently, for some positive constant $C$ we have

$$\left| \frac{N^{-1/3}}{e^{N^{-1/3}w'} - e^{N^{-1/3}z}} N^{-1/3} G_{\zeta_x, t}(N^{-1/3}w, N^{-1/3}z) \right| \leq C. \tag{2.4.31}$$

Observe that $e^{N^{-1/3}w} = O(1)$ when $|Im(w)| \leq \pi N^{1/3}$ and $w \in \gamma_W'(A)$. Combining the latter with (2.4.29), (2.4.30) and (2.4.31) we see that whenever $\max(|Im(w)|, |Im(w')|, |Im(z)|) \leq N^{1/3}\pi$, $z \in \gamma_Z'(A)$ and $w', w \in \gamma_W'(A)$ we have

$$|g_{w,w'}^{N,x}(z)| \leq C \exp(-c|w|^3 + |x\alpha^{-1}w|) \exp(-c|z|^3 + |x\alpha^{-1}z|), \tag{2.4.32}$$

where $C, c$ are positive constants. Since $g_{w,w'}^{N,x}(z) = 0$ when $\max(|Im(w)|, |Im(w')|, |Im(z)|) > N^{1/3}\pi$ we see that (2.4.32) holds for all $z \in \gamma_Z'(A)$ and $w', w \in \gamma_W'(A)$.

**Step 4.** We apply Lemma 2.2.12 to the functions $g_{w,w'}^{N,x}(z)$ with $F_1(w) = C \exp(-c|w|^3 + |x\alpha^{-1}w|) = F_2(w)$ and $\Gamma_1 = \gamma_W'(A)$, $\Gamma_2 = \gamma_Z'(A)$. Notice that the functions $F_i$ are integrable on $\Gamma_i$ by the cube in the exponential. As a consequence we see that if we set $\tilde{K}_x^\infty(w, w') := \int_{\gamma_Z'(A)} g_{w,w'}^{\infty,x}(z) \frac{dz}{2\pi\iota}$, then $\tilde{K}_x^N$ and $\tilde{K}_x^\infty$ satisfy the conditions of Lemma 2.2.11, from which we conclude that

$$\lim_{r \to 1^-} \det(I - \tilde{K}_{\zeta_x})_{L^2(\gamma_W(A))} = \det(I - \tilde{K}_x^\infty)_{L^2(\gamma_W'(A))}. \tag{2.4.33}$$

What remains to be seen is that $\det(I - \tilde{K}_x^\infty)_{L^2(\gamma_W'(A))} = F_{GUE}(x)$.

Changing variables, we have that $\det(I - \tilde{K}_x^\infty)_{L^2(\gamma_W')} = 1 + \sum_{n=1}^\infty \frac{(-1)^n}{n!} H(n)$, where

$$H(n) = \sum_{\sigma \in S_n} sign(\sigma) \int_{\gamma_W'} \cdots \int_{\gamma_W'} \int_{\gamma_Z'} \cdots \int_{\gamma_Z'} \prod_{i=1}^n \frac{\exp(z_i^3/3 - w_i^3/3 - xz_i + xw_i)}{(w_i - z_i)(w_{\sigma(i)} - z_i)} \frac{dw_i}{2\pi\iota} \frac{dz_i}{2\pi\iota}.$$

Consequently, we see that

$$\det(I - \tilde{K}_x^\infty)_{L^2(\gamma_W')} = \det(I + \tilde{K}_{Ai})_{L^2(\gamma_W')},$$

where

$$\tilde{K}_{Ai}(w, w') = \int_{\gamma_Z'} \frac{\exp(z^3/3 - w^3/3 - xz + xw)}{(w - z)(z - w')} \frac{dz}{2\pi\iota}. \tag{2.4.34}$$

The proof of Lemma 8.6 in [26] can now be repeated verbatim to show that

$$\det(I + \tilde{K}_{Ai})_{L^2(\gamma_W')} = \det(I - K_{Ai})_{L^2(x,\infty)} = F_{GUE}(x).$$

This suffices for the proof.

## 2.5 CDRP asymptotics

In this section, we obtain alternative formulas for the $t$-Laplace transform of $t^{1-\lambda_1'}$, with $\lambda$ distributed according to the Hall-Littlewood measure with parameters $a, r, t \in (0, 1)$ (see

Section 2.2.4), which are more suitable for asymptotics in the CDRP case. Subsequently, we analyze the formulas that we get in the limiting regime $r,t \to 1^-$, and prove Theorem 2.1.3. In what follows, we will denote by $\mathbb{P}_{a,r,t}$ and $\mathbb{E}_{a,r,t}$ the probability distribution and expectation with respect to the Hall-Littlewood measure with parameters $a,r,t \in (0,1)$.

### 2.5.1 A formula suitable for asymptotics: CDRP case

In this section we use Proposition 2.4.1 to derive an alternative representation for $\mathbb{E}_{a,r,t}\left[\frac{1}{(\zeta t^{1-\lambda_1'};t)_\infty}\right]$. In what follows we will make reference to the following contours

**Definition 2.5.1.** For $t \in (0,1)$ define

$$\gamma_-^t = \{-1/4 + \iota y : y \in [-\pi(-\log t)^{-1}, \pi(-\log t)^{-1}]\},\ \ \gamma_- = \{-1/4 + \iota y : y \in \mathbb{R}\},$$

$$\gamma_+^t = \{1/4 + \iota y : y \in [-\pi(-\log t)^{-1}, \pi(-\log t)^{-1}]\} \text{ and } \gamma_+ = \{1/4 + \iota y : y \in \mathbb{R}\}.$$

All contours are oriented upward.

The following proposition is very similar to Proposition 2.4.6 and will be the starting point of our proof of Theorem 2.1.3 the same way Proposition 2.4.6 was the starting point of the proof of Theorem 2.1.2.

**Proposition 2.5.2.** *Suppose $a,r,t \in (0,1)$ and let $\delta > 0$ be such that $a < (1-\delta)$. If $t$ is sufficiently close to $1^-$ then for $\zeta \in \mathbb{C}\backslash\mathbb{R}^+$ one has*

$$\mathbb{E}_{a,r,t}\left[\frac{1}{(\zeta t^{1-\lambda_1'};t)_\infty}\right] = \det(I - \hat{K}_\zeta)_{L^2(\gamma_-^t)}.$$

*The kernel $\hat{K}_\zeta(W,W')$ has the integral representation*

$$\hat{K}_\zeta(W,W') = \frac{t^{-W}}{2\pi\iota}\int_{\gamma_+^t} G_\zeta(W,Z)\frac{(-\log t)(-\zeta)^{Z-W}dZ}{t^{-W'} - t^{-Z}}\frac{\exp\left(S_{a,r}((-\log t)Z)\right)}{\exp\left(S_{a,r}((-\log t)W)\right)}, \tag{2.5.1}$$

*where $G_\zeta(W,Z) = \sum_{k\in\mathbb{Z}} \frac{\pi(-\zeta)^{-2\pi k\iota/\log t}}{\sin(\pi(W-Z)+2\pi k\iota/\log t)}$, and the contours $\gamma_-^t$ and $\gamma_+^t$ are as in Definition 2.5.1.*

*Proof.* We consider the contour $C := \{-t^{-3/4}e^{\iota\theta} : \theta \in [-\pi,\pi]\}$, which is a positively oriented smooth contour, contained in the annulus $A_{\delta,t}$ in the statement of Proposition 2.4.1 for $t$ sufficiently close to $1^-$. Consequently, from (2.4.1) we know that

$$\mathbb{E}_{a,r,t}\left[\frac{1}{((1-t)ut^{-\lambda_1'};t)_\infty}\right] = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int_C\cdots\int_C \det\left[K_u(w_i,w_j)\right]_{i,j=1}^n \prod_{i=1}^n \frac{dw_i}{2\pi\iota},$$

where $K_u(w,w')$ is as in (2.4.2) and the above sum is absolutely convergent. The $n$-th summand equals

$$\frac{1}{n!}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\det\left[K_u\left(-t^{-3/4}e^{\iota\theta_i}, -t^{-3/4}e^{\iota\theta_j}\right)\right]_{i,j=1}^n \prod_{i=1}^n \frac{-t^{-3/4}\iota e^{\iota\theta_i}d\theta_i}{2\pi\iota}.$$

Setting $y_i = (-1/4) + \iota\theta_i/(-\log t)$, the above becomes

$$\frac{(-1)^n}{n!}\int_{\gamma_-^t}\cdots\int_{\gamma_-^t}\det\left[K_u\left(-t^{-3/4}t^{-y_i-1/4}, -t^{-3/4}t^{-y_j-1/4}\right)\right]_{i,j=1}^n\prod_{i=1}^n\frac{t^{-3/4}t^{-y_i-1/4}(-\log t)dy_i}{2\pi\iota},$$

which can be rewritten as

$$\frac{(-1)^n}{n!}\int_{\gamma_-^t}\cdots\int_{\gamma_-^t}\det\left[(-\log t)t^{-1}t^{-y_i}K_u\left(-t^{-1}t^{-y_i}, -t^{-1}t^{-y_j}\right)\right]_{i,j=1}^n\prod_{i=1}^n\frac{dy_i}{2\pi\iota},$$

and the latter is still absolutely summable over $n$.

To conclude the proof it suffices to show that for $W, W' \in \gamma_-^t$ and $\zeta = (t^{-1}-1)u$ one has

$$(-\log t)t^{-1}t^{-W}K_u\left(-t^{-1}t^{-W}, -t^{-1}t^{-W'}\right) = \hat{K}_\zeta(W, W'). \tag{2.5.2}$$

We observe that the LHS of (2.5.2) equals

$$\frac{(-\log t)t^{-1}t^{-W}}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty}ds\frac{\Gamma(-s)\Gamma(1+s)(-\zeta)^s}{t^{-1}t^{-W'}-t^{-1}t^{-W}t^{-s}}\prod_{j=0}^{\infty}\frac{(1+ar^jt^W)(1+ar^jt^{-W}t^{-s})}{(1+ar^jt^Wt^s)(1+ar^jt^{-W})}.$$

We set $Z = s + W$, and use that $Re(W) = -\frac{1}{4}$ for $W \in \gamma_-^t$ together with Euler's Gamma reflection formula (2.3.16) to see that the above equals

$$\frac{t^{-W}}{2\pi\iota}\int_{\gamma_+}\frac{\pi dZ}{\sin(\pi(W-Z))}\frac{(-\log t)(-\zeta)^{Z-W}}{t^{-W'}-t^{-Z}}\prod_{j=0}^{\infty}\frac{(1+ar^jt^W)(1+ar^jt^{-Z})}{(1+ar^jt^Z)(1+ar^jt^{-W})}.$$

We observe that $t^{\iota s}$ is periodic in $s$ with period $T = \frac{2\pi}{-\log t}$. This allows us to rewrite the above formula as

$$\sum_{k\in\mathbb{Z}}\frac{t^{-W}}{2\pi\iota}\int_{\gamma_+^t}\frac{\pi(-\zeta)^{\iota kT}}{\sin(\pi(W-\iota kT-Z))}\frac{(-\log t)(-\zeta)^{Z-W}dZ}{t^{-W'}-t^{-Z}}\prod_{j=0}^{\infty}\frac{(1+ar^jt^W)(1+ar^jt^{-Z})}{(1+ar^jt^Z)(1+ar^jt^{-W})}.$$

Let $(-\zeta) = re^{\iota\theta}$ with $|\theta| < \pi$. Then, using a similar argument as in (2.3.18), we have for $|k| \geq 1$

$$\left|\frac{\pi(-\zeta)^{\iota kT}}{\sin(\pi(W+\iota kT-Z)}\right| = \left|\frac{\pi e^{-\theta kT}}{\sin(\pi(W+\iota kT-Z))}\right| \leq Ce^{|k|T(|\theta|-\pi)}, \tag{2.5.3}$$

where $C$ is some positive constant, independent of $Z$ and $W$, provided $Z \in \gamma_+^t$ and $W \in \gamma_-^t$. The latter is clearly summable over $k$, which allows us to change the order of the sum and the integrals above and conclude that the LHS of (2.5.2) equals

$$\frac{t^{-W}}{2\pi\iota}\int_{\gamma_+^t}\left[\sum_{k\in\mathbb{Z}}\frac{\pi(-\zeta)^{\iota kT}}{\sin(\pi(W+\iota kT-Z)}\right]\frac{(-\log t)(-\zeta)^{Z-W}dZ}{t^{-W'}-t^{-Z}}\prod_{j=0}^{\infty}\frac{(1+ar^jt^W)(1+ar^jt^{-Z})}{(1+ar^jt^Z)(1+ar^jt^{-W})}.$$

From Lemma 2.4.5 we have that if $t$ is sufficiently close to 1 (so that $(-\log t)z \in D_\delta$ when $|Re(z)| = 1/4$) we have

$$\prod_{j=0}^{\infty} \frac{(1+ar^jt^W)(1+ar^jt^{-Z})}{(1+ar^jt^Z)(1+ar^jt^{-W})} = \frac{\exp\left(S_{a,r}((-\log t)Z)\right)}{\exp\left(S_{a,r}((-\log t)W)\right)}.$$

Substituting this above we see that the LHS of (2.5.2) equals

$$\frac{t^{-W}}{2\pi\iota}\int_{\gamma_+^t}\left[\sum_{k\in\mathbb{Z}}\frac{\pi(-\zeta)^{\iota kT}}{\sin(\pi(W+\iota kT-Z)}\right]\frac{(-\log t)(-\zeta)^{Z-W}dZ}{t^{-W'}-t^{-Z}}\frac{\exp\left(S_{a,r}((-\log t)Z)\right)}{\exp\left(S_{a,r}((-\log t)W)\right)},$$

which equals the RHS of (2.5.2) once we identify the sum in the square brackets with $G_\zeta(W,Z)$. ☐

### 2.5.2 Convergence of the $t$-Laplace transform (CDRP case) and proof of Theorem 2.1.3

Here we state the regime, in which we scale parameters and obtain an asymptotic formula for $\mathbb{E}_{a,r,t}\left[\frac{1}{(\zeta t^{1-\lambda_1'};t)_\infty}\right]$ in the CDRP case. The formula is analyzed below and used to prove Theorem 2.1.3. In the CDRP case the $t$-Laplace transform asymptotically behaves like the usual Laplace transform. The latter (as will be shown carefully below) allows one to obtain the limiting CDF of the properly scaled first column of of a partition distributed according to the Hall-Littlewood measure with parameters $a, r, t$ and match it with $F_{CDRP}$ (see Definition 2.1.7).

We summarize the limiting regime and some relevant expressions.

1. We fix a positive parameter $\kappa$ and let $r \to 1^-$ and $t \to 1^-$ so that $\kappa = \frac{-\log t}{(1-r)^{1/3}}$.
2. We assume that $a$ depends on $r$ and for some $\delta > 0$ we have $\lim_{r\to 1^-} a(r) = a(1) \in (0, 1-\delta]$.
3. We denote by $N(r) = \frac{1}{1-r}$, $M(r) = 2\sum_{k=1}^{\infty}(-1)^{k+1}a(r)^k\frac{1}{1-r^k}$ and $\alpha = \left[\frac{a(1)}{(1+a(1))^2}\right]^{-1/3}$.

$$\text{For a given } x \in \mathbb{R} \text{ set } \zeta_x = -t^{M(r)-x\kappa^{-1}N(r)^{1/3}}. \tag{2.5.4}$$

The following result is the key fact for the limiting fluctuations of the first column of a partition distributed according to the Hall-Littlewood measure with parameters $a, r, t$ in the CDRP case. It shows that under the scaling regime described above the Fredholm determinant (and hence the $t$-Laplace transform) appearing in Proposition 2.5.2 converges to the Laplace transform of $\mathcal{F}(T, 0) + T/24$ (see Definition 2.1.7 and equation (2.1.7)). The latter, as demonstrated below, implies convergence of the usual Laplace transforms and leads to a weak convergence necessary for the proof of Theorem 2.1.3.

**Theorem 2.5.3.** *Let $x \in \mathbb{R}$ be given and let $\zeta_x$ be given as in (2.5.4). Then we have*

$$\lim_{r\to 1^-}\det(I-\hat{K}_{\zeta_x})_{L^2(\gamma_-^t)} = \det(I-K_{CDRP})_{L^2(\mathbb{R}^+)}, \tag{2.5.5}$$

*where $K_{CDRP}$ is given in (2.1.8) with $T = 2\kappa^3\alpha^{-3}$, $\gamma_-^t$ is as in Definition 2.5.1, and $\hat{K}_{\zeta_x}$ is as in (2.5.1).*

In what follows we prove Theorem 2.1.3, assuming the validity of Theorem 2.5.3, whose proof is postponed until the next section.

We begin by summarizing the key results from our previous work that we will use as well as stating a couple of lemmas. From Proposition 2.5.2 and Theorem 2.5.3 we have that under the scaling described in the beginning of this section and any $x \in \mathbb{R}$

$$\lim_{r \to 1^-} \mathbb{E}_{a,r,t}\left[\frac{1}{((-t) \cdot t^{M(r)-\kappa^{-1}xN(r)^{1/3}} t^{-\lambda_1'}; t)_\infty}\right] = \det(I - K_{CDRP})_{L^2(\mathbb{R}^+)}. \tag{2.5.6}$$

Set $\hat{\xi}_r := (-\log t)(\lambda_1' - M(r)) - \log(1-t)$ and observe that (2.5.6) is equivalent to

$$\lim_{r \to 1^-} \mathbb{E}_{a,r,t}\left[\frac{1}{((-t)(1-t) \cdot e^{\hat{\xi}_r + x}; t)_\infty}\right] = \det(I - K_{CDRP})_{L^2(\mathbb{R}^+)}. \tag{2.5.7}$$

The function that appears on the LHS under the expectation in (2.5.7) has the following asymptotic property.

**Lemma 2.5.4.** *For $t \in (0,1)$ and $x \ge 0$ let*

$$g_t(x) := \frac{1}{((-t)(1-t)x; t)_\infty} = \prod_{k=1}^{\infty} \frac{1}{1 + (1-t)xt^k}. \tag{2.5.8}$$

*Then $g_t(x) \to e^{-x}$ uniformly on $\mathbb{R}_{\ge 0}$ as $t \to 1^-$.*

*Proof.* From the monotonicity of $g_t(x)$ and $e^{-x}$ it suffices to show the result only for compact subsets of $\mathbb{R}_{\ge 0}$. Using (10.2.7) in [7] one has that $\frac{1}{(-(1-t)x;t)_\infty} \to e^{-x}$ uniformly on compact subsets of $\mathbb{R}_{\ge 0}$ as $t \to 1^-$. Consequently,

$$g_t(x) = \frac{1 + (1-t)x}{(-(1-t)x; t)_\infty} = \frac{1}{(-(1-t)x; t)_\infty} + \frac{(1-t)x}{(-(1-t)x; t)_\infty}$$

also converges uniformly to $e^{-x}$ on compact subsets $\mathbb{R}_{\ge 0}$ as $t \to 1^-$. □

We will use the following elementary probability lemma.

**Lemma 2.5.5.** *Suppose that $f_n$ is a sequence of functions, $f_n : \mathbb{R}_{\ge 0} \to [0,1]$, such that $f_n(x) \to e^{-x}$ uniformly on $\mathbb{R}_{\ge 0}$. Let $X_n$ be a sequence of non-negagive random variables such that for each $c > 0$ one has*

$$\lim_{n \to \infty} \mathbb{E}[f_n(cX_n)] = p(c),$$

*and assume that $p(c) = \mathbb{E}\left[e^{-cX}\right]$ for some non-negative random variable $X$. Then we have*

$$\lim_{n \to \infty} \mathbb{E}\left[e^{-cX_n}\right] = \mathbb{E}\left[e^{-cX}\right].$$

*In particular, $X_n$ converges in distribution to $X$ as $n \to \infty$.*

*Proof.* Let $\epsilon > 0$ be given. We observe that

$$\left|\mathbb{E}\left[e^{-cX_n}\right] - \mathbb{E}\left[f_n(cX_n)\right]\right| \leq \mathbb{E}\left[\left|e^{-cX_n} - f_n(cX_n)\right|\right] \leq \sup_{x \in \mathbb{R}_{\geq 0}} \left|e^{-x} - f_n(x)\right| \to 0 \text{ as } n \to \infty.$$

In the second inequality we used that $X_n$ are non-negative and the last statement holds by assumption.

It follows that for every $c > 0$ (and clearly also when $c = 0$)

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-cX_n}\right] = \mathbb{E}\left[e^{-cX}\right].$$

The above statement implies $X_n$ converges to $X$ in distribution by Theorem 4.3 in [55]. □

*Proof.* (Theorem 2.1.3) Let $r_n$ be a sequence converging to $1^-$ and set $t_n$ so that $(-\log t_n) = \kappa(1 - r_n)^{1/3}$. Define

$$f_n(x) = \frac{1}{((-t_n)(1 - t_n) \cdot x; t_n)_\infty} \text{ and } X_n = e^{\hat{\xi}_{r_n}}.$$

Lemma 2.5.4 shows that $f_n$ satisfy the conditions of Lemma 2.5.5. In addition, recall that by (2.1.7) we have

$$\det(I - K_{CDRP})_{L^2(\mathbb{R}_+)} = \mathbb{E}\left[e^{-e^x \exp(\mathcal{F}(T,0)+T/24)}\right].$$

where $\mathcal{F}$ is as in Definition 2.1.7 and $T = 2\kappa^3\alpha^{-3}$. Consequently, Lemma 2.5.5 and (2.5.7) show that for $x \in \mathbb{R}$ one has

$$\lim_{n\to\infty} \mathbb{E}_{a,r_n,t_n}\left[e^{-e^x \exp(\hat{\xi}_{r_n})}\right] = \mathbb{E}\left[e^{-e^x \exp(\mathcal{F}(T,0)+T/24)}.\right] \tag{2.5.9}$$

In particular, $\exp(\hat{\xi}_r)$ converges weakly to $\exp(\mathcal{F}(T,0) + T/24) = e^{T/24}\mathcal{Z}(T,0)$. In [67] it was shown that $\mathcal{Z}(T,0)$ is a.s. positive and has a smooth density, thus we conclude that for each $x \in \mathbb{R}_+$ we have

$$\lim_{r\to 1^-} \mathbb{P}_{a,r,t}(\exp(\hat{\xi}_r) \leq x) = \mathbb{P}(\exp(\mathcal{F}(T,0) + T/24) \leq x).$$

Taking logarithms we see that for each $x \in \mathbb{R}$ we have

$$\lim_{r\to 1^-} \mathbb{P}_{a,r,t}(\hat{\xi}_r \leq x) = \mathbb{P}(\mathcal{F}(T,0) + T/24 \leq x). \tag{2.5.10}$$

Consider $a(r) = r^{(1+|\lfloor \tau N(r) \rfloor|)/2}$. Since, $\lim_{r\to 1^-} r^{N(r)} = e^{-1}$, we see that $\lim_{r\to 1^-} a(r) = a(1) = e^{-|\tau|/2} < 1$ (whenever $\tau \neq 0$). This means that $\alpha^{-1} := \left[\frac{a(1)}{(1+a(1))^2}\right]^{1/3} = \left[\frac{e^{-|\tau|/2}}{(1+e^{-|\tau|/2})^2}\right]^{1/3} =:$

$\chi$. From Section 2.2.4 we conclude that

$$\mathbb{P}^{r,t}_{HL}\left(\frac{\lambda'_1(\lfloor \tau N(r)\rfloor) - M(r)}{\chi^{-1}N(r)^{1/3}(T/2)^{-1/3}} + \log(N(r)^{1/3}\chi^{-1}(T/2)^{-1/3}) \leq x\right) =$$
$$\mathbb{P}_{a,r,t}\left(\frac{\lambda'_1 - M(r)}{\alpha^{-1}N(r)^{1/3}(T/2)^{-1/3}} + \log(N(r)^{1/3}\alpha^{-1}(T/2)^{-1/3}) \leq x\right)$$

The latter implies that if we set $\kappa = (T/2)^{1/3}\alpha$ we will get

$$\mathbb{P}^{r,t}_{HL}\left(\frac{\lambda'_1(\lfloor \tau N(r)\rfloor) - M(r)}{\chi^{-1}N(r)^{1/3}(T/2)^{-1/3}} + \log(N(r)^{1/3}\chi^{-1}(T/2)^{-1/3}) \leq x\right) = \mathbb{P}_{a,r,t}(\hat{\xi}_r + \log((1-t)\kappa^{-1}N(r)^{1/3}) \leq x).$$

One observes that $(1-t)\kappa^{-1}N(r)^{1/3} = \frac{1-t}{-\log t} \to 1$ as $r \to 1^-$ and so from (2.5.10) we conclude that

$$\lim_{r\to 1^-} \mathbb{P}^{r,t}_{HL}\left(\frac{\lambda'_1(\lfloor \tau N(r)\rfloor) - M(r)}{\chi^{-1}N(r)^{1/3}(T/2)^{-1/3}} + \log(N(r)^{1/3}\chi^{-1}(T/2)^{1/3}) \leq x\right) = \mathbb{P}(\mathcal{F}(T,0) + T/24 \leq x).$$

From (2.6.8) we have $c_1 = M(r) = 2N(r)\log(1 + a(1)) + O(1) = 2N(r)\log(1 + e^{-|\tau|/2}) + O(1)$. Substituting this above concludes the proof of the theorem. □

### 2.5.3 Proof of Theorem 2.5.3

We split the proof of Theorem 2.5.3 into three steps. In the first step we rewrite the LHS of (2.5.5) in a suitable form for the application of Lemmas 2.2.11 and 2.2.12 and identify the pointwise limit of the integrands. In the second step we provide dominating functions, which are necessary to apply the lemmas. In the third step we obtain a limit for the LHS of (2.5.5), subsequently we use a result from [26], to show that the limit we obtained is in fact $\det(I - K_{CDRP})_{L^2(\mathbb{R}^+)}$.

In Steps 1 and 2 we will require some estimates, which we summarize in Lemmas 2.5.6 and 2.5.7 below. The proofs are postponed until Section 2.6.

**Lemma 2.5.6.** *Let $t$ be sufficiently close to $1^-$. Then for all large $N$ we have*

$$Re(S_{a,r}((-\log t)z) - M(r)(-\log t)z) \leq C - c|z|^2 \text{ for all } z \in \gamma^t_+ \text{ and} \tag{2.5.11}$$

$$Re(S_{a,r}((-\log t)z) - M(r)(-\log t)z) \geq c|z|^2 - C \text{ for all } z \in \gamma^t_-. \tag{2.5.12}$$

*In the above $C, c > 0$ depends on $\delta$. In addition, we have*

$$\lim_{N\to\infty} S_{a,r}((-\log t)u) - M(r)(-\log t)u = u^3\kappa^3\alpha^{-3}/3 \text{ for all } u \in \mathbb{C}. \tag{2.5.13}$$

**Lemma 2.5.7.** *Let $t \in (1/2, 1)$. Then we can find a universal constant $C$ such that*

$$\left|\frac{1}{e^z - e^w}\right| \leq C \text{ and } \sum_{k\in\mathbb{Z}} \left|\frac{1}{\sin(\pi(w - \frac{2\pi k\iota}{-\log t} - z))}\right| \leq C \text{ when } Re(z) = 1/4 \text{ and } Re(w) = -1/4. \tag{2.5.14}$$

**Step 1.** Observe that the LHS of (2.5.5) can be rewritten as $\det(I - \hat{K}_x^N)_{L^2(\gamma_-)}$, where

$$\hat{K}_x^N(w,w') = \int_{\gamma_+} g_{w,w'}^{N,x}(z)\frac{dz}{2\pi\iota}, \text{ and } g_{w,w'}^{N,x}(z) = \mathbf{1}_{\{\max(|Im(w)|,|Im(w')|,|Im(z)|)\leq(-\log t)^{-1}\pi\}} \times$$
$$t^{-w}G_{\zeta_x}(w,z)\frac{(-\log t)}{t^{-w'}-t^{-z}}\frac{\exp\left(S_{a,r}((-\log t)z)+M(\log t)z+xz\right)}{\exp\left(S_{a,r}((-\log t)w)+M(\log t)w+xw\right)}. \tag{2.5.15}$$

Let us now fix $w, w' \in \gamma_W'(A)$ and $z \in \gamma_Z'(A)$ and show that

$$\lim_{N\to\infty} g_{w,w'}^{N,x}(z) = g_{w,w'}^{\infty,x}(z), \text{ where } g_{w,w'}^{\infty,x}(z) := \frac{\pi}{\sin(\pi(z-w))}\frac{1}{z-w'}\frac{\exp(\alpha^{-3}\kappa^3z^3/3+xz)}{\exp(\alpha^{-3}\kappa^3w^3/3+xw)}. \tag{2.5.16}$$

One readily observes that

$$\lim_{N\to\infty} t^{-w}\mathbf{1}_{\{\max(|Im(w)|,|Im(w')|,|Im(z)|)\leq(-\log t)^{-1}\pi\}}\frac{(-\log t)}{t^{-w'}-t^{-z}} = \frac{1}{w'-z} \tag{2.5.17}$$

Using (2.5.13) we get

$$\lim_{N\to\infty}\frac{\exp\left(S_{a,r}((-\log t)z)+M(\log t)z+xz\right)}{\exp\left(S_{a,r}((-\log t)w)+M(\log t)w+xw\right)} = \frac{\exp(\alpha^{-3}\kappa^3z^3/3+xz)}{\exp(\alpha^{-3}\kappa^3w^3/3+xw)}. \tag{2.5.18}$$

From the definition of $G_{\zeta_x}$ we have

$$G_{\zeta_x}(w,z) = \sum_{k\in\mathbb{Z}}\frac{\pi(-\zeta_x)^{2\pi k\iota/(-\log t)}}{\sin(\pi(w-z)+2\pi k\iota/\log t)}. \tag{2.5.19}$$

Using a similar argument as in (2.3.18) we see that for $|k| \geq 1$ and all large $N$ one has

$$\left|\frac{\pi(-\zeta_x)^{2\pi k\iota/(-\log t)}}{\sin(\pi(w-z)+2\pi k\iota/\log t)}\right| \leq Ce^{-2|k|\pi/(-\log t)}.$$

The latter is summable over $|k| \geq 1$ and since $1/(-\log t)$ goes to infinity the sum goes to 0. We see that the only non-trivial contribution in (2.5.19) comes from $k = 0$ and so

$$\lim_{N\to\infty} G_{\zeta_x}(w,z) = \lim_{N\to\infty}\frac{\pi}{\sin(\pi(w-z))} = \frac{\pi}{\sin(\pi(w-z))}. \tag{2.5.20}$$

Equations (2.5.17), (2.5.18) and (2.5.20) imply (2.5.16).

**Step 2.** We now proceed to find estimates of the type necessary in Lemma 2.2.12 for the functions $g_{w,w'}^{N,x}(z)$. If $z \in \gamma_+$ and $|Im(z)| \leq \pi(-\log t)^{-1}$ the estimates of (2.5.11) are applicable and so we obtain

$$|\exp(S_{a,r}((-\log t)z)+M(\log t)z+xz)| \leq C\exp(-c|z|^2+|xz|), \tag{2.5.21}$$

where $C, c$ are positive constants.

If $w \in \gamma_-$ and $|Im(w)| \leq \pi(-\log t)^{-1}$ the estimates of (2.5.12) are applicable and we obtain

$$|\exp(-S_{a,r}((-\log t)w) - M(\log t)w - xw)| \leq C\exp(-c|w|^2 + |xw|), \tag{2.5.22}$$

for some $C, c > 0$.

From Lemma 2.5.7 we have for some $C > 0$ that

$$\left| G_{\zeta_x}(w,z) \frac{(-\log t)}{t^{-w'} - t^{-z}} \right| \leq C. \tag{2.5.23}$$

Observe that $t^{-w} = O(1)$ when $|Im(w)| \leq \pi(-\log t)^{-1}$ and $w \in \gamma_-$. Combining the latter with (2.5.21), (2.5.22) and (2.5.23) we see that whenever $\max(|Im(w)|, |Im(w')|, |Im(z)|) \leq (-\log t)^{-1}\pi$, $z \in \gamma_+$ and $w', w \in \gamma_-$ we have

$$|g^{N,x}_{w,w'}(z)| \leq C\exp(-c|w|^2 + |xw|)\exp(-c|z|^2 + |xz|), \tag{2.5.24}$$

where $C, c$ are positive constants. Since $g^{N,x}_{w,w'}(z) = 0$ when $\max(|Im(w)|, |Im(w')|, |Im(z)|) > (-\log t)^{-1}\pi$ we see that (2.5.24) holds for all $z \in \gamma_+$ and $w', w \in \gamma_+$.

**Step 3.** We may now apply Lemma 2.2.12 to the functions $g^{N,x}_{w,w'}(z)$ with $F_1(w) = C\exp(-c|w|^2 + |xw|) = F_2(w)$ and $\Gamma_1 = \gamma_-$, $\Gamma_2 = \gamma_+$. Notice that the functions $F_i$ are integrable on $\Gamma_i$ by the square in the exponential. As a consence we see that if we set $\hat{K}^{\infty}_x(w,w') := \int_{\gamma_-} g^{\infty,x}_{w,w'}(z)\frac{dz}{2\pi\iota}$, then $\hat{K}^N_x$ and $\tilde{K}^{\infty}_x$ satisfy the conditions of Lemma 2.2.11, from which we conclude that

$$\lim_{r \to 1^-} \det(I - \hat{K}_{\zeta_x})_{L^2(\gamma^t_-)} = \det(I - \hat{K}^{\infty}_x)_{L^2(\gamma_-)}. \tag{2.5.25}$$

What remains to be seen is that $\det(I - \tilde{K}^{\infty}_x)_{L^2(\gamma_-)} = \det(I - K_{CDRP})_{L^2(\mathbb{R}^+)}$.

We have that $\det(I - \tilde{K}^{\infty}_x)_{L^2(\gamma_-)} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}H(n)$, where

$$H(n) = \sum_{\rho \in S_n} sign(\rho) \int_{\gamma_-} \cdots \int_{\gamma_-} \int_{\gamma_+} \cdots \int_{\gamma_+} \prod_{i=1}^{n} \frac{\pi e^{\alpha^{-3}\kappa^3 Z_i^3/3 - \kappa^3\alpha^{-3}W_i^3/3 + xZ_i - xW_i}}{\sin(\pi(Z_i - W_i))(Z_i - W_{\rho(i)})} \frac{dW_i}{2\pi\iota}\frac{dZ_i}{2\pi\iota},$$

Put $\sigma = \alpha\kappa^{-1}$ and consider the change of variables $z_i = \sigma^{-1}Z_i$, $w_i = \sigma^{-1}W_i$. Then we have

$$H(n) = \sum_{\rho \in S_n} sign(\rho) \int_{\frac{-1}{4\sigma}+\iota\mathbb{R}} \cdots \int_{\frac{-1}{4\sigma}+\iota\mathbb{R}} \int_{\frac{1}{4\sigma}+\iota\mathbb{R}} \cdots \int_{\frac{1}{4\sigma}+\iota\mathbb{R}} \prod_{i=1}^{n} \frac{\sigma\pi e^{z_i^3/3 - w_i^3/3 + \sigma x z_i - \sigma x w_i}}{\sin(\sigma\pi(z_i - w_i))(z_i - w_{\rho(i)})} \frac{dw_i}{2\pi\iota}\frac{dz_i}{2\pi\iota}.$$

Consequently, we see that

$$\det(I - \hat{K}^{\infty}_x)_{L^2(\gamma_-)} = \det(I + \hat{K}_{CDRP})_{L^2(\frac{-1}{4}+\iota\mathbb{R})},$$

where

$$\hat{K}_{CDRP}(w,w') = \frac{-1}{2\pi\iota}\int_{\frac{1}{4\sigma}+\iota\mathbb{R}} dz \frac{\sigma\pi e^{\sigma x(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3 - w^3/3}}{z - w'} \tag{2.5.26}$$

The proof of Lemma 8.8 in [26] can now be repeated verbatim to show that

$$\det(I + \hat{K}_{CDRP})_{L^2(\frac{-1}{4}+\iota\mathbb{R})} = \det(I - K_{CDRP})_{L^2(\mathbb{R}_+)}.$$

This suffices for the proof.

## 2.6 The function $S_{a,r}$

In this section we isolate some of the more technical results that were implicitly used in the proofs of Theorems 2.1.2 and 2.1.3. We start by summarizing some of the analytic properties of the function $S_{a,r}$ (see Definition 2.4.4). Subsequently, we identify different ascent/descent contours and analyze the real part of the function along them. We finish with several estimates that played a central role in the proofs of Theorems 2.4.7 and 2.5.3.

### 2.6.1 Analytic properties

We summarize some of the properties of $S_{a,r}$ in a sequence of lemmas. For the reader's convenience we recall the definition of $S_{a,r}$.

$$S_{a,r}(z) := \sum_{j=0}^{\infty} \log(1 + ar^j e^z) - \sum_{j=0}^{\infty} \log(1 + ar^j e^{-z}),$$

where $a, r \in (0, 1)$.

**Lemma 2.6.1.** *Suppose that $\delta \in (0,1)$. Consider $r \in (0,1)$ and $a \in (0, 1-\delta]$. Then there exists $\Delta'(\delta) > 0$ such that $S_{a,r}(z)$ is well-defined and analytic on $D_\delta = \{z \in \mathbb{C} : |Re(z)| < \Delta'\}$ and satisfies*

$$\exp(S_{a,r}(z)) = \prod_{j=0}^{\infty} \frac{1 + ar^j e^z}{1 + ar^j e^{-z}}. \tag{2.6.1}$$

*Proof.* We let $\Delta' > 0$ be such that $(1-\delta)e^{\Delta'} < 1$. Since $r \in (0,1)$, we have that $|ar^j e^{\pm z}| < 1$ for $z \in D_\delta$ and $j \geq 0$. Consequently, $\log(1 + ar^j e^{\pm z})$ is a well-defined analytic function on $D_\delta$ for each $j \geq 0$.

Let $K \subset D_\delta$ be compact. Then there exists a constant $C(K) > 0$ such that $|e^{\pm z}| \leq C$ for all $z \in K$. It follows, that for all large $j$ one has $|e^{\pm z} ar^j| < 1/2$. Using that $|\log(1+w)| \leq 2|w|$ when $|w| < 1/2$ we see that $|(1 + ar^j e^{\pm z})| \leq 2Car^j$ for all large $j$, which are summable. This implies that the sums $\sum_{j=0}^{\infty} \log(1 + ar^j e^{\pm z})$ are absolutely convergent on $K$. This in particular shows $S_{a,r}$ is well-defined, but also, since the absolutely convergent sum of analytic functions is analytic, we conclude that $S_{a,r}(z)$ is analytic on $D_\delta$.

Next let $z \in D_\delta$. From our work above

$$S_{a,r}(z) = \lim_{M \to \infty} \left[ \sum_{j=0}^{M} \log(1 + ar^j e^z) - \sum_{j=0}^{M} \log(1 + ar^j e^{-z}) \right].$$

By continuity of the exponential we see that

$$\exp(S_{a,r}(z)) = \lim_{M\to\infty} \exp\left[\sum_{j=0}^{M} \log(1+ar^je^z) - \sum_{j=0}^{M} \log(1+ar^je^{-z})\right] = \lim_{M\to\infty} \prod_{j=0}^{M} \frac{1+ar^je^z}{1+ar^je^{-z}},$$

which equals the RHS of (2.6.1). □

**Lemma 2.6.2.** *Assume the notation in Lemma 2.6.1. Then $S_{a,r}(z)$ is an odd function on $D_\delta$ and the power series expansion of $S_{a,r}(z)$ near zero has the form*

$$S_{a,r} = c_1 z + c_3 z^3 + \cdots, \quad \textit{where } c_{2l+1} = \frac{2}{(1-r)(2l+1)!} \sum_{k=1}^{\infty} k^{2l}(-1)^{k+1} a^k \frac{1-r}{1-r^k} \in \mathbb{R}. \quad (2.6.2)$$

*Moreover, for each $l \geq 1$ one has that*

$$c_{2l+1} \leq \frac{1}{(1-r)\delta^{2l+1}}. \quad (2.6.3)$$

*Proof.* The fact that $S_{a,r}$ is odd follows from its definition and Lemma 2.6.1. Next we consider $G(z) = \sum_{j=0}^{\infty} \log(1+ar^je^z)$. On $D_\delta$ we have that $|ar^je^z| < 1$ so we can use the power-series expansion for $\log(1+x)$ to get

$$\sum_{j=0}^{\infty} \log(1+ar^je^z) = \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(ar^j)^k e^{kz}.$$

Power-expanding the exponential, the above becomes

$$\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{m!}\frac{(-1)^{k+1}}{k}(ar^j)^k k^m z^m. \quad (2.6.4)$$

We will show that the above sum is absolutely convergent (provided $|z|$ is sufficiently small), which would allow us to freely rearrange the sum.

Consider $f(x) = \frac{1}{1-x} = \sum_{j\geq 0} x^j$ for $|x| < 1$. We know that for $|x| < 1$ and $m \geq 0$ we have

$$f^{(m)}(x) = \sum_{j\geq 0}(j+m)(j+m-1)\cdots(j+1)x^j, \text{ and } f^{(m)}(x) = \frac{m!}{(1-x)^{m+1}}.$$

Putting $x = a$ we see that

$$\sum_{k=1}^{\infty} a^k k^{m-1} \leq \sum_{k=1}^{\infty} a^k k^m \leq \sum_{k\geq 1}(k+m)\cdots(k+1)a^k < \frac{m!}{(1-a)^{m+1}}. \quad (2.6.5)$$

The latter shows that

$$\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(ar^j)^k k^m |z|^m}{km!} \le \frac{1}{1-r}\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{k^{m-1}|z|^m}{m!}a^k < \frac{1}{1-r}\sum_{m=0}^{\infty}\frac{|z|^m}{(1-a)^{m+1}},$$

and the leftmost expression is finite for small enough $|z|$.

Rearranging (2.6.4) we see that the coefficient of $z^m$ in $G(z)$ is $\frac{1}{m!}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+1}}{k}(ar^j)^k k^m$. Since $S_{a,r}(z) = G(z) - G(-z)$ we see that the even coefficients of $S_{a,r}(z)$ are zero, while the odd ones equal

$$c_{2l+1} = \frac{2}{(2l+1)!}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+1}}{k}(ar^j)^k k^{2l+1} = \frac{2}{(1-r)(2l+1)!}\sum_{k=1}^{\infty}k^{2l}(-1)^{k+1}a^k\frac{1-r}{1-r^k},$$

as desired.

For the second part of the lemma observe that

$$\left|\sum_{k=1}^{\infty}k^{2l}(-1)^{k+1}a^k\frac{1-r}{1-r^k}\right| \le \sum_{k=1}^{\infty}k^{2l}a^k < \frac{(2l)!}{(1-a)^{2l+1}},$$

where in the last inequaity we used (2.6.5). If $l \ge 1$ and $a \in (0, 1-\delta]$ we conclude that

$$|c_{2l+1}| \le \frac{2}{(1-r)(2l+1)!}\frac{(2l)!}{(1-a)^{2l+1}} \le \frac{1}{(1-r)\delta^{2l+1}}.$$

□

**Lemma 2.6.3.** *Let $c_1$ and $c_3$ be as in Lemma 2.6.2. Also suppose that $a$, depends on $r$ and $\lim_{r\to 1^-} a(r) = a(1) \in (0, 1-\delta]$. Then*

$$\lim_{r\to 1^-}(1-r)c_1 = 2\log(1+a(1)) \text{ and } \lim_{r\to 1^-}(1-r)c_3 = \frac{1}{3}\frac{a(1)}{(1+a(1))^2}. \tag{2.6.6}$$

*Proof.* From Lemma 2.6.2 we know that $c_1 = \frac{2}{1-r}\sum_{k=1}^{\infty}(-1)^{k+1}a(r)^k\frac{1-r}{1-r^k}$. Consequently,

$$\lim_{r\to 1^-}(1-r)c_1 = 2\lim_{r\to 1^-}\sum_{k=1}^{\infty}(-1)^{k+1}a(r)^k\frac{1-r}{1-r^k} = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{a(1)^k}{k} = 2\log(1+a(1)),$$

where the middle equality follows from the Dominated Convergence Theorem with dominating function $(1-\delta/2)^k$.

Similarly, we have $c_3 = \frac{1}{3(1-r)}\sum_{k=1}^{\infty}k^2(-1)^{k+1}a(r)^k\frac{1-r}{1-r^k}$. Consequently,

$$\lim_{r\to 1^-}(1-r)c_3 = \frac{1}{3}\lim_{r\to 1^-}\sum_{k=1}^{\infty}k^2(-1)^{k+1}a(r)^k\frac{1-r}{1-r^k} = \frac{1}{3}\sum_{k=1}^{\infty}k(-1)^{k+1}a(1)^k = \frac{1}{3}\frac{a(1)}{(1+a(1))^2},$$

where the middle equality follows from the Dominated Convergence Theorem with dominating function $k^2(1-\delta/2)^k$. $\square$

**Lemma 2.6.4.** *Let $c_1$ and $c_3$ be as in Lemma 2.6.2. Let $\tau \in \mathbb{R}\backslash\{0\}$ and suppose $a(r) = \exp\left(\log r \left(1/2 + \frac{1}{2}\left|\lfloor \frac{\tau}{1-r} \rfloor\right|\right)\right)$, then*

$$\lim_{r\to 1^-}(1-r)c_1 = 2\log(1+e^{-|\tau|/2}) \text{ and } \lim_{r\to 1^-}(1-r)c_3 = \frac{1}{3}\frac{e^{-|\tau|/2}}{(1+e^{-|\tau|/2})^2}. \tag{2.6.7}$$

*Moreover, one has*

$$c_1 - \frac{2\log(1+e^{-|\tau|/2})}{1-r} = O(1), \text{ where the constant depends on } \tau. \tag{2.6.8}$$

*Proof.* Using that $r^{\frac{1}{1-r}} \to e^{-1}$ as $r \to 1^-$ we see that $a(1) = \lim_{r\to 1^-} a(r) = e^{-|\tau|/2}$. (2.6.7) now follows from Lemma 2.6.3.

We can rewrite

$$c_1 - \frac{2\log(1+a(1))}{1-r} = I_1 + I_2, \text{ where } I_1 = \frac{2}{1-r}\sum_{k=1}^{\infty} b_k \text{ and } I_2 = \frac{2}{1-r}\sum_{k=1}^{\infty} c_k,$$

with $b_k := (-1)^{k+1}\left[a(r)^k \frac{1-r}{1-r^k} - a(r)^k \frac{1}{k}\right]$ and $c_k := (-1)^{k+1}\left[a(r)^k \frac{1}{k} - a(1)^k \frac{1}{k}\right]$. We will show that $I_1 = O(1) = I_2$.

We begin with $I_1$. One observes that

$$\frac{1-r}{1-r^k} - \frac{1}{k} = \frac{1}{1+\cdots+r^{k-1}} - \frac{1}{k} = \frac{k-1-r-\cdots-r^{k-1}}{k(1+r+\cdots+r^{k-1})} = (1-r)\frac{r^{k-2}+2r^{k-3}+\cdots+(k-1)r^0}{k(1+r+\cdots+r^{k-1})}.$$

Consequently,

$$|b_k| \leq (1-r)a(r)^k \frac{1+2+\cdots+(k-1)}{k} \leq \frac{k}{2}(1-r)a(r)^k.$$

It follows that

$$|I_1| \leq \frac{1}{1-r}\sum_{k=1}^{\infty}(1-r)ka(r)^k \leq \frac{2}{(1-a(r))^3} \leq \frac{2}{(1-e^{-|\tau|/4})^3} = O(1),$$

where in the second inequality we used (2.6.5) and the last inequality holds for all $r$ close to $1^-$.

Next we turn to $I_2 = \frac{2}{1-r}\left[\log(1+a(r)) - \log(1+a(1))\right]$. Since $\log(1+x)$ is $C^1$ on $\mathbb{R}^+$, we see that $|I_2| \leq \frac{2C}{1-r}|a(r)-a(1)|$ for some constant $C$, independent of $r$ (provided it is sufficiently close to $1^-$, so that $|a(1)-a(r)| \leq 1/2$). Hence it suffices to show that

$a(1) - a(r) = O(1-r)$. We know that

$$a(1) - a(r) = e^{-|\tau|/2} - \exp\left(\log r \left(1/2 + \frac{1}{2}\left|\lfloor \frac{\tau}{1-r} \rfloor\right|\right)\right) \in [A(r), B(r)],$$

where $A(r) = e^{-|\tau|/2} - \exp\left(\log r/2 + \frac{\log r|\tau|}{2(1-r)}\right)$ and $B(r) = e^{-|\tau|/2} - \exp\left(\log r + \frac{\log r|\tau|}{2(1-r)}\right)$. Thus it suffices to show that $A(r) = O(1-r) = B(r)$. We know that $r^{1/2}e^{-|\tau|/2} - e^{-|\tau|/2} = O(1-r) = r^1 e^{-|\tau|/2} - e^{-|\tau|/2}$, thus it remains to show that $e^{-|\tau|/2} - \exp\left(-\frac{-\log r|\tau|}{2(1-r)}\right) = O(1-r)$. Using that $e^{-|\tau|u/2}$ is $C^1$ in $u$, we see that

$$\left|e^{-|\tau|/2} - \exp\left(-\frac{-\log r|\tau|}{2(1-r)}\right)\right| \leq C\left|1 - \frac{-\log r}{1-r}\right|,$$

and the latter is clearly $O(1-r)$ by power expanding the logarithm near 1. $\square$

**Lemma 2.6.5.** *Assume the notation in Lemma 2.6.1. On $D_\delta$ one has*

$$S'_{a,r}(z) = \sum_{j=0}^{\infty} \frac{ar^j e^z}{1+ar^j e^z} + \sum_{j=0}^{\infty} \frac{ar^j e^{-z}}{1+ar^j e^{-z}} = \sum_{j=0}^{\infty} ar^j \left[\frac{e^z}{1+ar^j e^z} + \frac{e^{-z}}{1+ar^j e^{-z}}\right]. \tag{2.6.9}$$

*Proof.* In the proof of Lemma 2.6.1 we showed that on $D_\delta$

$$S_{a,r}(z) = \sum_{j=0}^{\infty} \log(1+ar^j e^z) - \sum_{j=0}^{\infty} \log(1+ar^j e^{-z}),$$

the latter sum being absolutely convergent over compact subsets of $D_\delta$. From Theorem 5.2 in Chapter 2 of [77] it follows that

$$S'_{a,r}(z) = \sum_{j=0}^{\infty} \frac{d}{dz}\log(1+ar^j e^z) - \sum_{j=0}^{\infty} \frac{d}{dz}\log(1+ar^j e^{-z}) = \sum_{j=0}^{\infty} \frac{ar^j e^z}{1+ar^j e^z} + \sum_{j=0}^{\infty} \frac{ar^j e^{-z}}{1+ar^j e^{-z}}.$$

$\square$

### 2.6.2 Descent contours

In the following lemmas we demonstrate contours, along which the real part of $S_{a,r}(z) - zS'_{a,r}(0)$ varies monotonically. This monotonicity plays an important role in obtaining the estimates of Lemmas 2.4.10 and 2.5.6.

**Lemma 2.6.6.** *Assume the notation in Lemma 2.6.1. Set $\epsilon = \pm 1$ and $c_1 = S'_{a,r}(0)$. Then there exists an $A_0 > 0$ such that if $0 < A \leq A_0$, one has*

$$\frac{d}{dy} Re\left(S_{a,r}(Ay + \epsilon\iota y) - c_1(Ay + \epsilon\iota y)\right) \leq 0 \textit{ for all } y \in [0, \pi].$$

$$\frac{d}{dy} Re\left(S_{a,r}(-Ay + \epsilon\iota y) - c_1(-Ay + \epsilon\iota y)\right) \geq 0 \textit{ for all } y \in [0, \pi].$$

*Proof.* Choose $A_0 > 0$ sufficiently small so that $\{\pm Ay + \iota y : y \in [-\pi, \pi]\} \subset D_\delta$, whenever $0 < A \leq A_0$.

Set $b_j = ar^j$. We will focus on the first statement. We have (using Lemma 2.6.5) that

$$\frac{d}{dy} Re\left(S_{a,r}(Ay + \epsilon\iota y) - c_1(Ay + \iota y)\right) =$$

$$= \sum_{j=0}^{\infty} Re\left[b_j\left[\frac{e^{Ay+\epsilon\iota y}}{1 + b_j e^{Ay+\epsilon\iota y}} + \frac{e^{-(Ay+\epsilon\iota y)}}{1 + b_j e^{-(Ay+\epsilon\iota y)}} - \frac{2}{1+b_j}\right](A + \epsilon\iota)\right].$$

We will show that each summand is $\leq 0$, provided $A$ is small enough. The latter would follow provided we know that for every $b \in (0, 1-\delta]$ one has

$$Re\left[\left[\frac{e^{Ay+\epsilon\iota y}}{1 + be^{Ay+\epsilon\iota y}} + \frac{e^{-(Ay+\epsilon\iota y)}}{1 + be^{-(Ay+\epsilon\iota y)}} - \frac{2}{1+b}\right](A + \epsilon\iota)\right] \leq 0.$$

Multiplying denominators by their complex conjugates and extracting the real part, we see that the above is equivalent to $I_1 + I_2 \leq 0$, where

$$I_1 := A\left[\frac{be^{2Ay} + e^{Ay}\cos(y)}{|1 + be^{Ay+\epsilon\iota y}|^2} + \frac{be^{-2Ay} + e^{-Ay}\cos(y)}{|1 + be^{-Ay-\epsilon\iota y}|^2} - \frac{2}{1+b}\right] \text{ and}$$

$$I_2 := \frac{-e^{Ay}\epsilon\sin(\epsilon y)}{|1 + be^{Ay+\epsilon\iota y}|^2} + \frac{e^{-Ay}\epsilon\sin(\epsilon y)}{|1 + be^{-Ay-\epsilon\iota y}|^2}.$$

We show that $I_1 \leq 0$ and $I_2 \leq 0$, provided $A$ is small enough.

We start with $I_2$, which can be rewritten as

$$I_2 = \frac{-e^{Ay}\sin(y)}{1 + b^2e^{2Ay} + 2\cos(y)be^{Ay}} + \frac{e^{-Ay}\sin(y)}{1 + b^2e^{-2Ay} + 2\cos(y)be^{-Ay}}.$$

Since $y \in [0, \pi]$, we have that $\sin(y) \geq 0$. Hence it suffices to show that

$$0 \geq \frac{-e^{Ay}}{1 + b^2e^{2Ay} + 2\cos(y)be^{Ay}} + \frac{e^{-Ay}}{1 + b^2e^{-2Ay} + 2\cos(y)be^{-Ay}} \iff$$

$$u^{-1} + b^2u + 2b\cos(y) \geq u + b^2u^{-1} + 2b\cos(y)$$

where $u = e^{-Ay} \in (0, 1]$. The above now is equivalent to $(u^{-1} - u)(1 - b^2) \geq 0$, which clearly holds if $u \in (0, 1]$ and $b \in (0, 1]$, as is the case. Hence $I_2 \leq 0$ without any restrictions on $A$ except that it is positive.

Next we analyze $I_1$, which can be rewritten as

$$I_1 = A\left[\frac{be^{2Ay} + e^{Ay}\cos(y)}{1 + b^2e^{2Ay} + 2b\cos(y)e^{Ay}} + \frac{be^{-2Ay} + e^{-Ay}\cos(y)}{1 + b^2e^{-2Ay} + 2b\cos(y)e^{-Ay}} - \frac{2}{1+b}\right].$$

We see that (since $A > 0$)

$I_1 \le 0 \iff (1+b^2e^{-2Ay}+2b\cos(y)e^{-Ay})(be^{2Ay}+e^{Ay}\cos(y))(1+b)+(1+b^2e^{2Ay}+2b\cos(y)e^{Ay})\cdot$

$(be^{-2Ay}+e^{-Ay}\cos(y))(1+b)-2(1+b^2e^{-2Ay}+2b\cos(y)e^{-Ay})(1+b^2e^{2Ay}+2b\cos(y)e^{Ay}) \le 0 \iff$

$$\left(1+b^2e^{-2Ay}+2b\cos(y)e^{-Ay}\right)\left(be^{2Ay}+e^{Ay}\cos(y)-1-be^{Ay}\cos(y)\right)+$$
$$+\left(1+b^2e^{2Ay}+2b\cos(y)e^{Ay}\right)\left(be^{-2Ay}+e^{-Ay}\cos(y)-1-be^{-Ay}\cos(y)\right) \le 0 \iff$$

$f(y) = u(y)^2(b-b^2)+u(y)\cos(y)(1-b)^3+[-2b-2+2b^3+2b^2+4b\cos(y)^2-4b^2\cos(y)^2] \le 0,$

where $u(y) = e^{Ay}+e^{-Ay}$. We want to show $f(y) \le 0$ on $[0,\pi]$, provided $A$ is small enough.

First consider $y \in [0,\pi/2]$. We have

$f'(y) = 2uu'(b-b^2)+u'\cos(y)(1-b)^3-u\sin(y)(1-b)^3+[-8b\cos(y)\sin(y)+8b^2\cos(y)\sin(y)].$

The last term equals $8b\sin(y)\cos(y)(b-1)$ and is non-positive, when $y \in [0,\pi/2]$. Thus

$$f'(y) \le 2uu'(b-b^2)+u'\cos(y)(1-b)^3-u\sin(y)(1-b)^3.$$

For $A$ sufficiently small we have $u' \le 4Ay$, $u \le 3$ and $\sin(y) > y/5$ on $[0,\pi/2]$. Thus we see

$$f'(y) \le 24A(b-b^2)y+4(1-b)^3Ay-\frac{2}{5}(1-b)^3y.$$

For $A$ sufficiently small $f'(y) < 0$ on $(0,\pi/2)$ so $f$ is decreasing on $(0,\pi/2)$. But $f(0) = 0$ so we see $f(y) \le 0$ when $y \in [0,\pi/2]$.

Next we consider the case when $y \in [\pi/2,\pi]$. In that case $\cos(y) \le 0$ and we see

$$f(y) \le u(y)^2b(1-b)-2(1-b)(1+b)^2+4b\cos(y)^2(1-b).$$

The latter expression is non-positive exactly when

$$bu(y)^2-2(1+b)^2+4b\cos(y)^2 \le 0.$$

For $A$ sufficiently small we have $u^2 \in [4, 4+\epsilon_0)$ for all $y \in [\pi/2,\pi]$. Thus it suffices to show that we can find $\epsilon_0 > 0$ such that

$$4b+b\epsilon_0-2(1+b)^2+4b \le 0 \iff b\epsilon_0 \le 2(1-b)^2,$$

which is clearly possible as $b \in [0,1-\delta]$. Thus we conclude that there exists $A > 0$ small enough so that the first statement of the lemma holds. Using that $S_{a,r}(z)$ is an odd function, the second statement of the lemma follows from the first and the same $A$ may be chosen. □

**Lemma 2.6.7.** *Assume the notation in Lemma 2.6.1. Suppose $t$ is sufficiently close to $1^-$. If $\beta \ge 0$ and $z = (-\log t)(\beta + \iota s)$ then*

$$\frac{d}{ds}Re(S_{a,r}(z)) \le 0 \text{ when } s \in [0,\pi(-\log t)^{-1}] \text{ and } \frac{d}{ds}Re(S_{a,r}(z)) \ge 0 \text{ when } s \in [-\pi(-\log t)^{-1},0].$$

*If* $\beta \leq 0$ *and* $z = (-\log t)(\beta + \iota s)$ *then*

$$\frac{d}{ds}Re(S_{a,r}(z)) \geq 0 \text{ when } s \in [0, \pi(-\log t)^{-1}] \text{ and } \frac{d}{ds}Re(S_{a,r}(z)) \leq 0 \text{ when } s \in [-\pi(-\log t)^{-1}, 0].$$

*Proof.* The dependence on $t$ comes from our desire to make $|\beta|(-\log t) < \Delta'$ in the statement of Lemma 2.6.1. We assume this for the remainder of the proof.

Setting $z = (-\log t)(\beta + \iota s)$ we see from Lemma 2.6.5

$$\frac{d}{ds}S_{a,r}(z) = \sum_{j=0}^{\infty} \iota b_j(-\log t)\left[\frac{e^{(-\log t)(\beta+\iota s)}}{1 + b_j e^{(-\log t)(\beta+\iota s)}} - \frac{e^{(-\log t)(-\beta-\iota s)}}{1 + b_j e^{(-\log t)(-\beta-\iota s)}}\right],$$

where $b_j = ar^j$. Thus we see that

$$\frac{d}{ds}Re(S_{a,r}(z)) = \sum_{j=0}^{\infty}\left[-\frac{b_j(-\log t)\sin(\theta)t^{-\beta}}{1 + 2\cos(\theta)b_j t^{-\beta} + b_j^2 t^{-2\beta}} + \frac{b_j(-\log t)\sin(\theta)t^{\beta}}{1 + 2\cos(\theta)b_j t^{\beta} + b_j^2 t^{2\beta}}\right],$$

where $\theta = s(-\log t)$.

We now check that each summand has the right sign for the ranges of $s$ and $\beta$ in the statement in the lemma. We focus on $\beta \geq 0$ and $s \in [0, \pi(-\log t)^{-1}]$, all other cases can be handled similarly.

We want to show that

$$-\frac{b_j(-\log t)\sin(\theta)t^{-\beta}}{1 + 2\cos(\theta)b_j t^{-\beta} + b_j^2 t^{-2\beta}} + \frac{b_j(-\log t)\sin(\theta)t^{\beta}}{1 + 2\cos(\theta)b_j t^{\beta} + b_j^2 t^{2\beta}} \leq 0 \text{ for each } j.$$

Put $u = t^{-\beta}$ and $b_j = b$. Observe that for $s \in [0, \pi(-\log t)^{-1}]$, $\theta \in [0, \pi]$ so the above would follow from

$$-\frac{u}{1 + 2\cos(\theta)bu + b^2u^2} + \frac{u^{-1}}{1 + 2\cos(\theta)bu^{-1} + b^2u^{-2}} \leq 0 \iff$$

$$\iff u^{-1}(1 + 2\cos(\theta)bu + b^2u^2) \leq u(1 + 2\cos(\theta)bu^{-1} + b^2u^{-2}) \iff$$

$$u^{-1} + 2\cos(\theta)b + b^2u \leq u + 2\cos(\theta)b + b^2u^{-1} \iff (u^{-1} - u)(1 - b^2) \leq 0.$$

The latter is true since $u \geq 1$ and $b \in (0, 1)$. □

### 2.6.3 Proof of Lemmas 2.4.10 and 2.5.6

Suppose that $\delta > \epsilon > 0$ is sufficiently small so that $S_{a,r}$ has an analytic expansion in the disc of radius $\epsilon$ for $r \in (0, 1)$ and $a \in (0, 1 - \delta]$. From (2.6.3) we know that when $|z| < \epsilon$ one has

$$|S_{a,r}(z) - c_1 z - c_3 z^3| \leq \frac{|z|^4}{1 - r}\sum_{l \geq 2} \epsilon^{2l-3}\delta^{-2l-1}, \tag{2.6.10}$$

and the latter sum is finite by comparison with the geometric series. Suppose that $z = N^{-1/3}w$ where $N = \frac{1}{1-r}$. Clearly, the RHS of (2.6.10) is $O(N^{-1/3})$ and so

$$\lim_{N\to\infty} |S_{a,r}(N^{-1/3}w) - c_1 N^{-1/3}w - c_3 N^{-1}w^3| = 0.$$

Using that $\lim_{N\to\infty} c_3 N^{-1} = \frac{1}{3}\frac{a(1)}{(1+a(1))^2}$ (this is (2.6.6)) and the above we conclude that

$$S_{a,r}(N^{-1/3}w) - c_1 N^{-1/3}w = O(1) \text{ if } w = O(1) \text{ and } \lim_{N\to\infty} S_{a,r}(N^{-1/3}w) - c_1 N^{-1/3}w = \frac{a(1)w^3}{3(1+a(1))^2}.$$

This proves (2.4.20), (2.4.21) and once we set $(-\log t) = \kappa N^{-1/3}$ also (2.5.13).

Suppose $A$ sufficiently small so that the statement of Proposition 2.6.6 holds and so that $\phi = \arctan(A)$ is less than $10°$. By choosing a smaller $\epsilon$ than the one we had before we may assume that $\sum_{l\geq 2} \epsilon^{2l-3}\delta^{2l+1} \leq \frac{a(1)\sin(3\phi)}{12(1+a(1))^2} = c'$. In view of (2.6.10) and (2.6.6) we know that for all large $N$ and $|z| < \epsilon$

$$Re\,(S_{a,r}(z) - c_1 z) \geq c_3 Re(z^3) - c' N|z|^4 \geq N|z|^3 \frac{a(1)\sin(3\phi)}{6(1+a(1))^2} - c'N|z|^3 \geq c'N|z|^3 \text{ if } z \in \gamma_W.$$

This proves (2.4.19) when $|z| < \epsilon$. Put $K = \frac{\epsilon}{2\pi}$ and observe that if $z \in \gamma_W$ then $Kz \in \gamma_W$ and $K|z| < \epsilon$. The latter suggests that if $z \in \gamma_W$ we have

$$Re\,(S_{a,r}(z) - c_1 z) \geq Re(S_{a,r}(Kz) - M(r)Kz) \geq c'NK^3|z|^3,$$

where in the first inequality we used the first statement of Lemma 2.6.6, and in the second one we used that $K|z| < \epsilon$ and our earlier estimate. This proves (2.4.19) and using that $S_{a,r}(-z) = -S_{a,r}(z)$, while $\gamma_W = -\gamma_Z$ it also proves (2.4.18).

Let $z = 1/4 + \iota s$ and set $(-\log t) = \kappa N^{-1/3}$ for some positive $\kappa$. Suppose $|(-\log t)z| < \epsilon$ with $\epsilon$ as in the beginning of the section. We have the following equality

$$Re(c_{2l+1}(-\log t)^{2l+1} z^{2l+1}) = c_{2l+1}(-\log t)^{2l+1} \sum_{k=0}^{l} \binom{2l+1}{2k} s^{2k}(-1)^k \frac{1}{4^{2l-2k+1}}.$$

In particular, we see that

$$\begin{gathered}\left|Re(c_{2l+1}(-\log t)^{2l+1} z^{2l+1})\right| \leq c_{2l+1}(-\log t)^{2l+1}\left((|s| + 1/4)^{2l+1} - |s|^{2l+1}\right) \leq \\ c_{2l+1}(-\log t)^{2l+1}\frac{1}{4}\sum_{k=0}^{2l} |s|^k (1/4)^{2l-k} \leq (2l+1)c_{2l+1}(-\log t)^{2l+1}|z|^{2l}.\end{gathered} \tag{2.6.11}$$

Using (2.6.11) and (2.6.3) we have for $|(-\log t)z| < \epsilon$ that

$$\left|Re\left(\sum_{l\geq 2} c_{2l+1}(-\log t)^{2l+1} z^{2l+1}\right)\right| \leq \kappa^3 |z|^2 \sum_{l\geq 2}(2l+1)\delta^{-2l-1}\epsilon^{2l-2}. \tag{2.6.12}$$

On the other hand, we have that

$$Re(c_3(-\log t)^3 z^3) = -(3c_3/4)(-\log t)^3|z|^2 + (-\log t)^3/64 + (3c_3/64)(-\log t)^3. \quad (2.6.13)$$

Combining equations (2.6.12) and (2.6.13) we see that if $|(-\log t)z| < \epsilon$ then

$$Re(S_{a,r}((-\log t)z) - c_1(-\log t)z) \le -(3c_3/4)(-\log t)^3|z|^2 + (-\log t)^3/64 + (3c_3/64)(-\log t)^3 +$$
$$+\kappa^3|z|^2 \sum_{l\ge 2}(2l+1)\delta^{-2l-1}\epsilon^{2l-2}.$$

Notice that $(3c_3/4)(-\log t)^3 \to \kappa^3 \frac{a(1)}{4(1+a(1))^2} =: \rho$ as $N \to \infty$ from (2.6.6). Moreover if we pick $\epsilon$ small enough we can make $\kappa^3 \sum_{l\ge 2}(2l+1)\delta^{-2l-1}\epsilon^{2l-2} \le (\rho/4)$. It follows that for all large $N$ we have

$$Re(S_{a,r}((-\log t)z) - c_1(-\log t)z) \le -(\rho/2)|z|^2 + (\rho/8).$$

This proves (2.5.11) whenever $|(-\log t)z| < \epsilon$.

Suppose now that $z = 1/4 + \iota s$ and $s \in [-\pi(-\log t)^{-1}, \pi(-\log t)^{-1}]$. Put $K = \frac{\epsilon}{2\pi}$ and notice that for all large $N$ we have $\tilde{z} := 1/4 + \iota K s$ satisfies $|\tilde{z}(-\log t)| < \epsilon$. It follows from the first result of Lemma 2.6.7 and our estimate above that

$$Re(S_{a,r}((-\log t)z) - c_1(-\log t)z) \le Re(S_{a,r}((-\log t)\tilde{z}) - c_1(-\log t)\tilde{z}) \le -(\rho/2)|\tilde{z}|^2 + (\rho/8).$$

Observing that $|\tilde{z}|^2 \ge K^{-2}|z|^2$ we conclude (2.5.11) for all $z \in \gamma_+^t$. The result of (2.5.12) now follows from (2.5.11) once we use that $S_{a,r}(-z) = -S_{a,r}(z)$ and that $\gamma_-^t = -\gamma_+^t$.

### 2.6.4 Proof of Lemmas 2.4.11 and 2.5.7

Let $z = x + \iota p$ and $w = y + \iota q$ so that $x > 0$ and $y \le 0$ . Then we have

$$\left|\frac{1}{e^z - e^w}\right| = \left|\frac{1}{e^x - e^y e^{\iota(q-p)}}\right| \le \left|\frac{1}{e^x - e^y}\right| \le \frac{1}{e^x - 1} \le x^{-1},$$

where in the last inequality we used $e^c \ge c + 1$ for $c \ge 0$. This proves the first parts of (2.4.22) and (2.5.14).

Let $\sigma = (-\log t)^{-1}$. Then we have

$$\left|\frac{1}{\sin(-\pi\sigma(x - y + \iota(p-q)))}\right| = \left|\frac{2}{e^{-\iota\pi\sigma(x-y)}e^{\pi\sigma(p-q)} - e^{\iota\pi\sigma(x-y)}e^{\pi\sigma(q-p)}}\right|$$

If $q \ge p$ we see

$$\left|e^{-\iota\pi\sigma(x-y)}e^{\pi\sigma(p-q)} - e^{\iota\pi\sigma(x-y)}e^{\pi\sigma(q-p)}\right| = \left|e^{\pi\sigma(p-q)} - e^{2\iota\pi\sigma(x-y)}e^{\pi\sigma(q-p)}\right| \ge e^{\pi\sigma(q-p)}|\sin(2\pi\sigma(x-y))|.$$

Conversely, if $q < p$ we see

$$\left|e^{-\iota\pi\sigma(x-y)}e^{\pi\sigma(p-q)} - e^{\iota\pi\sigma(x-y)}e^{\pi\sigma(q-p)}\right| = \left|e^{-2\iota\pi\sigma(x-y)}e^{\pi\sigma(p-q)} - e^{\pi\sigma(q-p)}\right| \ge e^{\pi\sigma(p-q)}|\sin(2\pi\sigma(x-y))|.$$

We thus conclude that

$$\left|\frac{1}{\sin(-\pi\sigma(x-y+\iota(p-q))}\right| \leq e^{-\pi\sigma|p-q|}\frac{2}{|\sin(2\pi\sigma(x-y))|}. \tag{2.6.14}$$

In the assumption of Lemma 2.4.11 we have $x - y \in [u, 2U]$ and $2U \leq \sigma^{-1}/5$. Thus $2\pi\sigma(x-y) \in [2\pi\sigma u, 2\pi/5]$. This implies that

$$\left|\frac{1}{|\sin(2\pi\sigma(x-y))|}\right| \leq e^{-\pi\sigma|p-q|}\frac{1}{\sigma u}, \tag{2.6.15}$$

where we used that $\sin x$ is increasing on $[0, \pi/2]$ and satisfies $\pi \sin x \geq x$ there. In addition, we have from the above

$$\sum_{k\in\mathbb{Z}}\left|\frac{1}{\sin(-\pi\sigma(x-y+\iota(p+2\pi k-q))}\right| \leq \sum_{k\in\mathbb{Z}} e^{-\pi\sigma|p+2\pi k-q|}\sigma^{-1}u^{-1} \leq 2\sigma^{-1}u^{-1}\sum_{k\geq 0} e^{-2k\pi^2\sigma}.$$

This proves the second part of (2.4.22).

Finally, suppose that $x = 1/4$ and $y = -1/4$. Notice that if $\mathrm{dist}(s, \mathbb{Z}) > c$ for some constant $c > 0$ then $\left|\frac{1}{\sin(\pi s)}\right| \leq c' e^{-\pi|Im(s)|}$ for some $c'$, depending on $c$. Using this we get

$$\sum_{k\in\mathbb{Z}}\left|\frac{1}{\sin(\pi(w - \frac{2\pi k\iota}{-\log t} - z))}\right| = \sum_{k\in\mathbb{Z}}\left|\frac{1}{\sin(\pi/2 - \frac{2\pi^2 k\iota}{-\log t} + \pi\iota(q-p)))}\right| \leq$$

$$\leq c'\sum_{k\in\mathbb{Z}}\exp\left(-\left|-\frac{2\pi^2 k}{-\log t} + \pi(q-p)\right|\right) \leq 2c'\sum_{k\geq 0}\exp\left(-\frac{2\pi^2 k}{-\log t}\right).$$

The latter is uniformly bounded for $t \in (1/2, 1)$, by $\frac{2c'}{1-v}$ with $v = \exp\left(-\frac{2\pi^2}{-\log(1/2)}\right)$. This concludes the proof of the second part of (2.5.14).

## 2.7 Sampling of plane partitions

In this section, we describe a sampler of random plane partitions, based on *Glauber dynamics*. Subsequently, we formulate several conjectures about the convergence of the measure $\mathbb{P}_{HL}^{r,t}$ and provide some evidence about their validity.

### 2.7.1 Glauber dynamics

We start with a brief recollection of the (single-site) *Glauber dynamics* for probability measures on labelled graphs. In what follows, we will use Section 3.3 in [62] as a main reference and recommend the latter for more details.

Let $V$ and $S$ be finite sets and suppose that $\Omega$ is a subset of $S^V$. The elements of $S^V$, called *configurations*, are the functions from $V$ to $S$. One visualizes a configuration as a labeling of the vertex set $V$ of some graph by elements in $S$. Let $\pi$ be a probability distribution, whose support is $\Omega$. The (single-site) Glauber dynamics for $\pi$ is a reversible

Markov chain with state space $\Omega$, stationary distribution $\pi$ and transition probabilities as described below.

For $x \in \Omega$ and $v \in V$, let

$$\Omega(x,v) := \{y \in \Omega : y(w) = x(w) \text{ for all } w \neq v\} \text{ and } \pi^{x,v}(y) := \begin{cases} \frac{\pi(y)}{\pi(\Omega(x,v))} & \text{if } y \in \Omega(x,v), \\ 0 & \text{if } y \notin \Omega(x,v). \end{cases}$$

With the above notation, the Glauber chain moves from state $x$ as follows: a vertex $v$ is chosen uniformly at random from $V$, then one chooses a new configuration according to $\pi^{x,v}$.

One can show that $\pi$ is a stationary measure for the Glauber dynamics and that the chain is *ergodic*. This implies that if the chain is run for $T$ steps, started from any initial state, then the distribution of the state at step $T$ will converge to the stationary distribution $\pi$ as $T \to \infty$. The latter observation explains how one can use the Glauber dynamics to numerically (approximately) sample arbitrary distributions $\pi$ on $\Omega$. Namely, one constructs the Glauber dynamics and runs it for a very long time $T$, so that the distribution is close to the stationary distribution of the chain. This sampling method is called a *Gibbs sampler* and it belongs to a more general class of methods called *Markov chain Monte Carlo*. The time one has to wait for the chain to converge, is typically referred to as a *mixing time*; and finding estimates for mixing times is in general very hard.

In our case, we consider the measure $\mathbb{P}_{r,t}$ (here $r \in (0,1)$ and $t \in [-1,1]$ ) on plane partitions, which are contained in a big box $N \times N \times N$, satisfying

$$\mathbb{P}_{r,t}(\pi) \propto r^{|\pi|} B_\pi(t), \tag{2.7.1}$$

where $|\pi|$ is the volume of the partition and $B_\pi(t)$ is as in Section 2.2.4. Specifically, $\mathbb{P}_{r,t}$ is the same as the distribution $\mathbb{P}_{HL}^{r,t,N}$ of Section 2.2.4, conditioned on plane partitions not exceeding height $N$. We now describe a Gibbs sampler for the above measure.

Set $V = \{(x,y,z) : x,y,z \in \{1,...,N\}\}$ and $S = \{0,1\}$. A configuration $\omega \in S^V$ is interpreted as a placements of unit cubes inside the box $N \times N \times N$, so that $\omega((x,y,z)) = 1$ if an only if there is a cube at position $(x,y,z)$. We next let $\Omega$ be the subset of cube placements, corresponding to plane partitions. This describes the state space of our Glauber dynamics. Since $|S| = 2$, we see that if $\pi_a \in \Omega$ we have $|\Omega(\pi_a, v)| = 1$ or 2; hence, $\mathbb{P}_{r,t}^{\pi_a,v}$ is either a point mass at $\pi_a$ or a Bernoulli measure, whose support lies on $\pi_a$ and the partition $\pi_b$, which is obtained from $\pi_a$ by changing the value of $\pi_a$ at $v$ from 1 to 0 or vice versa.

At this time we introduce some terminology. Given a plane partition $\pi$, we call a cube *addable* if the the cube does not belong to $\pi$ and by placing the cube in the box we obtain a valid plane partition. Similarly, we call a cube *removable* if it belongs to $\pi$ and removing the cube from the box results in a valid plane partition. Denote by $Add_\pi$ and $Rem_\pi$ the (disjoint) sets of addable and removable cubes respectively. Some of these concepts are illustrated in Figure 2-8. We observe that $|\Omega(\pi,v)| = 2$ precisely when there is an element of $Add_\pi$ or $Rem_\pi$ at position $v$.

We now turn to finding $\mathbb{P}_{r,t}^{\pi,v}$ when $|\Omega(\pi,v)| = 2$. Let $\hat{\pi}$ be the plane partition obtained from $\pi$ by adding a cube at position $v$ if one is not already present there, otherwise $\hat{\pi} = \pi$. In addition, let $\check{\pi}$ be the plane partition obtained from $\pi$ by removing the cube at position

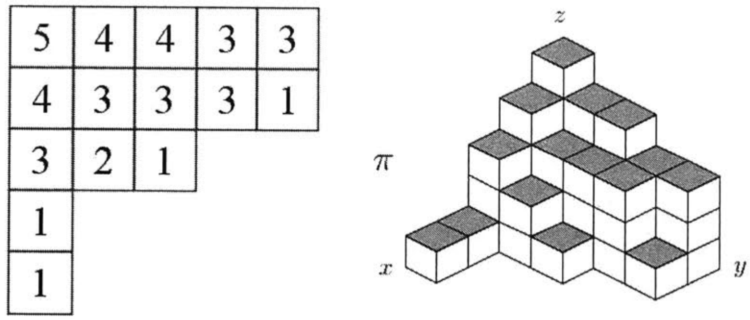


Figure 2-8: If $N = 5$, then the addable cubes in this example are at positions: $(4,1,2),(3,1,4),(2,1,5),(3,2,3),(2,2,4),(1,2,5),(3,3,2),(1,4,4),(2,5,2)$. The removable cubes are at positions: $(5,1,1),(3,1,3),(2,1,4),(1,1,5),(3,2,2),(3,3,1),$ $(1,3,4),(2,4,3),(2,5,1),(1,5,3)$.

$v$ if there is one, otherwise $\check{\pi} = \pi$. Observe that if $|\Omega(\pi, v)| = 2$, we have either $\hat{\pi} = \pi$, or $\check{\pi} = \pi$.

From our earlier discussion, $\mathbb{P}^{\pi,v}_{r,t}$ is a Bernoulli measure supported on $\hat{\pi}$ and $\check{\pi}$. Using results from Section 2.2.4 we have that if $\hat{\lambda}^k$ and $\check{\lambda}^k$ denote the diagonal slices of $\hat{\pi}$ and $\check{\pi}$ respectively, we have

$$\begin{aligned}\mathbb{P}_{r,t}(\hat{\pi}) &\propto r^{|\pi|} \prod_{n=-N+1}^{0} \psi_{\hat{\lambda}^n/\hat{\lambda}^{n-1}}(0,t) \times \prod_{n=1}^{N} \phi_{\hat{\lambda}^{n-1}/\hat{\lambda}^n}(0,t), \\ \mathbb{P}_{r,t}(\check{\pi}) &\propto r^{|\pi|} \prod_{n=-N+1}^{0} \psi_{\check{\lambda}^n/\check{\lambda}^{n-1}}(0,t) \times \prod_{n=1}^{N} \phi_{\check{\lambda}^{n-1}/\check{\lambda}^n}(0,t).\end{aligned} \tag{2.7.2}$$

We recall that $\hat{\lambda}^{-N} = \hat{\lambda}^N = \varnothing = \check{\lambda}^{-N} = \check{\lambda}^N$ and

$$\phi_{\lambda/\mu}(0,t) = \prod_{i \in I}(1 - t^{m_i(\lambda)}) \quad \text{and} \quad \psi_{\lambda/\mu}(0,t) = \prod_{j \in J}(1 - t^{m_j(\mu)}).$$

In the above formula we assume $\lambda \succ \mu$ otherwise both expressions equal 0. The sets $I, J$ are:

$$I(\lambda,\mu) = \{i \in \mathbb{N} : \lambda'_{i+1} = \mu'_{i+1} \text{ and } \lambda'_i > \mu'_i\} \text{ and } J(\lambda,\mu) = \{j \in \mathbb{N} : \lambda'_{j+1} > \mu'_{j+1} \text{ and } \lambda'_j = \mu'_j\}.$$

Set $k = x - y$ and observe that $\check{\lambda}_i = \hat{\lambda}_i = \lambda_i$ whenever $i \neq k$. By combining common factors this gives

1. $k = 0$: $\mathbb{P}^{\pi,v}_{r,t}(\hat{\pi}) \propto r\psi_{\hat{\lambda}^0/\lambda^{-1}}(0,t)\phi_{\hat{\lambda}^0/\lambda^1}(0,t)$ and $\mathbb{P}^{\pi,v}_{r,t}(\check{\pi}) \propto \psi_{\check{\lambda}^0/\lambda^{-1}}(0,t)\phi_{\check{\lambda}^0/\lambda^1}(0,t)$.
2. $k > 0$: $\mathbb{P}^{\pi,v}_{r,t}(\hat{\pi}) \propto r\phi_{\lambda^{k-1}/\hat{\lambda}^k}(0,t)\phi_{\hat{\lambda}^k/\lambda^{k+1}}(0,t)$ and $\mathbb{P}^{\pi,v}_{r,t}(\check{\pi}) \propto \phi_{\lambda^{k-1}/\check{\lambda}^k}(0,t)\phi_{\check{\lambda}^k/\lambda^{k+1}}(0,t)$.
3. $k < 0$: $\mathbb{P}^{\pi,v}_{r,t}(\hat{\pi}) \propto r\psi_{\hat{\lambda}^k/\lambda^{k-1}}(0,t)\psi_{\lambda^{k+1}/\hat{\lambda}^k}(0,t)$ and $\mathbb{P}^{\pi,v}_{r,t}(\check{\pi}) \propto \psi_{\check{\lambda}^k/\lambda^{k-1}}(0,t)\psi_{\lambda^{k+1}/\check{\lambda}^k}(0,t)$.

In the above $\check{\lambda}^k$ is obtained from $\hat{\lambda}^k$, by removing 1 box from row $\min(x, y)$. The above weights, while explicit, are difficult to calculate efficiently on a computer. Thus we will search for simpler formulas, utilizing that $\check{\lambda}^k$ is structurally similar to $\hat{\lambda}^k$.

For a partition $\lambda$ we introduce the following notation. Let $S(\lambda)$ be the multiset of positive row-lengths of $\lambda$, counted with multiplicities. One observes that if $\lambda \succ \mu$ one has $I(\lambda, \mu) = S(\lambda)\backslash S(\mu)$ and $J(\lambda, \mu) = S(\mu)\backslash S(\lambda)$ as multisets, in particular $S(\lambda)\backslash S(\mu)$ and $S(\mu)\backslash S(\lambda)$ are honest sets. Let us prove this briefly.

Since $\lambda \succ \mu$ we have $\lambda'_k = \mu'_k$ or $\mu'_k + 1$. Consequently, we have $i \in I(\lambda, \mu) \iff \lambda'_i > \mu'_i$ and $\lambda'_{i+1} = \mu'_{i+1} \iff \lambda'_i = \mu'_i + 1$ and $\lambda'_{i+1} = \mu'_{i+1} \iff \lambda'_i - \lambda'_{i+1} = \mu'_i - \mu'_{i+1} + 1$ and $\lambda'_{i+1} = \mu'_{i+1} \iff \lambda'_i - \lambda'_{i+1} = \mu'_i - \mu'_{i+1} + 1 \iff m_i(\lambda) = m_i(\mu) + 1 \iff i \in S(\lambda)/S(\mu)$ and has multiplicity 1.

Similarly, $j \in J(\lambda, \mu) \iff \lambda'_{j+1} > \mu'_{j+1}$ and $\lambda'_j = \mu'_j \iff \lambda'_{j+1} = \mu'_{j+1} + 1$ and $\lambda'_j = \mu'_j \iff \lambda'_{j+1} - \lambda'_j = \mu'_{j+1} - \mu'_j + 1$ and $\lambda'_j = \mu'_j \iff \lambda'_{j+1} - \lambda'_j = \mu'_{j+1} - \mu'_j + 1 \iff m_j(\mu) = m_j(\lambda) + 1 \iff j \in S(\mu)\backslash S(\lambda)$ and has multiplicity 1.

The above arguments show that

$$\phi_{\lambda/\mu}(0, t) = \prod_{i \in S(\lambda)\backslash S(\mu)} (1 - t^{m_i(\lambda)}) \text{ and } \psi_{\lambda/\mu}(0, t) = \prod_{i \in S(\mu)\backslash S(\lambda)} (1 - t^{m_i(\mu)}).$$

Suppose that $\lambda, \mu, \nu$ are plane partitions, such that $\lambda \succ \nu$, $\mu \succ \nu$ and $\mu$ is obtained from $\lambda$ by removing a single box from row $k$. In addtition, set $c = \mu_k$. Then we have $S(\lambda) = S(\mu) - \{c\} + \{c+1\}$ as multisets. Put $M = [S(\lambda)\backslash S(\nu)] \cap [S(\mu)\backslash S(\nu)]$ and observe that $m_i(\lambda) = m_i(\mu)$, whenever $i \in M$. Indeed, we have from our earlier work that $i \in M \iff i \in S(\lambda)\backslash S(\nu)$ and $i \in S(\mu)\backslash S(\nu) \iff m_i(\lambda) = 1 + m_i(\nu)$ and $m_i(\mu) = 1 + m_i(\nu) \implies m_i(\lambda) = m_i(\mu)$. Then we have

$$\begin{aligned} \phi_{\lambda/\nu}(0, t) &= \left(1 - \mathbf{1}_{c \in S(\lambda)\backslash S(\nu)} t^{m_c(\lambda)}\right) \left(1 - \mathbf{1}_{c+1 \in S(\lambda)\backslash S(\nu)} t^{m_{c+1}(\lambda)}\right) \prod_{i \in M} (1 - t^{m_i(\lambda)}), \\ \phi_{\mu/\nu}(0, t) &= \left(1 - \mathbf{1}_{c \in S(\mu)\backslash S(\nu)} t^{m_c(\mu)}\right) \left(1 - \mathbf{1}_{c+1 \in S(\mu)\backslash S(\nu)} t^{m_{c+1}(\mu)}\right) \prod_{i \in M} (1 - t^{m_i(\mu)}). \end{aligned} \tag{2.7.3}$$

A similar argument shows that if $L = [S(\nu)\backslash S(\lambda)] \cap [S(\nu)\backslash S(\mu)]$, then we have

$$\begin{aligned} \psi_{\lambda/\nu}(0, t) &= \left(1 - \mathbf{1}_{c \in S(\nu)\backslash S(\lambda)} t^{m_c(\nu)}\right) \left(1 - \mathbf{1}_{c+1 \in S(\nu)\backslash S(\lambda)} t^{m_{c+1}(\nu)}\right) \prod_{i \in L} (1 - t^{m_i(\nu)}), \\ \psi_{\mu/\nu}(0, t) &= \left(1 - \mathbf{1}_{c \in S(\nu)\backslash S(\mu)} t^{m_c(\nu)}\right) \left(1 - \mathbf{1}_{c+1 \in S(\nu)\backslash S(\mu)} t^{m_{c+1}(\nu)}\right) \prod_{i \in L} (1 - t^{m_i(\nu)}). \end{aligned} \tag{2.7.4}$$

Set

$$G(\lambda, \nu, c) := \begin{cases} 1 - \mathbf{1}_{\{m_c(\nu) > m_c(\lambda)\}} t^{m_c(\nu)} & \text{if } c > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.7.5}$$

Then the above work implies that when $v = (x, y, z)$ and $k = x - y$ we get

$$\begin{aligned}\mathbb{P}^{\pi,v}_{r,t}(\hat{\pi}) &\propto rG(\hat{\lambda}^k, \lambda^{k-1}, z-1)G(\hat{\lambda}^k, \lambda^{k-1}, z)G(\lambda^{k+1}, \hat{\lambda}^k, z-1)G(\lambda^{k+1}, \hat{\lambda}^k, z)\\ \mathbb{P}^{\pi,v}_{r,t}(\check{\pi}) &\propto G(\check{\lambda}^k, \lambda^{k-1}, z-1)G(\check{\lambda}^k, \lambda^{k-1}, z)G(\lambda^{k+1}, \check{\lambda}^k, z-1)G(\lambda^{k+1}, \check{\lambda}^k, z).\end{aligned} \tag{2.7.6}$$

In obtaining the above formulas we used (2.7.3) and (2.7.4) for the three different cases $k < 0$, $k > 0$ and $k = 0$. Some special care is needed when $k = N$ and in this case the terms in (2.7.6) involving $\lambda^{k+1}$ are replaced with 1's.

Summarizing our results, we see that the transition from $\pi$ is as follows: pick a position $v = (x, y, z)$ in the box $N \times N \times N$ uniformly at random; if the position $v$ does not correspond to an element in the sets $Add_\pi$ or $Rem_\pi$ then leave $\pi$ unchanged with probability 1; if the position $v \in Add_\pi \sqcup Rem_\pi$, then $\pi$ is goes to $\hat{\pi}$ with probability $p$ and to $\check{\pi}$ with probability $1 - p$, where

$$p := \frac{r}{r + \frac{G(\check{\lambda}^k,\lambda^{k-1},z-1)G(\check{\lambda}^k,\lambda^{k-1},z)G(\lambda^{k+1},\check{\lambda}^k,z-1)G(\lambda^{k+1},\check{\lambda}^k,z)}{G(\hat{\lambda}^k,\lambda^{k-1},z-1)G(\hat{\lambda}^k,\lambda^{k-1},z)G(\lambda^{k+1},\hat{\lambda}^k,z-1)G(\lambda^{k+1},\hat{\lambda}^k,z)}}. \tag{2.7.7}$$

As before if $k = N$ we replace the terms in the above formula involving $\lambda^{k+1}$ with 1's.

### 2.7.2 Gibbs sampler algorithm and simulations

In Section 2.7.1 we described a Gibbs sampler for the measure $\mathbb{P}_{r,t}$ and gave exact formulas for the transition probabilities in (2.7.7). Our goal now is to give an outline for an algorithm implementing the sampler and present some simulations of random plane partitions. The main difficulty in constructing Gibbs samplers for distributions involving symmetric functions is finding computationally efficient ways to calculate the transition probabilities, which we did in (2.7.7). Beyond this formula there are no particularly novel ideas in the algorithm below; however, as we could not find an adequate reference in the literature, we believe that an outline is in order. It is quite possible that different methods can be used to *exactly* sample the distribution $\mathbb{P}^{r,t}_{HL}$ or some variant of it, using ideas like those in [31], [11] or [14]. Unfortunately, we were unable to implement exact sampling algorithms efficiently, which is why we resort to the Gibbs sampler and leave the development of better samplers for future work.

One of the difficulties in making simulations is that the number of iterations necessary to obtain convergence is very large. In the cases described below we will need about $2 \times 10^{15}$ iterations to see a limit shape emerge. Part of the reason for needing so many iterations is that most of the time the uniformly sampled position $v$ in the $N \times N \times N$ box will not belong to the sets $Add_\pi$ and $Rem_\pi$ and thus the chain will stay in one place for extended periods of time. Let us call steps of the chain, where $v$ was not chosen inside $Add_\pi$ or $Rem_\pi$ *empty*; if $v \in Add_\pi \cup Rem_\pi$ we call the step *successful.* Empty steps, although individually computationally cheap, add up and significantly increase the runtime of a simulation. It is thus very important to come up with ways to circumvent spending so much time in empty steps.

We will now describe a neat idea that allows us to group together empty steps and thus greatly reduce the runtime of simulations. Let $add_\pi = |Add_\pi|$ and $rem_\pi = |Rem_\pi|$ and observe that the probability of making an empty step, starting from the plane partition $\pi$, is

$$\mathbb{P}_\pi(v \notin Add_\pi \cup Rem_\pi) = 1 - \frac{add_\pi + rem_\pi}{N^3} =: x_\pi.$$

Consequently, the number of empty steps $E_\pi$, before a successful one, is distributed according to the geometric distribution

$$\mathbb{P}_\pi(E_\pi = k) = x_\pi^k(1 - x_\pi) \text{ for } k \geq 0. \tag{2.7.8}$$

Using the latter observation, instead of sampling $v$ uniformly from the $N \times N \times N$ box, updating our chain and increasing the number of iterations by 1, we may sample a geometric random variable $X$ with the above distribution, sample $v$ uniformly from $Add_\pi \cup Rem_\pi$ update our chain and increase the number of iterations by $1 + X$. What we have done is calculate beforehand how many empty moves we need to make before we make a successful one and then do all of them together, which by definition means to just do the successful move.

Typically, the cost of drawing an integer-valued random variable $K$ according to some prescribed distribution is of the order of the value $k$ that is finally assigned to $K$ (see the discussion at the end of Section 3 in [14]). An exception is the geometric law, which is simpler. Indeed, to draw $X$ according to (2.7.8) it is enough to set $X = \lfloor \log U / \log(x_\pi) \rfloor$, where $U$ is uniform $(0, 1)$. Hence, the cost of drawing a geometric law is $O(1)$.

If $N$ is very large, one observes that $x_\pi$ is very close to 1. Indeed, $add_\pi$ and $rem_\pi$ are both bounded from above by $N^2$, since there can be at most one addable and removable cube in every column $(x, y, \cdot)$. Consequently, one expects to make on average at most 1 successful step every $N$ steps of the iteration. The upshot of our idea now is that we have replaced sampling a large number of uniform random variables, with sampling a single geometric random variable at cost $O(1)$. Moreover, we have reduced the number of jump commands in our loop, improving runtime further.

With the above discussion we are now prepared to describe our algorithm for the Gibbs sampler. We begin with a brief description of random number generators. `Bernoulli`$(p)$ samples a Bernoulli random variable $X$ with parameter $p$, i.e. $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$. `Geom`$(p)$ samples a geometric random variable $X$ with parameter $p$, i.e. $\mathbb{P}(X = k) = p^k(1 - p)$ for $k \geq 0$. `Uniform`$(n)$ samples a uniform random variable $X$ on $\{1, ..., n\}$, i.e. $\mathbb{P}(X = k) = 1/n$ for $k = 1, ..., n$. The random number generator algorithms are described below.

| `Bernoulli`$(p)$ |
|---|
| $U$ := uniform(0,1); |
| **if** $U$ <p **return** 1; |
| **else return** 0; |

| `Geom`$(p)$ |
|---|
| $U$ := uniform(0,1); |
| **return** $\lfloor \log(U)/\log p \rfloor$; |

| `Uniform`$(n)$ |
|---|
| $U$ := uniform(0,1); |
| **return** $1 + \lfloor nU \rfloor$; |

Next we consider the following functions, which perform the basic operations on plane partitions necessary for running the Glauber dynamics. In the functions below we recall that for a plane partition $\pi$, $add_\pi$ and $rem_\pi$ are the number of cubes that can be added to and removed from $\pi$ respectively, so that the result is a plane partition contained in $N \times N \times N$.

```
AddCube(π, k)
```
**Input:** $\pi$; index $k \in \{1, ..., add_\pi\}$.
Add the $k$-th addable cube to $\pi$.

```
RemCube(π, k)
```
**Input:** $\pi$; index $k \in \{1, ..., rem_\pi\}$.
Remove the $k$-th removable cube from $\pi$.

```
GetAdd(π, k)
```
**Input:** $\pi$; index $k \in \{1, ..., add_\pi\}$.
**Output:** The position $(x, y, z)$ of the $k$-th addable cube.

```
GetRem(π, k)
```
**Input:** $\pi$; index $k \in \{1, ..., rem_\pi\}$.
**Output:** The position $(x, y, z)$ of the $k$-th removable cube.

```
GetMult(π, k, c)
```
**Input:** $\pi$, $k$ - slice index, $c \geq 0$.
**Output:** $m_c(\lambda^k)$ - multiplicity of $c$ in the $k$-th slice of $\pi$. If $c = 0$ the output is $-2$.

```
WeightG(m, n, t)
```
**if** $((n < 0)$ **or** $(m < 0))$ **return** $1$;
**if** $m > n$ **return** $(1 - t^m)$;
**return** $1$;

With the above functions we now write an algorithm, which runs the Glauber dynamics for some predescribed number of iterations.

```
Algorithm GibbsSampler(π, N, T, r, t)
Input: π - initial plane partition, N - size of box, T - total number of iterations,
       r ∈ (0,1) and t ∈ [-1,1] - parameters of the distribution.
iter := 0;
while (iter < T) do
  X := Geom(1 - (add_π + rem_π)/N^3);
  iter = iter + X;
  if (iter ≥ T) break;
  u := Uniform(add_π + rem_π);
  if (u < add_π)
    (x, y, z) := GetAdd(π, u);
    k := x - y;
    w_1 := r* WeightG(GetMult(π, k - 1, z), GetMult(π, k, z) + 1, t);
    w_1 = w_1* WeightG(GetMult(π, k - 1, z - 1), GetMult(π, k, z - 1) - 1, t);
    w_2 := WeightG(GetMult(π, k - 1, z), GetMult(π, k, z), t);
    w_2 = w_2* WeightG(GetMult(π, k - 1, z - 1), GetMult(π, k, z - 1), t);
    if (k < N)
      w_1 = w_1* WeightG(GetMult(π, k, z) + 1, GetMult(π, k + 1, z), t);
      w_1 = w_1* WeightG(GetMult(π, k, z - 1)-1, GetMult(π, k + 1, z - 1), t);
      w_2 = w_2* WeightG(GetMult(π, k, z), GetMult(π, k + 1, z), t);
      w_2 = w_2* WeightG(GetMult(π, k, z - 1), GetMult(π, k + 1, z - 1), t);
    end
    p := w_1/(w_1 + w_2);
    B := Bernoulli(p);
    if (B == 1) AddCube(π, u);
  else
    (x, y, z) := GetRem(π, u - add_π);
    k := x - y;
    w_1 := WeightG(GetMult(π, k - 1, z), GetMult(π, k, z) - 1, t);
    w_1 = w_1* WeightG(GetMult(π, k - 1, z - 1), GetMult(π, k, z - 1) + 1, t);
    w_2 := r* WeightG(GetMult(π, k - 1, z), GetMult(π, k, z), t);
    w_2 = w_2* WeightG(GetMult(π, k - 1, z - 1), GetMult(π, k, z - 1), t);
    if (k < N)
      w_1 = w_1* WeightG(GetMult(π, k, z) - 1, GetMult(π, k + 1, z), t);
      w_1 = w_1* WeightG(GetMult(π, k, z - 1) + 1, GetMult(π, k + 1, z - 1), t);
      w_2 = w_2* WeightG(GetMult(π, k, z), GetMult(π, k + 1, z), t);
      w_2 = w_2* WeightG(GetMult(π, k, z - 1), GetMult(π, k + 1, z - 1), t)
    end
    p := w_1/(w_1 + w_2);
    B := Bernoulli(p);
    if (B == 1) RemCube(π, u - add_π);
  end
  iter = iter + 1;
end
Output: π.
```

*Remark* 2.7.1. In the above algorithm, an expression of the form `WeightG(GetMult(`$\pi,\cdot,\cdot$`), GetMult(`$\pi,\cdot,\cdot$`),` $t$`)` simulates the function $G$, given in (2.7.5). The case $z = 1$ is special, since $G$ is defined differently depending on $c > 0$ and $c = 0$. In order to make the algorithm more concise, and exclude additional checks of whether $z = 1$, we have rigged the functions `GetMult` and `WeightG` so that the end results agree with (2.7.6).

### 2.7.3 Discussion and extensions

In this section we discuss some of the implications of the results of the paper and some of their possible extensions.

We start by considering possible limit shape phenomena. In [36] it was shown that if each dimension of a plane partition $\pi$, distributed according to $\mathbb{P}^{r,t}_{HL}$ with $t = 0$, is scaled by $1 - r$ then as $r \to 1^-$ the distribution concentrates on a limit shape with probability 1. We expect that a similar phenomenon occurs for any value $t \in (0, 1)$. The limit shape, if it exists, should depend on $t$, which one observes by considering the volume of the plane partition. Specifically, we have that

$$\mathbb{E}\left[|\pi|\right] = \frac{r\frac{d}{dr}Z(r,t)}{Z} \text{ and } Var(|\pi|) = \mathbb{E}\left[|\pi|^2\right] - \mathbb{E}\left[|\pi|\right]^2 = r\frac{d}{dr}\mathbb{E}\left[|\pi|\right].$$

Using that $Z(r,t) = \prod_{n=1}^{\infty}\left(\frac{1-tr^n}{1-r^n}\right)^n$ one readily verifies that

$$\mathbb{E}\left[|\pi|\right] = \sum_{k=1}^{\infty}\frac{r^k(1+r^k)}{(1-r^k)^3} - \sum_{k=1}^{\infty}t^k\frac{r^k(1+r^k)}{(1-r^k)^3}.$$

The latter implies that $\lim_{r\to 1^-}\mathbb{E}\left[(1-r)^3|\pi|\right] = 2\zeta(3) - 2Li_3(t)$, where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ is the Riemann zeta function and $Li_3(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^3}$ is the polylogarithm of order 3. In addition, one verifies that $\lim_{r\to 1^-} Var\left((1-r)^3|\pi|\right) = 0$ and so the rescaled volume $(1-r)^3|\pi|$ converges in probability to $2\zeta(3) - 2Li_3(t)$. In particular, the volume decreases from $2\zeta(3)$ to 0 as $t$ varies from 0 to 1. When $t = 1$ the measure $\mathbb{P}^{r,t}_{HL}$ is concentrated on the empty plane partition for any value of $r$ and so convergence of the volume to 0 is expected.

In sharp contrast, the result of Theorem 2.1.2 suggests that while the volume of the plane partition decreases in $t$ the bottom slice asymptotically looks the same. Using `GibbsSampler` we can run different simulations, to verify this type of behavior. At this time we remark that we have not done any analysis to estimate the mixing time of the chain we have constructed, hence our choice of number of iterations below will be somewhat arbitrary. The major point to be made here is that we are only interested in qualitative information about the distribution, such as a limit-shape phenomenon, and the purpose of the iterations is to pictorially support statements for which we have analytic proofs.

In the simulations below, the sampler is started from $\pi = \varnothing$, the size of the box $N = 2000$, the number of iterations is $T = 2 \times 10^{15}$ and $r = 0.99$. The only parameter we will vary is $t$. Results are summarized in Figures 2-9 - 2-12, where the red curve indicates the limit shape $2\log(1 + e^{-|\tau|/2})$ in Theorem 2.1.2.

What happens as $t$ increases to 1 is that the mass from the top part of the plane partition $\pi$ decreases (so $\pi_{i,j}$ decrease), but the base (given by the non-zero $\pi_{i,j}$) remains asymptotically

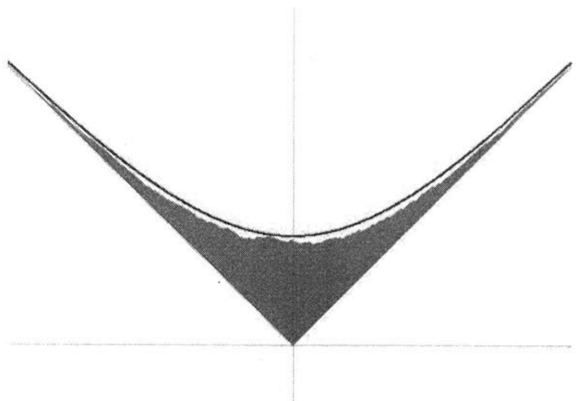

Figure 2-9: $t = 0$.

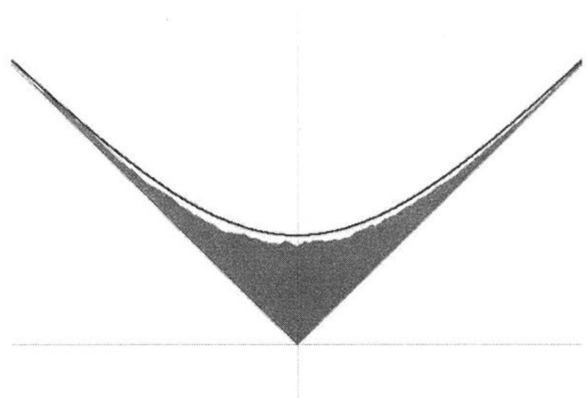

Figure 2-10: $t = 0.2$.

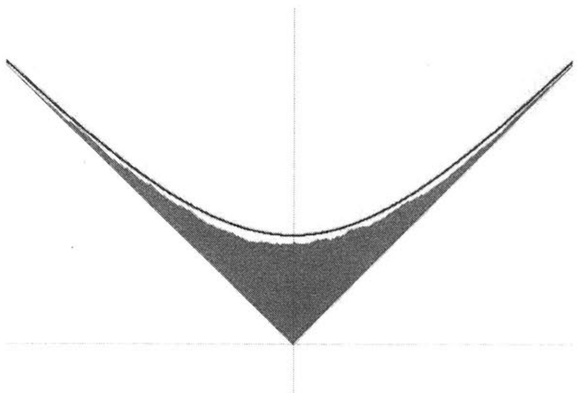

Figure 2-11: $t = 0.4$.

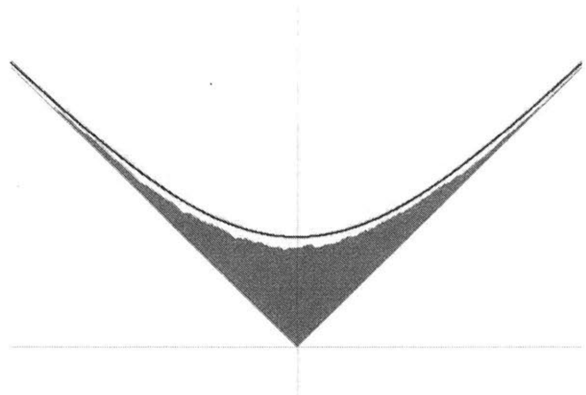

Figure 2-12: $t = 0.6$.

the same. The latter can be observed in the left parts of Figures 2-13 and 2-14 (we will get to the right parts shortly).

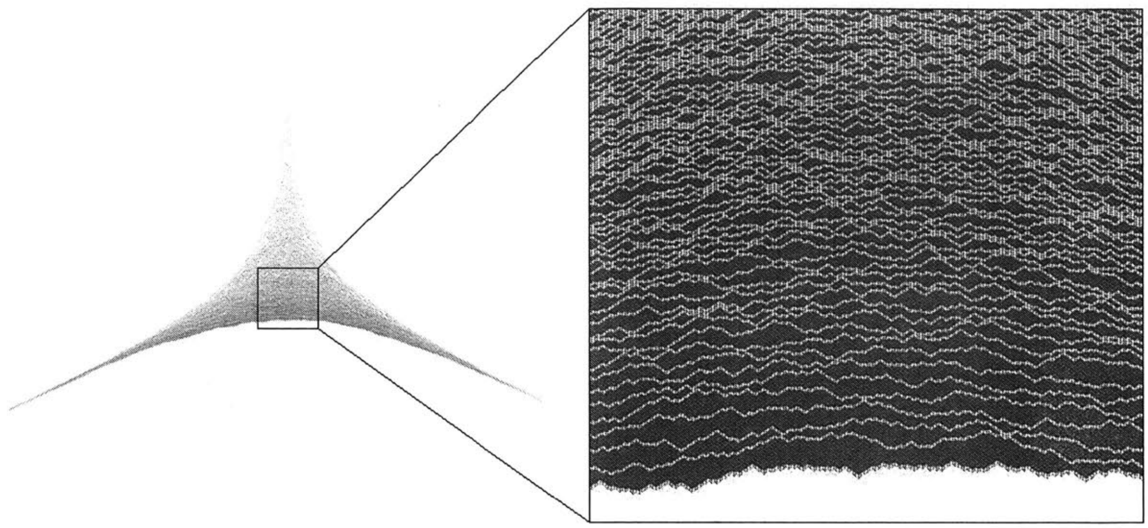

Figure 2-13: Simulation with $t = 0.4$.

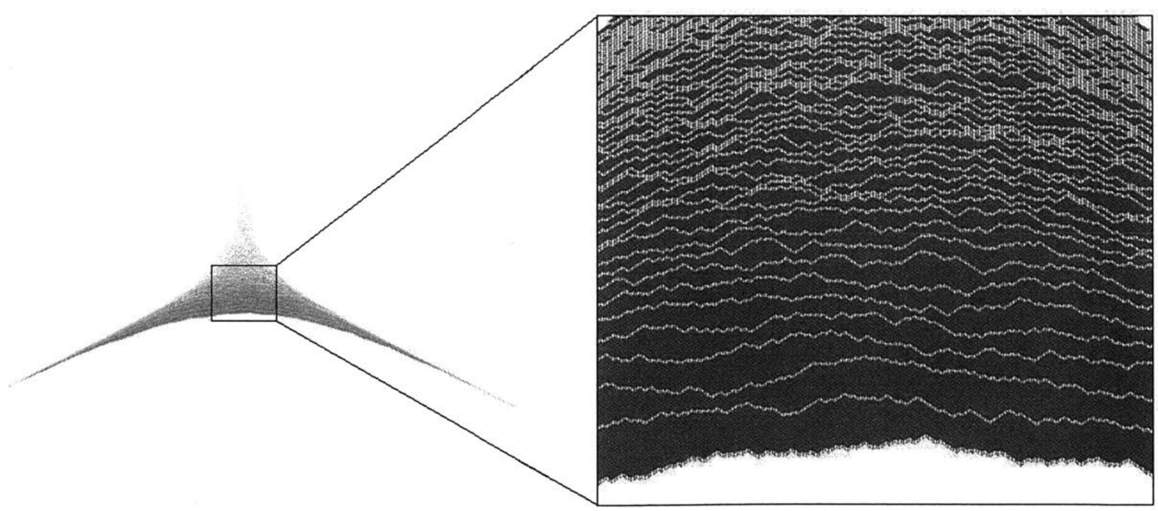

Figure 2-14: Simulation with $t = 0.8$.

We next turn to possible extensions of Theorems 2.1.2 and 2.1.3 and state a couple of conjectures about the convergence of $\mathbb{P}^{r,t}_{HL}$ that go beyond the results of this paper. At this time we do not have any clear strategy on how they can be proved, however, we will provide some evidence for their validity. We start by rather informally recalling the definitions and properties of the Airy and KPZ line ensembles. For more details about these objects the reader is encouraged to look at [42] and [43], where they were introduced and analyzed.

Let $B_1^N, ..., B_N^N$ be $N$ independent standard Brownian bridges on $[-N, N]$, $B_i^N(-N) = B_i^N(N) = 0$, conditioned on not intersecting in $(-N, N)$ and set $\Sigma_N = \{1, ..., N\}$. The latter object can be viewed as a *line ensemble*, i.e. a random variable with values in the space $X$ of continuous functions $f : \Sigma_N \times [-N, N] \to \mathbb{R}$ endowed with the topology of uniform convergence on compact subsets of $\Sigma_N \times [-N, N]$. In [42] these line ensembles are called *Dyson line ensembles* and it is shown that under suitable shifts and scaling they converge (in the sense of line ensembles - see the discussion at the beginning of Section 2.1 in [42]) to a continuous non-intersecting $\mathbb{N} \times \mathbb{R}$-indexed line ensemble. The limit is called the *Airy line ensemble* and is denoted by$\mathcal{A} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. The two properties of $\mathcal{A}$ that we will focus on are that $\mathcal{A}_1(t)$ is distributed according to the Airy process and that the $\mathbb{N}$-indexed line ensemble $\mathcal{L} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, given by $\mathcal{L}_i(x) := 2^{-1/2}(\mathcal{A}_i(x) - x^2)$ for each $i \in \mathbb{N}$ satisfies a certain *Brownian Gibbs property* that we describe below.

The Airy process first appeared in the paper of Prähofer and Spohn [73], as the scaling limit of the fluctuations of the PNG droplet and it is believed to be the universal scaling limit of a large class of stochastic growth models. It's single time distribution is given by the GUE Tracy-Widom distribution.

We now describe an instance of the Brownian Gibbs property, satisfied by $\mathcal{L}_i$. Let $k \geq 2$, and consider the curves $\mathcal{L}_{k-1}, \mathcal{L}_k$ and $\mathcal{L}_{k+1}$. Let $a, b \in \mathbb{R}$ and $a < b$ be given and put $x = \mathcal{L}_k(a)$, $y = \mathcal{L}_k(b)$. Then if we erase $\mathcal{L}_k([a, b])$ and sample an independent Brownian bridge on $[a, b]$ between the points $x$ and $y$, conditional on not intersecting $\mathcal{L}_{k-1}$ and $\mathcal{L}_{k+1}$, then the new line ensemble has the same distribution as the old one.

We shift our attention to the KPZ line ensemble. Let $N \in \mathbb{N}$ and $s > 0$ be given. For each sequence $0 < s_1 < \cdots < s_{N-1} < s$ we can associate an up/right path $\phi$ in $[0, s] \times \{1, ..., N\}$ that is the range of the unique non-decreasing surjective map $[0, s] \to \{1, ..., N\}$ whose set of jump times is $\{s_i\}_{i=1}^{N-1}$. Let $B_1, ..., B_N$ be independent standard Brownian motions and define

$$E(\phi) = B_1(s_1) + (B_2(s_2) - B_2(s_1)) + \cdots + (B_N(s) - B_N(s_{N-1})).$$

The *O'Connel-Yor polymer partition function line ensemble* is a $\{1, ..., N\} \times \mathbb{R}_+$-indexed line ensemble $\{Z_n^N(s) : n \in \{1, ..., N\}, s > 0\}$, defined by

$$Z_n^N(s) := \int_{D_n(s)} \exp\left(\sum_{i=1}^{n} E(\phi_i)\right) d\phi_1 \cdots d\phi_n,$$

where the integral is with respect to Lebesgue measure on the Euclidean set $D_n(s)$ of all $n$-tuples of non-intersecting (disjoint) up/right paths $\phi_1, ..., \phi_n$ with initial points $(0, 1), ..., (0, n)$ and endpoints $(s, N - n + 1), ..., (s, N)$. Setting $Z_0^N(s) \equiv 1$ we define the *O'Connel-Yor polymer free energy line ensemble* as the $\{1, ..., N\} \times \mathbb{R}_+$-indexed line ensemble $\{X_n^N(s) : n \in$

$\{1, ..., N\}, s > 0\}$ defined by

$$X_n^N(s) = \log\left(\frac{Z_n^N(s)}{Z_{n-1}^N(s)}\right).$$

In [43] it was shown that under suitable shifts and scaling the line ensembles $X_n^N(\sqrt{t}N + \cdot)$ are sequentially compact and hence have at least one weak limit, called the *$KPZ_t$ line ensemble* and denoted by $\mathcal{H}^t : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. The uniqueness of this limit is an open problem, however any weak limit has to satisfy the following two properties. The lowest index curve $\mathcal{H}_1^t : \mathbb{R} \to \mathbb{R}$ is equal in distribution to $\mathcal{F}(t, \cdot)$ - the time $t$ Hopf-Cole solution to the narrow wedge initial data KPZ equation (see Definition 2.1.6). In addition, the ensemble $\mathcal{H}^t$ satisfies a certain $\mathbf{H}_1$- *Brownian Gibbs property*, an instance of which we now describe.

Let $k \geq 2$, and consider the curves $\mathcal{H}_{k-1}^t, \mathcal{H}_k^t$ and $\mathcal{H}_{k+1}^t$. Let $a, b \in \mathbb{R}$ and $a < b$ be given and put $x = \mathcal{H}_k^t(a)$, $y = \mathcal{H}_k^t(b)$. We erase $\mathcal{H}_k^t([a,b])$ and sample an independent Brownian bridge on $[a, b]$ between the points $x$ and $y$. The new path is accepted with probability

$$\exp\left[-\int_a^b \mathbf{H}_1\left(\mathcal{H}_{k+1}^t(u) - \mathcal{H}_k^t(u)\right) du - \int_a^b \mathbf{H}_1\left(\mathcal{H}_k^t(u) - \mathcal{H}_{k-1}^t(u)\right) du\right], \quad \mathbf{H}_t(x) = e^{t^{1/3}x},$$

and if the path is not accepted we sample a new Brownian bridge and repeat. This procedure yields a new line ensemble and it has the same distribution as the old one.

The *Hamiltonian* $\mathbf{H}_t$ acts as a potential in which the Brownian paths evolve, assigning more weight to certain path configurations. Formally, setting $t = \infty$ we have $H_{+\infty}(x) = \infty$ if $x > 0$ and $0$ if $x < 0$. This Hamiltonian corresponds to conditioning consecutively labeled curves to not touch and hence reduces the $\mathbf{H}$- Brownian Gibbs property to the Brownian Gibbs property we had earlier.

For $\tau > 0$ let $f(\tau) = 2\log(1 + e^{-\tau/2})$, $f'(\tau) = -\frac{e^{-\tau/2}}{1+e^{-\tau/2}}$ and $f''(\tau) = \frac{1}{2}\frac{e^{-\tau/2}}{(1+e^{-\tau/2})^2}$. Also set $N(r) = \frac{1}{1-r}$. With this notation we have the following conjectures.

**Conjecture 2.7.2.** Consider the measure $\mathbb{P}_{HL}^{r,t}$ on plane partitions, given in (2.1.3), with $t \in (0, 1)$ fixed. For $\tau \in \mathbb{R}$ define the random $\mathbb{N} \times \mathbb{R}$-indexed line ensemble $\Lambda^\tau$ as

$$\Lambda_k^\tau(s) = \frac{\lambda_k'(\lfloor \tau N + sN^{2/3}\rfloor) - Nf(|\tau|) - sN^{2/3}f'(|\tau|) - (1/2)s^2N^{1/3}f''(|\tau|)}{\sqrt[3]{2f''(|\tau|)N}}. \tag{2.7.9}$$

Then as $r \to 1^-$ we have $\Lambda^\tau \implies \mathcal{A}^\tau$ (weak convergence in the sense of line ensembles), where $\mathcal{A}^\tau$ is defined as $\mathcal{A}_k^\tau(s) = \mathcal{A}_k(s\sqrt[3]{2f''(|\tau|)}/2)$ and $(\mathcal{A}_k)_{k\in\mathbb{N}}$ is the Airy line ensemble.

**Conjecture 2.7.3.** Consider the measure $\mathbb{P}_{HL}^{r,t}$ on plane partitions, given in (2.1.3). Suppose $T > 0$ is fixed and $\frac{-\log t}{(1-r)^{1/3}} = \frac{(T/2)^{1/3}}{\sqrt[3]{2f''(|\tau|)}}$. For $\tau \in \mathbb{R}$ define the random $\mathbb{N} \times \mathbb{R}$-indexed line ensemble $\Xi^\tau$ as

$$\Xi_k^\tau(s) = -T/24 + \frac{\lambda_k'(\lfloor \tau N + sN^{2/3} \rfloor) - Nf(|\tau|) - sN^{2/3}f'(|\tau|) - (1/2)s^2N^{1/3}f''(|\tau|)}{(T/2)^{-1/3}\sqrt[3]{2f''(|\tau|)N}} +$$
$$+ \log((T/2)^{-1/3}\sqrt[3]{2f''(|\tau|)N}) + (k-1)\log\left(\frac{NT^{-1}(2f''(|\tau|))^{-3/2}}{2\sqrt{2}}\right) - \frac{s^2T^{1/3}(2f''(|\tau|))^{2/3}}{8}. \tag{2.7.10}$$

Then as $r \to 1^-$ we have $\Xi^\tau \implies \mathcal{H}^{\tau,T}$ (weak convergence in the sense of line ensembles), where $\mathcal{H}^{\tau,T}$ is defined as $\mathcal{H}_k^{\tau,T}(s) = \mathcal{H}_k^T(sT^{2/3}\sqrt[3]{2f''(|\tau|)}/2)$ and $(\mathcal{H}_k^T)_{k\in\mathbb{N}}$ is the KPZ line ensemble.

*Remark* 2.7.4. We provide some motivation behind our choice of scaling in Conjecture 2.7.2. Since the lines in the Airy line ensemble a.s. do not intersect as do the lines $\lambda_k'(\lfloor \tau N + sN^{2/3} \rfloor)$ we expect that all lines undergo the same scaling and translation. This allows us to only concern ourselves with $\lambda_1'(\lfloor \tau N + sN^{2/3} \rfloor)$, whose limit should be some rescaled version of the Airy process (the distribution of $\mathcal{A}_1$). Arguments in the proof of Theorem 2.1.2 can be used to show that in distribution the expression on the RHS in (2.7.9) converges to the GUE Tracy-Widom distribution for each $s$. The latter still leaves the question of possible argument scaling since $\mathcal{A}_1(\kappa s)$ has the same one-point marginal distribution for all values of $\kappa$. In [51] an expression similar to $\Lambda_1^\tau(s)$ (related to setting $t = 0$ in $\mathbb{P}_{HL}^{r,t}$), was shown to converge to the Airy process, with a rescaled argument. Consequently, we have chosen to rescale the argument so that it matches this result.

*Remark* 2.7.5. The choice for scaling in Conjecture 2.7.3 is somewhat more involved. When $k = 1$ in equation (2.7.10) we run into the same argument scaling issue as in Conjecture 2.7.2; however, we no longer have results in the literature that we can use as a guide. Nevertheless, in [6] it was conjectured that $(T/2)^{-1/3}\left(\mathcal{F}(T, T^{2/3}X) + \frac{T^{4/3}X^2}{2T} + \frac{T}{24}\right)$ converges to the Airy process as $T \to \infty$. Consequently, we have picked a scaling of the argument in Conjecture 2.7.3 in such a way that under the scaling by $(T/2)^{-1/3}$ we would obtain the (argument rescaled) Airy process in Conjecture 2.7.2. Since the lines in the KPZ line ensemble are allowed to cross, we no longer expect that all lines $\lambda_k'(\lfloor \tau N + sN^{2/3} \rfloor)$ undergo the same translation and scaling and in equation (2.7.10) we see that each line is deterministically shifted by a $N^{1/3}\log(N)$ factor compared to the previously indexed line. The precise choice of this shift is explained below and it is related to the $\mathbf{H}_1$- Brownian Gibbs property, enjoyed by the KPZ line ensemble.

We will now present some evidence that supports the validity of the above conjectures, starting from the results of this paper. Theorems 2.1.2 and 2.1.3 only deal with $\lambda_1'$ and can be understood as one-point convergence results about the bottom slice of the partition $\pi$ as follows. The proof of Theorem 2.1.2 shows that

$$\lim_{r\to 1^-} \mathbb{P}_{HL}^{r,t}\left(\frac{\lambda_1'(\lfloor \tau N + sN^{2/3} \rfloor) - M(r)}{\sqrt[3]{2f''(|\tau|)N}} \le x\right) = F_{GUE}(x) = \mathbb{P}(\mathcal{A}_1^\tau(s) \le x).$$

In the last equality we used that the one-point distribution of the Airy process is given by the

Tracy-Widom GUE distribution [73]. In the above formula we have $M(r) = 2\sum_{k=1}^{\infty} a(r)^k \frac{(-1)^{k+1}}{1-r^k}$, where $a(r) = r^{\lfloor N(r)\tau + sN(r)^{2/3}\rfloor}$. Using ideas that are similar to those in Lemma 2.6.4 one obtains $M(r) = Nf(|\tau|) + sN^{2/3}f'(|\tau|) + (1/2)s^2N^{1/3}f''(|\tau|) + O(1)$. Consequently, Theorem 2.1.2 implies that the one-point distribution of $\Lambda_1^\tau$ converges to that of $\mathcal{A}_1^\tau$.

Similarly, the proof of Theorem 2.1.3 shows that

$$\lim_{r\to 1^-} \mathbb{P}_{HL}^{r,t}\left(\frac{\lambda_1'(\lfloor \tau N + sN^{2/3}\rfloor) - M(r)}{(T/2)^{-1/3}\sqrt[3]{2f''(|\tau|)N}} + \log((T/2)^{-1/3}\sqrt[3]{2f''(|\tau|)N}) \leq x\right) = F_{CDRP}(x) =$$

$$= \mathbb{P}\left(\mathcal{H}_1^T(sT^{2/3}\sqrt[3]{2f''(|\tau|)}/2) + \frac{s^2T^{1/3}(2f''(|\tau|))^{2/3}}{8} + T/24 \leq x\right).$$

In the last equality we used that $\mathcal{F}(T,X) + \frac{X^2}{2T}$ is a stationary process in $X$ and hence $\mathcal{F}(T,0) + T/24$ has the same distribution as $\mathcal{H}_1^T(sT^{2/3}\sqrt[3]{2f''(|\tau|)}/2) + \frac{s^2T^{1/3}(2f''(|\tau|))^{2/3}}{8} + T/24$. In the above formula we have $M(r) = 2\sum_{k=1}^{\infty} a(r)^k \frac{(-1)^{k+1}}{1-r^k}$, where $a(r) = r^{\lfloor N(r)\tau + sN(r)^{2/3}\rfloor}$. Using $M(r) = Nf(|\tau|) + sN^{2/3}f'(|\tau|) + (1/2)s^2N^{1/3}f''(|\tau|) + O(1)$ we see that Theorem 2.1.3 implies that the one-point distribution of $\Xi_1^\tau$ converges to that of $\mathcal{H}_1^{\tau,T}$.

The next observation that we make is that in the statement of Conjecture 2.7.2, the separation between consecutive horizontal slices of $\pi$, distributed according to $\mathbb{P}_{HL}^{r,t}$ is suggested to be of order $N^{1/3}$, which is the order of the fluctuations. On the other hand, in Conjecture 2.7.3 there is a deterministic shift of order $N^{1/3}\log N$, while fluctuations remain of order $N^{1/3}$. The latter phenomenon can be observed in simulations, as is shown in Figures 2-13 and 2-14. Namely, the conjectures suggest that as $t$ goes to 1, one should observe a larger spacing between the bottom slices of $\pi$, which is clearly visible.

Finally, we match the Brownian Gibbs and $\mathbf{H}_1$-Brownian Gibbs properties. Suppose that we fix the slices $\lambda_{k-1}'(m)$ and $\lambda_{k+1}'(m)$, $m \in \mathbb{Z}$ and consider the conditional distribution of $\lambda_k'([A,B])$. The weight $w(\lambda_k'([A,B])$ that each path obtains consists of two terms: an *entropy term*, which comes from the $r^{|\pi|}$ dependence of $\mathbb{P}_{HL}^{r,t}$, and a *potential term*, which comes from the dependence on $A_\pi(t)$. Specifically, if the number of cells between $\lambda_k'([A,B])$ and $\lambda_{k+1}'([A,B])$ is $P$ then the entropy term is given by $r^P$. The potential term is a bit more involved but depends only on the local structure of the paths. It is constructed as follows: start from $A$ and move to the right towards $B$, every time the distance between $\lambda_k'(m)$ and $\lambda_{k\pm1}'(m)$ decreases by 1 when we increase $m$ by 1 we obtain a factor of $(1 - t^{|\lambda_k'(m) - \lambda_{k\pm1}'(m)|})$; the potential term is now the product of these factors. The weight $w(\lambda_k'([A,B])$ is given by the product of the entropy and potential terms and the conditional probability is the ratio of the weight and the sum of all path weights. See Figure 2-15 for a pictorial depiction of the latter construction.

In the limit as $r \to 1^-$, the entropy term goes to 1 and if we ignore the potential, we see that the measure converges to the uniform measure on all paths from $A$ to $B$, which do not intersect the lines $\lambda_{k-1}'$ and $\lambda_{k+1}'$. This motivates the Brownian limit of the paths. When $t \in (0,1)$ is fixed, we have the conjectural separation of consecutive lines in $\Lambda^\tau$ being of order $N^{1/3}$. This implies that if $B - A$ is of order $N^{2/3}$, which is the conjectural scaling we have suggested, then the potential term is bounded from below by an expression of the

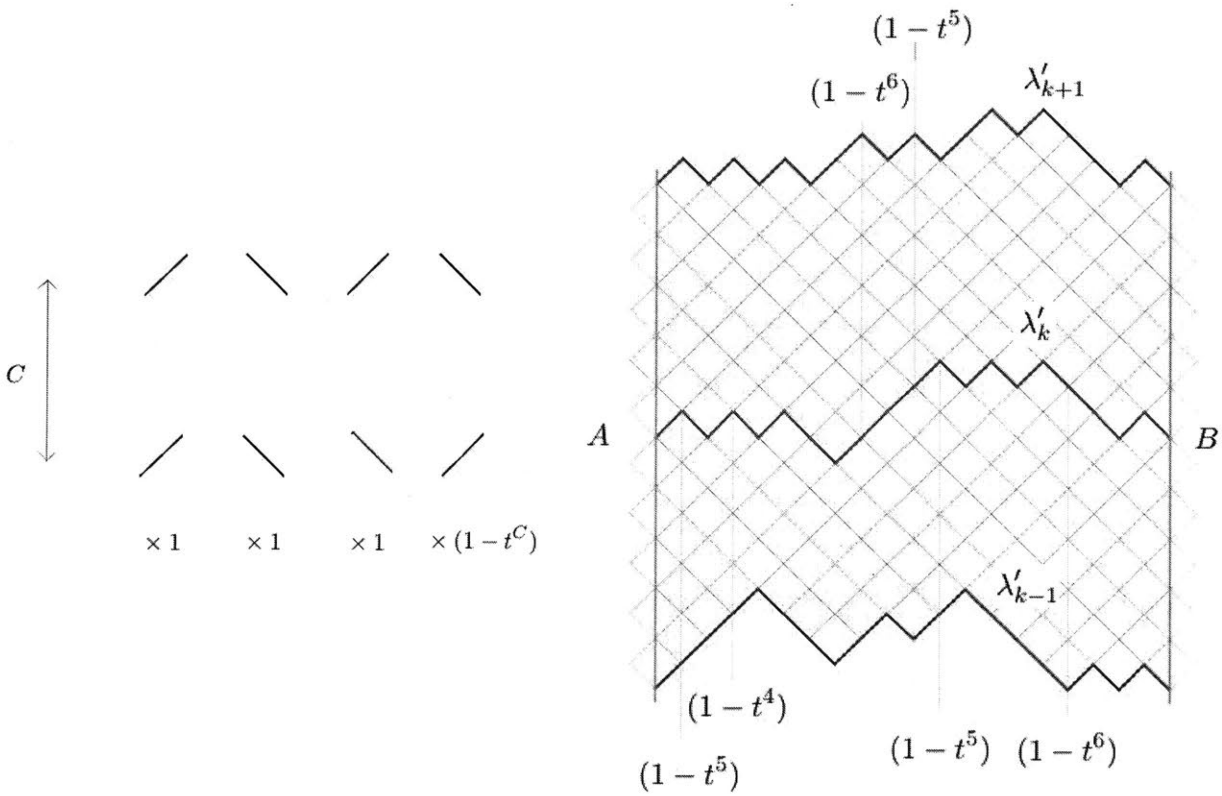


Figure 2-15: The left part of the figure shows that we get a non-trivial factor only when the distance between two slices decreases. For the path on the right we have $P = 16 \times 5 + 2 \times 6 + 1 \times 4 = 96$, hence the entropy term is $r^{96}$. The potential term is given by $(1-t^4) \times (1-t^5)^3 \times (1-t^6)^2$. The weight is the product of the entropy and potential terms and equals $w(\lambda'_k([A,B]) = r^{96}(1-t^4)(1-t^5)^3(1-t^6)^2$.

form $(1 - t^{cN^{1/3}})^{CN^{2/3}}$. The latter converges to 1 exponentially fast, and so we see that the contribution of the potential disappears in the limit. Consequently, the limit distribution of $\mathcal{A}_k^\tau$, at least heuristically, converges to a Brownian path, which is conditioned on not intersecting $\mathcal{A}_{k\pm1}^\tau$. This is precisely the Brownian Gibbs property.

When both $r$ and $t$ converge to $1^-$ as in Conjecture 2.7.3, the potential term can no longer be ignored. One can understand the contribution of the potential term as an acceptance probability similarly to the KPZ line ensemble. Specifically, suppose we fix the slices $\lambda'_{k-1}(m)$ and $\lambda'_{k+1}(m)$, $m \in \mathbb{Z}$ and consider the conditional distribution of $\lambda'_k([A,B])$. One way to obtain it is to draw a random path between the points $A$ and $B$ that does not intersect the slices $\lambda'_{k-1}(m)$ and $\lambda'_{k+1}(m)$ using the entropy term alone. Then with probability equal to the potential term we accept the path and otherwise we draw again and repeat. When $r$ and $t$ go to $1^-$ we have that the paths we sample converge to a uniform sampling of all paths, suggesting the Brownian nature of the limits; and what we would like to show is that the acceptance probability in the discrete case converges to the acceptance probability in the limit. Notice that the separation between slices being of order $N^{1/3}\log(N)$, while fluctuations remaining of order $N^{1/3}$ suggests that non-intersection of the lines automatically holds with large probability and hence can be ignored.

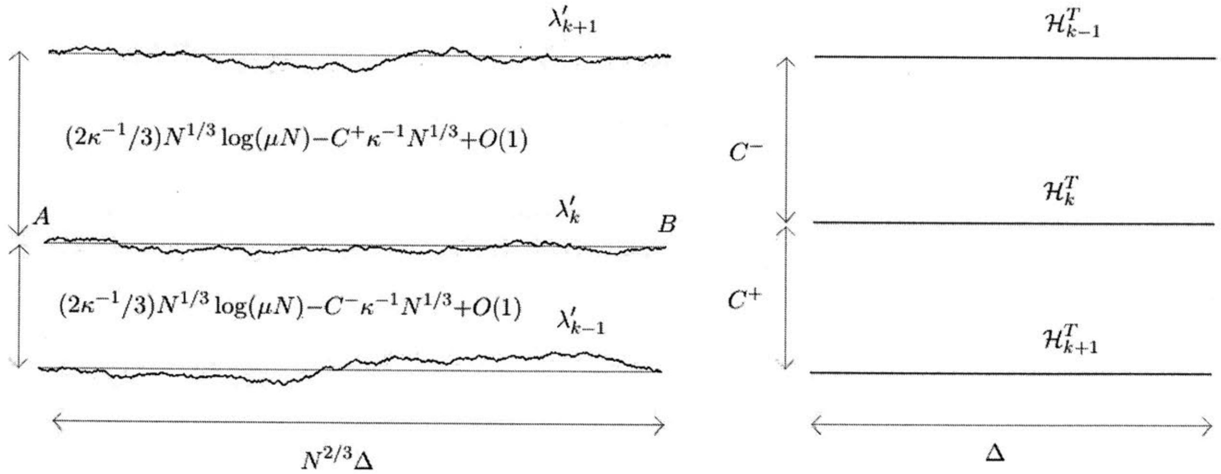


Figure 2-16: $\Xi^{\tau}_k$ and $\Xi^{\tau}_{k\pm1}$ converge to constant functions. Quantities increase downwards.

We will now proceed to match the acceptance probabilities, by considering a simple to analyze case, when the paths converge to constant lines. The situation is depicted in Figure 2-16. To simplify notation, let $\Delta = N^{-2/3}(B - A)$, $\chi^{-1} = \sqrt[3]{2f''(|\tau|)}$, $\kappa = (T/2)^{1/3}\chi^{-1} = (-\log t)N^{1/3}$ and $\mu = \frac{T^{-1}\chi^{3/2}}{2\sqrt{2}}$. Due to the Brownian nature of the limit of the paths, one expects roughly $\frac{\Delta N^{2/3}}{4}$ of the steps to lead to decreasing the distance between $\lambda'_k$ and $\lambda'_{k\pm1}$. Suppose that $|\lambda'_k(m) - \lambda'_{k\pm1}(m)| = (2\kappa^{-1}/3)N^{1/3}\log(\mu N) - C^{\pm}\kappa^{-1}N^{1/3} + O(1)$, for $m \in [A, B]$; then the acceptance probability is roughly equal to

$$p_N(t) = (1 - t^{(2\kappa^{-1}/3)N^{1/3}\log(\mu N) - C^{+}\kappa^{-1}N^{1/3}})^{\Delta N^{2/3}/4}(1 - t^{(2\kappa^{-1}/3)N^{1/3}\log(\mu N) - C^{-}\kappa^{-1}N^{1/3}})^{\Delta N^{2/3}/4}.$$

Taking logarithms we see that $\log(p_N(t)) = -\frac{\Delta N^{2/3}}{4}(e^{-(2/3)\log(\mu N)+C^{+}} + e^{-(2/3)\log(\mu N)+C^{-}}) + O(N^{-2/3})$. We thus see that $\lim_{N\to\infty}\log(p_N(t)) = -(\Delta/4)e^{-(2/3)\log(\mu)}(e^{C^+} + e^{C^-})$.

On the other hand, the acceptance probability for $\mathcal{H}^{\tau,T}$ is given by $\exp(-(\Delta T^{2/3}\chi^{-1}/2)(e^{C^+} + e^{C^-})$. Equality of the latter and $\lim_{N\to\infty} p_N(t)$ is equivalent to

$$-(\Delta/4)e^{-(2/3)\log(\mu)}(e^{C^+} + e^{C^-}) = -(\Delta T^{2/3}\chi^{-1}/2)(e^{C^+} + e^{C^-}) \iff e^{-(2/3)\log(\mu)} = 2T^{2/3}\chi^{-1}.$$

Substituting $\mu = \frac{T^{-1}\chi^{3/2}}{2\sqrt{2}}$ one readily verifies that the latter equality holds. This shows that the discrete acceptance probability, at least heuristically, converges to the limiting one, verifying the $\mathbf{H}_1$-Brownian Gibbs property.

# Chapter 3

# Six-vertex models and the GUE-corners process

## 3.1 Introduction

The six-vertex model is a well-known exactly solvable lattice model of equilibrium statistical mechanics. The study of its properties is a rich subject, which has enjoyed many exciting developments during the last half-century (see, e.g., [9], [74], and the references therein). Fixing particular boundary conditions and weights, connects the six-vertex model to a number of combinatorial objects like alternating sign matrices and domino tilings [49]. The six-vertex model and certain higher spin generalizations of it have been linked to a large class of integrable probabilistic models that belong to the KPZ universality class in $1 + 1$ dimensions - this was first observed in [53] and studied more recently in [27, 33, 44]. These recent advances have spurred new interest in vertex models and the development of tools to analyze them.

The main subject of this chapter is the (vertically inhomogeneous) six-vertex model in a half-infinite strip. We will work with a particular weight parametrization, introduced in [20], whose origin lies in the Yang-Baxter equation, and which corresponds to the so-called *ferroelectric regime* [9]. The partition function of this model is described by a remarkable family of symmetric rational functions $\mathsf{F}_\lambda$, parametrized by non-negative signatures $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. These functions form a one-parameter generalization of the classical Hall-Littlewood polynomials [64] and enjoy many of the same structural properties [20]. In a recent paper [33], the authors derive many useful features of the functions $\mathsf{F}_\lambda$, which allow them to obtain integral representations for certain multi-point $q$-moments of the inhomogeneous higher spin six vertex model in infinite volume. Such formulas are well-known to be a fruitful source of asymptotic results and were recently utilized to study the asymptotics of various stochastic six-vertex models [1, 4, 19].

In this chapter we develop a different approach to study the vertically inhomogeneous six-vertex model , which is based on a new class of operators $D_N^k$. These operators act diagonally on the functions $\mathsf{F}_\lambda$, whenever $\lambda$ has distinct parts and can be used to derive formulas for the probability of observing certain arrow configurations in different locations of the model. These observables were very recently investigated for the six-vertex model with domain wall boundary condition (DWBC) in [38] under the name of *generalized emptiness formation probability* (GEFP). The derivation of the formulas in [38] is based on the quantum inverse scattering method, which has been successfully used to derive large classes of correlation functions [16, 38, 39]. Whether our operators are related to the quantum inverse scattering method is unclear, and at this time the two approaches appear to be orthogonal. As discussed in [38] the GEFP can be used to understand macroscopic frozen regions in the six-vertex model with DWBC and it is our hope that the operators we develop can be used to address similar questions for more general six-vertex models.

Our main goal is to use the correlation functions obtained from our operators to analyze a particular class of homogeneous six-vertex models as the system size becomes large. There are two natural ways to understand the probability distributions that we analyze. On the one hand, one can view them as stochastic six-vertex models on the half-infinite strip with a particular choice of boundary data, which is related to a special class of symmetric functions, considered in [33]. Alternatively, these probabilities distributions describe the marginal law of a discrete time Markov process on vertex models, which is started from the stochastic six-vertex model of [27], and whose dynamics is described by certain sequential update rules. For the models we consider we show that certain configurations of holes (absence of arrows or empty edges) weakly converge to the GUE-corners process as the size of the system tends to infinity. We view the latter as the main result of this chapter and the exact statement is given in Theorem 3.1.3. The proof is based on the formulas obtained from our operators as well as a classification result, which identifies the GUE-corners process as the unique probability measure that satisfies the continuous Gibbs property (see Definition 3.5.4) and has the correct marginal distribution on the right edge.

We now turn to describing our model and main results.

### 3.1.1 Problem statement and main results

For $N \in \mathbb{N}$ we let $\mathcal{P}_N$ denote the collection of $N$ up-right paths drawn in the sector $D_N := \mathbb{Z}_{\geq 0} \times \{1, \dots, N\}$ of the square lattice, with all paths starting from a left-to-right arrow entering each of the points $\{(0, m) : 1 \leq m \leq N\}$ on the left boundary and all paths exiting from the top boundary. We assume that no two paths are allowed to share a horizontal or vertical piece. For $\omega \in \mathcal{P}_N$ and $k = 1, \dots, N$ we let $\lambda^k(\omega) = \lambda_1^k \geq \lambda_2^k \geq \cdots \geq \lambda_k^k$ be the ordered $x$-coordinates of the intersection points of $\omega$ with the horizontal line $y = k + 1/2$. Let $\mathsf{Sign}_k^+$ denote the set of signatures $\lambda$ of length $k$ with $\lambda_k \geq 0$, then $\lambda^k(\omega) \in \mathsf{Sign}_k^+$ for all $\omega \in \mathcal{P}_N$ and $k = 1, \dots, N$. The condition that no two paths share a horizontal piece, implies that $\lambda^k$ satisfy the interlacing property

$$\lambda_1^{k+1} \geq \lambda_1^k \geq \lambda_2^{k+1} \geq \cdots \geq \lambda_k^k \geq \lambda_{k+1}^{k+1} \text{ for } k = 1, \dots, N-1,$$

while the condition of no shared vertical edges implies $\lambda_1^k > \lambda_2^k > \cdots > \lambda_k^k$. See Figure 3-1.

We encode arrow configurations at a vertex through the numbers $(i_1, j_1; i_2, j_2)$, representing the number of arrows coming from the bottom and left of the vertex, and leaving from the top and right, respectively (see Figure 3-2). Let us fix a parameter $s$ and $N$ indeterminates $u_1, \dots, u_N$, called *spectral parameters*. For a spectral parameter $u$, we define the following vertex weights

$$\begin{aligned} &w_u(0,0;0,0) = 1, \; w_u(1,0;1,0) = \frac{1 - s^{-1}u}{1 - su}, \; w_u(1,0;0,1) = \frac{(1 - s^2)u}{1 - su} \\ &w_u(0,1;1,0) = \frac{1 - s^{-2}}{1 - su}, \; w_u(0,1;0,1) = \frac{u - s}{1 - su}, \; w_u(1,1;1,1) = \frac{u - s^{-1}}{1 - su}, \end{aligned} \tag{3.1.1}$$

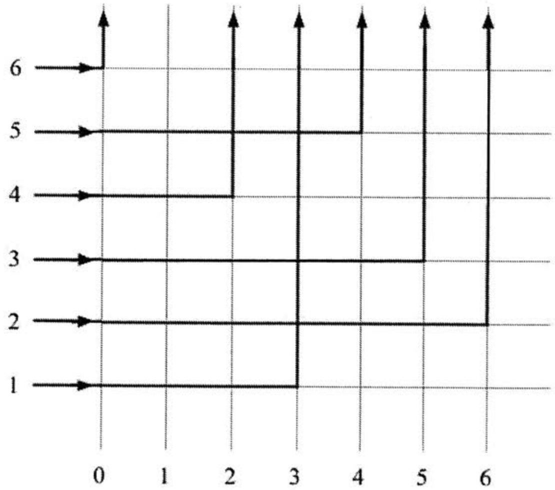


Figure 3-1: A path collection $\omega \in \mathcal{P}_N$ with $N = 6$. In this example $\lambda_1^4 = 6$, $\lambda_2^4 = 5$, $\lambda_3^4 = 3$ and $\lambda_4^4 = 2$

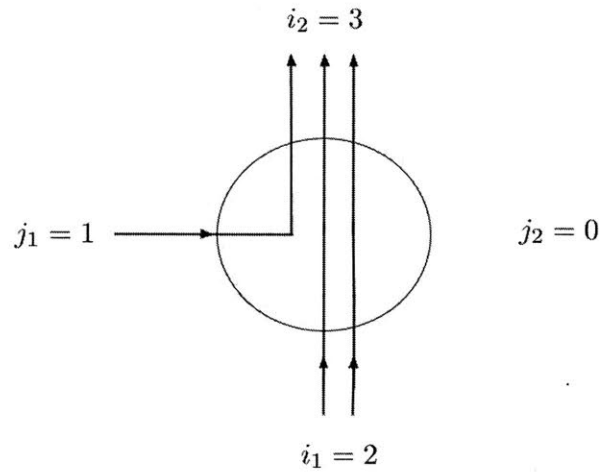


Figure 3-2: Incoming and outgoing vertical and horizontal arrows at a vertex, denoted by $(i_1, j_1; i_2, j_2) = (2, 1; 3, 0)$

and set all other vertex weights to zero. The choice of the above parametrization is made after [20], where higher spin versions of the above vertex weights were considered. Those weights depend on two parameters $s, q$ and they are closely related to the matrix elements of the higher spin $R$-matrix associated with $U_q(\widehat{sl_2})$. Formulas for the higher spin weights are present in (3.2.1) later in the text, and those in (3.1.1) are obtained by setting $q = s^{-2}$. Given $\omega \in \mathcal{P}_N$, we let $\omega(i,j)$ denote the arrow configuration at the vertex $(i,j) \in D_N$ and note that we have six possible arrow configurations for $\omega(i,j)$, corresponding to the weights in (3.1.1).

In addition, let us consider a function $f : \mathsf{Sign}_N^+ \to \mathbb{R}$. With the above data we define the *weight* of a path configuration $\omega$ as

$$\mathcal{W}^f(\omega) := \prod_{i=0}^{\infty} \prod_{j=1}^{N} w_{u_j}(\omega(i,j)) \times f(\lambda^N(\omega)). \tag{3.1.2}$$

We observe that all but finitely many of $\omega(i,j)$ equal $(0,0;0,0)$, which by (3.1.1) has weight 1 and so the product in (3.1.2) is a well-defined rational function. Suppose that for a choice of parameters and function $f$ the weights $\mathcal{W}^f(\omega)$ are non-negative, not all 0 and their sum

$$Z^f := \sum_{\omega \in \mathcal{P}_N} \mathcal{W}^f(\omega) < \infty,$$

then we may define a probability measure on $\mathcal{P}_N$ through $\mathbb{P}^f(\omega) = \frac{\mathcal{W}^f(\omega)}{Z^f}$. The function $f$ can be interpreted as a condition for the top boundary of an arrow configuration on $D_N$, complementing the other boundary conditions of no arrows entering from the bottom, all arrows entering from the left and no arrows propagating to infinity on the right. For example, taking $f(\lambda)$ to be zero unless $\lambda_{N-i+1} = i - 1$ for $i = 1, \ldots, N$ corresponds to the (vertically) inhomogeneous six-vertex model with domain wall boundary condition [58].

Our main algebraic tools are particular operators $D_m^k$, which can be used to extract a set of observables for measures on $\mathcal{P}_N$ of the form $\mathbb{P}^f$ above. The operators $D_m^k$ are inspired by the Macdonald difference operators, which have been used successfully in deriving asymptotic statements about random plane partitions, directed polymers and particle systems [24–26, 29, 47, 50]. To give an example, the first order operator $D_m^1$ acts on functions in $m$ variables and is given by

$$D_m^1 := \sum_{i=1}^{m} \prod_{j=1, j\neq i}^{m} \left( \frac{u_j - qu_i}{u_j - u_i} \frac{u_j - s}{u_j - sq} \right) \frac{1 - s^2}{1 - su_i} T_{u_i, s},$$

where $T_{u_i,s}F(u_1, \dots, u_m) = F(u_1, \dots, u_{i-1}, s, u_{i+1}, \dots, u_m)$. The formula for the general operator $D_m^k$ is given in Definition 3.3.2.

As will be explained later in Sections 3.2 and 3.3, the probability distribution $\mathbb{P}^f$ is related to certain symmetric functions $\mathsf{F}_\lambda(u_1, \dots, u_m)$, parametrized by non-negative signatures $\lambda$. The key property of $D_m^k$ is that they act diagonally on $\mathsf{F}_\lambda(u_1, \dots, u_m)$, whenever $\lambda$ has distinct parts and satisfy

$$D_m^k \mathsf{F}_\lambda(u_1, \dots, u_m) = \mathbf{1}_{\{\lambda_m = 0, \lambda_{m-1} = 1, \dots, \lambda_{m-k+1} = k-1\}} \mathsf{F}_\lambda(u_1, \dots, u_m).$$

The above relation is essentially sufficient to prove that for $1 \leq k \leq m \leq N$ we have

$$\mathbb{P}^f \left( \{ \omega \in \mathcal{P}_N : \lambda_m^m(\omega) = 0, \dots, \lambda_{m-k+1}^m(\omega) = k - 1 \} \right) = \frac{D_m^k Z^f}{Z^f}, \tag{3.1.3}$$

where we remark that the partition function $Z^f$ is a function of the variables $u_1, \dots, u_N$ and $D_m^k$ acts on the first $m$ variables. In words, the above expresses the probability of observing $k$ vertical arrows going from $(m, i)$ to $(m + 1, i)$ for $i = 0, \dots, k - 1$, in terms of the partition function $Z^f$ and the result of $D_m^k$ acting on it.

The validity of (3.1.3) can be established for a fairly general class of boundary functions $f$; however, in order for the formula to be useful one needs to understand the action of our operators on the partition function $Z^f$. For general boundary conditions the partition function may not have a closed form or the action of the operators might not be clear. One particular class of functions, on which $D_k^m$ act well are functions that have the product form $F(u_1, \dots, u_m) = \prod_{i=1}^m g(u_i)$. Such functions are eigenfunctions for $D_k^m$ with eigenvalues expressed through $k$-fold contour integrals - see Lemmas 3.3.10 and 3.3.12. Whenever a model has a partition function in such a form (this can be achieved by fixing appropriate boundary conditions $f$ and is the case for the models we study in this chapter) our method leads to contour integral representations for the probabilities in (3.1.3). In general, such representations are useful for asymptotic analysis as one has a lot of freedom in deforming contours and using steepest descent methods.

In what follows we write down the general form of a function $f$ that we will consider and explain the probabilistic meaning of this choice. Define

$$\mathsf{G}^c_\lambda(\rho) := (-1)^N \mathbf{1}_{\{n_0=0\}} \prod_{i=1}^{\infty} \mathbf{1}_{\{n_i \le 1\}} \prod_{j=1}^{N} (-s)^{\lambda_j},$$

where $\lambda = 0^{n_0} 1^{n_1} \cdots$ is the multiplicative expression for $\lambda$ (see Section 3.2.1). For an $M$-tuple of real parameters $(v_1, \dots, v_M)$ we define $f$ as

$$f(\lambda) = \mathsf{G}^c_\lambda(\rho, v_1, \dots, v_M) = \sum_{\mu \in \mathsf{Sign}^+_N} \mathsf{G}^c_\mu(\rho) \mathsf{G}^c_{\lambda/\mu}(v_1, \dots, v_M),$$

where $\mathsf{G}^c_{\lambda/\mu}(v_1, \dots, v_M)$ is given by Definition 3.2.1 below.

If $M = 0$ we have that $f(\lambda) = \mathsf{G}^c_\lambda(\rho)$ and

$$Z^f = \sum_{\omega \in \mathcal{P}_N} \mathcal{W}^f(\omega) = (s^{-2}; s^{-2})_N \prod_{i=1}^{N} \frac{1 - s^{-1} u_i}{1 - s u_i},$$

where we recall that $(a;q)_n$ denotes the $q$-Pochhammer symbol and equals $(a;q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$. The latter identity is understood as an equality of formal power series and was derived in [33]. Fixing $s > 1$ and $u_i > s$ has the effect that $\mathcal{W}^f(\omega) \ge 0$ and that the above identity holds numerically as well. In particular, for this choice of $f$, we have a well-defined probability distribution $\mathbb{P}^f$ on $\mathcal{P}_N$. The latter measure is the (vertically) inhomogeneous stochastic six-vertex model (see Section 6.5 in [33]). Further setting $u_i = u > s$ for $i = 1, \dots, N$, one arrives at the stochastic six-vertex model of [27] (see also [33]).

Given the above discussion, one can understand $f(\lambda) = \mathsf{G}^c_{\lambda/\mu}(v_1, \dots, v_M)$ as a certain many-parameter generalization of the boundary function of the previous models. As will be explained in Section 3.2 we have for this choice of $f$ that

$$Z^f = \sum_{\omega \in \mathcal{P}_N} \mathcal{W}^f(\omega) = (s^{-2}; s^{-2})_N \prod_{i=1}^{N} \left( \frac{1 - s^{-1} u_i}{1 - s u_i} \prod_{j=1}^{M} \frac{1 - s^{-2} u_i v_j}{1 - u_i v_j} \right),$$

where the equality is in the sense of formal power series. As before, we set $s > 1$, $u_i > s$ and, in addition, assume $v_j > 0$ are such that $u_i v_j < 1$ for $i = 1, \dots, N$ and $j = 1, \dots, M$. Under these conditions one can show that $\mathcal{W}^f(\omega) \ge 0$ and the above identity holds numerically as well. In particular, for this choice of $f$, we have a well-defined probability distribution $\mathbb{P}^f$ on $\mathcal{P}_N$, denoted by $\mathbb{P}_{\mathbf{u},\mathbf{v}}$. This is the main probabilistic object we will study.

For $m = 0, \dots, M$ we let $\mathbb{P}_{\mathbf{u},\mathbf{v}_m}$ denote the above probability distribution, where $\mathbf{v}_m = (v_1, \dots, v_m)$. Then one can interpret the distribution $\mathbb{P}_{\mathbf{u},\mathbf{v}_m}$ as the time $m$ distribution of a Markov chain $\{X_m\}_{m=0}^M$, whose dynamics is governed by sequential update rules. For more details and an exact formulation we refer the reader to Section 3.8 below as well as Section 6 in [33]. For a pictorial description of how the configurations $X_m$ evolve as time increases see Figure 3-3. Our primary interest is in understanding the large-time behavior of $X_m$ and we investigate this by studying the measure $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ as both $M$ and $N$ tend to infinity.

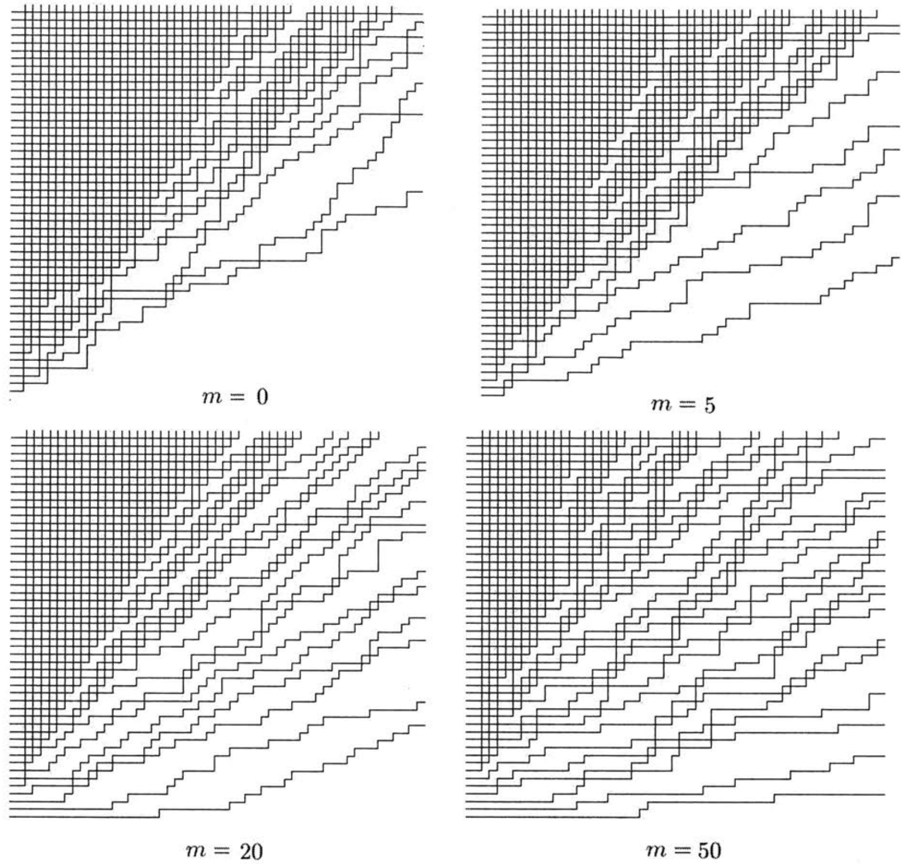


Figure 3-3: Random paths in $\mathcal{P}_N$, sampled from the Markov chain $X_m$ at times $m = 0, 5, 20$ and 50. In this example $N = M = 50$, $s^{-2} = 0.7$, $u_i = u$ for $i = 1, \dots, N$ and $v_j = v$ for $j = 1, \dots, M$, where $u = 1.5$ and $v = 0.4$

While most of the results we have in mind can readily be extended to more general parameter choices for $\mathbf{u}$ and $\mathbf{v}$ we keep our discussion simple and assume that $u_i = u$ for $i = 1, \dots, N$ and $v_j = v$ for $j = 1, \dots, M$. The resulting measure is denoted by $\mathbb{P}_{u,v}^{N,M}$ (the measure also depends on the parameter $s$ but we suppress it from the notation). The first result about this measure is the following.

**Theorem 3.1.1.** *Suppose $s > 1$, $\frac{s+s^3}{2} > u > s$ and $v \in (0, u^{-1})$. Let $a = \frac{v^{-1}(u-s^{-1})(u-s)}{u(v^{-1}-s^{-1})(v^{-1}-s)} > 0$ and suppose $\gamma > a$. Let $N(M) \geq \gamma \cdot M$ for all $M >> 1$ and consider the measure $\mathbb{P}_{u,v}^{N,M}$ on $\mathcal{P}_N$, defined above. Then for every $k \in \mathbb{N}$, we have that*

$$\lim_{M \to \infty} \mathbb{P}_{u,v}^{N,M}\left(\{\omega \in \mathcal{P}_N : \lambda_{N-i+1}^N(\omega) = i, 1 \leq i \leq k\}\right) = 1. \tag{3.1.4}$$

*Remark* 3.1.2. We choose $s > 1$ and $u > s$ to ensure non-negativity of the weights defining $\mathbb{P}_{u,v}^{N,M}$. This choice of parameters lands our Gibbs measure in the *ferroelectric regime* of the six-vertex model [9] and $s > 1$ covers the entire range of the ferroelectric region - see also the discussion in Section 3.6.1. One requires $v \in (0, u^{-1})$ in order to ensure finiteness of the partition function and non-negativity of the weights. The condition $\frac{s+s^3}{2} > u$ is technical and assumed in order to simplify some arguments later in the text.

Informally, Theorem 3.1.1 states that the probability $\mathbb{P}^{N,M}_{u,v}$ concentrates on path configurations, which have outgoing vertical arrows at locations $(i, N)$ for $i = 1, \ldots, k$, where $k \in \mathbb{N}$ is fixed but arbitrary, and no such arrow at $(0, N)$. Let us consider such a path configuration $\omega$ and denote by $A_j = \{(j, i) : i = 1, \ldots, N\}$ the vertical slice of $D_N$ at location $j$. We observe that the left and bottom boundary conditions on $\omega$ imply that there are exactly $N$ arrows going into the set $A_0$ and no vertical outgoing arrow from $(0, N)$. The conservation of arrows over the region $A_0$, implies that all $N$ arrows must leave from the right boundary of $A_0$, and so each arrow that enters $(0, i)$ must continue horizontally (see Figure 3-4). When

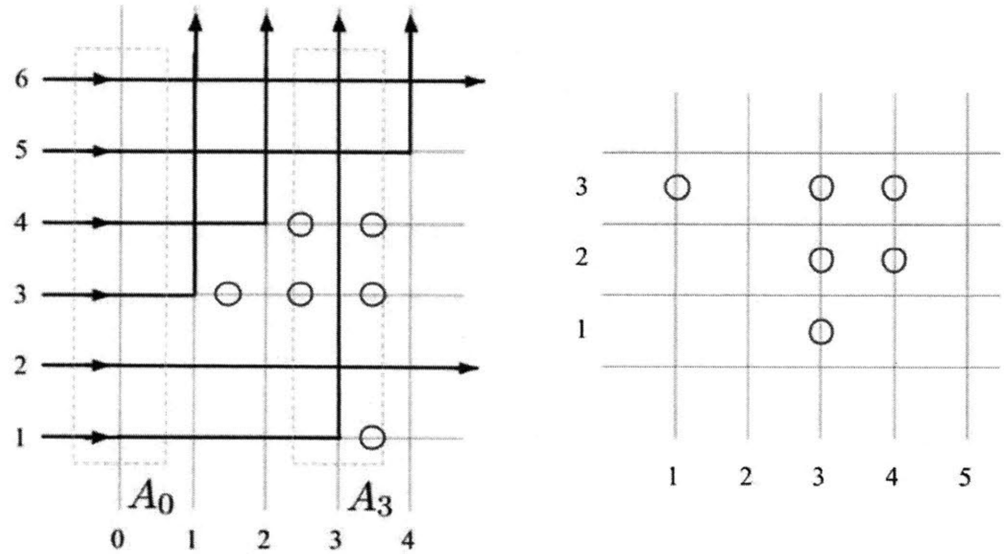


Figure 3-4: The left figure shows a path collection $\omega$, such that $\lambda^N_{N-i+1}(\omega) = i$ for $i = 1, \ldots, k$ with $N = 6$ and $k = 3$. Circles indicate the positions of the empty edges. The right figure shows the array $(Y^j_i)_{1 \leq i \leq j \leq 3}$; $j$ varies vertically and position is measured horizontally. In this case $Y^1_1 = Y^2_1 = Y^3_2 = 3$, $Y^2_2 = Y^3_3 = 4$, $Y^3_1 = 1$.

we consider $A_1$, we see that there are still $N$ arrows going in, however, one arrow leaves at $(1, N)$ and so the conservation of arrows implies that there are $N - 1$ arrows leaving $A_1$ to the right and entering $A_2$. In general, there will be $N - j + 1$ arrows going into region $A_j$ and one arrow leaving from the top, implying that there are $N - j$ arrows leaving from the right and entering $A_{j+1}$. Let us denote by $Y^j_1 < Y^j_2 < \cdots < Y^j_j$, the ordered vertical positions of the $j$ vertices in $A_j$, that have no outgoing horizontal arrow (alternatively, the vertical coordinates of the empty horizontal edges between $A_j$ and $A_{j+1}$) - see Figure 3-4. A direct consequence of the up-right path direction, implies that $Y^j_i$ satisfy the interlacing property

$$Y^{j+1}_1 \leq Y^j_1 \leq Y^{j+1}_2 \leq \cdots \leq Y^j_j \leq Y^{j+1}_{j+1} \text{ for } j = 1, \ldots, k - 1.$$

The above definition can readily be extended to $\omega \in \mathcal{P}_N$, which do not satisfy the condition $\lambda^N_{N-i+1}(\omega) = i, 1 \leq i \leq k$ as follows. We set $Y^j_i$ to be the $i$-th smallest $y$-coordinate of a vertex in $A_j$ with no horizontal outgoing arrow, or $Y^j_i = +\infty$ if the number of such vertices is less than $i$. In this way, we obtain an extended random vector $Y(N, M; k)(\omega) :=$

$(Y_i^j)_{1\leq i\leq j; 1\leq j\leq k} \in (\mathbb{N}\cup\{\infty\})^{\frac{k(k+1)}{2}}$. The statement of Theorem 3.1.1 is that with probability going to 1, the interlacing array $Y(N,M;k)(\omega)$ is actually finite.

Recall that the Gaussian Unitary Ensemble (GUE) of rank $k$ is the ensemble of random Hermitian matrices $X = \{X_{ij}\}_{i,j=1}^k$ with probability density (proportional to) $\exp(-\mathrm{Tr}(X^2)/2)$, with respect to Lebesgue measure. For $r = 1,\dots,k$ we let $\lambda_1^r \leq \lambda_2^r \leq \cdots \leq \lambda_r^r$ denote the eigenvalues of the top-left $r\times r$ corner $\{X_{ij}\}_{i,j=1}^r$. The joint distribution of $\lambda_i^j$ $i = 1,\dots,j$, $j = 1,\dots,k$ is known as the *GUE-corners* process of rank $k$ (sometimes called the GUE-minors process). The following theorem is the main result of this chapter.

**Theorem 3.1.3.** *Assume the same notation as in Theorem 3.1.1, put $q = s^{-2}$ and fix $k\in\mathbb{N}$. Consider the sequence $Y(N,M;k)(\omega)$ with $\omega$ distributed according to $\mathbb{P}_{u,v}^{N,M}$. Then the random vectors*

$$\frac{1}{c\sqrt{M}}\left(Y(N,M;k) - aM\cdot\mathbf{1}_{\frac{k(k+1)}{2}}\right) \tag{3.1.5}$$

*converge weakly to the GUE-corners process of rank $k$ as $M\to\infty$. In the above equation $\mathbf{1}_K$ is the vector of $\mathbb{R}^K$ with all entries equal to 1 and $c = (2a_2)^{1/2}b_1^{-1}$, with*

$$a_2 = \frac{(1-q)v^{-1}}{(v^{-1}-s)(v^{-1}-sq)}\left[\frac{(q+1)s-2v^{-1}}{(v^{-1}-s)(v^{-1}-sq)} - \frac{(q+1)s-2u}{(u-s)(u-sq)}\right],\ b_1 = \frac{1}{u-s} - \frac{1}{q^{-1}u-s}.$$

We end this section by briefly outlining the key ideas that go into proving Theorem 3.1.3. The first key observation is that for $\omega\in\mathcal{P}_N$ one has $\lambda_m^m(\omega) = 1,\dots,\lambda_{m-k+1}^m(\omega) = k$ if and only if $Y_k^k(\omega)\leq m$. Using this observation and our operators $D_m^k$, we express $\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_k^k\leq m)$ and more generally $\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_1^1\leq m_1,\dots,Y_k^k\leq m_k)$ in terms of certain $k$-fold contour integrals. These formulas for the joint cumulative distribution functions (CDFs) of the random vector $(Y_1^1,\dots,Y_k^k)$ are suitable for asymptotic analysis and can be used to show that under the translation and rescaling of Theorem 3.1.3, this vector converges weakly to $(\lambda_1^1,\dots,\lambda_k^k)$, where $\lambda_i^j$ $i = 1,\dots,j$, $j = 1,\dots,k$ is the GUE-corners process of rank $k$. Using the six-vertex Gibbs property (see Section 3.6.2) and our convergence result for $(Y_1^1,\dots,Y_k^k)$, we show that the sequence of random interlacing arrays $Y(N,M;k)$ under the translation and rescaling of Theorem 3.1.3 is tight and any subsequential limit satisfies the continuous Gibbs property (see Definition 3.5.4). The final ingredient, in the proof is a classification result, which identifies the GUE-corners process as the unique probability measure on interlacing arrays that satisfies the continuous Gibbs property and has the correct distribution on the right edge. This shows that any weak subsequential limit of $Y(N,M;k)$ is in fact the GUE-corners process of rank $k$, which together with tightness proves Theorem 3.1.3.

### 3.1.2 Outline

The introductory section above formulated the problem statement and gave the main results of the chapter. In Section 3.2 we study the measure $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ and derive formulas for its finite dimensional distributions. In Section 3.3 we define our operators $D_m^k$ and establish some of their properties. In Section 3.4 we obtain integral formulas for the probabilities $\mathbb{P}_{u,v}^{N,M}(Y_1^1\leq m_1,\dots,Y_k^k\leq m_k)$, study their asymptotics and prove Theorem 3.1.1. In Section 3.5 we discuss probability measures on Gelfand-Tsetlin cones, satisfying a continuous Gibbs property. In Section 3.6 we study probability measures on Gelfand-Tsetlin patterns, satisfying what we call the six-vertex Gibbs property. The proof of Theorem 3.1.3 is given in Section 3.7. In Section 3.8 we describe an exact sampling algorithm for the measure $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ and discuss some of their conjectural properties.

## 3.2 Measures on up-right paths

In this section we provide some results about $\mathbb{P}_{\mathbf{u},\mathbf{v}}$. In particular, we show that it arises as a limit of measures on non-negative signatures, studied in Section 6 of [33], and is a well-defined probability measure on the set of oriented up-right paths drawn in the region $D_N = \mathbb{Z}_{\geq 0} \times \{1, \dots, N\}$. We also provide explicit formulas for its marginal distributions. In what follows we adopt the notation from [33] and summarize some of the results there.

### 3.2.1 Symmetric rational functions

We start by introducing some necessary notation. A *signature* of length $N$ is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N)$, $\lambda_i \in \mathbb{Z}$. The set of all signatures of length $N$ is denoted by $\mathsf{Sign}_N$, and $\mathsf{Sign}_N^+$ is the set of signatures with $\lambda_N \geq 0$. We agree that $\mathsf{Sign}_0 = \mathsf{Sign}_0^+$ consists of the single empty signature $\varnothing$ of length 0. We also denote by $\mathsf{Sign}^+ := \sqcup_{N \geq 0} \mathsf{Sign}_N^+$ the set of all non-negative signatures. An alternative representation of a signature $\mu \in \mathsf{Sign}^+$ is through the multiplicative notation $\mu = 0^{m_0} 1^{m_1} 2^{m_2} \cdots$, which means that $m_j = |\{i : \mu_i = j\}|$ is the number of parts in $\mu$ that are equal to $j$ (also called *multiplicity* of $j$ in $\mu$). We also recall the $q$-Pochhammer symbol $(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$.

In what follows, we want to define the *weight* of a finite collection of up-right paths in some region $D$ of $\mathbb{Z}^2$, which will be given by the product of weights of all vertices that belong to the path collection. Throughout this chapter we will always assume that the weight of an empty vertex is 1 and so alternatively the weight of a path configuration can be defined through the product of the weights of all vertices in $D$. Figures 3-1 and 3-3 give examples of collections of up-right paths, see also Figure 3-5 below.

The configuration at a vertex is determined by four numbers $(i_1, j_1; i_2, j_2)$, representing the number of arrows that enter the vertex from below and right, and that leave from the top and left respectively (see Figure 2). Vertex weights are thus functions of those four variables. We postulate that a configuration $(i_1, j_1; i_2, j_2)$ must satisfy $i_1, j_1, i_2, j_2 \geq 0$, $j_1, j_2 \in \{0, 1\}$ and $i_1 + j_1 = i_2 + j_2$ (otherwise its weight is 0).

We will consider two sets of special vertex weights. They are both defined through two parameters $s, q$ (which are fixed throughout this section) as well as an additional *spectral parameter* $u$. We assume all parameters are generic complex numbers, and for the most part ignore possible singularities of the expressions below. The first set of vertex weights is explicitly given by

$$\begin{aligned} w_u(g,0;g,0) &= \frac{1-sq^g u}{1-su}, & w_u(g+1,0;g,1) &= \frac{(1-s^2q^g)u}{1-su} \\ w_u(g,1;g,1) &= \frac{u-sq^g}{1-su}, & w_u(g,1;g+1,0) &= \frac{1-q^{g+1}}{1-su}, \end{aligned} \tag{3.2.1}$$

where $g$ is any non-negative integer. All other weights are assumed to be zero. We also define the following *conjugated* vertex weights

$$\begin{aligned} w^c_u(g,0;g,0) &= \frac{1-sq^g u}{1-su}, & w^c_u(g+1,0;g,1) &= \frac{(1-q^{g+1})u}{1-su} \\ w^c_u(g,1;g,1) &= \frac{u-sq^g}{1-su}, & w^c_u(g,1;g+1,0) &= \frac{1-s^2q^g}{1-su}, \end{aligned} \tag{3.2.2}$$

where as before $g \in \mathbb{Z}_{\geq 0}$ and all other weights are zero. We remark that the weights are non-zero only if $j_1, j_2 \in \{0,1\}$, which implies that the multiplicity of the horizontal edges is bounded by 1. For more background and motivation for this particular choice of weights we refer the reader to Section 2 of [33].

Let us fix a number $n \in \mathbb{N}$, $n$ indeterminates $u_1, \dots, u_n$ and the region $D_n = \mathbb{Z}_{\geq 0} \times \{1, \dots, n\}$. Let $\omega$ be a finite collection of up-right paths in $D_n$, which end in the top boundary, but are allowed to start from the left or bottom boundary of $D_n$. By $\omega(i,j)$ we denote the arrow configuration of the vertex at location $(i,j) \in D_n$. Then the weight of $\omega$ with respect to the two sets of weights above is defined by

$$\mathcal{W}(\omega) = \prod_{i=0}^{\infty} \prod_{j=1}^{n} w_{u_j}(\omega(i,j)), \qquad \mathcal{W}^c(\omega) = \prod_{i=0}^{\infty} \prod_{j=1}^{n} w^c_{u_j}(\omega(i,j)).$$

We notice that by (3.2.1) and (3.2.2) $w_u(0,0;0,0) = 1 = w^c_u(0,0;0,0)$ and since all but finitely many vertices are empty, the products above are in fact finite. With the above notation we define the following partition functions.

**Definition 3.2.1.** Let $N, n \in \mathbb{Z}_{\geq 0}$, $\lambda, \mu \in \mathsf{Sign}^+_N$ and $u_1, \dots, u_n \in \mathbb{C}$ be given. Let $\mathcal{P}^c_{\lambda/\mu}$ be the collection of up-right paths $\omega$, which

- start with $N$ vertical edges $(\mu_i, 0) \to (\mu_i, 1)$, $i = 1, \dots, N$;
- end with $N$ vertical edges $(\lambda_i, n) \to (\lambda_i, n+1)$, $i = 1, \dots, N$.

Then we define

$$\mathsf{G}^c_{\lambda/\mu}(u_1, \dots, u_n) := \sum_{\omega \in \mathcal{P}^c_{\lambda/\mu}} \mathcal{W}^c(\omega).$$

We will also use the abbreviation $\mathsf{G}^c_\lambda$ for $\mathsf{G}^c_{\lambda/(0,0,\dots,0)}$. For the second set of weights we have a similar definition.

**Definition 3.2.2.** Let $N, n \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathsf{Sign}^+_N$, $\lambda \in \mathsf{Sign}^+_{N+n}$ and $u_1, \dots, u_n \in \mathbb{C}$ be given. Let $\mathcal{P}_{\lambda/\mu}$ be the collection of up-right paths, which

- start with $N$ vertical edges $(\mu_i, 0) \to (\mu_i, 1)$, $i = 1, \dots, N$ and with $n$ horizontal edges $(-1, y) \to (0, y)$, $y = 1, \dots, n$;
- end with $N + n$ vertical edges $(\lambda_i, n) \to (\lambda_i, n+1)$, $i = 1, \dots, N+n$.

Then we define

$$\mathsf{F}_{\lambda/\mu}(u_1, \dots, u_n) := \sum_{\omega \in \mathcal{P}_{\lambda/\mu}} \mathcal{W}(\omega).$$

We will also use the abbreviation $\mathsf{F}_\lambda = \mathsf{F}_{\lambda/\varnothing}$. Path configurations that belong to $\mathcal{P}_{\lambda/\mu}$ and $\mathcal{P}^c_{\lambda/\mu}$ are depicted in Figure 3-5.

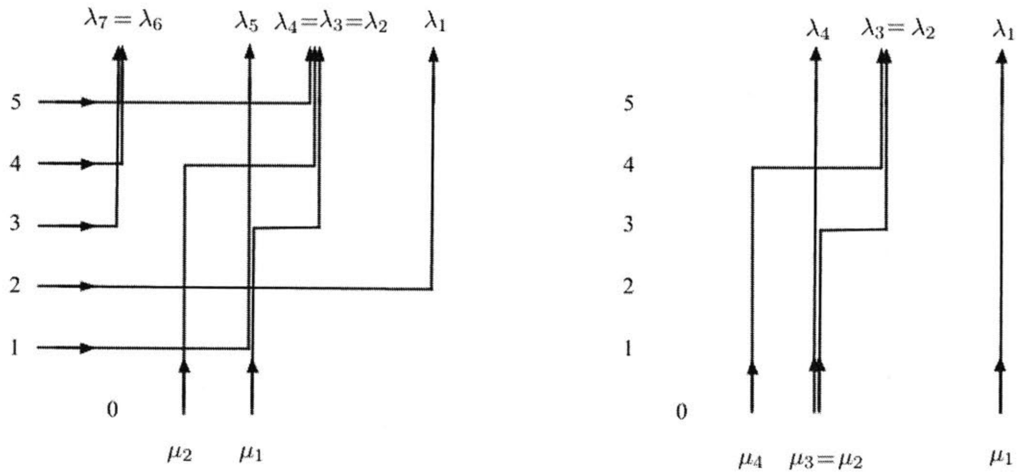


Figure 3-5: Path collections belonging to $\mathcal{P}_{\lambda/\mu}$ (left) and $\mathcal{P}^c_{\lambda/\mu}$ (right).

In the definitions above we define the weight of a collection of paths to be 1, if it has no interior vertices. Also, the weight of an empty collection of paths is 0. We now summarize some of the properties of the functions $\mathsf{G}^c_{\lambda/\mu}$ and $\mathsf{F}_{\lambda/\mu}$ in a sequence of propositions; see Section 4 of [33] for details.

**Proposition 3.2.3.** *Let $N, n, k \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathsf{Sign}^+_N$, $\lambda \in \mathsf{Sign}^+_{N+n}$ and $u_1, \dots, u_n \in \mathbb{C}$ be given. Suppose $\mu_N \geq k$ and $\lambda_{N+n} \geq k$, and denote by $\mu - (k)^N$ and $\lambda - (k)^{N+n}$ the signatures with parts $\mu_i - k$ and $\lambda_i - k$ respectively. Then we have*

$$\mathsf{F}_{\lambda/\mu}(u_1, \dots, u_n) = \left( \prod_{i=1}^{n} \frac{u_i - s}{1 - su_i} \right)^k \mathsf{F}_{\lambda-(k)^{N+n}/\mu-(k)^N}(u_1, \dots, u_n). \tag{3.2.3}$$

**Proposition 3.2.4.** *The functions $\mathsf{F}_{\lambda/\mu}(u_1, \dots, u_n)$ and $\mathsf{G}^c_{\lambda/\mu}(u_1, \dots, u_n)$ defined above are rational symmetric functions in the variables $u_1, \dots, u_n$.*

**Proposition 3.2.5. 1.** *For any $N, n_1, n_2 \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathsf{Sign}^+_N$ and $\lambda \in \mathsf{Sign}^+_{N+n_1+n_2}$, one has*

$$\mathsf{F}_{\lambda/\mu}(u_1, \dots, u_{n_1+n_2}) = \sum_{\kappa \in \mathsf{Sign}^+_{N+n_1}} \mathsf{F}_{\lambda/\kappa}(u_{n_1+1}, \dots, u_{n_1+n_2}) \mathsf{F}_{\kappa/\mu}(u_1, \dots, u_{n_1}). \tag{3.2.4}$$

**2.** *For any $N, n_1, n_2 \in \mathbb{Z}_{\geq 0}$ and $\lambda, \mu \in \mathsf{Sign}^+_N$, one has*

$$\mathsf{G}^c_{\lambda/\mu}(u_1, \dots, u_{n_1+n_2}) = \sum_{\kappa \in \mathsf{Sign}^+_N} \mathsf{G}^c_{\lambda/\kappa}(u_{n_1+1}, \dots, u_{n_1+n_2}) \mathsf{G}^c_{\kappa/\mu}(u_1, \dots, u_{n_1}). \tag{3.2.5}$$

The properties of the last proposition are known as *branching rules*.

**Definition 3.2.6.** We say that two complex numbers $u, v \in \mathbb{C}$ are *admissible* with respect to the parameter $s$ if $\left| \frac{u-s}{1-su} \cdot \frac{v-s}{1-sv} \right| < 1$.

**Proposition 3.2.7.** *Let $u_1, \ldots, u_N$ and $v_1, \ldots, v_K$ be complex numbers such that $u_i, v_j$ are admissible for all $i = 1, \ldots, N$ and $j = 1, \ldots, K$. Then for any $\lambda, \nu \in \mathsf{Sign}^+$ one has*

$$\sum_{\kappa \in \mathsf{Sign}^+} \mathsf{G}^c_{\kappa/\lambda}(v_1, \ldots, v_K)\mathsf{F}_{\kappa/\nu}(u_1, \ldots, u_N) = \prod_{i=1}^{N}\prod_{j=1}^{K} \frac{1 - qu_iv_j}{1 - u_iv_j} \sum_{\mu \in \mathsf{Sign}^+} \mathsf{F}_{\lambda/\mu}(u_1, \ldots, u_N)\mathsf{G}^c_{\nu/\mu}(v_1, \ldots, v_K). \tag{3.2.6}$$

*Remark* 3.2.8. Equation (3.2.6) is called the *skew Cauchy identity* for the symmetric functions $\mathsf{F}_{\lambda/\mu}$ and $\mathsf{G}^c_{\lambda/\mu}$ because of its similarity with the skew Cauchy identities for Schur, Hall-Littlewood, or Macdonald symmetric functions [64]. The sum on the right-hand side (RHS) of (3.2.6) has finitely many non-zero terms and is thus well-defined. The left-hand side (LHS) can have infinitely many non-zero terms, but part of the statement of the proposition is that if the variables are admissible, then this sum is absolutely converging and numerically equals the right side.

A special case of (3.2.6), when $\lambda = \varnothing$ and $\nu = (0, 0, \ldots, 0)$ leads us to the *Cauchy identity*

$$\sum_{\nu \in \mathsf{Sign}^+_N} \mathsf{F}_\nu(u_1, \ldots, u_N)\mathsf{G}^c_\nu(v_1, \ldots, v_K) = (q; q)_N \prod_{i=1}^{N} \left( \frac{1}{1 - su_i} \prod_{j=1}^{K} \frac{1 - qu_iv_j}{1 - u_iv_j} \right). \tag{3.2.7}$$

We end this section with the *symmetrization formulas* for $\mathsf{G}^c_\nu$ and $\mathsf{F}_\mu$ and also formulas for the functions when the variable set forms a geometric progression with parameter $q$.

**Proposition 3.2.9. 1.** *For any $N \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathsf{Sign}^+_N$ and $u_1, \ldots, u_N \in \mathbb{C}$, one has*

$$\mathsf{F}_\mu(u_1, \ldots, u_N) = \frac{(1 - q)^N}{\prod_{i=1}^{N}(1 - su_i)} \sum_{\sigma \in S_N} \sigma \left( \prod_{1 \leq \alpha < \beta \leq N} \frac{u_\alpha - qu_\beta}{u_\alpha - u_\beta} \left( \frac{u_i - s}{1 - su_i} \right)^{\mu_i} \right). \tag{3.2.8}$$

**2.** *Let $n \geq 0$ and $\mathsf{Sign}^+_n \ni \nu = 0^{n_0}1^{n_1}2^{n_2}\cdots$. Then for any $N \geq n - n_0$ and $u_1, \ldots, u_N \in \mathbb{C}$ we have*

$$\begin{aligned} \mathsf{G}^c_\nu(u_1, \ldots, u_N) &= \frac{(1 - q)^N(q; q)_n}{\prod_{i=1}^{N}(1 - su_i)(q; q)_{N-n+n_0}(q; q)_{n_0}} \prod_{k=1}^{\infty} \frac{(s^2; q)_{n_k}}{(q; q)_{n_k}} \\ &\times \sum_{\sigma \in S_N} \sigma \left( \prod_{1 \leq \alpha < \beta \leq N} \frac{u_\alpha - qu_\beta}{u_\alpha - u_\beta} \left( \frac{u_i - s}{1 - su_i} \right)^{\nu_i} \prod_{i=1}^{n-n_0} \frac{u_i}{u_i - s} \prod_{j=n-k+1}^{N} (1 - sq^{n_0}u_j) \right). \end{aligned} \tag{3.2.9}$$

*In both equations above, $S_N$ denotes the permutation group on $\{1, \ldots, N\}$ and an element $\sigma \in S_N$ acts on the expression by permuting the variable set to $u_{\sigma(1)}, \ldots, u_{\sigma(N)}$. By agreement, we set $\nu_j = 0$ if $j > n$. If $N < n - n_0$, then $\mathsf{G}^c_\nu(u_1, \ldots, u_N)$ is equal to 0.*

**Proposition 3.2.10. 1.** *For any $N \in \mathbb{Z}_{\geq 0}$, $\mu \in \mathsf{Sign}^+_N$ and $u \in \mathbb{C}$, one has*

$$\mathsf{F}_\mu(u, qu, \ldots, q^{N-1}u) = (q; q)_N \prod_{i=1}^{N} \left( \frac{1}{1 - sq^{i-1}u} \left( \frac{uq^{i-1} - s}{1 - sq^{i-1}u} \right)^{\mu_i} \right). \tag{3.2.10}$$

**2.** *Let $n \geq 0$ and $\mathsf{Sign}_n^+ \ni \nu = 0^{n_0}1^{n_1}2^{n_2}\cdots$. Then for any $N \geq n - n_0$ and $u \in \mathbb{C}$ we have*

$$\mathsf{G}^c_\nu(u, qu, \dots, q^{N-1}u) = \prod_{k=1}^{\infty} \frac{(s^2;q)_{n_k}}{(q;q)_{n_k}} \cdot \frac{(q;q)_N(su;q)_{N+n_0}(q;q)_n \prod_{i=1}^N \left(\frac{1}{1-sq^{i-1}u}\left(\frac{q^{i-1}u-s}{1-sq^{i-1}u}\right)^{\nu_j}\right)}{(q;q)_{N-n+n_0}(su;q)_n(q;q)_{n_0}(su^{-1};q^{-1})_{n-n_0}}. \tag{3.2.11}$$

### 3.2.2 The measure $\mathbb{P}_{\mathbf{u},\mathbf{v}}$

As discussed in Section 3.1.1 the main probabilistic object we study is the measure $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ on up-right paths in the half-infinite strip that share no horizontal or vertical pieces. The purpose of this section is to properly define it.

Let us briefly explain the main steps of the construction of $\mathbb{P}_{\mathbf{u},\mathbf{v}}$. We begin by considering the bigger space of all up-right paths in the half-infinite strip that share no horizontal piece but are allowed to share vertical pieces. For each such collection of paths we define its weight and show that these weights are absolutely summable and their sum has a product form. Afterwards we specialize one parameter in those weights and perform a limit transition for some of the other parameters. This procedure has the effect of killing the weight of those path configurations that share a vertical piece. Consequently, we are left with weights that are non-zero only for six-vertex configurations, are absolutely summable and their sum has a product form. We check that each weight is non-negative, and define $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ as the quotient of these weights with the partition function.

We fix positive integers $N, M$, $J$, and $K = M + J$, as well as real numbers $q \in (0,1)$ and $s > 1$. In addition, we suppose $\mathbf{u} = (u_1, \dots, u_N)$ and $\mathbf{w} = (w_1, \dots, w_K)$ are positive real numbers, such that $\max_{i,j} u_i w_j < 1$ and $u := \min_i u_i > s$. One readily verifies that the latter conditions ensure that $u_i, w_j$ are admissible with respect to $s$ for $i = 1, \dots, N$ and $j = 1, \dots, K$.

Let us go back to the setup of Section 3.1.1. We let $\mathcal{P}'_N$ be the collection of $N$ up-right paths drawn in the sector $D_N = \mathbb{Z}_{\geq 0} \times \{1, \dots, N\}$ of the square lattice, with all paths starting from a left-to-right arrow entering each of the points $\{(0,m) : 1 \leq m \leq N\}$ on the left boundary and all paths exiting from the top boundary of $D_N$. We still assume that no two paths share a horizontal piece, but sharing vertical pieces is allowed. As in Section 3.1.1 we let $\mathcal{P}_N \subset \mathcal{P}'_N$ be those collections of paths that do not share vertical pieces. For $\omega \in \mathcal{P}'_N$ and $k = 1, \dots, N$ we let $\lambda^k(\omega) \in \mathsf{Sign}_k^+$ denote the ordered $x$-coordinates of the intersection of $\omega$ with the horizontal line $y = k + 1/2$. We denote by $\omega(i,j)$ the arrow configuration at the vertex in position $(i,j) \in D_N$. We also let $f : \mathsf{Sign}_N^+ \to \mathbb{R}$ be given by

$$f(\lambda; \mathbf{w}) := \sum_{\mu \in \mathsf{Sign}_N^+} \mathsf{G}^c_\mu(w_1, \dots, w_J)\mathsf{G}^c_{\lambda/\mu}(w_{J+1}, \dots, w_K).$$

With the above data, we define the weight of a collection of paths $\omega$ by

$$\mathcal{W}^f_{\mathbf{u},\mathbf{w}}(\omega) = \prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j)) \times f(\lambda^N(\omega); \mathbf{w}).$$

If we perform the summation over $\mu$ and use Proposition 3.2.5 we see that $f(\lambda;\mathbf{w}) = \mathsf{G}^c_\lambda(w_1,\dots,w_K)$. This together with Definition 3.2.2 implies that

$$\mathcal{W}^f_{\mathbf{u},\mathbf{w}}(\omega) = \mathsf{F}_{\lambda^1(\omega)}(u_1)\mathsf{F}_{\lambda^2(\omega)/\lambda^1(\omega)}(u_2)\cdots\mathsf{F}_{\lambda^N(\omega)/\lambda^{N-1}(\omega)}(u_N)\times\mathsf{G}^c_{\lambda^N(\omega)}(w_1,\dots,w_K).$$

Using the branching relations for $\mathsf{F}_{\lambda/\mu}$ from Proposition 3.2.5 and performing the sum over $\lambda^1,\dots,\lambda^{N-1}$ we obtain $\mathsf{F}_{\lambda^N(\omega)}(u_1,\dots,u_N)\mathsf{G}^c_{\lambda^N(\omega)}(w_1,\dots,w_K)$. A final summation over $\lambda^N$ and application of the Cauchy identity (3.2.7) leads us to

$$\sum_{\omega\in\mathcal{P}'_N}\mathcal{W}^f_{\mathbf{u},\mathbf{w}}(\omega) = (q;q)_N\prod_{i=1}^N\left(\frac{1}{1-su_i}\prod_{j=1}^K\frac{1-qu_iw_j}{1-u_iw_j}\right) =: Z^f(\mathbf{u};\mathbf{w}).$$

In view of the admissability conditions satisfied by $\mathbf{u}$ and $\mathbf{w}$, the above sum is in fact absolutely convergent, hence the particular order of summation we chose is irrelevant. We remark that the weights $\mathcal{W}^f_{\mathbf{u},\mathbf{w}}(\omega)$ are real and not necessarily non-negative, but they are absolutely summable and their sum equals the above expression.

We next wish to specialize some of the variables $w_i$ and relabel the others, in addition we fix $s = q^{-1/2}$. Set $w_i = q^{i-1}\epsilon$ for $i = 1,\dots,J$ and put $v_j = w_{j+J}$ for $j = 1,\dots,M$. Here $\epsilon > 0$ is chosen sufficiently small so that the admissibility condition is maintained.

*Remark* 3.2.11. Choosing $s = q^{-1/2}$ has the effect that if $\mu\in\mathsf{Sign}^+_N$ has distinct parts and $\lambda\in\mathsf{Sign}^+_N$ then $\mathsf{G}^c_{\lambda/\mu}(u_1,\dots,u_n) = 0$, unless $\lambda$ has distinct parts. Indeed, suppose that $k = \lambda_i = \lambda_{i+1}$ for some $i\in\{1,\dots,N-1\}$. Let $\omega'\in\mathcal{P}^c_{\lambda/\mu}$ (see Definition 3.2.1). For $j = 1,\dots,n+1$ denote by $a_j$ the number of arrows from $(k,j-1)$ to $(k,j)$. As the number of horizontal arrows entering or leaving a given vertex is 0 or 1, we see that $a_{j+1}\in\{a_j,a_j-1,a_j+1\}$ for $j = 1,\dots,n$. Our assumption on $\lambda$ and $\mu$ implies that $a_1\le 1$, while $a_{n+1}\ge 2$, thus for some $j\in\{1,\dots,n\}$ we must have $a_j = 1$ and $a_{j+1} = 2$. Consequently, any $\omega'\in\mathcal{P}^c_{\lambda/\mu}$ contains a vertex of the form $(1,1;2,0)$. By (3.2.2) the conjugated weight of such a vertex equals $w^c_u(1,1;2,0) = \frac{1-s^2q}{1-su} = 0$ if $s = q^{-1/2}$. We conclude that $\mathcal{W}^c(\omega') = 0$ for any $\omega'\in\mathcal{P}^c_{\lambda/\mu}$, which by Definition 3.2.1 implies $\mathsf{G}^c_{\lambda/\mu}(u_1,\dots,u_n) = 0$.

*Remark* 3.2.12. A similar argument to the one presented in Remark 3.2.11 shows that $s = q^{-1/2}$ has the effect that if $\mu\in\mathsf{Sign}^+_N$, $\lambda\in\mathsf{Sign}^+_{N+n}$ and $\lambda$ has distinct parts then $\mathsf{F}_{\lambda/\mu}(u_1,\dots,u_n) = 0$ unless $\mu$ has distinct parts.

We next investigate how the new choice of parameters affects the function $f$.

**Lemma 3.2.13.** *Suppose $J\ge N$, $q = (0,1)$, $s = q^{-1/2}$ and $\nu\in Sign^+_N$ with $\nu = 0^{n_0}1^{n_1}2^{n_2}\cdots$. Then for any $v\in(0,s^{-1})$ we have*

$$\begin{aligned}\mathsf{G}^c_\nu(v,qv,q^2,\dots,q^{J-1}v) = {}&\frac{(q;q)_N(-q)^{n_0-N}}{(q;q)_{n_0}}\frac{(sv;q)_{N-n_0}}{(sv;q)_N}\frac{1}{(sv^{-1};q^{-1})_{N-n_0}}\times\\ &(q^{J-N+n_0+1};q)_{N-n_0}(svq^J;q)_{n_0}\prod_{j=1}^{N-n_0}\left(\frac{1}{1-svq^{j-1}}\left(\frac{vq^{j-1}-s}{1-svq^{j-1}}\right)^{\nu_j}\right),\end{aligned}\tag{3.2.12}$$

*when $n_i\le 1$ for $i\ge 1$ and 0 otherwise.*

*Proof.* We begin by dropping the assumption that $s = q^{-1/2}$ and consider $\mathsf{G}^c_\nu(v, qv, q^2, \ldots, q^{J-1}v; s)$, where we record the dependence on $s$ in the notation. The latter is a *finite* sum of finite products of weights $w^c_{vq^j}$ and by continuity of the weights (see (3.2.2)) we have

$$\mathsf{G}^c_\nu(v, qv, q^2, \ldots, q^{J-1}v; q^{-1/2}) = \lim_{s \to q^{-1/2}} \mathsf{G}^c_\nu(v, qv, q^2, \ldots, q^{J-1}v; s).$$

Using Proposition 3.2.10 we have

$$\mathsf{G}^c_\nu(u, qu, \ldots, q^{J-1}u; s) = \prod_{k=1}^{\infty} \frac{(s^2;q)_{n_k}}{(q;q)_{n_k}} \cdot \frac{(q;q)_J (su;q)_{J+n_0} (q;q)_N}{(q;q)_{J-N+n_0} (su;q)_N (q;q)_{n_0} (su^{-1};q^{-1})_{N-n_0}} \times$$

$$\prod_{i=1}^{N-n_0} \left( \frac{1}{1 - sq^{i-1}u} \left( \frac{q^{i-1}u - s}{1 - sq^{i-1}u} \right)^{\nu_j} \right) \cdot \frac{1}{(svq^{N-n_0};q)_{J-N+n_0}}.$$

Some cancellations and rearrangements (see also the proof of Proposition 6.7 in [33]) give

$$\mathsf{G}^c_\nu(v, qv, q^2, \ldots, q^{J-1}v; s) = \frac{(q;q)_N}{(q;q)_{n_0}} \prod_{k=1}^{\infty} \frac{(s^2;q)_{n_k}}{(q;q)_{n_k}} \frac{1}{(sv;q)_N} \frac{1}{(sv^{-1};q^{-1})_{N-n_0}} \times$$

$$(q^{J-M+n_0+1};q)_{M-n_0} (sv;q)_{N-n_0} (svq^J;q)_{n_0} \prod_{j=1}^{N-n_0} \left( \frac{1}{1 - svq^{j-1}} \left( \frac{vq^{j-1} - s}{1 - svq^{j-1}} \right)^{\nu_j} \right).$$

Finally, letting $s \to q^{-1/2}$ we see that $\prod_{k=1}^{\infty} (s^2;q)_{n_k} \to 0$ unless $n_k \leq 1$ for all $k \geq 1$, i.e. unless the non-zero parts of $\nu$ are all distinct. If $n_k \leq 1$ for all $k \geq 1$, then $\prod_{k=1}^{\infty} \frac{(s^2;q)_{n_k}}{(q;q)_{n_k}} \to (-q)^{n_0 - N}$, which proves the lemma. □

Let us denote $q^J$ by $X$. Then, in view of Lemma 3.2.13, $f$ becomes

$$f_\epsilon(\lambda; \mathbf{v}, X) := \sum_{\nu \in \mathsf{Sign}^+_N} \frac{(q;q)_N (-q)^{n_0 - N} (s\epsilon;q)_{N-n_0}}{(q;q)_{n_0} (s\epsilon;q)_N (s\epsilon^{-1};q^{-1})_{N-n_0}} (Xq^{-N+n_0+1};q)_{N-n_0} (s\epsilon X;q)_{n_0} \times$$

$$\prod_{i=1}^{\infty} \mathbf{1}_{\{n_i \leq 1\}} \prod_{j=1}^{N-n_0} \left( \frac{1}{1 - s\epsilon q^{j-1}} \left( \frac{\epsilon q^{j-1} - s}{1 - s\epsilon q^{j-1}} \right)^{\nu_j} \right) \mathsf{G}^c_{\lambda/\nu}(v_1, \ldots, v_M), \tag{3.2.13}$$

where $\nu = 0^{n_0} 1^{n_1} 2^{n_2} \cdots$ and $\mathbf{1}_E$ is the indicator function of an event $E$. In addition, specializing our $w$ variables in $Z^f(\mathbf{u}; \mathbf{w})$ and replacing $q^J$ with $X$, we get

$$Z^{f_\epsilon}(\mathbf{u}; \mathbf{v}, X) := (q;q)_N \prod_{i=1}^{N} \left( \frac{1}{1 - su_i} \frac{1 - X\epsilon u_i}{1 - \epsilon u_i} \prod_{j=1}^{M} \frac{1 - qu_iv_j}{1 - u_iv_j} \right).$$

Our earlier results now yield

$$\sum_{\omega\in\mathcal{P}'_N}\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j))\times f_\epsilon(\lambda^N(\omega);\mathbf{v},X) = (q;q)_N\prod_{i=1}^{N}\left(\frac{1}{1-su_i}\frac{1-X\epsilon u_i}{1-\epsilon u_i}\prod_{j=1}^{M}\frac{1-qu_iv_j}{1-u_iv_j}\right), \tag{3.2.14}$$

provided $\epsilon$ is sufficiently small and $X = q^J$ with $J \geq N$.

In view of Lemma 3.2.13, we have that $f_\epsilon(\lambda;\mathbf{v},X)$ is a *polynomial* in $X$. Moreover, one readily observes that as $X$ varies over compact sets in $\mathbb{C}$ the weights $\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j))\times f(\lambda^N(\omega);\mathbf{v},X)$ are absolutely summable (this is a consequence of the admissibility conditions and our choice for $\epsilon$). Hence the LHS of (3.2.14) is an entire function in $X$. The RHS of (3.2.14) is also clearly entire in $X$ and the two sides agree whenever $X = q^J$ with $J \geq N$. Since $q^J$ is a sequence with a limit point in $\mathbb{C}$, we conclude that (3.2.14) holds for all $X$ and we will set $X = (s\epsilon)^{-1}$.

When we subsitute $X = (s\epsilon)^{-1}$ in the expression for $f_\epsilon(\lambda;\mathbf{v},X)$ we see that the factor $(s\epsilon X;q)_{n_0}$ vanishes unless $n_0 = 0$, in which case it equals 1. Denoting $f_\epsilon(\lambda;\mathbf{v},(s\epsilon)^{-1})$ by $f_\epsilon(\lambda;\mathbf{v})$ we thus obtain

$$f_\epsilon(\lambda;\mathbf{v}) = \sum_{\nu\in\mathsf{Sign}_N^+}\frac{(q;q)_N(-q)^{n_0-N}(s\epsilon;q)_{N-n_0}}{(q;q)_{n_0}(s\epsilon;q)_N(s\epsilon^{-1};q^{-1})_{N-n_0}}((s\epsilon)^{-1}q^{-N+n_0+1};q)_{N-n_0}\times$$

$$\mathbf{1}_{\{n_0=0\}}\prod_{i=1}^{\infty}\mathbf{1}_{\{n_i\leq 1\}}\prod_{j=1}^{N-n_0}\left(\frac{1}{1-s\epsilon q^{j-1}}\left(\frac{\epsilon q^{j-1}-s}{1-s\epsilon q^{j-1}}\right)^{\nu_j}\right)\mathsf{G}^c_{\lambda/\nu}(v_1,\ldots,v_M),$$

and equation (3.2.14) takes the form

$$\sum_{\omega\in\mathcal{P}'_N}\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j))\times f_\epsilon(\lambda^N(\omega);\mathbf{v}) = (q;q)_N\prod_{i=1}^{N}\left(\frac{1}{1-su_i}\frac{1-s^{-1}u_i}{1-\epsilon u_i}\prod_{j=1}^{M}\frac{1-qu_iv_j}{1-u_iv_j}\right). \tag{3.2.15}$$

Since $\mathsf{G}^c_{\lambda/\nu}(v_1,\ldots,v_M) = 0$ unless $\lambda_i \geq \nu_i$ for $i = 1,\ldots,N$, we conclude that the sum, defining $f_\epsilon(\lambda;\mathbf{v})$ is finite and taking the limit as $\epsilon$ goes to zero we have

$$f(\lambda;\mathbf{v},\rho) := \lim_{\epsilon\to 0} f_\epsilon(\lambda;\mathbf{v}) = (-1)^N(q;q)_N\sum_{\nu\in\mathsf{Sign}_N^+}\mathbf{1}_{\{n_0=0\}}\prod_{i=1}^{\infty}\mathbf{1}_{\{n_i\leq 1\}}\prod_{j=1}^{N}(-s)^{\nu_j}\,\mathsf{G}^c_{\lambda/\nu}(v_1,\ldots,v_M), \tag{3.2.16}$$

where we used that $s^2 = q^{-1}$. Taking the limit as $\epsilon \to 0^+$ in equation (3.2.15) we conclude

$$\sum_{\omega\in\mathcal{P}'_N}\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j))\times f(\lambda^N(\omega);\mathbf{v},\rho) = (q;q)_N\prod_{i=1}^{N}\left(\frac{1-s^{-1}u_i}{1-su_i}\prod_{j=1}^{M}\frac{1-qu_iv_j}{1-u_iv_j}\right). \tag{3.2.17}$$

The change of the order of the limit and the sum is justified, because $u_i$ and $v_j$ are admissible for $i = 1,\ldots,N$ and $j = 1,\ldots,M$, and $u_i > s$.

With $f(\lambda; \mathbf{v}, \rho)$ given by (3.2.16), we define the following weight of a collection of paths in $\mathcal{P}'_N$

$$\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega) = \prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j)) \times f(\lambda^N(\omega); \mathbf{v}, \rho). \tag{3.2.18}$$

So far, we only know that $\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega)$ are finite real numbers, which are absolutely summable and their sum equals the RHS of (3.2.17). We will show below that $\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega) = 0$ unless $\omega \in \mathcal{P}_N$, in which case it is non-negative. This will show that one can define an honest probability measure on $\mathcal{P}_N$ through these weights.

We first investigate when such a weight vanishes. Since $\mathsf{G}^c_{\lambda/\nu}(v_1, \dots, v_M)$ vanishes unless $\lambda_i \geq \nu_i$ for $i = 1, \dots, N$, we see that $f(\lambda; \mathbf{v}, \rho) = 0$ unless $\lambda_N > 0$. Combining this with Remark 3.2.11, we see that $f(\lambda; \mathbf{v}, \rho) = 0$ unless $\lambda^N(\omega)$ has all distinct and positive parts. Let $\omega \in \mathcal{P}'_N$, be such that $\lambda^N(\omega)$ has distinct and non-zero parts. Using that

$$\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j)) = \mathsf{F}_{\lambda^1(\omega)}(u_1)\mathsf{F}_{\lambda^2(\omega)/\lambda^1(\omega)}(u_2)\cdots \mathsf{F}_{\lambda^N(\omega)/\lambda^{N-1}(\omega)}(u_N),$$

together with Remark 3.2.12, we conclude that $\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega) = 0$ unless $\lambda^i$ have distinct parts for all $i = 1, \dots, N$, i.e. unless $\omega \in \mathcal{P}_N$.

We next investigate the sign of $\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega)$. Since the weight is 0 otherwise, we may assume that $\omega \in \mathcal{P}_N$. Hence we have six possible choices for the vertices $\omega(i,j)$: $(0,0;0,0)$, $(0,1;1,0)$, $(1,0;1,0)$, $(0,1;1,0)$, $(0,1;0,1)$ and $(1,1;1,1)$. Using the formulas in (3.2.1) we see that the sign of the weight of a vertex is precisely $(-1)^{j_1}$. Consequently, the sign of $\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j))$ is precisely $(-1)^{K(\omega)}$, where $K(\omega)$ is the number of horizontal arrows in the configuration $\omega$. One readily observes that the number of horizontal arrows in $D_N$ is precisely $\sum_{i=1}^{N} \lambda_i^N(\omega)$. In addition, we have $N$ horizontal arrows from $(-1,i)$ to $(0,i)$ for $i = 1, \dots, N$. Thus we conclude that $sign\left(\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j))\right) = (-1)^{N+\sum_{i=1}^{N}\lambda_i^N(\omega)}$.

We next consider the sign of $\mathcal{W}^c(\omega')$, where $\omega' \in \mathcal{P}^c_{\lambda/\nu}$ and $\nu$ has distinct and positive parts. Arguing as in Remark 3.2.11, we can assume that no paths in $\omega'$ share a vertical piece, otherwise $\mathcal{W}^c(\omega') = 0$. Consequently, we may assume that $\omega'(i,j)$ is among the six vertex types we had before for all $(i,j) \in D_M$. From (3.2.2) the sign of the conjugated weight of a vertex is again $(-1)^{j_1}$, and so the sign equals $(-1)^{K(\omega')}$, where $K(\omega')$ is the total number of horizontal arrows in $\omega'$. One readily observes that $K(\omega') = \sum_{i=1}^{N}\lambda_i - \sum_{i=1}^{N}\nu_i$ (notice that in this case we do not have horizontal arrows entering the 0-th column). We conclude that all weights $\mathcal{W}^c(\omega')$ for $\omega' \in \mathcal{P}^c_{\lambda/\nu}$ have the same sign, which implies that $sign\left(\mathsf{G}^c_{\lambda/\nu}(v_1, \dots, v_M)\right) = (-1)^{\sum_{i=1}^{N}\lambda_i - \sum_{i=1}^{N}\nu_i}$.

The last paragraph implies that each summand in (3.2.16) has sign $(-1)^{\sum_{i=1}^{N}\lambda_i}$ and so we conclude that $sign\left(f(\lambda; \mathbf{v}, \rho)\right) = (-1)^{N+\sum_{i=1}^{N}\lambda_i}$. Since $sign\left(\prod_{i=1}^{N}\prod_{j=1}^{\infty} w_{u_i}(\omega(i,j))\right)$ $= (-1)^{N+\sum_{i=1}^{N}\lambda_i^N(\omega)}$, we conclude that $\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega) \geq 0$ for all $\omega \in \mathcal{P}_N$. As $\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega) = 0$ for

$\omega \in \mathcal{P}'_N/\mathcal{P}_N$, equation (3.2.17) can be rewritten as

$$\sum_{\omega \in \mathcal{P}_N} \mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega) = (q;q)_N \prod_{i=1}^{N} \left( \frac{1 - s^{-1}u_i}{1 - su_i} \prod_{j=1}^{M} \frac{1 - qu_iv_j}{1 - u_iv_j} \right) =: Z^f(\mathbf{u}). \tag{3.2.19}$$

As weights are non-negative and the partition function $Z^f(\mathbf{u})$ is positive and finite, we see that

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(\omega) := \frac{\mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega)}{Z^f(\mathbf{u})},$$

defines an honest probability measure on $\mathcal{P}_N$. For future reference we summarize the parameter choices we have made in the following definition.

**Definition 3.2.14.** Let $N, M \in \mathbb{N}$. We fix $q \in (0,1)$ and $s = q^{-1/2}$, $\mathbf{u} = (u_1, \dots, u_N)$ with $u_i > s$ and $\mathbf{v} = (v_1, \dots, v_M)$ with $v_j > 0$, and $\max_{i,j} u_iv_j < 1$. With these parameters, we denote $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ to be the probability measure on $\mathcal{P}_N$, defined above.

### 3.2.3 Projections of $\mathbb{P}_{\mathbf{u},\mathbf{v}}$

We assume the same notation as in the previous section. Let us fix $k \in \mathbb{N}$, $1 \leq m_1 < m_2 < \dots < m_k \leq N$ and $\mu^{m_i} \in \mathsf{Sign}^+_{m_i}$. Our goal in this section is to derive formulas for the following probabilities

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}\left(\lambda^{m_1}(\omega) = \mu^{m_1}, \dots, \lambda^{m_k}(\omega) = \mu^{m_k}\right).$$

Let $A = \{\omega \in \mathcal{P}_N : \lambda^{m_1}(\omega) = \mu^{m_1}, \dots, \lambda^{m_k}(\omega) = \mu^{m_k}\}$. Then we have that

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(A) = Z^f(\mathbf{u})^{-1} \sum_{\omega \in A} \mathcal{W}^f_{\mathbf{u},\mathbf{v}}(\omega) = Z^f(\mathbf{u})^{-1} \sum_{\omega \in A} \prod_{i=1}^{N} \prod_{j=1}^{\infty} w_{u_i}(\omega(i,j)) \times f(\lambda^N(\omega); \mathbf{v}, \rho) =$$

$$= Z^f(\mathbf{u})^{-1} \sum_{\omega \in A} \mathsf{F}_{\lambda^1(\omega)}(u_1) \mathsf{F}_{\lambda^2(\omega)/\lambda^1(\omega)}(u_2) \cdots \mathsf{F}_{\lambda^N(\omega)/\lambda^{N-1}(\omega)}(u_N) f(\lambda^N(\omega); \mathbf{v}, \rho).$$

Let $M = \{m_1, \dots, m_k\}$. We observe that the rightmost sum above may be replaced with the sum over all $\lambda^i \in \mathsf{Sign}^+_i$, where $i \in \{1, \dots, N\}/M$. Indeed, from our work in the previous section, the extra terms that we are summing over are all 0. We thus conclude that

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(A) = Z^f(\mathbf{u})^{-1} \sum_{i \in \{1,\dots,N\}/M} \sum_{\lambda^i \in \mathsf{Sign}^+_i} \mathsf{F}_{\lambda^1}(u_1) \mathsf{F}_{\lambda^2/\lambda^1}(u_2) \cdots \mathsf{F}_{\lambda^N/\lambda^{N-1}}(u_N) f(\lambda^N; \mathbf{v}, \rho), \tag{3.2.20}$$

where $\lambda^j = \mu^j$ for $j \in M$ are fixed.

Let us first assume that $m_k = N$. Then the branching relations (3.2.4) yield

$$\sum_{i=m_r+1}^{m_{r+1}-1} \sum_{\lambda^i \in \mathsf{Sign}^+_i} \mathsf{F}_{\lambda^{m_r+1}/\lambda^{m_r}}(u_{m_r+1}) \mathsf{F}_{\lambda^{m_r+2}/\lambda^{m_r+1}}(u_{m_r+2}) \cdots \mathsf{F}_{\lambda^{m_{r+1}}/\lambda^{m_{r+1}-1}}(u_{m_{r+1}}) =$$

$$\mathsf{F}_{\lambda^{m_{r+1}}/\lambda^{m_r}}(u_{m_r+1}, \dots, u_{m_{r+1}}),$$

when $r = 0, \dots, k-1$ with the convention that $m_0 = 0$ and $\lambda^0 = \varnothing$. Substituting these expressions in (3.2.20) we see that if $m_k = N$ we have

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(A) = Z^f(\mathbf{u})^{-1} \times \prod_{r=0}^{k-1} \mathsf{F}_{\mu^{m_{r+1}}/\mu^{m_r}}(u_{m_r+1}, \dots, u_{m_{r+1}}) \times f(\mu^{m_k}; \mathbf{v}, \rho).$$

In the remainder, we assume that $m_k < N$. In this case we may still apply the branching relations as above to conclude that

$$\begin{aligned} &\mathbb{P}_{\mathbf{u},\mathbf{v}}(A) = Z^f(\mathbf{u})^{-1} \times \prod_{r=0}^{k-1} \mathsf{F}_{\mu^{m_{r+1}}/\mu^{m_r}}(u_{m_r+1}, \dots, u_{m_{r+1}}) \times F(\mu^{m_k}; \mathbf{v}, \rho), \text{ where} \\ &F(\mu^{m_k}; \mathbf{v}, \rho) := \sum_{\lambda \in \mathsf{Sign}_N^+} \mathsf{F}_{\lambda/\mu^{m_k}}(u_{m_k+1}, \dots, u_N) f(\lambda; \mathbf{v}, \rho). \end{aligned} \tag{3.2.21}$$

An alternative formula for $F(\mu^{m_k}; \mathbf{v}, \rho)$ is derived in the following lemma.

**Lemma 3.2.15.** *Let $N, m \in \mathbb{N}$, $q \in (0,1)$, $s = q^{-1/2}$, $\lambda \in \mathsf{Sign}_N^+$ and $\mu \in \mathsf{Sign}_{m-1}^+$. Assume $u_m, \dots, u_N$ and $v_1, \dots, v_M$ are positive, $u_i > s$ and $u_i, v_j$ admissible with respect to $s$. Then*

$$\sum_{\lambda \in \mathsf{Sign}_N^+} \mathsf{F}_{\lambda/\mu}(u_m, \dots, u_N) f(\lambda; \mathbf{v}, \rho) = \prod_{i=m}^{N} \frac{(1-q^i)(1-s^{-1}u_i)}{1-su_i} \prod_{j=1}^{M} \frac{1-qu_iv_j}{1-u_iv_j} f(\mu; \mathbf{v}, \rho). \tag{3.2.22}$$

*Proof.* We start by considering the expression

$$\sum_{\lambda \in \mathsf{Sign}_N^+} \mathsf{F}_{\lambda/\mu}(u_{m+1}, \dots, u_N) \mathsf{G}_\lambda^c(w_1, \dots, w_{J+M}),$$

where as in the previous section $w_i = \epsilon q^{i-1}$ for $i = 1, \dots, J$ and $w_{j+J} = v_j$ for $j = 1, \dots, M$. The skew Cauchy identity in (3.2.6) yields (see also Corollary 4.11 in [33]):

$$\sum_{\lambda \in \mathsf{Sign}_N^+} \mathsf{F}_{\lambda/\mu}(u_m, \dots, u_N) \mathsf{G}_\lambda^c(w_1, \dots, w_{J+M}) = \prod_{i=m}^{N} \frac{1-q^i}{1-su_i} \prod_{j=1}^{J+M} \frac{1-qu_iw_j}{1-u_iw_j} \mathsf{G}_\mu^c(w_1, \dots, w_{J+M}).$$

Substituting $w_i$ in the above expression and denoting $q^J$ by $X$ we arrive at

$$\sum_{\lambda \in \mathsf{Sign}_N^+} \mathsf{F}_{\lambda/\mu}(u_m, \dots, u_N) f_\epsilon(\lambda; \mathbf{v}, X) = \prod_{i=m}^{N} \frac{1-q^i}{1-su_i} \frac{1-\epsilon X u_i}{1-\epsilon u_i} \prod_{j=1}^{M} \frac{1-qu_iv_j}{1-u_iv_j} f_\epsilon(\mu; \mathbf{v}, X), \tag{3.2.23}$$

where $f_\epsilon(\mu; \mathbf{v}, X)$ is given in (3.2.13). As in the previous section we argue that both sides of (3.2.23) are entire functions in $X$, which are equal on a sequence with a limit point in $\mathbb{C}$, hence equality holds for all $X$. If we set $X = (s\epsilon)^{-1}$ and let $\epsilon$ go to zero we get (3.2.22). □

Substituting $F(\mu^{m_k}; \mathbf{v}, \rho)$ into (3.2.21) with the expression in (3.2.22) and performing a bit of cancellations we see that

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}\left(\lambda^{m_1}(\omega)=\mu^{m_1},\ldots,\lambda^{m_k}(\omega)=\mu^{m_k}\right)=\prod_{r=0}^{k-1}\mathsf{F}_{\mu^{m_{r+1}}/\mu^{m_r}}(u_{m_r+1},\ldots,u_{m_{r+1}})\times f(\mu^{m_k};\mathbf{v},\rho)\times$$
$$Z^f(\mathbf{u},\mathbf{v};m_k)^{-1},\ \text{where } Z^f(\mathbf{u},\mathbf{v};m_k)=(q;q)_{m_k}\prod_{i=1}^{m_k}\left(\frac{1-s^{-1}u_i}{1-su_i}\prod_{j=1}^{M}\frac{1-qu_iv_j}{1-u_iv_j}\right). \tag{3.2.24}$$

## 3.3 The operators $D_m^k$

In this section we fix a positive integer $m \geq 1$ and provide operators $D_m^k$ for $k = 1, \ldots, m$ that act diagonally on the functions $\mathsf{F}_\lambda(u_1, \ldots, u_m)$ with $\lambda \in \mathsf{Sign}_m^+$ and $\lambda_i > \lambda_{i+1}$ for $i = 1, \ldots, m-1$. Specifically, we will show that

$$D_m^k\mathsf{F}_\lambda(u_1,\ldots,u_m)=\mathbf{1}_{\{\lambda_m=0,\lambda_{m-1}=1,\ldots,\lambda_{m-k+1}=k-1\}}\mathsf{F}_\lambda(u_1,\ldots,u_m).$$

In addition, we explain how the operators $D_m^k$ can be used to extract formulas for a set of observables and prove several properties that are relevant to the problem we consider.

### 3.3.1 Definition of $D_m^k$

We start with the symmetrization formula for $\mathsf{F}_\lambda(u_1, \ldots, u_m)$ (here $\lambda \in \mathsf{Sign}_m^+$), given in Proposition 3.2.9:

$$\mathsf{F}_\lambda(u_1,\ldots,u_m)=\frac{(1-q)^m}{\prod_{i=1}^m(1-su_i)}\sum_{\sigma\in S_m}\prod_{1\leq\alpha<\beta\leq m}\frac{u_{\sigma(\alpha)}-qu_{\sigma(\beta)}}{u_{\sigma(\alpha)}-u_{\sigma(\beta)}}\prod_{i=1}^m\left(\frac{u_{\sigma(i)}-s}{1-su_{\sigma(i)}}\right)^{\lambda_i}. \tag{3.3.1}$$

We are interested in setting $u_m = \cdots = u_{m-k+1} = s$ for each $k \in \{0, \ldots, m\}$ in the above expression, which is the content of the following lemma.

**Lemma 3.3.1.** *Let $m \geq 1$, $k \in \{0, \ldots, m\}$ and $\lambda \in \mathsf{Sign}_m^+$ with $\lambda_i > \lambda_{i+1}$ for $i = 1, \ldots, m-1$. Then we have that*

$$\mathsf{F}_\lambda(u_1,\ldots,u_{m-k},s,\ldots,s)=\frac{1}{(1-s^2)^k}\frac{(1-q)^m}{\prod_{i=1}^{m-k}(1-su_i)}\left(\frac{s(1-q)}{1-s^2}\right)^{k(k-1)/2}\times$$
$$\left(\prod_{i=1}^{m-k}\frac{u_i-qs}{1-su_i}\right)^k\sum_{\sigma\in S_{m-k}}\prod_{1\leq\alpha<\beta\leq m-k}\frac{u_{\sigma(\alpha)}-qu_{\sigma(\beta)}}{u_{\sigma(\alpha)}-u_{\sigma(\beta)}}\prod_{i=1}^{m-k}\left(\frac{u_{\sigma(i)}-s}{1-su_{\sigma(i)}}\right)^{\lambda_i-k}, \tag{3.3.2}$$

*if $\lambda_m = 0$, $\lambda_{m-1} = 1, \ldots, \lambda_{m-k+1} = k-1$ (if $k = 0$ this condition is empty). Otherwise $\mathsf{F}_\lambda(u_1, \ldots, u_{m-k}, s, \ldots, s) = 0$. If $k = m$ the sum over $S_{m-k}$ is replaced by 1.*

*Proof.* We proceed by induction on $k$ with base case $k = 0$ true by (3.3.1). Supposing the result for $k$ we now show it for $k + 1$.

By induction hypothesis we may assume that $\lambda_m = 0$, $\lambda_{m-1} = 1, \ldots, \lambda_{m-k+1} = k-1$, for otherwise the expression is 0 for all $u_{m-k}$ in particular for $u_{m-k} = s$ and there is nothing to prove. Consequently, we have that

$$\mathsf{F}_\lambda(u_1, \dots, u_{m-k}, s, \dots, s) = \frac{1}{(1-s^2)^k} \frac{(1-q)^m}{\prod_{i=1}^{m-k}(1-su_i)} \left(\frac{s(1-q)}{1-s^2}\right)^{k(k-1)/2} \times$$

$$\left(\prod_{i=1}^{m-k} \frac{u_i - qs}{1-su_i}\right)^k \sum_{\sigma \in S_{m-k}} \prod_{1 \le \alpha < \beta \le m-k} \frac{u_{\sigma(\alpha)} - qu_{\sigma(\beta)}}{u_{\sigma(\alpha)} - u_{\sigma(\beta)}} \prod_{i=1}^{m-k} \left(\frac{u_{\sigma(i)} - s}{1 - su_{\sigma(i)}}\right)^{\lambda_i - k}.$$

Since $\lambda_i > \lambda_{i+1}$ we know $\lambda_{m-k} \ge \lambda_{m-k+1} + 1 = k$. We notice that $(u_{m-k} - s)^{\lambda_{m-k} - k}$ divides each summand and so the total sum will be 0 unless $\lambda_{m-k} = k$. Let us assume that $\lambda_{m-k} = k$, which means $\lambda_i > k$ for $i < m-k$. The latter implies that each summand for which $\sigma(m-k) \ne m-k$ is divisible by $(u_{m-k} - s)$ and so vanishes when $u_{m-k} = s$. This reduces the sum over $S_{m-k}$ to a sum over $S_{m-k-1}$ and if we substitute $u_{m-k} = s$ we see that

$$\mathsf{F}_\lambda(u_1, \dots, u_{m-k-1}, s, s, \dots, s) = \frac{1}{(1-s^2)^{k+1}} \frac{(1-q)^m}{\prod_{i=1}^{m-k-1}(1-su_i)} \left(\frac{s(1-q)}{1-s^2}\right)^{k(k-1)/2} \left(\frac{s-qs}{1-s^2}\right)^k \times$$

$$\left(\prod_{i=1}^{m-k-1} \frac{u_i - qs}{1-su_i}\right)^k \sum_{\sigma \in S_{m-k-1}} \prod_{1 \le \alpha < \beta \le m-k-1} \frac{u_{\sigma(\alpha)} - qu_{\sigma(\beta)}}{u_{\sigma(\alpha)} - u_{\sigma(\beta)}} \prod_{i=1}^{m-k-1} \frac{u_{\sigma(i)} - qs}{u_{\sigma(i)} - s} \prod_{i=1}^{m-k-1} \left(\frac{u_{\sigma(i)} - s}{1 - su_{\sigma(i)}}\right)^{\lambda_i - k}.$$

Upon rearrangement the above equals the expression in (3.3.2) with $k+1$. The general result now proceeds by induction on $k$. □

Put $M = \{1, \dots, m\}$. We record the following alternative representation of $\mathsf{F}_\lambda(u_1, \dots, u_m)$, which can be obtained from (3.3.1) by splitting the sum over the possible variable subsets formed by $\{u_{\sigma(m)}, \dots, u_{\sigma(m-k+1)}\}$ (these correspond to sets $I$ below and $I^c = M/I$)

$$\mathsf{F}_\lambda(u_1, \dots, u_m) = \frac{(1-q)^m}{\prod_{i=1}^m (1-su_i)} \sum_{\sigma \in S_k} \sum_{\tau \in S_{m-k}} \sum_{\substack{I = \{i_1, \dots, i_k\} \subset M \\ I^c = \{j_1, \dots, j_{m-k}\}}} \prod_{\alpha=1}^{k} \prod_{\beta=1}^{m-k} \frac{u_{j_{\tau(\beta)}} - qu_{i_{\sigma(\alpha)}}}{u_{j_{\tau(\beta)}} - u_{i_{\sigma(\alpha)}}} \times$$

$$\prod_{1 \le \alpha_1 < \alpha_2 \le k} \frac{u_{i_{\sigma(\alpha_1)}} - qu_{i_{\sigma(\alpha_2)}}}{u_{i_{\sigma(\alpha_1)}} - u_{i_{\sigma(\alpha_2)}}} \prod_{x=1}^{k} \left(\frac{u_{i_{\sigma(x)}} - s}{1 - su_{i_{\sigma(x)}}}\right)^{\lambda_{m-x+1}} \prod_{1 \le \beta_1 < \beta_2 \le m-k} \frac{u_{j_{\tau(\beta_1)}} - qu_{j_{\tau(\beta_2)}}}{u_{j_{\tau(\beta_1)}} - u_{j_{\tau(\beta_2)}}} \prod_{y=1}^{m-k} \left(\frac{u_{j_{\tau(y)}} - s}{1 - su_{i_{\tau(y)}}}\right)^{\lambda_y}. \tag{3.3.3}$$

We introduce some necessary notation. Define operators $T_{s,u_i}$ that act on functions of $m$ variables $(u_1, \dots, u_m)$, by setting $u_i$ to $s$. I.e.

$$T_{s,u_i} F(u_1, \dots, u_m) = F(u_1, \dots, u_{i-1}, s, u_{i+1}, \dots, u_m).$$

We also consider the function

$$F_k(u_1, \dots, u_k) = \frac{(1-q)^k}{\prod_{i=1}^k (1-su_i)} \sum_{\sigma \in S_k} \prod_{1 \le \alpha < \beta \le k} \frac{u_{\sigma(\alpha)} - qu_{\sigma(\beta)}}{u_{\sigma(\alpha)} - u_{\sigma(\beta)}} \prod_{i=1}^{k} \left(\frac{u_{\sigma(i)} - s}{1 - su_{\sigma(i)}}\right)^{i-1}.$$

Notice that $F_k(u_1,\ldots,u_k) = \mathsf{F}_\lambda(u_1,\ldots,u_k)$ with $\lambda = (k-1,k-2,\ldots,0)$. In particular, $F_k$ is a symmetric rational function.

Let $k \in M$ be fixed. For a subset $I \subset M$ with $I = \{i_1,\ldots,i_k\}$ we write $F(u_I)$ to mean $F(u_{i_1},\ldots u_{i_k})$, whenever $F$ is a symmetric function in $k$ variables. We also write $F_k(S)$ to be $F_k(s,s,\ldots,s)$ and from Lemma 3.3.1 we have

$$F_k(S) = s^{k(k-1)/2}\left(\frac{1-q}{1-s^2}\right)^{k(k+1)/2}.$$

With the above notation we define the following operators.

**Definition 3.3.2.** Let $m \in \mathbb{N}$ and $M = \{1,\ldots,m\}$. For $1 \le k \le m$ we define the operator $D_m^k$ on functions of $m$ variables to be

$$D_m^k := \sum_{I\subset M:|I|=k}\prod_{i\in I;j\notin I}\frac{u_j-qu_i}{u_j-u_i}\prod_{j\notin I}\left(\frac{u_j-s}{u_j-sq}\right)^k\frac{F_k(u_I)}{F_k(S)}\prod_{i\in I}T_{u_i,s}. \tag{3.3.4}$$

*Remark* 3.3.3. One readily observes that $D_m^k$ is a linear operator on the set of functions in $m$-variables, and also satisfies the property that if $f_r(u_1,\ldots,u_m)$ converge pointwise to $f(u_1,\ldots,u_m)$, then $D_m^k f_r$ converge pointwise to $D_m^k f$ away from the points $u_i = u_j$ for $i \neq j$.

The key property of $D_m^k$ is given in the following lemma.

**Lemma 3.3.4.** *Let $m \ge 1$, $k \in \{1,\ldots,m\}$ and $\lambda \in Sign_m^+$ with $\lambda_i > \lambda_{i+1}$ for $i = 1,\ldots,m-1$. Then we have that*

$$D_m^k\mathsf{F}_\lambda(u_1,\ldots,u_m) = \mathbf{1}_{\{\lambda_m=0,\lambda_{m-1}=1,\ldots,\lambda_{m-k+1}=k-1\}}\mathsf{F}_\lambda(u_1,\ldots,u_m). \tag{3.3.5}$$

*Proof.* Using Lemma 3.3.1 and that $\mathsf{F}_\lambda$ is symmetric we have that $D_m^k\mathsf{F}_\lambda(u_1,\ldots,u_m) = 0$ unless $\lambda_m = 0$, $\lambda_{m-1} = 1,\ldots,$ $\lambda_{m-k+1} = k-1$. We thus assume that $\lambda_m = 0$, $\lambda_{m-1} = 1,\ldots,$ $\lambda_{m-k+1} = k-1$. Let $\mu \in Sign_{m-k}^+$ be given by $\mu_i = \lambda_i - k$. It follows from Lemma 3.3.1 and (3.3.1) that

$$D_m^k\mathsf{F}_\lambda(u_1,\ldots,u_m) = \sum_{I\subset M:|I|=k}\prod_{i\in I;j\notin I}\frac{u_j-qu_i}{u_j-u_i}\prod_{j\notin I}\left(\frac{u_j-s}{u_j-sq}\right)^k F_k(u_I)\left(\prod_{j\notin I}\frac{u_j-qs}{1-su_j}\right)^k\mathsf{F}_\mu(u_{I^c})$$

$$= \sum_{I\subset M:|I|=k}\prod_{i\in I;j\notin I}\frac{u_j-qu_i}{u_j-u_i}F_k(u_I)\left(\prod_{j\notin I}\frac{u_j-s}{1-su_j}\right)^k\mathsf{F}_\mu(u_{I^c}),\text{ where } I^c = M/I.$$

Using Proposition 3.2.3 and (3.3.1) we can rewrite the above as

$$\frac{(1-q)^m}{\prod_{i=1}^m(1-su_i)}\sum_{\substack{I=\{i_1,\ldots,i_k\}\subset M\\ I^c=\{j_1,\ldots,j_{m-k}\}}}\sum_{\sigma\in S_k}\sum_{\tau\in S_{m-k}}\prod_{\alpha=1}^{k}\prod_{\beta=1}^{m-k}\frac{u_{j_{\tau(\beta)}}-qu_{i_{\sigma(\alpha)}}}{u_{j_{\tau(\beta)}}-u_{i_{\sigma(\alpha)}}}\times$$

$$\prod_{1\le\alpha_1<\alpha_2\le k}\frac{u_{i_{\sigma(\alpha_1)}}-qu_{i_{\sigma(\alpha_2)}}}{u_{i_{\sigma(\alpha_1)}}-u_{i_{\sigma(\alpha_2)}}}\prod_{x=1}^{k}\left(\frac{u_{i_{\sigma(x)}}-s}{1-su_{i_{\sigma(x)}}}\right)^{x-1}\prod_{1\le\beta_1<\beta_2\le m-k}\frac{u_{j_{\tau(\beta_1)}}-qu_{j_{\tau(\beta_2)}}}{u_{j_{\tau(\beta_1)}}-u_{j_{\tau(\beta_2)}}}\prod_{y=1}^{m-k}\left(\frac{u_{j_{\tau(y)}}-s}{1-su_{i_{\tau(y)}}}\right)^{\lambda_y}.$$

By virtue of (3.3.3) the latter is exactly $\mathsf{F}_\lambda(u_1, \dots, u_m)$ as desired. □

### 3.3.2 Observables from $D_m^k$

This section is devoted to explaining how one can use the operators $D_m^k$ to analyze the probability measures $\mathbb{P}^f$ on $\mathcal{P}_N$. These measures were discussed in the beginning of Section 3.1.1 and $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ is a particular example. In addition, we will prove an interesting property for the first operator $D_m^1$, which we believe to be of separate interest. Throughout this section we require that $q = s^{-2}$.

Let us summarize the assumptions we need to make the statements in this section valid.
**Assumptions**:

- $N \in \mathbb{N}$ and $u_1, \dots, u_N$ are pairwise distinct complex numbers;
- $f : \mathsf{Sign}_N^+ \to \mathbb{R}$ is supported on signatures with distinct parts;
- for $\omega \in \mathcal{P}_N$ we define the weights

$$\mathcal{W}^f(\omega; \mathbf{z}) := \prod_{i=0}^{\infty} \prod_{j=1}^{N} w_{z_j}(\omega(i,j)) \times f(\lambda^N(\omega)); \tag{3.3.6}$$

- the weights in (3.3.6) are absolutely summable in some neighborhood of the point $(u_1, \dots, u_N)$ and we denote

$$\sum_{\omega \in \mathcal{P}_N} \mathcal{W}^f(\omega; \mathbf{z}) =: Z^f(z_1, \dots, z_N) = Z^f(\mathbf{z});$$

- for every $\omega \in \mathcal{P}_N$ we have $\mathcal{W}^f(\omega; \mathbf{u}) \geq 0$ and $Z^f(\mathbf{u}) > 0$.

Notice that under the above assumptions $\mathbb{P}^f(\omega) := \frac{\mathcal{W}^f(\omega;\mathbf{u})}{Z^f(\mathbf{u})}$ is a probability measure on $\mathcal{P}_N$. For the remainder of this section we will work under the above assumptions.

Let us introduce the following definitions

**Definition 3.3.5.** For $m, r \geq 0$ we define

$$\mathsf{Sign}_m^* = \{\lambda \in \mathsf{Sign}_m^+ : \lambda_m < \lambda_{m-1} < \cdots < \lambda_1\}, \;\; \mathsf{Sign}_{m,r}^* = \{\lambda \in \mathsf{Sign}_m^* : \lambda_m = 0, \dots, \lambda_{m-r+1} = r-1\}.$$

Suppose $k \in \{1, \dots, N\}$ and $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq N$ are given. Set $S_i = \mathsf{Sign}_{m_i,i}^*$ for $i = 1, \dots, k$ and define

$$A(\mathbf{m}) = A(m_1, \dots, m_k) = \{\omega : \lambda^{m_i}(\omega) \in S_i, \; i = 1, \dots, k\} \text{ and } \mathcal{W}^f(\mathbf{m}; \mathbf{z}) = \sum_{\omega \in A(\mathbf{m})} \mathcal{W}^f(\omega; \mathbf{z}). \tag{3.3.7}$$

**Lemma 3.3.6.** *Assume the same notation as in Definition 3.3.5. Then we have*

$$\mathbb{P}^f(A(\mathbf{m})) = \frac{\mathcal{W}^f(\mathbf{m}; \mathbf{u})}{Z^f(\mathbf{u})} = \frac{(D_{m_1}^1 D_{m_2}^2 \cdots D_{m_k}^k Z^f)(\mathbf{u})}{Z^f(\mathbf{u})}. \tag{3.3.8}$$

We view Lemma 3.3.6 as one of the main results of this article. Under very mild conditions on the function $f$ it provides formulas for the observables $\mathbb{P}^f(A(\mathbf{m}))$, which form a large class of correlation functions that can be used to analyze the six-vertex model. In the context of this chapter (3.3.8) plays the role of a starting point for our asymptotic analysis, and we hope that it will be useful for studying other six-vertex models in the future.

*Proof.* Repeating some of the arguments from Section 3.2.3, we have that

$$\mathcal{W}^f(\mathbf{m};\mathbf{z}) = \sum_{\mu^k \in S_k} \sum_{\mu^{k-1}\in S_{k-1}} \cdots \sum_{\mu^1 \in S_1} \prod_{i=1}^{k} \mathsf{F}_{\mu^i/\mu^{i-1}}(z_{m_{i-1}+1},\dots,z_{m_i}) F(\mu^k), \text{ and}$$
$$Z^f(\mathbf{z}) = \sum_{\mu \in \mathsf{Sign}^*_{m_k}} \mathsf{F}_{\mu}(z_1,\dots,z_{m_k})F(\mu), \text{ where } F(\mu) = \sum_{\lambda\in \mathsf{Sign}^*_N} \mathsf{F}_{\lambda/\mu}(z_{m_k+1},\dots,z_N) f(\lambda). \tag{3.3.9}$$

The statement of the lemma will be produced if we apply $D^1_{m_1}D^2_{m_2}\cdots D^k_{m_k}$ (in the $z$-variables) to both sides of the second line of (3.3.9), set $\mathbf{z}=\mathbf{u}$ and divide by $Z^f(\mathbf{u})$. We provide the details below.

We start by applying $D^k_{m_k}$ to get

$$D^k_{m_k} \sum_{\mu\in \mathsf{Sign}^*_{m_k}} \mathsf{F}_\mu(z_1,\dots,z_{m_k})F(\mu) = \sum_{\mu\in \mathsf{Sign}^*_{m_k}} D^k_{m_k}\mathsf{F}_\mu(z_1,\dots,z_{m_k})F(\mu) = \sum_{\mu \in S_k} \mathsf{F}_\mu(z_1,\dots,z_{m_k})F(\mu).$$

The change of the order of the sum and the operator is allowed by the linearity of $D^k_{m_k}$ and the absolute convergence of the sum (see Remark 3.3.3), while the second equality follows from Lemma 3.3.4. We next use Proposition 3.2.5 and rewrite the above as

$$\sum_{\mu^k\in S_k} \sum_{\lambda \in \mathsf{Sign}^+_{m_{k-1}}} \mathsf{F}_\lambda(z_1,\dots,z_{m_{k-1}}) \mathsf{F}_{\mu^k/\lambda}(z_{m_{k-1}+1},\dots,z_{m_k})F(\mu^k). \tag{3.3.10}$$

If $\mu^k \in S_k$, we know that it has all distinct parts. The latter implies by Remark 3.2.12 that $\mathsf{F}_{\mu^k/\lambda}(z_{m_{k-1}+1},\dots,z_{m_k}) = 0$ unless $\lambda$ has distinct parts. Consequently we may rewrite (3.3.10) as

$$\sum_{\mu^k\in S_k} \sum_{\lambda \in \mathsf{Sign}^*_{m_{k-1}}} \mathsf{F}_\lambda(z_1,\dots,z_{m_{k-1}}) \mathsf{F}_{\mu^k/\lambda}(z_{m_{k-1}+1},\dots,z_{m_k})F(\mu^k). \tag{3.3.11}$$

Applying $D^{k-1}_{m_{k-1}}$ to (3.3.11), using its linearity and Lemma 3.3.4, we get

$$\sum_{\mu^k\in S_k} \sum_{\mu^{k-1} \in S_{k-1}} \mathsf{F}_{\mu^{k-1}}(z_1,\dots,z_{m_{k-1}}) \mathsf{F}_{\mu^k/\mu^{k-1}}(z_{m_{k-1}+1},\dots,z_{m_k})F(\mu^k). \tag{3.3.12}$$

Repeating the above argument for $k-2,\dots,1$, we see the result of applying $D^1_{m_1}D^2_{m_2}\cdots D^k_{m_k}$ to the RHS of (3.3.9) to be

$$\sum_{\mu^k \in S_k} \sum_{\mu^{k-1}\in S_{k-1}} \cdots \sum_{\mu^1 \in S_1} \prod_{i=1}^{k} \mathsf{F}_{\mu^i/\mu^{i-1}}(z_{m_{i-1}+1},\dots,z_{m_i}) F(\mu^k), \tag{3.3.13}$$

with the convention that $m_0 = 0$ and $\mu^0 = \varnothing$. From (3.3.9) the latter equals $\mathcal{W}^f(\mathbf{m};\mathbf{z})$ and so $(D^1_{m_1} D^2_{m_2} \cdots D^k_{m_k} Z^f)(\mathbf{u}) = \mathcal{W}^f(\mathbf{m};\mathbf{u})$. Dividing both sides by $Z^f(\mathbf{u})$ and recalling that $\mathbb{P}^f(A(\mathbf{m})) = Z^f(\mathbf{u})^{-1}\mathcal{W}^f(\mathbf{m};\mathbf{u})$ proves the lemma. □

In the remainder of this section, we explain how our first order operator $D^1_m$ can be used to derive an interesting recurrence relation for $\mathcal{W}^f(\mathbf{m};\mathbf{z})$ in terms of the same quantity for a system of fewer parameters. The exact statement is given in the following lemma.

**Lemma 3.3.7.** *Assume the same notation as in Definition 3.3.5. Let $\hat{\mathbf{m}} = (m_2-1,\dots,m_k-1)$ and $\mathbf{z}/\{z_i\}$ be the variable set $(z_1,\dots,z_{i-1},z_{i+1},\dots,z_N)$. Then we have*

$$\mathcal{W}^f(\mathbf{m};\mathbf{z}) = \prod_{j=m_1+1}^{N} \frac{z_j - sq}{1-sz_j} \sum_{i=1}^{m_1} \frac{1-q}{1-sz_i} \prod_{j=1,j\neq i}^{m_1} \left( \frac{z_j - qz_i}{z_j - z_i} \frac{z_j - s}{1-sz_j}\right) \mathcal{W}^g(\hat{\mathbf{m}};\mathbf{z}/\{z_i\}), \quad (3.3.14)$$

*where $g : \mathsf{Sign}^+_{N-1} \to \mathbb{R}$ is given by $g(\mu) = f(\mu + 1^{N-1})$ and $\lambda = \mu + 1^{N-1}$ is such that $\lambda_i = \mu_i + 1$ for $i \le N-1$ and $\lambda_N = 0$.*

This result will not be used in the remainder of the chapter, but we believe it to be of separate interest as we explain now. In order to use $\mathbb{P}^f(A(\mathbf{m}))$ to analyze a six-vertex model it is desirable to have closed formulas for these quantities. In this chapter we will work with a particular model, for which $Z^f$ has a product form. This will allow us to find contour integral formulas for the RHS of (3.3.8) as will be explained in the next section. For other boundary conditions; however, one might not be able to use (3.3.8) to derive formulas for $\mathbb{P}^f(A(\mathbf{m}))$ and a different approach needs to be taken. Having a recurrence relation for $\mathcal{W}^f(\mathbf{m};\mathbf{z})$ provides a possible route for finding closed formulas for these correlation functions. In the base case, which occurs when $k = 0$ or equivalently $\mathbf{m} = \varnothing$, we have that $\mathcal{W}^f(\varnothing;\mathbf{z}) = Z^f(\mathbf{z})$. If one has a closed formula for $Z^f(\mathbf{z})$ then (3.3.14) can be potentially used to guess a formula for $\mathcal{W}^f(\mathbf{m};\mathbf{z})$, by matching the base case and showing it satisfies the above recurrence relation. A similar approach was used in [38], where a determinant formula for $\mathcal{W}^f(\mathbf{m};\mathbf{z})$ was guessed for the six-vertex model with DWBC and shown to satisfy such a recurrence relation. The key point is that the recurrence relation we prove holds for general boundary conditions.

*Proof.* For $\mu \in \mathsf{Sign}^*_m$ we define $\hat{\mu} \in \mathsf{Sign}^*_{m-1}$ by $\hat{\mu}_i = \mu_i - 1$ for $i = 1,\dots,m-1$. We apply $D^1_{m_1}$ (in the $z$-variables) to both sides of the first line of (3.3.9) and get

$$D^1_{m_1}\mathcal{W}^f(\mathbf{m};\mathbf{z}) = \sum_{\mu^k\in S_k} \sum_{\mu^{k-1}\in S_{k-1}} \cdots \sum_{\mu^1 \in S_1} \prod_{i=2}^{k} \mathsf{F}_{\mu^i/\mu^{i-1}}(z_{m_{i-1}+1},\dots,z_{m_i}) F(\mu^k) D^1_{m_1}\mathsf{F}_{\mu^1}(z_1,\dots,z_{m_1}).$$

In obtaining the above we used the linearity of $D^1_{m_1}$ and the convergence of the sum to change the order of the sum and operator. Using that $D^1_{m_1}\mathsf{F}_{\mu^1}(z_1,\dots,z_{m_1}) = \mathsf{F}_{\mu^1}(z_1,\dots,z_{m_1})$ whenever $\mu^1 \in S_1$, we deduce

$$D^1_{m_1}\mathcal{W}^f(\mathbf{m};\mathbf{z}) = \mathcal{W}^f(\mathbf{m};\mathbf{z}). \quad (3.3.15)$$

On the other hand, using the definition of $D^1_{m_1}$ and Lemma 3.3.1, we have

$$D^1_{m_1}\mathsf{F}_{\mu^1}(z_1,\dots,z_{m_1}) = \sum_{i=1}^{m_1}\frac{1-q}{1-sz_i}\prod_{j=1,j\neq i}^{m_1}\left(\frac{z_j-qz_i}{z_j-z_i}\frac{z_j-s}{1-sz_j}\right)\mathsf{F}_{\hat{\mu}^1}(\mathbf{z}_{m_1}/\{z_i\}), \tag{3.3.16}$$

where $\mathbf{z}_{m_1}/\{z_i\}$ stands for the variable set $(z_1,\dots,z_{i-1},z_{i+1},\dots,z_{m_1})$. Replacing (3.3.16) in our earlier expression for $D^1_{m_1}\mathcal{W}^f(\mathbf{m};\mathbf{z})$ and utilizing (3.3.15) we conclude that

$$\mathcal{W}^f(\mathbf{m};\mathbf{z}) = \sum_{i=1}^{m_1}\frac{1-q}{1-sz_i}\prod_{j=1,j\neq i}^{m_1}\left(\frac{z_j-qz_i}{z_j-z_i}\frac{z_j-s}{1-sz_j}\right)\times$$
$$\sum_{\mu^k\in S_k}\sum_{\mu^{k-1}\in S_{k-1}}\cdots\sum_{\mu^1\in S_1}\prod_{i=2}^{k}\mathsf{F}_{\mu^i/\mu^{i-1}}(z_{m_{i-1}+1},\dots,z_{m_i})F(\mu^k)\mathsf{F}_{\hat{\mu}^1}(\mathbf{z}_{m_1}/\{z_i\}). \tag{3.3.17}$$

We notice from the definition of $\mathsf{F}_{\lambda/\mu}$ that for $\lambda\in\mathsf{Sign}^*_{b,1}$ and $\mu\in\mathsf{Sign}^*_{a,1}$ we have

$$\mathsf{F}_{\lambda/\mu}(z_{a+1},\dots,z_b) = \prod_{j=a+1}^{b}\frac{z_j-sq}{1-sz_j}\times\mathsf{F}_{\hat{\lambda}/\hat{\mu}}(z_{a+1},\dots,z_b).$$

Substituting this and the definition of $F$ in (3.3.17), we arrive at

$$\mathcal{W}^f(\mathbf{m};\mathbf{z}) = \prod_{j=m_1+1}^{N}\frac{z_j-sq}{1-sz_j}\sum_{i=1}^{m_1}\frac{1-q}{1-sz_i}\prod_{j=1,j\neq i}^{m_1}\left(\frac{z_j-qz_i}{z_j-z_i}\frac{z_j-s}{1-sz_j}\right)\sum_{\hat{\mu}^k\in T_{k-1}}\sum_{\hat{\mu}^{k-1}\in T_{k-2}}\cdots\sum_{\hat{\mu}^2\in T_1}$$
$$\sum_{\hat{\mu}^1\in\mathsf{Sign}^*_{m_1-1}}\prod_{i=2}^{k}\mathsf{F}_{\hat{\mu}^i/\hat{\mu}^{i-1}}(z_{m_{i-1}+1},\dots,z_{m_i})G(\hat{\mu}^k)\mathsf{F}_{\hat{\mu}^1}(\mathbf{z}_{m_1}/\{z_i\}).$$

Above $T_i=\mathsf{Sign}^*_{m_{i+1}-1,i}$ and $G(\hat{\mu})=\sum_{\lambda\in\mathsf{Sign}^*_N}\mathsf{F}_{\hat{\lambda}/\hat{\mu}}(z_{m_k+1},\dots,z_N)f(\lambda)$. Using the branching relations (3.2.4), (3.3.9) and the definition of $g$ we recognize the above identity as (3.3.14). ☐

*Remark* 3.3.8. So far in this chapter we have considered the vertically inhomogeneous six-vertex model; however, one can introduce horizontal inhomogeneities as well. A particular way to do this is given in [33], where the weights depend on an additional set $\Xi=\{\xi_j\}_{j=0,1,\dots}$ of inhomogeneity parameters (our model corresponds to setting $\xi_i=1$ for all $i$). We denote the partition function in this case by $\mathsf{F}_\lambda(u_1,\dots,u_m|\Xi)$ and refer the reader to (1.4) in [33] for the exact formula (the variables $s_x$ in that formula need to be set to $q^{-1/2}$). In a certain sense, one can interpret $D^k_m$ as acting on the first $k$ columns of the six-vertex model. If the first $k$ inhomogeneity parameters $\xi_0,\dots,\xi_{k-1}$ are all the same, then we can find an equivalent to Lemma 3.3.4, but in general no such extension seems possible. Let us explain how this can be done in the case $k=1$. If we set

$$D^1_m := \sum_{i=1}^{n}\prod_{j\neq i}\frac{u_j-qu_i}{u_j-u_i}\prod_{j\neq i}\left(\frac{u_j-s\xi_0^{-1}}{u_j-sq\xi_0^{-1}}\right)\frac{\phi_0(u_i)}{\phi_0(s\xi_0^{-1})}T_{u_i,s\xi_0^{-1}},$$

then one readily verifies, as done above, that $D^1_m \mathsf{F}_\lambda(u_1, \dots, u_m|\Xi) = \mathbf{1}_{\{\lambda_m = 0\}} \mathsf{F}_\lambda(u_1, \dots, u_m|\Xi)$, whenever $\lambda$ has distinct parts. The latter can be used to derive a recurrence relation for $\mathcal{W}^f(\mathbf{m}; \mathbf{u}|\Xi)$ in terms of $\mathcal{W}^g(\hat{\mathbf{m}}; \mathbf{u}/\{u_i\}|\tau_1\Xi)$ (here $\tau_1\Xi = \{\xi_j\}_{j=1,2,\dots}$), which generalizes (3.3.14). The proof is essentially the same as the one presented above.

*Remark* 3.3.9. In the case of the domain wall boundary condition for the six-vertex model, which corresponds to $f(\lambda) = \mathbf{1}_{\{\lambda_1 = N-1, \dots, \lambda_N = 0\}}$ above, the quantity $Z^f(\mathbf{u})^{-1}\mathcal{W}^f(\mathbf{m}; \mathbf{u})$ was investigated in [38] under the name generalized emptiness formation probability (GEFP). In this setting, (3.3.14) naturally corresponds to equation (3.6) of [38], which is the key ingredient in finding closed determinant formulas for the GEFP. The derivation of (3.6) in [38] is based on the quantum inverse scattering method, and we see that the operators $D^1_m$ (and their generalization outlined in Remark 3.3.8) provide an alternative route for establishing the recurrence relation.

### 3.3.3 Action on product functions

Equation (3.3.8) shows that understanding $\mathbb{P}^f(A(\mathbf{m}))$ requires knowledge of how $D^k_m$ act on the partition function $Z^f$. In this section, we will see that if $Z^f$ has a product form, then the action of the operators is relatively simple.

In the following sequence of lemmas we investigate how $D^1_{m_1} D^2_{m_2} \cdots D^k_{m_k}$ acts on a function $F(\mathbf{z})$ of the form $F(\mathbf{z}) = F(z_1, \dots, z_m) = \prod_{i=1}^m f(z_i)$.

**Lemma 3.3.10.** *Let $m \geq 1$ and $1 \leq k \leq m$ be given. Suppose that $q \in (0,1)$, $s > 1$, $u_1, \dots, u_m > s$ and $u_i \neq u_j$ when $i \neq j$. Let $f(z)$ be a holomorphic non-vanishing function in a neighborhood of an interval containing $s, u_1, \dots, u_m$. Put $F(\mathbf{z}) = F(z_1, \dots, z_m) = \prod_{i=1}^m f(z_i)$. Then we have that*

$$(D^k_m F)(\mathbf{u}) = F(\mathbf{u}) \cdot q^{\frac{-k(k-1)}{2}} \prod_{i=1}^m \left( \frac{u_i - s}{u_i - sq} \right)^k \frac{f(s)^k}{(2\pi\iota)^k k!} \times \\ \int_\gamma \cdots \int_\gamma \det \left[ \frac{1}{qz_i - z_j} \right]_{i,j=1}^k \frac{F_k(z_1, \dots, z_k)}{F_k(S)} \prod_{j=1}^k \left( \prod_{i=1}^m \frac{qz_j - u_i}{z_j - u_i} \right) \left( \frac{z_j - sq}{z_j - s} \right)^k \frac{dz_j}{f(z_j)}. \tag{3.3.18}$$

*The contour $\gamma$ is a positively oriented contour around the points $u_1, \dots, u_m$, and does not contain other singularities of the integrand. Such a contour will exist, provided $u_i$ are sufficiently close to each other.*

*Proof.* The proof is essentially the same as that of Proposition 2.11 in [24]. Firstly, we notice that the contours will always exist, provided $u_i$ are sufficiently close to each other. Indeed, the singularities of the integrand that are not singularities of $f$ are precisely at $u_i$, $s$, 0, $z_i = q^{-1}z_j$ and at $s^{-1}$ (the latter one is a singularity of $F_k$). Since $u_i$ are bounded away to the right from $s$ (and hence $s^{-1}$ and 0 ) and the function $f$ does not vanish in a neighborhood of an interval containing $u_i$ we may pick the contour $\gamma$ so as to exclude all singularities of the integrand, except possibly for $z_i = q^{-1}z_j$. However, if $u_i$ are sufficiently close then we can choose $\gamma$ to be a small circle around those points, which is disjoint from $q \cdot \gamma$. This excludes the remaining singularities.

We substitute in (3.3.18) the Cauchy determinant identity

$$\det\left[\frac{1}{z_i - qz_j}\right]_{i,j=1}^{k} = \frac{q^{\frac{k(k-1)}{2}}\prod_{1\le i<j\le k}(z_i - z_j)(z_j - z_i)}{\prod_{i,j=1}^{k}(qz_i - z_j)}$$

and calculate the residues at $z_j = u_{l_j}$. The Vandermonde determinants in the numerator prevent any of the $l_j$'s to be the same. If they are distinct and $I = \{l_1, \dots, l_k\}$ one calculates the residue to be

$$\frac{1}{k!}\cdot\prod_{j\notin I}\left(\frac{u_j - s}{u_j - sq}\right)^k \frac{F_k(u_I)}{F_k(S)}\prod_{i\in I; j\notin I}\frac{u_j - qu_i}{u_j - u_i}\left[\prod_{i\in I}\frac{f(s)}{f(u_{l_i})}F(\mathbf{u})\right].$$

The expression in the bracket is precisely $\prod_{i\in I} T_{u_i,s}F(\mathbf{u})$. Summing over all permutations of $I$ removes the $k!$ above and summing over $I$ we recognize precisely $(D_m^k F)(\mathbf{u})$ as desired. □

For $k \le r \le m$ we let $D_r^k$ be the operator that acts on the variables $u_1, \dots, u_r$. Then we have the following result.

**Lemma 3.3.11.** *Suppose* $1 \le k \le m_1 \le m_2 \le \cdots \le m_k \le m$. *Denote by* $M_i = \{1, \dots, m_i\}$ *for* $i = 1, \dots, k$. *Then*

$$D_{m_1}^1 D_{m_2}^2 \cdots D_{m_k}^k = \sum_{i_1\in M_1}\sum_{i_2\in M_2/I_1}\cdots\sum_{i_k\in M_k/I_{k-1}}\prod_{r=1}^{k}\left(\prod_{j\in M_r/I_r}\frac{u_j - qu_{i_r}}{u_j - u_{i_r}}\frac{u_j - s}{u_j - sq}\right)\times \tag{3.3.19}$$
$$\frac{\prod_{r=1}^{k}\frac{1-q}{1-su_{i_r}}\left(\frac{u_{i_r}-sq}{1-su_{i_r}}\right)^{r-1}}{F_k(S)}\prod_{r=1}^{k}T_{u_{i_r},s} \quad, \text{ where } I_r = \{i_1, \dots, i_r\}.$$

*The above is understood as an equality of operators on functions in m variables.*

*Proof.* We proceed by induction on $k$ with base case $k = 1$ being just the definition of $D_{m_1}^1$. Suppose the result is known for $k$ and we wish to show it for $k+1$. Substituting the definition of $D_{m_{k+1}}^{k+1}$ and the induction hypothesis we have

$$D_{m_1}^1 D_{m_2}^2 \cdots D_{m_k}^k D_{m_{k+1}}^{k+1} = \sum_{i_1\in M_1}\sum_{i_2\in M_2/I_1}\cdots\sum_{i_k\in M_k/I_{k-1}}\prod_{r=1}^{k}\left(\prod_{j\in M_r/I_r}\frac{u_j - qu_{i_r}}{u_j - u_{i_r}}\frac{u_j - s}{u_j - sq}\right)\times$$

$$\frac{\prod_{r=1}^{k}\frac{1}{1-su_{i_r}}\left(\frac{u_{i_r}-sq}{1-su_{i_r}}\right)^{r-1}}{F_k(S)}\prod_{r=1}^{k}T_{u_{i_r},s}\sum_{\substack{I\subset M_{k+1}\\ |I|=k+1}}\prod_{\substack{i\in I\\ j\in M_{k+1}/I}}\frac{u_j - qu_i}{u_j - u_i}\prod_{j\in M_{k+1}/I}\left(\frac{u_j - s}{u_j - sq}\right)^{k+1}\frac{F_{k+1}(u_I)}{F_{k+1}(S)}\prod_{i\in I}T_{u_i,s}.$$

Suppose that $i_k \notin I$. Then

$$T_{u_{i_k},s}\prod_{i\in I; j\in M_{k+1}/I}\frac{u_j - qu_i}{u_j - u_i}\prod_{j\in M_{k+1}/I}\left(\frac{u_j - s}{u_j - sq}\right)^{k+1}\frac{F_{k+1}(u_I)}{F_{k+1}(S)}\prod_{i\in I}T_{u_i,s} = 0,$$

since one of the factors in the above expressions is $(u_{i_k} - s)^{k+1}$ and it vanishes when $u_{i_k} = s$. It follows that to get a non-zero contribution we must have $i_k \in I$. Repeating the argument we see that $i_r \in I$ for all $r = 1, \dots, k$. Thus $I = I_k \sqcup \{i_{k+1}\}$ for some $i_{k+1} \in M_{k+1}$ are the only cases that lead to a non-zero contribution. If $I$ does have this form we see that

$$\prod_{r=1}^{k} T_{u_{i_r},s} \prod_{i \in I; j \in M_{k+1}/I} \frac{u_j - qu_i}{u_j - u_i} \prod_{j \in M_{k+1}/I} \left( \frac{u_j - s}{u_j - sq} \right)^{k+1} \frac{F_{k+1}(u_I)}{F_{k+1}(S)} \prod_{i \in I} T_{u_i,s} =$$

$$= \prod_{j \in M_{k+1}/I} \frac{u_j - qu_{i+1}}{u_j - u_{i+1}} \left( \frac{u_j - qs}{u_j - s} \right)^{k} \prod_{j \in M_{k+1}/I} \left( \frac{u_j - s}{u_j - sq} \right)^{k+1} \frac{F_{k+1}(s, \dots, s, u_{i_{k+1}})}{F_{k+1}(S)} \prod_{r=1}^{k+1} T_{u_{i_r},s} =$$

$$= \prod_{j \in M_{k+1}/I} \frac{u_j - qu_{i+1}}{u_j - u_{i+1}} \prod_{j \in M_{k+1}/I} \left( \frac{u_j - s}{u_j - sq} \right) \frac{F_{k+1}(s, \dots, s, u_{i_{k+1}})}{F_{k+1}(S)} \prod_{r=1}^{k+1} T_{u_{i_r},s}.$$

From Lemma 3.3.1 we know that $F_{k+1}(s, \dots, s, u_{i_{k+1}}) = F_k(S) \frac{1-q}{1-su_{i_{k+1}}} \left( \frac{u_{i_{k+1}} - sq}{1-su_{i_{k+1}}} \right)^k$. Substituting this above and cancelling $F_k(S)$ we get

$$D^1_{m_1} D^2_{m_2} \cdots D^k_{m_k} D^{k+1}_{m_{k+1}} = \sum_{i_1 \in M_1} \sum_{i_2 \in M_2/I_1} \cdots \sum_{i_k \in M_k/I_{k-1}} \sum_{i_{k+1} \in M_{k+1}/I_k} \prod_{r=1}^{k+1} \left( \prod_{j \in M_r/I_r} \frac{u_j - qu_{i_r}}{u_j - u_{i_r}} \frac{u_j - s}{u_j - sq} \right)$$

$$\times \frac{\prod_{r=1}^{k+1} \frac{1-q}{1-su_{i_r}} \left( \frac{u_{i_r} - sq}{1-su_{i_r}} \right)^{r-1}}{F_{k+1}(S)} \prod_{r=1}^{k+1} T_{u_{i_r},s}.$$

This proves the case $k+1$ and the general result now follows by induction. □

**Lemma 3.3.12.** *Suppose $1 \leq k \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq m$. Suppose that $q \in (0,1)$, $s > 1$, $u_1, \dots, u_m > s$ and $u_i \neq u_j$ when $i \neq j$. Let $f(z)$ be a holomorphic non-vanishing function in a neighborhood of an interval containing $s, u_1, \dots, u_m$. Put $F(\mathbf{z}) = F(z_1, \dots, z_m) = \prod_{i=1}^m f(z_i)$. Then we have that*

$$(D^1_{m_1} D^2_{m_2} \cdots D^k_{m_k} F)(\mathbf{u}) = F(\mathbf{u}) \cdot \prod_{r=1}^{k} \left( \prod_{i=1}^{m_r} \frac{u_i - s}{u_i - sq} \right) \frac{f(s)^k}{(2\pi \iota)^k} \int_\gamma \cdots \int_\gamma \prod_{1 \leq i < j \leq k} \frac{z_i - z_j}{z_i - qz_j} \times$$
$$\frac{\prod_{r=1}^{k} \frac{1-q}{1-sz_r} \left( \frac{z_r - sq}{1-sz_r} \right)^{r-1}}{F_k(S)} \prod_{r=1}^{k} \left( \prod_{i=1}^{m_r} \frac{qz_r - u_i}{z_r - u_i} \right) \left( \frac{z_r - sq}{z_r - s} \right)^{k-r+1} \frac{dz_r}{f(z_r) z_r (q-1)}. \tag{3.3.20}$$

*The contour $\gamma$ is a positively oriented contour around the points $u_1, \dots, u_m$, and does not contain other singularities of the integrand. Such a contour will exist, provided $u_i$ are sufficiently close to each other.*

*Proof.* The proof is similar to that of Lemma 3.3.10 and by the same arguments we know that the contour $\gamma$ exists, provided $u_i$ are sufficiently close.

We calculate the residues at $z_r = u_{i_r}$. The Vandermonde determinant in the numerator prevents any of the $i_r$'s to be the same. The residue at $z_1 = u_{i_1}, \dots, z_k = u_{i_k}$ is given by

$$\prod_{r=1}^{k}\left(\prod_{i=1}^{m_r}\frac{u_i - s}{u_i - sq}\right)\prod_{1\leq r<p\leq k}\frac{u_{i_r} - u_{i_p}}{u_{i_r} - qu_{i_p}}\prod_{r=1}^{k}\frac{1}{u_{i_r}(q-1)}\frac{\prod_{r=1}^{k}\frac{1-q}{1-su_{i_r}}\left(\frac{u_{i_r}-sq}{1-su_{i_r}}\right)^{r-1}}{F_k(S)}\times$$
$$\prod_{r=1}^{k}(q-1)u_{i_r}\left(\prod_{i=1, i\neq i_r}^{m_r}\frac{qu_{i_r} - u_i}{u_{i_r} - u_i}\right)\left(\frac{u_{i_r} - sq}{u_{i_r} - s}\right)^{k-r+1}\left[\frac{F(\mathbf{u})f(s)^k}{\prod_{r=1}^{k} f(u_{i_r})}\right].$$

Performing some cancellations and recognizing the term inside the square brackets as $\prod_{r=1}^{k} T_{u_{i_r},s}F(\mathbf{u})$ we recognize precisely the term on the RHS of (3.3.19) corresponding to $i_1, \dots, i_k$. Summing over all the residues we arrive at the desired identity. □

## 3.4 Weak convergence of $(Y_1^1, \dots, Y_k^k)$

In this section we use our results from Section 3.3 to derive formulas for $\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_1^1 \leq m_1, \dots, Y_k^k \leq m_k)$. Afterwards we specialize our formulas to the case when all $u$ and all $v$ parameters are the same and show that under the scaling of Theorem 3.1.3 the joint CDFs of the vectors $(Y_1^1, \dots, Y_k^k)$ converge to a fixed function as the size of the six-vertex model increases. We finish by identifying the limit as the joint CDF of the right edge of the GUE-corners process of rank $k$ and proving Theorem 3.1.1.

### 3.4.1 Pre-limit formulas

The goal of this section is to use the results from Section 3.3 to obtain formulas for $\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_1^1 \leq m_1, \dots, Y_k^k \leq m_k)$, where $m_i \in \mathbb{N}$ for $i = 1, \dots, k$ and $Y_i^j$ are defined in Section 3.1.1. We summarize the result in the following proposition.

**Proposition 3.4.1.** *Fix parameters as in Definition 3.2.14. Let $k$ and $m_i$ for $i = 1, \dots, k$ be positive integers such that $1 \leq k \leq m_1 \leq m_2 \leq \dots \leq m_k \leq N$. Then we have*

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_1^1 \leq m_1, \dots, Y_k^k \leq m_k) = \left(\prod_{j=1}^{M}\frac{1-qsv_j}{1-sv_j}\right)^k\prod_{r=1}^{k}\left(\prod_{i=1}^{m_r}\frac{u_i - s}{u_i - sq}\right)\frac{q^{-k}}{(2\pi\iota)^k}\int_\gamma\cdots\int_\gamma$$
$$\prod_{1\leq i<j\leq k}\frac{z_i - z_j}{z_i - qz_j}\prod_{r=1}^{k}\left(\prod_{i=1}^{m_r}\frac{qz_r - u_i}{z_r - u_i}\right)\left(\frac{z_r - sq}{z_r - s}\right)^{k-r+1}\prod_{j=1}^{M}\frac{1-z_rv_j}{1-qz_rv_j}\frac{dz_r}{z_r(1-sz_r)}. \tag{3.4.1}$$

*The contour $\gamma$ is a positively oriented contour that contains $u_i$'s and excludes all other singularities of the integrand. Such a contour will exist, provided $u_i$ are sufficiently close to each other.*

*Proof.* In what follows we adopt notation from Sections 3.1.1 and 3.2.2.

Let $E = \{\omega \in \mathcal{P}_N : \omega(0,j) = (0,1;0,1) \text{ for } i = 1, \dots, N\}$. From our discussion in Section 3.2, we know that $\mathbb{P}_{\mathbf{u},\mathbf{v}}(E) = 1$. Consider the map $h : E \to \mathcal{P}_N$, given by $h(\omega)(i,j) = \omega(i+1,j)$. I.e. $h(\omega)$ is just the collection of up-right paths $\omega$, with the zeroth column deleted. One readily observes that $h$ is a bijection and the distribution of $h(\omega)$, induced by the distribution of $\omega$, is given by $\mathbb{P}^g$, where $g(\mu) = \frac{(-s)^N}{(q;q)_N} \times f(\mu + (1)^N; \mathbf{v}, \rho)$. We recall that $\lambda = \mu + (1)^N$ is the signature with $\lambda_i = \mu_i + 1$ for $i = 1, \dots, N$ and $f(\lambda; \mathbf{v}, \rho)$ is given in (3.2.16).

Indeed, we have for $\omega \in E$, that

$$\mathcal{W}^g(h(\omega); \mathbf{u}) = \frac{(-s)^N}{(q;q)_N} \prod_{i=1}^{\infty} \prod_{j=1}^{N} w_{u_j}(\omega(i,j)) f(\lambda^N(\omega); \mathbf{v}, \rho) = \frac{(-s)^N}{(q;q)_N} \prod_{i=1}^{N} \frac{1 - su_i}{u_i - s} \times \mathcal{W}^f(\omega; \mathbf{u}).$$

The above shows that the weights $\mathcal{W}^g(h(\omega); \mathbf{u})$ are constant multiples of $\mathcal{W}^f(\omega; \mathbf{u})$, and so the probability distributions they define are the same. The partition function $Z^g(\mathbf{u})$ differs from $Z^f(\mathbf{u})$ by the same constant factor $\frac{(-s)^N}{(q;q)_N} \prod_{i=1}^N \frac{1-su_i}{u_i - s}$, and by (3.2.19) equals

$$Z^g(\mathbf{u}) = \prod_{i=1}^{N} \left( \prod_{j=1}^{M} \frac{1 - qu_iv_j}{1 - u_iv_j} \right). \tag{3.4.2}$$

One easily observes the following equality of events

$$\{\omega \in E : \lambda^{m_i}(h(\omega)) \in \mathsf{Sign}^*_{m_i,i},\ i = 1, \dots, k\} = \{\omega \in E : Y_i^i(\omega) \le m_i,\ i = 1, \dots, k\}.$$

For example $\lambda^{m_1}(h(\omega)) \in \mathsf{Sign}^*_{m_1,1}$ is equivalent to $\lambda^{m_1}_{m_1}(\omega) = 1$, which by the conservation of arrows in the region $\{(1, y) \in D_N : y = 1, \dots, m_1\}$ is equivalent to $Y_1^1(\omega) \le m_1$. The above equality of events, coupled with $\mathbb{P}_{\mathbf{u},\mathbf{v}}(E) = 1$ and the previous two paragraphs implies

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_1^1 \le m_1, \dots, Y_k^k \le m_k) = \mathbb{P}^g(A(\mathbf{m})), \tag{3.4.3}$$

where $A(\mathbf{m})$ is as in (3.3.7). In view of (3.3.8) and (3.4.2), we conclude that if $u_1, \dots, u_N$ are pairwise distinct

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_1^1 \le m_1, \dots, Y_k^k \le m_k) = \frac{(D^1_{m_1} D^2_{m_2} \cdots D^k_{m_k} Z^g)(\mathbf{u})}{Z^g(\mathbf{u})}, \text{ where } Z^g(\mathbf{u}) = \prod_{i=1}^{N} \left( \prod_{j=1}^{M} \frac{1 - qu_iv_j}{1 - u_iv_j} \right). \tag{3.4.4}$$

The result of the proposition now follows from (3.4.4) and Lemma 3.3.12 when $u_1, \dots, u_N$ are pairwise distinct. By continuity it also holds if some are equal. □

### 3.4.2 Asymptotic analysis

While most of the results below can be extended to a more general choice of parameters, we keep discussion simple and assume that all $u$ and all $v$ parameters are the same, and that $\frac{s+s^3}{2} > u > s$. With this in mind we have the following definition.

**Definition 3.4.2.** Let $N, M \in \mathbb{N}$ and fix $q \in (0, 1)$, $s = q^{-1/2}$, $\frac{s+s^3}{2} > u > s$ and $v \in (0, u^{-1})$. We denote by $\mathbb{P}^{N,M}_{u,v}$ the probability measure $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ of Definition 3.2.14, with $u_i = u$ and $v_j = v$ for $i = 1, \dots, N$ and $j = 1, \dots, M$.

With the above definition, we have the following consequence of Proposition 3.4.1. If

$1 \le k \le m_1 \le m_2 \le \cdots \le m_k \le N$, then

$$\mathbb{P}_{\mathbf{u},\mathbf{v}}(Y_1^1 \le m_1, \dots, Y_k^k \le m_k) = \frac{q^{-k}}{(2\pi\iota)^k} \int_\gamma \cdots \int_\gamma \prod_{1\le i<j\le k} \frac{z_i - z_j}{z_i - qz_j} \times$$
$$\prod_{r=1}^{k} \left(\frac{qz_r - u}{z_r - u}\frac{u-s}{u-sq}\right)^{m_r} \left(\frac{z_r - sq}{z_r - s}\right)^{k-r+1} \left(\frac{1-z_r v}{1-qz_r v}\frac{1-qsv}{1-sv}\right)^{M} \frac{dz_r}{z_r(1-sz_r)}, \tag{3.4.5}$$

where $\gamma$ is a contour, containing $u$ and excluding other singularities of the integrand. Equation (3.4.5) is prime for asymptotic analysis and we use it to prove the following proposition.

**Proposition 3.4.3.** *Let $\mathbb{P}_{u,v}^{N,M}$ be as in Definition 3.4.2. Put $a = \frac{v^{-1}(u-sq)(u-s)}{u(v^{-1}-sq)(v^{-1}-s)}$ and $c = (2a_2)^{1/2}b_1^{-1}$, where*

$$a_2 = \frac{(1-q)v^{-1}}{(v^{-1}-s)(v^{-1}-sq)}\left[\frac{(q+1)s - 2v^{-1}}{(v^{-1}-s)(v^{-1}-sq)} - \frac{(q+1)s-2u}{(u-s)(u-sq)}\right] \text{ and } b_1 = \frac{1}{u-s} - \frac{1}{q^{-1}u - s}.$$

*Let $\gamma > a$ and assume that $N(M) \ge \gamma \cdot M$ for all $M >> 1$. Then for any $k \ge 1$ and $x_1 \le \cdots \le x_k$, $x_i \in \mathbb{R}$ we have*

$$\lim_{M\to\infty} \mathbb{P}_{u,v}^{N,M}\left(\frac{Y_i^i - aM}{c\sqrt{M}} \le x_i; i = 1,\dots,k\right) = \det\left[\frac{1}{2\pi\iota}\int_{1+\iota\mathbb{R}} y^{j-i-1} e^{y^2/2 + x_j y} dy\right]_{i,j=1}^{k}. \tag{3.4.6}$$

*Proof.* Put $m_i = aM + cx_i\sqrt{M} + h_i$ for $i = 1,\dots,k$, where $h_i \in (-1,1)$ are such that $m_i \in \mathbb{N}$ and $m_1 \le m_2 \le \cdots \le m_k$ for $M$ sufficiently large. Using (3.4.5) we reduce the proof of (3.4.6) to the following statement

$$\lim_{M\to\infty} \frac{q^{-k}}{(2\pi\iota)^k}\int_{\gamma_M}\cdots\int_{\gamma_M} \prod_{1\le i<j\le k}\frac{z_i - z_j}{z_i - qz_j}\prod_{r=1}^{k}\left(\frac{z_r - sq}{z_r - s}\right)^{k-r+1} \frac{e^{MG(z_r) + c\sqrt{M}x_r g(z_r) + h_r g(z_r)}dz_r}{z_r(1-sz_r)}$$
$$= \det\left[\frac{1}{2\pi\iota}\int_{1+\iota\mathbb{R}} y^{j-i-1}e^{y^2/2 + x_j y}dy\right]_{i,j=1}^{k}, \tag{3.4.7}$$

where $\gamma_M$ are contours that contain $u$ and do not include $0$, $q^{-1}v^{-1}$, $s$ or points from $q \cdot \gamma_M$ and

$$G(z) = a\log\left(\frac{qz-u}{z-u}\frac{u-s}{u-sq}\right) + \log\left(\frac{v^{-1}-z}{v^{-1}-qz}\frac{v^{-1}-sq}{v^{-1}-s}\right) \text{ and } g(z) = \log\left(\frac{qz-u}{z-u}\frac{u-s}{u-sq}\right). \tag{3.4.8}$$

Our goal is to find the $M \to \infty$ limit of the LHS of (3.4.7) and match it with the RHS. Let us briefly explain what the strategy is. We will find specific contours $\gamma_M = \gamma_M(0) \sqcup \gamma_M(1)$, such that $Re[G(z)] < 0$ on $\gamma_M(1)$ and the integrand (upon a change of variables) has a clear limit on $\gamma_M(0)$. The condition $Re[G(z)] < 0$ will show that the integral over $\gamma_M(1)$ decays exponentially fast, and hence does not contribute to the limit. The non-vanishing contribution, coming from $\gamma_M(0)$, will then be shown to equal the RHS of (3.4.7). The latter approach is typically referred to as the *method of steepest descent* in the literature.

To simplify formulas in the sequel we denote $v^{-1}$ by $w$. We start by analyzing the functions $G$ and $g$. From (3.4.8) we have $G'(z) = a\frac{(1-q)u}{(z-u)(qz-u)} - \frac{(1-q)w}{(z-w)(qz-w)}$ and so $G(s) = 0$ and $G'(s) = 0$ by our choice of $a$. We observe

$$\frac{d}{dy}Re\left[G(s+\iota y)\right] = Im\left[\frac{(1-q)w}{(s+\iota y - w)(q(s+\iota y)-w)} - \frac{a(1-q)u}{(s+\iota y-u)(q(s+\iota y)-u)}\right] = w(1-q)yA(y),$$

where

$$A(y) = \frac{(u-s)(u-sq)}{(w-s)(w-sq)}\frac{2q-(q+1)u}{((s-u)^2+y^2)((qs-u)^2+q^2y^2)} - \frac{2q-(q+1)w}{((s-w)^2+y^2)((qs-w)^2+q^2y^2)}.$$

We observe that

$$A(0) = \frac{1}{(w-s)(w-sq)}\left[\frac{2q-(q+1)u}{(u-s)(u-sq)} - \frac{2q-(q+1)w}{(w-s)(w-sq)}\right] < 0,$$

where we used $u, w > s$, $q \in (0,1)$ and $w > u$. In addition, if we put the two fractions in the definition of $A(y)$ under a common denominator, we see that the sign of $A(y)$ agrees with the sign of a certain quadratic polynomial in $y^2$ with a positive leading coefficient. This implies that as $y$ goes from 0 to $\infty$, $A(y)$ is initially negative and then becomes positive, i.e. $Re\left[G(s+\iota y)\right]$ initially decreases and then increases in $y > 0$. A similar statement holds when $y < 0$. In particular, we can find $\epsilon > 0$ small such that $Re\left[G(s+\iota y)\right] \leq 0$ for $y \in [-\epsilon, \epsilon]$ and $Re\left[G(s \pm \iota\epsilon)\right] < 0$.

Using $u, w > s$, $q \in (0,1)$ and $w > u$, we notice

$$G(0) = a\log\left(\frac{u-s}{u-sq}\right) + \log\left(\frac{w-sq}{w-s}\right) =: -c_0 < 0.$$

We next observe that

$$a_2 = G''(s) = \frac{(1-q)w}{(w-s)(w-sq)}\left[\frac{(q+1)s-2w}{(w-s)(w-sq)} - \frac{(q+1)s-2u}{(u-s)(u-sq)}\right] > 0.$$

Consequently, we have that near $s$ we have $G(z) = a_2(z-s)^2 + a_3(z-s)^3 + \cdots$ and $g(z) = b_1(z-s) + b_2(z-s)^2 + \cdots$. In particular, if we choose $\epsilon$ sufficiently small we can ensure that

$$|G(z) - a_2(z-s)^2| \leq R|z-s|^3 \text{ and } |g(z) - b_1(z-s)| \leq R|z-s|^2, \text{ when } |z-s| < \epsilon, \tag{3.4.9}$$

where $R$ can be taken to be $|b_2| + |a_3| + 1$. For the remainder we fix $\epsilon > 0$ sufficiently small so that (3.4.9) holds and $c_\epsilon := -Re\left[G(s \pm \iota\epsilon)\right] > 0$.

In what follows we define the contour $\gamma_M$. Let $B_1$ and $C_1$ be the points $u$ and $q^{-1}u$ in the complex plane respectively, and also denote $w$ and $q^{-1}w$ by $B_2$ and $C_2$ respectively. For

$i = 1, 2$ we let $\omega_i$ be the Appolonius circle[1] of the segment $B_iC_i$, which passes through the origin. By properties of the Appolonius circle, we know that $X_iY_i$ is a diameter for $\omega_i$, where $X_i = 0$, $Y_1 = \frac{2u}{1+q}$ and $Y_2 = \frac{2w}{1+q}$. Observe that since $w > u$ we have that $\omega_1$ is internally tangent to $\omega_2$ at 0.

Let $s \pm \iota y_1$ be the points on $\omega_1$ that lie on the vertical line through $s$, with $y_1 > 0$. $\gamma_M$ starts from $s - \iota y_1$ and follows $\omega_1$ to $s + \iota y_1$ counterclockwise, afterwards it goes down to to $s + \iota M^{-1/2}$, follows the right half of the circle of radius $M^{-1/2}$ around $s$ to $s - \iota M^{-1/2}$, and then continues down to $s - \iota y_1$. See the left part of Figure 3-6. Observe that by construction $u$ is enclosed by $\gamma_M$ and $0, s, s^{-1}$ are not. In addition, we notice that since $q \cdot \omega_1$ lies to the left of $q \cdot Y_1 = \frac{2uq}{1+q}$ and the latter is less than $s$ if $u < \frac{s+s^3}{2}$, then $q \cdot \gamma_M$ lies to the left of $\gamma_M$. This means that $\gamma_M$ satisfies the conditions we stated after (3.4.7).

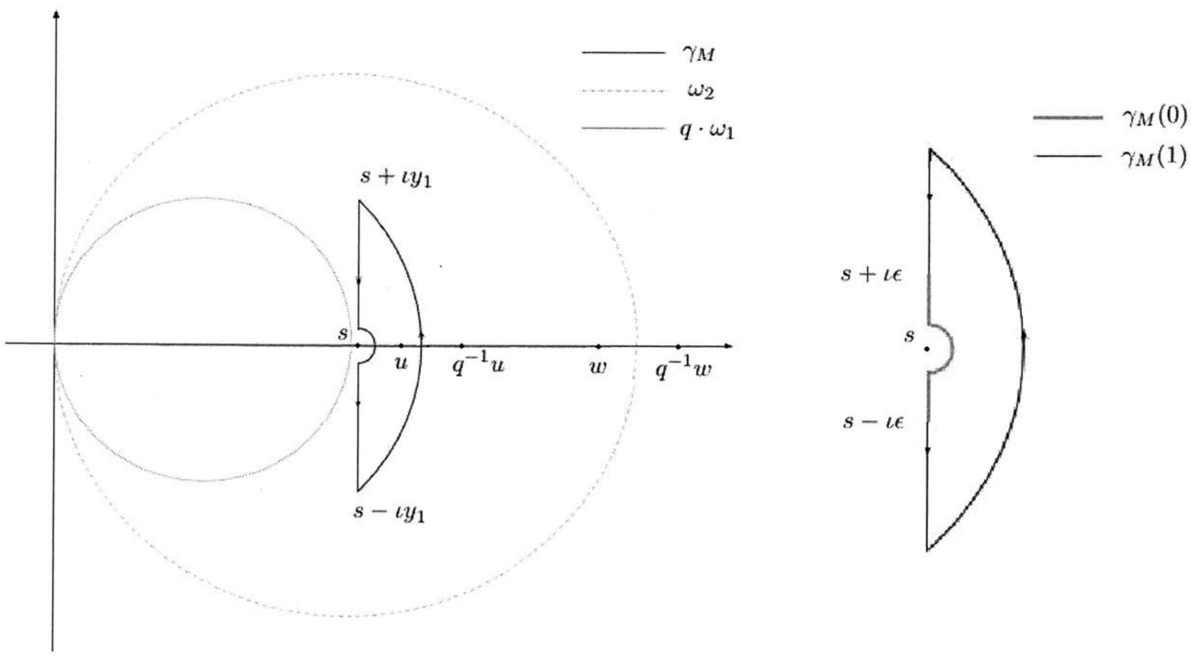


Figure 3-6: The contour $\gamma_M$ (left) and $\gamma_M(0)$ and $\gamma_M(1)$ (right).

We now investigate the real part of $G(z)$ on $\gamma_M$. Using the properties of the Appolonius circle, we see that for $z \in \omega_1$ we have

$$Re\left[a \log\left(\frac{qz-u}{z-u}\frac{u-s}{u-sq}\right)\right] = a\log\left(\frac{|z-q^{-1}u|}{|z-u|}\right) + a\log\left(\frac{qu-qs}{u-sq}\right) =$$
$$a\log\left(\frac{|X_2C_2|}{|X_2B_2|}\right) + a\log\left(\frac{qu-qs}{u-sq}\right), \text{ while on the other hand}$$

$$Re\left[\log\left(\frac{w-z}{w-qz}\frac{w-sq}{w-s}\right)\right] = \log\left(\frac{|w-z|}{|q^{-1}w-z|}\right) + \log\left(\frac{w-sq}{qw-qs}\right) \le \log\left(\frac{|X_1B_1|}{|X_1C_1|}\right) + \log\left(\frac{w-sq}{qw-qs}\right).$$

[1]For $r \in (0, 1)$, the Appolonius circle of a segment $BC$ with ratio $r$ is the set of points $X$ such that $\frac{XB}{XC} = r$. For points inside the circle we have $\frac{XB}{XC} < r$ and for those outside $\frac{XB}{XC} > r$. If $X$ and $Y$ denote the (unique) points on the line $BC$, which satisfy $\frac{XB}{XC} = r = \frac{YB}{YC}$, with $X$ lying inside and $Y$ outside the segment $BC$, the Appolonius circle of $BC$ with ratio $r$, is the circle with diamater $XY$.

Adding the above inequalities we see that for $z \in \omega_1$, we have

$$Re\left[G(z)\right] \leq Re\left[G(0)\right] = -c_0 < 0. \tag{3.4.10}$$

Equation (3.4.10) in particular says that $Re[G(s \pm \iota y_1)] \leq -c_0$, and since $Re[G(s + \iota y)]$ decreases and then increases in $|y|$, while $Re[G(s \pm \iota\epsilon)] \leq -c_\epsilon < 0$, we know that $Re[G(s \pm \iota y)] \leq -\min(c_0, c_\epsilon)$, for $|y| \in [\epsilon, y_2]$. Let us denote by $\gamma_M(0)$ the portion of $\gamma_M^-$, which connects $s \pm \iota\epsilon$ near $s$, and by $\gamma_M(1)$, the rest of $\gamma_M$ - see the right part of Figure 3-6.

The above estimates show that $Re[G(z)] \leq -\min(c_0, c_\epsilon)$ for $z \in \gamma_M(1)$. This suggests, that asymptotically, we may ignore $\gamma_M(1)$, as its contribution goes to zero exponentially fast. Explicitly, if we denote by $H(z_1, \dots, z_k)$ the integrand in (3.4.7) then we have

$$\lim_{M\to\infty} \left| \int_{\gamma_M} \cdots \int_{\gamma_M} H(z_1, \dots, z_k) d\mathbf{z} - \int_{\gamma_M(0)} \cdots \int_{\gamma_M(0)} H(z_1, \dots, z_k) d\mathbf{z} \right| = 0. \tag{3.4.11}$$

We isolate the proof of the above statement in Proposition 3.4.4 below and continue with the proof of (3.4.7).

In view of (3.4.11), the limit as $M \to \infty$ of the LHS of (3.4.7) is the same as that of

$$\frac{q^{-k}}{(2\pi\iota)^k} \int_{\gamma_M(0)} \cdots \int_{\gamma_M(0)} \prod_{1 \leq i < j \leq k} \frac{z_i - z_j}{z_i - qz_j} \prod_{r=1}^{k} \left( \frac{z_r - sq}{z_r - s} \right)^{k-r+1} \frac{e^{MG(z_r) + c\sqrt{M} x_r g(z_r) - a_r g(z_r)} dz_r}{z_r(1 - sz_r)}, \tag{3.4.12}$$

We do the change of variables $y_i = (z_i - s)M^{1/2}$ and set $\Gamma$ to be the contour that goes up from $-\iota\infty$ to $-\iota$, follows the right half of the circle of radius 1 around 0 to $\iota$, and then continues up to $\iota\infty$. Using (3.4.9), we observe that (3.4.12) equals

$$\begin{gathered} \frac{(-q)^{-k}}{(2\pi\iota)^k} \int_\Gamma \cdots \int_\Gamma \prod_{1 \leq i < j \leq k} \frac{y_i - y_j}{M^{-1/2}(y_i - qy_j) + s(1-q)} \times \\ \prod_{r=1}^{k} \left( \frac{M^{-1/2} y_r + s(1-q)}{y_r} \right)^{k-r+1} \frac{e^{a_2 y_r^2 + cb_1 x_r y_r + O(M^{-1/2})} \mathbf{1}_{\{|y_r| \leq M^{-1/2}\epsilon\}} dy_r}{(1 - s^2 - M^{-1/2} s y_r)(M^{-1/2} y_r + s)}. \end{gathered} \tag{3.4.13}$$

The pointwise limit of the integrand as $M \to \infty$ is given by

$$(-q)^k \prod_{1 \leq i < j \leq k} (y_i - y_j) \prod_{r=1}^{k} e^{a_2 y_r^2 + cb_1 x_r y_r} \frac{dy_r}{y_r^{k-r+1}}$$

Since $a_2 > 0$ we see that the integrand in (3.4.13) is dominated by $C \prod_{r=1}^{k} e^{-a_2 |y_r|^2/2}$. From the Dominated Convergence Theorem the $M \to \infty$ limit of (3.4.12), and hence (3.4.7) is

$$\frac{1}{(2\pi\iota)^k} \int_\Gamma \cdots \int_\Gamma \prod_{1 \leq i < j \leq k} (y_i - y_j) \prod_{r=1}^{k} e^{a_2 y_r^2 + cb_1 x_r y_r} \frac{dy_r}{y_r^{k-r+1}}. \tag{3.4.14}$$

What remains is to show that (3.4.14) and the RHS of (3.4.7) agree. We perform the change of variables $y_i \to (2a_2)^{-1/2}y_i$, replace $\prod_{1\le i<j\le k}(y_i-y_j)\prod_{r=1}^{k}\frac{1}{y_r^{k-r+1}}$ with $\det[y_i^{i-j-1}]_{i,j=1}^{k}$, set $\Gamma' = (2a_2)^{1/2}\Gamma$ and use $(2a_2)^{-1/2}b_1c = 1$. This allows us to rewrite (3.4.14) as

$$\frac{1}{(2\pi\iota)^k}\int_{\Gamma'}\cdots\int_{\Gamma'}\det[y_i^{i-j-1}]_{i,j=1}^{k}\prod_{r=1}^{k}e^{y_r^2/2+x_ry_r}dy_r.$$

Using properties of determinants we rewrite the above as

$$\det\left[\frac{1}{2\pi\iota}\int_{\Gamma'}y^{i-j-1}e^{y^2/2+x_iy}dy\right]_{i,j=1}^{k}. \tag{3.4.15}$$

By Cauchy's theorem and the rapid decay of $e^{y^2/2}$ near $\pm\iota\infty$, we may deform $\Gamma'$ to $1+\iota\mathbb{R}$, without changing the value of the integral. Replacing the matrix in the determinant with its transpose, finally transforms (3.4.15) into the RHS of (3.4.7).

□

**Proposition 3.4.4.** *Denote by $H(z_1,\dots,z_k)$ the integrand in (3.4.7). Then we have*

$$\lim_{M\to\infty}\left|\int_{\gamma_M}\cdots\int_{\gamma_M}H(z_1,\dots,z_k)d\mathbf{z}-\int_{\gamma_M(0)}\cdots\int_{\gamma_M(0)}H(z_1,\dots,z_k)d\mathbf{z}\right|=0 \tag{3.4.16}$$

*Proof.* We adopt the same notation as in the proof of Proposition 3.4.3. We write

$$\int_{\gamma_M}\cdots\int_{\gamma_M}H(z_1,\dots,z_k)d\mathbf{z}=\sum_{\epsilon_1,\dots,\epsilon_k\in\{0,1\}}\int_{\gamma_M(\epsilon_1)}\cdots\int_{\gamma_M(\epsilon_k)}H(z_1,\dots,z_k)d\mathbf{z},$$

and so we observe that the expression in the absolute value in (3.4.16) is a finite sum of terms

$$\int_{\gamma_M(\epsilon_1)}\cdots\int_{\gamma_M(\epsilon_k)}H(z_1,\dots,z_k)d\mathbf{z},$$

where $\epsilon_i$ are not all equal to 1. Recall from (3.4.7)

$$H(z_1,\dots,z_k)=\prod_{1\le i<j\le k}\frac{z_i-z_j}{z_i-qz_j}\prod_{r=1}^{k}\left(\frac{z_r-sq}{z_r-s}\right)^{k-r+1}\frac{e^{MG(z_r)+c\sqrt{M}x_rg(z_r)+h_rg(z_r)}}{z_r(1-sz_r)}$$

When $z_i\in\gamma_M$, we know that

$$\left|\prod_{1\le i<j\le k}\frac{z_i-z_j}{z_i-qz_j}\prod_{r=1}^{k}\frac{(z_r-sq)^{k-r+1}}{z_r(1-sz_r)}\right|\le C \text{ and } \left|\prod_{r=1}^{k}\left(\frac{1}{z_r-s}\right)^{k-r+1}\right|\le CM^{k(k+1)/4}, \tag{3.4.17}$$

for some constant $C>0$, where we used that $z\in\gamma_M$ is at least a distance $M^{-1/2}$ from the point $s$, and is uniformly bounded away from other singularities.

Further, from our earlier analysis of the real part of $G(z)$ on $\gamma_M$, we know that when

$z \in \gamma_M(1)$, we have for some (maybe different than before) constant $C > 0$

$$\left| e^{MG(z_r)+c\sqrt{M}x_r g(z_r)+h_r g(z_r)} \right| \leq Ce^{-c'M}, \text{ where } c' = \min(c_0, c_\epsilon). \tag{3.4.18}$$

Finally, if $z \in \gamma_M(0)$, we know that

$$\begin{aligned} \left| e^{MG(z_r)+c\sqrt{M}x_r g(z_r)+h_r g(z_r)} \right| &\leq Ce^{K\sqrt{M}} \quad \text{if } M^{-1/2} \leq |Im(z)| \leq \epsilon, \text{ and} \\ \left| e^{MG(z_r)+c\sqrt{M}x_r g(z_r)+h_r g(z_r)} \right| &\leq C \quad \text{if } |Im(z)| \leq M^{-1/2}. \end{aligned} \tag{3.4.19}$$

In (3.4.19), $K$ is a constant that dominates $|cx_r g(z)|$, for $z \in \gamma_M$ and $r = 1, \dots, k$. In obtaining the first estimate in (3.4.19), we used that $Re[G(s + \iota y)] \leq 0$ for $|y| \in [M^{-1/2}, \epsilon]$, while for the second one we used (3.4.9).

If we combine the statements in (3.4.17), (3.4.18) and (3.4.19) and use the compactness of $\gamma_M$, we see that

$$\left| \int_{\gamma_M(\epsilon_1)} \cdots \int_{\gamma_M(\epsilon_k)} H(z_1, \dots, z_k) d\mathbf{z} \right| \leq Ce^{-c'(\epsilon_1 + \cdots + \epsilon_k)M} M^{k(k+1)/4} e^{k \cdot K\sqrt{M}},$$

and if $\epsilon_i$ are not all 0, we see that the above decays to 0 as $M \to \infty$. □

### 3.4.3 Limit identification and proof of Theorem 3.1.1

We start this section by showing that the RHS of (3.4.6) equals $\mathbb{P}(\lambda_1^1 \leq x_1, \cdots, \lambda_k^k \leq x_k)$ when $x_1 \leq x_2 \leq \cdots \leq x_k$ and $x_i \in \mathbb{R}$. Here $\lambda_i^j$ $i = 1, \dots, j$, $j = 1, \dots, k$ is the GUE-corners process (see Section 3.1.1). The density of $\lambda_1^1, \dots, \lambda_k^k$ was calculated in [82] to equal

$$\rho(x_1, \dots, x_k) = \mathbf{1}_{\{x_1 \leq x_2 \leq \cdots \leq x_k\}} \det \left[ \Phi^{i-j}(x_j) \right]_{i,j=1}^k.$$

In the above we have that $\Phi^n$ for $n \geq 1$ is the $n$-th order iterated integral of the Gaussian density $\phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$$\Phi^n(y) = \int_{-\infty}^{y} \frac{(y-x)^{n-1}}{(n-1)!} \phi(x) dx, \tag{3.4.20}$$

and when $n \geq 0$, $\Phi^{-n}$ denotes the $n$-th order derivative of $\phi$. Let us denote

$$\Psi^m(y) := \frac{1}{2\pi\iota} \int_{1+\iota\mathbb{R}} x^m e^{x^2/2 + yx} dx.$$

Then to show that the RHS of (3.4.6) equals $\mathbb{P}(\lambda_1^1 \leq x_1, \cdots, \lambda_k^k \leq x_k)$, it suffices to show that, when $x_1 \leq x_2 \leq \cdots \leq x_k$,

$$\int_{-\infty}^{x_1} dy_1 \int_{y_1}^{x_2} dy_2 \cdots \int_{y_{n-1}}^{x_n} dy_n \det \left[ \Phi^{i-j}(y_j) \right]_{i,j=1}^k = \det \left[ \Psi^{j-i-1}(x_j) \right]_{i,j=1}^k. \tag{3.4.21}$$

The rapid decay of $e^{y^2/2}$ near $\pm\iota\infty$ shows that $\Psi^m(y)$ is differentiable, and its derivative equals

$$\frac{d}{dy}\Psi^m(y) = \frac{1}{2\pi\iota}\int_{1+\iota\mathbb{R}} x^m \frac{d}{dy} e^{x^2/2+yx}dx = \frac{1}{2\pi\iota}\int_{1+\iota\mathbb{R}} x^{m+1}e^{x^2/2+yx}dx = \Psi^{m+1}(y).$$

The other properties of $\Psi^m$ that we will need are that $\Psi^0(y) = \phi(y)$ and $\lim_{y\to-\infty}\Psi^m(y) = 0$. To see the former, we complete the square in the exponential of $\Psi^0(y)$ and change variables $x = 1 + \iota z$ to see

$$\Psi^0(y) = \frac{e^{-y^2/2}}{2\pi}\int_{\mathbb{R}} e^{(1+y+\iota z)^2/2}dz = \frac{e^{-y^2/2}}{2\pi}\int_{\mathbb{R}} e^{-(z-\iota(y+1))^2/2}dz = \frac{e^{-y^2/2}}{2\pi}\int_{\mathbb{R}} e^{-z^2/2}dz = \phi(y).$$

The middle equality follows from the usual shift of $\mathbb{R}$ to $\mathbb{R}+\iota(y+1)$, which does not change the integral by Cauchy's theorem. Performing the same change of variables we see that for any $m \in \mathbb{Z}$ and $y \leq -1$, we have that

$$\Psi^m(y) = \frac{e^{-y^2/2}}{2\pi}\int_{\mathbb{R}}(1+\iota z)^m e^{-(z-\iota(y+1))^2/2}dz = \frac{e^{-y^2/2}}{2\pi}\int_{\mathbb{R}}(\iota z - y)^m e^{-z^2/2}dz,$$

where the last equality follows from the shift of $\mathbb{R}$ to $\mathbb{R}+\iota(y+1)$, which does not change the integral by Cauchy's theorem, as the possible pole at $z = \iota$ is never crossed when $y < 0$. When $m \leq 0$, we notice that $|(\iota z - y)^m| \leq 1$, when $y \leq -1$, while when $m \geq 0$, we can bound the same expression by $C(|y|^m + |z|^m + 1)$, uniformly in $z \in \mathbb{R}$ and $y \leq -1$. The upshot is that

$$|\Psi^m(y)| \leq \frac{e^{-y^2/2}}{2\pi}\int_{\mathbb{R}} C(|y|^m+|z|^m+1)e^{-z^2/2}dz \leq C(m)|y|^m e^{-y^2/2}, \text{ and hence } \lim_{y\to-\infty}\Psi^m(y) = 0.$$

Similar arguments also show that $\lim_{y\to-\infty}\Phi^m(y) = 0$ for any $m \in \mathbb{Z}$.

We next show that $\Phi^{-m}(y) = \Psi^m(y)$ for all $m \in \mathbb{Z}$. From the previous paragraph we know this to be the case when $m = 0$. Since $\Psi^{m+1}(y) = \frac{d}{dy}\Psi^m(y)$ and $\Phi^{-m-1}(y) = \frac{d}{dy}\Phi^{-m}(y)$, when $m \geq 0$, we have equality when $m \geq 0$. Finally, we prove the result for $-m \geq 0$ by induction on $-m$. Suppose, we know that $\Phi^k(y) = \Psi^{-k}(y)$, for $k \geq 0$. Then we have

$$\frac{d}{dy}\Phi^{k+1}(y) = \frac{d}{dy}\Psi^{-k-1}(y) \text{ and so } \Phi^{k+1}(y) - \Psi^{-k-1}(y) \text{ is constant.}$$

As both $\Phi^{k+1}(y)$ and $\Psi^{-k-1}(y)$ vanish as $y \to -\infty$, we see that the constant is 0, and we have $\Phi^{k+1}(y) = \Psi^{-k-1}(y)$. The general result now follows by induction.

We now turn to the proof of (3.4.21). From our discussion above we know that both sides define continuously differentiable functions in $x_1$. When $x_1$ goes to $-\infty$, we have that the first column in the matrix on the RHS goes to 0 and so the determinant vanishes. The LHS also vanishes, as it is dominated by $\mathbb{P}(\lambda_1^1 \leq x_1)$. Consequently, it suffices to show that the derivatives w.r.t. $x_1$ on both sides agree. Replacing $\Phi^m$ with $\Psi^{-m}$, what we want is to show that when $x_1 \leq x_2 \leq \cdots \leq x_k$ and $y_1 = x_1$

$$\int_{x_1}^{x_2} dy_2 \int_{y_2}^{x_3} dy_3 \cdots \int_{y_{n-1}}^{x_n} dy_n \det \left[\Psi^{j-i}(y_j)\right]_{i,j=1}^k = \frac{d}{dx_1} \det \left[\Psi^{j-i-1}(x_j)\right]_{i,j=1}^k.$$

Using that $\frac{d}{dy}\Psi^m(y) = \Psi^{m+1}(y)$, we see that RHS above is the determinant of a matrix, whose first column is $\Psi^0(x_1), \dots, \Psi^{k-1}(x_1)$ and its $j$-th column for $2 \leq j \leq k$ is $\Psi^{j-2}(x_j), \Psi^{j-3}(x_j), \dots, \Psi^{j-k-1}(x_j)$. In particular, when $x_2 = x_1$ the first two columns are the same and so the determinant vanishes. The LHS also vanishes because of the integral $\int_{x_1}^{x_2} dy_2$, and so to show equality it suffices to show equality of the derivatives w.r.t. $x_2$. I.e. we want when $x_1 \leq x_2 \leq \cdots \leq x_k$ and $y_1 = x_1$, $y_2 = x_2$

$$\int_{x_2}^{x_3} dy_3 \cdots \int_{y_{n-1}}^{x_n} dy_n \det \left[\Psi^{j-i}(y_j)\right]_{i,j=1}^k = \frac{d}{dx_2}\frac{d}{dx_1} \det \left[\Psi^{j-i-1}(x_j)\right]_{i,j=1}^k.$$

In this case, when $x_3 = x_2$, the RHS vanishes as the second and third column of the matrix become the same, while the LHS vanishes because of $\int_{x_2}^{x_3} dy_3$. Thus it is enough to show that the derivatives w.r.t. $x_3$ are equal. Continuing in this fashion for $x_3, \dots, x_k$, we see that (3.4.21) will follow if we know that

$$\det \left[\Psi^{j-i}(x_j)\right]_{i,j=1}^k = \frac{d}{dx_k} \cdots \frac{d}{dx_2}\frac{d}{dx_1} \det \left[\Psi^{j-i-1}(x_j)\right]_{i,j=1}^k.$$

The above is now a trivial consequence of $\frac{d}{dy}\Psi^m(y) = \Psi^{m+1}(y)$ and so we conclude the validity of (3.4.21).

Our work above together with Proposition 3.4.3 show that when $x_1 \leq x_2 \leq \cdots \leq x_k$

$$\lim_{M\to\infty} \mathbb{P}_{u,v}^{N,M}\left(\frac{Y_i^i - aM}{c\sqrt{M}} \leq x_i; i = 1, \dots, k\right) = \mathbb{P}(\lambda_1^1 \leq x_1, \cdots, \lambda_k^k \leq x_k).$$

Since with probability 1, we have $Y_1^1 \leq Y_2^2 \leq \cdots \leq Y_k^k$ and $\lambda_1^1 \leq \lambda_2^2 \leq \cdots \leq \lambda_k^k$, the above equality readily extends to all $x_1, \dots, x_k \in \mathbb{R}$. In particular, we obtain the following lemma.

**Lemma 3.4.5.** *Assume the same notation as in Theorem 3.1.3. For any $k \geq 1$, we have that*

$$\frac{1}{c\sqrt{M}}(Y_1^1(M) - aM, \cdots, Y_k^k(M) - aM),$$

*converge weakly to the vector $(\lambda_1^1, \dots, \lambda_k^k)$, where $\lambda_i^j$ for $i = 1, \dots, j$ and $j = 1, \dots, k$ is the GUE-corners process of rank $k$.*

The above lemma will be one of the central ingredients necessary for the proof of Theorem 3.1.3 and we use it below to prove Theorem 3.1.1

*Proof.* (Theorem 3.1.1) Assume the same notation as in Theorem 3.1.1. It follows from our discussion in the proof of Proposition 3.4.1 that

$$\mathbb{P}_{u,v}^{N,M}\left(\{\omega : \lambda_{N-i+1}^N(\omega) = i, 1 \leq i \leq k\}\right) = \mathbb{P}_{u,v}^{N,M}(Y_1^1 \leq N, \dots, Y_k^k \leq N) = \mathbb{P}_{u,v}^{N,M}(Y_k^k \leq N).$$

Let $x \in \mathbb{R}$ and notice that as $N \geq \gamma \cdot M$ with $\gamma > a$, we have that for all large $M$,

$$\mathbb{P}^{N,M}_{u,v}(Y_1^1 \leq N, \dots, Y_k^k \leq N) \geq \mathbb{P}^{N,M}_{u,v}\left(\frac{Y_i^i - aM}{c\sqrt{M}} \leq x; i = 1, \dots, k\right).$$

By Lemma 3.4.5, the latter expression converges to $\mathbb{P}\left(\lambda_i^i \leq x; i = 1, \dots, k\right) = \mathbb{P}\left(\lambda_k^k \leq x\right)$ as $M \to \infty$. Thus we have

$$\liminf_{M \to \infty} \mathbb{P}^{N,M}_{u,v}\left(\{\omega : \lambda^N_{N-i+1}(\omega) = i, 1 \leq i \leq k\}\right) \geq \mathbb{P}\left(\lambda_k^k \leq x\right).$$

The above holds for all $x \in \mathbb{R}$, and sending $x \to \infty$ we conclude the statement of the theorem.

□

## 3.5 Gibbs measures on Gelfand-Tsetlin cones

In this section we investigate probability measures on Gelfand-Tsetlin cones in $\mathbb{R}^{n(n+1)/2}$, which satisfy what is known as the continuous Gibbs property (see Definition 3.5.4 below). An example of such a measure is given by the GUE-corners process $\lambda_i^j$, $i = 1, \dots, j$, $j = 1, \dots, n$ of rank $n$. The main result of this section is Proposition 3.5.6, which can be understood as a classification result for the GUE-corners process. Essentially, it distinguishes the GUE-corners process as the unique probability measure on the Gelfand-Tsetlin cone $GT^n$ (defined in Section 3.5.1 below), which satisfies the continuous Gibbs property and has a certain marginal distribution. A similar result, which we also use, is given by Proposition 6 in [52].

It is well known that Gibbs measures on $\mathcal{C}_n$ are related to measures on $n \times n$ Hermitian matrices, that are invariant under the action of the unitary group $U(n)$ (see e.g. [45]). The study of unitarily invariant measures on Hermitian matrices is a rich subject with connections to many branches of mathematics. A towering result in this area is the classification of the ergodic unitarily invariant Borel probability measures on infinite Hermitian matrices [71], which can be viewed as the origin of our GUE-corners process classification result.

### 3.5.1 The continuous Gibbs property

In what follows we adopt some of the terminology from [45] and [52]. Let $\mathcal{C}_n$ be the *Weyl chamber* in $\mathbb{R}^n$ i.e.

$$\mathcal{C}_n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq x_2 \leq \dots \leq x_n\}.$$

For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{n-1}$ we write $x \succeq y$ to mean that

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \dots \leq x_{n-1} \leq y_{n-1} \leq x_n.$$

For $x = (x_1, \dots, x_n) \in \mathcal{C}_n$ we define the *Gelfand-Tsetlin polytope* to be

$$GT_n(x) := \{(x^1, \dots, x^n) : x^n = x, x^k \in \mathbb{R}^k, x^k \succeq x^{k-1}, 2 \leq k \leq n\}.$$

We explain what we mean by the uniform measure on a Gelfand-Tsetlin polytope $GT_n(x)$. The latter set is a bounded convex set $C$ of a real vector space. We define its volume, as we do for any bounded convex set, to be to be its measure according to the Lebesgue measure on the real affine subspace that it spans (if the subspace is of dimension 0, i.e. $x_1 = \cdots = x_n$ the Lebesgue measure is given by the delta mass at $x_1 = \cdots = x_n$) and denote it by $vol(C)$. We define the Lebesgue measure on $C$ as this Lebesgue measure restricted to $C$ and the uniform probability measure on $C$ as the normalized Lebesgue measure on $C$ by $vol(C)$. The inclusion $x^k \in \mathbb{R}^k$ identifies $GT_n(x)$ as a subset of $\mathbb{R}^{n(n-1)/2}$ and we can naturally think of measures on $GT_n(x)$ as measures on $\mathbb{R}^{n(n-1)/2}$.

If $\lambda \in \mathcal{C}_n$ we denote by $\mu_\lambda$ the image of the uniform measure on $GT_n(\lambda)$ by the map $p_{n-1} : x \in GT_n(\lambda) \to x^{n-1} \in \mathcal{C}_{n-1}$. Let $l_\lambda$ be the Lebesgue measure on the convex set $p_{n-1}(GT_n(\lambda))$. Then Lemma 3.8 of [45] shows that $\mu_\lambda$ is a probability measure on the set $\{x^{n-1} \in \mathcal{C}_{n-1} : \lambda \succeq x^{n-1}\}$ and

$$\mu_\lambda(d\beta) = \frac{d_{n-1}(\beta)}{d_n(\lambda)} l_\lambda(d\beta),$$

where $d_k(\lambda)$ for $\lambda \in \mathcal{C}_k$ denotes $vol(GT_k(\lambda))$. Lemma 3.7 in [45] shows that $d_n(\lambda)$ is explicitly given by

$$d_n(\lambda) = \prod_{\substack{1 \le i < j \le n \\ \lambda_i \neq \lambda_j}} \frac{\lambda_j - \lambda_i}{j - i}.$$

For $\lambda \in \mathcal{C}_n$ we define $\mathbb{E}^{\mu_\lambda}$ to be the expectation with respect to $\mu_\lambda$ as defined above and we also set $\mathbb{E}^\lambda$ to be the expectation with respect to the uniform measure on $GT_n(\lambda)$ as defined above. We summarize some of the properties of these expectations in a sequence of lemmas, whose proof is deferred to Section 3.5.2.

**Lemma 3.5.1.** *Fix $n \ge 2$. Let $\lambda \in \mathcal{C}_n$ and $\lambda^k \in \mathcal{C}_n$ be such that $\lim_{k\to\infty} |\lambda - \lambda^k| = 0$. Suppose $f : \mathbb{R}^{n-1} \to \mathbb{C}$ is a bounded continuous function. Then we have*

$$\lim_{k\to\infty} \mathbb{E}^{\mu_{\lambda^k}} [f(x)] = \mathbb{E}^{\mu_\lambda} [f(x)].$$

**Lemma 3.5.2.** *Let $n \ge 2$ and $f : \mathbb{R}^{n(n+1)/2} \to \mathbb{C}$ be bounded and continuous. Then the function*

$$g(y) := \mathbb{E}^y \left[ f(y, x^{n-1}, \dots, x^1) \right], \text{ is bounded and continuous on } \mathcal{C}_n.$$

**Lemma 3.5.3.** *Let $n \ge 2$, $\lambda \in \mathcal{C}_n$ and $\lambda^k \in \mathcal{C}_n$ be such that $\lim_{k\to\infty} |\lambda - \lambda^k| = 0$. Suppose $f : \mathbb{R}^{n(n-1)/2} \to \mathbb{C}$ is a bounded continuous function. Then we have*

$$\lim_{k\to\infty} \mathbb{E}^{\lambda^k} \left[ f(x^{n-1}, \dots, x^1) \right] = \mathbb{E}^\lambda \left[ f(x^{n-1}, \dots, x^1) \right].$$

We define the *Gelfand-Tsetlin cone* $GT^n$ to be

$$GT^n = \{ y \in \mathbb{R}^{n(n+1)/2} : y_i^{j+1} \le y_i^j \le y_{i+1}^{j+1}, \ 1 \le i \le j \le n-1 \}.$$

Alternatively, we have $GT^n = \cup_{\lambda \in \mathcal{C}_n} GT_n(\lambda)$. We make the following definition after [52].

**Definition 3.5.4.** A probability measure $\mu$ on $GT^n$ is said to satisfy the *continuous Gibbs property* if conditioned on $y^n$ the distribution of $(y^1, \dots, y^{n-1})$ under $\mu$ is uniform on $GT_n(y^n)$. Equivalently, for any bounded continuous function $f : \mathbb{R}^{n(n+1)/2} \to \mathbb{C}$ we have that

$$\mathbb{E}^{\mu}\left[f(y^n, \dots, y^1)\right] = \mathbb{E}^{\mu^n}\left[\mathbb{E}^{y^n}\left[f(y^n, y^{n-1}, \dots, y^1)\right]\right],$$

where $\mu^n$ is the pushforward of $\mu$ to the top row $y^n$ of the Gelfand-Tsetlin cone $GT^n$.

*Remark* 3.5.5. It follows from Lemma 3.5.2 that $\mathbb{E}^{y^n}\left[f(y^n, y^{n-1}, \dots, y^1\right]$ is a continuous function of $y^n$ and so its expectation with respect to $\mu^n$ is a well-defined quantity.

The main result of this section is as follows.

**Proposition 3.5.6.** *Suppose that $\mu$ is a probability distributions on $GT^n$, which satisfies the continuous Gibbs property (Definition 3.5.4). Suppose that the joint distribution of $(y_1^1, \dots, y_n^n)$ under $\mu$ agrees with the law of $(\lambda_1^1, \dots, \lambda_n^n)$, where $\lambda_i^j$, $i = 1, \dots, j$, $j = 1, \dots, n$ is the GUE-corners process of rank $n$. Then $\mu$ is the GUE-corners process of rank $n$.*

The above proposition relies on the following lemmas, whose proof is deferred to Section 3.5.3.

**Lemma 3.5.7.** *For $x^i \in C_i$, $i = 1, .., n$ and $t = (t_1, \dots, t_n)$ with $t_i \in \mathbb{R}$ define*

$$f_n(t, x^n, x^{n-1}, \dots, x^1) := \prod_{i=1}^{n} \exp(\iota t_i(|x^i| - |x^{i-1}|)),$$

*where $|x^k| = x_1^k + \dots + x_k^k$ and $|x^0| = 0$. Suppose $n \geq 2$ and $x^n \in C_n$ with $x_n^n > x_{n-1}^n \cdots > x_1^n$ and $t = (t_1, \dots, t_n)$ with $t_i$ pairwise distinct. Then*

$$d_n(x^n) \cdot \mathbb{E}^{x^n}\left[f_n(t, x^n, \dots, x^1)\right] = \prod_{1 \leq i < j \leq n} \frac{1}{\iota(t_j - t_i)} \times \sum_{\sigma \in S_n} sign(\sigma) \exp\left(\iota \sum_{i=1}^{n} t_{\sigma(i)} x_i^n\right).$$

**Lemma 3.5.8.** *Suppose $n \geq 2$ and $x^n \in C_n$ with $x_n^n > x_{n-1}^n \cdots > x_1^n$. Let $t = (t_1, \dots, t_n)$ with $t_i \in \mathbb{R}$. For $\sigma \in S_n$ we define $t_\sigma := (t_{\sigma(1)}, \dots, t_{\sigma(n)})$ and we set*

$$g_n(t, x^n, x^{n-1}, \dots, x^1) := \prod_{i=1}^{n} \exp(\iota t_i x_i^i).$$

*If $t_i$ are all nonzero we have*

$$d_n(x^n) \cdot \sum_{\sigma \in S_n} sign(\sigma) \mathbb{E}^{x^n}\left[g_n(t_\sigma, x^n, \dots, x^1)\right] \prod_{i=1}^{n} (\iota t_{\sigma(i)})^{n-i} = (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in S_n} sign(\sigma) \exp\left(\iota \sum_{i=1}^{n} t_{\sigma(i)} x_i^n\right)$$

*Proof.* (Proposition 3.5.6) Suppose $t = (t_1, \dots, t_n)$ with $t_i \in \mathbb{R}$ is such that $t_i$ are pairwise distinct and non-zero. It follows from Lemmas 3.5.7 and 3.5.8 that if $x_n^n > x_{n-1}^n > \dots > x_1^n$, we have

$$\mathbb{E}^{x^n}\left[f_n(t,x^n,\dots,x^1)\right] = \prod_{1\le i<j\le n}\frac{(-1)}{\iota(t_j-t_i)}\sum_{\sigma\in S_n} sign(\sigma)\mathbb{E}^{x^n}\left[g_n(t_\sigma,x^n,\dots,x^1)\right]\prod_{i=1}^{n}(\iota t_{\sigma(i)})^{n-i}.$$

From Lemma 3.5.3 we know that both sides of the above equality are continuous in $x^n$ and so the equality holds for all $x^n \in \mathcal{C}_n$.

Taking the expectation with respect to $\mu$ on both sides we recognize the LHS as the characteristic function of $(|x^n|-|x^{n-1}|,\dots,|x^2|-|x^1|,|x^1|)$ under the law $\mu$. The RHS is a linear combination of the characteristic functions of $(x_1^1,\dots,x_n^n)$ under the law $\mu$. By assumption, $(x_1^1,\dots,x_n^n)$ has the same law under $\mu$ as $(\lambda_1^1,\dots,\lambda_n^n)$, from which we conclude that

$$\mathbb{E}^{\mu}\left[\exp\left(\sum_{i=1}^{n}\iota t_i(|x^i|-|x^{i-1}|\right)\right] = \mathbb{E}\left[\exp\left(\sum_{i=1}^{n}\iota t_i(|\lambda^i|-|\lambda^{i-1}|\right)\right],$$

whenever $t_i$ are pairwise distinct and non-zero (recall $|x^0|=0$). Since the characteristic functions are continuous in $t$ it follows that the above equality holds for all $t\in\mathbb{R}^n$. As the characteristic function of a distribution uniquely defines it we conclude that $(|x^n|-|x^{n-1}|,\dots,|x^2|-|x^1|,|x^1|)$ are i.i.d. Gaussian random variables with mean 0 and variance 1. The latter together with the continuous Gibbs property, satisfied by $\mu$, implies that $\mu$ is the GUE corners process by Proposition 6 in [52]. □

### 3.5.2 Proof of Lemmas 3.5.1, 3.5.2 and 3.5.3

We adopt the same notation as in Section 3.5.1.

*Proof.* (Lemma 3.5.1) We begin by first assuming that $f(x)=\prod_{i=1}^{n-1}f_i(x_i)$ where $f_i$ are bounded, continuous and non-negative real-valued functions. Let $1\le n_1<n_2<\dots<n_r<n-1$ and $m_1,\dots,m_r>1$ be such that

- $\lambda_i=\lambda_j$ if $i,j\in M_q$ for some $q=1,\dots,r$;
- $\lambda_i<\lambda_j$ if $i<j$ and $\{i,j\}\not\subset M_q$ for any $q=1,\dots,r$,

where $M_q=\{n_q,\dots,n_q+m_q-1\}$. We also set $J:=\{j:1\le j\le n-1,\text{ and }\{j,j+1\}\not\subset M_q\text{ for any }q=1,\dots,r\}$ and $M_q'=\{n_q,\dots,n_q+m_q-2\}$. Then by the definition of $\mu_\lambda$

$$\mathbb{E}^{\mu_\lambda}\left[\prod_{i=1}^{n-1}f_i(x_i)\right] = \prod_{\substack{1\le i<j\le n\\ \lambda_i\ne\lambda_j}}\frac{j-i}{\lambda_j-\lambda_i}\times\prod_{q=1}^{r}\prod_{j\in M_q'}f_j(\lambda_{n_q})\times\prod_{1\le i<j\le r}\prod_{s\in M_i'}\prod_{t\in M_j'}\frac{\lambda_t-\lambda_s}{t-s}\times F(\lambda),$$

$$\text{where } F(\lambda)=\left(\prod_{j\in J}\int_{\lambda_j}^{\lambda_{j+1}}dx_j\right)\prod_{j\in J}f_j(x_j)\prod_{\substack{1\le i<j\le n-1\\ i,j\in J}}\frac{x_j-x_i}{j-i}\prod_{\substack{1\le i<j\le n-1\\ i\in J,j\notin J}}\frac{x_j-\lambda_i}{j-i}\prod_{\substack{1\le i<j\le n-1\\ i\notin J,j\in J}}\frac{\lambda_j-x_i}{j-i}.$$

Let us assume that for each $k$ we have $\lambda_1^k<\lambda_2^k<\dots<\lambda_n^k$. Then the above formula yields

$$\mathbb{E}^{\mu_{\lambda^k}}\left[\prod_{i=1}^{n-1} f_i(x_i)\right] = \prod_{1\le i<j\le n} \frac{j-i}{\lambda_j^k-\lambda_i^k}\left(\prod_{j=1}^{n-1}\int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j\right)\prod_{j=1}^{n-1} f_j(x_j) \prod_{1\le i<j\le n-1}\frac{x_j-x_i}{j-i}.$$

Suppose $\epsilon>0$ is given. Then if $k$ is sufficiently large we know by the continuity of the functions that for all $j\in J$ we have $|f_j(x_j)-f(\lambda_j)|<\epsilon$ for all $x_j\in[\lambda_j^k,\lambda_{j+1}^k]$. Using that $f_i$ are uniformly bounded by some $M$ we conclude that

$$\left|\mathbb{E}^{\mu_{\lambda^k}}\left[\prod_{i=1}^{n-1} f_i(x_i)\right] - \prod_{1\le i<j\le n} \frac{j-i}{\lambda_j^k-\lambda_i^k}\left(\prod_{j=1}^{n-1}\int_{\lambda_{j+1}^k}^{\lambda_j^k} dx_j\right)\prod_{j\notin J} f_j(\lambda_j)\prod_{j\in J} f_j(x_j) \prod_{1\le i<j\le n-1}\frac{x_j-x_i}{j-i}\right| < C\epsilon, \tag{3.5.1}$$

for all sufficiently large $k$, where $C$ can be taken to be $(n-1)(1+M)^{n-1}$. Observing that $\prod_{j\notin J} f_j(\lambda_j)=\prod_{q=1}^{r}\left(\prod_{j=0}^{m_q-2} f_{j+n_q}(\lambda_{n_q})\right)$ and using (3.5.1) we get

$$\limsup_{k\to\infty}\left|\mathbb{E}^{\mu_\lambda}\left[\prod_{i=1}^{n-1} f_i(x_i)\right] - \mathbb{E}^{\mu_{\lambda^k}}\left[\prod_{i=1}^{n-1} f_i(x_i)\right]\right| \le C\epsilon + (M+1)^n \limsup_{k\to\infty}|G_1(\lambda)-G_2(\lambda^k)|, \tag{3.5.2}$$

where $G_1(\lambda)=\left(\prod_{\substack{i,j=1,\dots,r\\ i<j}}\prod_{s\in M_i'}\prod_{t\in M_j'}\frac{\lambda_t-\lambda_s}{t-s}\right)\prod_{\substack{1\le i<j\le n\\ \lambda_i\ne\lambda_j}}\frac{j-i}{\lambda_j-\lambda_i}F(\lambda)$, and

$$G_2(\lambda^k) = \prod_{1\le i<j\le n} \frac{j-i}{\lambda_j^k-\lambda_i^k}\left(\prod_{j=1}^{n-1}\int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j\right)\prod_{j\in J} f_j(x_j) \prod_{1\le i<j\le n-1}\frac{x_j-x_i}{j-i}.$$

For $j\notin J$ denote by $\phi(j)$ the $q$ such that $\{j,j+1\}\subset M_q$. We define $G_2^+$ and $G_2^-$ as follows

$$G_2^+(\lambda^k) = \prod_{1\le i<j\le n} \frac{j-i}{\lambda_j^k-\lambda_i^k}\left(\prod_{j=1}^{n-1}\int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j\right)\prod_{j\in J} f_j(x_j) \prod_{\substack{1\le i<j\le n-1\\ i,j\in J}}\frac{x_j-x_i}{j-i}$$

$$\times \prod_{\substack{j\in J, i\notin J\\ i<j}}\frac{x_j-\lambda_i^k}{j-i}\prod_{\substack{i\in J, j\notin J\\ i<j}}\frac{\lambda_{j+1}^k-x_i}{j-i}\prod_{q=1}^{r}\prod_{\substack{i<j,\\ i,j\in M_q'}}\frac{x_j-x_i}{j-i}\prod_{\substack{i,j\notin J, i<j,\\ j\notin M_{\phi(i)}'}}\frac{\lambda_{j+1}^k-\lambda_i^k}{j-i};$$

$$G_2^-(\lambda^k) = \prod_{1\le i<j\le n} \frac{j-i}{\lambda_j^k-\lambda_i^k}\left(\prod_{j=1}^{n-1}\int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j\right)\prod_{j\in J} f_j(x_j) \prod_{\substack{1\le i<j\le n-1\\ i,j\in J}}\frac{x_j-x_i}{j-i}$$

$$\times \prod_{\substack{j\in J, i\notin J\\ i<j}}\frac{x_j-\lambda_{i+1}^k}{j-i}\prod_{\substack{i\in J, j\notin J\\ i<j}}\frac{\lambda_j^k-x_i}{j-i}\prod_{q=1}^{r}\prod_{\substack{i<j,\\ i,j\in M_q'}}\frac{x_j-x_i}{j-i}\prod_{\substack{i,j\notin J, i<j,\\ j\notin M_{\phi(i)}'}}\frac{\lambda_j^k-\lambda_{i+1}^k}{j-i};$$

Using the non-negativity of $f_i$ we observe that $G_2^-(\lambda^k)\le G_2(\lambda^k)\le G_2^+(\lambda^k)$.

Performing the integration over $x_j$ for $j \notin J$ we may rewrite $G_2^+(\lambda^k)$ as

$$G_2^+(\lambda^k) = \prod_{\substack{1 \le i < j \le n \\ \lambda_i \neq \lambda_j}} \frac{j-i}{\lambda_j^k - \lambda_i^k} \prod_{\substack{i,j \notin J, i<j, \\ j \notin M'_{\phi(i)}}} \frac{\lambda_{j+1}^k - \lambda_i^k}{j-i} \left( \prod_{j \in J} \int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j \right) \prod_{j \in J} f_j(x_j) \times$$

$$\prod_{\substack{1 \le i < j \le n-1 \\ i,j \in J}} \frac{x_j - x_i}{j-i} \prod_{\substack{j \in J, i \notin J \\ i<j}} \frac{x_j - \lambda_i^k}{j-i} \prod_{\substack{i \in J, j \notin J \\ i<j}} \frac{\lambda_{j+1}^k - x_i}{j-i}.$$

Similarly, we have

$$G_2^-(\lambda^k) = \prod_{\substack{1 \le i < j \le n \\ \lambda_i \neq \lambda_j}} \frac{j-i}{\lambda_j^k - \lambda_i^k} \prod_{\substack{i,j \notin J, i<j, \\ j \notin M'_{\phi(i)}}} \frac{\lambda_j^k - \lambda_{i+1}^k}{j-i} \left( \prod_{j \in J} \int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j \right) \prod_{j \in J} f_j(x_j) \times$$

$$\prod_{\substack{1 \le i < j \le n-1 \\ i,j \in J}} \frac{x_j - x_i}{j-i} \prod_{\substack{j \in J, i \notin J \\ i<j}} \frac{x_j - \lambda_{i+1}^k}{j-i} \prod_{\substack{i \in J, j \notin J \\ i<j}} \frac{\lambda_j^k - x_i}{j-i}.$$

We observe that

$$\lim_{k \to \infty} \prod_{\substack{1 \le i < j \le n \\ \lambda_i \neq \lambda_j}} \frac{j-i}{\lambda_j^k - \lambda_i^k} \prod_{\substack{i,j \notin J, i<j, \\ j \notin M'_{\phi(i)}}} \frac{\lambda_{j+1}^k - \lambda_i^k}{j-i} = \lim_{k \to \infty} \prod_{\substack{1 \le i < j \le n \\ \lambda_i \neq \lambda_j}} \frac{j-i}{\lambda_j^k - \lambda_i^k} \prod_{\substack{i,j \notin J, i<j, \\ j \notin M'_{\phi(i)}}} \frac{\lambda_j^k - \lambda_{i+1}^k}{j-i} =$$
$$\prod_{\substack{1 \le i < j \le n \\ \lambda_i \neq \lambda_j}} \frac{j-i}{\lambda_j - \lambda_i} \times \prod_{\substack{i,j=1,\ldots,r \\ i<j}} \prod_{s \in M'_i} \prod_{t \in M'_j} \frac{\lambda_t - \lambda_s}{t-s}. \tag{3.5.3}$$

Moreover, by the Bounded Convergence Theorem we conclude that

$$\lim_{k \to \infty} \left( \prod_{j \in J} \int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j \right) \prod_{j \in J} f_j(x_j) \prod_{\substack{1 \le i < j \le n-1 \\ i,j \in J}} \frac{x_j - x_i}{j-i} \prod_{\substack{j \in J, i \notin J \\ i<j}} \frac{x_j - \lambda_i^k}{j-i} \prod_{\substack{i \in J, j \notin J \\ i<j}} \frac{\lambda_{j+1}^k - x_i}{j-i} = F(\lambda),$$
$$\lim_{k \to \infty} \left( \prod_{j \in J} \int_{\lambda_j^k}^{\lambda_{j+1}^k} dx_j \right) \prod_{j \in J} f_j(x_j) \prod_{\substack{1 \le i < j \le n-1 \\ i,j \in J}} \frac{x_j - x_i}{j-i} \prod_{\substack{j \in J, i \notin J \\ i<j}} \frac{x_j - \lambda_{i+1}^k}{j-i} \prod_{\substack{i \in J, j \notin J \\ i<j}} \frac{\lambda_j^k - x_i}{j-i} = F(\lambda). \tag{3.5.4}$$

From (3.5.3) and (3.5.4) we conclude that $\lim_{k \to \infty} G_2^{\pm}(\lambda^k) = G_1(\lambda)$ and since $G_2^-(\lambda^k) \le G_2(\lambda^k) \le G_2^+(\lambda^k)$ we conclude that $\lim_{k \to \infty} G_2(\lambda^k) = G_1(\lambda)$. The latter implies from (3.5.2) that

$$\limsup_{k \to \infty} \left| \mathbb{E}^{\mu_\lambda} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] - \mathbb{E}^{\mu_{\lambda^k}} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] \right| \le C\epsilon.$$

Since $\epsilon > 0$ was arbitrary we conclude that

$$\limsup_{k\to\infty} \left| \mathbb{E}^{\mu_\lambda} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] - \mathbb{E}^{\mu_{\lambda^k}} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] \right| = 0.$$

We next suppose that $\lambda^k$ do not necessarily satisfy $\lambda_1^k < \lambda_2^k < \cdots < \lambda_n^k$. If we are given a $\lambda^k$, then from our earlier work we may find $\nu^k$ such that

1. $|\nu^k - \lambda^k| < 1/k$,
2. $\nu_1^k < \nu_2^k < \cdots < \nu_n^k$,
3. $\left|\mathbb{E}^{\mu_{\lambda^k}} \left[\prod_{i=1}^{n-1} f_i(x_i)\right] - \mathbb{E}^{\mu_{\nu^k}} \left[\prod_{i=1}^{n-1} f_i(x_i)\right]\right| < 1/k$.

Condition (1) above implies that $\nu^k$ converges to $\lambda$ and by (2) our earlier work applies so we get

$$\limsup_{k\to\infty} \left| \mathbb{E}^{\mu_\lambda} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] - \mathbb{E}^{\mu_{\nu^k}} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] \right| = 0.$$

Finally, by the triangle inequality and condition (3) we conclude that

$$\limsup_{k\to\infty} \left| \mathbb{E}^{\mu_\lambda} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] - \mathbb{E}^{\mu_{\lambda^k}} \left[ \prod_{i=1}^{n-1} f_i(x_i) \right] \right| = 0.$$

This proves the statement of the lemma, whenever $f(x) = \prod_{i=1}^{n-1} f_i(x_i)$ with $f_i$ bounded, continuous and non-negative real-valued functions.

Using linearity of expectation and our earlier result we concude the statement of the lemma, whenever $f(x)$ is a finite linear combination of functions of the form $\prod_{i=1}^{n-1} f_i(x_i)$ with $f_i$ bounded and continuous. In particular, we know the result whenever $f$ equals $P(x) \cdot \mathbf{1}_{B_R}$, where $R > 0$, $B_R = \{x \in \mathbb{R}^{n-1} | |x_i| \leq R \text{ for } i = 1, \dots, n-1\}$ and $P(x)$ is a polynomial.

If $f(x)$ is any bounded continuous function, we may replace it with $f(x)\mathbf{1}_{B_R}$, where $R = 1 + \max(|\lambda_1|, |\lambda_n|)$, without affecting the statement of the lemma, since for large $k$, the support of $\mu_{\lambda^k}$ lies in $B_R$. By the Stone-Weierstrass Theorem we can find a polynomial $g(x)$ such that $\sup_{x \in \mathbb{R}^{n-1}} |f(x)\mathbf{1}_{B_R} - g(x)\mathbf{1}_{B_R}| < \epsilon$. The triangle inequality and our result for polynomials now show

$$\limsup_{k\to\infty} |\mathbb{E}^{\mu_\lambda}[f(x)] - \mathbb{E}^{\mu_{\lambda^k}}[f(x)]| = \limsup_{k\to\infty} |\mathbb{E}^{\mu_\lambda}[f(x)\mathbf{1}_{B_R}] - \mathbb{E}^{\mu_{\lambda^k}}[f(x)\mathbf{1}_{B_R}]| \leq$$

$$\limsup_{k\to\infty}(|\mathbb{E}^{\mu_\lambda}[f(x)\mathbf{1}_{B_R}] - \mathbb{E}^{\mu_\lambda}[g(x)\mathbf{1}_{B_R}]| + |\mathbb{E}^{\mu_{\lambda^k}}[f(x)\mathbf{1}_{B_R}] - \mathbb{E}^{\mu_{\lambda^k}}[g(x)\mathbf{1}_{B_R}]| +$$

$$|\mathbb{E}^{\mu_\lambda}[g(x)\mathbf{1}_{B_R}] - \mathbb{E}^{\mu_{\lambda^k}}[g(x)\mathbf{1}_{B_R}]|) \leq 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary we conclude that $\limsup_{k\to\infty} |\mathbb{E}^{\mu_\lambda}[f(x)] - \mathbb{E}^{\mu_{\lambda^k}}[f(x)]| = 0$.

□

*Proof.* (Lemma 3.5.2) We begin by assuming that $f(x^n,\dots,x^1) = f_1(x^n)f_2(x^{n-1},\dots,x^1)$ with $f_1, f_2$ bounded and continuous. Fix $\beta \in \mathcal{C}_n$ and suppose $\mathcal{C}_n \ni \beta^k \to \beta$ as $k \to \infty$. From Lemma 3.5.1 we have

$$\lim_{k\to\infty} \mathbb{E}^{\beta^k}[f(\beta^k, x^{n-1},\dots,x^1)] = \lim_{k\to\infty} f_1(\beta^k)\mathbb{E}^{\beta^k}[f_2(x^{n-1},\dots,x^1)] = f_1(\beta)\mathbb{E}^{\beta}[f_2(x^{n-1},\dots,x^1)].$$

Using linearity of expectation and the above we have that $\mathbb{E}^{\beta}[f(\beta, x^{n-1},\dots,x^1)]$ is a continuous function in $\beta$, whenever $f$ is of the form $P(x^n,x^{n-1},\dots,x^1)\cdot \mathbf{1}_{B_R}$, where $R > 0$, $B_R = \{x \in \mathbb{R}^{n(n+1)/2} | |x_i^j| \leq R \text{ for } i = 1,\dots,j; j = 1,\dots,n\}$ and $P(x)$ is a polynomial.

Suppose now $f$ is any bounded continuous function, fix $\beta \in \mathcal{C}_n$ and suppose $\mathcal{C}_n \ni \beta^k \to \beta$ as $k \to \infty$. For all large $k$ we have that $\beta^k$ lie in the compact set $B_R$, with $R = 1 + \max(|\beta_1|, |\beta_n|)$. By the Stone-Weierstrass Theorem we can find a polynomial $g(x)$ such that $\sup_{x\in\mathbb{R}^{n(n+1)/2}} |f(x)\mathbf{1}_{B_R} - g(x)\mathbf{1}_{B_R}| < \epsilon$. The triangle inequality and our result for polynomials now show

$$\limsup_{k\to\infty} \left|\mathbb{E}^{\beta^k}[f(\beta^k, x^{n-1},\dots,x^1)] - \mathbb{E}^{\beta}[f(\beta, x^{n-1},\dots,x^1)]\right| =$$

$$\limsup_{k\to\infty} \left|\mathbb{E}^{\beta^k}[f(\beta^k, x^{n-1},\dots,x^1)\mathbf{1}_{B_R}] - \mathbb{E}^{\beta}[f(\beta, x^{n-1},\dots,x^1)\mathbf{1}_{B_R}]\right| \leq$$

$$\limsup_{k\to\infty} \left|\mathbb{E}^{\beta^k}[g(\beta^k, x^{n-1},\dots,x^1)\mathbf{1}_{B_R}] - \mathbb{E}^{\beta}[g(\beta, x^{n-1},\dots,x^1)\mathbf{1}_{B_R}]\right| + 2\epsilon = 2\epsilon.$$

As $\epsilon > 0$ was arbitrary we conclude continuity, while boundedness is immediate from the boundedness of $f$. $\square$

*Proof.* (Lemma 3.5.3) We proceed by induction on $n$ with $n = 2$ being true by Lemma 3.5.1. Suppose the result holds for $n - 1 \geq 2$ and we want to prove it for $n$.

For any $\nu \in \mathcal{C}_n$ we have

$$\mathbb{E}^{\nu}\left[f(x^{n-1},\dots,x^1)\right] = \int_{\mathcal{C}_{n-1}} \mu_\nu(d\beta)\mathbb{E}^{\beta}[f(\beta, x^{n-2},\dots,x^1)] = \mathbb{E}^{\mu_\nu}\left[\mathbb{E}^{\beta}[f(\beta, x^{n-2},\dots,x^1)]\right].$$

By Lemma 3.5.2, we have $\mathbb{E}^{\beta}[f(\beta, x^{n-2},\dots,x^1)]$ is a bounded and continuous function in $\beta \in \mathcal{C}_n$. From Lemma 3.5.1 we conclude that

$$\lim_{k\to\infty} \mathbb{E}^{\mu_{\lambda^k}}\left[\mathbb{E}^{\beta}[f(\beta, x^{n-2},\dots,x^1)]\right] = \mathbb{E}^{\mu_\lambda}\left[\mathbb{E}^{\beta}[f(\beta, x^{n-2},\dots,x^1)]\right].$$

This proves the result for $n$ and the general result follows by induction. $\square$

### 3.5.3 Proof of Lemmas 3.5.7 and 3.5.8

We adopt the same notation as in Section 3.5.1.

*Proof.* (Lemma 3.5.7) We proceed by induction on $n$. When $n = 2$ we have that

$$d_2(x^n)\cdot \mathbb{E}^{x^2}\left[f_2(t, x^2, x^1)\right] = e^{\iota t_2(x_1^2+x_2^2)}\int_{x_1^2}^{x_2^2} e^{\iota(t_1-t_2)x}dx = e^{\iota t_2(x_1^2+x_2^2)}\frac{e^{\iota(t_1-t_2)x_2^2} - e^{\iota(t_1-t_2)x_1^2}}{\iota(t_1-t_2)} =$$

$$\frac{1}{\iota(t_1-t_2)} \times \left[\exp(\iota(t_2x_1^2+t_1x_2^2)) - \exp(\iota(t_1x_1^2+t_2x_2^2))\right],$$

which proves the base case.

Suppose the result holds for $n-1 \geq 2$ and we wish to prove it for $n$. We have

$$d_n(x)\cdot\mathbb{E}^{x^n}\left[f(t,x^n,\ldots,x^1)\right] = \int_{x_{n-1}^n}^{x_n^n}\cdots\int_{x_1^n}^{x_2^n} e^{it_n|x^n|}dy_{n-1}\cdots dy_1 d_{n-1}(y)\cdot\mathbb{E}^y\left[f_{n-1}(s,y,x^{n-2},\ldots,x^1)\right],$$

where $s=(t_1-t_n,\ldots,t_{n-2}-t_n,t_{n-1}-t_n)$. By induction hypothesis the above becomes

$$e^{\iota t_n|x^n|}\int_{x_{n-1}^n}^{x_n^n}\cdots\int_{x_1^n}^{x_2^n} dy_{n-1}\cdots dy_1 \prod_{1\leq i<j\leq n-1}\frac{1}{\iota(t_j-t_i)} \times \sum_{\sigma\in S_{n-1}} sign(\sigma)\exp\left(\iota\sum_{i=1}^{n-1}s_iy_{\sigma(i)}\right) =$$

$$\prod_{1\leq i<j\leq n-1}\frac{1}{\iota(t_j-t_i)}\sum_{\sigma\in S_{n-1}} sign(\sigma)e^{it_n|x^n|}\prod_{i=1}^{n-1}\frac{\exp\left(\iota s_i x_{\sigma(i)}^n\right)-\exp\left(\iota s_i x_{\sigma(i)+1}^n\right)}{\iota(t_n-t_i)} =$$

$$e^{\iota t_n|x^n|}\prod_{1\leq i<j\leq n}\frac{1}{\iota(t_j-t_i)}\sum_{\sigma\in S_{n-1}} sign(\sigma)\prod_{i=1}^{n-1}\left(\exp\left(\iota s_{\sigma(i)}x_i^n\right)-\exp\left(\iota s_{\sigma(i)}x_{i+1}^n\right)\right),$$

where in the last equality we used that $sign(\sigma)=sign(\sigma^{-1})$.

The above equality reduces the induction step to showing

$$\sum_{\sigma\in S_{n-1}} sign(\sigma)\prod_{i=1}^{n-1}\left(\exp\left(\iota s_{\sigma(i)}x_i^n\right)-\exp\left(\iota s_{\sigma(i)}x_{i+1}^n\right)\right) = \sum_{\sigma\in S_n} sign(\sigma)\exp\left(\iota\sum_{i=1}^{n}s_{\sigma(i)}x_i^n\right), \tag{3.5.5}$$

where $s_n=0$.

Put $A_{i,\sigma}=\exp\left(\iota s_{\sigma(i)}x_i^n\right)$ and $B_{i,\sigma}=-\exp\left(\iota s_{\sigma(i)}x_{i+1}^n\right)$. We open the brackets on the LHS of (3.5.5) and obtain a sum of words $sign(\sigma)C_{1,\sigma}\cdots C_{n-1,\sigma}$, where $C=A$ or $B$. We consider the words that have $B$ followed by an $A$ at positions $r,r+1$ and set $\tau$ to be the transposition $(r,r+1)$. Observe that

$$sign(\sigma)B_{r,\sigma}A_{r+1,\sigma}+sign(\tau\sigma)B_{r,\tau\sigma}A_{r+1,\tau\sigma}=0, \text{ and hence}$$

$$\sum_{\sigma\in S_{n-1}} sign(\sigma)C_{1,\sigma}\cdots C_{r-1,\sigma}B_{r,\sigma}A_{r+1,\sigma}C_{r+2,\sigma}\cdots C_{n-1,\sigma}=0.$$

The latter implies that the only words that contribute to the LHS of (3.5.5) are $k$ $A$'s followed by $n-k-1$ $B$'s for $k=0,\ldots,n-1$. We conclude that the LHS of (3.5.5) equals

$$\sum_{k=0}^{n-1}(-1)^{n-1-k}\sum_{\sigma\in S_{n-1}} sign(\sigma)\prod_{i=1}^{k}\exp\left(\iota s_{\sigma(i)}x_i^n\right)\prod_{i=k+1}^{n-1}\exp\left(\iota s_{\sigma(i)}x_{i+1}^n\right) \tag{3.5.6}$$

and the latter now clearly equals the RHS of (3.5.5) by inspecting the signs of the summands

$\exp\left(\iota \sum_{i=1}^{n} s_{\sigma(i)} x_i^n\right)$ on both sides for $\sigma \in S_n$. □

*Proof.* (Lemma 3.5.8) We proceed by induction on $n$. When $n = 2$ we have that

$$d_n(x^2)\mathbb{E}^{x^2}\left[g_n(t, x^2, x^1)\right] = e^{\iota t_2 x_2^2} \int_{x_1^2}^{x_2^2} e^{\iota t_1 x} dx = e^{\iota t_2 x_2^2} \frac{e^{\iota t_1 x_2^2} - e^{\iota t_1 x_1^2}}{\iota t_1}.$$

Consequently, we have

$$d_n(x^2) \cdot \sum_{\sigma \in S_2} sign(\sigma)(\iota t_{\sigma(1)})\mathbb{E}^{x^2}\left[g_2(t_\sigma, x^2, \ldots, x^1)\right] = e^{\iota t_1 x_2^2 + \iota t_2 x_1^2} - e^{\iota t_2 x_2^2 + \iota t_1 x_1^2},$$

from which we conclude the base case.

Suppose we know the result for $n - 1 \geq 2$ and we wish to prove it for $n$. We have

$$d_n(x^n) \cdot \sum_{\sigma \in S_n} sign(\sigma)\mathbb{E}^{x^n}\left[g_n(t_\sigma, x^n, \ldots, x^1)\right] \prod_{i=1}^{n} (\iota t_{\sigma(i)})^{n-i} =$$

$$\sum_{\sigma \in S_n} sign(\sigma) e^{\iota t_{\sigma(n)} x_n^n} \prod_{i=1}^{n} (\iota t_{\sigma(i)})^{n-i} \int_{x_{n-1}^n}^{x_n^n} \cdots \int_{x_1^n}^{x_2^n} dy_{n-1} \cdots dy_1 \cdot d_{n-1}(y)\mathbb{E}^y\left[g_{n-1}(s_\sigma, y, x^{n-2}, \ldots, x^1)\right],$$

where $s_\sigma = (t_{\sigma(1)}, \ldots, t_{\sigma(n-1)})$. Splitting the above sum over permutations of $t_{\sigma(1)}, \ldots, t_{\sigma(n-1)}$ and applying the induction hypothesis we see that the above equals

$$(-1)^{\frac{(n-1)(n-2)}{2}} \sum_{k=1}^{n} (-1)^{n-k} e^{\iota t_k x_n^n} \prod_{r \neq k} (\iota t_r) \int_{x_{n-1}^n}^{x_n^n} \cdots \int_{x_1^n}^{x_2^n} dy_{n-1} \cdots dy_1 \sum_{\tau \in S_{n-1}} sign(\tau) \exp\left(\iota \sum_{i=1}^{n-1} s_{\tau(i)}^k y_i^n\right)$$

$$= (-1)^{\frac{(n-1)(n-2)}{2}} \sum_{k=1}^{n} (-1)^{n-k} e^{\iota t_k x_n^n} \sum_{\tau \in S_{n-1}} sign(\tau) \prod_{i=1}^{n-1} \left(\exp\left(\iota s_{\tau(i)}^k x_{i+1}^n\right) - \exp\left(\iota s_{\tau(i)}^k x_i^n\right)\right)$$

where $s^k = (t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$.

Using equation (3.5.6), we may rewrite the above as

$$(-1)^{\frac{(n-1)(n-2)}{2}} \sum_{k=1}^{n} (-1)^{n-k} e^{\iota t_k x_1^n} \sum_{l=0}^{n-1} (-1)^l \sum_{\tau \in S_{n-1}} sign(\tau) \prod_{i=1}^{l} \exp\left(\iota s_{\tau(i)}^k x_i^n\right) \prod_{i=l+1}^{n-1} \exp\left(\iota s_{\tau(i)}^k x_{i+1}^n\right).$$

If $l < n - 1$ we have

$$\sum_{k=1}^{n} (-1)^{n-k} e^{\iota t_k x_n^n} \sum_{\tau \in S_{n-1}} sign(\tau) \prod_{i=1}^{l} \exp\left(\iota s_{\tau(i)}^k x_i^n\right) \prod_{i=l+1}^{n-1} \exp\left(\iota s_{\tau(i)}^k x_{i+1}^n\right) =$$

$$\sum_{\sigma \in S_n} sign(\sigma) \exp\left(\iota(t_{\sigma(l+1)} + t_{\sigma(n)}) x_n^n\right) \prod_{i=1}^{l} \exp\left(\iota t_{\sigma(i)} x_i^n\right) \prod_{i=l+1}^{n-2} \exp\left(\iota t_{\sigma(i)} x_{i+1}^n\right) = 0.$$

To see the last equality we may swap $l + 1$ and $n$ in the above sum by a transposition and observe that we get the same sum but with a flipped sign due to the factors $sign(\sigma)$. Hence, the sum is invariant under change of sign and must be 0. The last argument shows that only $l = n-1$ contributes in our earlier formula and so we conclude that

$$d_n(x^n) \cdot \sum_{\sigma \in S_n} sign(\sigma) \mathbb{E}^{x^n} \left[ g_n(t_\sigma, x^n, \dots, x^1) \right] \prod_{i=1}^{n} (\iota t_{\sigma(i)})^{n-i} =$$

$$= (-1)^{\frac{n(n-1)}{2}} \sum_{k=1}^{n} (-1)^{n-k} e^{\iota t_k x_n^n} \sum_{\tau \in S_{n-1}} sign(\tau) \prod_{i=1}^{n-1} \exp\left( \iota s^k_{\tau(i)} x_i^n \right).$$

The latter expression is clearly equal to $(-1)^{\frac{n(n-1)}{2}} \sum_{\sigma \in S_n} sign(\sigma) \exp\left( \iota \sum_{i=1}^n t_{\sigma(i)} x_i^n \right)$, which proves the case $n$. The general result now follows by induction. □

## 3.6 Gibbs measures on Gelfand-Tsetlin patterns

The purpose of this section is to analyze probability measures on half-strict Gelfand-Tsetlin patterns $\mathsf{GT}_n^+$, which satisfy what we call the *six-vertex Gibbs property* (see Definition 3.6.2). An example of such a measure is given by the distribution function of $(Y_i^j)_{1 \le i \le j; 1 \le j \le n}$ (see Section 3.1.1). The main result of this section is Proposition 3.6.7, which roughly states that under weak limits the six-vertex Gibbs property becomes the continuous Gibbs property (Definition 3.5.4).

### 3.6.1 Gibbs measures on the six-vertex model

In this section we define the Gibbs property for the six-vertex model on a domain $D$. We also explain how to symmetrize such a model when $D$ is finite and relate the weight choice in this chapter to the ferroelectric phase of the six-vertex model. In what follows we will adopt some of the notation from Appendix A in [1].

Suppose we have a finite domain $D \subset \mathbb{Z}^2$. For $\Lambda \subset \mathbb{Z}^2$, we let $\partial\Lambda$ denote the *boundary* of $\Lambda$, which consists of all vertices in $\mathbb{Z}^2/\Lambda$, which are adjacent to some vertex in $\Lambda$. We consider the six-vertex model on $D$ with fixed boundary condition. This is a probability measure on up-right paths in $D$ with fixed endpoints and we explain its construction below.

We start by assigning certain arrow configurations to the vertices in $\partial D$ and consider all up-right path configurations in $D$, which match the arrow assignments in $\partial D$. Call the latter set $\mathcal{P}(D, \partial D)$. Paths are not allowed to share horizontal or vertical pieces and as in Section 3.1.1 we encode the arrow configuration at a vertex through the four-tuple $(i_1, j_1; i_2, j_2)$, representing the number of incoming and outgoing vertical and horizontal arrows. For $(i,j) \in D$ and $\omega \in \mathcal{P}(D, \partial D)$ we let $\omega(i,j)$ denote the arrow configuration at the corresponding vertex. We have six possible arrow configurations and we define corresponding positive vertex weights as follows

$$\begin{aligned} &w(0,0;0,0) = w_1, \;\; w(1,1;1,1) = w_2, \;\; w(1,0;1,0) = w_3, \\ &w(0,1;0,1) = w_4, \;\; w(1,0;0,1) = w_5, \;\; w(0,1;1,0) = w_6. \end{aligned} \tag{3.6.1}$$

The weight of a path configuration $\omega$ is defined through $\mathcal{W}(\omega) := \prod_{(i,j)\in D} w(\omega(i,j))$, and we define the six-vertex model as the the probability measure $\mu$ on $\mathcal{P}(D, \partial D)$ with probability proportional to $\mathcal{W}(\omega)$. As weights are positive and $D$ is finite this is well-defined.

For $\omega \in \mathcal{P}(D, \partial D)$, $\Lambda \subset D$ and an arrow configuration $(i_1, j_1; i_2, j_2)$ we let $N_{\omega;\Lambda}(i_1, j_1; i_2, j_2)$ denote the number of vertices $(x, y) \in \Lambda$ with arrow configuration $(i_1, j_1; i_2, j_2)$. We abbreviate $N_1 = N_{\omega;\Lambda}(0, 0; 0, 0)$, $N_2 = N_{\omega;\Lambda}(1, 1; 1, 1)$, $N_3 = N_{\omega;\Lambda}(1, 0; 1, 0)$, $N_4 = N_{\omega;\Lambda}(0, 1; 0, 1)$, $N_5 = N_{\omega;\Lambda}(1, 0; 0, 1)$, and $N_6 = N_{\omega;\Lambda}(0, 1; 1, 0)$. With this notation we make the following definition.

**Definition 3.6.1.** Fix $w_1, w_2, w_3, w_4, w_5, w_6 > 0$. A probability measure $\rho$ on $\mathcal{P}(D; \partial D)$ is said to satisfy the *Gibbs property* (for the six-vertex model on $D$ with weights $(w_1, w_2, w_3, w_4, w_5, w_6)$) if for any finite subset $\Lambda \subset D$ the conditional probability $\rho_\Lambda(\omega)$ of selecting $\omega \in \mathcal{P}(D, \partial D)$ conditioned on $\omega|_{D/\Lambda}$ is proportional to $w_1^{N_1} w_2^{N_2} w_3^{N_3} w_4^{N_4} w_5^{N_5} w_6^{N_6}$.

Notice that Definition 3.6.1 makes sense even if $D$ is not finite. It is easy to see that the measure $\mu$ we defined earlier satisfies the Gibbs property with weights $(w_1, w_2, w_3, w_4, w_5, w_6)$. Similarly, let us consider the measure $\mathbb{P}_{u,v}^{N,M}$ from Definition 3.4.2 conditioned on the top row $\lambda^N(\omega)$ being fixed. The latter satisfies the Gibbs property for the domain $D_N = \mathbb{Z}_{\geq 0} \times \{1, \dots, N\}$ with weights

$$(w_1, w_2, w_3, w_4, w_5, w_6) = \left(1, \frac{u - s^{-1}}{us - 1}, \frac{us^{-1} - 1}{us - 1}, \frac{u - s}{us - 1}, \frac{u(s^2 - 1)}{us - 1}, \frac{1 - s^{-2}}{us - 1}\right). \tag{3.6.2}$$

The change of sign above compared to (3.1.1) is made so that the above weights are positive (recall $u > s > 1$ in our case).

If we have $w_1 = w_2 = a$, $w_3 = w_4 = b$ and $w_5 = w_6 = c$ we call the resulting model a *symmetric* six-vertex model. Otherwise, we call the model *asymmetric*. An important point we want to make is that a single measure $\rho$ on $\mathcal{P}(D, \partial D)$ can satisfy a Gibbs property for many different 6-tuples of weights $(w_1, w_2, w_3, w_4, w_5, w_6)$. The latter is a consequence of certain *conservation laws* satisfied by the quantities $N_{\omega;\Lambda}(i_1, j_1; i_2, j_2)$. As discussed in Appendix A of [1] we have the following conservation laws (see also Section 3 in [21]).

1. The quantity $N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = |\Lambda|$ is constant.
2. Conditioned on $\omega|_{D/\Lambda}$, the quantity $N_2 + N_4 + N_5$ is constant.
3. Conditioned on $\omega|_{D/\Lambda}$, the quantity $N_2 + N_3 + N_6$ is constant.
4. Conditioned on $\omega|_{D/\Lambda}$, the quantity $N_5 - N_6$ is constant.

The latter imply that if a measure $\rho$ satisfies the Gibbs property with weights $(w_1, w_2, w_3, w_4, w_5, w_6)$ then $\rho$ also satisfies the Gibbs property with weights $(xw_1, xyzw_2, xzw_3, xyw_4, xytw_5, xzt^{-1}w_6)$ for any $x, y, z, t > 0$.

Let us fix $x = \frac{w_2}{\sqrt{w_1 w_2}}$, $y = \frac{\sqrt{w_1 w_2 w_3 w_4}}{w_2 w_4}$, $z = \frac{\sqrt{w_1 w_2 w_3 w_4}}{w_3 w_4}$ and $t = \frac{\sqrt{w_4 w_6}}{\sqrt{w_3 w_5}}$. Then one directly checks that

$$(xw_1, xyzw_2, xzw_3, xyw_4, xytw_5, xzt^{-1}w_6) = (a, a, b, b, c, c),$$

where $a = \sqrt{w_1 w_2}$, $b = \sqrt{w_3 w_4}$ and $c = \sqrt{w_5 w_6}$. The latter shows that any six-vertex model on a finite domain with prescribed boundary condition can be realized as a symmetric six vertex model.

The above arguments can be repeated for other (e.g. periodic) boundary conditions and the consequence is that when working in a finite domain, one can always assume that the six-vertex model is symmetric. This is how the model typically appears in the literature. An important parameter for the symmetric six-vertex model with weights $(a, a, b, b, c, c)$ is

$$\Delta := \frac{a^2 + b^2 - c^2}{2ab}.$$

As discussed in Chapters 8 and 9 in [9] (see also [74]) the symmetric six-vertex model has several phases called *ferroelectric* ($\Delta > 1$), *disordered* ($|\Delta| < 1$) and *antiferroelectric* ($\Delta < -1$).

Based on our earlier discussion, we may extend the definition of $\Delta$ to any (not necessarily symmetric) six-vertex model by

$$\tilde{\Delta} := \frac{w_1 w_2 + w_3 w_4 - w_5 w_6}{2\sqrt{w_1 w_2 w_3 w_4}}.$$

Observe that the latter quantity is invariant under the transformation of $(w_1, w_2, w_3, w_4, w_5, w_6)$ into $(xw_1, xyzw_2, xzw_3, xyw_4, xytw_5, xzt^{-1}w_6)$. This implies that the parameter $\tilde{\Delta}$ for a six-vertex model on a finite domain agrees with the parameter $\Delta$ for its symmetric realization.

For the six-vertex model we defined in Section 3.1.1 a crucial assumption is that $w_1 = 1$, since our configurations contain infinitely many vertices of type $(0, 0; 0, 0)$. This restriction forbids us from freely rescaling our vertex weights and forces us to work with an asymmetric six-vertex model. However, the above extension of $\Delta$ allows us to investigate to which phase our parameter choice $u > s > 1$ corresponds. As remarked $\mathbb{P}^{N,M}_{u,v}$ satisfies the Gibbs property for the domain $D_N = \mathbb{Z}_{\geq 0} \times \{1, \dots, N\}$ with weights as in (3.6.2). For these weights we find that $\tilde{\Delta} = (s + s^{-1})/2$. The latter expression covers $(1, \infty)$ when $s > 1$ and so our parameter choice $u > s > 1$ corresponds to the ferroelectric phase of the six-vertex model.

A natural question that arises from the above discussion is whether we can find different parameter choices for $u$ and $s$, which would land us in the disordered or antiferroelectric phase. If this is achieved one could potentially use the methods of this chapter to study the macroscopic behavior of this new model. It would be very interesting to see if the limit shape in Figure 3-12 changes when we move to a different phase - like in the six-vertex model with periodic (or domain wall) boundary condition. We leave these questions outside of the scope of this chapter.

### 3.6.2 The six-vertex Gibbs property

We define several important concepts, adopting some of the notation from [52]. Let $\mathsf{GT}_n$ denote the set of $n$-tuples of *distinct* integers

$$\mathsf{GT}_n = \{\lambda \in \mathbb{Z}^n : \lambda_1 < \lambda_2 < \cdots < \lambda_n\}.$$

We let $\mathsf{GT}_n^+$ be the subset of $\mathsf{GT}_n$ with $\lambda_1 \geq 0$. We say that $\lambda \in \mathsf{GT}_n$ and $\mu \in \mathsf{GT}_{n-1}$ *interlace* and write $\mu \preceq \lambda$ if

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

Let $\mathsf{GT}^n$ denote the set of sequences

$$\mu^1 \preceq \mu^2 \preceq \cdots \preceq \mu^n, \quad \mu^i \in \mathsf{GT}_i, \ \ 1 \leq i \leq n.$$

We call elements of $\mathsf{GT}^n$ *half-strict* Gelfand-Tsetlin patterns (also known as monotonous triangles, cf. [65]). We also let $\mathsf{GT}^{n+}$ be the subset of $\mathsf{GT}^n$ with $\mu^n \in \mathsf{GT}_n^+$. For $\lambda \in \mathsf{GT}_n$ we let $\mathsf{GT}_\lambda \subset \mathsf{GT}^n$ denote the set of half-strict Gelfand-Tsetlin patterns $\mu^1 \preceq \cdots \preceq \mu^n$ such that $\mu^n = \lambda$.

We turn back to the notation from Section 3.1.1 and consider $\omega \in \mathcal{P}_n$. For $k = 1, \dots, n$ we have that $\mu_i^k(\omega) = \lambda^k_{k-i+1}(\omega)$ for $i = 1, .., k$ satisfy $\mu^n \in \mathsf{GT}_n^+$ and $\mu^{k+1} \succeq \mu^k$ for $k = 1, \dots, n-1$. Consequently, the sequence $\mu^1, \dots, \mu^n$ defines an element of $\mathsf{GT}^{n+}$. It is easy to see that the map $h : \mathcal{P}_n \to \mathsf{GT}^{n+}$, given by $h(\omega) = \mu^1(\omega) \preceq \cdots \preceq \mu^n(\omega)$, is a bijection. For $\lambda \in \mathsf{GT}_n^+$ we let

$$\mathcal{P}_n^\lambda = \{\omega \in \mathcal{P}_n : \lambda_i^n(\omega) = \lambda_{n-i+1} \text{ for } i = 1, \dots, n\}.$$

One observes that the map $h$ by restriction is a bijection between $\mathsf{GT}_\lambda$ and $\mathcal{P}_n^\lambda$. With the above notation we make the following definition.

**Definition 3.6.2.** Fix $w_1, w_2, w_3, w_4, w_5, w_6 > 0$. A probability distribution $\rho$ on $\mathsf{GT}^{n+}$ is said to satisfy the *six-vertex Gibbs property* (with weights $(w_1, w_2, w_3, w_4, w_5, w_6)$) if the following holds. For any $\lambda \in \mathsf{GT}_n^+$ such that $\rho(\mu^n(\omega) = \lambda) > 0$ we have that the measure $\nu$ on $\mathcal{P}_n^\lambda$ defined through

$$\nu(h^{-1}(\omega)) = \rho(\omega | \mu^n = \lambda)$$

satisfies the Gibbs property for the six-vertex model on $D_n$ with weights $(w_1, w_2, w_3, w_4, w_5, w_6)$. In the above $\rho(\cdot | \mu^n = \lambda)$ stands for the measure $\rho$ conditioned on $\mu^n = \lambda$.

*Remark* 3.6.3. If $w_1 = \cdots = w_6 = 1$ and $\rho$ satisfies the six-vertex Gibbs property with these weights then the conditional distribution $\rho(\cdot | \mu^n = \lambda)$ becomes the uniform distribution on $\mathsf{GT}_\lambda$. In this case the six-vertex Gibbs property reduces to the *discrete Gibbs property* of [52].

For a probability distribution $\rho$ on $\mathsf{GT}^{n+}$ and an element $\lambda \in \mathsf{GT}_n^+$ such that $\rho(\mu^n(\omega) = \lambda) > 0$, we denote by $\rho_\lambda$ the distribution on $\mathsf{GT}_{n-1}^+$ given by $\rho(\mu^1, \dots, \mu^{n-1} | \mu^n = \lambda)$. We let $\rho^k$ denote the projection of $\rho$ onto $\mu^k$ for $k = 1, \dots, n$. Then the six-vertex Gibbs property is equivallent to the following statement. If $f : \mathbb{Z}^{n(n+1)/2} \to \mathbb{C}$ is a bounded function, then

$$\mathbb{E}^\rho \left[ f(\mu^n, \dots, \mu^1) \right] = \mathbb{E}^{\rho^n} \left[ \mathbb{E}^{\rho_{\mu^n}} \left[ f(\mu^n, \dots, \mu^1) \right] \right]. \tag{3.6.3}$$

We record two lemmas whose proof is deferred to Section 3.6.3.

**Lemma 3.6.4.** *Fix $w_1, w_2, w_3, w_4, w_5, w_6 > 0$. Let $n \in \mathbb{N}$ and $\rho$ be a measure on $\mathsf{GT}^{n+}$, which satisfies the six-vertex Gibbs property. Then we can find a positive constant $c \in (0,1)$ (depending on $n$ and $w_1, \dots, w_6$) such that for all $\lambda \in \mathsf{GT}_n^+$ with $\rho(\mu^n = \lambda) > 0$, and $(\alpha^1, \dots, \alpha^{n-1}, \lambda), (\beta^1, \dots, \beta^{n-1}, \lambda) \in \mathsf{GT}_\lambda$ we have*

$$c^{-1} \geq \frac{\rho_\lambda(\alpha^1, \dots, \alpha^{n-1})}{\rho_\lambda(\beta^1, \dots, \beta^{n-1})} \geq c.$$

**Definition 3.6.5.** We consider sequences $\lambda^k \in \mathsf{GT}_n$. We call the sequence *very good* if for $i = 1, \dots, n-1$ each sequence $\lambda_{i+1}^k - \lambda_i^k$ has a limit in $\mathbb{N} \cup \{\infty\}$ and for $i = 1, \dots, n-2$, each sequence $\lambda_{i+2}^k - \lambda_i^k$ goes to $\infty$. We call the sequence *good* if every subsequence of $\lambda^k$ has a further subsequence that is very good.

**Lemma 3.6.6.** *Fix $n \in \mathbb{N}$. Let $a(k)$ and $b(k)$ be sequences in $\mathbb{R}$ such that $b(k) \to \infty$ as $k \to \infty$. Suppose that $\lambda^k$ is a good sequence in $\mathsf{GT}_n^+$ and $f$ is a bounded uniformly continuous function on $\mathbb{R}^{n(n+1)/2}$. Put $g_k : \mathbb{R}^{n(n+1)/2} \to \mathbb{R}^{n(n+1)/2}$ to be*

$$g_k(x) = \frac{1}{b(k)} \left( x - a(k) \cdot \mathbf{1}_{\frac{n(n+1)}{2}} \right).$$

*Then we have*

$$\lim_{k\to\infty} \mathbb{E}^{\rho_{\lambda^k}} \left[ f \circ g_k(\lambda^k, \mu^{n-1}, \mu^{n-2}, \dots, \mu^1) \right] - \mathbb{E}^{\lambda^k} \left[ f \circ g_k(\lambda^k, x^{n-1}, x^{n-2}, \dots, x^1) \right] = 0, \quad (3.6.4)$$

*where $\mathbb{E}^{\rho_{\lambda^k}}$ is defined above while $\mathbb{E}^{\lambda^k}$ is as in Section 3.5.1.*

With the above lemma we can prove the main result of this section.

**Proposition 3.6.7.** *Fix $w_1, w_2, w_3, w_4, w_5, w_6 > 0$ and $n \in \mathbb{N}$. Let $\rho(k)$ be a sequence of probability measures on $\mathsf{GT}^{n+}$, satisfying the six-vertex Gibbs property with weights $(w_1, w_2, w_3, w_4, w_5, w_6)$. Let $a(k)$ and $b(k)$ be sequences in $\mathbb{R}$ such that $b(k) \to \infty$ as $k \to \infty$. Put $g_k : \mathbb{R}^{n(n+1)/2} \to \mathbb{R}^{n(n+1)/2}$ to be*

$$g_k(x) = \frac{1}{b(k)} \left( x - a(k) \cdot \mathbf{1}_{\frac{n(n+1)}{2}} \right),$$

*and suppose that $\rho(k) \circ g_k^{-1}$ converges weakly to a probability distribution $\mu$ on $GT^n$ (Gelfand-Tsetlin cone), such that*

$$\mathbb{P}^{\mu}(y_i^n = y_{i+1}^n = y_{i+2}^n \text{ for some } i = 1, \dots, n-2) = 0. \quad (3.6.5)$$

*Then $\mu$ satisfies the continuous Gibbs property (Definition 3.5.4).*

*Remark* 3.6.8. The statement of the proposition remains true if we remove the condition (3.6.5) on $\mu$; however, its proof requires a stronger statement than Lemma 3.6.6. For the applications we have in mind Proposition 3.6.7 is sufficient and we will not pursue the most general possible result here.

*Proof.* By Skorohod's theorem, we may find random vectors $Y(k)$ for $k \in \mathbb{N}$ and $X$, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $Y(k)$ have distribution $\rho(k)$, $X$ has distribution $\mu$, and

$$\mathbb{P}\left( \{\omega \in \Omega | \lim_{k\to\infty} g_k(Y(k)(\omega)) = X(\omega)\} \right) = 1.$$

Let $f$ be a bounded continuous function on $\mathbb{R}^{n(n+1)/2}$. As usual we write $Y(k) = Y^1(k) \preceq \cdots \preceq Y^n(k)$ and $X = X^1 \preceq \cdots \preceq X^n$. We want to show that

$$\mathbb{E}\left[ f(X) \right] = \mathbb{E}\left[ \mathbb{E}^{X^n} \left[ f(X) \right] \right].$$

From the Bounded Convergence Theorem we know that

$$\mathbb{E}\left[f(X)\right] = \lim_{k\to\infty} \mathbb{E}\left[f(g_k(Y(k)))\right]. \tag{3.6.6}$$

We now let $A = \{\omega \in \Omega | \lim_{k\to\infty} g_k(Y(k)(\omega)) = X(\omega)$ and $X_i^n(\omega) = X_{i+1}^n(\omega) = X_{i+2}^n(\omega)$ for no $i\}$. One observes that for $\omega \in A$, $Y(k)(\omega)$ is a good sequence and so by Lemma 3.6.6

$$\lim_{k\to\infty} \mathbb{E}^{Y^n(k)(\omega)}\left[f(g_k(Y(k)))\right] - \mathbb{E}^{\rho_{Y^n(k)}(\omega)}\left[f(g_k(Y(k)))\right] = 0.$$

Taking expectations on both sides above (which is justified by the Bounded convergence theorem) and using that $\rho(k)$ satisfy the six-vertex Gibbs property (see also (3.6.3)), we conclude that

$$\lim_{k\to\infty} \mathbb{E}\left[\mathbb{E}^{Y^n(k)(\omega)}\left[f(g_k(Y(k)))\right]\right] - \mathbb{E}\left[f(g_k(Y(k)))\right] = 0. \tag{3.6.7}$$

Finally, if $\omega \in A$ and $Z(k) = g_k(Y(k))$, we have by Lemma 3.5.2 that

$$\lim_{k\to\infty} \mathbb{E}^{Y^n(k)(\omega)}\left[f(g_k(Y(k)))\right] = \lim_{k\to\infty} \mathbb{E}^{Z^n(k)(\omega)}\left[f(Z(k))\right] = \mathbb{E}^{X^n(\omega)}\left[f(X)\right].$$

Taking expectations on both sides above (which is justified by the Bounded convergence theorem) we conclude that

$$\lim_{k\to\infty} \mathbb{E}\left[\mathbb{E}^{Y^n(k)(\omega)}\left[f(g_k(Y(k)))\right]\right] = \mathbb{E}\left[\mathbb{E}^{X^n(\omega)}\left[f(X)\right]\right]. \tag{3.6.8}$$

Combining (3.6.6), (3.6.7) and (3.6.8) proves the proposition.

□

### 3.6.3 Proof of Lemmas 3.6.4 and 3.6.6

We adopt the same notation as in Section 3.6.2.

*Proof.* (Lemma 3.6.4) Introduce vertex weights as in (3.6.1). For $\lambda \in \mathsf{Sign}_n^+$ we fix $\omega_\lambda \in \mathcal{P}_n$, such that $\lambda_i^j(\omega_\lambda) = \lambda_i$ for $i = 1, \dots, j$ and $j = 1, \dots, n$. We also define for $\omega \in \mathcal{P}_n$ the weight $\mathcal{W}(\omega) := \prod_{i=1}^n \prod_{j=1}^{\lambda_1} w(\omega(i,j))$.

Since $\rho$ satisfies the conditions of Definition 3.6.2, it is enough to show that for each $\lambda \in \mathsf{Sign}_n^+$, and any collection of paths $\omega \in \mathcal{P}_n$, with $\lambda_i^n(\omega) = \lambda_i$ for $i = 1, \dots, n$, we have

$$c^{-1} \geq \frac{\mathcal{W}(\omega)}{\mathcal{W}(\omega_\lambda)} \geq c,$$

for some $c \in (0,1)$, which depends on $n$ and $w_1, \dots, w_6$. The strategy is to apply elementary moves to the configuration $\omega$ that transform it to $\omega_\lambda$, and record how the weight changes at each step. We will see that the number of changes is at most $n(n-1)$ and each change is given by a multiplication by some factor, which can take finitely many values, depending on $w_1, \dots, w_6$. This will show that $\frac{\mathcal{W}(\omega)}{\mathcal{W}(\omega_\lambda)}$ belongs to a finite set of numbers, which then can be upper and lower bounded, proving the lemma.

Let $\mathcal{P}_n^\lambda$ denote the set of $\omega \in \mathcal{P}_n$ such that $\lambda^n(\omega) = \lambda$. Starting from any $\omega \in \mathcal{P}_n^\lambda$ an elementary move consists of increasing one of $\lambda_i^j(\omega)$ by 1 so that the resulting element still lies in $\mathcal{P}_n^\lambda$. If we apply an elementary move to $\omega$, increasing $m = \lambda_i^j(\omega)$ by 1 and obtain $\omega^+ \in \mathcal{P}_n^\lambda$ as a result, we observe that

$$\frac{\mathcal{W}(\omega)}{\mathcal{W}(\omega^+)} = \frac{w(\omega(m,j))w(\omega(m,j+1))w(\omega(m+1,j))w(\omega(m+1,j+1))}{w(\omega^+(m,j))w(\omega^+(m,j+1))w(\omega^+(m+1,j))w(\omega^+(m+1,j+1))}.$$

Since we have only finitely many possible vertex weights we see that $\frac{\mathcal{W}(\omega)}{\mathcal{W}(\omega^+)}$ can take finitely many values.

The way we transform $\omega$ to $\omega_\lambda$ is as follows. We consider the complete order on pairs $(x,y)$ given by $(x,y) < (x',y')$ if and only if $x < x'$ or $x = x'$ and $y > y'$. We traverse the pairs $(i,j)$: $i = 1,\dots,j$, $j = 1,..,n-1$ in increasing order, and for each $(i,j)$ we increase $\lambda_i^j(\omega)$ by 1 until it reaches $\lambda_i^j(\omega_\lambda)$. One readily observes that each such move is elementary and the result of applying all these moves to $\omega$ is indeed $\omega_\lambda$. We continue to denote the result of applying an elementary move to $\omega$ by $\omega$ - this should cause no confusion.

An important situation occurs when prior to the application of the move $m = \lambda_i^j(\omega) \to m+1$ we have that there are no vertical arrows coming in $(m,j)$ and $(m+1,j)$ and coming out of $(m,j+1)$ and $(m+1,j+1)$. The latter situation determines the types of the four vertices:

$$\omega(m,j) = (0,1;1,0),\ \omega(m+1,j) = (0,0;0,0),$$
$$\omega(m,j+1) = (1,0;0,1),\ \omega(m+1,j+1) = (0,1;0,1).$$

After the application of the move they become

$$\omega(m,j) = (0,1;0,1),\ \omega(m+1,j) = (0,1;1,0),$$
$$\omega(m,j+1) = (0,0;0,0),\ \omega(m+1,j+1) = (1,0;0,1).$$

We thus see that the product of these weights stays the same and so $\mathcal{W}(\omega)$ remains unchanged. We call such a situation *good*.

Suppose that in the string of elementary moves, transforming $\omega$ to $\omega_\lambda$, we have reached the pair $(i,j)$, and we are increasing $\lambda_i^j(\omega)$ to $\lambda_i^j(\omega_\lambda)$. Let us denote $A = \lambda_i^j(\omega)$ and $B = \lambda_i^j(\omega_\lambda)$. The condition that we can increase $\lambda_i^j(\omega)$ to $B$ via elementary moves, implies that there are no arrows from $(k,j)$ to $(k,j+1)$ or from $(k,j-1)$ to $(k,j)$ for $k = A+1,\dots,B-1$. Consequently, in the process of increasing $\lambda_i^j(\omega)$ to $B$, we encounter at most two non-good situations (corresponding to the first and last move). As we have $n(n-1)/2$ pairs $(i,j)$, we see that in our string of elementary moves the situation is good in all but at most $n(n-1)$ moves. This proves our desired result.

□

Before we go to the proof of Lemma 3.6.6, we introduce some notation and prove a couple of facts. Let $\overline{\mathsf{GT}}_n$ denote the set of $n$-tuples of integers

$$\overline{\mathsf{GT}}_n = \{\lambda \in \mathbb{Z}^n : \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\}.$$

Let $\overline{\mathsf{GT}}^n$ denote the set of sequences

$$\mu^1 \preceq \mu^2 \preceq \cdots \preceq \mu^n, \quad \mu^i \in \overline{\mathsf{GT}}_i, \ 1 \le i \le n.$$

We call elements of $\overline{\mathsf{GT}}^n$ Gelfand-Tsetlin patterns. For $\lambda \in \overline{\mathsf{GT}}_n$ we let $\overline{\mathsf{GT}}_\lambda \subset \overline{\mathsf{GT}}^n$ denote the set of Gelfand-Tsetlin patterns $\mu^1 \preceq \cdots \preceq \mu^n$ such that $\mu^n = \lambda$.

We say that $\lambda \in \mathsf{GT}_n$ and $\mu \in \mathsf{GT}_{n-1}$ *strictly interlace* and write $\mu \prec \lambda$ if

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n.$$

Let $\widehat{\mathsf{GT}}^n$ denote the set of sequences

$$\mu^1 \prec \mu^2 \prec \cdots \prec \mu^n, \quad \mu^i \in \mathsf{GT}_i, \ 1 \le i \le n.$$

We call elements of $\widehat{\mathsf{GT}}^n$ *strict* Gelfand-Tsetlin patterns. For $\lambda \in \mathsf{GT}_n$ we let $\widehat{\mathsf{GT}}_\lambda \subset \widehat{\mathsf{GT}}^n$ denote the set of Gelfand-Tsetlin patterns $\mu^1 \prec \cdots \prec \mu^n$ such that $\mu^n = \lambda$.

For $\lambda \in \overline{\mathsf{GT}}_n$ we consider the size of $\overline{\mathsf{GT}}_\lambda$, which is equal to the dimension of the representation of the unitary griup $U(n)$ with the highest weight $\lambda$ and is given by the well-known formula

$$\left|\overline{\mathsf{GT}}_\lambda\right| = \prod_{i<j} \left( \frac{\lambda_j - \lambda_i + j - i}{j-i} \right). \tag{3.6.9}$$

For $\lambda \in \overline{\mathsf{GT}}_n$ we let $\lambda^* = (\lambda_1 + (n-1), \lambda_2 + (n-3), \ldots, \lambda_n + (-n+1))$. It is easy to check that if $\lambda \in \overline{\mathsf{GT}}_n$ and $\mu \in \overline{\mathsf{GT}}_{n-1}$ then $\mu \preceq \lambda$ if and only if $\mu^* \prec \lambda^*$. Consequently, we have that the map $f : \overline{\mathsf{GT}}_\lambda \to \widehat{\mathsf{GT}}_{\lambda^*}$, given by $f(\mu^1, \ldots, \mu^{n-1}, \lambda) = ((\mu^1)^*, \ldots, (\mu^{n-1})^*, \lambda^*)$, is a bijection.

It follows from (3.6.9) that for $\lambda \in \mathsf{GT}_n$, the size of $\widehat{\mathsf{GT}}_\lambda$ is given by

$$\left|\widehat{\mathsf{GT}}_\lambda\right| = \prod_{i<j} \left( \frac{\lambda_j - \lambda_i - j + i}{j-i} \right). \tag{3.6.10}$$

Let us recall from Definition 3.6.5 that a sequence in $\mathsf{GT}_n$ is very good if for $i = 1, \ldots, n-1$ each sequence $\lambda^k_{i+1} - \lambda^k_i$ has a limit in $\mathbb{N} \cup \{\infty\}$, while for $i = 1, \ldots, n-2$, each sequence $\lambda^k_{i+2} - \lambda^k_i$ goes to $\infty$. For each very good sequence $\lambda^k$, we let $M \subset \{1, \ldots, n-1\}$, be the set of indices $i$, such that $\lambda^k_{i+1} - \lambda^k_i$ is bounded, and for $i \in M$, denote by $m_i \in \mathbb{N}$ the limit of the sequence $\lambda^k_{i+1} - \lambda^k_i$, which exists by assumption.

Given a subset $M \subset \{1, \ldots, n-1\}$ and $\lambda \in \mathsf{GT}_n$, we let

$$\widehat{\mathsf{GT}}_\lambda(M) := \{(\mu^1, \ldots, \mu^n) \in \mathsf{GT}_\lambda : \mu^{n-1} \succ \mu^{n-2} \succ \cdots \succ \mu^1, \mu^{n-1}_i \in (\lambda_i, \lambda_{i+1}) \text{ for } i \notin M\}.$$

For $(\mu^1, \ldots, \mu^n) \in \widehat{\mathsf{GT}}_\lambda(M)$ and numbers $x_i$ for $i \in M$, we define the function

$$f^{\mathbf{x}}_M(\mu^1, \ldots, \mu^n) = (\nu^1, \ldots, \nu^n) \text{ with } \nu^j_i = \begin{cases} x_i & \text{if } j = n-1 \text{ and } i \in M, \\ \mu^j_i & \text{else.} \end{cases}$$

We now define

$$\widehat{\mathsf{GT}}_\lambda(M; f_M) := \{(\mu^1, \ldots, \mu^n) \in \widehat{\mathsf{GT}}_\lambda(M) : f_M^{\mathbf{x}}(\mu^1, \ldots, \mu^n) \in \widehat{\mathsf{GT}}_\lambda(M) \text{ if } x_i \in [\lambda_i, \lambda_{i+1}], i \in M\}.$$

The first key result we need is the following.

**Lemma 3.6.9.** *Let $\lambda^k \in \mathsf{GT}_n$ be a very good sequence and $M$ as above. As $k \to \infty$, we have*

$$|\mathsf{GT}_{\lambda^k}| \sim \left|\overline{\mathsf{GT}}_{\lambda^k}\right| \sim \left|\widehat{\mathsf{GT}}_{\lambda^k}(M; f_M)\right| \sim \prod_{\substack{1 \le i < j \le n \\ i+1 < j}} \left(\frac{\lambda_j^k - \lambda_i^k}{j - i}\right) \times \prod_{i \notin M, i < n} (\lambda_{i+1}^k - \lambda_i^k) \times \prod_{i \in M} (m_i + 1).$$

*Proof.* We observe that $\widehat{\mathsf{GT}}_{\lambda^k}(M; f_M) \subset \mathsf{GT}_{\lambda^k} \subset \overline{\mathsf{GT}}_{\lambda^k}$, and so by (3.6.9), it suffices to show that

$$\limsup_{k \to \infty} \left|\widehat{\mathsf{GT}}_{\lambda^k}(M; f_M)\right|^{-1} \prod_{\substack{1 \le i < j \le n \\ i+1 < j}} \left(\frac{\lambda_j^k - \lambda_i^k}{j - i}\right) \times \prod_{i \notin M, i < n} (\lambda_{i+1}^k - \lambda_i^k) \times \prod_{i \in M} (m_i + 1) \le 1. \quad (3.6.11)$$

For $i \in \{1, \ldots, n\}$, we let $X_i = \{(x, y) : i \le x \le y \le n\}$, and $Y_i = \{(x, y) : 1 \le x \le y \le n \text{ and } y - x < n - i - 1\}$. Let $(\mu^1, \ldots, \mu^n) \in \mathsf{GT}^n$ and let $i \in \{0, \ldots, n\}$ be given. Then it is easy to see that if we increase $\mu_x^y$ by 1 for all $(x, y) \in X_i$ or, alternatively, for all $(x, y) \in Y_i$, we still get an element that belongs to $\mathsf{GT}^n$. See Figure 3-7.

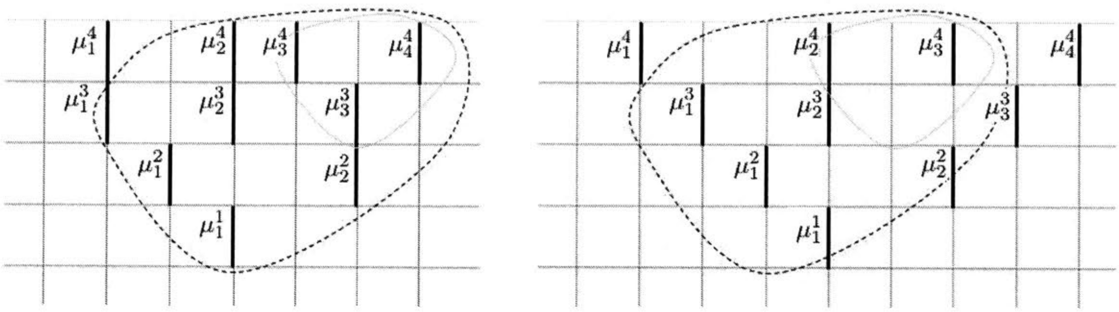

Figure 3-7: An element $\mu \in \mathsf{GT}^4$. In left picture the grey curve encloses $\mu_x^y$ for $(x, y) \in X_3$ and the dashed curve for $(x, y) \in Y_0$. The right picture is the result of increasing $\mu_x^y$ by 1 for $(x, y) \in X_3$ and then for $(x, y) \in Y_0$.

Let $M \subset \{1, \ldots, n-1\}$, be such that $M$ does not contain adjacent elements. Suppose that $\lambda \in \mathsf{GT}_n$ is such that $\lambda_{i+1} - \lambda_i = 2$ when $i \in M$. Suppose $m_i \in \mathbb{N}$ for $i \in M$ are given. Let $c_i \in \{0, \ldots, m_i\}$ for $i \in M$. Starting from an element $\mu^1, \ldots, \mu^n$ in $\widehat{\mathsf{GT}}_\lambda$, and given $c_i$ for $i \in M$ as above, we construct a new element in $\mathsf{GT}^n$ as follows:

1. We traverse the elements in $M$ in increasing order.

2. For each element $i \ge 2, i \in M$, we increase the values $\mu_x^y$ for each $(x, y) \in X_i$ by 1.

3. Afterwards we increase $\mu_x^y$ by $m_i$ for each $(x, y) \in Y_i$.

4. Finally, we set $\mu_{i+1}^n = \mu_i^n + m_i$ and set $\mu_i^{n-1}$ to equal $\mu_i^n + c_i$.

For a simple application of the above algorithm see Figure 3-8.

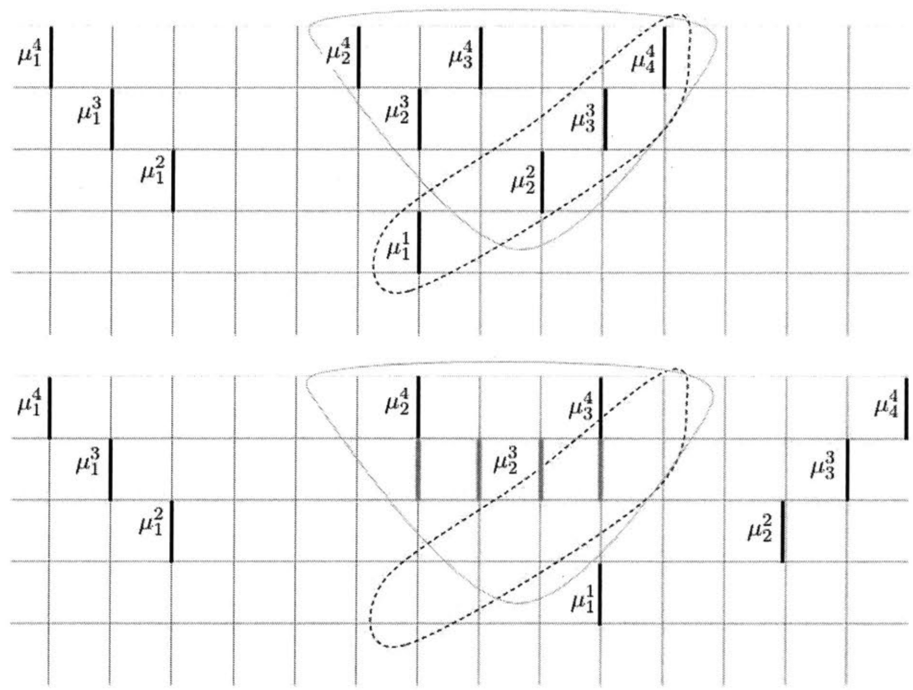

Figure 3-8: The top picture gives an element $\mu \in \widehat{\mathsf{GT}}^4$; the grey curve encloses $\mu_x^y$ for $(x,y) \in X_2$ and the dashed curve for $(x,y) \in Y_2$. If $M = \{2\}$ and $m_2 = 3$ the bottom picture gives the output of applying our algorithm to $\mu$. The position $\mu_2^3$ can be any element in $[\mu_2^4, \mu_3^4]$.

One readily verifies that each element that was constructed with the above algorithm belongs to $\widehat{\mathsf{GT}}_\nu(M; f_M)$, where $\nu_1 = \lambda_1$ and for $i = 2, \dots, n$, we have

$$\nu_{i+1} - \nu_i = \begin{cases} m_i & \text{if } i \in M, \\ \lambda_{i+1} - \lambda_i + 1 & \text{if } i+1 \in M \text{ and } i-1 \notin M, \\ \lambda_{i+1} - \lambda_i + 2 & \text{if } i+1 \notin M \text{ and } i-1 \in M, \\ \lambda_{i+1} - \lambda_i + 3 & \text{if } i+1 \in M \text{ and } i-1 \in M \\ \lambda_{i+1} - \lambda_i & \text{if } i, i+1, i-1 \notin M. \end{cases}$$

We thus obtain a map from $\widehat{\mathsf{GT}}_\lambda \times \prod_{i \in M}\{0, \dots, m_i\}$ into $\widehat{\mathsf{GT}}_\nu(M; f_M)$, and it is easy to see that it is injective. The latter implies that

$$\left|\widehat{\mathsf{GT}}_\nu(M; f_M)\right| \geq \prod_{i \in M}(m_i + 1) \times \left|\widehat{\mathsf{GT}}_\lambda\right|.$$

Combining the above with (3.6.10) we see that if $\nu \in \mathsf{GT}_n$ is such that $\nu_{i+1} - \nu_i = m_i$ for

$i \in M$ and $\nu_{i+1} - \nu_i \geq 4$ for $i \notin M$, then

$$\left|\widehat{\mathsf{GT}}_\nu(M; f_M)\right| \geq \prod_{i \in M}(m_i + 1) \times \prod_{\substack{1 \leq i < j \leq n \\ i+1<j}} \left(\frac{\nu_j - \nu_i - 4(j-i)}{j-i}\right) \times \prod_{i \notin M, i<n} (\nu_{i+1} - \nu_i - 4).$$

This readily implies (3.6.11) and hence the lemma. □

An important property of $\widehat{\mathsf{GT}}_\lambda(M)$ is contained in the following lemma.

**Lemma 3.6.10.** *Fix $w_1, w_2, w_3, w_4, w_5, w_6 > 0$ and $n \in \mathbb{N}$. Suppose $\rho$ is a probability distribution on $\mathsf{GT}^{n+}$, which satisfies the six-vertex Gibbs property with weights $(w_1, w_2, w_3, w_4, w_5, w_6)$. Let $\lambda \in \mathsf{GT}_n$ be such that $\rho(\mu^n(\omega) = \lambda) > 0$ and define $\rho_\lambda$ as in Section 3.6.2. Let $M \subset \{1, \dots, n-1\}$ and suppose that $(\mu^1, \dots, \mu^n)$, $(\nu^1, \dots, \nu^n) \in \widehat{\mathsf{GT}}_\lambda(M)$, are such that $\mu_i^{n-1} = \nu_i^{n-1}$ for $i \in M$. Then $\rho_\lambda(\mu^1, \dots, \mu^{n-1}) = \rho_\lambda(\nu^1, \dots, \nu^{n-1})$.*

*Proof.* We set $\omega_1 = h^{-1}((\mu^1, \dots, \mu^{n-1}, \lambda)$ and $\omega_2 = h^{-1}((\nu^1, \dots, \nu^{n-1}, \lambda)$ (the function $h$ was defined in Section 3.6.2). By definition we know that $\lambda_i^n(\omega_1) = \lambda_i^n(\omega_2) = \lambda_{n-i+1}$ for $i = 1, \dots, n$. As in the proof of Lemma 3.6.4 we introduce vertex weights as in (3.6.1) and define for $\omega \in \mathcal{P}_n$ the weight $\mathcal{W}(\omega) := \prod_{i=1}^n \prod_{j=1}^{\lambda_n} w(\omega(i,j))$. Since $\rho$ satisfies the six-vertex Gibbs property we see that to prove the lemma it suffices to show that $\mathcal{W}(\omega_1) = \mathcal{W}(\omega_2)$.

Recalling the proof of Lemma 3.6.4, we see that it suffices to show that we can transform $\omega_1$ to $\omega_2$ via good elementary moves. I.e. we wish to show that any two elements in $h^{-1}(\widehat{\mathsf{GT}}_\lambda(M))$, that satisfy $\lambda_i^{n-1}(\omega_1) = \lambda_i^{n-1}(\omega_2)$ for $n-i \in M$ are connected via good elementary moves. We prove the latter by induction on $|\omega_1 - \omega_2| := \sum_{j=1}^{n-1} \sum_{i=1}^{j} |\lambda_i^j(\omega_1) - \lambda_i^j(\omega_2)|$, the base case $|\omega_1 - \omega_2| = 0$ being obvious.

Suppose we know the result for $|\omega_1 - \omega_2| = k - 1 \geq 0$, and we wish to show it for $k$. Since $|\omega_1 - \omega_2| = k \geq 1$, we know that there exist $(x, y)$ such that $\lambda_x^y(\omega_1) - \lambda_x^y(\omega_2) \neq 0$. Let $(x, y)$ be the smallest such index (in the order considered in the proof of Lemma 3.6.4), and without loss of generality we assume that $\lambda_x^y(\omega_1) > \lambda_x^y(\omega_2)$. Notice that by assumption $(x, y) \neq (i, n-1)$ for any $n - i \in M$ and also $y \leq n - 1$.

We want to increase $\lambda_x^y(\omega_2)$ by 1 and show that this is a good elementary move. In order for this to be the case we must have that $\lambda_{x-1}^y(\omega_2)$, $\lambda_{x-1}^{y-1}(\omega_2)$ (if $x > 1$) and $\lambda_x^{y+1}(\omega_2)$ are all strictly bigger than $\lambda_x^y(\omega_2) + 1$. Observe that

$$\lambda_x^{y+1}(\omega_2) = \lambda_x^{y+1}(\omega_1) \geq \lambda_x^y(\omega_1) + 1 \geq \lambda_x^y(\omega_2) + 2,$$

where in the first equality we used the minimality of $(x, y)$, in the second one we used that $(x, y) \neq (i, n-1)$ for any $n - i \in M$ and in the third that $\lambda_x^y(\omega_1) > \lambda_x^y(\omega_2)$. Similarly, we have for $x > 1$ that

$$\lambda_{x-1}^y(\omega_2) = \lambda_{x-1}^y(\omega_1) \geq \lambda_x^y(\omega_1) + 1 \geq \lambda_x^y(\omega_2) + 2,$$

and

$$\lambda_{x-1}^{y-1}(\omega_2) = \lambda_{x-1}^{y-1}(\omega_1) \geq \lambda_x^y(\omega_1) + 1 \geq \lambda_x^y(\omega_2) + 2.$$

Thus increasing $\lambda_x^y(\omega_2)$ by 1 is a good elementary move, and does not change $\mathcal{W}(\omega_2)$, while it reduces $|\omega_1 - \omega_2|$ by 1. Applying the induction hypothesis proves the result for $k$, and the general result follows by induction. □

With the above two results, we now turn to the proof of Lemma 3.6.6.

*Proof.* (Lemma 3.6.6) Clearly it suffices to prove the lemma when $\lambda^k$ is very good. As before we let $M \subset \{1, \dots, n-1\}$, be the set of indices $i$, such that $\lambda^k_{i+1} - \lambda^k_i$ is bounded, and for $i \in M$, denote by $m_i \in \mathbb{N}$ the limit of the sequence $\lambda^k_{i+1} - \lambda^k_i$. By ignoring finitely many elements of the sequence $\lambda^k$, we may assume that $\lambda^k_{i+1} - \lambda^k_i = m_i$ for all $k$. We denote by $M_k = \prod_{\substack{1 \le i < j \le n \\ i+1<j}} \left(\frac{\lambda^k_j - \lambda^k_i}{j-i}\right) \times \prod_{i \notin M, i<n} (\lambda^k_{i+1} - \lambda^k_i)$.

For $y = y^j_i$, $i = 1, \dots, j$ and $j = 1, \dots, n$, we let $Q(y)$ be the cube in $\mathbb{R}^{n(n-1)/2}$, given by $\prod_{j=1}^{n-1} \prod_{i=1}^{j} (y^j_i, y^j_i + 1)$. For $\lambda \in \mathsf{GT}_n$, we define

$$\mathsf{GT}^*_\lambda := \{(\mu^1, \dots, \mu^n) \in \widehat{\mathsf{GT}}_\lambda(M; f_M) : \mu^{n-1}_i < \lambda_i \text{ for } i \in M\}.$$

It is easy to see that if $(\mu^1, \dots, \mu^n) \in \mathsf{GT}^*_\lambda$, and $y^j_i = \mu^j_i$ for $i = 1, \dots, j$ and $j = 1, \dots, n$, then

$$x^1 \preceq x^2 \preceq \cdots \preceq x^{n-1} \preceq \lambda, \text{ for any } (x^1, \dots, x^{n-1}) \in Q(y).$$

Fix $c_i \in \{0, \dots, m_i\}$ for $i \in M$, and set $\mathsf{GT}^*_{\lambda^k}(\mathbf{c}) = \{(\mu^1, \dots, \mu^n) \in \mathsf{GT}^*_\lambda : \mu^{n-1}_i = \lambda^k_i + c_i \text{ for } i \in M\}$. Then from the proof of Lemma 3.6.4 we know that

$$|\mathsf{GT}^*_{\lambda^k}(\mathbf{c})| \sim M_k \text{ and } |\mathsf{GT}^*_{\lambda^k}| \sim M_k \prod_{i \in M} m_i \sim d_n(\lambda^k) = vol(GT_n(\lambda^k)), \text{ as } k \to \infty. \tag{3.6.12}$$

Let us denote $B(M) = \prod_{i \in M} \{0, \dots, m_i\}$ and $B^*(M) = \prod_{i \in M} \{0, \dots, m_i - 1\}$. In view of (3.6.12) and the boundedness of $f$, we know that

$$\lim_{k \to \infty} \mathbb{E}^{\lambda^k} \left[f \circ g_k(\lambda^k, x^{n-1}, x^{n-2}, \dots, x^1)\right] - \frac{\sum\limits_{\mathbf{c} \in B^*(M)} \sum\limits_{y \in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} \int_{Q(y)} f \circ g_k(\lambda^k, x^{n-1}, \dots, x^1) dx}{M_k \times \prod_{i \in M} m_i} = 0.$$

Moreover, using the uniform continuity of $f$ and the fact that $b(k) \to \infty$ as $k \to \infty$, we conclude

$$\lim_{k \to \infty} \mathbb{E}^{\lambda^k} \left[f \circ g_k(\lambda^k, x^{n-1}, x^{n-2}, \dots, x^1)\right] - \frac{\sum_{\mathbf{c} \in B^*(M)} \sum_{y \in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} f \circ g_k(y)}{M_k \times \prod_{i \in M} m_i} = 0. \tag{3.6.13}$$

Given $c_i \in \{0, \dots, m_i\}$ for $i \in M$, we let $w(\mathbf{c}, k) = \rho_{\lambda^k}(\mu^1, \dots, \mu^{n-1})$, where $(\mu^1, \dots, \mu^{n-1}, \mu^n) \in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})$. By Lemma 3.6.10, we know that $w(\mathbf{c}, k)$ is well-defined. The boundedness of $f$, Lemma 3.6.9, and Lemma 3.6.4 now imply that

$$\lim_{k \to \infty} \mathbb{E}^{\rho_{\lambda^k}} \left[f \circ g_k(\lambda^k, \mu^{n-1}, \mu^{n-2}, \dots, \mu^1)\right] - \frac{\sum_{\mathbf{c} \in B(M)} w(\mathbf{c}, k) \sum_{y \in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} f \circ g_k(y)}{M_k \times \sum_{\mathbf{c} \in B(M)} w(\mathbf{c}, k)} = 0. \tag{3.6.14}$$

Let $\mathbf{c}^0$, be such that $c^0_i = 0$ for $i \in M$. From the uniform continuity of $f$ and the fact that $b(k) \to \infty$, we note that for any $\mathbf{c}$ we have

$$\lim_{k\to\infty} \frac{\sum_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c}^0)} f\circ g_k(y)}{M_k} - \frac{\sum_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} f\circ g_k(y)}{M_k} = 0.$$

The latter, together with the boundedness of $f$ and Lemma 3.6.4, implies that

$$\lim_{k\to\infty} \frac{\sum_{\mathbf{c}\in B^*(M)} \sum_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} f\circ g_k(y)}{M_k \times \prod_{i\in M} m_i} - \frac{1}{\prod_{i\in M} m_i} \sum_{\mathbf{c}\in B^*(M)} \frac{\sum_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c}^0)} f\circ g_k(y)}{M_k} = 0, \text{ and}$$

$$\lim_{k\to\infty} \frac{\sum\limits_{\mathbf{c}\in B(M)} w(\mathbf{c},k) \sum\limits_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} f\circ g_k(y)}{M_k \times \sum\limits_{\mathbf{c}\in B(M)} w(\mathbf{c},k)} - \frac{1}{\sum\limits_{\mathbf{c}\in B(M)} w(\mathbf{c},k)} \sum_{\mathbf{c}\in B(M)} w(\mathbf{c},k) \frac{\sum_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c}^0)} f\circ g_k(y)}{M_k} = 0.$$

and so

$$\lim_{k\to\infty} \frac{\sum_{\mathbf{c}\in B^*(M)} \sum_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} f\circ g_k(y)}{M_k \times \prod_{i\in M} m_i} - \frac{\sum_{\mathbf{c}\in B(M)} w(\mathbf{c},k) \sum_{y\in \mathsf{GT}^*_{\lambda^k}(\mathbf{c})} f\circ g_k(y)}{M_k \times \sum_{\mathbf{c}\in B(M)} w(\mathbf{c},k)} = 0. \quad (3.6.15)$$

Combining (3.6.13), (3.6.14) and (3.6.15) concludes the proof. □

## 3.7 Proof of Theorem 3.1.3

In this section we give the proof of Theorem 3.1.3. We will split the proof into several steps and outline here the flow of the argument. We assume the same notation as in Section 3.1.1 and define $g_M : \mathbb{R}^{k(k+1)/2} \to \mathbb{R}^{k(k+1)/2}$ as

$$g_M(x) = \frac{1}{c\sqrt{M}} \left( x - aM \cdot \mathbf{1}_{\frac{k(k+1)}{2}} \right).$$

In addition, we replace $Y(N, M; k)$ with $Y(M)$ for brevity. The statement of Theorem 3.1.3 is that $g_M(Y(M))$ converge weakly to the GUE-corners process or rank $k$.

In the first step of the proof we show that we may replace the distribution of $Y(M)$ with the distribution $\nu_M$, given by the distribution of $Y(M)$ conditioned on $Y(M)^k_k \leq N$, without affecting the statement of the theorem. The latter is a consequence of Theorem 3.1.1. The measures $\nu_M$ are probability measures on $\mathsf{GT}^{k+}$, which satisfy the six-vertex Gibbs property with certain weights.

In the second step we check that the sequence of measures $\nu_M \circ g_M^{-1}$ on $\mathbb{R}^{k(k+1)/2}$ is tight. This is shown by using the six-vertex Gibbs property satisfied by $\nu_M$ and Lemma 3.4.5. The proof we present is similar to the proof of Proposition 7 in [52].

In the third step we prove that $\nu_M \circ g_M^{-1}$ converge weakly to the GUE-corners process of rank $k$ by induction on $k$. The base case is proved via Lemma 3.4.5. When going from $k$ to $k+1$ we use the induction hypothesis and Proposition 3.6.7 to show that any weak limit of $\nu_M \circ g_M^{-1}$ satisfies the continuous Gibbs property. The latter is combined with Proposition 3.5.6 to prove the result for $k+1$.

**Step 1.** Let $E_M$ be the event that $Y(M)_k^k \leq N$. It follows from Theorem 3.1.1 (see also (3.4.3)) that $\mathbb{P}_{u,v}^{N,M}(E_M) \to 1$ as $M \to \infty$. Let $\nu_M$ be the distribution of $Y(M)$, conditioned on $E_M$. Since $\mathbb{P}_{u,v}^{N,M}(E_M) \to 1$ , we see that it suffices to prove that $\nu_M \circ g_M^{-1}$ converge weakly to the GUE-corners process of rank $k$.

We will show that $\nu_M$ is a probability distribution on $\mathsf{GT}^{k+}$, which satisfies the six-vertex Gibbs property with weights

$$w_1 = \frac{u - s^{-1}}{us - 1}, \ w_2 = 1, \ w_3 = \frac{u - s}{us - 1}, \ w_4 = \frac{us^{-1} - 1}{us - 1}, \ w_5 = \frac{(s^2 - 1)u}{us - 1}, \ w_6 = \frac{1 - s^{-2}}{us - 1}. \tag{3.7.1}$$

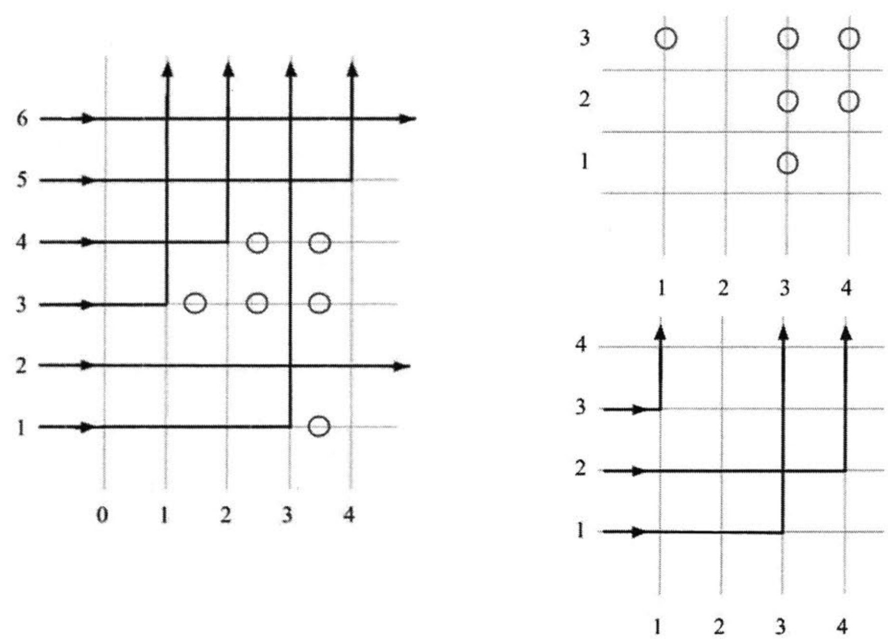


Figure 3-9: The left figure shows a path collection $\omega \in E_N$ with $N = 6$ and $k = 3$. Circles indicate the positions of the empty edges. The top right figure shows the array $(Y_i^j)_{1 \leq i \leq j \leq 3}$; $j$ varies vertically and position is measured horizontally. The bottom right figure shows the image of $(Y_i^j)_{1 \leq i \leq j \leq 3}$ under $h^{-1}$.

Recall from Section 3.1.1 that for $\omega \in E_N$, $(Y_i^j)_{1 \leq i \leq j \leq k}$ were the vertical positions of the empty horizontal edges in the first $k$ columns of $\omega$ (see the left part of Figure 3-9). The condition $\omega \in E_M$ ensures that no $Y(M)_i^j$ are infinite, so that $\nu_M$ is a valid probability distribution on $\mathsf{GT}^{k+}$.

The fact that $\nu_M$ satisfies a six-vertex Gibbs property is a consequence of the fact that $\mathbb{P}_{u,v}^{N,M}$ satisfies a Gibbs property for the six-vertex model on $D_N$ with weights $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5, \tilde{w}_6) = \left(1, \frac{u-s^{-1}}{us-1}, \frac{us^{-1}-1}{us-1}, \frac{u-s}{us-1}, \frac{u(s^2-1)}{us-1}, \frac{1-s^{-2}}{us-1}\right)$ (see Section 3.6.2). We observe that there is a simple relationship between $\omega$ and $h^{-1}((Y_i^j)(\omega))$ (here $h$ is as in Section 3.6.2). Namely, $h^{-1}((Y_i^j)(\omega))$ is obtained by reflecting $\omega$ with respect to the line $x = y$ and then flipping filled and empty edges (see Figure 3-9). This transformation has the followng effect on arrow configurations at a vertex

$$(0,0;0,0) \leftrightarrow (1,1;1,1) \text{ and } (1,0;1,0) \leftrightarrow (0,1;0,1),$$

while the vertices $(0,1;1,0)$ and $(1,0;0,1)$ are sent to themselves. This vertex transformation implies that the measure $h^{-1}((Y_i^j)(\omega))$ satisfies the Gibbs property for the six-vertex model on $D_k$ with weights $(\tilde{w}_2, \tilde{w}_1, \tilde{w}_4, \tilde{w}_3, \tilde{w}_5, \tilde{w}_6)$, which are the weights in (3.7.1).

**Step 2.** In this step we show that $\nu_M \circ g_M^{-1}$ is tight, which is equivalent to showing that $\eta(M)_i^j := \frac{Y(M)_i^j - aM}{c\sqrt{M}}$ is tight for each $i = 1, \dots, j$ and $j = 1, \dots, k$. We proceed by induction on $k$, with base case $k = 1$, true by Lemma 3.4.5.

Suppose the result is known for $k-1 \geq 1$ and we wish to show it for $k$. By induction hypothesis $\eta(M)_i^j$ is tight for each $i = 1, \dots, j$ and $j = 1, \dots, k-1$. In addition, by Lemma 3.4.5 $\eta(M)_k^k$ is also tight. Using the interlacing condition $Y(M)_{i-1}^{k-1} \leq Y(M)_i^k \leq Y(M)_i^{k-1}$ for $i = 2, \dots, k-1$, and the induction hypothesis we conclude that $\eta(M)_i^k$ is tight for $i = 2, \dots, k-2$. What remains to be seen is that $\eta(M)_1^k$ is tight.

We argue by contradiction and suppose that $\eta(M)_1^k$ is not tight as $M \to \infty$. Then we may find a positive number $p > 0$, a subsequence $M_r$ and an increasing sequence $L_r$ going to $\infty$ such that

$$\mathbb{P}\left(\left|\eta(M_r)_1^k\right| > L_r\right) > p. \tag{3.7.2}$$

Since $Y(M_r)_1^k \leq Y(M_r)_1^{k-1}$ and by induction hypothesis $\eta(M_r)_1^{k-1}$ is tight, we see that if (3.7.2) holds then we must have (by potentially passing to a further subsequence) that

$$\mathbb{P}\left(\eta(M_r)_1^k < -L_r\right) > p/2.$$

Let us denote by $B(M) = \min\left(Y(M)_2^k, Y(M)_2^{k-1} - 1, Y(M)_1^{k-2}\right)$ (if $k = 2$, $B(M) = Y(M)_2^k$). $B(M)$ is the rightmost positition that $Y_1^{k-1}(M)$ can take. From the tightness result established for $\eta(M)_2^k$, $\eta(M)_2^{k-1}$ and $\eta(M)_1^{k-2}$ we know that

$$\lim_{r\to\infty} \mathbb{P}\left(\left|\frac{B(M_r) - aM_r}{c\sqrt{M_r}}\right| < \sqrt{L_r}\right) = 1.$$

Thus by further passing to a subsequence we know that

$$\mathbb{P}\left(\eta(M_r)_1^k < -L_r; \left|\frac{B(M_r) - aM_r}{c\sqrt{M_r}}\right| < \sqrt{L_r}\right) > p/4. \tag{3.7.3}$$

We know that $Y(M)_1^{k-1}$ is supported on $A(M), A(M)+1, \dots, B(M)$, where $A(M) = Y_1^k(M)$ and $B(M)$ is as above. Moreover, if

$$p_i(M) = \mathbb{P}\left(Y_1^{k-1}(M) = i | Y_1^k(M), Y_2^k(M), Y_2^{k-1}(M), Y_1^{k-2}(M)\right) \text{ for } i = A(M), \dots, B(M),$$

then by Lemma 3.6.4 $c^{-1} > \frac{p_i(M)}{p_j(M)} > c$ for some constant $c \in (0,1)$ that depends on $k$. The latter implies that $p_i(M) \geq c^2/(B - A + 1)$ and so

$$\mathbb{P}\left(Y_1^{k-1}(M) \leq \frac{A(M) + B(M)}{2} | Y_1^k(M), Y_2^k(M), Y_2^{k-1}(M), Y_1^{k-2}(M)\right) \geq \frac{c^2}{2}.$$

This together with (3.7.3) implies

$$\mathbb{P}\left(\eta(M_r)_1^{k-1} < -L_r/3\right) > (p/4)(c^2/2), \tag{3.7.4}$$

which clearly contradicts the tightness of $\eta(M_r)_1^{k-1}$. The contradiction arose from our assumption that $\eta(M)_1^k$ is not tight as $M \to \infty$. This proves the induction step. By induction we conclude that $\nu_M \circ g_M^{-1}$ is tight for any $k \in \mathbb{N}$.

**Step 3.** In this step we prove that $g_M(Y(M))$ converge to $\Lambda = \lambda_i^j$ $i = 1, \dots, j$, $j = 1, \dots, k$ as $M \to \infty$, where $\Lambda$ is the GUE-corners process of rank $k$. We proceed by induction on $k$, with base case $k = 1$ true by Lemma 3.4.5.

Suppose the result is known for $k - 1 \geq 1$ and we wish to show it for $k$. From our earlier work we know that $\nu_M \circ g_M^{-1}$ is tight. Let $\mu$ be any subsequential limit and $\nu_{M_r} \circ g_{M_r}^{-1}$ converge weakly to $\mu$ for some sequence $M_r \to \infty$ as $r \to \infty$.

We observe that $\mu$ is a probability measure on $GT^k = \{y \in \mathbb{R}^{k(k+1)/2} : y_i^{j+1} \leq y_i^j \leq y_{i+1}^{j+1},\ 1 \leq i \leq j \leq k-1\}$. We have by induction hypothesis that the restriction to $GT^{k-1}$ of $\mu$ is the GUE-corners process of rank $k - 1$. In particular, we have

$$\mu(y \in GT^k : y_i^{k-1} = y_{i+1}^{k-1} \text{ for some } i = 1, \dots, k-2) = 0.$$

The above together with the interlacing property of elements in $GT^k$, shows that $\mu$ satisfies

$$\mathbb{P}^\mu(y_i^n = y_{i+1}^n = y_{i+2}^n \text{ for some } i = 1, \dots, n-2) = 0.$$

From Step 1. in this proof we know that $\nu_{M_r}$ satisfy the six-vertex Gibbs property with weights as in (3.7.1). We may thus apply Proposition 3.6.7 to conclude that $\mu$ satisfies the continuous Gibbs property.

From Lemma 3.4.5, we know that under $\mu$ the distribution of $(y_1^1, \dots, y_k^k)$ is the same as $(\lambda_1^1, \dots, \lambda_k^k)$. This together with Proposition 3.5.6 shows that $\mu$ is the GUE-corners process of rank $k$. The above work shows that any subsequential limit of $\nu_M \circ g_M^{-1}$ has the same law as $\Lambda$. As $\nu_M \circ g_M^{-1}$ is tight we conclude it (and hence $g_M(Y(M))$) weakly converge to the GUE-corners process of rank $k$. The general result now follows by induction.

## 3.8 Exact sampling algorithm for $\mathbb{P}_{\mathbf{u},\mathbf{v}}$

In this section, we describe an exact sampling algorithm for $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ (see Definition 3.2.14), which is based on discrete time dynamics on $\mathcal{P}_N$. We provide details on how this algorithm can be implemented efficiently and give some examples of typical path collections sampled from $\mathbb{P}_{\mathbf{u},\mathbf{v}}$.

### 3.8.1 Markov kernels and sequential update

We start by recalling some notation from Section 6.2 in [33]. For any $n$ we define

$$\Lambda^-_{u|\mathbf{u}}(\nu \to \mu) := \frac{\mathsf{F}_\mu(u_1, \dots, u_n)}{\mathsf{F}_\nu(u_1, \dots, u_n, u)} \mathsf{F}_{\nu/\mu}(u), \tag{3.8.1}$$

where $\mathbf{u} = (u_1, \dots, u_n)$, and $\nu \in \mathsf{Sign}^+_{n+1}$, $\mu \in \mathsf{Sign}^+_n$. Let us also define

$$Q^\circ_{\mathbf{u};v}(\mu \to \nu) := \left( \prod_{i=1}^m \frac{1 - u_i v}{1 - q u_i v} \right) \frac{\mathsf{F}_\nu(u_1, \dots, u_n)}{\mathsf{F}_\mu(u_1, \dots, u_n)} \mathsf{G}^c_{\nu/\mu}(v), \tag{3.8.2}$$

where $u_i$ and $v$ are admissible with respect to $s = q^{-1/2}$ for all $i$ and $\mu, \nu \in \mathsf{Sign}^+_n$. It follows from Propositions 3.2.5 and 3.2.7 that $\Lambda^-_{u|\mathbf{u}} : \mathsf{Sign}^+_{m+1} \dashrightarrow \mathsf{Sign}^+_m$ and $Q^\circ_{\mathbf{u};v} : \mathsf{Sign}^+_m \dashrightarrow \mathsf{Sign}^+_m$ define Markov kernels.[2]

For $\omega \in \mathcal{P}_N$, we let $\lambda^n = \lambda^n(\omega)$ for $n = 1, \dots, N$ be as in Section 3.1.1, we also let $\lambda^0(\omega) = \varnothing$. Let $\mathbb{P}^n_{\mathbf{u},\mathbf{v}}$ be the projection of $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ on $\lambda^n$. As direct consquences of Proposition 3.2.7, we have

$$\mathbb{P}^n_{\mathbf{u}\cup u,\mathbf{v}} \Lambda^-_{u|\mathbf{u}} = \mathbb{P}^n_{\mathbf{u},\mathbf{v}}, \quad \mathbb{P}^n_{\mathbf{u},\mathbf{v}} Q^\circ_{\mathbf{u};v} = \mathbb{P}^n_{\mathbf{u},\mathbf{v}\cup v}, \text{ and } \quad Q^\circ_{\mathbf{u}\cup u;v} \Lambda^-_{u|\mathbf{u}} = \Lambda^-_{u|\mathbf{u}} Q^\circ_{\mathbf{u};v}, \tag{3.8.3}$$

where for a variable set $\mathbf{w} = (w_1, \dots, w_k)$ we write $\mathbf{w} \cup w = (w_1, \dots, w_k, w)$.

Our next goal is to define a stochastic dynamics on $\mathcal{P}_N$. The construction we use is parallel to those of [30] (see also [17, 24, 32]) and it is based on an idea going back to [46], which allows to couple the dynamics on signatures of different sizes.

Suppose that $\omega$ is distributed accoriding to $\mathbb{P}_{\mathbf{u},\mathbf{v}}$ as in Definition 3.2.14, and let $v$ be such that $0 < v$ and $u_i v < 1$ for all $i$. We consider a random $\mu^1 \preceq \mu^2 \preceq \cdots \preceq \mu^N$, with $\mu^i \in \mathsf{Sign}^+_i$, whose distribution depends on $\lambda^n(\omega)$ for $n = 1, \dots, N$ and the parameters $\mathbf{u}$, $v$, and is defined through the following *sequential update* rule.

We start with $\mu^1$ and sample it according to the distribution

$$\mathbb{P}(\mu^1 = \nu | \lambda^1 = \lambda) = \frac{\mathsf{F}_\nu(u_1) \mathsf{G}^c_{\nu/\lambda}(v)}{\sum_{\kappa \in \mathsf{Sign}^+_1} \mathsf{F}_\kappa(u_1) \mathsf{G}^c_{\kappa/\lambda}(v)}. \tag{3.8.4}$$

If $\mu^1, \dots, \mu^{k-1}$ are sampled, we sample $\mu^k$ for $k \geq 2$ according to

$$\mathbb{P}(\mu^k = \nu | \lambda^k = \lambda, \mu^{k-1} = \mu) = \frac{\mathsf{F}_{\nu/\mu}(u_k) \mathsf{G}^c_{\nu/\lambda}(v)}{\sum_{\kappa \in \mathsf{Sign}^+_k} \mathsf{F}_{\kappa/\mu}(u_k) \mathsf{G}^c_{\kappa/\lambda}(v)}. \tag{3.8.5}$$

We now let $\omega'$ be the resulting element in $\mathcal{P}_N$, i.e., $\lambda^n(\omega') = \mu^n$ for $n = 1, \dots, N$. The key observation is that if $\omega$ is distributed accoridng to $\mathbb{P}_{\mathbf{u},\mathbf{v}}$, then $\omega'$ is distributed according to $\mathbb{P}_{\mathbf{u},\mathbf{v}\cup v}$. The latter is a consequence of (3.8.3) and a Gibbs property satisfied by $\mathbb{P}_{\mathbf{u},\mathbf{v}}$, which

[2] We use the notation "$\dashrightarrow$" to indicate that $\Lambda^-_{u|\mathbf{u}}$ and $Q^\circ_{\mathbf{u};v}$ are Markov kernels, i.e., they are functions in the first variable (belonging to the space on the left of "$\dashrightarrow$')' and probability distributions in the second variable (belonging to the space on the right of "$\dashrightarrow$").

states that conditioned on $\lambda^k$, the distribution of $\lambda^1, \dots, \lambda^{k-1}$ is independent of $\mathbf{v}$ and is given by

$$\Lambda^-_{u_k|(u_1,\dots,u_{k-1})}(\lambda^k \to \lambda^{k-1}) \cdots \Lambda^-_{u_3|(u_1,u_2)}(\lambda^3 \to \lambda^2)\Lambda^-_{u_2|(u_1)}(\lambda^2 \to \lambda^1). \tag{3.8.6}$$

For a more detailed description of the above procedure in analogous contexts we refer the reader to Section 2 of [30] and Section 2 of [24].

For $m = 0, \dots, M$ we let $\mathbb{P}_{\mathbf{u},\mathbf{v}_m}$, denote the probability distribution as in Definition 3.2.14 with $\mathbf{v}_m = (v_1, \dots, v_m)$. Equations (3.8.4) and (3.8.5) provide a mechanism for sampling $\omega'$ distributed according to $\mathbb{P}_{\mathbf{u},\mathbf{v}_{m+1}}$, given $\omega$ distributed as $\mathbb{P}_{\mathbf{u},\mathbf{v}_m}$. Our strategy to sample $\mathbb{P}_{\mathbf{u},\mathbf{v}} = \mathbb{P}_{\mathbf{u},\mathbf{v}_M}$ is to first sample $\mathbb{P}_{\mathbf{u},\mathbf{v}_0}$ and then use the above mechanism to sequentially sample $\mathbb{P}_{\mathbf{u},\mathbf{v}_{m+1}}$ for $m = 0, \dots, M-1$.

We now turn to an algorithmic description of the above strategy. We assume we have the following samplers, which will be described in the following section. For $N \geq 1$, $q \in (0,1)$ and $\mathbf{u} = (u_1, \dots, u_N)$ such that $u_i > q^{-1/2}$ for $i = 1, \dots, N$, we let $\texttt{ZeroSampler}(N, q, \mathbf{u})$ produce a random element $\omega \in \mathcal{P}_N$, distributed according to $\mathbb{P}_{\mathbf{u},\mathbf{v}_0}$. For $k \in \{1, \dots, N\}$, $v > 0$ such that $u_i v < 1$ for all $i$, $\lambda \in \mathsf{Sign}^+_k$ and $\mu \in \mathsf{Sign}^+_{k-1}$, we let $\texttt{RowSampler}(k, q, \mathbf{u}, v, \lambda, \mu)$ produce a random signature $\mu^k \in \mathsf{Sign}^+_k$, distributed according to (3.8.5). With this notation we have the following exact sampler for $\mathbb{P}_{\mathbf{u},\mathbf{v}}$.

**Algorithm** $\texttt{SixVerexSampler}(N, M, q, \mathbf{u}, \mathbf{v})$

**Input:** $q \in (0,1)$, $\mathbf{u} = (u_1, \dots, u_N)$ and $\mathbf{v} = (v_1, \dots, v_M)$ - parameters of the distribution.

```
ω := ZeroSampler(N, q, u);
initialize μ^i for i = 0, ..., N;
μ^0 = ∅;
for (i = 1, i ≤ M, i = i + 1) do
    for (k = 1, k ≤ N, k = k + 1) do
        μ^k = RowSampler(k, q, u_k, v_i, λ^k(ω), μ^{k-1});
    end
  ω = (μ^1 ⪯ μ^2 ⪯ ··· ⪯ μ^N);
end
```

**Output:** $\omega$.

### 3.8.2 The algorithms ZeroSampler and RowSampler

From the definition of $\mathbb{P}_{\mathbf{u},\mathbf{v}_0}$, we know that it agrees with the distribution of the vertically inhomogeneous stochastic six vertex model of Section 6.5 in [33], except that all columns are shifted by 1 to the right so that all vertices in the 0-th column are of the form $(0,1;0,1)$. The vertically inhomogeneous six vertex model has a known sampling procedure, which we now describe - see Section 6.5 [33] and [27] for details.

For $u > q^{-1/2}$ and $q \in (0,1)$, we let

$$b_1(u) = \frac{1 - uq^{1/2}}{1 - uq^{-1/2}} \text{ and } b_2(u) = \frac{-uq^{-1/2} + q^{-1}}{1 - uq^{-1/2}}.$$

Notice that $b_1(u), b_2(u) \in (0,1)$. We construct a random element $\omega \in \mathcal{P}_N$ by choosing the types of vertices sequentially: we start from the corner vertex at $(1,1)$, then proceed to $(1,2)$ and $(2,1), \dots$, then proceed to all vertices $(x,y)$ with $x+y=k$, then with $x+y=k+1$ and so forth. The combinatorics of the model implies that when we choose the type of the vertex $(x,y)$, either it is uniquely determined by the types of its previously chosen neighbors, or we need to choose between vertices of type $(1,0;1,0)$ and $(1,0;0,1)$, or we need to choose between vertices of type $(0,1;0,1)$ and $(0,1;1,0)$. We do all choiced independently and choose type $(1,0;1,0)$ with probability $b_1(u_y)$ and type $(1,0;0,1)$ with probability $1-b_1(u_y)$. Similarly, we choose type $(0,1;0,1)$ with probability $b_2(u_y)$ and $(0,1;1,0)$ with probability $1-b_2(u_y)$. We denote by $\texttt{Bernoulli}(p)$ a Bernoulli random variable sampler with parameter $p \in (0,1)$. For a vertex $\alpha = (i_1, j_1; i_2, j_2)$, we let $\texttt{I2}(\alpha) = i_2$ and $\texttt{J2}(\alpha) = j_2$. With this notation we have the following algorithm for `ZeroSampler`.

**Algorithm** ZeroSampler$(N, q, \mathbf{u})$

**Input:** $q \in (0,1)$, $\mathbf{u} = (u_1, \ldots, u_N)$ - parameters of the distribution.

```
initialize ω;
c := 0;
k := 2;
while (c < N) do
    for (x = 1, x < k, x = x + 1) do
        y = k − x;
        if  (y > N) do nothing
        else if  (x == 1 and y == 1)
            if  (Bernoulli(b_2(u_y)) == 1) ω(x, y) = (0, 1; 0, 1);
            else ω(x, y) = (0, 1; 1, 0);
            end
        else if  (x == 1)
            if  (I2(ω(x, y − 1)) == 1) ω(x, y) = (1, 1; 1, 1);
            else if (Bernoulli(b_2(u_y)) == 1) ω(x, y) = (0, 1; 0, 1);
            else ω(x, y) = (0, 1; 1, 0);
            end
        else if (y == 1)
            if  (J2(ω(x − 1, y)) == 0) ω(x, y) = (0, 0; 0, 0);
            else if (Bernoulli(b_2(u_y)) == 1) ω(x, y) = (0, 1; 0, 1);
            else ω(x, y) = (0, 1; 1, 0);
            end
        else
            if  (I2(ω(x, y − 1)) == 0 and J2(ω(x − 1, y)) == 0) ω(x, y) = (0, 0; 0, 0);
            else if  (I2(ω(x, y − 1)) == 1 and J2(ω(x − 1, y)) == 1) ω(x, y) = (1, 1; 1, 1);
            else if  (I2(ω(x, y − 1)) == 0 and J2(ω(x − 1, y)) == 1)
                if  (Bernoulli(b_2(u_y)) == 1) ω(x, y) = (0, 1; 0, 1);
                else ω(x, y) = (0, 1; 1, 0);
                end
            else
                if (Bernoulli(b_1(u_y)) == 1) ω(x, y) = (1, 0; 1, 0);
                else ω(x, y) = (1, 0; 0, 1);
                end
            end
        end
        if  (y == N and I2(ω(x, y)) == 1) c = c + 1;
    end
    k = k + 1;
end
initialize μ^i for i = 1, ..., N;
for (i = 1, i ≤ M, i = i + 1) do
    μ^i = λ^i(ω) + 1^i;
end
```

**Output:** $(\mu^1 \preceq \mu^2 \preceq \cdots \preceq \mu^N)$.

Let us fix $k \geq 1$, parameters $q, u, v$ such that $q \in (0,1)$, $u > q^{-1/2}$, $v > 0$ and $uv < 1$. We also fix $\mu \in \mathsf{Sign}_{k-1}^+$ and $\lambda \in \mathsf{Sign}_k^+$. We now discuss how to sample the distribution from (3.8.5) with the above parameters, which we denote by $\mathbb{P}$ for brevity. Let us define the numbers $a_i = \max(\mu_i, \lambda_i, 0)$ and $b_i = \min(\mu_{i-1}, \lambda_{i-1})$, where we agree that $\mu_k = -\infty$, $\lambda_0 = \mu_0 = \infty$. We also set $A(\mathbf{a}, \mathbf{b}) = \cup_{i=1}^k [a_i, b_i]$. The definition of $\mathbb{P}$ implies that $\mathbb{P}(\{\nu \in \mathsf{Sign}_k^+ : \nu_i \in [a_i, b_i] \text{ for } i = 1, \dots, k\}) = 1$. Moreover, if $\nu \in \mathsf{Sign}_k^+$ is such that $\nu_i \in [a_i, b_i]$ for $i = 1, \dots, k$ then $\mathcal{P}^c_{\nu/\lambda}$ and $\mathcal{P}_{\nu/\mu}$ (see Definitions 3.2.1 and 3.2.2) consist of single elements $\omega$ and $\omega^c$, which implies that

$$\mathbb{P}(\{\nu\}) \propto \prod_{j \in A(\mathbf{a},\mathbf{b}) \cap [0, \nu_1]} w_u(\omega(j,1)) w_v^c(\omega^c(j,1)), \text{ where } w_u \text{ and } w_u^c \text{ are as in (3.2.1) and (3.2.2).}$$

Sampling $\mathbb{P}$ is rather hard because there are infinite possible signautes $\nu$ that are allowed. Even if we consider only signatures, whose parts are bounded by some large constant $L$, their number is still exponentially large in $L$ and we cannot hope to efficiently enumerate possible cases and calculate their weight.

The key observation that allows one to sample this distribution is that if $l \in \{1, \dots, k\}$, then conditional on $\nu_l$, the distributions of $\nu_1, \dots, \nu_{l-1}$ and $\nu_{l+1}, \dots, \nu_k$ are independent and similar to the one of $\nu_1, \dots, \nu_k$. Let us make the last statement more precise. Fix an integer $l \in \{1, \dots, k\}$, suppose we have fixed $\nu_l = x \in [a_l, b_l]$ and that there is at least one possible signature $\nu_1, \dots, \nu_k$ with $\nu_l = x$. We modify $a_i$ and $b_i$ as follows

$$b_i^x = \begin{cases} b_i & \text{if } b_i \neq x \\ b_i - 1 & \text{else,} \end{cases} \qquad a_i^x = \begin{cases} a_i & \text{if } a_i \neq x \\ a_i + 1 & \text{else.} \end{cases}$$

Let us fix $y_i \in [a_i^x, b_i^x]$ for $i \neq l$, put $y_l = x$ and denote $A_R = \cup_{i=1}^{l-1} [a_i^x, b_i^x]$ and $A_L = \cup_{i=l+1}^{k} [a_i^x, b_i^x]$. Then we have

$$\mathbb{P}(\{\nu \in \mathsf{Sign}_k^+ : \nu_i = y_i \text{ for } i \neq l | \nu_l = x) = \mathbb{P}(\{\nu \in \mathsf{Sign}_k^+ : \nu_i = y_i \text{ for } i = 1, \dots, l-1 | \nu_l = x) \times$$

$$\mathbb{P}(\{\nu \in \mathsf{Sign}_{k-l}^+ : \nu_i = y_i \text{ for } i = l+1, \dots, k | \nu_l = x).$$

Moreover, we have

$$\mathbb{P}(\{\nu \in \mathsf{Sign}_k^+ : \nu_i = y_i \text{ for } i = 1, \dots, l-1\} | \nu_l = x) \propto \prod_{j \in A_R \cap [0, y_1]} w_u(\omega(j,1)) w_v^c(\omega^c(j,1)),$$

$$\mathbb{P}(\{\nu \in \mathsf{Sign}_{k-l}^+ : \nu_i = y_i \text{ for } i = l+1, \dots, k\} | \nu_l = x) \propto \prod_{j \in A_L \cap [0, y_1]} w_u(\omega(j,1)) w_v^c(\omega^c(j,1)).$$

The above arguments imply that we can sample $\mathbb{P}$, by first sampling $\nu_{\lfloor k/2 \rfloor}$, conditioning on its value and recursively sampling $\nu_1, \dots, \nu_{\lfloor k/2 \rfloor - 1}$ and $\nu_{\lfloor k/2 \rfloor + 1}, \dots, \nu_k$. The recursion reduces runtime from exponentially large to polynomial in $N$.

We begin by explaining how to sample $\nu_l$ for $l \in \{1, \dots, k\}$. Suppose that $I = \{i_1, \dots, i_r\} \subset$

$\{1, \ldots, k\}$, $[c_i, d_i] \subset [a_i, b_i]$ for $i \in I$ and $x_i \in [c_i, d_i]$. We define

$$\mathcal{W}(\mathbf{c}, \mathbf{d}, I, \mathbf{x}) := \prod_{i \in I} \prod_{j=c_i}^{x_i} w_u(\omega(j,1)) w_v^c(\omega^c(j,1)),$$

where $\omega^c$ and $\omega$ are the single elements of $\mathcal{P}^c_{\nu/\lambda}$ and $\mathcal{P}_{\nu/\mu}$, where $\nu_i = x_i$ for $i \in I$ and $\nu_j < c_i$ if $j > i$, $i \in I$ and $j \notin I$ (if no such $\nu \in \mathsf{Sign}_k^+$ exists $\mathcal{W}(\mathbf{c}, \mathbf{d}, I, \mathbf{x}) = 0$). We also define

$$\mathcal{W}(\mathbf{c}, \mathbf{d}; I) := \sum_{x_{i_1}=c_{i_1}}^{d_{i_1}} \cdots \sum_{x_{i_r}=c_{i_r}}^{d_{i_r}} \mathcal{W}(\mathbf{c}, \mathbf{d}, I, \mathbf{x}).$$

Let us denote by $p = w_u((0,1;0,1)) w_v^c((0,1;0,1))$, and suppose $[c_l, d_l] \subset [a_l, b_l]$. We wish to sample $\nu_l$ conditioned on it belonging to $[c_l, d_l]$ according to $\mathbb{P}$. We define

$$\begin{aligned} wL1 &= \mathcal{W}(\mathbf{c}, \mathbf{d}; \{1, \ldots, l-1\}), \text{ where } c_i = a_i \text{ and } d_i = b_i; \\ wL2 &= \mathcal{W}(\mathbf{c}, \mathbf{d}; \{1, \ldots, l-1\}), \text{ where } c_i = a_i \text{ and } d_i = b_i, \text{ except } c_{l-1} = a_{l-1} + 1; \\ wR1 &= \mathcal{W}(\mathbf{c}, \mathbf{d}; \{l+1, \ldots, k\}), \text{ where } c_i = a_i \text{ and } d_i = b_i; \\ wR2 &= \mathcal{W}(\mathbf{c}, \mathbf{d}; \{l+1, \ldots, k\}), \text{ where } c_i = a_i \text{ and } d_i = b_i, \text{ except } d_{l+1} = d_{l+1} - 1. \end{aligned} \tag{3.8.7}$$

$$\begin{aligned} bLL &= 1 \text{ if } c_l = \lambda_l \text{ and } bLL = 0 \text{ otherwise;} \\ bRL &= 1 \text{ if } d_l = \lambda_{l-1} \text{ and } bRL = 0 \text{ otherwise;} \\ bLM &= 1 \text{ if } c_l = \mu_l \text{ and } bLM = 0 \text{ otherwise;} \\ bRM &= 1 \text{ if } d_l = \mu_{l-1} \text{ and } bRM = 0 \text{ otherwise;} \end{aligned} \tag{3.8.8}$$

The conditional distribution of $\nu_l$ depends on $c_l, d_l$ and the above four variables $bLL, bRL, bLM$ and $bRM$.

If $c_l = d_l$ then we have $\nu_l = c_l$ with probability 1. If $d_l = c_l + 1$, then we have sixteen possible cases for the variables $bLL, bRL, bLM$ and $bRM$, which lead to different probability distributions. To give one example, if $bLL = bRL = bLM = bRM = 0$, then

$$\mathbb{P}(\nu_l = c_l) \propto wL2 \cdot wR1, \quad \mathbb{P}(\nu_l = d_l) \propto p \cdot wL1 \cdot wR2.$$

If $d_l - c_l = n \geq 2$, then we have sixteen possible cases for the variables $bLL, bRL, bLM$ and $bRM$, which lead to different probability distributions. To give one example, if $bLL = bRL = bLM = bRM = 0$, then

$$\begin{aligned} &\mathbb{P}(\nu_l = c_l) \propto w_u(0,1;1,0) w_u(0,1;1,0) \cdot wL2 \cdot wR1, \\ &\mathbb{P}(\nu_l = d_l) \propto w_u(0,1;1,0) w_v^c(0,1;1,0) w_u(0,1;0,1) w_v^c(0,1;0,1) p^{n-1} \cdot wL1 \cdot wR2, \\ &\mathbb{P}(\nu_l = c_l + i) \propto w_u(0,1;1,0) w_v^c(0,1;1,0) w_u(0,1;0,1) w_v^c(0,1;0,1) p^{i-1} \cdot wL1 \cdot wR1, \ 1 \leq i \leq k-1. \end{aligned}$$

There are altogether thirty-three cases (sixteen corresponding to $d_l = c_l + 1$, sixteen for $d_l > c_l + 1$ and the trivial case of $c_l = d_l$) and we will not write them out explicitly. The important point is that the conditional distribution of $\nu_l$, given $wL1, wL2, wR1$ and $wR2$ is explicit and can be sampled. We let $\texttt{ArrowSampler}(u, v, c, d, bLL, bRL, bLM, bRM, wL1, wL2, wR1, wR2)$

denote an algorithm that samples the above probability distribution when $c_l = c$, $d_l = d$.

Suppose that we have an algorithm $\texttt{Weight}(u, v, \mathbf{c}, \mathbf{d}, x, y, \lambda, \mu) := \mathcal{W}(\mathbf{c}, \mathbf{d}; \{x, x{+}1 \ldots, y\})$, then we have the following algorithm for `RowSampler`.

---

**Algorithm** $\texttt{RowSampler}(u, v, \mathbf{c}, \mathbf{d}, x, y, \lambda, \mu)$

---

**Input:** $u, v$ - parameters, $\mathbf{c} = (c_x, c_{x+1}, \ldots, c_y)$, $\mathbf{d} = (d_x, d_{x+1}, \ldots, d_y)$, $\lambda \in \mathsf{Sign}_k^+$, $\mu \in \mathsf{Sign}_{k-1}^+$.

**if** $(x == y)$
    $bLL := 0$; **if** $(c_x == \lambda_x)$ $bLL = 1$; **end**
    $bRL := 0$; **if** $(d_x == \lambda_{x-1})$ $bRL = 1$; **end**
    $bLM := 0$; **if** $(c_x == \mu_x)$ $bLM = 1$; **end**
    $bRM := 0$; **if** $(d_x == \mu_{x-1})$ $bRM = 1$; **end**
    $\nu_x = \texttt{ArrowSampler}(u, v, c_x, d_x, bLL, bRL, bLM, bRM, 1, 1, 1, 1)$;
**else**
    $s := \lfloor (x + y)/2 \rfloor$;
    $\mathbf{c}' := \{c_x, \ldots, c_{s-1}\}$; $\mathbf{c}'' := \{c_{s+1}, \ldots, c_y\}$; $\mathbf{d}' := \{d_x, \ldots, d_{s-1}\}$; $\mathbf{d}'' := \{d_{s+1}, \ldots, d_y\}$;
    $wR1 := \texttt{Weight}(u, v, \mathbf{c}', \mathbf{d}', x, s-1, \lambda, \mu)$; $wR2 := wR1$;
    **if** $(d_s == c_{s-1})$
        $\mathbf{c}' = \{c_x, \ldots, c_{s-2}, c_{s-1} + 1\}$; $wR2 = \texttt{Weight}(u, v, \mathbf{c}', \mathbf{d}', x, s-1, \lambda, \mu)$;
    **end**
    $wL1 := \texttt{Weight}(u, v, \mathbf{c}'', \mathbf{d}'', s+1, y)$; $wL2 := wL1$;
    **if** $(c_s == d_{s+1})$
        $\mathbf{d}'' = \{d_{s+1} - 1, d_{s+2}, \ldots, d_y\}$; $wL2 = \texttt{Weight}(u, v, \mathbf{c}'', \mathbf{d}'', s+1, y, \lambda, \mu)$;
    **end**
    $bLL := 0$; **if** $(c_s == \lambda_s)$ $bLL = 1$; **end**
    $bRL := 0$; **if** $(d_s == \lambda_{s-1})$ $bRL = 1$; **end**
    $bLM := 0$; **if** $(c_s == \mu_s)$ $bLM = 1$; **end**
    $bRM := 0$; **if** $(d_s == \mu_{s-1})$ $bRM = 1$; **end**
    $\nu_s = \texttt{ArrowSampler}(u, v, c_s, d_s, bLL, bRL, bLM, bRM, wL1, wL2, wR1, wR2)$;
    $\mathbf{c}' = \{c_x, \ldots, c_{s-1}\}$; $\mathbf{c}'' = \{c_{s+1}, \ldots, c_y\}$; $\mathbf{d}' = \{d_x, \ldots, d_{s-1}\}$; $\mathbf{d}'' = \{d_{s+1}, \ldots, d_y\}$;
    **if** $(\nu_s == d_{s+1})$ $\mathbf{d}'' = \{d_{s+1} - 1, d_{s+2}, \ldots, d_y\}$; **end**
    **if** $(\nu_s == c_{s-1})$ $\mathbf{c}' = \{c_x, \ldots, c_{s-2}, c_{s-1} + 1\}$; **end**
    $\texttt{RowSampler}(u, v, \mathbf{c}', \mathbf{d}', x, s-1, \lambda, \mu)$;
    $\texttt{RowSampler}(u, v, \mathbf{c}'', \mathbf{d}'', s+1, y, \lambda, \mu)$;
**Output:** $\nu$

---

In the algorithm `RowSampler` $\nu = \nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$ is a global variable that we are updating through the recursive calls to the same algorithm. Going back to the notation of `SixVerexSampler`, we have that $\texttt{RowSampler}(k, q, u, v, \lambda, \mu) = \texttt{RowSampler}(u, v, \mathbf{a}, \mathbf{b}, 1, k, \lambda, \mu)$. Thus what remains is to show how to calculate $\texttt{Weight}(u, v, \mathbf{c}, \mathbf{d}, x, y)$.

The function `Weight` can be calculated recursively by again conditioning on the middle arrow and summing over the weights corresponding to its possible positions. We first discuss the base case of having a single interval $[c_l, d_l]$. We will consider a reweighted version of $\mathcal{W}(\mathbf{c}, \mathbf{d}; \{l\})$, where we have additional four weights $wL1, wL2, wR1$ and $wR2$, which are fixed. By definition $\mathcal{W}(\mathbf{c}, \mathbf{d}; \{l\})$ is the sum of weights over the possible positions of the arrow in $[c_l, d_l]$. Our reweighed version will be the same sum, however we will multiply each term by $wL1 \cdot wR1$, $wL1 \cdot wR2$, $wL2 \cdot wR1$ or $wL2 \cdot wR2$ according to the following rules.

We multiply the weight by $wL1$ unless the arrow is in location $c_l$, in which case we multiply it by $wL2$, we then multiply the weight by $wR1$ unless the arrow is in location $d_l$, in which case we multiply it by $wR2$. One observes that the weight of the interval $[c_l, d_l]$, depends on $d_l - c_l$. We have three cases for $d_l - c_l$ - when it is 0, 1 and $\geq 2$, and the weights are as follows.

- $d_l - c_l = 0 : \mathcal{W}(\mathbf{c}, \mathbf{d}; \{l\}) = w_u(1,1;1,1) \cdot w_v^c(1,0;1,0) \cdot wL2 \cdot wR2;$
- $d_l - c_l = 1 : \mathcal{W}(\mathbf{c}, \mathbf{d}; \{l\}) = w_u(0,1;1,0) \cdot w_u(1,0;0,1) \cdot w_v^c(1,0;1,0) \cdot wL2 \cdot wR1 + w_u(0,1;0,1) \cdot w_u(1,1;1,1) \cdot w_v^c(1,0;0,1) \cdot w_v^c(0,1;1,0) \cdot wL1 \cdot wR2;$
- $d_l - c_l = n \geq 2 : \mathcal{W}(\mathbf{c}, \mathbf{d}; \{l\}) = w_u(0,1;1,0) \cdot w_u(1,0;0,1) \cdot w_v^c(1,0;1,0) \cdot wL2 \cdot wR1 + w_u(0,1;0,1) \cdot w_u(1,1;1,1) \cdot w_v^c(1,0;0,1) \cdot w_v^c(0,1;1,0) \cdot p^{n-1} \cdot wL1 \cdot wR2 + w_u(0,1;1,0) \cdot w_u(1,0;0,1) \cdot w_u(0,1;0,1) \cdot w_v^c(1,0;0,1) \cdot w_v^c(0,1;1,0) \cdot \frac{1-p^{n-1}}{1-p} \cdot wL1 \cdot wR1.$

We let $\texttt{BaseWeight}(u, v, c, d, wL1, wL2, wR1, wR2)$ denote the above single interval weight function and with it we define $\texttt{Weight}(u, v, \mathbf{c}, \mathbf{d}, x, y, \lambda, \mu)$ as follows.

**Algorithm** $\texttt{Weight}(u, v, \mathbf{c}, \mathbf{d}, x, y, \lambda, \mu)$

**Input:** $u, v$ - parameters, $\mathbf{c} = (c_x, c_{x+1}, \dots, c_y)$, $\mathbf{d} = (d_x, d_{x+1}, \dots, d_y)$, $\lambda \in \mathsf{Sign}_k^+$, $\mu \in \mathsf{Sign}_{k-1}^+$.

```
initialize w;
if (x == y)
    w = BaseWeight(u, v, c_x, d_x, 1, 1, 1, 1);
else
    s := ⌊(x + y)/2⌋;
    c' := {c_x, ..., c_{s-1}};  c'' := {c_{s+1}, ..., c_y};  d' := {d_x, ..., d_{s-1}};  d'' := {d_{s+1}, ..., d_y};
    wR1 := Weight(u, v, c', d', x, s - 1, λ, μ);  wR2 := wR1;
    if  (d_s == c_{s-1})
        c' = {c_x, ..., c_{s-2}, c_{s-1} + 1};   wR2 = Weight(u, v, c', d', x, s - 1, λ, μ);
    end
    wL1 := Weight(u, v, c'', d'', s + 1, y);  wL2 := wL1;
    if  (c_s == d_{s+1})
        d'' = {d_{s+1} - 1, d_{s+2}, ..., d_y};   wL2 = Weight(u, v, c'', d'', s + 1, y, λ, μ);
    end
    w = BaseWeight(u, v, c_s, d_s, bLL, bRL, bLM, bRM, wL1, wL2, wR1, wR2);
end
Output: w
```

### 3.8.3 Discussion and extensions

In this section we discuss some of the implications of the results of the paper and some of their possible extensions. We also use the sampling algorithm developed above to produce some simulations. We will be interested in demonstrating that there is a limit shape for the six-vertex model that we have considered. In addition, we will provide some empirical evidence supporting the validity of Theorem 3.1.3.

Results similar to Theorem 3.1.3 are known for models of random Young diagrams and random tilings, see [8, 54, 68, 70]. Moreover, for random lozenge tilings the GUE-corners process is believed to be a universal scaling limit near the point separating two frozen regions (also called a *turning point*) [54,70]. We believe, although we cannot prove, that in our model the GUE-corners process also appears near the point separating two frozen regions. At this time, our methods do not seem to be strong enough to verify a limit-shape phenomenon; however, simulation results seem to indicate that this is indeed the case. For the simulations we fix $N = 100$ and consider different choices for $q, u$ and $v$. From Theorem 3.1.3 we know that $Y_1^1$ asymptotically looks like $a \cdot M$, with $a$ as in Theorem 3.1.1. We pick the parameter $M$ in our simulations so that $a \cdot M$ is roughly $N/2$. The results are summarized in Figures 3-10, 3-11and 3-12.

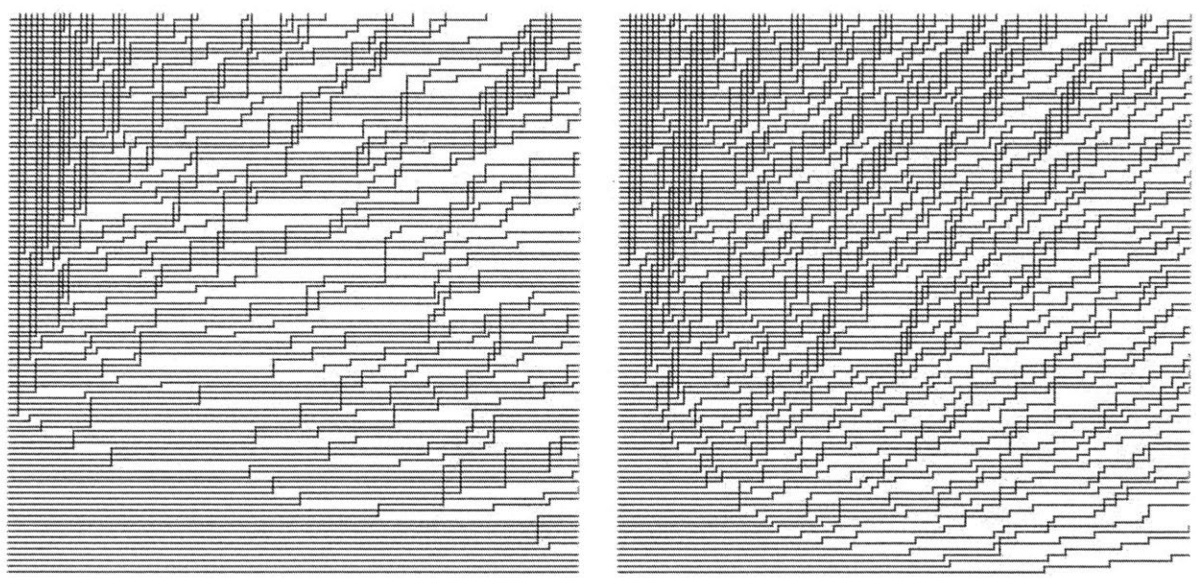

Figure 3-10: Random paths in $\mathcal{P}_N$, sampled according to $\mathbb{P}_{u,v}^{N,M}$ with $N = 100$. For the left picture $s^{-2} = q = 0.5$, $u = 5$, $v = 0.1$ and $M = 100$; for the right $s^{-2} = q = 0.5$, $u = 2$, $v = 0.1$ and $M = 1000$.

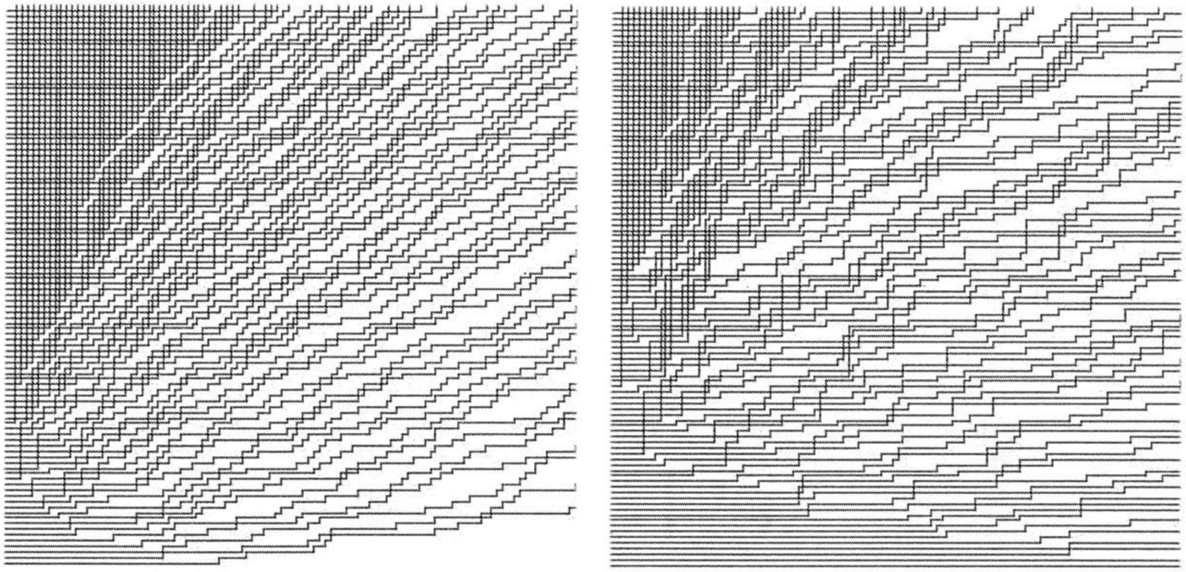

Figure 3-11: Random paths in $\mathcal{P}_N$, sampled according to $\mathbb{P}_{u,v}^{N,M}$ with $N = M = 100$. For the left picture $s^{-2} = q = 0.25$, $u = 2.5$ and $v = 0.25$; for the right $s^{-2} = q = 0.25$, $u = 5$, and $v = 0.1$ .

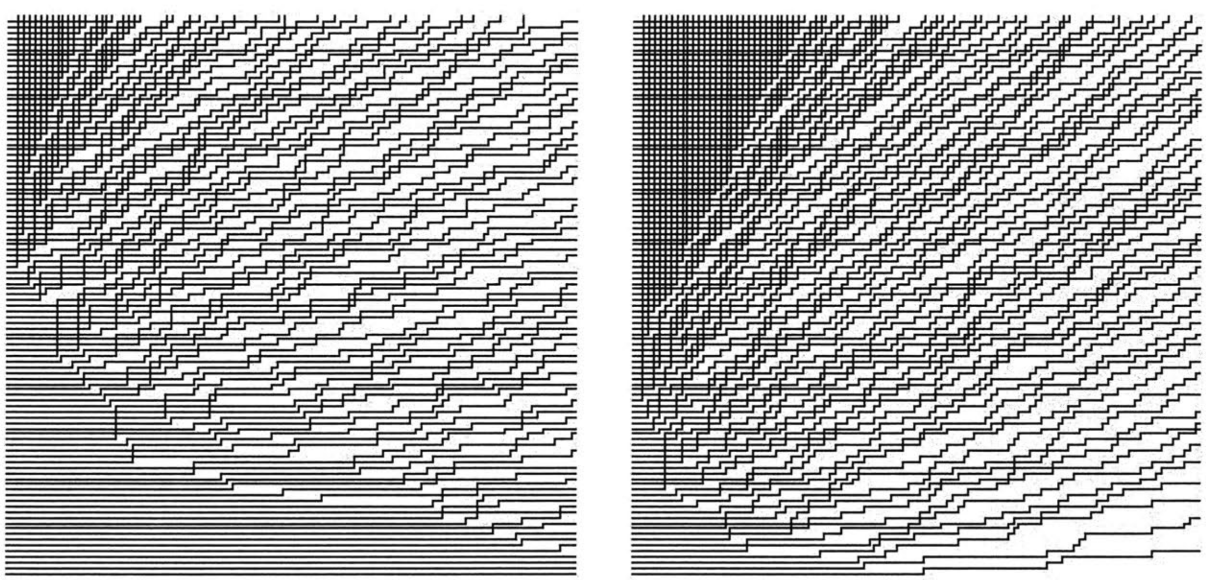

Figure 3-12: Random paths in $\mathcal{P}_N$, sampled according to $\mathbb{P}_{u,v}^{N,M}$ with $N = M = 100$. For the left picture $s^{-2} = 0.8$, $u = 1.2$ and $v = 0.8$; for the right $s^{-2} = 0.5$, $u = 1.5$ and $v = 0.6$.

As can be seen on Figures 3-10, 3-11and 3-12 , there is a macroscopic frozen region, made of $(0,1;0,1)$ vertices in the bottom left corner and another one, made of $(1,1;1,1)$ vertices in the top left corner. The two regions are separated by a disordered region containing all six types of vertices. It would be interesting to see if the methods of this chapter can be utilized to rigorously confirm the existence of a limit shape, and to find parametrizations for it.

A particular implication of Theorem 3.1.3 is that $\frac{1}{c\sqrt{M}}\left(Y_1^1 - aM\right)$ converges to the standard Gaussian distribution as $M \to \infty$. In what follows, we provide some numerical simulations supporting this fact. We took 1000 samples from $\mathbb{P}_{u,v}^{N,M}$ with $N = M = 200$ and different values for $q, u, v$, and calculated $\frac{1}{c\sqrt{M}}\left(Y_1^1 - aM\right)$. The empirical distribution of the samples is compared with the standard normal cdf, and the results are given in Figure 3-13. As can be seen, the distributions appear to be quite close, as is expected.

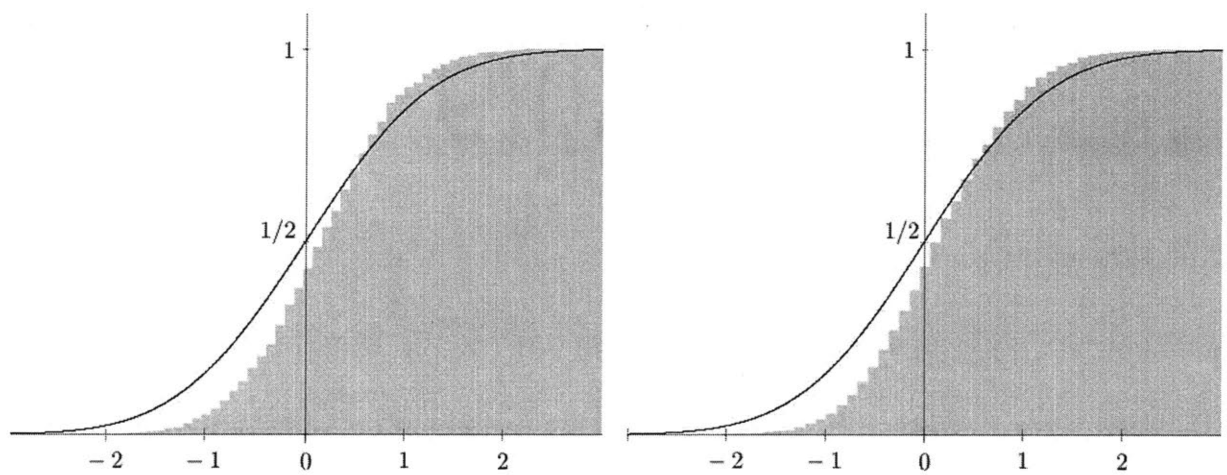


Figure 3-13: Empirical distribution of 1000 samples of $\frac{1}{c\sqrt{M}}\left(Y_1^1(\omega) - aM\right)$ with $\omega$ distributed as $\mathbb{P}_{u,v}^{N,M}$ with $N = M = 200$. For the left picture $s^{-2} = q = 0.8$, $u = 1.2$ and $v = 0.8$; for the right $s^{-2} = q = 0.5$, $u = 1.5$, $v = 0.6$.

Recall that one way to interpret the measure $\mathbb{P}_{u,v}^{N,m}$ is as the time $m$ distribution of a certain discrete time Markov chain, which at time 0 is distributed as the stochastic six-vertex model of [27]. In [27] it was shown that configurations sampled from $\mathbb{P}_{u,v}^{N,0}$ converge to a certain deterministic cone-like limit shape (see Figure 3-14 for sample simulations). Comparing Figures 3-12 and 3-14, we see that the stochastic dynamics has lead to a change in the limit shape. What is remarkable is that Theorem 3.1.3 indicates that the bulk fluctuations change as well. For the stochastic six-vertex model it is known that the fluctuations of the height function[3] in the bulk are governed by the GUE Tracy-Widom distribution [27]. On the other hand, the bulk fluctuations of the GUE-corners process are described by the Gaussian Free Field (GFF) [18]. Theorem 3.1.3 suggests that the stochastic dynamics has transformed height fluctuations from KPZ-like to GFF-like.

A possible explanation of the above phenomenon was suggested to us by Alexei Borodin and Fabio Toninelli and goes as follows. At large times one has both KPZ and GFF statistics

[3]The height function $h(x,y)$ of the six-vertex model is defined as the number of paths that cross the horizontal line through $y$ to the right or at the point $x$.

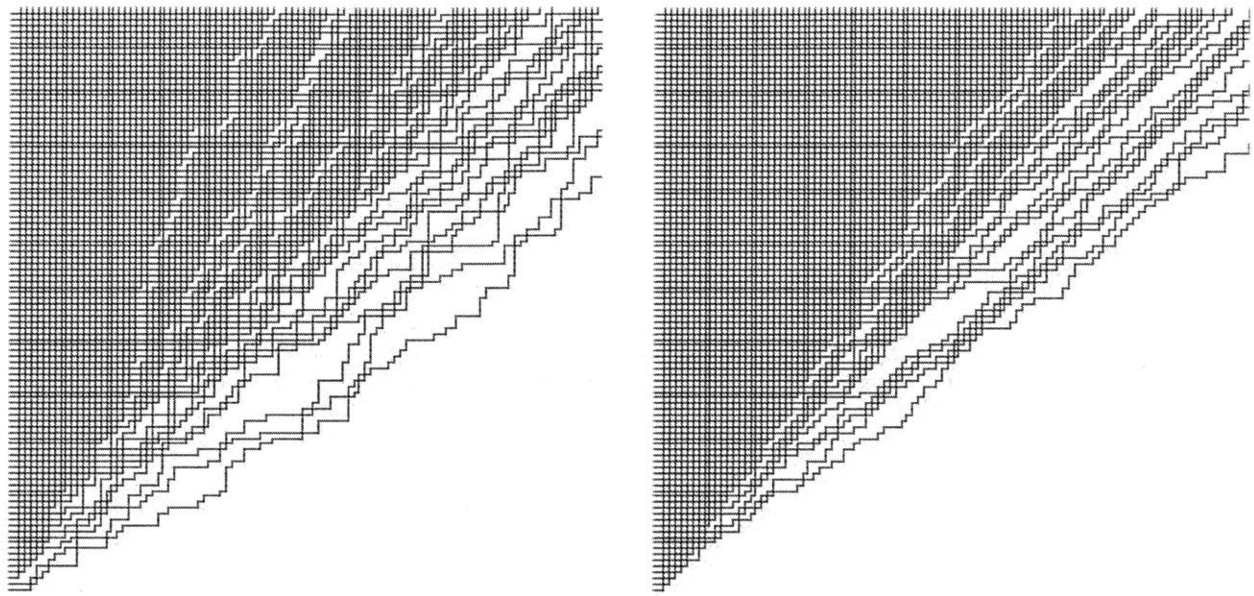

Figure 3-14: Random paths in $\mathcal{P}_N$, sampled according to $\mathbb{P}^f$ with $N = 100$ when $f(\lambda) = \mathsf{G}^c_\lambda(\rho)$. For the left picture $s^{-2} = 0.8$, $u = 1.5$; for the right $s^{-2} = 0.5$, $u = 2$

within the model, but they manifest themselves in different portions of the configurations. As path configurations evolve, the KPZ region is pushed away from the origin and in its place GFF statistics emerge. We motivate the latter explanation with some simulations in Figure 3-15. One distinguishing feature between KPZ and GFF statistics is the order of growth of the fluctuations, which are algebraic in the former and logarithmic in the latter case. We expect that the variance of the height function in the KPZ region to be of order $N^{2/3}$, while in the GFF region to be of order $\log(N)$. The latter implies that we can use the height variance as a proxy for distinguishing the different regions in our model and the results are presented in Figure 3-15. As can be seen, there is indeed a high-variance cone, which is moving away from the origin and a very low variance region takes its place. It would be very interesting to verify that both GFF and KPZ fluctuations coexist in our model, since to our knowledge such a phenomenon has not been observed in other settings.

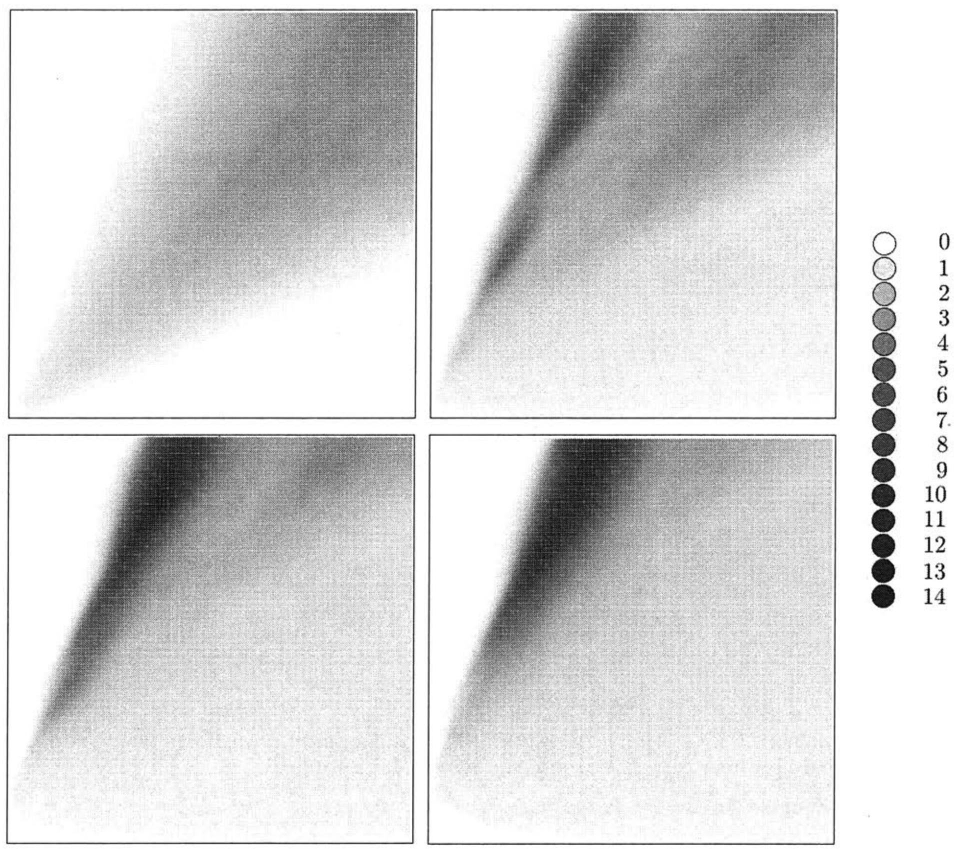


Figure 3-15: Variance of the height function at different locations for 2000 samples from $\mathbb{P}_{u,v}^{N,M}$ . For the above simulations $s^{-2} = 0.5, u = 3, v = 0.2$ and $N = 200$ and $M = 0, 30, 60$ and 90 for the top-left, top-right, bottom-left and bottom-right diagrams respectively. The variance-to-shade correspondence is indicated on the right.

# Chapter 4

# Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood Gibbsian line ensembles

## 4.1 Introduction

In this chapter we prove, as Theorem 4.3.13 and Corollary 4.3.11, the long predicted transversal 2/3 exponent for the asymmetric simple exclusion process (ASEP) [63,76] and the stochastic six vertex (S6V) model [53] – two (closely related) 1 + 1 dimensional random interface growth models / interacting particle systems in the Kardar-Parisi-Zhang (KPZ) universality class. We work with step initial data for both models and demonstrate that their height functions, scaled in space by $T^{2/3}$ and in fluctuation size by $T^{1/3}$, are tight as spatial processes as time $T$ goes to infinity (we use $T$ for time since $t \in (0,1)$ will be reserved for the Hall-Littlewood parameter). We also show as Corollary 4.7.4, that all subsequential limits of the scaled height function (shifted by a parabola) have increments, which are absolutely continuous with respect to a Brownian bridge measure. Conjecturally the limit process should be the Airy$_2$ process and we provide further evidence for this conjecture by uncovering a Gibbsian line ensemble structure behind these models, which formally limits to that of the Airy line ensemble [42].

### 4.1.1 Main results

We now state our main results concerning the ASEP. Precise definitions of this model and further discussion can be found in Section 4.2.3. We forgo stating the S6V model result until the main text – Corollary 4.3.11 – since it requires more notation to define the model.

In the ASEP, particles occupy sites indexed by $\mathbb{Z}$ with at most one particle per site (the exclusion rule) and jump according to independent exponential clocks to the right and left with rates $R$ and $L$ respectively ($R > L$ is assumed). Jumps that would violate the exclusion rule are suppressed. Step initial data means that particles start at every site in $\mathbb{Z}_{\leq 0}$ (and no particles start elsewhere). The height function $\mathfrak{h}_T(x)$ records the number of particles at or to the right of position $x \in \mathbb{Z}$ at time $T$. For $x \notin \mathbb{Z}$ we linearly interpolate to make the height function continuous. With this notation we can state our main theorem (Theorem 4.3.13 and Corollary 4.7.4 in the main text).

**Theorem 4.1.1.** *Suppose $r > 0$, $R = 1$, $L \in (0,1)$, $\gamma = R - L$ and fix $\alpha \in (0,1)$. For $s \in [-r, r]$ set*

$$f_N^{ASEP}(s) = \sigma_\alpha^{-1} N^{-1/3} \left( f_3(\alpha) N + f_3'(\alpha) s N^{2/3} + (1/2) s^2 f_3''(\alpha) N^{1/3} - \mathfrak{h}_{N/\gamma}\left(\alpha N + s N^{2/3}\right)\right), \tag{4.1.1}$$

*The constants above are given by $\sigma_\alpha = 2^{-4/3}(1-\alpha^2)^{2/3}$, $f_3(\alpha) = \frac{(1-\alpha)^2}{4}$, $f_3'(\alpha) = -\frac{1-\alpha}{2}$, $f_3''(\alpha) = \frac{1}{2}$. If $\mathbb{P}_N$ denotes the law of $f_N^{ASEP}(s)$ as a random variable in $(C[-r,r], \mathcal{C})$ — the space of continuous functions on $[-r,r]$ with the uniform topology and Borel $\sigma$-algebra $\mathcal{C}$ (see e.g. Chapter 7 in [13]) — then the sequence $\mathbb{P}_N$ is tight.*

*Moreover, if $\mathbb{P}_\infty$ denotes any subsequential limit of $\mathbb{P}_N$ and $f_\infty^{ASEP}$ has law $\mathbb{P}_\infty$, then $g_\infty^{ASEP}$ defined by*

$$g_\infty^{ASEP}(x) = \sigma_\alpha f_\infty^{ASEP}(x) - \frac{x^2 f_3''(\alpha)}{2}, \text{ for } x \in [-r, r],$$

*is absolutely continuous with respect to a Brownian bridge of variance $-2r f_3'(\alpha)[1 + f_3'(\alpha)]$ in the sense of Definition 4.7.2.*

Our approach for proving Theorem 4.1.1 is to (1) embed the ASEP height function into a line ensemble, which enjoys a certain 'Hall-Littlewood Gibbs' resampling property, and (2) use the known one-point tightness in the $T^{1/3}$ fluctuation scale to obtain the $T^{2/3}$ transversal tightness. These two points are discussed more extensively in the section below. Here we mention that the Gibbs property implies that conditional on the second curve in the line ensemble, the top curve (i.e. the height function) has a law expressible in terms of an explicit Radon-Nikodym derivative with respect to the trajectory of a random walk. By controlling this Radon-Nikodym derivative as $T$ goes to infinity, we are able to control quantities like the maximum, minimum and modulus of continuity of the prelimit continuous curves, which translates into a tightness statement in the space of continuous curves. By exploiting a strong coupling of random walk and Brownian bridges we can further deduce the absolute continuity of subsequential limits with respect to Brownian bridges of appropriate variance.

### 4.1.2 Hall-Littlewood Gibbsian line ensembles

#### Line ensembles and resampling

The central objects that we study in this chapter are discrete line ensembles, which satisfy what we call the Hall-Littlewood Gibbs property. In what follows we describe the general setup informally, and refer the reader to Section 4.3.1 for the details.

A discrete line ensemble is a finite collection of up-right paths $\{L_i\}_{i=1}^k$ drawn on the integer lattice, which we assume to be weakly ordered, meaning that $L_i(x) \geq L_{i+1}(x)$ for $i = 1, ..., k-1$, and all $x$. The up-right paths $L_i$ are understood to be continuous curves on some interval $I = [a, b]$, and to be piecewise constant or have slope 1 (see Figure 4-1 for examples). Suppose we are given a probability distribution $\mu$ on the set of ensembles $\{L_i\}_{i=1}^k$. We will consider the following resampling procedure. Fix any $i \in \{1, \dots, k-1\}$ and denote by $f = L_{i-1}$ and $g = L_{i+1}$ with the convention that $L_0 = +\infty$. Sample $\{L_i\}_{i=1}^k$ according to $\mu$ and afterwards erase the line $L_i$, between its endpoints $A = L_i(a)$ and $B = L_i(b)$. Sample a new path $L_i'$, connecting the points $(a, A)$ and $(b, B)$ from the uniform distribution on all

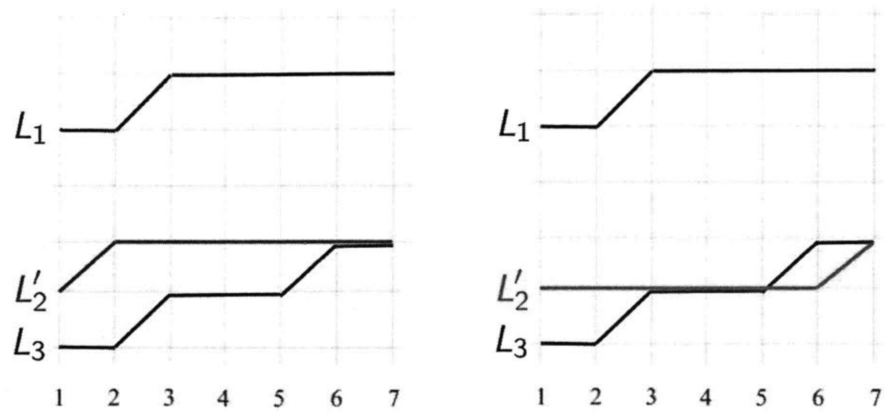


Figure 4-1: The black lines are a sample from a discrete line ensemble $\{L_i\}_{i=1}^k$ with $k = 3$ ($L_2$ is not drawn and coincides with the blue line above). Each line is a continuous curve on $I = [1, 7]$ that is piecewise constant or has slope 1. The red and blue lines are uniformly sampled up-right paths connecting the endpoints $(1, 1)$ and $(7, 2)$ of $L_2$.

up-right paths that connect these points, and independently accept the path with probability $W_t(L_i', f, g)$. If the new path is not accepted the same procedure is repeated until a path is accepted. We say that $\mu$ has the Hall-Littlewood Gibbs property with parameter $t \in (0, 1)$ if given $\{L_i\}_{i=1}^k$ distributed according to $\mu$, the random path ensemble obtained from the above resampling procedure again has distribution $\mu$. The acceptance probability is

$$W_t(L_i', f, g) = \prod_{s=a+1}^{b} \left(1 - \mathbf{1}_{\{\Delta^+(s-1) - \Delta^+(s) = 1\}} \cdot t^{\Delta^+(s-1)}\right) \cdot \left(1 - \mathbf{1}_{\{\Delta^-(s-1) - \Delta^-(s) = 1\}} \cdot t^{\Delta^-(s-1)}\right), \tag{4.1.2}$$

where $\Delta^+(s) = f(s) - L_i'(s)$ and $\Delta^-(s) = L_i'(s) - g(s)$. The above expression can be understood as follows. Follow the path $L_i'$ from left to right and any time $f - L_i'$ decreases from $\Delta^+$ to $\Delta^+ - 1$ at location $s - 1$ we multiply by a new factor $1 - t^{\Delta^+(s-1)}$. Similarly, any time $L_i' - g$ decreases from $\Delta^-$ to $\Delta^- - 1$ at location $s - 1$ we multiply by a new factor $1 - t^{\Delta^-(s-1)}$. Observe that by our assumption on $t$ we have that $W_t(L_i', f, g) \in [0, 1]$, which is why we can interpret it as a probability.

We make a couple of additional observations about the acceptance probability $W_t(L_i', f, g)$. By assumption $f(a) \geq L_i'(a) \geq g(a)$ and $f(b) \geq L_i'(b) \geq g(b)$. If for some $s$ we fail to have $f(s) \geq L_i'(s) \geq g(s)$, we see that one of the factors in $W_t(L_i', f, g)$ is zero and we will never accept such a path. Consequently, the resampling procedure always maintains the relative order of lines in the ensemble. An additional point we make is that if $L_i'$ is very well separated from $f$ and $g$ (in particular, when $f = +\infty$) we have that $\Delta^{\pm}$ is very large and so the factors in the definition of $W_t(L_i', f, g)$ are close to 1. In this sense, we can interpret $W_t(L_i', f, g)$ as a deformed indicator function of the paths $f, L_i', g$ having the correct order, the deformation being very slight if the paths are well-separated.

**Example:** We give a short example of resampling $L_2$ to explain the resampling procedure, using Figure 4-1 as a reference. We will calculate the acceptance probability if the uniform path we sampled is the red or blue one in Figure 4-1. If $L'_{red}$ denotes the red line, we have $W_t(L'_{red}, L_1, L_3) = 0$ because the lines $L_3$ and $L'_{red}$ go out of order. In particular, we see that $\Delta^-(s-1) = 0$ and $\Delta^-(s) = -1$ when $s = 6$, which means that the factor $\left(1 - \mathbf{1}_{\{\Delta^-(s-1)-\Delta^-(s)=1\}} \cdot t^{\Delta^-(s-1)}\right)$ is zero. Such a path is never accepted in the resampling procedure.

If $L'_{blue}$ denotes the blue line, we have $W_t(L'_{blue}, L_1, L_3) = (1-t)(1-t^2)(1-t^3)$. To see the latter notice that $\Delta^+$ decreases at location 1 from 3 to 2, producing the factor $(1-t^3)$. On the other hand, $\Delta^-$ decreases from 2 to 1 and from 1 to 0 at locations 2 and 5 respectively, producing factors $(1-t^2)$ and $(1-t)$. The product of all these factors equals $W_t(L'_{blue}, L_1, L_3)$ and with this probability we accept the new path.

The main result we prove for the Hall-Littlewood Gibbsian line ensembles appears as Theorem 4.3.8 in the main text. It is a general result showing how one-point tightness for the top curve of a sequence of Hall-Littlewood Gibbsian line ensembles translates into tightness for the entire top curve. This theorem can be considered the main technical contribution of this work, and we deduce tightness statements for different models like the ASEP by appealing to it. It is possible that under some stronger (than tightness) assumptions, one might be able to extend the results of that theorem to tightness of the entire ensemble (i.e. all subsequent curves too) – but since we do not need this for our applications, we do not pursue it here.

This idea of using the Gibbs property to propagate one-point tightness to tightness of the entire ensemble was developed in [42, 43]. In those works, the Gibbs property was either non-intersecting or an exponential repulsion. In other words, curves are penalized by either an infinite energetic cost or an exponential energetic cost for moving out of their indexed order. Those works rely fundamentally upon certain stochastic monotonicity enjoyed by such Gibbsian line ensemble. Namely, if you consider a given curve and either shift the starting/ending points of that curve up, or shift the above/below curves up, then the conditional measure of the given curve will stochastically shift up too. Since the Hall-Littlewood Gibbs property relies on not just the distance between curves, but on their relative slope (or derivative of the distance), this type of monotonicity is lost. Indeed, it is not just the proof of the monotonicity, but the actual result which no longer holds true in our present setting (see Remark 4.4.2).

Faced with the loss of the above form of monotonicity, we had to find a weak enough variant of it which would actually be true, while being strong enough to allow us to rework various types of arguments from [42,43]. Lemma 4.4.1 (and its corollaries) ends up fitting this need. In essence, it says that the acceptance probability of the top curve increases (though only in terms of its expected value and up to a factor of $c(t) = \prod_{i=1}^{\infty}(1-t^i)$) as the curve is raised. Informally, this result is a weaker version of the stochastic monotonicity of [42, 43] in that pointwise inequalities are replaced with ones that hold on average and upto an additional factor. Armed with this result, we are able to redevelop a route to prove tightness of the entire top line of the ensemble from its one-point tightness. Our approach should apply for more general Gibbs properties which rely upon not just the relative separation of lines, but also their relative slopes. Indeed, the constant $c(t)$ arises in our case as a relatively crude estimate needed to handle the dependence of our weights on the derivative of the distance between the top two curves. If the dependence of the weights becomes different, one should be able to reproduce the same arguments, with only the constant $c(t)$ changing its value.

### The homogeneous ascending Hall-Littlewood process

The prototypical model behind the Hall-Littlewood Gibbsian line ensemble of the previous section is the (homogeneous ascending) *Hall-Littlewood process* (HAHP). The HAHP (a special case of the ascending Macdonald processes [24]) is a probability distribution on interlacing sequences $\varnothing \prec \lambda(1) \prec \lambda(2) \prec \cdots \prec \lambda(M)$, where $\lambda(i)$ are partitions (see the beginning of Section 4.2.1 for some background on partitions, Young diagrams etc.). It depends on two positive integers $M$ and $N$ as well as two parameters $t, \zeta \in (0,1)$. We will provide a careful definition in terms of symmetric functions in Section 4.2.1 later, but here we want to give a more geometric interpretation of this measure. In what follows we will describe a measure on interlacing sequences of partitions $\varnothing \prec \lambda^{-M+1} \prec \cdots \prec \lambda^0 \succ \lambda^1 \succ \cdots \succ \lambda^{N-1} \succ \varnothing$. The HAHP is then recovered by restriction to the first $M$ partitions of this sequence. The description we give dates back to [80], and we emphasize it here as it is the origin of the Hall-Littlewood Gibbs property that we use.

We can associate an interlacing sequence of partitions with a boxed plane partition or 3d Young diagram, which is contained in the $M \times N$ rectangle – Figure 4-2 provides an illustration of this correspondence. Consequently, measures on interlacing sequences are equivalent to measures on boxed plane partitions and we focus on the latter. If a plane partition $\pi$ is given, we define its weight by

$$W(\pi) = A_\pi(t) \times \zeta^{diag(\pi)}, \tag{4.1.3}$$

where $diag(\pi)$ denotes the sum of the entries on the main diagonal of $\pi$ (alternatively this is the sum of the parts of $\lambda^0$ or the number of cubes on the diagonal $x = y$ in the 3d Young diagram). The function $A_\pi(t)$ depends on the geometry of $\pi$ and is described in Figure 4-2 (see also Section 2.1.1 where the same $A_\pi$ appears in a slightly different measure on plane partitions). With the above notation, we have that the probability of a plane partition is given by the weight $W(\pi)$, divided by the sum of the weights of all plane partitions.

Let us denote $\lambda(i) = \lambda^{i-M}$ for $i = 1, ..., M$. Then the HAHP is the probability distribution induced from the weights (4.1.3) and projected to the first $M$ terms $\varnothing \prec \lambda(1) \prec \lambda(2) \prec \cdots \prec \lambda(M)$. Denoting by $\lambda'$ the transpose of a partition $\lambda$ we observe that $\{\lambda'_j(\cdot)\}_{j=1}^N$ defines a discrete line ensemble on the interval $[0, M]$. In the above geometric setting, the lines in the discrete line ensemble $\{\lambda'_j(\cdot)\}_{j=1}^N$ can be associated to the level lines of $\pi$ (in particular, $\lambda'_1(\cdot)$ corresponds to the bottom slice of the plane partition $\pi$). The important point we emphasize is that the geometric interpretation of $A_\pi(t)$ above can be seen to be equivalent with the statement that the line ensemble $\{\lambda'_j(\cdot)\}_{j=1}^N$ satisfies the Hall-Littlewood Gibbs property of the previous section. The latter is proved in Proposition 4.3.9 in the main text.

The main result we prove for the HAHP is that as $M, N$ tend to infinity the top line $\lambda'_1(\cdot)$ (or alternatively the bottom slice of $\pi$), appropriately shifted and scaled, is tight – this is Theorem 4.3.10 in the text. In Theorem 4.2.2 we combine arguments from Chapter 2 as well as [27] to show that the analogue of Theorem 2.1.2 is true for the model we described above. This convergence implies in particular one-point tightness for the top line of the ensemble

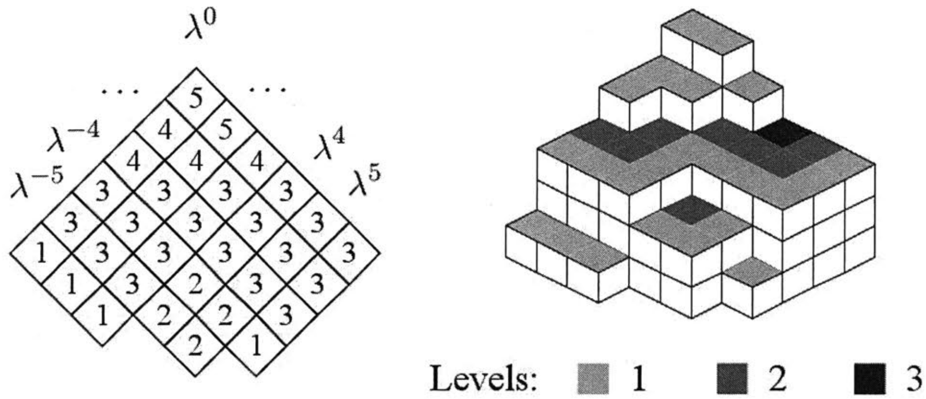


Figure 4-2: If given a sequence $\varnothing \prec \lambda^{-M+1} \prec \cdots \prec \lambda^0 \succ \cdots \succ \lambda^{N-1} \succ \varnothing$ we write the parts of $\lambda^i$ downward – in this way we obtain a plane partition. The left part of the figure shows how to do this when $\lambda^{-5} = (3)$, $\lambda^{-4} = (3,1)$, $\lambda^{-3} = (3,3,1)$ and so on. In this example $N = M = 6$. The right part of the figure shows the corresponding 3d Young diagram. The entry in a cell of the plane partition corresponds to the number of cubes in a vertical stack of the 3d diagram.
For the above diagram we have $diag(\pi) = 5 + 4 + 3 + 2 + 2 = 16$.
To find $A_\pi(t)$ we do the coloring in the right part of the figure. Each cell gets a level, which measures the distance of the cell to the boundary of the terrace on which it lies. We consider connected components (formed by cells of the same level that share a side) and for each one we have a factor $(1 - t^i)$, where $i$ is the level of the cells in the component. The product of all these factors is $A_\pi(t)$. For the example above we have 7 components of level 1, 3 of level 2 and one of level 3 – thus $A_\pi(t) = (1-t)^7(1-t^2)^3(1-t^3)$.

$\{\lambda'_j(\cdot)\}_{j=1}^N$. Once the one-point tightness and Hall-Littlewood Gibbs property are established we enter the setup Theorem 4.3.8, from which Theorem 4.3.10 is deduced.

### Connection to the ASEP and S6V model

In this section we explain how the ASEP and S6V model fit into the setup of Hall-Littlewood Gibbsian line ensembles.

For the S6V model, the key ingredient comes from the remarkable recent work in [23]. In particular, Theorem 4.1 in [23] (recalled as Theorem 4.2.4 in the main text), shows that the top curve $\lambda'_1$ of the line ensemble $\{\lambda'_j(\cdot)\}_{j=1}^N$ of the previous section has the same distribution as the height function on a horizontal slice of the S6V model, with appropriately matched parameters. This equivalence relies on the use of the $t$-Boson vertex model, as well as the infinite volume limit of the Yang-Baxter equation (as developed, for instance, in [12,20,59]). Alternatively, [34] relates this distributional equality to a Hall-Littlewood version of the RSK correspondence. Through this identification one deduces the predicted transversal $2/3$ exponent for the height function of the S6V model as a corollary of the HAHP result Theorem 4.3.10 – the exact statement is given in Corollary 4.3.11 in the text.

We now explain how to relate the ASEP to our line ensemble framework. Recall from Section 4.1.1 that $\mathfrak{h}_T(x)$ denotes the height function of the ASEP with rates $R$ and $L$, started from step initial condition at time $T$. Set $R = 1$ and $L = t \in (0,1)$. Since we use linear interpolation to define $\mathfrak{h}_T(x)$ for non-integer $x$, one observes that $-\mathfrak{h}_T(x)$ either stays constant or goes up linearly with slope 1 as $x$ increases, i.e. $-\mathfrak{h}_T(x)$ is an up-right path. In Proposition 4.3.12 we show that for any $T > 0$ and $k, K \in \mathbb{N}$ there is a random discrete line ensemble $\{L_i^{ASEP}\}_{i=1}^k$ on $I = [-K, K]$ such that (1) the law of $\{L_i^{ASEP}\}_{i=1}^k$ satisfies the Hall-Littlewood Gibbs property and (2) $L_1^{ASEP}$ has the same law as $-\mathfrak{h}_T(x)$, restricted to $x \in [-K, K]$. The realisation of $-\mathfrak{h}_T(x)$ as the top line in a Hall-Littlewood Gibbsian line ensemble is an important step in our arguments and we will provide some details how this is accomplished in a moment. For now let us explain the implications of this fact.

Once we have that $\{L_i^{ASEP}\}_{i=1}^k$ satisfies the Hall-Littlewood Gibbs property, we can use Theorem 4.3.8 to reduce the spatial tightness of the top curve $L_1^{ASEP}$ (i.e. the negative height function $-\mathfrak{h}_T(\cdot)$) to the one-point tightness of its height function. The latter is a well-known fact – it follows from the celebrated theorem of Tracy-Widom [79, Theorem 3], and is recalled as Theorem 4.2.5 in the main text. Consequently, once $-\mathfrak{h}_T(x)$ is understood as the top line of a discrete line ensemble with the Hall-Littewood Gibbs property, the general machinery of Theorem 4.3.8 takes over and produces the tightness statement of Theorem 4.1.1.

Let us briefly explain how we construct the line ensemble $\{L_i^{ASEP}\}_{i=1}^k$ from earlier – see Proposition 4.3.12 for the details. One starts from a sequence of HAHP with parameters $\zeta_N = 1 - \frac{1-t}{N}$. Under suitable shifts and truncations, these line ensembles give rise to a sequence of line ensembles $\{L_i^N\}_{i=1}^k$, which one can show to be tight. One defines $\{L_i^{ASEP}\}_{i=1}^k$ as a subsequential limit of this sequence. Since the HAHP satisfies the Hall-Littlewood Gibbs property one deduces the same for $\{L_i^{ASEP}\}_{i=1}^k$. The property that $L_1^{ASEP}$ has the same law as $-\mathfrak{h}_T(x)$ follows from the connection between the HAHP and the S6V model height function we discussed above and the convergence of the height function of the S6V model to $\mathfrak{h}_T(x)$. The fact that one can obtain the ASEP height function through a limit transition of the S6V model was suggested in [27, 53] with a complete proof given in [3].

We end this section with a brief discussion on possible extensions of our results. In Theorem 4.3.10, Corollary 4.3.11 and Theorem 4.3.13 we construct sequences of random continuous curves, which are tight in the space of continuous curves. We believe that the same sequences should converge to the Airy$_2$ process – that is how the particular scaling constants in those results were chosen. The missing ingredient necessary to establish this is the convergence of several-point marginals of these curves (currently only one-point convergence is known). It is possible that such several point-convergence will come from integrable formulas for these models but we also mention here a possible alternative approach. One could try to enhance the arguments of this chapter to show that the one-point convergence of the top line of a Hall-Littlewood Gibbsian line ensemble in fact implies tightness of the entire line ensemble (not just the top curve). This was done in a continuous setting in [42, 43]. If one achieves the latter and [42, Conjecture 3.2] were proved, this would provide a means to prove that the entire line ensemble corresponding to the ascending Hall-Littlewood process converges to the Airy line ensemble. In particular, this would demonstrate the Airy$_2$ process limit for the ASEP and S6V height functions too.

### 4.1.3 Outline

The introductory section above provided background context for our work and a general overview of the chapter. In Section 4.2 we define the HAHP, S6V model and the ASEP and supply some known one-point convergence results for the latter. Section 4.3 introduces the necessary definitions in the chapter, states the main technical result – Theorem 4.3.8, as well as the main results we prove about the HAHP, the S6V model and the ASEP in Theorem 4.3.10, Corollary 4.3.11 and Theorem 4.3.13 respectively. Section 4.4 summarizes the primary set of results we need to prove Theorem 4.3.8. In Section 4.5 we give the proof of Theorem 4.3.8 by reducing it to three key lemmas, whose proofs are presented in Section 4.6. In Section 4.7 we demonstrate that all subsequential limits of the tight sequence of Theorem 4.3.8 are absolutely continuous with respect to Brownian bridges of appropriate variance. Section 4.8 is an appendix, which contains the proof of a strong coupling between random walks and Brownian bridges, used in Section 4.4.

## 4.2 Three stochastic models

The results of our chapter have applications to three different but related probabilistic objects – the ascending Hall-Littlewood process, the stochastic six-vertex model in a quadrant and the ASEP. In this section we recall the definitions of these models, some known one-point convergence results about them and explain how they are connected.

### 4.2.1 The ascending Hall-Littlewood process

In this section we briefly recall the definition of the Hall-Littlewood process (a special case of the Macdonald process [24]). We will isolate a particular case that will be important for us, which we call the *homogeneous ascending Hall-Littlewood process* (HAHP) and derive a certain one-point convergence result for it. We adopt the same notation on partitions and Hall-Littlewood symmetric functions as in Section 2.2.

Fix $t \in (0,1)$. For partitions $\mu, \lambda \in \mathbb{Y}$ we let $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$ denote the (skew) Hall-Littlewood symmetric functions with parameter $t$. Let us fix $M, N \in \mathbb{N}$ and suppose $(\mathcal{L}, \mathcal{M})$ is a pair of sequences of partitions $\mathcal{L} = \{\lambda^k\}_{k=-M+1}^{N-1}$ and $\mathcal{M} = \{\mu^k\}_{k=-M+1}^{N-2}$. Define the weight of such a pair as

$$\mathcal{W}(\mathcal{L}, \mathcal{M}) := \prod_{n=-M+1}^{N-1} P_{\lambda^n/\mu^{n-1}}(x_n) Q_{\lambda^n/\mu^n}(y_n), \tag{4.2.1}$$

where $x_i, y_i \in [0,1]$ for all $i \in \{-M+1, ..., N-1\}$ and we have $\mu^{-M} = \mu^{N-1} = \varnothing$. From (5.8) and (5.8′) in Chapter III of [64] we have

$$\begin{gathered} P_{\lambda/\mu}(x;t) = \psi_{\lambda/\mu}(t) x^{|\lambda|-|\mu|} \text{ and } Q_{\lambda/\mu}(x;t) = \phi_{\lambda/\mu}(t) x^{|\lambda|-|\mu|}, \text{ where} \\ \psi_{\lambda/\mu}(t) = \mathbf{1}_{\lambda \succ \mu} \prod_{j \in J} (1 - t^{m_j(\mu)}) \quad \text{and } \phi_{\lambda/\mu}(t) = \mathbf{1}_{\lambda \succ \mu} \prod_{i \in I} (1 - t^{m_i(\lambda)}); \\ I = \{i \in \mathbb{N} : \lambda'_{i+1} = \mu'_{i+1} \text{ and } \lambda'_i > \mu'_i\} \text{ and } J = \{j \in \mathbb{N} : \lambda'_{j+1} > \mu'_{j+1} \text{ and } \lambda'_j = \mu'_j\}. \end{gathered} \tag{4.2.2}$$

Observe that the weights are non-negative (as $t \in (0,1)$) and provided $Z := \sum_{\mathcal{L},\mathcal{M}} \mathcal{W}(\mathcal{L},\mathcal{M})$, is finite we have that $\mathbb{P}(\mathcal{L},\mathcal{M}) := Z^{-1} \cdot \mathcal{W}(\mathcal{L},\mathcal{M})$ defines probability measure on $(\mathcal{L},\mathcal{M})$, which we call a *Hall-Littlewood process.*

In this chapter we will consider the following variable specialization

$$x_{n+1} = 1,\ y_n = 0 \text{ if } n \leq -1;\ x_{n+1} = 0,\ y_n = \zeta \text{ if } 0 \leq n, \text{ where } \zeta \in (0,1) \text{ is fixed.} \quad (4.2.3)$$

Using (4.2.2) and Proposition 2.4 in [24] we conclude that for the above variables we have

1. $Z = \left(\frac{1-t\zeta}{1-\zeta}\right)^{NM} < \infty$ so that the measure is well defined;
2. $\mu^n = \lambda^n$ for $n < 0$ and $\mu^n = \lambda^{n+1}$ for $n \geq 0$;
3. $\varnothing \prec \lambda^{-M+1} \cdots \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \cdots \succ \lambda^{N-1} \succ \varnothing$.

The last statement shows that $\mathcal{L}$ defines a plane partition $\pi$, whose base is contained in an $M \times N$ rectangle (i.e. such that $\pi_{i,j} = 0$ for $i \geq M$ or $j \geq N$). Denoting the set of such plane partitions by $\mathcal{P}(M,N)$ we see that the projection of the Hall-Littlewood process on $\mathcal{L}$ induces a measure on $\mathcal{P}(M,N)$.

Substituting $P_{\lambda/\mu}(x)$ and $Q_{\lambda/\mu}(x)$ from (4.2.2) one arrives at

$$\mathbb{P}(\mathcal{L}) = \left(\frac{1-\zeta}{1-t\zeta}\right)^{NM} \cdot \zeta^{|\lambda^0|} \cdot B_{\mathcal{L}}(t), \text{ where } B_{\mathcal{L}}(t) = \prod_{n=-M+1}^{0} \psi_{\lambda^n/\lambda^{n-1}}(t) \times \prod_{n=1}^{N} \phi_{\lambda^{n-1}/\lambda^n}(t).$$

What is remarkable is that if $\pi$ is the plane partition associated to $\mathcal{L}$, then $B_{\mathcal{L}}(t) = A_\pi(t)$ from (4.1.3), i.e. $B_{\mathcal{L}}$ admits the geometric interpretation from Figure 4-2. The latter is very far from obvious from the definition of $B_{\mathcal{L}}$, since the functions $\phi$ and $\psi$ are somewhat involved, and we refer the reader to [81] where this identification was first discovered.

The above formulation aimed to reconcile the definition of the Hall-Littlewood process in terms of symmetric functions with the geometric formulation given in Section 4.1.2. In the remainder of the chapter; however, we will be mostly interested in the projection of this measure to the partitions $\lambda^{-M+1}, ..., \lambda^0$. We perform a shift of the indices by $M$ and denote the latter by $\lambda(1), ..., \lambda(M)$. Using results from Section 2.2 in [24] we have the following (equivalent) definition of the measure on these sequences, which we isolate for future reference.

**Definition 4.2.1.** Let $M, N \in \mathbb{N}$ and $\zeta \in (0,1)$. The *homogeneous ascending Hall-Littlewood process* (HAHP) is a probability distribution on sequences of partitions $\varnothing \prec \lambda(1) \prec \lambda(2) \prec \cdots \prec \lambda(M)$ such that

$$\mathbb{P}_\zeta^{M,N}(\lambda(1), ..., \lambda(M)) = \left(\frac{1-\zeta}{1-t\zeta}\right)^{NM} \times \prod_{i=1}^{M} P_{\lambda(i)/\lambda(i-1)}(1) \times Q_{\lambda(M)}(\zeta^N), \quad (4.2.4)$$

where we use the convention that $\lambda(0) = \varnothing$ is the empty partition and $\zeta^N$ denotes the specialization of $N$ variables to $\zeta$. We also write $\mathbb{E}_\zeta^{M,N}$ for the expectation with respect to

$\mathbb{P}_\zeta^{M,N}$.

We end this section with an important asymptotic statement for the measures $\mathbb{P}_\zeta^{M,N}$.

**Theorem 4.2.2.** *Let $r > 0$, $\zeta, t \in (0,1)$ be given and fix $\mu \in (\zeta, \zeta^{-1})$. Suppose $N, M \in \mathbb{N}$ are sufficiently large so that $\mu N > (r+2)N^{2/3}$ and $M > \mu N + (r+2)N^{2/3}$. Let $\lambda_1'(\cdot)$ be sampled from $\mathbb{P}_\zeta^{M,N}$ and set for $x \in [-r-1, r+1]$*

$$f_N^{HL}(x) := \sigma_\mu^{-1} N^{-1/3}\left(\lambda_1'(\mu N + xN^{2/3}) - f_1(\mu)N - f_1'(\mu)xN^{2/3} - \frac{x^2}{2}f_1''(\mu)N^{1/3}\right), \quad (4.2.5)$$

*where we define $\lambda_1'$ at non-integer points by linear interpolation. The constants above are given by $\sigma_\mu = \frac{(\zeta\mu)^{1/6}(1-\sqrt{\zeta\mu})^{2/3}(1-\sqrt{\zeta/\mu})^{2/3}}{1-\zeta}$, $f_1(\mu) = 1 - \frac{(1-\sqrt{\zeta\mu})^2}{1-\zeta}$, $f_1'(\mu) = \frac{\sqrt{\zeta}(1-\sqrt{\zeta\mu})}{\sqrt{\mu}(1-\zeta)}$, $f_1''(\mu) = \frac{-\sqrt{\zeta}}{2\mu^{3/2}(1-\zeta)}$. Then for any $x \in [-r-1, r+1]$ and $y \in \mathbb{R}$ we have*

$$\lim_{N\to\infty} \mathbb{P}_\zeta^{M,N}\left(f_N^{HL}(x) \le y\right) = F_{GUE}(y), \quad (4.2.6)$$

*where $F_{GUE}$ is the GUE Tracy-Widom distribution [78].*

*Remark* 4.2.3. Owing to the recent work in [19], the result of Theorem 4.2.2 can be established by reduction to the Schur process (corresponding to $t = 0$). For the Schur process a proof of the convergence in (4.2.6) for the case $s = 0$ can be found in the proof of Theorem 6.1 in [19]. For the sake of completeness we will present a different (more direct) proof below, relying on ideas from [27] and Chapter 2.

*Proof.* Fix $x \in [-r-1, r+1]$ and $y \in \mathbb{R}$ throughout. For clarity we split the proof into several steps.

**Step 1.** From Section 2.2 in [24] we know that for $1 \le K \le M$ we have

$$\mathbb{P}_\zeta^{M,N}(\lambda(K) = \nu) = \mathbb{P}_\zeta^{K,N}(\lambda(K) = \nu) = \left(\frac{1-\zeta}{1-t\zeta}\right)^{KN} P_\nu(1^K) \cdot Q_\nu(\zeta^N),$$

In the last equality we used the homogeneity of $P_\nu$ and $Q_\nu$. Setting $\lambda_1' = \lambda_1'(K)$, we have as a consequence of Proposition 2.3.10 that if $\phi \in \mathbb{C}\backslash\mathbb{R}^+$ then

$$\mathbb{E}_\zeta^{M,N}\left[\frac{1}{(\phi t^{1-\lambda_1'}; t)_\infty}\right] = \det(I + K_\phi^{K,N})_{L^2(C_\rho)}. \quad (4.2.7)$$

The contour $C_\rho$ is the positively oriented circle of radius $\rho \in (\zeta t^{-1}, t^{-1})$, centered at 0, and the operator $K_\phi^{K,N}$ is defined in terms of its integral kernel

$$K_\phi^{K,N}(w, w') = \frac{1}{2\pi\iota}\int_{1/2-\iota\infty}^{1/2+\iota\infty} ds\Gamma(-s)\Gamma(1+s)(-\phi)^s g_{w,w'}^{K,N}(t^s),$$

where $\Gamma$ is the Euler gamma function and

$$g_{w,w'}^{K,N}(t^s) = \frac{1}{wt^{-s} - w'} \left( \frac{1 - \zeta(wt)^{-1}}{1 - \zeta(wt)^{-1} t^s} \right)^K \left( \frac{1 - (wt)t^{-s}}{1 - (wt)} \right)^N.$$

We also recall that $(x;t)_\infty = \prod_{i=1}^{\infty}(1 - xt^{i-1})$ is the $t$-Pochhammer symbol and

$$\det(I + K_\phi^{K,N})_{L^2(C_\rho)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{C_\rho} \cdots \int_{C_\rho} \det \left[ K_\phi^{K,N}(w_i, w_j) \right]_{i,j=1}^n \prod_{i=1}^n dw_i$$

is the Fredholm determinant of the kernel $K_\phi^{K,N}$ (see Section 2.2 for details).

**Step 2.** For the remainder of the proof we set

$$K = \mu N + xN^{2/3} + O(1) \text{ and } \phi(N) = (-t^{-1}) \times t^{f_1(\mu)N + f_1'(\mu)xN^{2/3} + (1/2)f_1''(\mu)x^2N^{1/3} + y\sigma_\mu},$$

where $\sigma_\mu$ and $f(\mu)$ are as in the statement of the theorem. Our goal in this step is to show that

$$\lim_{N \to \infty} \det(I + K_\phi^{K,N})_{L^2(C_\rho)} = F_{GUE}(y). \tag{4.2.8}$$

We use the following change of variables and functional identities

$$w_i \to \frac{-1}{\tilde{w}_i}, \quad \rho \to \tilde{\rho}^{-1}, \quad \Gamma(-s)\Gamma(1+s) = \frac{\pi}{\sin(\pi s)}$$

to rewrite

$$\det(I + K_\phi^{K,N})_{L^2(C_\rho)} = \det(I + \tilde{K}_\phi^{K,N})_{L^2(C_{\tilde{\rho}})}.$$

In the above we have that

$$\tilde{K}_\phi^{K,N}(w, w') = \frac{1}{2\iota} \int_{1/2 - \iota\infty}^{1/2 + \iota\infty} \frac{t^{-\tilde{m}_\nu K + y\tilde{\sigma}_\nu K^{1/3}}}{\sin(\pi s)} \frac{g(\tilde{w}; \zeta, \nu K, K)}{g(t^s\tilde{w}; \zeta, \nu K, K)} \cdot \frac{ds}{\tilde{w}t^s - \tilde{w}'}, \text{ where}$$

$$g(\tilde{w}; b_1, b_2, x, t) = \left(1 + zt^{-1}\zeta\right)^x \left( \frac{1}{1 + t^{-1}\tilde{z}} \right)^t, \; \tilde{\sigma}_\nu = \frac{\zeta^{1/2}\nu^{-1/6}}{1 - \zeta} \left( (1 - \sqrt{\nu\zeta})(\sqrt{\nu/\zeta} - 1) \right)^{2/3}$$

$$\tilde{m}_\nu = \frac{(\sqrt{\nu} - \sqrt{\zeta})^2}{1 - \zeta}, \text{ and } \nu = \mu^{-1} - x\mu^{-5/3}K^{-1/3} + \frac{2x^2}{3}\mu^{-7/3}K^{-2/3} + O(1). \tag{4.2.9}$$

The validity of (4.2.8) is now equivalent to Proposition 5.3 in [27]. To make the connection clearer we reconcile the notation from equation (65) in that paper with our own below:

$$y \leftrightarrow h, \quad t \leftrightarrow \tau, \quad \tilde{\rho} \leftrightarrow r, \quad K \leftrightarrow L, \quad b_1 \leftrightarrow \frac{1 - \zeta}{1 - t\zeta}, \quad b_2 \leftrightarrow t\frac{1 - \zeta}{1 - t\zeta}, \quad \zeta \leftrightarrow \kappa.$$

We remark that in [27] the variable $\nu$ is constant, while in our case it changes with $K$ and quickly converges to $\mu^{-1}$ – this does not affect the validity of Proposition 5.3 and the same arguments can be repeated verbatim.

**Step 3.** Combining (4.2.7) and (4.2.8) we see that

$$\lim_{N\to\infty} \mathbb{E}_{\zeta}^{M,N}\left[g_N(X_N - y)\right] = F_{GUE}(y), \text{ where } g_N(z) = \frac{1}{(-t^{-N^{1/3}z};t)_\infty} \text{ and}$$
$$X_N = \sigma_\mu^{-1}N^{-1/3}(\lambda_1'(\mu N + xN^{2/3}) - f_1(\mu)N - f_1'(\mu)xN^{2/3} - (1/2)f_1''(\mu)x^2N^{1/3}). \tag{4.2.10}$$

As discussed in the proof of Theorem 5.1 in [27], we have that $g_N(z)$ satisfy the conditions of Lemma 5.2 of the same paper, which implies that $X_N$ weakly converges to a random variable $X$ such that $\mathbb{P}(X \leq y) = F_{GUE}(y)$. This suffices for the proof.

□

### 4.2.2 The stochastic six-vertex model in a quadrant

In this section we recall the definition of a stochastic inhomogeneous six-vertex model in a quadrant, considered in [27, 33, 73]. There are several (equivalent) ways to define the model and we will, for the most part, adhere to the one presented in Section 1.1.2 in [2]. We also refer the reader to Section 1 of [33] for the definition of a more general higher spin version of this model.

A *six-vertex directed path ensemble* is a family of up-right directed paths drawn in the first quadrant $\mathbb{Z}^2_{\geq 1}$ of the square lattice, such that all the paths start from a left-to-right arrow entering each of the points $\{(1,m) : m \geq 1\}$ on the left boundary (no path enters from the bottom boundary) and no two paths share any horizontal or vertical edge (but common vertices are allowed); see Figure 4-3. In particular, each vertex has six possible *arrow configurations*, presented in Figure 4-4.

The stochastic inhomogeneous six-vertex model is a probability distribution $\mathcal{P}$ on six-vertex directed path ensembles, which depends on a set of parameters $\{\xi_x\}_{x\geq 1}$, $\{u_y\}_{y\geq 1}$ and $q$, which satisfy

$$q \in (0,1), \quad \xi_x > 0, u_y > 0, \quad \xi_x u_y > q^{-1/2} \text{ for all } x, y \geq 1. \tag{4.2.11}$$

It is defined as the infinite-volume limit of a sequence of probability measures $\mathcal{P}_n$, which are constructed as follows.

For $n \geq 1$ we consider the triangular regions $T_n = \{(x,y) \in \mathbb{Z}^2_{\geq 1} : x + y \leq n\}$ and let $P_n$ denote the set of six-vertex directed path ensembles whose vertices are all contained in $T_n$. By convention, the set $P_1$ consists of a single empty ensemble. We construct a consistent family of probability distributions $\mathcal{P}_n$ on $P_n$ (in the sense that the restriction of a random element sampled from $\mathcal{P}_{n+1}$ to $T_n$ has law $\mathcal{P}_n$) by induction on $n$, starting from $\mathcal{P}_1$, which is just the delta mass at the single element in $P_1$.

For any integer $n \geq 1$ we define $\mathcal{P}_{n+1}$ from $\mathcal{P}_n$ in the following Markovian way. Start by sampling a directed path ensemble $\mathcal{E}_n$ on $T_n$ according to $\mathcal{P}_n$. This gives arrow configurations (of the type presented in Figure 4-4) to all vertices in $T_{n-1}$. In addition, each vertex in $D_n = \{(x,y) \in \mathbb{Z}^2_{\geq 1} : x + y = n\}$ is given "half" of an arrow configuration, meaning that the arrows entering the vertex from the bottom or left are specified, but not those leaving from the top or right; see the right part of Figure 4-3.

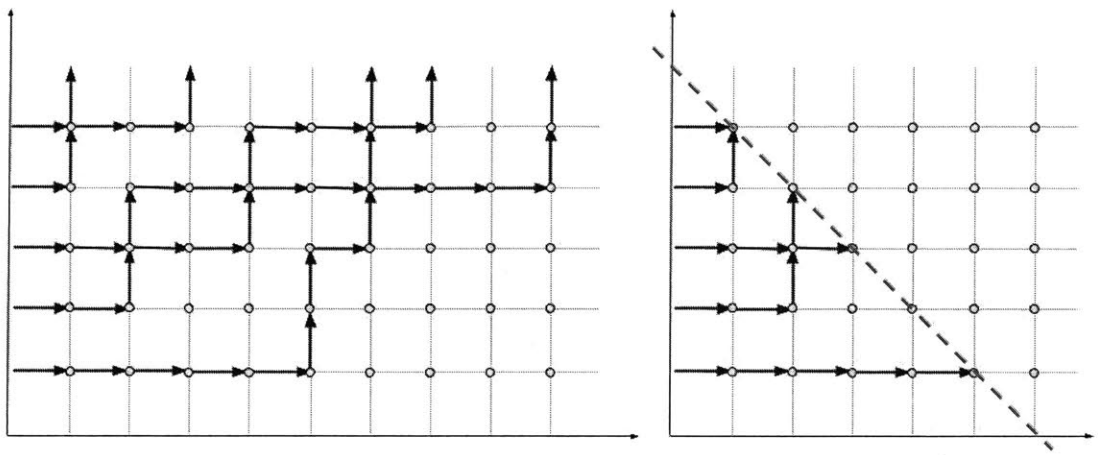

Figure 4-3: The left picture shows an example of a six-vertex directed path ensemble. The right picture shows an element in $P_n$ for $n = 6$. The vertices on the dashed line belong to $D_n$ and are given half of an arrow configuration if a directed path ensemble from $P_n$ is drawn. Vertices in $D_n$ with zero (two) incoming arrows from the left and bottom can be completed in a unique way - by having zero (two) outgoing arrows. Compare vertices $(4, 2)$ in both pictures, also vertices $(1, 5)$. Vertices in $D_n$ with a single incoming arrow can be completed by having exactly one outgoing arrow, which can go either to the right or up. Compare vertices $(5, 1)$ in both pictures, also vertices $(2, 4)$.

To extend $\mathcal{E}_n$ to a path ensemble on $T_{n+1}$, we must "complete" the configurations, i.e. specify the top and right arrows, for the vertices on $D_n$. Any half-configuration at a vertex $(x, y)$ can be completed in at most two ways; selecting between these completions is done independently for each vertex in $D_n$ at random according to the probabilities given in the second row of Figure 4-4, where the probabilities $b_1(x, y)$ and $b_2(x, y)$ are defined as

$$b_1(x,y) = \frac{1 - q^{1/2}\xi_x u_y}{1 - q^{-1/2}\xi_x u_y} \quad b_2(x,y) = \frac{q^{-1} - q^{-1/2}\xi_x u_y}{1 - q^{-1/2}\xi_x u_y}. \tag{4.2.12}$$

In this way we obtain a random ensemble $\mathcal{E}_{n+1}$ in $P_{n+1}$ and we denote its law by $\mathcal{P}_{n+1}$. One readily verifies that the distributions $\mathcal{P}_n$ are consistent and then we define $\mathcal{P} = \lim_{n\to\infty} \mathcal{P}_n$.

A particular case that will be of interest to us is setting $\xi_x = \xi$ and $u_y = u$ for all $x \geq 1$ and $y \geq 1$, where $\xi, u > 0$ are such that $\xi u > q^{-1/2}$. We refer to this model as the *homogeneous stochastic six-vertex model* and denote the corresponding measure as $\mathbb{P}_{\xi,u,q}$. Let us remark that (upto a reflection with respect to the diagonal $x = y$) this model was investigated in [53] and more recently in [27] under the name "stochastic six-vertex model".

Given a six-vertex directed path ensemble on $\mathbb{Z}^2_{\geq 1}$, we define the *height function* $h(x, y)$ as the number of up-right paths, which intersect the horizontal line through $y$ at or to the right of $x$. We end this section by recalling the following important connection between the height function of the homogeneous stochastic six-vertex model and the homogeneous ascending Hall-Littewood process. The following result is a special case of Theorem 4.1

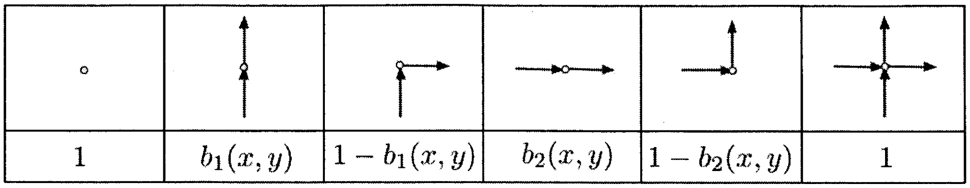


Figure 4-4: The top row shows the six possible arrow configurations at a vertex $(x, y)$. The bottom row shows the probabilities of top-right completion, given the bottom-left half of a configuration. The probabilities $b_1(x, y)$ and $b_2(x, y)$ depend on $\xi_x, u_y$ and $q$ and are given in (4.2.12).

in [23] and plays a central role in our arguments.

**Theorem 4.2.4** (Theorem 4.1 in [23]). *Let $\xi, u, q > 0$ be given such that $q \in (0,1)$, $\zeta = \xi^{-1}u^{-1}q^{-1/2} < 1$ and fix $\mu \in (\zeta, \zeta^{-1})$. Let $h(x,y)$ denote height function sampled from $\mathbb{P}_{\xi,u,q}$ and $\varnothing \prec \lambda(1) \prec \cdots \prec \lambda(M)$ be distributed as $\mathbb{P}^{M,N}_{\zeta}$ from Definition 4.2.1, where $t = q$. Then we have the following equality in distribution of random vectors*

$$(N - \lambda_1'(0), ..., N - \lambda_1'(M)) \stackrel{d}{=} (h(1,N), \cdots, h(M+1,N)), \text{ where by convention } \lambda_1'(0) = 0.$$

### 4.2.3 The asymmetric simple exclusion process

The *asymmetric simple exclusion process* (ASEP) is a continuous time Markov process, which was introduced in the mathematical community by Spitzer in [76]. In this chapter we consider ASEP started from the so-called step initial condition, which can be described as follows. Particles are initially (at time 0) placed on $\mathbb{Z}$ so that there is a particle at each location in $\mathbb{Z}_{\leq 0}$ and all positions in $\mathbb{Z}_{\geq 1}$ are vacant. There are two exponential clocks, one with rate $L$ and one with rate $R$, associated to each particle; we assume that $R > L \geq 0$ and that all clocks are independent. When some particle's left clock rings, it attempts to jump to the left by one; similarly when its right clock rings, it attempts to jump to the right by one. If the adjacent site in the direction of the jump is unoccupied, the jump is performed; otherwise it is not. For a more careful description of the model, as well as a proper definition of this dynamics with infinitely many particles, we refer the reader to [63].

Given a particle configuration on $\mathbb{Z}$, we define the *height function* $\mathfrak{h}(x)$ as the number of particles at or to the right of the position $x$, when $x \in \mathbb{Z}$. For non-integral $x$, we define $\mathfrak{h}(x)$ by linear interpolation of $\mathfrak{h}(\lfloor x \rfloor)$ and $\mathfrak{h}(\lceil x \rceil)$. For $R > L \geq 0$ and $T \geq 0$ we denote by $\mathbb{P}^T_{L,R}$ the law of the height function $\mathfrak{h}$ of the random particle configuration sampled from the ASEP (started from the step initial condition) with parameters $R$ and $L$ after time $T$.

We isolate the following one-point convergence result for future use.

**Theorem 4.2.5.** *Suppose $r > 0$, $R = 1$, $L \in (0,1)$, $\gamma = R - L$ and fix $\alpha \in (0,1)$. Let $\mathfrak{h}(x)$ denote height function sampled from $\mathbb{P}^{N/\gamma}_{L,R}$ and for $s \in [-r-1, r+1]$ set*

$$f_N^{ASEP}(s) = \sigma_\alpha^{-1} N^{-1/3} \left( f_3(\alpha)N + f_3'(\alpha)sN^{2/3} + (1/2)s^2 f_3''(\alpha)N^{1/3} - \mathfrak{h}\left(\alpha N + sN^{2/3}\right)\right), \tag{4.2.13}$$

*where we define $\mathfrak{h}(\cdot)$ at non-integer points by linear interpolation. The constants above are* *given by* $\sigma_\alpha = 2^{-4/3}(1-\alpha^2)^{2/3}$, $f_3(\alpha) = \frac{(1-\alpha)^2}{4}$, $f_3'(\alpha) = -\frac{1-\alpha}{2}$, $f_3''(\alpha) = \frac{1}{2}$. *Then for any* $s \in [-r-1, r+1]$ *and* $y \in \mathbb{R}$ *we have*

$$\lim_{N\to\infty} \mathbb{P}_{L,R}^{N/\gamma}\left(f_N^{ASEP}(s) \le y\right) = F_{GUE}(y), \tag{4.2.14}$$

*where* $F_{GUE}$ *is the GUE Tracy-Widom distribution [78].*

*Proof.* The above result follows immediately from the celebrated theorem of Tracy-Widom [79, Theorem 3], which says that

$$\lim_{T\to\infty} \mathbb{P}_{L,R}^{T/\gamma}\left(\frac{c_1 T - x_m}{c_2 T^{1/3}} \le y\right) = F_{GUE}(y), \tag{4.2.15}$$

where $\sigma = \frac{m}{T} \in (0,1)$, $c_1 = 1 - 2\sqrt{\sigma}$, $c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}$. In the above relation $x_m$ denotes the position of the $m$-th right-most ASEP particle (notice there is a sign change with the result in [79], due to the fact that in that paper $L > R$ and the particles initially occupy the positive integers). Below we briefly explain why the above statement implies the theorem.

One observes that at each fixed time and for any positive integers $m, n$ we have the equality of events $\{\mathfrak{h}(n) \ge m\} = \{x_m \ge n\}$, which implies that for any $\tau \ge 0$ we have

$$\mathbb{P}_{L,R}^{\tau}(\mathfrak{h}(n) \ge m) = \mathbb{P}_{L,R}^{\tau}(x_m \ge n) \tag{4.2.16}$$

Let $m(N) = \lfloor f_3(\alpha)N + f_3'(\alpha)sN^{2/3} + (1/2)s^2 f_3''(\alpha)N^{1/3} - yN^{1/3}\sigma_\alpha \rfloor$, $n(N) = \lfloor \alpha N + sN^{2/3} \rfloor$ and observe that

$$\mathbb{P}_{L,R}^{N/\gamma}\left(\mathfrak{h}(n(N)) \ge m(N)+1\right) \le \mathbb{P}_{L,R}^{N/\gamma}\left(f_N^{ASEP}(s) \le y\right) \le \mathbb{P}_{L,R}^{N/\gamma}\left(\mathfrak{h}(n(N)+1) \ge m(N)\right). \tag{4.2.17}$$

From (4.2.16) we have

$$\mathbb{P}_{L,R}^{N/\gamma}\left(\mathfrak{h}(n(N)+1) \ge m(N)\right) = \mathbb{P}_{L,R}^{N/\gamma}\left(x_{m(N)} \ge c_1 N - c_2 N^{1/3} + O(1)\right), \tag{4.2.18}$$

where we also used that $\sigma = \frac{m}{N} = f_3(\alpha) + f_3'(\alpha)sN^{-1/3} + (1/2)s^2N^{-2/3}f_3''(\alpha) - y\sigma_\alpha N^{-2/3} + O(N^{-1})$. One similarly obtains

$$\mathbb{P}_{L,R}^{N/\gamma}\left(\mathfrak{h}(n(N)) \ge m(N)+1\right) = \mathbb{P}_{L,R}^{N/\gamma}\left(x_{m(N)+1} \ge c_1 N - c_2 N^{1/3} + O(1)\right). \tag{4.2.19}$$

The right sides in (4.2.18) and (4.2.19) converge to $F_{GUE}(y)$ from (4.2.15) as $N \to \infty$, which together with (4.2.17) proves the theorem. □

We end this section by recalling the following important connection between the height function of the homogeneous stochastic six-vertex model and the height function of the ASEP started from step initial condition. This connection was observed in [27, 53] and carefully proved for general initial conditions in [3].

**Theorem 4.2.6** (Theorem 1 in [3]). *Let* $\xi(N), u(N), q > 0$ *be given such that* $q \in (0,1)$, $\zeta(N) = \xi(N)^{-1}u(N)^{-1}q^{-1/2} < 1$, $b_1(N) = \frac{1-q^{1/2}\xi(N)u(N)}{1-q^{-1/2}\xi(N)u(N)} = qN^{-1} + O(N^{-2})$ *and* $b_2(N) =$

$\frac{q^{-1}-q^{-1/2}\xi(N)u(N)}{1-q^{-1/2}\xi(N)u(N)} = N^{-1}+O(N^{-2})$. *In addition, fix* $K \in \mathbb{N}$, $T > 0$ *and set* $N_T = \lfloor N \cdot T \rfloor$. *Let* $h^N(x, y)$ *denote height function sampled from* $\mathbb{P}_{\xi(N),u(N),q}$ *and* $\mathfrak{h}$ *have law* $\mathbb{P}^T_{L,R}$, *where* $R = 1$ *and* $L = q$. *Then we have the following convergence in distribution of random vectors*

$$(h^N(N_T - K + 1, N_T), ..., h^N(N_T + K + 1, N_T)) \implies (\mathfrak{h}(-K + 1), ..., \mathfrak{h}(K + 1)) \text{ as } N \to \infty.$$

## 4.3 Definitions, notations and main results

In this section we introduce the necessary definitions and notations that will be used in the chapter as well as our main technical result – Theorem 4.3.8 below. Afterwards we give several applications of Theorem 4.3.8 to the three models discussed in the previous section.

### 4.3.1 Discrete line ensembles and the Hall-Littlewood Gibbs property

In this section we introduce the concept of a discrete line ensemble and the Hall-Littlewood Gibbs property. Subsequently, we state the main result of the chapter.

**Definition 4.3.1.** Let $N \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$ and denote $\Sigma = \{1, \dots, N\}$, $[\![T_0, T_1]\!] = \{T_0, T_0 + 1, \dots, T_1\}$. Consider the set $Y$ of functions $f : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(j, i + 1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$ and let $\mathcal{D}$ denote the discrete topology on $Y$. We call elements in $Y$ *up-right paths.*

A $\Sigma \times [\![T_0, T_1]\!]$*-indexed (up-right) discrete line ensemble* $\mathfrak{L}$ is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in $Y$ such that $\mathfrak{L}$ is a $(\mathcal{B}, \mathcal{D})$-measurable function.

*Remark* 4.3.2. Notice that the definition of an up-right path we use here differs from the one in the six-vertex model. Namely, for the six-vertex model an up-right path is one that moves either to the right or up, while in discrete line ensembles up-right paths move to the right or with slope 1. This should cause no confusion as it will be clear from context, which paths we mean.

The way we think of discrete line ensembles is as random collections of up-right paths on the integer lattice, indexed by $\Sigma$ (see Figure 4-5). Observe that one can view a path $L$ on $[\![T_0, T_1]\!] \times \mathbb{Z}$ as a continuous curve by linearly interpolating the points $(i, L(i))$. This allows us to define $(\mathfrak{L}(\omega))(i, s)$ for non-integer $s \in [T_0, T_1]$ and to view discrete line ensembles as line ensembles in the sense of [42]. In particular, we can think of $L(s), s \in [T_0, T_1]$ as a random variable in $(C[T_0, T_1], \mathcal{C})$ – the space of continuous functions on $[T_0, T_1]$ with the uniform topology and Borel $\sigma$-algebra $\mathcal{C}$ (see e.g. Chapter 7 in [13]).

We will often slightly abuse notation and write $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$, even though it is not $\mathfrak{L}$ which is such a function, but rather $\mathfrak{L}(\omega)$ for each $\omega \in \Omega$. Furthermore we write $L_i = (\mathfrak{L}(\omega))(i, \cdot)$ for the index $i \in \Sigma$ path.

In what follows we fix a parameter $t \in (0, 1)$ and make several definitions. Suppose we are given three up-right paths $f, g, L$ on $[\![T_0, T_1]\!] \times \mathbb{Z}$. Given a (finite) subset $S \subset [\![T_0 + 1, T_1]\!]$

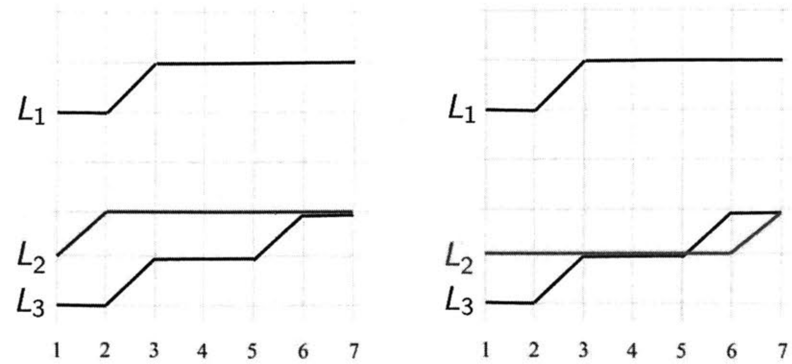


Figure 4-5: Two samples of $\{1,2,3\} \times [\![1,7]\!]$-indexed discrete line ensembles.

we define the following weight function

$$W_t(T_0, T_1, L, f, g; S) = \prod_{i \in S} \left(1 - \mathbf{1}_{\{\Delta^+(i-1) - \Delta^+(i) = 1\}} \cdot t^{\Delta^+(i-1)}\right) \times \\ \prod_{i \in S} \left(1 - \mathbf{1}_{\{\Delta^-(i-1) - \Delta^-(i) = 1\}} \cdot t^{\Delta^-(i-1)}\right), \tag{4.3.1}$$

if $f(i) \geq L(i) \geq g(i)$ for $i \in S$ and 0 otherwise. In the above $\Delta^+(s) = f(s) - L(s)$ and $\Delta^-(s) = L(s) - g(s)$. In words (4.3.1) means that we follow the paths $f, g, L$ from left to right and any time $f - L$ (resp. $L - g$) decreases from $\Delta^+$ to $\Delta^+ - 1$ (resp. $\Delta^-$ to $\Delta^- - 1$) at a location in the set $S$ we multiply by a factor of $1 - t^{\Delta^+}$ (resp. $1 - t^{\Delta^-}$). Observe that by our assumption on $t$ we have that $W_t \in (0, 1]$ unless $L(i) > f(i)$ or $g(i) > L(i)$ for some $i \in S$, in which case the weight is 0. Typically $S$ will be a finite union of disjoint intervals (i.e. consecutive integer points).

*Remark* 4.3.3. Observe that (4.3.1) makes sense even if $f = \infty$. In the latter case as $t \in (0, 1)$ the product on the first line of (4.3.1) becomes 1 – in fact, this will be the most common way $W_t(T_0, T_1, L, f, g; S)$ will appear in the text.

**Example.** Take the left sample in Figure 4-5. If $S = \{2, \dots, 7\}$ then we have $W_t(1, 7, L_2, L_1, L_3; S) = (1 - t)(1 - t^2)(1 - t^3)$ and $W_t(1, 7, L_1, \infty, L_2; S) = (1 - t^3)$. If $S = \{3, \dots, 5\}$ then $W_t(1, 7, L_2, L_1, L_3; S) = (1 - t^2)$ and $W_t(1, 7, L_1, \infty, L_2; S) = 1$. If we take the right sample in Figure 4-5 with $S = \{2, \dots, 7\}$ then we have $W_t(1, 7, L_2, L_1, L_3; S) = 0$ and $W_t(1, 7, L_1, \infty, L_2; S) = (1 - t^4)$.

Let $t_i, z_i \in \mathbb{Z}$ for $i = 1, 2$ be given such that $t_1 < t_2$ and $0 \leq z_2 - z_1 \leq t_2 - t_1$. We denote by $\Omega(t_1, t_2; z_1, z_2)$ the collection of up-right paths that start from $(t_1, z_1)$ and end at $(t_2, z_2)$, by $\mathbb{P}_{free}^{t_1, t_2; z_1, z_2}$ the uniform distribution on $\Omega(t_1, t_2; z_1, z_2)$ and write $\mathbb{E}_{free}^{t_1, t_2; z_1, z_2}$ for the expectation with respect to this measure. One thinks of the distribution $\mathbb{P}_{free}^{t_1, t_2; z_1, z_2}$ as the law of a simple random walk with i.i.d. Bernoulli increments with parameter $p \in (0, 1)$ that starts from $z_1$ at time $t_1$ and is conditioned to end in $z_2$ at time $t_2$. Notice that by our assumptions on the parameters the state space is non-empty.

The key definition of this section is the following.

**Definition 4.3.4.** Fix $N \geq 2$, $t \in (0,1)$, two integers $T_0 < T_1$ and set $\Sigma = \{1, \dots, N\}$. Suppose $\mathbb{P}$ is a probability distribution on $\Sigma \times [\![T_0, T_1]\!]$-indexed discrete line ensembles $\mathfrak{L} = (L_1, \dots, L_N)$ and adopt the convention $L_0 = \infty$. We say that $\mathbb{P}$ satisfies the *Hall-Littlewood Gibbs property* with parameter $t$ for a subset $S \subset [\![T_0 + 1, T_1]\!]$ if the following holds. Fix an arbitrary index $i \in \{1, \dots, N-1\}$ and let $\ell_{i-1}, \ell_i, \ell_{i+1}$ be three paths drawn in $\{(r, z) \in \mathbb{Z}^2 : T_0 \leq r \leq T_1\}$ such that $\mathbb{P}(L_{i-1} = \ell_{i-1}, L_{i+1} = \ell_{i+1}) > 0$ (if $i = 1$ we set $\ell_0 = \infty$). Then for any path $\ell$ such that $\ell(T_0) = a = \ell_i(T_0)$ and $\ell(T_1) = b = \ell_i(T_1)$ we have

$$\mathbb{P}(L_i = \ell | L_i(T_0) = a, L_i(T_1) = b, L_{i-1} = \ell_{i-1}, L_{i+1} = \ell_{i+1}) = \frac{W_t(T_0, T_1, \ell, \ell_{i-1}, \ell_{i+1}; S)}{Z_t(T_0, T_1, a, b, \ell_{i-1}, \ell_{i+1}; S)}, \tag{4.3.2}$$

where $Z_t(T_0, T_1, a, b, \ell_{i-1}, \ell_{i+1}; S)$ is a normalization constant. We refer to the measure in (4.3.2) as $\mathbb{P}_S^{T_0, T_1, a, b}(\cdot | \ell_{i-1}, \ell_{i+1})$.

*Remark* 4.3.5. An equivalent formulation of the above definition is that the law of $L_i$, conditioned on its endpoints $a = L_i(T_0)$ and $b = L_i(T_1)$, $L_{i-1} = \ell_{i-1}$ and $L_{i+1} = \ell_{i+1}$ is given by the Radon-Nikodym derivative

$$\frac{d\mathbb{P}}{d\mathbb{P}_{free}^{T_0, T_1; a, b}}(\ell) = \frac{W_t(T_0, T_1, \ell, \ell_{i-1}, \ell_{i+1}; S)}{Z_t(T_0, T_1, a, b, \ell_{i-1}, \ell_{i+1}; S)}.$$

With the above reformulation we get that

$$Z_t(T_0, T_1, a, b, \ell_{i-1}, \ell_{i+1}; S) = \mathbb{E}_{free}^{T_0, T_1; a, b}\left[W_t(T_0, T_1, \ell, \ell_{i-1}, \ell_{i+1}; S)\right],$$

where the expectation is over $\ell$, distributed according to $\mathbb{P}_{free}^{T_0, T_1; a, b}$.

If a measure $\mathbb{P}$ satisfies the Hall-Littlewood Gibbs property, it enjoys the following sampling property. Start by (jointly) sampling $L_i(T_0), L_i(T_1)$ and $L_j(r)$ for $j \neq i$ and $r \in [\![T_0, T_1]\!]$ according to $\mathbb{P}$ (i.e. according to the restriction of $\mathbb{P}$ to these random variables). Set $a = L_i(T_0)$ and $b = L_i(T_1)$ and let $L_i^N, N \in \mathbb{N}$ be a sequence of i.i.d. up-right paths distributed according to $\mathbb{P}_{free}^{T_0, T_1; a, b}$. Let $U$ be a uniform random variable on $(0,1)$, which is independent of all else. For each $N \in \mathbb{N}$ we check if $W_t(T_0, T_1, L_i^N, L_{i-1}, L_{i+1}; S) > U$ and set $Q$ to be the minimal index $N$ for which the inequality holds. Observe that $Q$ is a geometric random variable with parameter $Z_t(T_0, T_1, a, b, L_{i-1}, L_{i+1}; S)$, which we call the *acceptance probability.* In view of the above Radon-Nikodym derivative formulation, it is clear that the random ensemble of up-right paths $(L_1, \dots, L_{i-1}, L_i^Q, L_{i+1}, \dots, L_N)$ is distributed according to $\mathbb{P}$.

*Remark* 4.3.6. We mention that the resampling property of Remark 4.3.5 for a $\{1, \dots, N\} \times [\![T_0, T_1]\!]$-indexed line ensemble $\{L_i\}_{i=1}^N$ only holds for the first $N-1$ lines. The latter, in particular, implies that for $M \leq N$, we have that the induced law on $\{L_i\}_{i=1}^M$ also satisfies the Hall-Littlewood Gibbs property with parameter $t$ and subset $S$ as an $\{1, \dots, M\} \times [\![T_0, T_1]\!]$-indexed line ensemble.

In this chapter, we will be primarily concerned with the case when $\Sigma = \{1, 2\}$ and the discrete line ensemble is *non-crossing*, meaning that $L_1(r) \geq L_2(r)$ for all $r \in [\![T_0, T_1]\!]$. For brevity we will call $\{1, 2\} \times [\![T_0, T_1]\!]$-indexed non-crossing discrete line ensembles *simple.*

These line ensembles will typically arise by restricting a discrete line ensemble with many lines to the top two lines. If the original line ensemble satisfies a Hall-Littlewood Gibbs property with parameter $t$ and set $S$, the same will be true for the restriction to the simple line ensemble at the top (see Remark 4.3.6). To simplify notation, whenever we are working with a simple discrete line ensemble we will omit the $i-1$ index from all of the earlier formulas and notation, as $L_0, \ell_0$ are deterministically $\infty$.

In the remainder of this section we describe a general framework that can be used to prove tightness for the top curve of a sequence of simple discrete line ensembles. We start with the following useful definition.

**Definition 4.3.7.** Fix $t \in (0,1)$, $\alpha > 0$, $p \in (0,1)$ and $T > 0$. Suppose we are given a sequence $\{T_N\}_{N=1}^{\infty}$ with $T_N \in \mathbb{N}$ and that $\{\mathfrak{L}^N\}_{N=1}^{\infty}$, $\mathfrak{L}^N = (L_1^N, L_2^N)$ is a sequence of simple discrete line ensembles on $[\![-T_N, T_N]\!]$. We call the sequence $(\alpha, p, T)$-*good* if there exists $N_0(\alpha, p, T)$ such that for $N \geq N_0$ we have

- $T_N > TN^{\alpha}$ and $\mathfrak{L}^N$ satisfies the Hall-Littlewood Gibbs property with parameter $t$ for $S = [\![-T_N + 1, T_N]\!]$;
- for each $s \in [-T, T]$ the sequence of random variables $\{N^{-\alpha/2}(L_1^N(sN^{\alpha}) - psN^{\alpha})\}$ is tight (i.e. we have one-point tightness of the top curves).

The main technical result of the chapter is as follows.

**Theorem 4.3.8.** *Fix $\alpha, r > 0$ and $p \in (0,1)$ and let $\mathfrak{L}^N = (L_1^N, L_2^N)$ be an $(\alpha, p, r+1)$-good sequence. For $N \geq N_0(\alpha, p, r+1)$ (as in Definition 4.3.7) set*

$$f_N(s) = N^{-\alpha/2}(L_1^N(sN^{\alpha}) - psN^{\alpha}), \text{ for } s \in [-r, r]$$

*and denote by $\mathbb{P}_N$ the law of $f_N(s)$ as a random variable in $(C[-r,r], \mathcal{C})$. Then the sequence $\mathbb{P}_N$ is tight.*

Roughly, Theorem 4.3.8 states that if a process can be viewed as the top curve of a simple discrete line ensemble and under some shift and scaling the process's one-point marginals are tight, then under the same shift and scaling the trajectory of the process is tight in the space of continuous curves. We will show later in Theorem 4.7.3 that any subsequential limit of the measures $\mathbb{P}_N$ in Theorem 4.3.8 is absolutely continuous with respect to a Brownian bridge of a certain variance – see Section 4.7 for the details. We also want to remark that both Theorem 4.3.8 and Theorem 4.7.3 do not depend strongly on any particular structure of the Hall-Littlewood Gibbs property. Indeed, the main ingredient that is used in deriving these results is a lower bound on the acceptance probability $Z_t(T_0, T_1, a, b, L_2; S)$ (see Remark 4.3.5), which is the content of Proposition 4.5.1. It is our belief that our arguments can be extended to other (similar) discrete Gibbs properties without significant modifications.

### 4.3.2 Applications to the three models

In this section we use Theorem 4.3.8 to prove our main results for the three models in Section 4.2, given in Theorem 4.3.10, Corollary 4.3.11 and Theorem 4.3.13 below. In order to apply

Theorem 4.3.8 we will need to rephrase the ascending Hall-Littlewood process and the ASEP in the language of discrete line ensembles, to which we first turn.

Suppose we are given a sequence $\varnothing = \lambda(0) \prec \lambda(1) \prec \lambda(2) \prec \cdots \prec \lambda(M)$. The condition $\lambda(i) \prec \lambda(i+1)$ is equivalent to $\lambda_j'(i+1) - \lambda_j'(i) \in \{0,1\}$ for any $j \geq 1$. The latter implies that we can view the sequence $\varnothing = \lambda(0) \prec \lambda(1) \prec \lambda(2) \prec \cdots \prec \lambda(M)$ as a collection of up-right paths $\{\lambda_j'(\cdot)\}_{j=1}^N$ drawn in the sector $\{0, \dots, M\} \times \mathbb{Z}$ (see Figure 4-6). In particular, this allows us to interpret the ascending Hall-Littlewood process as a probability distribution of $\{1, \dots, N\} \times [\![0, M]\!]$-indexed discrete line ensembles in the sense of Definition 4.3.1, where $L_j(i) = \lambda_j'(i)$ for $i = 0, \dots, M$ and $j = 1, \dots, N$.

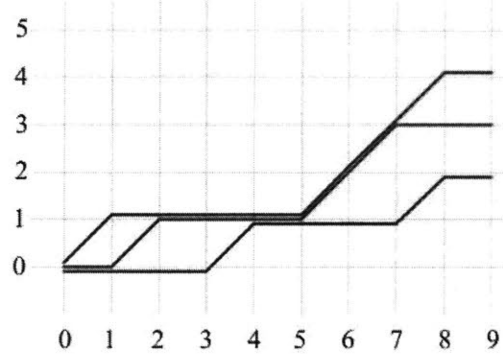


Figure 4-6: The up-right paths corresponding to $\lambda_1'(i), \lambda_2'(i), \lambda_3'(i)$, for $0 \leq i \leq 9$, where $\lambda(i)$ is the $i$-the element in the sequence $\varnothing \prec (1) \prec (2) \prec (2) \prec (4) \prec (4,2) \prec (5,2,2) \prec (5,3,2) \prec (8,5,2,1)$.

The key observation we make is that if $\varnothing = \lambda(0) \prec \lambda(1) \prec \lambda(2) \prec \cdots \prec \lambda(M)$ is distributed according to $\mathbb{P}_\zeta^{M,N}$ from Definition 4.2.1, then the discrete line ensemble $L_j(i) = \lambda_j'(i)$ for $i = 0, \dots, M$ and $j = 1, \dots, N$ satisfies the Hall-Littlewood Gibbs property (this is the origin of the name of this property). We isolate this in the following proposition.

**Proposition 4.3.9.** *Fix $M, N \in \mathbb{N}$ and $\zeta, t \in (0,1)$. Let $\varnothing = \lambda(0) \prec \lambda(1) \prec \lambda(2) \prec \cdots \prec \lambda(M)$ be sampled from $\mathbb{P}_\zeta^{M,N}$ (see Definition 4.2.1). Then $(\lambda_1'(\cdot), \lambda_2'(\cdot), \dots, \lambda_N'(\cdot))$ satisfies the Hall-Littlewood Gibbs property with parameter $t$ for $S = [\![1, \dots, M]\!]$.*

*Proof.* By Definition 4.2.1 we know that

$$\mathbb{P}_\zeta^{M,N}(\lambda(1), \dots, \lambda(M)) = \left(\frac{1-\zeta}{1-t\zeta}\right)^{NM} \times \prod_{i=1}^{M} P_{\lambda(i)/\lambda(i-1)}(1) \times Q_{\lambda(M)}(\zeta^N).$$

The latter equation implies that $\lambda_1'(0) = 0$ and $0 \leq \lambda_1'(M) \leq \min(M, N)$ with probability 1. Using (4.2.2) we see that

$$\begin{aligned} &\mathbb{P}_\zeta^{M,N}(\lambda(1), \dots, \lambda(M)) = Q_{\lambda(M)}(\zeta^N) \cdot \left(\frac{1-\zeta}{1-t\zeta}\right)^{NM} \cdot \prod_{i=1}^{M} \psi_{\lambda(i)/\lambda(i-1)}(t), \text{ where} \\ &\psi_{\lambda/\mu}(t) = \mathbf{1}_{\{\lambda \succ \mu\}} \cdot \prod_{j=1}^{\infty} \left(1 - \mathbf{1}_{\{\Delta(\mu,j) - \Delta(\lambda,j) = 1\}} t^{\Delta(\mu,j)}\right), \text{ and } \Delta(\mu, j) = \mu_j' - \mu_{j+1}'. \end{aligned} \tag{4.3.3}$$

Fix $i \in \{1, \dots, N-1\}$ and notice that (4.3.3) and (4.3.1) imply that for any $\ell \in \Omega(0, M; 0, k)$ with $0 \le k \le \min(M, N)$ we have

$$\mathbb{P}_{\zeta}^{M,N}\left(\lambda_i'(\cdot) = \ell | \mathcal{F}_{ext}(\{i\}, (0, M))\right) = C \cdot W_t(0, M, \ell, \lambda_{i-1}'(\cdot), \lambda_{i+1}'(\cdot); S),$$

where $\mathcal{F}_{ext}(\{i\}, (0, M))$ is the $\sigma$-algebra generated by $\lambda_j'(a)$ for $a = 0, \dots, M$ and $j \neq i$ as well as $\lambda_i'(0) = 0$ and $\lambda_i'(M) = k$, and $C$ is an $\mathcal{F}_{ext}(\{i\}, (0, M))$-measurable normalization constant. Let $\mathcal{F}_1$ be the $\sigma$-algebra generated by $\lambda_i'(0) = 0$, $\lambda_i'(M) = k$, $\lambda_{i-1}'(\cdot)$ and $\lambda_{i+1}'(\cdot)$ and observe that $\mathcal{F}_1 \subset \mathcal{F}_{ext}(\{i\}, (0, M))$. It follows from the tower property for conditional expectation that

$$\mathbb{P}_{\zeta}^{M,N}\left(\lambda_i'(\cdot) = \ell | \lambda_i'(0) = 0, \lambda_i'(M) = k, \lambda_{i-1}'(\cdot), \lambda_{i+1}'(\cdot)\right) =$$

$$\mathbb{P}_{\zeta}^{M,N}\left(CW_t(0, M, \ell, \lambda_{i-1}'(\cdot), \lambda_{i+1}'(\cdot); S) | \mathcal{F}_1\right) = W_t(0, M, \ell, \lambda_{i-1}'(\cdot), \lambda_{i+1}'(\cdot); S) \cdot \mathbb{P}_{\zeta}^{M,N}\left(C | \mathcal{F}_1\right),$$

where in the last equality we used that $W_t(0, M, \ell, \lambda_{i-1}'(\cdot), \lambda_{i+1}'(\cdot); S)$ is $\mathcal{F}_1$-measurable. The latter equation is equivalent to (4.3.2), which proves the proposition. □

With the help of Proposition 4.3.9 we deduce the following results for the homogeneous ascending Hall-Littlewood process and stochastic six-vertex model.

**Theorem 4.3.10.** *Assume the same notation as in Theorem 4.2.2. If $\mathbb{P}_N$ denotes the law of $f_N^{HL}(\cdot)$ as a random variable in $(C[-r, r], \mathcal{C})$, then the sequence $\mathbb{P}_N$ is tight.*

*Proof.* Consider the $\{1, 2\} \times [\![-T_N, T_N]\!]$-indexed simple discrete line ensemble with $T_N = \lfloor (r+2)N^{2/3} \rfloor$, given by

$$(L_1^N(i), L_2^N(i)) = (\lambda_1'(\lfloor \mu N \rfloor + i) - \lfloor f_1(\mu)N \rfloor, \lambda_2'(\lfloor \mu N \rfloor + i) - \lfloor f_1(\mu)N \rfloor).$$

It follows from Proposition 4.3.9 that $(L_1^N, L_2^N)$ is a simple discrete line ensemble, which satisfies the Hall-Littlewood Gibbs property with parameter $t$ for $S = [\![-T_N + 1, T_N]\!]$. In addition, by Theorem 4.2.2 we know that for each $s \in [-r-1, r+1]$ the sequence of random variables $N^{-1/3}\left(L_1^N(sN^{2/3}) - sN^{2/3} f_1'(\mu)\right)$ is tight. The latter statements imply that the sequence $(L_1^N, L_2^N)$ is $(2/3, f_1'(\mu), r+1)$-good. It follows from Theorem 4.3.8 that if

$$g_N^{HL}(s) = N^{-1/3}\left(\lambda_1'(\lfloor \mu N \rfloor + sN^{2/3}) - \lfloor f_1(\mu)N \rfloor - f_1'(\mu)sN^{2/3}\right), \text{ for } s \in [-r, r],$$

then $g_N^{HL}(\cdot)$ form a tight sequence of random variables in $(C[-r, r], \mathcal{C})$. The latter clearly implies the statement of the theorem. □

**Corollary 4.3.11.** *Let $\xi, u, q, r > 0$ be given such that $q \in (0, 1)$, $\zeta = \xi^{-1}u^{-1}q^{-1/2} < 1$ and fix $\mu \in (\zeta, \zeta^{-1})$. Let $h(x, y)$ denote height function sampled from $\mathbb{P}_{\xi,u,q}$ and set for $s \in [-r, r]$*

$$f_N^{SV}(s) = \sigma_\mu^{-1} N^{-1/3}\left(f_2(\mu)N + f_2'(\mu)sN^{2/3} + (1/2)s^2 f_2''(\mu)N^{1/3} - h(1 + \mu N + sN^{2/3}, N)\right), \tag{4.3.4}$$

*where we define $h(\cdot, N)$ at non-integer points by linear interpolation. The constants above are given by* $\sigma_\mu = \frac{(\zeta\mu)^{1/6}(1-\sqrt{\zeta\mu})^{2/3}\left(1-\sqrt{\zeta/\mu}\right)^{2/3}}{1-\zeta}$, $f_2(\mu) = \frac{(1-\sqrt{\zeta\mu})^2}{1-\zeta}$, $f_2'(\mu) = -\frac{\sqrt{\zeta}(1-\sqrt{\zeta\mu})}{\sqrt{\mu}(1-\zeta)}$, $f_2''(\mu) =$

$\frac{\sqrt{\zeta}}{2\mu^{3/2}(1-\zeta)}$. *If $\mathbb{P}_N$ denotes the law of $f_N^{SV}(s)$ as a random variable in $(C[-r,r],\mathcal{C})$, then the sequence $\mathbb{P}_N$ is tight.*

*Proof.* From Theorem 4.2.4 we know that the law of $f_N^{HL}$ as in the statement of Theorem 4.3.10 is the same as $f_N^{SV}$. The result now follows from Theorem 4.3.10. □

Before we apply Theorem 4.3.8 to the ASEP, we need to rephrase the latter in the language of discrete line ensembles that satisfy the Hall-Littlewood Gibbs property. We achieve this in the following proposition, whose proof is deferred to the next section.

**Proposition 4.3.12.** *Suppose $R = 1$, $L = t \in (0,1)$ are given, fix $K_1, K_2 \in \mathbb{N}$, $T > 0$ and set $\Sigma = \{1,\dots,K_1\}$. Then there exists a probability space, on which a $\Sigma \times [\![-K_2, K_2]\!]$-indexed discrete line ensemble $(L_1, L_2, \dots, L_{K_1})$ is defined such that*

- *the law of $(L_1, L_2, \dots, L_{K_1})$ satisfies the Hall-Littlewood Gibbs property with parameter $t$ for the set $S = [\![-K_2+1, K_2]\!]$;*
- *the law of $(L_1(-K_2),\dots,L_1(K_2))$ is the same as $(-\mathfrak{h}(-K_2+1),\dots,-\mathfrak{h}(K_2+1))$, viewed as random vectors in $\mathbb{R}^{2K_2+1}$, where $\mathfrak{h}$ has law $\mathbb{P}^T_{L,R}$ (see Section 4.2.3).*

With the help of Proposition 4.3.12 we deduce the following results for the ASEP.

**Theorem 4.3.13.** *Assume the same notation as in Theorem 4.2.5. If $\mathbb{P}_N$ denotes the law of $f_N^{ASEP}(s)$ as a random variable in $(C[-r,r],\mathcal{C})$, then the sequence $\mathbb{P}_N$ is tight.*

*Proof.* Consider the $\{1,2\} \times [\![-T_N, T_N]\!]$-indexed simple discrete line ensemble with $T_N = \lfloor (r+2)N^{2/3}\rfloor$, given by

$$(\tilde{L}_1^N(i), \tilde{L}_2^N(i)) = \left(L_1(\lfloor \alpha N\rfloor + i) + \lfloor f_3(\alpha)N\rfloor, L_2(\lfloor \alpha N\rfloor + i) + \lfloor f_3(\alpha)N\rfloor\right),$$

with $(L_1, L_2)$ defined as in Proposition 4.3.12 with $K_1 = 2$, $K_2 = \alpha N + T_N$ and $T = N/\gamma$.

By construction, we have that $(\tilde{L}_1^N, \tilde{L}_2^N)$ satisfies the Hall-Littlewood Gibbs property with parameter $t$ for $S = [\![-T_N+1, T_N]\!]$. In addition, by Theorem 4.2.5 and the fact that $L_1$ has the same law as $-\mathfrak{h}$, we know that for each $s \in [-r-1, r+1]$ the sequence of random variables $N^{-1/3}\left(\tilde{L}_1^N(sN^{2/3}) + sN^{2/3}f_3'(\alpha)\right)$ is tight. The latter statements imply that the sequence $(\tilde{L}_1^N, \tilde{L}_2^N)$ is $(2/3, -f_3'(\alpha), r+1)$-good. It follows from Theorem 4.3.8 that if

$$g_N^{ASEP}(s) = N^{-1/3}\left(L_1(\lfloor \alpha N\rfloor + sN^{2/3}) + \lfloor f_3(\alpha)N\rfloor + f_3'(\alpha)sN^{2/3}\right), \text{ for } s \in [-r,r],$$

then $g_N^{ASEP}(\cdot)$ form a tight sequence of random variables in $(C[-r,r],\mathcal{C})$. The latter clearly implies the statement of the theorem. □

*Remark* 4.3.14. In Corollary 4.7.4 we show that any subsequential limit of either of the sequences $f_N^{HL}$, $f_N^{SV}$ and $f_N^{ASEP}$ as in the text above, when shifted by an appropriate parabola, is absolutely continuous with respect to a Brownian bridge of appropriate variance. This, in particular, implies that the subsequential limits of these random curves are non-trivial.

### 4.3.3 Proof of Proposition 4.3.12

In this section we present the proof of Proposition 4.3.12, which we split into several steps for clarity. Before we go into the main argument let us briefly outline the main ideas of the proof. We begin by considering a particular sequence of $\{1, \dots, K_1\} \times [\![-K_2, K_2]\!]$-indexed discrete line ensemble $(\Lambda_1^N, \dots, \Lambda_{K_1}^N)$. The latter are defined through appropriately truncated and shifted discrete line ensembles associated to ascending Hall-Littlewood processes with parameters $\zeta(N)$ such that $\zeta(N)$ converges to 1. In Step 1 below we carefully explain the construction of $(\Lambda_1^N, \dots, \Lambda_{K_1}^N)$ and assume that the sequence is tight and that $(\Lambda_1^N(-K_2), \dots, \Lambda_1^N(K_2))$ weakly converges to $(-\mathfrak{h}(-K_2+1), \dots, -\mathfrak{h}(K_2+1))$. Using the tightness assumption we can pick some subsequential limit $(\Lambda_1^\infty, \dots, \Lambda_{K_1}^\infty)$ and show it satisfies the conditions of the proposition. The weak convergence of $(\Lambda_1^N(-K_2), \dots, \Lambda_1^N(K_2))$ to $(-\mathfrak{h}(-K_2+1), \dots, -\mathfrak{h}(K_2+1))$ is proved in Step 2 and it relies on Theorems 4.2.4 and 4.2.6. The tightness of $(\Lambda_1^N, \dots, \Lambda_{K_1}^N)$ is demonstrated in Steps 3, 4, 5 and 6, by combining the already known tightness of $\Lambda_1^N$ and the Hall-Littlewood Gibbs property.

**Step 1.** For each $N \in \mathbb{N}$ consider the homogeneous ascending Hall-Littlewood process $\mathbb{P}^{M,N_T}_{\zeta(N)}$ where $N_T = \lfloor N \cdot T \rfloor$, $\zeta(N) = 1 - \frac{1-t}{N}$ and $M = N_T + K$. For $N$ such that $N_T \geq K_1$ we let $(\Lambda_1^N, \dots, \Lambda_{K_1}^N)$ be the $\Sigma \times [\![-K_2, K_2]\!]$-indexed discrete line ensemble, given by

$$\Lambda_j^N(i) = \lambda_j'(i + N_T) - N_T, \text{ for } i \in \{-K_2, -K_2+1, \dots, K_2\} \text{ and } j \in \{1, \dots, K_1\} \quad (4.3.5)$$

where $(\lambda_1'(\cdot), \dots, \lambda_{K_1}'(\cdot))$ is sampled from $\mathbb{P}^{M,N_T}_{\zeta(N)}$. We isolate the following claims.
**Claims:**

- the sequence $(\Lambda_1^N, \dots, \Lambda_{K_1}^N)$ is tight as random vectors in $\mathbb{Z}^{K_1 \cdot (2K_2+1)}$
- the sequence $(\Lambda_1^N(-K_2), \dots, \Lambda_1^N(K_2))$ weakly converges to $(-\mathfrak{h}(-K_2+1), \dots, -\mathfrak{h}(K_2+1))$ as random vectors in $\mathbb{Z}^{2K_2+1}$ as $N \to \infty$.

The latter statements are proved in the steps below. In what follows we assume their validity and finish the proof of the proposition.

Let $(\Lambda_1^\infty, \dots, \Lambda_{K_1}^\infty)$ be any subsequential limit of $(\Lambda_1^N, \dots, \Lambda_{K_1}^N)$ and assume that $N_k$ is an increasing sequence of integers such that

$$(\Lambda_1^{N_k}, \dots, \Lambda_{K_1}^{N_k}) \implies (\Lambda_1^\infty, \dots, \Lambda_{K_1}^\infty) \text{ as } k \to \infty, \quad (4.3.6)$$

We know that $(\Lambda_1^{N_k}, \dots, \Lambda_{K_1}^{N_k})$ is a $\Sigma \times [\![-K_2, K_2]\!]$-indexed discrete line ensemble, which by Proposition 4.3.9 satisfies the Hall-Littlewood Gibbs property with parameter $t$ on $S$ and we conclude that the same is true for $(\Lambda_1^\infty, \dots, \Lambda_{K_1}^\infty)$. By our earlier assumptions we know that $(\Lambda_1^\infty(-K_2), \dots, \Lambda_1^\infty(K_2))$ has the same law as $(-\mathfrak{h}(-K_2+1), \dots, -\mathfrak{h}(K_2+1))$ and so $(\Lambda_1^\infty, \dots, \Lambda_{K_1}^\infty)$ satisfies the conditions of the proposition.

**Step 2.** We show that $(\Lambda_1^N(-K_1), \dots, \Lambda_1^N(K_1))$ weakly converges to $(-\mathfrak{h}(-K_1+1), \dots, -\mathfrak{h}(K_1+1))$. Let us put $q = t$, $\xi(N) = t^{1/2}$ and $u = t^{-1}\zeta^{-1}$. From Theorem 4.2.4 we have the following equality in distribution

$$(\Lambda_1^N(-K_2),\dots,\Lambda_1^N(K_2)) \stackrel{d}{=} (-h(N_T-K_2+1,N_T),\dots,-h(N_T+K_2+1,N_T)),$$

where $h$ is the height function of a homogeneous stochastic six-vertex model sampled from $\mathbb{P}_{\xi(N),u(N),q}$. From (4.2.12) we have

$$b_1(N)=\frac{1-q^{1/2}\xi(N)u(N)}{1-q^{-1/2}\xi(N)u(N)}=tN^{-1}+O(N^{-2})\quad b_2(N)=\frac{q^{-1}-q^{-1/2}\xi(N)u(N)}{1-q^{-1/2}\xi(N)u(N)}=N^{-1}+O(N^{-2}).$$

As a consequence of Theorem 4.2.6 we have that $(-h(N_T-K_2+1,N_T),\dots,-h(N_T+K_2+1,N_T))$ converges weakly to $(-\mathfrak{h}(-K_2+1),\dots,-\mathfrak{h}(K_2+1))$, where $\mathfrak{h}$ has law $\mathbb{P}^T_{L,R}$.

**Step 3.** In this step we show that $(\Lambda_1^N,\dots,\Lambda_{K_1}^N)$ is tight, by showing that $\Lambda_k^N$ is tight for each $k=1,\dots,K_1$. We proceed by induction on $k$ with base case $k=1$ being true by Step 2. In what follows assume that $\Lambda_1^N,\dots,\Lambda_k^N$ are tight and want to show that $\Lambda_{k+1}^N$ is also tight. Notice that because $L_i^N(j)-L_i^N(j+1)\in\{0,1\}$ it is enought to show that $\lambda'_{k+1}(N_T)-N_T$ is tight.

Let $\epsilon>0$ be given. Set $D_N(B):=\{|\lambda'_{k-1}(N_T)-N_T|\geq B\}$. If $k\geq 2$ we have from the tightness of the sequence $\lambda'_{k-1}(N_T)-N_T$ that there exists $B\in\mathbb{N}$ sufficiently large so that

$$\mathbb{P}\left(D_N^c(B)\right)<\epsilon/16. \tag{4.3.7}$$

By convention, $\lambda_0=\infty$ and so $D_N(B)$ is a set of full measure and (4.3.7) holds even if $k=1$.

From the tightness of the sequence $\lambda'_k(N_T)-N_T$, we know that there exists $A\in\mathbb{N}$ sufficiently large so that

$$\mathbb{P}\left(|\lambda'_k(N_T)-N_T|\geq A\right)<\epsilon(1-t)^B/16 \text{ and } 1\geq(1-t^A)^{2A}\geq 1/2. \tag{4.3.8}$$

We make the following definitions

$$E_N:=\{\lambda'_k(N_T-2A)-N_T>-4A\} \text{ and } F_N:=\{\lambda'_{k+1}(N_T)-N_T<-8A\}.$$

Let us denote by $\mathcal{F}_N^k=\mathcal{F}_{ext}\left(\{k\}\times(N_T-2A,N_T]\right)$ the $\sigma$-algebra generated by the up-right paths $\lambda'_j(\cdot)$ for $j\neq k$ and $\lambda'_k(\cdot)$ on the interval $[0,N_T-2A]$. Observe that all three events $D_N(B)$, $E_N$ and $F_N$ are $\mathcal{F}_N^k$-measurable. Using the above notation we claim that for all $N$ sufficiently large we have

$$4\cdot\mathbb{E}\left[\mathbf{1}\{\lambda'_k(N_T)\leq N_T-A\}|\mathcal{F}_N^k\right]\geq(1-t)^B\cdot\mathbf{1}_{D_N\cap E_N\cap F_N}. \tag{4.3.9}$$

The above statement will be proved in Step 4 below. For now we assume it and finish the proof.

Taking expectations on both sides of (4.3.9) and using (4.3.8), we conclude that $\epsilon/4\geq\mathbb{P}(D_N\cap E_N\cap F_N)$. Notice that $E_N\subset\{0\geq\lambda'_k(N_T)-N_T>-2A\}$, which implies by (4.3.8) that $\mathbb{P}(E_N^c)\leq\epsilon/16$. Combining the last two estimates with (4.3.7) we see that for all large

$N$ we have

$$\mathbb{P}(F_N) \leq \mathbb{P}(D_N \cap E_N \cap F_N) + \mathbb{P}(E_N^c) + \mathbb{P}(D_N^c(B)) \leq \epsilon/4 + \epsilon/16 + \epsilon/16 < \epsilon.$$

The latter means that for all large $N$ we have

$$\mathbb{P}\left(0 \geq \lambda'_{k+1}(N_T) - N_T \geq -8A\right) > 1 - \epsilon.$$

Since $\epsilon > 0$ was arbitrary this proves that $\lambda'_{k+1}(N_T) - N_T$ is tight.

**Step 4.** For $t_1, t_2, x \in \mathbb{Z}$ and $t_1 < t_2$ we let $\Omega_x(t_1, t_2)$ denote the set of up-right paths drawn in $\{t_1, \dots, t_2\} \times \mathbb{Z}$, which start from $(t_1, x)$. In addition, we fix two up-right path $\ell_{bot} \in \Omega_y(t_1, t_2)$ and $\ell_{top} \in \Omega_z(t_1, t_2)$, where $y < x - 4A$, $y \leq z$ and $K(\ell_{top}) \leq B$ where $K(\ell_{top}) := |\{N_T - 2A + 1 \leq i \leq N_T : \ell_{top}(i) - \ell_{top}(i-1) = 0\}|$. If $k = 1$ we set $\ell_{top} = \infty$ and $K(\ell_{top}) = 0$.

For $N \in \mathbb{N}$ we consider the measure $\mathbb{P}_N^{x,\ell_{top},\ell_{bot}}$ on $\Omega_x(N_T - 2A, N_T)$, given by

$$\mathbb{P}_N^{x,\ell_{top},\ell_{bot}}(\ell) = Z_N^{-1} \cdot W_t(N_T - 2A, N_T, \ell, \ell_{top}, \ell_{bot}; S_N) \cdot \zeta(N)^{\ell(N_T) - x},$$

where $S_N = [\![N_T - 2A + 1, N_T]\!]$ and $Z_N$ is a normalization constant. With the above notation we define $P(x, N, \ell_{top}, \ell_{bot}) = \mathbb{P}_N^{x,\ell_{top},\ell_{bot}}(\ell(N_T) \leq x + A)$ and claim that for all $N$ sufficiently large (depending on $t$ and $A$) we have that

$$P(x, N, \ell_{top}, \ell_{bot}) \geq (1 - t)^B/4. \tag{4.3.10}$$

The latter will be proved in Step 5 below. For now we assume its true and finish the proof of (4.3.9).

Let $\ell_{k\pm1}^N \in \Omega_{\lambda'_{k\pm1}(N_T - 2A)}(N_T - 2A, N_T)$ be such that $\ell_{k\pm1}^N(i) = \lambda'_{k\pm1}(i)$ for $i = N_T - 2A, \dots, N_T$, where $\ell_{k-1}^N = \ell_0^N = \infty$ when $k = 1$. As a consequence of Proposition 4.3.9 (see also (4.3.3)) we have the following a.s. equality of $\mathcal{F}_N^k$ random variables

$$\begin{aligned} &\mathbf{1}_{D_N(B) \cap E_N \cap F_N} \cdot \mathbb{E}\left[\mathbf{1}\{\lambda'_k(N_T) \leq \lambda'_k(N_T - 2A) + A\} | \mathcal{F}_N^k\right] = \\ &\mathbf{1}_{D_N(B) \cap E_N \cap F_N} \cdot P(\lambda'_k(N_T - 2A), N, \ell_{k-1}^N, \ell_{k+1}^N). \end{aligned}$$

In deriving the above equality we used that for $\omega \in D_N(B)$ we have $K(\ell_{k-1}^N(\omega)) \leq B$ by definition of $D_N(B)$.

Notice that a.s. $\lambda'_k(N_T - 2A) + A \leq N_T - A$, from which we conclude that we have the following a.s. inequality

$$\begin{aligned} &\mathbf{1}_{D_N(B) \cap E_N \cap F_N} \cdot \mathbb{E}\left[\mathbf{1}\{\lambda'_k(N_T) \leq N_T - A\} | \mathcal{F}_N^k\right] \geq \\ &\mathbf{1}_{D_N(B) \cap E_N \cap F_N} \cdot P(\lambda'_k(N_T - 2A), N, \ell_{k-1}^N, \ell_{k+1}^N). \end{aligned} \tag{4.3.11}$$

From (4.3.10) we have for all large $N$ that $P(\lambda'_k(N_T - 2A), N, \ell_{k-1}^N, \ell_{k+1}^N) \geq (1 - t)^B/4$, which together with $1 \geq \mathbf{1}_{E_N \cap F_N}$ and (4.3.11) imply (4.3.9).

**Step 5.** In this step we establish (4.3.10), but first we briefly explain our idea. By assumption, we know that $\ell$ is a random path that lies at least a distance $A$ above $\ell_{bot}$ and that $\ell_{top}(i)$ increases by 1 when $i$ increases by 1 on $[N_T - 2A, N_T]$ with at most $B$ exceptions. The latter implies that

$$1 \geq W_t(N_T - 2A, N_T, \ell, \ell_{top}, \ell_{bot}; S_N) \geq (1-t)^B(1-t^A)^{2A} \geq (1-t)^B/2,$$

where in the last inequality we used (4.3.8). On the other hand, we know that $\zeta(N) \to 1$ as $N \to \infty$. This implies that $\mathbb{P}_N^{x,\ell_{top},\ell_{bot}}(\ell)$ is essentially the uniform measure on up-right paths of length $2A$ started from $x$, conditioned to stay below $\ell_{top}$ and distorted by a well-behaved Radon-Nikodym derivative. At least half of the paths that start from $x$ and have length $2A$ end in a position below $x + A$, and since each path carries roughly the same weight we can obtain the desired estimate.

We make the following definitions

$$\Omega^+(\ell_{top}) := \{\ell \in \Omega_x(N_T - 2A, N_T) : \ell(N_T) > x + A \text{ and } \ell_{top}(i) \geq \ell(i) \text{ for } N_T - 2A \leq i \leq N_T\},$$
$$\Omega^-(\ell_{top}) := \{\ell \in \Omega_x(N_T - 2A, N_T) : \ell(N_T) \leq x + A \text{ and } \ell_{top}(i) \geq \ell(i) \text{ for } N_T - 2A \leq i \leq N_T\}.$$

We claim that we have

$$|\Omega^-| \geq |\Omega^+|. \tag{4.3.12}$$

The latter will be proved in Step 6 below. For now we assume it and finish the proof of (4.3.10).

Write $\mathbb{P}_N$ instead of $\mathbb{P}_N^{x,\ell_{top},\ell_{bot}}$ for brevity. We can find $N_0$ (depending on $t$ and $A$ ) such that for all $N \geq N_0$ we have $1 \geq \zeta(N)^{2A} \geq 1/2$. The latter together with our assumption on $\ell_{top}$ implies

$$1 \geq W_t(N_T - 2A, N_T, \ell, \ell_{top}, \ell_{bot}; S_N) \geq (1-t)^B(1-t^A)^{2A} \geq (1-t)^B/2$$

Consequently, for any $\ell_1, \ell_2 \in \Omega_x(N_T - 2A, N_T)$ we have

$$\mathbb{P}_N(\ell_1) \geq \left[(1-t)^B/2\right] \cdot \mathbb{P}_N(\ell_2).$$

In view of (4.3.12) we have

$$\mathbb{P}_N(\Omega^-) = \sum_{\ell \in \Omega^-} \mathbb{P}_N(\ell) \geq \left[(1-t)^B/2\right] \cdot \sum_{\ell \in \Omega^+} \mathbb{P}_N(\ell) = \left[(1-t)^B/2\right] \cdot \mathbb{P}_N(\Omega^+).$$

The latter implies that

$$\mathbb{P}_N(\Omega^-) \geq (1/2) \cdot \mathbb{P}_N(\Omega^-) + \left[(1-t)^B/4\right] \cdot \mathbb{P}_N(\Omega^+) \geq \left[(1-t)^B/4\right].$$

**Step 6.** In this final step we establish the validity of (4.3.12). It is easy to see that (4.3.12) is equivalent to the following purely probabilistic question:

Let $X_i$ be i.i.d. random variables such that $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = 1/2$ and $S_k = X_1 + \cdots + X_k$ be a random walk with increments $X_i$. Fix an up-right path $\ell_{top}$ such that

$\ell_{top}(0) \geq 0$ and $A \in \mathbb{N}$. Then we have the following inequality

$$\mathbb{P}(S_{2A} \leq A | S_k \leq \ell_{top}(k) \text{ for } k = 1, \ldots, 2A) \geq 1/2. \tag{4.3.13}$$

Observe that if $\ell_{top} = \infty$ then the above is trivial by symmetry. For finite $\ell_{top}$, conditioning the walk to stay below $\ell_{top}$ stochastically pushes the walk lower and so the probability it ends up below $A$ only increases.

A rigorous way to prove the above is using the FKG inequality. To be more specific, let $L_{2A}$ be the set of up-right paths starting from 0 of length $2A$. The natural partial order on $L_{2A}$ is given by

$$\ell_1 \leq \ell_2 \iff \ell_1(i) \leq \ell_2(i) \text{ for } i = 1, \ldots, 2A.$$

With this $L_{2A}$ has the structure of a lattice and so the FKG inequality reads

$$\left( \sum_{\ell \in L_{2A}} \frac{\mathbf{1}\{\ell \leq \ell_{top}\} \mathbf{1}\{\ell(2A) \leq A\}}{|L_{2A}|} \right) \geq \left( \sum_{\ell \in L_{2A}} \frac{\mathbf{1}\{\ell(2A) \leq A\}}{|L_{2A}|} \right) \cdot \left( \sum_{\ell \in L_{2A}} \frac{\mathbf{1}\{\ell \leq \ell_{top}\}}{|L_{2A}|} \right),$$

and clearly implies (4.3.13). This concludes the proof of the proposition.

## 4.4 Basic lemmas

This section contains the primary set of results we will need to prove Theorem 4.3.8. For the remainder of the chapter we will only work with simple discrete line ensembles and as discussed in Section 4.3.1 we will drop references to $\ell_0$ and $L_0$ from our notation.

### 4.4.1 Monotone weight lemma

In this section we isolate the key result that allows us to analyze measures that satisfy the Hall-Littlewood Gibbs property – Lemma 4.4.1 below. In addition, we derive two easy corollaries, which are more suitable for our arguments later in the text.

Let $z_1, z_2, t_1, t_2 \in \mathbb{Z}$ be such that $t_1 < t_2$ and $0 \leq z_2 - z_1 \leq t_2 - t_1$ and recall from Section 4.3.1 that $\Omega(t_1, t_2; z_1 z_2)$ denotes the set of up-right paths from $(t_1, z_1)$ to $(t_2, z_2)$. Each $\ell \in \Omega(t_1, t_2; z_1 z_2)$ can be encoded by a sequence $R(\ell)$ of $t_2 - t_1$ signs: $+$'s and $-$'s indexed from $t_1 + 1$ to $t_2$, so that $R(i) = +$ if and only if $\ell(i) - \ell(i-1) = 1$. The latter is depicted in Figure 4-7. The total number of $+$'s is fixed and equals $z_2 - z_1$ and the number of $-$'s equals $t_2 - t_1 - z_2 + z_1$.

The main result of this section is the following.

**Lemma 4.4.1.** *Fix $t \in (0,1)$ and let $c(t) = \prod_{i=1}^{\infty}(1 - t^i) \in (0,1)$. Suppose $a, b, z_1, z_2, t_1, t_2$ are given such that $t_1 + 1 < t_2$, $0 \leq z_2 - z_1 \leq t_2 - t_1$, $0 \leq b - a \leq t_2 - t_1$, $z_1 \leq a$, $z_2 \leq b$. Fix any $\ell_{bot} \in \Omega(t_1, t_2; z_1, z_2)$, $S \subset \{t_1 + 1, ..., t_2\}$ and $T \in \{t_1 + 1, ..., t_2 - 1\}$. Let $m(T)$ and $M(T)$ denote the minimal and maximal values of the set $\{\ell(T) : \ell \in \Omega(t_1, t_2; a, b)\}$ and let $m(T) \leq k_1 \leq k_2 \leq M(T)$. Then we have*

$$c(t) \cdot \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W_t(t_1, t_2, \ell, \ell_{bot}; S) | \ell(T) = k_1\right] \leq \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W_t(t_1, t_2, \ell, \ell_{bot}; S) | \ell(T) = k_2\right]. \tag{4.4.1}$$

*Proof.* For brevity we write $W(\ell)$ for $W_t(t_1, t_2, \ell, \ell_{bot}; S)$. Let $\ell_1$ be a random path sampled

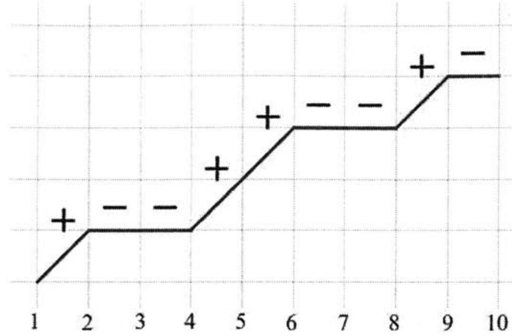


Figure 4-7: A path identified with a sequence of $+$ and $-$ signs. For the above path we have $z_2 - z_1 = 4$, $t_2 - t_1 = 9$ and $R(\ell) = (+,-,-,+,+,-,-,+,-)$.

according to $\mathbb{P}^{t_1,t_2;a,b}_{free}$, conditioned on $\ell_1(T) = k_1$. We identify this path with a sequence of $+$'s and $-$'s and observe that we have $(k_1 - a)$ $+$'s in the first $T - t_1$ slots and the rest are filled with $-$'s. Similarly, we have exactly $(b - k_2)$ $+$'s in the rest $t_2 - T$ slots. Let us pick uniformly at random $(k_2 - k_1)$ $-$'s in the first $T - t_1$ slots and change them to $+$, and also we pick uniformly at random $(k_2 - k_1)$ $+$'s in the last $t_2 - T$ slots and change them to $-$. In this way we build a new sequence of $+$'s and $-$'s that naturally corresponds to an element $\ell_2 \in \Omega(t_1, t_2; a, b)$ such that $\ell_2(T) = k_2$. Moreover it is clear that the random path $\ell_2$ is distributed according to $\mathbb{P}^{t_1,t_2;a,b}_{free}$, conditioned on $\ell_2(T) = k_2$. We are interested in proving the following statement

$$W(\ell_1) \leq c(t)^{-1} \cdot W(\ell_2). \tag{4.4.2}$$

The statement of the lemma is obtained by taking expectations on both sides of (4.4.2).

Since $W(\ell_1) = 0$ otherwise (and then (4.4.2) is immediate) we may assume that $\ell_1(i) \geq \ell_{bot}(i)$ for all $i \in S$. Let $r = k_2 - k_1$ and denote by $x_1 < x_2 < \cdots < x_r$ and $y_1 > y_2 \cdots > y_r$ the positions of $-$'s and $+$'s respectively that we changed when we transformed $\ell_1$ to $\ell_2$. We also let $\ell^j$ for $j = 0, ..., r$ denote the paths in $\Omega(t_1, t_2; a, b)$ obtained by flipping only the signs at locations $x_1, ..., x_j$ and $y_1, ..., y_j$ (in particular $\ell^0 = \ell_1$ and $\ell^r = \ell_2$). An example is depicted in Figure 4-8.

Recall from (4.3.1) that $W(\ell) = \prod_{j \in S}\left(1 - \mathbf{1}_{\{\Delta(j-1)-\Delta(j)=1\}} \cdot t^{\Delta(j-1)}\right)$, where $\Delta(j) = \ell(j) - \ell_{bot}(j)$. Let us explain how $W(\ell^{j+1})$ differs from $W(\ell^j)$. When we flip the signs at $x_{j+1}$ and $y_{j+1}$, we raise the path $\ell^j$ by 1 in the interval $[x_{j+1}, y_{j+1} - 1]$, while outside $(x_{j+1} - 1, y_{j+1})$ it remains the same (see Figure 4-8). The latter operation modifies the factors in $W(\ell^j)$ as follows.

- If $x_{j+1} \in S$ then $W(\ell^j)$ has a factor $\left(1 - \mathbf{1}_{\{\Delta(x_{j+1})-\Delta(x_{j+1}-1)=1\}} \cdot t^{\Delta(x_{j+1}-1)}\right)$, which changes to 1.

- All the factors $\left(1 - \mathbf{1}_{\{\Delta(i)-\Delta(i-1)=1\}} \cdot t^{\Delta(i-1)}\right)$ become $\left(1 - \mathbf{1}_{\{\Delta(i)-\Delta(i-1)=1\}} \cdot t^{\Delta(i-1)+1}\right)$ whenever $i \in S \cap [x_{j+1}, y_{j+1} - 1]$.

- If $y_{j+1} \in S$ then $W(\ell^j)$ has a factor $\left(1 - \mathbf{1}_{\{\Delta(y_{j+1})-\Delta(y_{j+1}-1)=1\}} \cdot t^{\Delta(y_{j+1}-1)}\right)$, which becomes $\left(1 - \mathbf{1}_{\{\Delta(y_{j+1})-\Delta(y_{j+1}-1)=0\}} \cdot t^{\Delta(y_{j+1}-1)+1}\right)$.

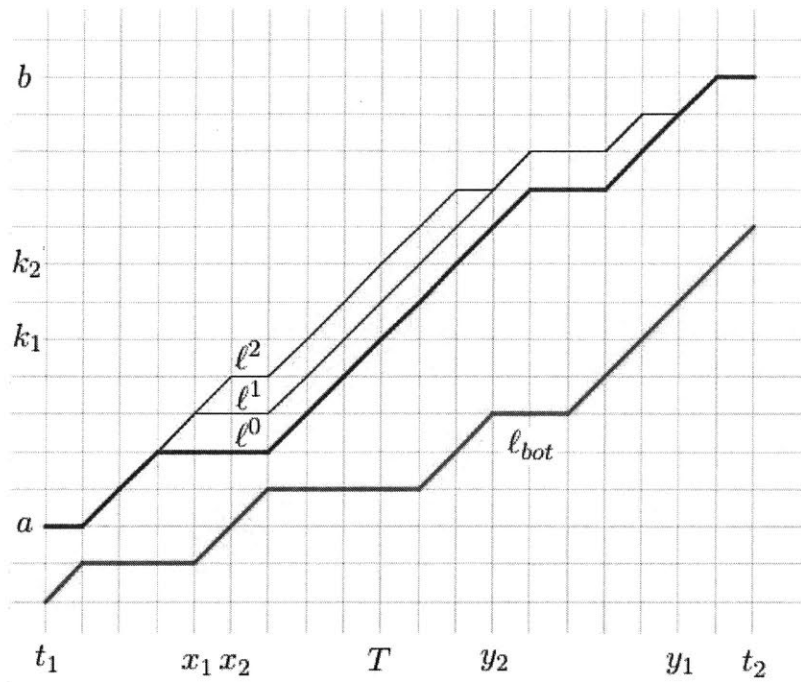


Figure 4-8: An example of $\ell^0, \ell^1$ and $\ell^2$ for the case $k_2 - k_1 = 2$.

The first two changes only increase the weight $W(\ell^j)$, while the last can decrease it but at most by a factor $1 - t^{m_j}$, where $m_j = 1 + \min_{i \in S \cap [x_{j+1}, y_{j+1}-1]} [\ell^j(i) - \ell_{bot}(i)]$. This implies

$$W(\ell^j) \leq (1 - t^{m_j})^{-1} \cdot W(\ell^{j+1}).$$

Notice that $m_0 \geq 1$ since we assumed that $\ell^0(i) = \ell_1(i) \geq \ell_{bot}(i)$ for $i \in S$. In addition, since at step $j+1$ we raise the path on $[x_{j+1}, y_{j+1} - 1]$ by 1 it is clear that $m_{j+1} \geq 1 + m_j$, which implies that $m_j \geq j + 1$ for each $j \geq 0$. We conclude that

$$W(\ell^0) \leq \prod_{j=1}^{r} (1 - t^j)^{-1} \cdot W(\ell^r) \leq c(t)^{-1} \cdot W(\ell^r).$$

As $\ell^0 = \ell_1$ and $\ell^r = \ell_2$ the above proves (4.4.2) and hence the lemma. □

*Remark* 4.4.2. If $t = 0$ the acceptance probability $W_0(t_1, t_2, \ell, \ell_{bot}; S)$ is equal to 1 if $\ell$ does not cross $\ell_{bot}$ on the set $S$, and 0 otherwise. In this case one can use the arguments in the proof of Lemmas 2.6 and 2.7 in [42] to show that we can construct on the same probability space $\ell'$ and $\ell''$ such that

$$\mathbb{P}(\ell' = \ell) = \mathbb{P}_{free}^{t_1,t_2;a,b}(\ell | \ell(T) = k_1), \qquad \mathbb{P}(\ell'' = \ell) = \mathbb{P}_{free}^{t_1,t_2;a,b}(\ell | \ell(T) = k_2)$$

and $\ell'(j) \leq \ell''(j)$ for $t_1 \leq j \leq t_2$ with probability 1. The latter statement implies that we have the following almost sure inequality $W_0(t_1, t_2, \ell', \ell_{bot}; S) \leq W_0(t_1, t_2, \ell'', \ell_{bot}; S)$, which means that higher curves are accepted with higher probability. This statement fits well with the continuous setup in [42].

For general $t \in (0,1)$ we no longer have the above inequality almost surely. For example, we can take $t_1 = 0$, $t_2 = 2n$, $a = k_1 = 0$, $b = k_2 = n$, $S = [\![t_1 + 1, t_2]\!]$, $\ell_{bot} = \ell'$ to be the path that is flat on the interval $[0, n]$ and goes up on $[n, 2n]$, while $\ell''$ the path that goes up on $[0, n]$ and is flat on $[n, 2n]$. For this choice one calculates

$$W_t(t_1, t_2, \ell', \ell_{bot}; S) = 1 > \prod_{i=1}^{n}(1 - t^i) = W_t(t_1, t_2, \ell'', \ell_{bot}; S).$$

Consequently, even though $\ell'$ is below $\ell''$ it is accepted with higher probability and the reason is that the acceptance probability depends not only on the distance between lines but also on their relative slope. In this context, the result of Lemma 4.4.1 is that the acceptance probability of the top line does increase as it is raised, although only in terms of its expected value and up to a factor of $c(t) = \prod_{i=1}^{\infty}(1 - t^i)$. This monotonicity is much weaker than the almost sure monotonicity in the $t = 0$ case, but it turns out to be sufficient for our methods to work.

Using the above lemma we prove two useful corollaries.

**Corollary 4.4.3.** *Assume the same notation as in Lemma 4.4.1. Suppose $A, B$ are non-empty subsets of $\{m(T), m(T) + 1, ..., M(T)\}$, such that $\alpha \geq \beta$ for all $\alpha \in A$ and $\beta \in B$. Then we have*

$$c(t) \cdot \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W_t(t_1, t_2, \ell, \ell_{bot}; S) | \ell(T) \in B\right] \leq \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W_t(t_1, t_2, \ell, \ell_{bot}; S) | \ell(T) \in A\right]. \quad (4.4.3)$$

*Proof.* For brevity we write $W(\ell)$ for $W_t(t_1, t_2, \ell, \ell_{bot}; S)$. We have that

$$c(t) \cdot \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W(\ell) | \ell(T) \in B\right] = c(t) \cdot \sum_{\beta \in B} \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W(\ell) | \ell(T) = \beta\right] \cdot \frac{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) = \beta)}{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) \in B)} =$$

$$c(t) \cdot \sum_{\beta \in B} \sum_{\alpha \in A} \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W(\ell) | \ell(T) = \beta\right] \cdot \frac{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) = \alpha)}{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) \in A)} \cdot \frac{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) = \beta)}{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) \in B)} \leq$$

$$\sum_{\beta \in B} \sum_{\alpha \in A} \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W(\ell) | \ell(T) = \alpha\right] \cdot \frac{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) = \alpha)}{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) \in A)} \cdot \frac{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) = \beta)}{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) \in B)} =$$

$$\sum_{\alpha \in A} \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W(\ell) | \ell(T) = \alpha\right] \cdot \frac{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) = \alpha)}{\mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) \in A)} = \mathbb{E}_{free}^{t_1,t_2;a,b}\left[W(\ell) | \ell(T) \in A\right].$$

The middle inequality follows from Lemma 4.4.1. ☐

**Corollary 4.4.4.** *Assume the same notation as in Lemma 4.4.1 and let $\alpha \leq M(T)$. Denote by $\mathbb{P}$ the probability distribution $\mathbb{P}_S^{t_1,t_2,a,b}(\cdot|\ell_{bot})$ from Definition 4.3.4. Then we have*

$$\mathbb{P}(\ell(T) \geq \alpha) \geq c(t) \cdot \mathbb{P}_{free}^{t_1,t_2;a,b}(\ell(T) \geq \alpha). \quad (4.4.4)$$

*Proof.* If $\alpha \leq m(T)$ then (4.4.4) becomes $1 \geq c(t)$, which is clearly true. We thus may assume that $M(T) \geq \alpha > m(T)$. Let $A = [\alpha, M(T)]$ and $B = [m(T), \alpha)$. Define $D_1 := \{\ell \in$

$\Omega(t_1, t_2; a, b) : \ell(T) \in A\}$ and $D_2 := \{\ell \in \Omega(t_1, t_2; a, b) : \ell(T) \in B\}$. Observe that $A$ and $B$ satisfy the conditions of Corollary 4.4.3 and hence

$$\sum_{\ell \in D_1} W_t(t_1, t_2, \ell, \ell_{bot}; S) \geq c(t) \cdot \frac{|D_1|}{|D_2|} \sum_{\ell \in D_2} W_t(t_1, t_2, \ell, \ell_{bot}; S).$$

Dividing both sides by $\sum_{\ell \in \Omega(t_1, t_2; a, b)} W_t(t_1, t_2, \ell, \ell_{bot}; S)$ we see that

$$\mathbb{P}(\ell(T) \geq \alpha) \geq c(t) \cdot \frac{|D_1|}{|D_2|}(1 - \mathbb{P}(\ell(T) \geq \alpha)) \text{ or equivalently } \mathbb{P}(\ell(T) \geq \alpha) \geq c(t) \cdot \frac{|D_1|}{|D_2| + c(t)|D_1|}.$$

Since $c(t) \in (0,1)$ we can increase the denominator by replacing it with $|D_1| + |D_2|$, which makes the RHS above precisely $c(t) \cdot \mathbb{P}_{free}^{t_1, t_2; a, b}(\ell(T) \geq \alpha)$ as desired. □

### 4.4.2 Properties of random paths

In this section we derive several lemmas about random paths distributed as $\mathbb{P}_{free}^{0,n;0,z}$ for $z \in \{0, ..., n\}$, which are essential for the proof of our main results. Recall that if $L$ is such a path, we define $L(s)$ for non-integral $s$ by linear interpolation (see Section 4.3.1). The key ingredient we use to derive the lemmas below is a strong coupling between random walk bridges and Brownian bridges, which is presented as Theorem 4.4.5 below.

If $W_t$ denotes a standard one-dimensional Brownian motion and $\sigma > 0$, then the process

$$B_t^\sigma = \sigma^2 (W_t - tW_1), \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (conditioned on $B_0 = 0, B_1 = 0$) with variance $\sigma^2$*. With this notation we state the main result we use and defer its proof to Section 4.8.

**Theorem 4.4.5.** *Let $p \in (0,1)$. There exist constants $0 < C, a, \alpha < \infty$ (depending on $p$) such that for every positive integer $n$, there is a probability space on which are defined a Brownian bridge $B^\sigma$ with variance $\sigma^2 = p(1-p)$ and a family of random paths $\ell^{(n,z)} \in \Omega(0, n; 0, z)$ for $z = 0, ..., n$ such that $\ell^{(n,z)}$ has law $\mathbb{P}_{free}^{0,n;0,z}$ and*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq Ce^{\alpha(\log n)^2} e^{|z-pn|^2/n}, \text{ where } \Delta(n,z) := \sup_{0 \leq t \leq n} \left| \sqrt{n} B_{t/n}^\sigma + \tfrac{t}{n} z - \ell^{(n,z)}(t) \right|. \tag{4.4.5}$$

*Remark* 4.4.6. When $p = 1/2$ the above theorem follows (after a trivial affine shift) from Theorem 6.3 in [60]. The proof we present in Section 4.8 for the more general $p \in (0,1)$ case is based on (suitably adapted) arguments from the same paper.

We will also need the following monotone coupling lemma for random walks, which can readily be established from the arguments used in the proof of Lemma 2.6 in [42].

**Lemma 4.4.7.** *Suppose $a_1, b_1, a_2, b_2, t_1, t_2$ are given such that $t_1 < t_2$, $0 \leq b_2 - a_2 \leq t_2 - t_1$, $0 \leq b_1 - a_1 \leq t_2 - t_1$, $a_1 \leq a_2$, $b_1 \leq b_2$. Then there exists a probability space on which are defined random paths $\ell_1$ and $\ell_2$ such that the law of $\ell_i$ is $\mathbb{P}_{free}^{t_1, t_2, a_i, b_i}$ for $i = 1, 2$ and $\mathbb{P}(\ell_1(s) \leq \ell_2(s),\ for\ s = t_1, ..., t_2) = 1$.*

Using facts about Brownian motion and the above coupling results we establish the following statements for random paths.

**Lemma 4.4.8.** *Let $M > 0$ and $p \in (0,1)$ be given. Then we can find $N_0(M,p)$ such that for $N \geq N_0$, $N \geq z \geq pN + MN^{1/2}$ and $s \in [0,N]$ we have*

$$\mathbb{P}^{0,N;0,z}_{free}\left(\ell(s) \geq \frac{s}{N}(pN + MN^{1/2}) - N^{1/4}\right) \geq 1/3. \tag{4.4.6}$$

*Proof.* Assume that $N_0 \geq 2M^2$ and $N \geq N_0$. In view of Lemma 4.4.7, we know that

$$\mathbb{P}^{0,N;0,z_2}_{free}\left(\ell(s) \geq \frac{s}{N}(pN + MN^{1/2}) - N^{1/4}\right) \geq \mathbb{P}^{0,N;0,z_1}_{free}\left(\ell(s) \geq \frac{s}{N}(pN + MN^{1/2}) - N^{1/4}\right),$$

whenever $z_2 \geq z_1$ and so it suffices to prove the lemma when $z = \lceil pN + MN^{1/2}\rceil$. Suppose we have the same coupling as in Theorem 4.4.5 and let $\mathbb{P}$ denote the probability measure on the space afforded by the theorem. Then we have for $\sigma^2 = p(1-p)$ that

$$\mathbb{P}^{0,N;0,z}_{free}\left(\ell(s) \geq \frac{s}{N}(pN + MN^{1/2}) - N^{1/4}\right) = \mathbb{P}\left(\ell^{(N,z)}(s) \geq \frac{s}{N}(pN + MN^{1/2}) - N^{1/4}\right) \geq$$

$$\geq \mathbb{P}\left(N^{1/2}B^{\sigma}_{s/N} \geq 0 \text{ and } \Delta(N,z) \leq (N^{1/4}-1)\right) \geq 1/2 - \mathbb{P}\left(\Delta(N,z) > N^{1/4}-1\right).$$

In the next to last inequality we used that $|z - (pN + MN^{1/2})| \leq 1$ and in last inequality we used that $\mathbb{P}(B^v_{s/N} \geq 0) = 1/2$ for every $v > 0$ and $s \in [0,N]$. Next by Theorem 4.4.5 and Chebyshev's inequality we know

$$\mathbb{P}\left(\Delta(N,z) > N^{1/4}-1\right) \leq Ce^{\alpha(\log N)^2}e^{M^2}e^{-aN^{1/4}}.$$

The latter is at most $1/6$ if we take $N_0$ sufficiently large and $N \geq N_0$, which would imply that $\mathbb{P}^{0,N;0,z}_{free}\left(\ell(s) \geq (s/N)(pN + MN^{1/2}) - N^{1/4}\right) \geq 1/3$ for such $N$, as desired. ☐

**Lemma 4.4.9.** *Let $M_1, M_2 > 0$ and $p \in (0,1)$ be given. Then we can find $N_0(M_1,M_2,p)$ such that for $N \geq N_0$, $z_1 \geq -M_1N^{1/2}$, $z_2 \geq pN - M_1N^{1/2}$ we have*

$$\mathbb{P}^{0,N;z_1,z_2}_{free}\left(\ell(N/2) \geq \frac{M_2N^{1/2} + pN}{2} - N^{1/4}\right) \geq (1/2)(1 - \Phi^{p(1-p)/2}(M_1 + M_2)), \tag{4.4.7}$$

*where $\Phi^v$ is the cdf of a Gaussian random variable with mean $0$ and variance $v$.*

*Proof.* Assume that $N_0 \geq 2(M_1+M_2)^2$ and $N \geq N_0$. In view of Lemma 4.4.7 it suffices to prove the lemma when $z_1 = \lceil -M_1N^{1/2}\rceil$ and $z_2 = \lceil pN - M_1N^{1/2}\rceil$. Set $\Delta z = z_2 - z_1$ and observe that

$$\mathbb{P}^{0,N;z_1,z_2}_{free}\left(\ell(N/2) \geq \frac{M_2N^{1/2} + pN}{2} - N^{1/4}\right) = \mathbb{P}^{0,N;0,\Delta z}_{free}\left(\ell(N/2) \geq \frac{M_2N^{1/2} + pN}{2} - z_1 - N^{1/4}\right).$$

Suppose we have the same coupling as in Theorem 4.4.5 and let $\mathbb{P}$ denote the probability measure on the space afforded by the theorem. Then we have

$$\mathbb{P}_{free}^{0,N;0,\Delta z}\left(\ell(N/2) \geq \frac{M_2N^{1/2}+pN}{2} - z_1 - N^{1/4}\right) = \mathbb{P}\left(\ell^{(N,\Delta z)}(N/2) \geq \frac{M_2N^{1/2}+pN}{2} - z_1 - N^{1/4}\right)$$

$$\geq \mathbb{P}\left(\ell^{(N,\Delta z)}(N/2) \geq \frac{(2M_1+M_2)N^{1/2}+\Delta z}{2} - N^{1/4} + 2\right),$$

where we used that $|z_1 + M_1N^{1/2}| \leq 1$ and $|z_2 + M_1N^{1/2} - pN| \leq 1$. We now note that the expression in the second line above is bounded from below by

$$\mathbb{P}\left(B^{\sigma}_{1/2} \geq \frac{M_2+2M_1}{2} \text{ and } \Delta(N,z) \leq N^{1/4} - 2\right), \text{ where } \sigma^2 = p(1-p).$$

Since $B^{\sigma}_{1/2}$ has the distribution of a normal random variable with mean 0 and variance $\sigma^2/2$, and $\Phi^v$ is decreasing on $\mathbb{R}_{>0}$ we conclude that the last expression is bounded from below by

$$1-\Phi^{p(1-p)/2}(M_1+M_2) - \mathbb{P}\left(\Delta(N,z) > N^{1/4}-2\right) \geq 1-\Phi^{p(1-p)/2}(M_1+M_2) - Ce^{\alpha(\log N)^2}e^{M^2}e^{-aN^{1/4}}.$$

In the last inequality we used Theorem 4.4.5 and Chebyshev's inequality. The above is at least $(1/2)(1-\Phi^{p(1-p)/2}(M_1+M_2))$ if $N_0$ is taken sufficiently large and $N \geq N_0$. □

**Lemma 4.4.10.** *Let $p \in (0,1)$ be given. Then we can find $N_0(p)$ such that for $N \geq N_0$, $z_1 \geq N^{1/2}$, $z_2 \geq pN + N^{1/2}$ we have*

$$\mathbb{P}_{free}^{0,N;z_1,z_2}\left(\min_{s\in[0,N]}[\ell(s) - ps] + N^{1/4} \geq 0\right) \geq \frac{1}{2}\left(1 - \exp\left(\frac{-2}{p(1-p)}\right)\right). \tag{4.4.8}$$

*Proof.* In view of Lemma 4.4.7 it suffices to prove the lemma when $z_1 = \lceil N^{1/2}\rceil$ and $z_2 = \lceil pN + N^{1/2}\rceil$. Set $\Delta z = z_2 - z_1$ and observe that

$$\mathbb{P}_{free}^{0,N;z_1,z_2}\left(\min_{s\in[0,N]}[\ell(s) - ps] + N^{1/4} \geq 0\right) = \mathbb{P}_{free}^{0,N;0,\Delta z}\left(\min_{s\in[0,N]}[\ell(s) - ps] + N^{1/4} \geq -z_1\right).$$

Suppose we have the same coupling as in Theorem 4.4.5 and let $\mathbb{P}$ denote the probability measure on the space afforded by the theorem. Then we have

$$\mathbb{P}_{free}^{0,N;0,\Delta z}\left(\min_{s\in[0,N]}[\ell(s) - ps] + N^{1/4} \geq -z_1\right) = \mathbb{P}\left(\min_{s\in[0,N]}\left[\ell^{(N,\Delta z)}(s) - ps\right] \geq -N^{1/4} - z_1\right) \geq$$

$$\mathbb{P}\left(\min_{s\in[0,N]}\left[\ell^{(N,\Delta z)}(s) - \frac{s}{N}\Delta z\right] \geq -N^{1/4} - N^{1/2} + 2\right),$$

where in the last inequality we used that $|z_1 - N^{1/2}| \leq 1$ and $|z_2 - pN - N^{1/2}| \leq 1$. We now note that the expression in the second line above is bounded from below by

$$\mathbb{P}\left(\min_{s\in[0,1]} B^{\sigma}_s \geq -1 \text{ and } \Delta(N,z) \leq N^{1/4} - 2\right), \text{ where } \sigma^2 = p(1-p).$$

We can lower-bound the above expression by $\mathbb{P}\left(\min_{s\in[0,1]} B_s^\sigma \geq -1\right) - \mathbb{P}\left(\Delta(N,z) \leq N^{1/4} - 2\right)$. By basic properties of Brownian bridges we know that

$$\mathbb{P}\left(\min_{s\in[0,1]} B_s^\sigma \geq -1\right) = \mathbb{P}\left(\min_{s\in[0,1]} B_s^1 \geq -\sigma^{-1}\right) = \mathbb{P}\left(\max_{s\in[0,1]} B_s^1 \leq \sigma^{-1}\right) = 1 - e^{-2\sigma^{-2}},$$

where the last equality can be found for example in (3.40) of Chapter 4 of [56]. On the other hand, by Theorem 4.4.5 and Chebyshev's inequality we have

$$\mathbb{P}\left(\Delta(N,z) > N^{1/4} - 2\right) \leq C e^{\alpha(\log N)^2} e^{M^2} e^{-aN^{1/4}},$$

and the latter is at most $(1/2)(1 - e^{-2\sigma^{-2}})$ if $N_0$ is taken sufficiently large and $N \geq N_0$. Combining the above estimates we conclude that if $N_0$ is sufficiently large and $N \geq N_0$, we have $\mathbb{P}_{free}^{0,N;z_1,z_2}\left(\min_{s\in[0,N]}[\ell(s) - ps] + N^{1/4} \geq 0\right) \geq (1/2)(1 - e^{-2\sigma^{-2}})$ as desired.

□

### 4.4.3 Modulus of continuity for random paths

For a function $f \in C[a,b]$ we define the *modulus of continuity* by

$$w(f,\delta) = \sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}} |f(x) - f(y)|. \tag{4.4.9}$$

In this section we derive estimates on the modulus of continuity of paths distributed according to $\mathbb{P}_{free}^{0,n;0,z}$ for $z \in \{0, ..., n\}$, which are essential for the proof of Theorem 4.3.8. Recall that if $L$ is such a path, we define $L(s)$ for non-integral $s$ by linear interpolation (see Section 4.3.1). The main result we want to show is as follows.

**Lemma 4.4.11.** *Let $M > 0$ and $p \in (0,1)$ be given. For each positive $\epsilon$ and $\eta$, there exist a $\delta > 0$ and an $N_0 \in \mathbb{N}$ (depending on $\epsilon, \eta, M$ and $p$) such that for $N \geq N_0$ we have*

$$\mathbb{P}_{free}^{0,N;0,z}\left(w(f^\ell,\delta) \geq \epsilon\right) \leq \eta, \tag{4.4.10}$$

*where $f^\ell(x) = N^{-1/2}(\ell(xN) - pxN)$ for $x \in [0,1]$ and $|z - pN| \leq MN^{1/2}$.*

*Proof.* The strategy is to use the strong coupling between $\ell$ and a Brownian bridge afforded by Theorem 4.4.5. This will allow us to argue that with high probability the modulus of continuity of $f^\ell$ is close to that of a Brownian bridge, and since the latter is continuous a.s., this will lead to the desired statement of the lemma. We now turn to providing the necessary details.

Let $\epsilon, \eta > 0$ be given and fix $\delta \in (0,1)$, which will be determined later. Suppose we have the same coupling as in Theorem 4.4.5 and let $\mathbb{P}$ denote the probability measure on the space afforded by the theorem. Then we have

$$\mathbb{P}_{free}^{0,N;0,z}\left(w(f^\ell,\delta) \geq \epsilon\right) = \mathbb{P}\left(w(f^{\ell^{(N,z)}},\delta) \geq \epsilon\right). \tag{4.4.11}$$

By definition, we have

$$w(f^{\ell^{(N,z)}}, \delta) = N^{-1/2} \sup_{\substack{x,y\in[0,1]\\ |x-y|\leq\delta}} \left|\ell^{(N,z)}(xN) - pxN - \ell^{(N,z)}(yN) + pyN\right|.$$

From Theorem 4.4.5 and the above we conclude that for $\sigma^2 = p(1-p)$ we have

$$w(f^{\ell^{(N,z)}}, \delta) \leq N^{-1/2} \sup_{\substack{x,y\in[0,1]\\ |x-y|\leq\delta}} \left|N^{1/2}B^\sigma_x - N^{1/2}B^\sigma_y + (z-pN)(x-y)\right| + 2N^{-1/2}\Delta(N,z). \tag{4.4.12}$$

From (4.4.11), (4.4.12), the triangle inequality and our assumption that $|z - pN| \leq MN^{1/2}$ we see that

$$\mathbb{P}^{0,N;0,z}_{free}\left(w(f^\ell,\delta) \geq \epsilon\right) \leq \mathbb{P}\left(w(B^\sigma,\delta) + \delta M + 2N^{-1/2}\Delta(N,z) \geq \epsilon\right). \tag{4.4.13}$$

Let $(I) = \mathbb{P}\left(w(B^\sigma,\delta) \geq \epsilon/3\right)$, $(II) = \mathbb{P}\left(\delta M \geq \epsilon/3\right)$ and $(III) = \mathbb{P}\left(2N^{-1/2}\Delta(N,z) \geq \epsilon/3\right)$, then we have

$$\mathbb{P}\left(w(B^\sigma,\delta) + \delta M + 2N^{-1/2}\Delta(N,z) \geq \epsilon\right) \leq (I) + (II) + (III).$$

By Theorem 4.4.5 and Chebyshev's inequality we know

$$\mathbb{P}\left(\Delta(N,z) > N^{1/4}\right) \leq Ce^{\alpha(\log N)^2}e^{M^2}e^{-aN^{1/4}}.$$

Consequently, if we pick $N_0$ sufficiently large and $N \geq N_0$ we can ensure that $2N^{-1/4} < \epsilon/3$ and $Ce^{\alpha(\log N)^2}e^{M^2}e^{-aN^{1/4}} < \eta/3$, which would imply $(III) \leq \eta/3$.

Since $B^\sigma$ is a.s. continuous we know that $w(B^\sigma,\delta)$ goes to 0 as $\delta$ goes to 0, hence we can find $\delta_0$ sufficiently small so that if $\delta < \delta_0$, we have $(I) < \eta/3$. Finally, if $\delta M < \epsilon/3$ then $(II) = 0$. Combining all the above estimates with (4.4.13) we see that for $\delta$ sufficiently small, $N_0$ sufficiently large and $N \geq N_0$, we have $\mathbb{P}^{0,N;0,z}_{free}\left(w_{f^\ell}(\delta) \geq \epsilon\right) \leq (2/3)\eta < \eta$ as desired. □

## 4.5 Proof of Theorem 4.3.8

The goal of this section is to prove Theorem 4.3.8 and for the remainder we assume that $\mathfrak{L}^N = (L_1^N, L_2^N)$ is an $(\alpha, p, r+1)$-good sequence for some $r > 0$, defined on a probability space with measure $\mathbb{P}$. The main technical result we will require is contained in Proposition 4.5.1 below and its proof is the content of Section 4.5.1. The proof of Theorem 4.3.8 is given in Section 4.5.2 and relies on Proposition 4.5.1 and Lemma 4.4.11.

### 4.5.1 Bounds on acceptance probabilities

The main result in this section is the following.

**Proposition 4.5.1.** *Fix $r > 0$ and denote $s_1 = \lfloor rN^\alpha \rfloor$. Then for any $\epsilon > 0$ there exist $\delta > 0$*

*and $N_1$ (both depending on $r, \epsilon, t, \alpha, p$) such that for all $N \geq N_1$ we have*

$$\mathbb{P}\left(Z_t(-s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2; S') < \delta\right) < \epsilon,$$

*where $S' = [\![-s_1+1, s_1]\!]$ and $Z_t$ is the acceptance probability defined in Definition 4.3.4 (see also Remark 4.3.5).*

The general strategy we use to prove Proposition 4.5.1 is inspired by the proof of Proposition 6.5 in [43]. We begin by stating three key lemmas that will be required. Their proofs are postponed to Section 4.6. All constants in the statements below will, in addition, depend on $\alpha, t$ and $p$, which are fixed throughout. We will not list this dependence explicitly.

**Lemma 4.5.2.** *For each $\epsilon > 0$ there exist $R(r, \epsilon) > 0$ and $N_1(r, \epsilon)$ such that for all $N \geq N_1$ we have*

$$\mathbb{P}\left(\sup_{s \in [-(r+1)N^\alpha, (r+1)N^\alpha]} \left[L_1^N(s) - ps\right] \geq RN^{\alpha/2}\right) < \epsilon.$$

Set $s_1 = \lfloor rN^\alpha \rfloor$ and $t_1 = \lfloor (r+1)N^\alpha \rfloor$ and assume $a, b, z_1, z_2, t_1$ satisfy, $0 \leq z_2 - z_1 \leq 2t_1$, $0 \leq b - a \leq 2t_1$, $z_1 \leq a$, $z_2 \leq b$. Let $\ell_{bot}$ be a fixed path in $\Omega(-t_1, t_1; z_1, z_2)$ and denote $S = [\![-t_1+1, t_1]\!]$, $\tilde{S} = [\![-t_1+1, -s_1]\!] \cup [\![s_1+1, t_1]\!]$. Let $L$ and $\tilde{L}$ be two random paths in $\Omega(-t_1, t_1; a, b)$ , with laws $\mathbb{P}_L$ and $\mathbb{P}_{\tilde{L}}$ respectively such that

$$\mathbb{P}_L(L = \ell) = \mathbb{P}_S^{-t_1, t_1, a, b}(\ell | \ell_{bot}) \text{ and } \mathbb{P}_{\tilde{L}}(\tilde{L} = \ell) = \mathbb{P}_{\tilde{S}}^{-t_1, t_1, a, b}(\ell | \ell_{bot}).$$

where the definition of $\mathbb{P}_S^{T_0, T_1, a, b}(\cdot | \ell_{bot})$ was given in Definition 4.3.4. From (4.3.1) we know that $L$ will not cross $\ell_{bot}$ with probability 1. On the other hand, $\tilde{L}$ can cross $\ell_{bot}$ multiple times in the interval $(-s_1, s_1+1)$ but it will stay above it on $[-t_1, -s_1] \cup [s_1+1, t_1]$.

**Lemma 4.5.3.** *Fix $M_1, M_2 > 0$, $S' = [\![-s_1+1, s_1]\!]$ and suppose*

1. $\sup_{s \in [-t_1, t_1]} \left[\ell_{bot}(s) - ps\right] \leq M_2 N^{\alpha/2}$,
2. $a \geq \max(\ell_{bot}(-t_1), -pt_1 - M_1 N^{\alpha/2})$,
3. $b \geq \max(\ell_{bot}(t_1), pt_1 - M_1 N^{\alpha/2})$.

*There exists $N_2 \in \mathbb{N}$ and explicit functions $g$ and $h$ (depending on $r, M_1, M_2$) such that for $N \geq N_2$*

$$\mathbb{P}_{\tilde{L}}\left(Z_t\left(-s_1, s_1, \tilde{L}(-s_1), \tilde{L}(s_1), \ell_{bot}; S'\right) \geq g\right) \geq h. \tag{4.5.1}$$

*The functions $g$ and $h$ are given by*

$$g = \frac{1}{4}\left(1 - \exp\left(\frac{-2}{p(1-p)}\right)\right) \text{ and } h = (c(t)^3/18)(1 - \Phi^{p(1-p)/2}\left(10(1+r)^2(M_1+M_2+1)\right),$$

*where $c(t) = \prod_{i=1}^\infty (1 - t^i)$ and $\Phi^v$ is the cdf of a Gaussian random variable with mean zero and variance $v$.*

**Lemma 4.5.4.** *Fix $M_1, M_2 > 0$, $S' = [\![-s_1+1, s_1]\!]$ and suppose*

*1.* $\sup_{s\in[-t_1,t_1]}\left[\ell_{bot}(s) - ps\right] \le M_2 N^{\alpha/2}$,

*2.* $a \ge \max(\ell_{bot}(-t_1), -pt_1 - M_1 N^{\alpha/2})$,

*3.* $b \ge \max(\ell_{bot}(t_1), pt_1 - M_1 N^{\alpha/2})$.

*Let $N_2, g, h$ be as in Lemma 4.5.3 and for any $\tilde{\epsilon} > 0$ set $\delta(\tilde{\epsilon}) = \tilde{\epsilon} \cdot g \cdot h$. Then for $N \ge N_2$ we have*

$$\mathbb{P}_L\left(Z_t\left(-s_1, s_1, L(-s_1), L(s_1), \ell_{bot}; S'\right) \le \delta(\tilde{\epsilon})\right) \le \tilde{\epsilon}. \tag{4.5.2}$$

In the remainder we prove Proposition 4.5.1 assuming the validity of Lemmas 4.5.2 and 4.5.4. The arguments we present are similar to those used in the proof of Proposition 6.5 in [43].

*Proof.* (Proposition 4.5.1) Define the event

$$E_N = \left\{L_1^N(-t_1) \ge \max(L_2^N(-t_1), -pt_1 - M_1 N^{\alpha/2})\right\} \cap$$
$$\left\{L_1^N(t_1) \ge \max(L_2^N(t_1), pt_1 - M_1 N^{\alpha/2})\right\} \cap \left\{\sup_{s\in[-t_1,t_1]}\left[L_2^N(s) - ps\right] \le M_2 N^{\alpha/2}\right\},$$

where $M_1$ and $M_2$ are sufficiently large so that for all large $N$ we have $\mathbb{P}(E_N^c) < \epsilon/2$. The existence of such $M_1$ and $M_2$ is assured from Lemma 4.5.2 (since $L_1^N$ dominates $L_2^N$ pointwise) and the fact that $\mathfrak{L}^N$ is $(\alpha, p, r+1)$ - good.

Let $\delta(\tilde{\epsilon})$ be as in Lemma 4.5.4 for the values $\tilde{\epsilon} = \epsilon/2$, $r, M_1, M_2$ in the statement of the lemma. Consider the probability

$$\mathbb{P}\left(\left\{Z_t(-s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2^N; S') < \delta(\tilde{\epsilon})\right\} \cap E_N\right) =$$
$$= \mathbb{E}\left[\mathbf{1}_{E_N}\mathbb{E}\left[\mathbf{1}\{Z_t(-s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2^N; S') < \delta(\tilde{\epsilon})\}\middle|\mathcal{F}_{ext}\left(\{1\}\times(-t_1,t_1)\right)\right]\right]. \tag{4.5.3}$$

In the above equation we have $\mathcal{F}_{ext}\left(\{1\}\times(-t_1,t_1)\right)$ is the $\sigma$-algebra generated by the up-right paths $L_2^N$ and $L_1^N$ outside the interval $(-t_1, t_1)$. The equality in (4.5.3) is justified by the tower property since $E_N$ is measurable with respect to $\mathcal{F}_{ext}\left(\{1\}\times(-t_1,t_1)\right)$ . We next notice that we have the following a.s. equality of $\mathcal{F}_{ext}\left(\{1\}\times(-t_1,t_1)\right)$-measurable random variables

$$\mathbb{E}\left[\mathbf{1}\{Z_t(-s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2^N; S') < \delta(\tilde{\epsilon})\}\middle|\mathcal{F}_{ext}\left(\{1\}\times(-t_1,t_1)\right)\right] =$$
$$\mathbb{P}_L\left(Z_t(-s_1, s_1, L(-s_1), L(s_1), L_2^N; S') < \delta(\tilde{\epsilon})\right),$$

where $\mathbb{P}_L$ is specified as in the setup after Lemma 4.5.2 with respect to $a = L_1^N(-t_1)$, $b = L_1^N(t_1)$, $\ell_{bot} = L_2^N$ on $[-t_1, t_1]$.

When the $\mathcal{F}_{ext}\left(\{1\}\times(-t_1,t_1)\right)$-measurable event $E_N$ holds we have that $\sup_{s\in[-t_1,t_1]}$ $\left[\ell_{bot}(s) - ps\right] \le M_2 N^{\alpha/2}$ and $a \ge \max(\ell_{bot}(-t_1), -pt_1 - M_1 N^{\alpha/2})$, $b \ge \max(\ell_{bot}(t_1), pt_1 - M_1 N^{\alpha/2})$ (recall that $\mathfrak{L}^N$ is a simple discrete line ensemble by definition so that $L_1^N$ lies above $L_2^N$). Thus we may apply Lemma 4.5.4 on $E_N$ and obtain that

$$\mathbb{P}_L\left(Z_t(S', -s_1, s_1, L(-s_1), L(s_2), L_2^N) < \delta(\tilde{\epsilon})\right) \leq \tilde{\epsilon}\mathbf{1}_{E_N} + \mathbf{1}_{E_N^c},$$

where the inequality is understood in the a.s. sense. Putting this into (4.5.3) we conclude that

$$\mathbb{P}\left(\{Z_t(S', -s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2^N) < \delta(\epsilon/2)\} \cap E_N\right) \leq \epsilon/2.$$

Using this and $\mathbb{P}(E_N^c) < \epsilon/2$, we see that for all large $N$ we have

$$\mathbb{P}\left(Z_t(S', -s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2^N) < \delta(\epsilon/2)\right) < \epsilon.$$

□

### 4.5.2 Concluding the proof of Theorem 4.3.8

For clarity we split the proof of Theorem 4.3.8 into several steps. In the first two steps we reduce the statement of the theorem to establishing a certain estimate on the modulus of continuity of the paths $L_1^N$. In the next two steps we show that it is enough to establish these estimates under the additional assumption that $(L_1^N, L_2^N)$ are well-behaved (in particular, well-behaved implies that the acceptance probability $Z_t(-s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2; S')$ is lower bounded and it is here that we use Proposition 4.5.1). The fact that the acceptance probability is lower bounded is exploited in Step 5, together with the resampling property of Remark 4.3.5, to effectively reduce the estimates on the modulus of continuity of $L_1^N$ to those of a uniform random path. The latter estimates are then derived in Step 6, by appealing to Lemma 4.4.11.

**Step 1.** Recall from (4.4.9) that the modulus of continuity of $f \in C[-r, r]$ is defined by

$$w(f, \delta) = \sup_{\substack{x,y \in [-r,r] \\ |x-y| \leq \delta}} |f(x) - f(y)|.$$

As an immediate generalization of Theorem 7.3 in [13], in order to prove the theorem it suffices for us to show that the sequence of random variables $f_N(0)$ is tight and that for each positive $\epsilon$ and $\eta$ there exist $\delta' > 0$ and $N_1 \in \mathbb{N}$ such that for $N \geq N_1$, we have

$$\mathbb{P}(w(f_N, \delta') \geq \epsilon) \leq \eta. \tag{4.5.4}$$

The tightness of $f_N(0)$ is immediate from our assumption that $\{\mathfrak{L}^N\}_{N=1}^{\infty}$ is an $(\alpha, p, r+1)$-good sequence (in fact we know from Definition 4.3.7 that $f_N(s)$ is tight for each $s \in [-r-1, r+1]$). Consequently, we are left with verifying (4.5.4).

**Step 2.** Suppose $\epsilon, \eta > 0$ are given and also denote $s_1 = \lfloor rN^\alpha \rfloor$. We claim that we can find $\delta > 0$ such that for all $N$ sufficiently large we have

$$\mathbb{P}\left( \sup_{\substack{x,y\in[-s_1,s_1] \\ |x-y|\leq 2\delta s_1}} \left|L_1^N(x) - L_1^N(y) - p(x-y)\right| \geq \frac{\epsilon(2s_1)^{1/2}}{2(2r)^{1/2}} \right) \leq \eta. \tag{4.5.5}$$

Let us assume the validity of (4.5.5) and deduce (4.5.4).

Let $\delta' = r\delta$. Suppose that $x, y \in [-r, r]$ are such that $|x - y| \leq \delta'$ and without loss of generality assume that $x < y$. Let $X = \lceil xN^\alpha \rceil$ and $Y = \lfloor yN^\alpha \rfloor$. One readily observes that if $N$ is sufficiently large then $|X - Y| \leq 2\delta s_1$, and $X, Y \in [-s_1, s_1]$. In addition, we have that

$$|f_N(x) - f_N(y)| = N^{-\alpha/2} \left|L_1^N(xN^\alpha) - L_1^N(yN^\alpha) - pN^\alpha(x-y)\right| \leq$$

$$N^{-\alpha/2} \left|L_1^N(X) - L_1^N(Y) - p(X-Y)\right| + 2N^{-\alpha/2}(1+p),$$

where we used that $|X - xN^\alpha| < 1$, $|Y - yN^\alpha| < 1$, the slope of $L_1$ is in absolute value at most 1, and the triangle inequality. The above inequality shows that for all $N$ sufficiently large we have

$$\mathbb{P}\left(w(f_N, \delta') \geq \epsilon\right) \leq \mathbb{P}\left( \sup_{\substack{x,y\in[-s_1,s_1] \\ |x-y|\leq 2\delta s_1}} \left|L_1^N(x) - L_1^N(y) - p(x-y)\right| \geq \epsilon N^{\alpha/2} - 2(1+p) \right).$$

Since $s_1 = \lfloor rN^\alpha \rfloor$ we see that $\frac{\epsilon(2s_1)^{1/2}}{2(2r)^{1/2}} \sim (\epsilon/2)N^{\alpha/2}$ as $N$ becomes large and so we conclude that for all sufficiently large $N$ we have $\frac{\epsilon(2s_1)^{1/2}}{2(2r)^{1/2}} \leq \epsilon N^{\alpha/2} - 2(1+p)$. This together with (4.5.5) implies that the RHS in the last equation is bounded from above by $\eta$, which is what we wanted.

**Step 3.** The first two steps above reduce the proof of the theorem to establishing (4.5.5), which is the core statement we want to show. In order to prove it we will need additional notation that we summarize in this step.

From the tightness of $N^{-\alpha/2}\left[L_1^N(xN^\alpha) - xpN^\alpha\right]$ at $x = -r$ and $x = r$ we can find $M_1 > 0$ sufficiently large so that for all large $N$ we have

$$\mathbb{P}\big((E_1(M_1, N)\big) \geq 1 - \eta/4, \text{ where } E_1(M_1, N) = \left\{\max\left(\left|L_1^N(-s_1) + ps_1\right|, \left|L_1^N(s_1) - ps_1\right|\right) \leq M_1 N^{\alpha/2}\right\}.$$

In addition, we know from Proposition 4.5.1 that we can find $\delta_1 > 0$ such that for all sufficiently large $N$ we have

$$\mathbb{P}\big(E_2(\delta_1, N)\big) \geq 1 - \eta/4, \text{ where } E_2(\delta_1, N) = \left\{Z_t(-s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2; S') > \delta_1\right\}.$$

Suppose $a, b, z_1, z_2$ are given such that $0 \leq z_2 - z_1 \leq 2s_1$, $0 \leq b - a \leq 2s_1$, $z_1 \leq a$, $z_2 \leq b$.

For a given $\ell_{bot} \in \Omega(-s_1, s_1; z_1, z_2)$, we let

$$E(a, b, \ell_{bot}, N) := \left\{ (L_1^N, L_2^N) : L_2^N = \ell_{bot} \text{ on } [-s_1, s_1],\ L_1^N(-s_1) = a \text{ and } L_1^N(s_1) = b \right\}.$$

Observe that $E_1(M_1, N) \cap E_2(\delta_1, N)$ can be written as a *countable disjoint* union of sets of the form $E(a, b, \ell_{bot}, N)$, where the triple $(a, b, \ell_{bot})$ satisfies:

1. $0 \leq b - a \leq 2s_1$, $|a + ps_1| \leq M_1 N^{\alpha/2}$ and $|b - ps_1| \leq M_1 N^{\alpha}$,
2. $z_1 \leq a$, $z_2 \leq b$ and $\ell_{bot} \in \Omega(-s_1, s_1, z_1, z_2)$,
3. $Z_t(S', -s_1, s_1, a, b, \ell_{bot}) > \delta_1$.

Clearly, there are only finitely many choices for $a, b$ that satisfy the conditions above. Then the number of $z_1, z_2$ for each given pair $(a, b)$ is countable, while the cardinality of $\Omega(-s_1, s_1, z_1, z_2)$ is finite. This means that the number of triplets $(a, b, \ell_{bot})$ is indeed countable. The fact that $E(a, b, \ell_{bot}, N)$ are disjoint is again clear, while the first and third condition above show that their union is indeed $E_1(M_1, N) \cap E_2(\delta_1, N)$. Let us denote by $F(\delta_1, M_1, s_1, N)$ the set of triplets $(a, b, \ell_{bot})$ that satisfy the three conditions above.

**Step 4.** Let us write $L_1^N([-s_1, s_1])$ as the restriction of $L_1^N$ to $[-s_1, s_1]$. For $\delta > 0$ and $\ell \in \Omega(-s_1, s_1; a, b)$ we define

$$V(\delta, \ell) = \sup_{\substack{x, y \in [-s_1, s_1] \\ |x - y| \leq 2\delta s_1}} |\ell(x) - \ell(y) - p(x - y)|.$$

We assert that we can find $\delta > 0$ such that for all large $N$ and $(a, b, \ell_{bot}) \in F(\delta_1, M_1, s_1, N)$, we have

$$\mathbb{P}\Big( V(\delta, L_1^N([-s_1, s_1])) \geq A \Big| E(a, b, \ell_{bot}, N) \Big) \leq \eta/4, \text{ where } A = \frac{\epsilon (2s_1)^{1/2}}{2(2r)^{1/2}}. \tag{4.5.6}$$

Let us assume the validity of (4.5.6) and deduce (4.5.5). We have

$$\mathbb{P}\Big( V(\delta, L_1^N([-s_1, s_1])) \geq A \Big) \leq \mathbb{P}\Big( \{ V(\delta, L_1^N([-s_1, s_1])) \geq A \} \cap E_1(M_1, N) \cap E_2(\delta_1, N) \Big) + \eta/2,$$

where we used that $\mathbb{P}(E_1^c(M_1, N)) \leq \eta/4$ and $\mathbb{P}(E_2^c(\delta_1, N)) \leq \eta/4$. In addition, we have

$$\mathbb{P}\Big( \{ V(\delta, L_1^N([-s_1, s_1])) \geq A \} \cap E_1(M_1, N) \cap E_2(\delta_1, N) \Big) =$$

$$\sum_{(a, b, \ell_{bot}) \in F(\delta_1, M_1, s_1, N)} \mathbb{P}\Big( \{ V(\delta, L_1^N([-s_1, s_1])) \geq A \} \cap E(a, b, \ell_{bot}, N) \Big),$$

where we used that $E_1(M_1, N) \cap E_2(\delta_1, N)$ is a disjoint union of $E(a, b, \ell_{bot}, N)$. Finally, we have from (4.5.6) above that

$$\mathbb{P}\Big( \{ V(\delta, L_1^N([-s_1, s_1])) \geq A \} \cap E(a, b, \ell_{bot}, N) \Big) =$$

$$\mathbb{P}\Big(V(\delta, L_1^N([-s_1,s_1])) \geq A \Big| E(a,b,\ell_{bot},N)\Big)\mathbb{P}\big(E(a,b,\ell_{bot},N)\big) \leq (\eta/4)\cdot \mathbb{P}\big(E(a,b,\ell_{bot},N)\big).$$

Summing the latter over $(a,b,\ell_{bot}) \in F(\delta_1, M_1, s_1, N)$ and combining it with the earlier inequalities we see that

$$\mathbb{P}\Big(V(\delta, L_1^N([-s_1,s_1])) \geq A\Big) \leq \eta/2 + \eta/4 \cdot \sum_{(a,b,\ell_{bot})\in F(\delta_1,M_1,s_1,N)} \mathbb{P}\big(E(a,b,\ell_{bot},N)\big) =$$

$$= \eta/2 + (\eta/4)\cdot \mathbb{P}\big(E_1(M_1,N) \cap E_2(\delta_1,N)\big) < \eta,$$

where in the middle equality we again used that $E_1(M_1,N) \cap E_2(\delta_1,N)$ is a disjoint union of $E(a,b,\ell_{bot},N)$. The last equation implies (4.5.5).

**Step 5.** In this step we establish (4.5.6) and begin by fixing $(a,b,\ell_{bot}) \in F(\delta_1, M_1, s_1, N)$. Since $\mathfrak{L}^N$ satisfies the Hall-Littlewood Gibbs property on $[\![-s_1,s_1]\!]$ with respect to $S' = [\![-s_1+1,s_1]\!]$ for $N$ sufficiently large we know that

$$\mathbb{P}\big(L_1^N([-s_1,s_1]) = \ell \big| E(a,b,\ell_{bot},N)\big) = \mathbb{P}_{S'}^{-s_1,s_1,a,b}(\ell|\ell_{bot}) \text{ for any } \ell \in \Omega(-s_1,s_1;a,b). \quad (4.5.7)$$

We now recall the sampling property we explained in Remark 4.3.5. Let $\ell^K$ be a sequence of i.i.d. up-right paths distributed according to $\mathbb{P}_{free}^{-s_1,s_1;a,b}$. Also let $U$ be a uniform random variable on $(0,1)$, independent of all else. For each $K \in \mathbb{N}$ we check if $W_t(-s_1,s_1,\ell^K,\ell_{bot};S') > U$ and set $Q$ to be the minimal index $K$, which satisfies the inequality. Then we have that $Q$ is a geometric random variable with parameter $Z_t(-s_1,s_1,a,b,\ell_{bot};S')$ and

$$\tilde{\mathbb{P}}\left(\ell^Q = \ell\right) = \mathbb{P}_{S'}^{-s_1,s_1,a,b}(\ell|\ell_{bot}) \text{ for any } \ell \in \Omega(-s_1,s_1;a,b), \quad (4.5.8)$$

where $\tilde{\mathbb{P}}$ is the probability measure on a space on which $\ell^K$ and $U$ are defined, we also write $\tilde{\mathbb{E}}$ for the expectation with respect to $\tilde{\mathbb{P}}$.

By our assumption that $(a,b,\ell_{bot}) \in F(\delta_1,M_1,s_1,N)$, we know that $Z_t(-s_1,s_1,a,b,\ell_{bot};S') > \delta_1$ and so $\tilde{\mathbb{E}}[Q] = Z_t(-s_1,s_1,a,b,\ell_{bot};S')^{-1} \leq \delta_1^{-1}$. It follows that if we take $R = 8\delta_1^{-1}\eta^{-1}$, then by Chebyshev's inequality we have

$$\tilde{\mathbb{P}}(Q > R) \leq \eta/8. \quad (4.5.9)$$

Fix $A = \frac{\epsilon(2s_1)^{1/2}}{2(2r)^{1/2}}$ and observe that

$$\tilde{\mathbb{P}}\left(V(\delta,\ell^Q) \geq A\right) = \tilde{\mathbb{P}}\left(V(\delta,\ell^Q) \geq A, Q > R\right) + \tilde{\mathbb{P}}\left(V(\delta,\ell^Q) \geq A, Q \leq R\right) \leq \tilde{\mathbb{P}}(Q > R) +$$

$$\tilde{\mathbb{P}}\left(\max_{1\leq i\leq R} V(\delta,\ell^i) \geq A\right) = \tilde{\mathbb{P}}(Q>R) + 1 - \tilde{\mathbb{P}}\left(\max_{1\leq i\leq R} V(\delta,\ell^i) < A\right) \leq 1 - \tilde{\mathbb{P}}\left(V(\delta,\ell^1) < A\right)^{\lfloor R\rfloor} + \eta/8.$$

In the last inequality we used (4.5.9) and the independence of $\ell^i$. Combining the latter inequality with (4.5.7) and (4.5.8) we see that

$$\mathbb{P}\left(V(\delta, L_1^N([-s_1,s_1])) \geq A | E(a,b,\ell_{bot},N)\right) \leq 1 - \tilde{\mathbb{P}}\left(V(\delta,\ell^1) < A\right)^{\lfloor R\rfloor} + \eta/8. \quad (4.5.10)$$

Equation (4.5.6) would now follow from (4.5.10) if we can show that for any $\epsilon' > 0$ we can find $\delta > 0$ (depending on $M_1$, $\epsilon'$, $\eta$, $r$ and $p$), such that for all large $N$ we have

$$\tilde{\mathbb{P}}\left(V(\delta, \ell^1) < A\right) \geq 1 - \epsilon'. \tag{4.5.11}$$

**Step 6.** In this final step we establish (4.5.11), which is the remaining statement we require. Notice that $A = \tilde{\epsilon}(2s_1)^{1/2}$, where $\tilde{\epsilon} = \frac{\epsilon}{2(2r)^{1/2}}$. The key observation we make is the following

$$\tilde{\mathbb{P}}\left(V(\delta, \ell^1) < A\right) = \mathbb{P}_{free}^{0,2s_1;0,b-a}\left(w(f^{\ell^1}, \delta) < \tilde{\epsilon}\right), \tag{4.5.12}$$

where $f^{\ell}(x) = (2s_1)^{-1/2}(\ell(2xs_1) - 2pxs_1))$ for $x \in [0,1]$ and $w(f, \delta)$ denotes the modulus of continuity on $[0,1]$ as in (4.4.9).

Notice that since $(a, b, \ell_{bot}) \in F(\delta_1, M_1, s_1, N)$, we know that $|b - a - 2ps_1| \leq 2M_1N^{\alpha/2} \leq \frac{4M_1}{(2r)^{1/2}}(2s_1)^{1/2}$ for all large $N$. The latter means that we can apply Lemma 4.4.11, and find $\delta > 0$ (depending on $M_1$, $\epsilon'$, $\tilde{\epsilon}$, $\eta$ and $p$), such that for all large $N$ we have

$$\mathbb{P}_{free}^{0,2s_1;0,b-a}\left(w(f^{\ell^1}, \delta) < \tilde{\epsilon}\right) \geq 1 - \epsilon'.$$

Combining the latter with (4.5.12) concludes the proof of (4.5.11).

*Remark* 4.5.5. An important idea in our arguments above is to condition on $E(a, b, \ell_{bot}, N)$ and obtain estimates on these events, where additional structure is available to us. The latter is possible because of the discrete nature of our problem and substitutes the more involved notions of *stopping domains* and *strong Brownian Gibbs properties* that were used in [42] and [43].

## 4.6 Proof of three key lemmas

Here we prove the three key lemmas from Section 4.5.1. The arguments we use below heavily depend on the results from Section 4.4.

### 4.6.1 Proof of Lemma 4.5.2

Let us start by fixing notation. As in Section 4.5.1 we set $s_1 = \lfloor rN^{\alpha} \rfloor$ and $t_1 = \lfloor (r+1)N^{\alpha} \rfloor$. Define the events

$$E(M) = \left\{\left|L_1^N(-t_1) + pt_1\right| > MN^{\alpha/2}\right\}, F(M) = \left\{L_1(-s_1) > -ps_1 + MN^{\alpha/2}\right\} \text{ and}$$

$$G(M) = \left\{\sup_{s \in [0, t_1]}\left[L_1^N(s) - ps\right] > (6r + 10)(M + 1)N^{\alpha/2}\right\}.$$

For $a, b \in \mathbb{Z}$ and $s \in \{0, 1, ..., t_1\}$ as well as a path $\ell_{bot}$ in $\Omega(-t_1, s; z_1, z_2)$, where $z_1 \leq a$ and $z_2 \leq b$ we define $E(a, b, s, \ell_{bot})$ to be the event that $L_1^N(-t_1) = a$, $L_1^N(s) = b$, and $L_2^N$ agrees with $\ell_{bot}$ on $[-t_1, s]$. We will also write $L_1^N([m, n])$ for the restriction of $L_1^N$ to the interval $[m, n]$.

Observe that $E^c(M) \cap G(M)$ can be written as a *countable disjoint* union of sets of the form $E(a,b,s,\ell_{bot})$, where the quadruple $(a,b,s,\ell_{bot})$ satisfies:

1. $0 \leq s \leq t_1$,
2. $0 \leq b - a \leq t_1 + s$, $|a + pt_1| \leq MN^{\alpha/2}$ and $b - ps > (6r+10)(M+1)N^{\alpha/2}$,
3. $z_1 \leq a$, $z_2 \leq b$ and $\ell_{bot} \in \Omega(-t_1, s, z_1, z_2)$,

Clearly, there are only finitely many choices for $s$ and for any $s$ there are countably many $a, b$ that satisfy the conditions above. Then the number of $z_1, z_2$ for each given pair $(a,b)$ is again countable, while the cardinality of $\Omega(-t_1, s, z_1, z_2)$ is finite. This means that the number of quadruples $(a,b,s,\ell_{bot})$ is indeed countable. The fact that $E(a,b,s,\ell_{bot})$ are disjoint is again clear, while the first and second condition above show that their union is indeed $E^c(M) \cap G(M)$. Let us denote by $D(M)$ the set of quadruples $(a,b,s,\ell_{bot})$ that satisfy the three conditions above.

By 1-point tightness of $L_1^N$ we know that there exists $M > 0$ sufficiently large so that for every $N \in \mathbb{N}$ we have

$$\mathbb{P}\big(E(M)\big) < \epsilon/4 \text{ and } \mathbb{P}\big(F(M)\big) < \frac{\epsilon c(t)}{12}, \tag{4.6.1}$$

where we recall that $c(t) = \prod_{i=1}^{\infty}(1-t^i)$. Suppose $(a,b,s,\ell_{bot}) \in D(M)$ and observe that we have

$$\mathbb{P}_{free}^{-t_1,s;a,b}\left(\ell(-s_1) \geq -ps_1 + MN^{\alpha/2}\right) = \mathbb{P}_{free}^{0,t_1+s;0,b-a}\left(\ell(t_1-s_1) + a \geq -ps_1 + MN^{\alpha/2}\right) \geq \mathbb{P}_{free}^{0,t_1+s;0,b-a}\left(\ell(t_1-s_1) \geq p(t_1-s_1) + 2MN^{\alpha/2}\right), \tag{4.6.2}$$

where in the last inequality we used that $a + pt_1 \geq -MN^{\alpha/2}$. Since $|a + pt_1| \leq MN^{\alpha/2}$ and $b - ps \geq (6r+10)(M+1)N^{\alpha/2}$, we conclude that $b - a \geq p(t_1+s) + (6r+9)(M+1)N^{\alpha/2}$. It follows from Lemma 4.4.8 that for all large $N$ we have

$$\mathbb{P}_{free}^{0,t_1+s;0,b-a}\left(\ell(t_1-s_1) \geq \frac{t_1-s_1}{t_1+s}[p(t_1+s) + (6r+9)(M+1)N^{\alpha/2}] - (t_1+s)^{1/4}\right) \geq 1/3. \tag{4.6.3}$$

Notice that since $s \in [0,t_1]$, $s_1 = \lfloor rN^{\alpha}\rfloor$ and $t_1 = \lfloor (r+1)N^{\alpha}\rfloor$, we have $\frac{t_1-s_1}{t_1+s} \geq \frac{1}{2r+3}$ for all large $N$. These estimates together imply that for all large $N$ we have $\frac{t_1-s_1}{t_1+s}[p(t_1+s) + (6r+9)(M+1)N^{\alpha/2}] - (t_1+s)^{1/4} \geq p(t_1-s_1) + 2MN^{\alpha/2}$ and so from (4.6.2) and (4.6.3) we conclude that

$$\mathbb{P}_{free}^{-t_1,s;a,b}\left(\ell(-s_1) \geq -ps_1 + MN^{\alpha/2}\right) \geq 1/3. \tag{4.6.4}$$

Since the sequence $\mathcal{L}^N$ is $(\alpha,p,r+1)$-good, we know that for any $\ell \in \Omega(-t_1,s;a,b)$ we have

$$\mathbb{P}(L_1^N([-t_1,s]) = \ell | E(a,b,s,\ell_{bot})) = \frac{W_t(-t_1,s,\ell,\ell_2;S)}{Z_t(-t_1,s,a,b,\ell_{bot};S)},$$

where $S = [\![-t_1+1, s]\!]$. The latter together with (4.6.4) and Corollary 4.4.4 allow us to conclude that

$$\mathbb{P}\left(L_1(-s_1) + ps_1 > MN^{\alpha/2} | E(a,b,s,\ell_{bot})\right) \geq c(t)/3. \tag{4.6.5}$$

We now observe that

$$\mathbb{P}\big(F(M)\big) \geq \sum_{(a,b,s,\ell_{bot})\in D(M)} \mathbb{P}\big(F(M)\cap E(a,b,s,\ell_{bot})\big) =$$

$$= \sum_{(a,b,s,\ell_{bot})\in D(M)} \mathbb{P}\big(F(M)\big|E(a,b,s,\ell_{bot})\big)\mathbb{P}\big(E(a,b,s,\ell_{bot})\big) \geq (c(t)/3)\mathbb{P}\big(E^c(M)\cap G(M)\big),$$

where in the last inequality we used (4.6.5). Combining the above inequality with the inequalities in (4.6.1) we see that for all large $N$ we have

$$\epsilon/2 > \mathbb{P}(G(M)) = \mathbb{P}\left(\sup_{s\in[0,t_1]}\big[L_1^N(s) - ps\big] > (6r+10)(M+1)N^{\alpha/2}\right). \tag{4.6.6}$$

A similar argument shows that for all large $N$ we have

$$\epsilon/2 > \mathbb{P}\left(\sup_{s\in[-t_1,0]}\big[L_1^N(s) - ps\big] > (6r+10)(M+1)N^{\alpha/2}\right). \tag{4.6.7}$$

Combining (4.6.6) and (4.6.7) we conclude the statement of the lemma for $R = (6r+10)(M+1)$.

### 4.6.2 Proof of Lemma 4.5.3

For clarity we will split the proof into two steps.
**Step 1.** Define $F = \Big\{\min\Big(\tilde{L}(-s_1) + ps_1, \tilde{L}(s_1) - ps_1\Big) \geq (M_2+1)N^{\alpha/2} + (2s_1)^{1/2}\Big\}$. We claim that for all $N$ sufficiently large we have

$$\mathbb{P}_{\tilde{L}}(F) \geq (c(t)^3/18)\left(1 - \Phi^{p(1-p)/2}\left(10(1+r)^2(M_1+M_2+1)\right)\right). \tag{4.6.8}$$

Establishing the validity of (4.6.8) will be done in the second step, and in what follows we assume it is true and finish the proof of the lemma.

We assert that if $N_2$ is sufficiently large and $N \geq N_2$ we have

$$F \subset A = \left\{Z\left(-s_1, s_1, \tilde{L}(-s_1), \tilde{L}(s_1), \ell_{bot}; S'\right) > \frac{1}{4}\left(1 - \exp\left(\frac{-2}{p(1-p)}\right)\right)\right\}. \tag{4.6.9}$$

Observe that (4.6.9) and (4.6.8) prove the lemma and so it suffices to verify (4.6.9). The details are presented below (see also Figure 4-9).

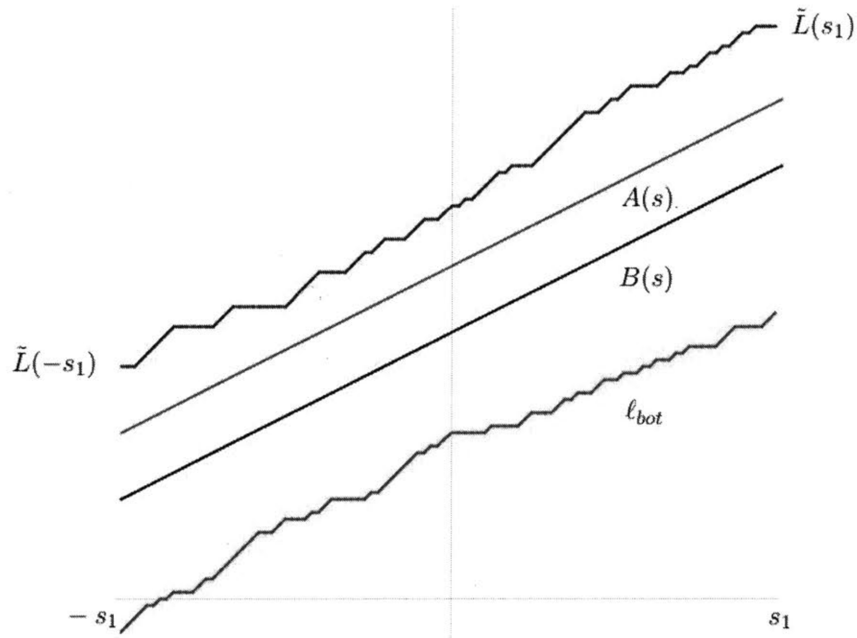


Figure 4-9: Overview of the arguments in Step 1:
We want to prove that on the event $F$, we have a lower bound on the acceptance probability $Z_t(\tilde{L}(-s_1), \tilde{L}(s_1)) = Z_t(-s_1, s_1, \tilde{L}(-s_1), \tilde{L}(s_1), \ell_{bot}; S')$. As explained in (4.6.10) the acceptance probability is just the average of the weights $W_t(-s_1, s_1, \ell, \ell_{bot}; S')$ over all up-right paths in $\Omega = \Omega(-s_1, s_1; \tilde{L}(-s_1), \tilde{L}(s_1))$. Consequently, to show that $Z_t(\tilde{L}(-s_1), \tilde{L}(s_1))$ is lower-bounded it suffices to find a big subset $\Omega' \subset \Omega$, such that the weights $W_t(-s_1, s_1, \ell, \ell_{bot}; S')$ for $\ell \in \Omega'$ are lower-bounded.
Let $A(s)$ and $B(s)$ denote the lines $ps + (M_2 + 1)N^{\alpha/2} - (2s_1)^{1/4}$ and $ps + M_2 N^{\alpha/2}$, drawn in grey and black respectively above. Then $\Omega'$ denotes the set of up-right paths in $\Omega$, which lie above $A(s)$ on $[-s_1, s_1]$. On the event $F$ we have that $\tilde{L}(\pm s_1)$ are at least a distance $(2s_1)^{1/2} + (2s_1)^{1/4}$ above the points $A(\pm s_1)$ respectively. Since the endpoints of paths in $\Omega$ are well above those of $A(s)$ this means that some positive fraction of these paths will stay above $A(s)$ on the entire interval $[-s_1, s_1]$; i.e. $|\Omega'|/|\Omega|$ is lower bounded. This is what we mean by $\Omega'$ being big and the exact relation is given in (4.6.11).
To see that $W_t(-s_1, s_1, \ell, \ell_{bot}; S')$ for $\ell \in \Omega'$ are lower bounded, we notice that elements $\ell \in \Omega'$ are well-above $B(s)$, which dominates $\ell_{bot}$ by assumption. This means that $\ell$ is well above $\ell_{bot}$ and for such paths $W_t(-s_1, s_1, \ell, \ell_{bot}; S')$ is lower bounded. The exact relation is given in (4.6.12).

From Definition 4.3.4 (see also Remark 4.3.5) we have

$$Z\left(-s_1, s_1, \tilde{L}(-s_1), \tilde{L}(s_1), \ell_{bot}; S'\right) = \mathbb{E}_{free}^{-s_1,s_1;\tilde{L}(-s_1),\tilde{L}(s_1)}\left[W_t(-s_1, s_1, \cdot, \ell_{bot}; S')\right].$$

If we set $\Omega = \Omega(-s_1, s_1; \tilde{L}(-s_1), \tilde{L}(s_1))$ and $Z_t(\tilde{L}(-s_1), \tilde{L}(s_1)) = Z\left(-s_1, s_1, \tilde{L}(-s_1), \tilde{L}(s_1), \ell_{bot}; S'\right)$ then the above implies

$$Z_t(\tilde{L}(-s_1), \tilde{L}(s_1)) = |\Omega|^{-1} \sum_{\ell \in \Omega} W_t(-s_1, s_1, \ell, \ell_{bot}; S'). \tag{4.6.10}$$

Denote $\Omega' = \left\{\ell \in \Omega : \ell(s) - ps \geq (M_2 + 1)N^{\alpha/2} - (2s_1)^{1/4} \text{ for } s \in [-s_1, s_1]\right\}$. It follows from Lemma 4.4.10 that on the event $F$, provided $N_2$ is sufficiently large and $N \geq N_2$ we have

$$\frac{|\Omega'|}{|\Omega|} \geq \frac{1}{2}\left(1 - \exp\left(\frac{-2}{p(1-p)}\right)\right). \tag{4.6.11}$$

Since $|s_1 - rN^{\alpha}| < 1$ we know that for $N_2$ sufficiently large and $N \geq N_2$, we have that $\ell \in \Omega'$ satisfies $\ell(s) - ps \geq (M_2 + 1/2)N^{\alpha/2} \geq \ell_{bot}(s) - ps + (1/2)N^{\alpha/2}$, where the last inequality holds true by our assumption on $\ell_{bot}$. The conclusion is that for $\ell \in \Omega'$, we have that $\ell(s) - \ell_{bot}(s) \geq m$, where $m = (1/2)N^{\alpha/2}$. In view of (4.3.1) we conclude that for $N_2$ sufficiently large, $N \geq N_2$ and $\ell \in \Omega'$, we have

$$W_t(-s_1, s_1, \ell, \ell_{bot}; S') \geq (1 - t^m)^{2s_1} \geq (1 - t^{(1/2)N^{\alpha/2}})^{2rN^{\alpha}} \geq \frac{1}{2}. \tag{4.6.12}$$

Combining (4.6.10), (4.6.11) and (4.6.12) we conclude that provided $N_2$ is sufficiently large and $N \geq N_2$ on the event $F$ we have

$$Z_t(\tilde{L}(-s_1), \tilde{L}(s_1)) \geq |\Omega|^{-1} \sum_{\ell \in \Omega'} W_t(-s_1, s_1, \ell, \ell_{bot}; S') \geq \frac{|\Omega'|}{|\Omega|} \cdot \frac{1}{2} \geq \frac{1}{4}\left(1 - \exp\left(\frac{-2}{p(1-p)}\right)\right).$$

**Step 2.** In this step we prove (4.6.8). We refer the reader to Figure 4-10 for an overview of the main ideas in this step and a graphical representation of the notation we use.

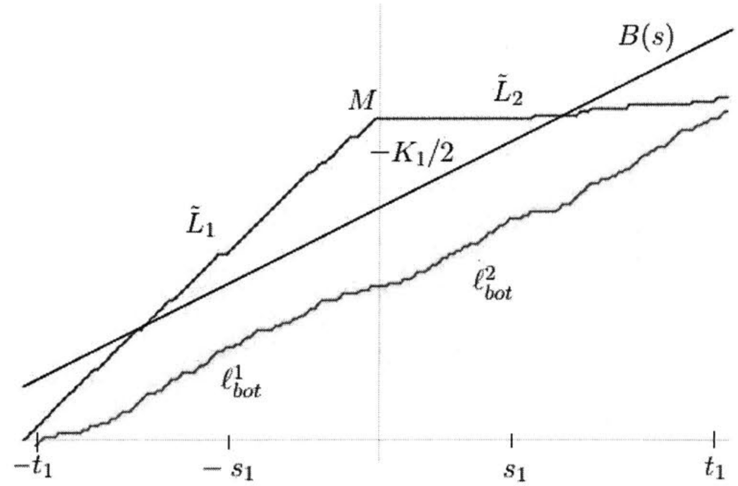


Figure 4-10: Overview of the arguments in Step 2:
$\tilde{L}_1$ and $\tilde{L}_2$ are the restrictions of $\tilde{L}$ to $[-t_1, 0]$ and $[0, t_1]$ respectively. $\ell^1_{bot}$ and $\ell^2_{bot}$ denote the restrictions of $\ell_{bot}$ to $[-t_1, 0]$ and $[0, t_1]$ respectively. Let $B(s)$ denote the line $ps + M_2N^{\alpha/2}$, drawn in black above. We have that $F$ denotes the event that $\tilde{L}$ is at least a distance $N^{\alpha/2} + (2s_1)^{1/2}$ above the line $B(s)$ at the points $\pm s_1$ and we want to find a lower bound on $\mathbb{P}_{\tilde{L}}(F)$.
We first let $E$ denote the event that $\tilde{L}(0)$ is much higher than $B(0)$, and prove that $\mathbb{P}_{\tilde{L}}(E)$ is lower bounded. The exact statement is given in (4.6.13). Afterwards, we show that on the event that the midpoint $\tilde{L}(0)$ is very high, the points $\tilde{L}(\pm s_1)$ are also very high with positive probability. The exact statement is given in (4.6.17).
In a sense, by conditioning on the midpoint $\tilde{L}(0)$ we split our problem into two independent subproblems for the left and right half of $\tilde{L}$ - see (4.6.14). Establishing the required estimates for each of the subproblems is then a relatively straightforward application of Lemma 4.4.8 and Corollary 4.4.4 - see (4.6.15).

Let $K_1 = 8(1+r)^2(M_1 + M_2 + 1)N^{\alpha/2}$. Define $E = \sqcup_{M \in X} E_M$ for

$$E_M = \{\tilde{L}(0) = M\} \text{ and } X = \left\{M \in \mathbb{N} : M \geq (1/2)K_1 - [2(r+1)N]^{\alpha/4} \text{ and } \mathbb{P}_{\tilde{L}}(E_M) > 0\right\}.$$

It follows from Lemma 4.4.9 that we can find $N_2$, depending on $r, M_1, M_2$ such that for $N \geq N_2$ we have

$$\mathbb{P}^{-t_1,t_1;a,b}_{free}\left(\ell(0) \geq (1/2)K_1 - [2(r+1)N]^{\alpha/4}\right) \geq (1/2)(1 - \Phi^{p(1-p)/2}(M_1 + K_1)).$$

Then by Corollary 4.4.4 we conclude

$$\mathbb{P}_{\tilde{L}}(E) \geq (c(t)/2)(1 - \Phi^{p(1-p)/2}(M_1 + K_1)). \tag{4.6.13}$$

Denote by $\tilde{L}_1$ and $\tilde{L}_2$ the restriction of $\tilde{L}$ to $[-t_1, 0]$ and $[0, t_1]$ respectively. Similarly, we let $\ell^1_{bot}$ and $\ell^2_{bot}$ denote the restriction of $\ell_{bot}$ to $[-t_1, 0]$ and $[0, t_1]$ respectively. The key observation we make is that if $M \in X$ then

$$\mathbb{P}_{\tilde{L}}(\tilde{L}_1 = \ell_1, \tilde{L}_2 = \ell_2 | E_M) = \mathbb{P}_{S_1}^{-t_1,0,a,M}(\ell_1 | \ell_{bot}^1) \cdot \mathbb{P}_{S_2}^{0,t_1,M,b}(\ell_2 | \ell_{bot}^2), \tag{4.6.14}$$

where $S_1 = [\![-t_1+1, -s_1]\!]$, $S_2 = [\![s_1+1, t_1]\!]$ and $\ell_1 \in \Omega(-t_1, 0; a, M)$, $\ell_2 \in \Omega(0, t_1; M, b)$.

From Lemma 4.4.8, we know that

$$\mathbb{P}_{free}^{-t_1,0;a,M}\left(\ell(-s_1) \geq M\frac{t_1 - s_1}{t_1} + a\frac{s_1}{t_1} - [(r+1)N]^{\alpha/4}\right) \geq 1/3,$$

provided $N_2$ is large enough and $N \geq N_2$. Since $a \geq -pt_1 - M_1 N^{\alpha/2}$, $s_1 = \lfloor rN^\alpha \rfloor$, $t_1 = \lfloor (r+1)N^\alpha \rfloor$, $M \geq (1/2)K_1 - [2(r+1)N]^{\alpha/4}$ and $K_1 = 8(1+r)^2(1+M_1+M_2)N^{\alpha/4}$ we conclude that if $N_2$ is sufficiently large and $N \geq N_2$ then

$$\mathbb{P}_{free}^{-t_1,0;a,M}\left(\ell(-s_1) + ps_1 \geq (M_2+1)N^{\alpha/2} + (2s_1)^{1/2}\right) \geq 1/3.$$

From Corollary 4.4.4 and the above inequality we conclude

$$\mathbb{P}_{S_1}^{-t_1,0,a,M}\left(\ell_1(-s_1) + ps_1 \geq (M_2+1)N^{\alpha/2} + (2s_1)^{1/2}\right) \geq c(t)/3. \tag{4.6.15}$$

Similar arguments show that

$$\mathbb{P}_{S_2}^{0,t_1,M,b}\left(\ell_2(s_1) - ps_1 \geq (M_2+1)N^{\alpha/2} + (2s_1)^{1/2}\right) \geq c(t)/3. \tag{4.6.16}$$

Combining (4.6.14), (4.6.15) and (4.6.16), we see that for $M \in X$, we have

$$\mathbb{P}_{\tilde{L}}\left(F | E_M\right) \geq c(t)^2/9. \tag{4.6.17}$$

The above inequality implies that

$$\mathbb{P}_{\tilde{L}}(F) \geq \mathbb{P}_{\tilde{L}}\left(F \cap E\right) = \sum_{M \in X} \mathbb{P}_{\tilde{L}}\left(F | E_M\right) \mathbb{P}_{\tilde{L}}(E_M) \geq (c(t)^2/9) \cdot \mathbb{P}_{\tilde{L}}\left(E\right).$$

The latter inequality together with (4.6.13) and the monotonicity of $\Phi^v$ on $\mathbb{R}_{>0}$ prove (4.6.8).

### 4.6.3 Proof of Lemma 4.5.4

Define $\mathbb{P}_{L'}$ and $\mathbb{P}_{\tilde{L}'}$ as the measure on up-right paths $L'$ and $\tilde{L}' : [-t_1, -s_1] \cup [s_1, t_1] \to \mathbb{R}$ (with $L'(-t_1) = \tilde{L}'(-t_1) = a$ and $L'(t_1) = \tilde{L}'(t_1) = b$ ) induced by the restriction of the measures $\mathbb{P}_L$ and $\mathbb{P}_{\tilde{L}}$ to these intervals. The Radon-Nikodym derivative between these two restricted measures is given on up-right paths $B : [-t_1, -s_1] \cup [s_1, t_1] \to \mathbb{R}$ by

$$\frac{d\mathbb{P}_{L'}}{d\mathbb{P}_{\tilde{L}'}}(B) = (Z')^{-1} Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S'), \tag{4.6.18}$$

where $Z' = \mathbb{E}_{\tilde{L}'}\left[Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S')\right]$.

Observe that $Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S')$ is a (deterministic) function of $(B(-s_1), B(s_1))$. In addition, the law of $(B(-s_1), B(s_1))$ under $\mathbb{P}_{\tilde{L}'}$ is the same as $(\tilde{L}(-s_1), \tilde{L}(s_1))$

under $\mathbb{P}_{\tilde{L}}$ (this is because $\mathbb{P}_{\tilde{L}'}$ is a restriction of $\mathbb{P}_{\tilde{L}}$ to intervals that contain $\pm s_1$). The latter and Lemma 4.5.3 imply

$$Z' = \mathbb{E}_{\tilde{L}'}\left[Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S')\right] = \mathbb{E}_{\tilde{L}}\left[Z_t(-s_1, s_1, \tilde{L}(-s_1), \tilde{L}(s_1), \ell_{bot}; S')\right] \geq gh.$$

Similarly, the law of $(B(-s_1), B(s_1))$ under $\mathbb{P}_{L'}$ is the same as $(L(-s_1), L(s_1))$ under $\mathbb{P}_L$ (this is because $\mathbb{P}_{L'}$ is a restriction of $\mathbb{P}_L$ to intervals that contain $\pm s_1$). Since $Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S')$ is a (deterministic) function of $(B(-s_1), B(s_1))$, we conclude that

$$\mathbb{P}_L\left(Z_t(-s_1, s_1, L(-s_1), L(s_1), \ell_{bot}; S') \leq \delta(\tilde{\epsilon})\right) = \mathbb{P}_{L'}\left(Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S') \leq \delta(\tilde{\epsilon})\right).$$

Let us denote $E = \{Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S') \leq \delta(\tilde{\epsilon})\} \subset \Omega$ (here $\Omega$ is the space of paths $B$). Then we have that

$$\mathbb{P}_{L'}(E) = \int_\Omega \mathbf{1}_E \cdot d\mathbb{P}_{L'}(B) = (Z')^{-1} \int_\Omega \mathbf{1}_E \cdot Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S') \cdot d\mathbb{P}_{\tilde{L}'}(B).$$

The above is immediate from (4.6.18). On $E$ we have that $Z_t(-s_1, s_1, B(-s_1), B(s_1), \ell_{bot}; S') \leq \delta(\tilde{\epsilon})$ and so the above is bounded by

$$(Z')^{-1} \int_\Omega \mathbf{1}_E \cdot \delta(\tilde{\epsilon}) \cdot d\mathbb{P}_{\tilde{L}'}(B) \leq \frac{1}{gh} \int_\Omega \mathbf{1}_E \cdot \delta(\tilde{\epsilon}) \cdot d\mathbb{P}_{\tilde{L}'}(B) \leq \tilde{\epsilon}.$$

The first inequality used that $Z' \geq gh$ and the second one that $\delta(\tilde{\epsilon}) = \tilde{\epsilon} \cdot gh$ and $\mathbf{1}_E \leq 1$. This concludes the proof of the lemma.

## 4.7 Absolute continuity with respect to Brownian bridges

In Theorem 4.3.8 we showed that under suitable shifts and scalings $(\alpha, p, r+1)$-good sequences give rise to tight sequences of continuous random curves. In this section, we aim to obtain some qualitative information about their subsequential limits and we will show that any subsequential limit is absolutely continuous with respect to a Brownian bridge with appropriate variance. In particular, this demonstrates that we have non-trivial limits and do not kill fluctuations with our rescaling. In Section 4.7.1 we present the main result of the section – Theorem 4.7.3 and explain how it relates to the other results in the chapter. The proof of Theorem 4.7.3 is given in Section 4.7.2 and for the most part relies on our control of the acceptance probability in Proposition 4.5.1 and the Hall-Littlewood Gibbs property.

### 4.7.1 Formulation of result and applications

We begin by introducing some relevant notation and defining what it means for a random curve to be absolutely continuous with respect to a Brownian bridge.

**Definition 4.7.1.** Let $X = C([0,1])$ and $Y = C([-r,r])$ be the spaces of continuous functions on $[0,1]$ and $[-r,r]$ respectively with the uniform topology. Denote by $d_X$ and $d_Y$ the metrics on the two spaces and by $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ their Borel $\sigma$-algebras. Given $z_1, z_2 \in \mathbb{R}$

we define $F_{z_1,z_2} : X \to Y$ and $G_{z_1,z_2} : Y \to X$ by

$$[F_{z_1,z_2}(g)](x) = z_1 + g\left(\frac{x+r}{2r}\right) + \frac{x+r}{2r}(z_2 - z_1) \quad [G_{z_1,z_2}(h)](\xi) = h(2r\xi - r) - z_1 - (z_2 - z_1)\xi, \tag{4.7.1}$$

for $x \in [-r, r]$ and $\xi \in [0, 1]$.

One observes that $F_{z_1,z_2}$ and $G_{z_1,z_2}$ are bijective homomorphisms between $X$ and $Y$ that are mutual inverses. Let $X_0 = \{f \in X : f(0) = f(1) = 0\}$ with the subspace topology and define $G : Y \to X$ through $G(h) = G_{h(-r),h(r)}(h)$. Let us make some observations.

1. $G$ is a continuous function. Indeed, from the triangle inequality we have $d_X\left(G_{h_1(-r),h_1(r)}(h_1), G_{h_2(-r),h_2(r)}(h_2)\right) \leq 2d_Y(h_1, h_2)$.
2. If $L$ is a random variable in $(Y, \mathcal{B}(Y))$ then $G(L)$ is a random variable in $(X, \mathcal{B}(X))$, which belongs to $X_0$ with probability 1. The measurability of $G(L)$ follows from the continuity of $G$, everything else is clearly true.

Recall from Section 4.4.2 that $B^\sigma$ stands for the Brownian bridge on $[0, 1]$, with variance $\sigma^2$ – this is a random variable in $(X, \mathcal{B}(X))$, which belongs to $X_0$ with probability 1.

With the above notation we make the following definition.

**Definition 4.7.2.** Let $L$ be a random variable in $(Y, \mathcal{B}(Y))$ with law $\mathbb{P}_L$. We say that $L$ is *absolutely continuous* with respect to a Brownian bridge with variance $\sigma^2$ if for any $K \in \mathcal{B}(X)$ we have

$$\mathbb{P}(B^\sigma \in K) = 0 \implies \mathbb{P}_L(G(L) \in K) = 0.$$

The main result of this section is as follows.

**Theorem 4.7.3.** *Assume the same notation as in Theorem 4.3.8 and let $\mathbb{P}_\infty$ be any subsequential limit of $\mathbb{P}_N$. If $f_\infty$ has law $\mathbb{P}_\infty$ then it is absolutely continuous with respect to a Brownian bridge with variance $2rp(1-p)$ in the sense of Definition 4.7.2.*

We have the following immediate corollary to Theorem 4.7.3 about the three stochastic models of Section 4.2.

**Corollary 4.7.4.** *Assume the same notation as in Theorem 4.2.2, Corollary 4.3.11 and Theorem 4.2.5 respectively and define for $x \in [-r, r]$*

$$g_N^{HL}(x) = \sigma_\mu f_N^{HL}(x) + \frac{x^2 f_1''(\mu)}{2}, \quad g_N^{SV}(x) = \sigma_\mu f_N^{SV}(x) - \frac{x^2 f_2''(\mu)}{2}, \quad g_N^{ASEP}(x) = \sigma_\alpha f_N^{ASEP}(x) - \frac{x^2 f_3''(\alpha)}{2}.$$

*If $g_\infty^{HL}$, $g_\infty^{SV}$ and $g_\infty^{ASEP}$ are any subsequential limits of $g_N^{HL}$, $g_N^{SV}$ and $g_N^{ASEP}$ respectively as $N \to \infty$ then $g_\infty^{HL}$, $g_\infty^{SV}$ and $g_\infty^{ASEP}$ are absolutely continuous with respect to a Brownian bridge of variance $2rf_1'(\mu)[1 - f_1'(\mu)]$, $-2rf_3'(\mu)[1 + f_2'(\mu)]$ and $-2rf_3'(\alpha)[1 + f_3'(\alpha)]$ respectively in the sense of Definition 4.7.2.*

*Proof.* From the proof of Theorem 4.3.10 we know that

$$g_N^{HL}(s) = N^{-1/3}\left(L_1^N(sN^{2/3}) - f_1'(\mu)sN^{2/3}\right), \text{ for } s \in [-r, r],$$

where the sequence $(L_1^N, L_2^N)$ is $(2/3, f_1'(\mu), r+1)$-good. By Theorem 4.7.3 we conclude the statement for $g_\infty^{HL}$. In addition, by Theorem 4.2.4 we know that for each $N \in \mathbb{N}$, $f_N^{HL}$ has the same distribution as $f_N^{SV}$ and so we conclude the statement for $g_\infty^{SV}$ as well.

From the proof of Theorem 4.3.13 we know that

$$g_N^{ASEP}(s) = N^{-1/3}\left(\tilde{L}_1^N(sN^{2/3}) + f_3'(\alpha)sN^{2/3}\right), \text{ for } s \in [-r, r],$$

where the sequence $(\tilde{L}_1^N, \tilde{L}_2^N)$ is $(2/3, -f_3'(\alpha), r+1)$-good. By Theorem 4.7.3 we conclude the statement for $g_\infty^{ASEP}$. □

*Remark* 4.7.5. Conjecturally, $f_N^{HL}$, $f_N^{SV}$ and $f_N^{ASEP}$ should converge to the Airy$_2$ process. Corollary 4.7.4 provides further evidence for this result as it is known that the Airy$_2$ process minus a parabola has Brownian paths [42]. See also the discussion at the end of Section 4.1.2.

### 4.7.2 Proof of Theorem 4.7.3

In this section we give the proof of Theorem 4.7.3, which for clarity is split into several steps. Before we go into the main argument we introduce some useful notation and give an outline of our main ideas.

Throughout we assume we have the same notation as in the statement of Theorem 4.3.8 as well as the notation from Section 4.7.1 above. We Denote $\sigma_1^2 = 2rp(1-p)$, $s_1 = \lfloor rN^\alpha \rfloor$, $r_N = s_1 N^{-\alpha}$ and $S' = [\![-s_1+1, s_1]\!]$. In addition, we define three probability spaces $\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3$ as well as a big probability space $\tilde{\mathbb{P}}$, which is the product space of $\mathbb{P}^1, \mathbb{P}^2$ and $\mathbb{P}^3$. The three spaces will carry different stochastic objects and we will use the superscript to emphasize, which properties we are using in different steps of the proof. We also reserve $\mathbb{P}$ to refer to the law of universal probabilistic objects like a Brownian bridge of a fixed variance.

From Theorem 4.4.5, we know that for each $n \in \mathbb{N}$ we have a probability space, on which we have a Brownian bridge $B^\sigma$ with variance $\sigma^2 = p(1-p)$ and a family of random paths $\ell^{(n,z)} \in \Omega(0, n; 0, z)$ for $z = 0, \dots, n$ such that $\ell^{(n,z)}$ has law $\mathbb{P}_{free}^{0,n;0,z}$ and

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \le Ce^{\alpha(\log n)^2} e^{|z-pn|^2/n}, \text{ where } \Delta(n,z) = \sup_{0\le t\le n}\left|\sqrt{n}B^\sigma_{t/n} + \tfrac{t}{n}z - \ell^{(n,z)}(t)\right|,$$

where the constants $C, a, \alpha$ depend only on $p$ and are fixed. By taking products of countably many of the above spaces we can construct a probability space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$, on which we have defined independent Brownian bridges $B^{\sigma,k,n}$ and independent families of random paths $\ell^{(n,k,z)} \in \Omega(0, n; 0, z)$ for $z = 0, \dots, n$ such that $\ell^{(n,k,z)}$ has law $\mathbb{P}_{free}^{0,n;0,z}$ for each $k$ and

$$\mathbb{E}_{\mathbb{P}^1}\left[e^{a\Delta(n,k,z)}\right] \le Ce^{\alpha(\log n)^2} e^{|z-pn|^2/n}, \text{ where } \Delta(n,k,z) := \sup_{0\le t\le n}\left|\sqrt{n}B^{\sigma,k,n}_{t/n} + \tfrac{t}{n}z - \ell^{(n,k,z)}(t)\right|.$$

In words, for each pair $(k, n) \in \mathbb{N} \times \mathbb{N}$ we have an independent copy of the probability space afforded by Theorem 4.4.5 sitting inside $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$. In addition, we assume that $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ carries a uniform random variable $U \in (0, 1)$, which is independent of all else.

Since $\mathbb{P}_\infty$ is a subseqential limit of $\mathbb{P}_N$ we know that we can find an increasing sequence

$N_j$ such that $\mathbb{P}_{N_j}$ weakly converge to $\mathbb{P}_\infty$. By Skorohod's representation theorem (see e.g. Theorem 6.7 in [13]) we can find a probability space $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$, on which are defined random variables $\tilde{f}_{N_j}$ and $\tilde{f}_\infty$ that take values in $(Y, \mathcal{B}(Y))$ such that the laws of $\tilde{f}_{N_j}$ and $\tilde{f}_\infty$ are $\mathbb{P}_{N_j}$ and $\mathbb{P}_\infty$ respectively and such that $d_Y\left(\tilde{f}_{N_j}(\omega^2), \tilde{f}_\infty(\omega^2)\right) \to 0$ as $j \to \infty$ for each $\omega^2 \in \Omega^2$.

We consider a probability space $(\Omega^3, \mathcal{F}^3, \mathbb{P}^3)$, on which we have defined the original $(\alpha, p, r+1)$-good sequence $\mathfrak{L}^N = (L_1^N, L_2^N)$ and so

$$f_N(s) = N^{-\alpha/2}(L_1^N(sN^\alpha) - psN^\alpha), \text{ for } s \in [-r, r]$$

has law $\mathbb{P}_N$ for each $N \geq 1$. Let us briefly explain the difference between $\mathbb{P}^2$ and $\mathbb{P}^3$ and why we need both. The space $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$ carries the random variables $\tilde{f}_{N_j}$ of law $\mathbb{P}_{N_j}$ and what is crucial is that the latter converge *almost surely* to $\tilde{f}_\infty$, whose law is $\mathbb{P}_\infty$. The space $(\Omega^3, \mathcal{F}^3, \mathbb{P}^3)$ carries the *entire* discrete line ensembles $\mathfrak{L}^N = (L_1^N, L_2^N)$ (and not just the top curve), which is needed to perform the resampling procedure of Section 4.3.1. Finally, we define $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ as the product of the three probability spaces we defined above.

At this time we give a brief outline of the steps in our proof. In the first step we fix $K \in \mathcal{B}(X)$ such that $\mathbb{P}(B^{\sigma_1} \in K) = 0$ and find an open set $O$, which contains $K$, and such that $B^{\sigma_1}$ is *extremely* unlikely to belong to $O$. Our goal is then to show that $G(\tilde{f}_\infty)$ is also unlikely to belong to $O$, the exact statement is given in (4.7.4) below. Using that $O$ is open and that $\tilde{f}_{N_j}$ converge to $\tilde{f}_\infty$ almost surely we can reduce our goal to showing that it is unlikely that $G(\tilde{f}_{N_j})$ belongs to $O$ *and* $\tilde{f}_{N_j}$ is at least a small distance away from the complement of $G^{-1}(O)$ for large $j$. Our gain from the almost sure convergence is that we have bounded ourselves away from $G^{-1}(O)^c$, which implies that by performing small perturbations we do not leave $G^{-1}(O)$. As the laws of $\tilde{f}_{N_j}$ and $f_{N_j}$ are the same we can switch from $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$ to $(\Omega^3, \mathcal{F}^3, \mathbb{P}^3)$, reducing the goal to showing that it is unlikely that $G(f_N)$ belongs to $O$ and $f_N$ is at least a small distance away from the complement of $G^{-1}(O)$ for large $N$. The exact statement is given in (4.7.5) and the reduction happens in Step 2. The benefit of this switch is that we can perform the resampling of Section 4.3.1 in $(\Omega^3, \mathcal{F}^3, \mathbb{P}^3)$ as the latter carries an entire line ensemble.

In the third step we use $U$ and $\ell^{(2s_1,k,z)}$ for $k = 1, 2, 3...$ to resample $f_N$ on the interval $[-s_1, s_1]$. If we denote by $Q$ the index $k$ of the first line we accept from the resampling, we can rephrase our statements for $f_N$ to equivalent statements that involve the path $\ell^{(2s_1,Q,z)}$ – this is (4.7.7). The benefit of working with $\ell^{(2s_1,k,z)}$ is that they are already strongly coupled with Brownian bridges by construction. In Step 4 we construct an event $F(N)$, on which our coupling of $\ell^{(2s_1,Q,z)}$ and the Browniand bridge $B^{\sigma,Q,2s_1}$ is good and on which $B^{\sigma,Q,2s_1}$ is well-behaved (its supremum and modulus of continuity are controlled). Provided we are on $F(N)$ (where the coupling is good) we see that $\ell^{(2s_1,Q,z)}$ belonging to a certain set (we want to show is unlikely) implies that $\sqrt{2r}B^{\sigma,Q,2s_1}$ belongs to $O$. Here it is crucial, that we have the extra distance to the complement of $G^{-1}(O)$ so that when we approximate our discrete paths with Brownian bridges we do not leave the set $G^{-1}(O)$.

The above steps reduce the problem to showing that it is unlikely that $\sqrt{2r}B^{\sigma,Q,2s_1}$ belongs to $O$ or that we are outside the event $F(N)$ – the exact statement is in (4.7.15). The control of $\sqrt{2r}B^{\sigma,Q,2s_1}$ is obtained by arguing that with high probability $Q$ is bounded – this requires our estimate on the acceptance probability from Proposition 4.5.1 and is the focus of Step 5. By having $Q$ bounded we reduce the question to a regular Brownian bridge, for which the event it belongs to $O$ is unlikely by definition of $O$. We demonstrate that $F(N)^c$ is unlikely in Step 6. As before we use the estimate on the acceptance probability to reduce the question to one involving a regular Brownian bridge. In addition, we use that with high probability we have uniform control of the coupling of our paths with Brownian bridges for all large $N$.

We now turn to the proof of the theorem.

**Step 1.** Suppose that $K \in \mathcal{B}(X)$ is given such that $\mathbb{P}(B^{\sigma_1} \in K) = 0$. We wish to show that

$$\mathbb{P}^2\left(G(\tilde{f}_\infty) \in K\right) = 0. \tag{4.7.2}$$

Let $\epsilon \in (0,1)$ be given and note that by Proposition 4.5.1 and Theorem 4.3.8 , we can find $\delta \in (0,1)$ and $M > 0$ such that for all large $N$ one has

$$\mathbb{P}^3\left(E(\delta, M, N)\right) < \epsilon, \text{ where } E(\delta, M, N) = \left\{Z_t(-s_1, s_1, L_1^N(-s_1), L_1^N(s_1), L_2; S') < \delta\right\} \cup \left\{\sup_{s \in [-rN^\alpha, rN^\alpha]} \left|L_1^N(s) - ps\right| \geq MN^{\alpha/2}\right\}. \tag{4.7.3}$$

We observe that since $C([-r,r])$ is a metric space we have by Theorem II.2.1 in [72] that the measure of $B^{\sigma_1}$ is outer-regular. In particular, we can find an open set $O$ such that $K \subset O$ and $\mathbb{P}(B^{\sigma_1} \in O) < \epsilon \cdot \frac{\log(1-\delta)}{\log(\epsilon)}$. The set $O$ will not be constructed explicitly and we will not require other properties from it other than it is open and contains $K$.

We will show that

$$\mathbb{P}^2\left(G(\tilde{f}_\infty) \in O\right) \leq 6\epsilon. \tag{4.7.4}$$

Notice that the above implies that $\mathbb{P}^2\left(G(\tilde{f}_\infty) \in K\right) \leq 6\epsilon$ and hence we reduce the proof of the theorem to establishing (4.7.4).

**Step 2.** Our goal in this step is to reduce (4.7.4) to a statement involving finite indexed curves.

We first observe $G^{-1}(O)$ is open since $G$ is continuous (this was proved in Section 4.7.1). The latter implies that

$$\mathbb{P}^2\left(G(\tilde{f}_\infty) \in O\right) = \mathbb{P}^2\left(\tilde{f}_\infty \in G^{-1}(O)\right) = \lim_{j \to \infty} \mathbb{P}^2\left(\left\{\tilde{f}_{N_j} \in G^{-1}(O)\right\} \cap \left\{d_Y(\tilde{f}_{N_j}, G^{-1}(O)^c) > r_j\right\}\right),$$

where $r_j$ is any sequence that converges to 0 as $j \to \infty$. The first equality is by definition. The second one follows from the fact that $\tilde{f}_{N_j}$ converge to $\tilde{f}_\infty$ in the uniform topology $\mathbb{P}^2$-almost surely and that $G^{-1}(O)$ is open. To be more specific we take $r_j = N_j^{-\alpha/8}$ for the sequel.

Since $f_N$ has law $\mathbb{P}_N$ for each $N \geq 1$, we observe that to get (4.7.4) it suffices to show that

$$\limsup_{N\to\infty} \mathbb{P}^3 \left( \{f_N \in G^{-1}(O)\} \cap \{d_Y(f_N, G^{-1}(O)^c) > N^{-\alpha/8}\} \right) \leq 6\epsilon. \tag{4.7.5}$$

**Step 3.** At this time we recall the resampling procedure from Remark 4.3.5 in the setting of our probability spaces $\mathbb{P}^i$. The goal of this step is to rephrase (4.7.5) into a statement involving the paths $\ell^{(n,k,z)}$ that are defined on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$.

Denote by $a = L_1^N(-s_1)$, $b = L_1^N(s_1)$, $z = b - a$, $n = 2s_1$ and $\ell_{bot} = L_2^N$ restricted to $[-s_1, s_1]$. We resample the top curve $L_1^N$ as follows. We start by erasing the curve in the interval $[-s_1, s_1]$. For $k = 1, 2, 3, \ldots$ we take $\ell^{(n,k,z)}$ (these were defined on the space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$), check if $W_t(-s_1, s_1, (-s_1, a) + \ell^{(n,k,z)}, \ell_{bot}; S') > U$ and set $Q$ to be the minimal index $k$, which satisfies the inequality. Here $(-s_1, a) + \ell^{(n,k,z)}$ is just the up-right path $\ell^{(n,k,z)}$ shifted so that it starts from the point $(-s_1, a)$.

Notice that by construction the path $(-s_1, a) + \ell^{(n,k,z)}$ are independent identically distributed as $\mathbb{P}_{free}^{-s_1, s_1; a, b}$. Because $\mathfrak{L}^N$ satisfies the Hall-Littlewood Gibbs property we have

$$\tilde{\mathbb{P}} \left( (-s_1, a) + \ell^{(2s_1, Q, b-a)} = \ell \right) = \mathbb{P}^3 \left( L_1^N[-s_1, s_1] = \ell \right), \tag{4.7.6}$$

for every $\ell \in \cup_{z_1 \leq z_2} \Omega(-s_1, s_1; z_1, z_2)$ where $L_1^N[-s_1, s_1]$ stands for the restriction of $L_1^N$ to the interval $[-s_1, s_1]$. If we denote

$$h_N(s) = \begin{cases} N^{-\alpha/2} \left( a + \ell^{(2s_1, Q, b-a)}(sN^\alpha + s_1) \right), & \text{for } s \in [-r_N, r_N] \\ f_N(s) & \text{for } s \in [-r, r] \backslash [-r_N, r_N], \end{cases}$$

we have that $h_N$ has the same law as $f_N$. Consequently it suffices to show that

$$\limsup_{N\to\infty} \tilde{\mathbb{P}} \left( \{h_N \in G^{-1}(O)\} \cap \{d_Y(h_N, G^{-1}(O)^c) > N^{-\alpha/8}\} \right) \leq 6\epsilon. \tag{4.7.7}$$

**Step 4.** Let $B^k(s) := B_s^{\sigma, k, 2s_1}$ for $s \in [0, 1]$ and consider the event

$$F(N) = \left\{ \Delta(2s_1, Q, b - a) < N^{\alpha/4} \right\} \cap \left\{ \sup_{s \in [0,1]} \left| B^Q(s) \right| \leq N^{\alpha/4} \right\} \cap \left\{ w(B^Q, N^{-\alpha}) \leq N^{-\alpha/4} \right\}. \tag{4.7.8}$$

In the above $w$ stands for the modulus of continuity of a function on $[0, 1]$ as defined in (4.4.9).

In this step we verify the following statement: There exists $N_0 \in \mathbb{N}$ and $C$ both depending on $r$ such that for $N \geq N_0$ and on the event $F(N)$ we have

$$d_Y \left( h_N, H_1^Q \right) \leq C N^{-\alpha/4}, \text{ where } H_1^Q = F_{h_N(-r), h_N(r)} \left( \sqrt{2r} B^Q \right) \tag{4.7.9}$$

Before we prove (4.7.9) we give a brief summary of the ideas. By definition, we have that $H_1^Q$ is given by an appropriate shift and rescaling of $B^Q$, which interpolates the points $(-r, h_N(-r))$ and $(r, h_N(r))$. To better understand how $H_1^Q$ differs from $h_N$ we first do an auxillary rescaling $H_2^Q$ by erasing the part of $h_N$ on the interval $[-r_N, r_N]$ and interpolating the points $(-r_N, h_N(-r_N)), (r_N, h_N(r_N))$ with an appropriate shift and rescaling of $B^Q$. The distance $d_Y(h_N, H_2^Q)$ is easily shown to be $O\left(N^{-\alpha/4}\right)$ using only the strong coupling of $B^Q$ and $\ell^{(2s_1,Q,b-a)}$ on $F(N)$ (this is the first event in (4.7.8)). Since $r_N$ is close to $r$ and $h_N(\pm r_N)$ is close to $h_N(\pm r)$ one can show that $d_Y(H_2^Q, H_1^Q) = O\left(N^{-\alpha/4}\right)$. The latter estimate uses the bounds on $B^Q$ and $w\left(B^Q, N^{-\alpha}\right)$ from the second and third event in (4.7.8), since what is involved is a certain stretching of the Brownian bridge $B^Q$. In what follows we supply the details of the above strategy.

We start by defining

$$H_2^Q(s) = \begin{cases} N^{-\alpha/2}\left(a + \sqrt{2s_1}B^Q\left(\frac{sN^\alpha + s_1}{2s_1}\right) + \frac{sN^\alpha+s_1}{2s_1}(b-a)\right) & \text{for } s \in [-r_N, r_N], \\ f_N(s) & \text{for } s \in [-r, r]\backslash[-r_N, r_N], \end{cases}$$

where we recall that $r_N = s_1 N^{-\alpha}$. Notice that

$$d_Y\left(h_N, H_2^Q\right) = N^{-\alpha/2} \cdot \Delta\left(2s_1, Q, b-a\right),$$

and so on the event $F(N)$ for all $N \geq 1$ we have

$$d_Y\left(h_N, H_2^Q\right) < N^{-\alpha/4}. \tag{4.7.10}$$

We next estimate $H_2^Q(s) - H_1^Q(s)$ on the interval $[-r, r]$. Whenever $s \in [-r_N, r_N]$ and we are on the event $F(N)$ we have

$$\begin{aligned} H_2^Q(s) - H_1^Q(s) = N^{-\alpha/2}\left(a + \sqrt{2s_1}B^Q\left(\frac{sN^\alpha + s_1}{2s_1}\right) + \frac{sN^\alpha + s_1}{2s_1}(b-a)\right) - \\ - \left(h_N(-r) + \sqrt{2r}B^Q\left(\frac{s+r}{2r}\right) + \frac{s+r}{2r}(h_N(r) - h_N(-r))\right) = O(N^{-\alpha/4}), \end{aligned} \tag{4.7.11}$$

where the constant in the big $O$ notation depends on $r$. In obtaining the second equality above we used that:

1. $s_1 = \lfloor rN^\alpha \rfloor = rN^\alpha + O(1)$, $b - a \leq 2rN^\alpha$,
2. $\left|h_N(-r) - N^{-\alpha/2} \cdot a\right| = \left|h_N(-r) - h_N\left(-r_N\right)\right| \leq N^{-\alpha/2}$,
3. $\left|h_N(r) - N^{-\alpha/2} \cdot b\right| = \left|h_N(r) - h_N\left(r_N\right)\right| \leq N^{-\alpha/2}$,
4. on $F(N)$ we have $\sup_{s\in[0,1]}\left|B^Q(s)\right| \leq N^{\alpha/4}$ and $w(B^Q, N^{-\alpha}) \leq N^{-\alpha/4}$.

For $s \in [-r, r]\backslash[-r_N, r_N]$, we know that

$$\left|H_1^Q(s) - H_2^Q(s)\right| \leq \left|H_1^Q(\pm r_N) - H_2^Q(\pm r_N)\right| + \left|H_1^Q(s) - H_1^Q\left(\pm r_N\right)\right| + \left|H_2^Q(s) - H_2^Q\left(\pm r_N\right)\right|,$$

where we choose the top sign if $s > r_N$ and the bottom sign otherwise. Note that the first term above is $O\left(N^{-\alpha/4}\right)$ by (4.7.11). Substituting the definitions of $H_1^Q$ and $H_2^Q$ we get for

$s \in [-r,r]\backslash[-r_N, r_N]$

$$\left|H_1^Q(s) - H_2^Q(s)\right| \leq O\left(N^{-\alpha/4}\right) + N^{-\alpha/2}\left|\sqrt{2s_1}B^Q\left(\frac{sN^\alpha + s_1}{2s_1}\right) + \frac{sN^\alpha \mp s_1}{2s_1}(b-a)\right| +$$

$$\left|\sqrt{2r}\left[B^Q\left(\frac{s+r}{2r}\right) - B^Q\left(\frac{\pm s_1 N^{-\alpha} + r}{2r}\right)\right] + \frac{s \mp s_1 N^{-\alpha}}{2r}(h_N(r) - h_N(-r))\right| = O\left(N^{-\alpha/4}\right), \tag{4.7.12}$$

where again we take the top sign if $s > r_N$ and the bottom sign otherwise and the constant in the big $O$ notation depends only on $r$. In obtaining the last equality we used the same estimates above together with the inequality $|h_N(r) - h_N(-r)| \leq 2rN^{\alpha/2}$. Combining (4.7.11) and (4.7.12) we deduce that

$$d_Y\left(H_1^Q, H_2^Q\right) = O\left(N^{-\alpha/4}\right), \tag{4.7.13}$$

where the constant in the big $O$ notation depends on $r$. Combining (4.7.10) and (4.7.13) we deduce (4.7.9).

**Step 5.** In this step we first show we have the following inclusion of events for all large $N$ (depending on $r$)

$$I(N) := F(N) \cap \{h_N \in G^{-1}(O)\} \cap \{d_Y(h_N, G^{-1}(O)^c) > N^{-\alpha/8}\} \subset \left\{\sqrt{2r}B^Q \in O\right\}. \tag{4.7.14}$$

Recall from (4.7.9) that there exists $N_0$ and $C$ depending on $r$ such that for $N \geq N_0$ and on the event $F(N)$

$$H_1^Q = F_{h_N(-r),h_N(r)}\left(\sqrt{2r}B^Q\right) \text{ and } d_Y\left(h_N, H_1^Q\right) \leq CN^{-\alpha/4}.$$

By increasing $N_0$ we can also ensure that $CN^{-\alpha/4} < N^{-\alpha/8}$ for $N \geq N_0$.

Fix $N \geq N_0$ and assume we are on the event $I(N)$. Since $h_N \in G^{-1}(O)$ and $d_Y(h_N, G^{-1}(O)^c) > N^{-\alpha/8}$, we see that $H_1^Q \in G^{-1}(O)$. Observe that $G(H_1^Q) = \sqrt{2r}B^Q$ by definition and so we conclude that $\sqrt{2r}B^Q \in O$ on $I(N)$. This proves (4.7.14).

From (4.7.14) we know that the LHS of (4.7.7) is bounded by

$$\limsup_{N\to\infty}\left[\tilde{\mathbb{P}}\left(\sqrt{2r}B^Q \in O\right) + \tilde{\mathbb{P}}\left(F(N)^c\right)\right].$$

In order to finish the proof it suffices to show

$$\limsup_{N\to\infty}\tilde{\mathbb{P}}\left(\sqrt{2r}B^Q \in O\right) \leq 3\epsilon \text{ and } \limsup_{N\to\infty}\tilde{\mathbb{P}}\left(F(N)^c\right) \leq 3\epsilon. \tag{4.7.15}$$

In the second part of this step we verify the first inequality in (4.7.15) and for brevity we set $W = \frac{\log(\epsilon)}{\log(1-\delta)}$. Observe that

$$\tilde{\mathbb{P}}\left(\sqrt{2r}B^Q \in O\right) \leq \mathbb{P}^3(E(\delta, M, N)) + \tilde{\mathbb{P}}\left(E(\delta, M, N)^c \cap \left\{\sqrt{2r}B^Q \in O\right\}\right),$$

where we recall that $E(\delta, M, N)$ was defined in Step 1. By assumption on $E(\delta, M, N)$ it suffices to show that

$$\limsup_{N\to\infty} \tilde{\mathbb{P}}\left(E(\delta, M, N)^c \cap \left\{\sqrt{2r}B^Q \in O\right\}\right) \leq 2\epsilon. \tag{4.7.16}$$

Notice that on $E(\delta, M, N)^c$ we have that $Q$ is a geometric random variable with parameter $Z_t(-s_1, s_1, a, b, \ell_{bot}; S') \geq \delta$. In particular, we have the a.s. inequality

$$\tilde{\mathbb{P}}\left(Q > W | E(\delta, M, N)^c\right) \leq (1-\delta)^W. \tag{4.7.17}$$

The above suggests that

$$\tilde{\mathbb{P}}\left(E(\delta, M, N)^c \cap \left\{\sqrt{2r}B^Q \in O\right\}\right) \leq$$
$$\tilde{\mathbb{P}}\left(E(\delta, M, N)^c\right)\left[(1-\delta)^W + \tilde{\mathbb{P}}\left(\{Q \leq W\} \cap \left\{\sqrt{2r}B^Q \in O\right\} \middle| E(\delta, M, N)^c\right)\right] \leq$$
$$(1-\delta)^W + \tilde{\mathbb{P}}\left(\bigcup_{i=1}^{W}\left\{\sqrt{2r}B^i \in O\right\}\right) \leq (1-\delta)^W + W \cdot \tilde{\mathbb{P}}\left(\sqrt{2r}B^1 \in O\right),$$

where we used in the last inequality that $B^k$ are identically distributed.

Now notice that $\tilde{\mathbb{P}}\left(\sqrt{2r}B^1 \in O\right) = \mathbb{P}(B^{\sigma_1} \in O) \leq \epsilon \cdot \frac{\log(1-\delta)}{\log(\epsilon)}$ by our choice of $O$ and so we conclude that

$$(1-\delta)^W + W \cdot \tilde{\mathbb{P}}\left(\sqrt{2r}B^1 \in O\right) \leq (1-\delta)^W + \epsilon \leq 2\epsilon.$$

This establishes (4.7.16).

**Step 6.** In this final step we establish the second inequality in (4.7.15) and as in Step 5 set $W = \frac{\log(\epsilon)}{\log(1-\delta)}$. Observe that

$$\tilde{\mathbb{P}}\left(F(N)^c\right) \leq \mathbb{P}^3(E(\delta, M, N)) + \tilde{\mathbb{P}}\left(E(\delta, M, N)^c \cap F(N)^c\right)$$

and so by assumption on $E(\delta, M, N)$ it suffices to show that

$$\limsup_{N\to\infty} \tilde{\mathbb{P}}\left(E(\delta, M, N)^c \cap F(N)^c\right) \leq 2\epsilon. \tag{4.7.18}$$

Using (4.7.17) we see that

$$\tilde{\mathbb{P}}\left(E(\delta, M, N)^c \cap F(N)^c\right) \leq \tilde{\mathbb{P}}\left(E(\delta, M, N)^c\right)\left[(1-\delta)^W + \tilde{\mathbb{P}}\left(\{Q \leq W\} \cap F(N)^c \middle| E(\delta, M, N)^c\right)\right] \leq$$

$$(1-\delta)^W + \sum_{i=1}^{W} \tilde{\mathbb{P}}\left(\{Q=i\} \cap F(N)^c \cap E(\delta, M, N)^c\right).$$

Since $(1-\delta)^W \leq \epsilon$, we reduce (4.7.18) to establishing

$$\limsup_{N\to\infty} \sum_{i=1}^{W} \tilde{\mathbb{P}}\Big( \{Q=i\} \cap F(N)^c \cap E(\delta, M, N)^c\Big) \leq \epsilon. \tag{4.7.19}$$

One clearly has that

$$\tilde{\mathbb{P}}\Big( \{Q=i\} \cap F(N)^c \cap E(\delta, M, N)^c\Big) \leq \tilde{\mathbb{P}}\Big(A_i^N \cap E(\delta, M, N)^c\Big) + \tilde{\mathbb{P}}\Big(B_i^N \cap E(\delta, M, N)^c\Big) + \tilde{\mathbb{P}}\Big(C_i^N \cap E(\delta, M, N)^c\Big),$$

where

$$A_i^N = \left\{\Delta\left(2s_1, i, b-a\right) \geq N^{\alpha/4}\right\}, B_i^N = \left\{ \sup_{s\in[0,1]} \left|B^i(s)\right| < N^{\alpha/4}\right\}, C_i^N = \left\{w(B^i, N^{-\alpha}) > N^{-\alpha/4}\right\}.$$

In addition, we know that since $B^i$ are identically distributed

$$\sum_{i=1}^{W} \tilde{\mathbb{P}}\left(\{Q=i\} \cap F(N)^c \cap E(\delta, M, N)^c\right) \leq$$

$$W \cdot \left[\tilde{\mathbb{P}}\left(A_1^N \cap E(\delta, M, N)^c\right) + \tilde{\mathbb{P}}\left(B_1^N \cap E(\delta, M, N)^c\right) + \tilde{\mathbb{P}}\left(C_1^N \cap E(\delta, M, N)^c\right)\right].$$

The above inequality reduces (4.7.19) to showing that

$$\begin{aligned} &\limsup_{N\to\infty} \tilde{\mathbb{P}}\left(A_1^N \cap E(\delta, M, N)^c\right) = 0, \\ &\limsup_{N\to\infty} \tilde{\mathbb{P}}\left(B_1^N \cap E(\delta, M, N)^c\right) = 0, \quad \limsup_{N\to\infty} \tilde{\mathbb{P}}\left(C_1^N \cap E(\delta, M, N)^c\right) = 0. \end{aligned} \tag{4.7.20}$$

Notice that by construction

$$\tilde{\mathbb{P}}\left(B_1^N\right) = \sum_{a\leq b} \mathbb{P}^1\left( \sup_{s\in[0,1]} \left|B_s^{\sigma, b-a, 2s_1}\right| < N^{\alpha/4}\right) \mathbb{P}^3\left(L_1^N(-s_1) = a, L_1^N(s_1) = b\right)$$

$$= \sum_{a\leq b} \mathbb{P}\left( \sup_{s\in[0,1]} |B^\sigma(s)| < N^{\alpha/4}\right) \mathbb{P}^3\left(L_1^N(-s_1) = a, L_1^N(s_1) = b\right) = \mathbb{P}\left( \sup_{s\in[0,1]} |B^\sigma(s)| < N^{\alpha/4}\right),$$

and the latter clearly converges to 0 as $N \to \infty$.

A similar argument shows that

$$\tilde{\mathbb{P}}\left(C_1^N\right) = \mathbb{P}\left(w(B^\sigma, N^{-\alpha}) > N^{-\alpha/4}\right),$$

and the latter converges to 0 as $N \to \infty$ by the almost sure Hölder-1/3 continuity of the Brownian bridge (see e.g. Proposition 7.8 in Chapter 8 of [37]). The above estimates establish the second line in (4.7.20).

In the remainder we study $\tilde{\mathbb{P}}\left(A_1^N \cap E(\delta, M, N)^c\right)$ and notice that by assumption on $E(\delta, M, N)$ we have that on the event $E(\delta, M, N)^c$ the values $a = L_1(-s_1)$ and $b = L(s_1)$ satisfy

$$|a + ps_1| \le MN^{\alpha/2} \text{ and } |b - ps_1| \le MN^{\alpha/2}.$$

The latter implies that

$$\tilde{\mathbb{P}}\left(A_1^N \cap E(\delta, M, N)^c\right) \le \sum_{|z| \le 2MN^{\alpha/2}} \tilde{\mathbb{P}}\left(A_1^N \cap \{b - a = \lfloor 2ps_1 + z \rfloor\}\right) =$$

$$= \sum_{|z| \le 2MN^{\alpha/2}} \mathbb{P}^1\left(\Delta\left(2s_1, 1, \lfloor 2ps_1 + z \rfloor\right) \ge N^{\alpha/4}\right) \mathbb{P}^3(\{b - a = \lfloor 2ps_1 + z \rfloor\}).$$

By Chebyshev's inequality and Theorem 4.4.5 we know that

$$\mathbb{P}^1\left(\Delta\left(2s_1, 1, \lfloor 2ps_1 + z \rfloor\right) \ge N^{\alpha/4}\right) \le C' N^{-\alpha/4} e^{c'(\log N)^2},$$

for some constants $C'$ and $c'$ that are independent of $N$ but depend on $M$. The latter inequalities show that

$$\tilde{\mathbb{P}}\left(A_1^N \cap E(\delta, M, N)^c\right) \le C' N^{-\alpha/4} e^{c'(\log N)^2} \sum_{|z| \le 2MN^{\alpha/2}} \mathbb{P}^3(\{b - a = \lfloor 2ps_1 + z \rfloor\}) \le C' N^{-\alpha/4} e^{c'(\log N)^2}.$$

Since the latter clearly converges to 0 as $N \to \infty$, we conclude (4.7.20), which finishes the proof.

## 4.8 Appendix: Strong coupling of random walks and Brownian bridges

In this section we prove a certain generalization of Theorem 6.3 in [60], given in Theorem 4.8.1 below, which we will use to prove Theorem 4.4.5 in the main text.

### 4.8.1 Proof of Theorem 4.4.5

Fix $p \in (0, 1)$ throughout this and the next sections. Let $X_i$ be i.i.d. random variables with $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = 0) = 1 - p$. We also let $S_n = X_1 + \cdots + X_n$ denote the random walk with increments $X_i$. For $z \in L_n = \{0, ..., n\}$ we let $S^{(n,z)} = \{S_m^{(n,z)}\}_{m=0}^n$ denote the process with the law of $\{S_m\}_{m=0}^n$, conditioned so that $S_n = z$. Finally, recall from Section 4.4.2 that $B^\sigma$ stands for the Brownian bridge (conditioned on $B_0 = 0, B_1 = 0$) with variance $\sigma^2$. We are interested in proving the following result.

**Theorem 4.8.1.** *For every $b > 0$, there exist constants $0 < C, a, \alpha < \infty$ (depending on $b$ and $p$) such that for every positive integer $n$, there is a probability space on which are defined* *a Brownian bridge $B^\sigma$ with variance $\sigma^2 = p(1-p)$ and the family of processes $S^{(n,z)}$ for $z \in L_n$ such that*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq Ce^{\alpha(\log n)^2}e^{b|z-pn|^2/n}, \tag{4.8.1}$$

*where $\Delta(n,z) = \Delta(n,z,B^\sigma,S^{(n,z)}) = \sup_{0\leq t\leq n}\left|\sqrt{n}B^\sigma_{t/n} + \frac{t}{n}z - S^{(n,z)}_t\right|$. We define $S^{(n,z)}_t$ for non-integer $t$ by linear interpolation.*

We observe that conditional on $S_n = z$ the law of the path determined by $S_n$ is precisely $\mathbb{P}^{0,n;0,z}_{free}$. Consequently, Theorem 4.8.1 implies Theorem 4.4.5 and in the remainder we focus on establishing the former. Our arguments will follow closely those in Section 6 of [60].

The proof of Theorem 4.8.1 relies on two lemmas, which we state below and whose proofs are deferred to Section 4.8.2. We begin by introducing some necessary notation. Suppose that $Z$ is a continuous random variable with strictly increasing cumulative distribution function $F$ and $G$ is the distribution function of a discrete random variable, whose support is $\{a_1, a_2, ...\}$. Then $(Z, W)$ are *quantile-coupled (with distribution functions $(F,G)$)* if $W$ is defined by

$$W = a_j \quad \text{if} \quad r_{j-} < Z \leq r_j,$$

where $r_{j-}, r_j$ are defined by

$$F(r_{j-}) = G(a_j-), \qquad F(r_j) = G(a_j).$$

The quantile-coupling has the following property. If

$$F(a_k - x) \leq G(a_k-) < G(a_k) \leq F(a_k + x),$$

then

$$|Z - W| = |Z - a_k| \leq x \qquad \text{on the event } \{W = a_k\}. \tag{4.8.2}$$

With the above notation we state the following two lemmas.

**Lemma 4.8.2.** *There exists $\epsilon_0$ (depending on $p$) such that for every $b_1 > 0$ there exist constants $0 < c_1, a_1 < \infty$ such that the following holds. Let $N$ be an $N(0,1)$ random variable. For each integers $m, n$ such that $n > 1$ and $|2m - n| \leq 1$ and every $z \in L_n$, let*

$$Z = Z^{(m,n,z)} = \frac{m}{n}z + \sqrt{p(1-p)m\left(1-\frac{m}{n}\right)}N, \text{ so that } Z \sim N\left(\frac{m}{n}z, p(1-p)m\left(1-\frac{m}{n}\right)\right).$$

*Let $W = W^{(m,n,z)}$ be the random variable, whose law is the same as that of $S^{(n,z)}_m$ and which is quantile-coupled with $Z$. Then if $|z - pn| \leq \epsilon_0 n$ and $\mathbb{P}(W = w) > 0$,*

$$\mathbb{E}\left[e^{a_1|Z-W|}\middle|W = w\right] \leq c_1 \cdot \sqrt{n} \cdot \exp\left(b_1\frac{(w-pm)^2 + (z-pn)^2}{n}\right). \tag{4.8.3}$$

**Lemma 4.8.3.** *There exist positive constants $\epsilon_0, c_2, b_2$ (depending on $p$) such that for every integers $m, n$ such that $n \geq 2$ and $|2m - n| \leq 1$, every $z \in L_n$ with $|z - pn| \leq \epsilon_0 n$ and every*

$w \in \mathbb{Z}$,

$$\mathbb{P}(S_m = w | S_n = z) \leq c_2 n^{-1/2} \exp\left(-b_2 \frac{(w-(z/2))^2}{n}\right).$$

*Proof.* (Theorem 4.8.1) It suffices to prove the theorem when $b$ is sufficiently small. For the remainder we fix $b > 0$ such that $b < b_2/37$, where $b_2$ is the constant from Lemma 4.8.3. Let $\epsilon_0$ be the smaller of the two values of $\epsilon_0$ in Lemmas 4.8.2 and 4.8.3.

In this proof, by an *n-coupling* we will mean a probability space on which are defined a Brownian bridge $B^\sigma$ and the family of processes $\{S^{(n,z)} : z \in L_n\}$. Notice that for any $n$-coupling if $z \in L_n$, $S_t = S_t^{(n,z)}$ then

$$\Delta(n,z) = \sup_{0 \leq t \leq n} \left| \sqrt{n} B^\sigma_{t/n} + \frac{t}{n} z - S_t^{(n,z)} \right| \leq 2n + \sup_{0 \leq t \leq n} |\sqrt{n} B^\sigma_{t/n}|.$$

The above together with the fact that there are positive constants $\tilde{c}$ and $u$ such that $\mathbb{E}\left[\exp\left(\sup_{0 \leq t \leq 1} y |B_t^\sigma|\right)\right] \leq \tilde{c} e^{uy^2}$ for any $y > 0$ (see e.g. (6.5) in [60]) imply that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq \tilde{c} e^{(2a+ua^2)n}.$$

Clearly, there exists $a_0 = a_0(b)$ such that if $0 < a < a_0$ then $2a + ua^2 \leq b\epsilon_0^2$.

The latter has the following implications. Firstly, (4.8.1) will hold for any $n$-coupling with $C = \tilde{c}$, $\alpha = 0$ and $a \in (0, a_0)$ if $z \in L_n$ satisfies $|z - pn| \geq \epsilon_0 n$. For the remainder of the proof we assume that $a < a_0$. Let $b_1 = b/20$ and let $a_1, c_1$ be as in Lemma 4.8.2 for this value of $b_1$. We assume that $a < a_1$ and show how to construct the $n$-coupling so that (4.8.1) holds for some $C, \alpha$.

We proceed by induction and note that we can find $C \geq \max(1, \tilde{c})$ sufficiently large so that for any $n$-coupling with $n \leq 2$ we have

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] e^{-b|z-pn|^2/n} \leq C, \quad \forall z \in L_n, n \leq 2.$$

With the above we have fixed our choice of $a$ and $C$.

We will show that for every positive integer $s$, we have that there exist $n$-couplings for all $n \leq 2^s$ such that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] e^{-b|z-pn|^2/n} \leq A_n^{s-1} \cdot C, \quad \forall z \in L_n, \tag{4.8.4}$$

where $A_n = 2c_1c_2 n + 2c_1\sqrt{n}$. The theorem clearly follows from this claim.

We proceed by induction on $s$ with base case $s = 1$ being true by our choice of $C$ above. We suppose our claim is true for $s$ and let $2^s < n \leq 2^{s+1}$. We will show how to construct a probability space on which we have a Brownian bridge and a family of processes $\{S^{(n,z)} : |z - pn| \leq \epsilon_0 n\}$, which satisfy (4.8.4). Afterwards we can adjoin (after possibly enlarging the probability space) the processes for $|z| > n\epsilon_0$. Since $C > \tilde{c}$ and $a < a_0$ we know that (4.8.4) will continue to hold for these processes as well. Hence, we assume that $|z - pn| \leq \epsilon_0 n$. For simplicity we assume that $n = 2k$, where $k$ is an integer such that $2^{s-1} < k \leq 2^s$ (if $n$ is odd we write $n = k + (k+1)$ and do a similar argument).

We define the $n$-coupling as follows:

- Choose two independent $k$-couplings

$$\left(\{S^{1(k,z))}\}_{z\in L_k}, B^1\right), \quad \left(\{S^{2(k,z))}\}_{z\in L_k}, B^2\right), \text{ satisfying (4.8.4).}$$

  Such a choice is possible by the induction hypothesis.

- Let $N \sim N(0,1)$ and define the translated *normal* variables $Z^z = \frac{z}{2} + \sqrt{\frac{p(1-p)n}{4}}N$ as well as the quantile-coupled random variables $W^z$ as in Lemma 4.8.2. Assume, as we may, that all of these random variables are independent of the two $k$-couplings chosen above. Observe that by our choice of $a$ we have that

$$\mathbb{E}\left[e^{a|Z^z-W^z|}\middle|W^z = w\right] \le c_1 \cdot \sqrt{n} \cdot \exp\left(\frac{b}{20} \cdot \frac{(w-kp)^2 + (z-np)^2}{n}\right). \quad (4.8.5)$$

- Let

$$B_t = \begin{cases} 2^{-1/2}B^1_{2t} + t\sqrt{p(1-p)}N & 0 \le t \le 1/2, \\ 2^{-1/2}B^2_{2(t-1/2)} + (1-t)\sqrt{p(1-p)}N & 1/2 \le t \le 1. \end{cases} \quad (4.8.6)$$

  By Lemma 6.5 in [60], $B_t$ is a Brownian bridge with variance $\sigma^2$.

- Let $S^{(n,z)}_k = W^z$, and

$$S^{(n,z)}_m = \begin{cases} S^{1(k,W^z)}_m & 0 \le m \le k, \\ W^z + S^{2(k,z-W^z)}_{m-k}, & k \le m \le n. \end{cases}$$

  What we have done is that we first chose the value of $S^{(n,z)}_k$ from the conditional distribution of $S_k$, given $S_n = z$. Conditioned on the midpoint $S^{(n,z)}_k = W^z$ the two halves of the random walk bridge are independent and upto a trivial shift we can use $S^{1(k,W^z)}$ and $S^{2(k,z-W^z)}$ to build them.

The above defines our coupling and what remains to be seen is that it satisfies (4.8.4) with $s+1$.

Note that

$$\Delta(n,z,S^{(n,z)},B) \le |Z^z - W^z| + \max\left(\Delta(k,W^z,S^{1(k,W^z)},B^1), \Delta(k,z-W^z,S^{2(k,z-W^z)},B^2)\right)$$

and therefore for any $w$ such that $\mathbb{P}(W^z = w) > 0$ we have

$$\mathbb{E}\left[e^{a\Delta(n,z)}\middle|W^z = w\right] \le \mathbb{E}\left[e^{a|Z^z-W^z|}\middle|W^z = w\right] \times CA^{s-1}_n\left(e^{b|w-kp|^2/k} + e^{b|z-w-kp|^2/k}\right).$$

In deriving the last expression we used that our two $k$-couplings satisfy (4.8.4) and the simple inequality $\mathbb{E}[e^{\max(Z_1,Z_2)}] \le \mathbb{E}[e^{Z_1}] + \mathbb{E}[e^{Z_2}]$. Taking expectation on both sides above we see that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq C \cdot (2c_1\sqrt{n}) \cdot A_n^{s-1} \sum_{w=0}^{k} \mathbb{P}(W^z = w) \exp\left(\frac{9}{4} \cdot \frac{b \max(|w-kp|^2, |z-w-kp|^2)}{n}\right). \tag{4.8.7}$$

In deriving the last expression we used (4.8.5) and the simple inequality $x^2+y^2 \leq 5\max(x^2, (x-y)^2)$ as well as that $k = n/2$.

We finally estimate the sum in (4.8.7) by splitting it over the $w$ such that $|w - z/2| > |z - pn|/6$ and $|w - z/2| \leq |z - pn|/6$. Notice that if $|w - z/2| \leq |z - pn|/6$ we have $\max(|w - pk|^2, |z - w - pk|^2) \leq (2|z - pn|/3)^2$; hence

$$\sum_{w:|w-z/2|\leq|z-pn|/6} \mathbb{P}(W^z = w) \exp\left(\frac{9}{4} \cdot \frac{\max(|w-kp|^2, |z-w-kp|^2)}{n}\right) \leq \exp\left(\frac{|z-pn|^2}{n}\right). \tag{4.8.8}$$

To handle the case $|w - z/2| > |z - pn|/6$ we use Lemma 4.8.3, from which we know that

$$\mathbb{P}(W^z = w) = \mathbb{P}(S_k = w | S_n = z) \leq c_2 n^{-1/2} \exp\left(-b_2 \frac{(w - (z/2))^2}{n}\right).$$

Using the latter together with the fact that for $|w-z/2| > |z-pn|/6$ we have that $(w-z/2)^2 > \frac{1}{16}\max\left((w-kp)^2, |z-w-kp|^2\right)$ we see that

$$\begin{aligned}\sum_{w:|w-z/2|>|z-pn|/6} \mathbb{P}(W^z = w) \exp\left(\frac{9}{4} \cdot \frac{b\max(|w-kp|^2, |z-w-kp|^2)}{n}\right) \leq \\ \sum_{w=1}^{k} c_2 n^{-1/2} \exp\left(-\frac{b}{16} \cdot \frac{(w-kp)^2}{n}\right) \leq c_2\sqrt{n}.\end{aligned} \tag{4.8.9}$$

Combining the above estimates we see that

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \leq C \cdot (2c_1\sqrt{n}) \cdot A_n^{s-1} \left[\exp\left(\frac{|z-pn|^2}{n}\right) + c_2\sqrt{n}\right] \leq C \cdot A_n^s \exp\left(\frac{|z-pn|^2}{n}\right).$$

□

### 4.8.2 Proof of Lemmas 4.8.2 and 4.8.3

Our proofs of Lemma 4.8.2 and 4.8.3 will mostly follow (appropriately adapted) arguments from Sections 6.4 and 6.5 in [60]. We begin with two technical lemmas.

**Lemma 4.8.4.** *There is a constant $c > 0$ (depending on $p$) such that for integers $m, n, z$ and real $w$ with $n \geq 2$, $|2m - n| \leq 1$, $|z - pn| \leq cn$, $|w| \leq cn$ and $w + \frac{m}{n}z \in \mathbb{N}$ one has*

$$\mathbb{P}\left(S_m = w + \frac{m}{n}z \middle| S_n = z\right) = \frac{1}{\sqrt{2\pi\sigma_{n,z}^2}} \exp\left(-\frac{w^2}{2\sigma_{n,z}^2} + O\left(\frac{1}{\sqrt{n}} + \frac{|w|^3}{n^2}\right)\right), \tag{4.8.10}$$

*where* $\sigma_{n,z}^2 = (n/4)(z/n)(1-z/n)$.

*Proof.* The result is similar to Lemma 6.7 in [60] and we only sketch the main ideas. The statement of the lemma will follow if we can show that if $|j| \le cn$ we have

$$p(j,m,n,z) = \mathbb{P}\left(\lfloor S_m - \frac{m}{n}z\rfloor = j \middle| S_n = z\right) = \frac{1}{\sqrt{2\pi\sigma_{n,z}^2}}\exp\left(-\frac{j^2}{2\sigma_{n,z}^2} + O\left(\frac{1}{\sqrt{n}} + \frac{j^3}{n^2}\right)\right).$$

Using Stirling's approximation formula $A! = \sqrt{2\pi}A^{A+1/2}e^{-A}[1+O(A^{-1})]$, we see that

$$p(0,m,n,z) = \mathbb{P}\left(\lfloor S_m - \frac{m}{n}z\rfloor = 0 \middle| S_n = z\right) = \frac{1}{\sqrt{2\pi\sigma_{n,z}^2}}\left(1 + O\left(n^{-1}\right)\right).$$

Let us remark that in order to apply Stirling's approximation, we needed to choose $c$ sufficiently small so that $\frac{m}{n}z$, $z$, $m - \frac{m}{n}z$, $n-z$ all tend to infinity faster than $\epsilon n$ for some $\epsilon > 0$ fixed (depending on $p$) as $n \to \infty$. For the remainder we assume such a $c$ is chosen and the constant in the big $O$ notation above depends on it.

Let us focus on the case $j > 0$ (if $j < 0$ a similar argument can be applied). For $j > 0$ and $A(j,m,n,z) = \frac{\left(m+z-2\lceil \frac{m}{n}z\rceil - 2j\right)^2 - (m-z)^2}{\left(2\lceil\frac{m}{n}z\rceil + 2j + 2 + m - z\right)^2 - (m-z)^2}$ we have

$$p(j+1,m,n,z) = p(j,m,n,z) \times A(j,m,n,z)$$

and so

$$p(j,m,n,z) = p(0,m,n,z) \times \prod_{i=1}^{j} A(i,m,n,z).$$

Given our earlier result for $p(0,m,n,z)$ to finish the proof it remains to show that

$$\sum_{i=1}^{j}\log\left[A(j,m,n,z)\right] = -\frac{j^2}{2\sigma_{n,z}^2} + O\left(\frac{1}{\sqrt{n}} + \frac{j^3}{n^2}\right). \tag{4.8.11}$$

Notice that if we choose $c$ sufficiently small, we have that

$$A(j,m,n,z) = 1 - B(j,m,n,z), \text{ where } B(j,m,n,z) = \frac{8jm}{m^2 - (m-z)^2} + O\left(\frac{j}{n^2} + \frac{1}{n}\right)$$

and $0 \le B(j,m,n,z) \le \frac{1}{2}$. Using the latter together with the fact that $\log(1+x) = x + O(x^2)$ for $|x| \le 1/2$ we get

$$\sum_{i=1}^{j}\log\left[A(j,m,n,z)\right] = -\sum_{i=1}^{j}\frac{8im}{m^2-(m-z)^2} + O\left(\frac{j^2}{n^2} + \frac{j}{n}\right) = -\frac{4j^2m}{m^2-(m-z)^2} + O\left(\frac{j^3}{n^2} + \frac{1}{\sqrt{n}}\right).$$

To conclude the proof we observe that

$$\frac{4j^2m}{m^2-(m-z)^2} = \frac{j^2}{m\cdot\frac{z}{2m}\cdot\left(1-\frac{z}{2m}\right)} = \frac{j^2}{\frac{n}{2}\cdot\frac{z}{n}\cdot\left(1-\frac{z}{n}\right)} + O\left(\frac{j^2}{n^2}\right) = \frac{j^2}{2\sigma_{n,z}^2} + O\left(\frac{j^3}{n^2} + \frac{1}{\sqrt{n}}\right).$$

□

We now state without proof an easy large deviation estmiate, which can be established in the same way one establishes large deviations for binomial random variables.

**Lemma 4.8.5.** *There exists an $\eta > 0$ (depending on $p$) such that, for any $a > 0$, there exist $C = C(a) < \infty$ and $\gamma = \gamma(a) > 0$ with the following properties. For any integers $m, n, z$ with $n \geq 2$, $|2m - n| \leq 1$, $|z - pn| \leq \eta n$ one has*

$$\mathbb{P}\left(\left|S_m - \frac{m}{n}\right| > am \middle| S_n = z\right) \leq Ce^{-\gamma m}. \tag{4.8.12}$$

It is clear that Lemmas 4.8.4 and 4.8.5 imply Lemma 4.8.3. What remains is to prove Lemma 4.8.2, to which we now turn.

*Proof.* (Lemma 4.8.2) Notice that we only need to prove the lemma for $n$ sufficiently large. In order to simplify the notation we will assume that $n$ is even and so $m = n/2$ (the case $n$ odd can be handled similarly).

We start by choosing $\epsilon_0 \leq \min(c, \eta)$ with $c$ and $\eta$ as in Lemmas 4.8.4 and 4.8.5 respectively. We denote

$$Z = Z_{n,z} = z/2 + \sqrt{p(1-p)n/4}N, \quad \hat{Z} = \hat{Z}_{n,z} = z/2 + \sigma_{n,z}N,$$

where we recall that $\sigma_{n,z}^2 = (n/4)(z/n)(1 - z/n)$ and let $W = W_{n,z}$ be the random variable with distribution $S_{n/2}^{(n,z)}$ that is quantile coupled with $N$. Notice that $W$ is also quantile coupled with $Z$ and $\hat{Z}$. We write $F = F_{n,z}$ for the distribution function of $\hat{Z}$ and $G = G_{n,z}$ for the distribution function of $W$. We observe that from Lemmas 4.8.4 and 4.8.5, the random variable $W - \lfloor z/2 \rfloor$ satisfies the conditions of Lemma 6.9 in [60], from which we deduce that there are constants $c', \epsilon' > 0$ and $N' \in \mathbb{N}$ such that for $n \geq N'$ and $|x - z/2| \leq \epsilon' n$ we have

$$F\left(x - c'\left[1 + \frac{(x - z/2)^2}{n}\right]\right) \leq G(x-1) \leq G(x+1) \leq F\left(x + c'\left[1 + \frac{(x - z/2)^2}{n}\right]\right). \tag{4.8.13}$$

In the remainder we assume $\epsilon_0 \leq \epsilon'$ as well. It follows from (4.8.2) and (4.8.13) that

$$|\hat{Z} - W| \leq c'\left[1 + \frac{(W - z/2)^2}{n}\right], \tag{4.8.14}$$

for all $n \geq N'$, provided that $|z - pn| \leq \epsilon_0 n$, $|W - z/2| \leq \epsilon_0 n$. In addition, we have the following string of inequalities for any $a > 0$

$$\mathbb{E}\left[e^{a|Z-\hat{Z}|}\middle| W = w\right] \leq \mathbb{E}\left[e^{a(Z-\hat{Z})} + e^{-a(Z-\hat{Z})}\middle| W = w\right] \leq \frac{\mathbb{E}\left[e^{a(Z-\hat{Z})} + e^{-a(Z-\hat{Z})}\right]}{\mathbb{P}(W = w)} = \frac{2e^{a^2\sigma(n,p)^2/2}}{\mathbb{P}(W = w)},$$

where $\sigma(n,p) = \sqrt{n/4} \cdot \left|\sqrt{p(1-p)} - \sqrt{(z/n)(1 - z/n)}\right|$. It follows from Lemma 4.8.4 that if $|w - z/2| \leq \epsilon_0 n$ and $|z - pn| \leq \epsilon_0 n$ then we have for some $C > 0$ and all $n \geq 2$ that

$$\mathbb{E}\left[e^{a|Z-\hat{Z}|}\middle|W = w\right] \leq Ce^{a^2\sigma(n,p)^2/2}\sqrt{n} \cdot \exp\left(C\frac{(w - z/2)^2}{n}\right). \tag{4.8.15}$$

Combining (4.8.14) and (4.8.15) we see that for some (possibly larger than before) $C > 0$ we have

$$\mathbb{E}\left[e^{a|W-Z|}\middle|W = w\right] \leq \mathbb{E}\left[e^{a|W-\hat{Z}|}e^{a|Z-Z|}\middle|W = w\right] \leq Ce^{a^2\sigma(n,p)^2/2}\sqrt{n} \cdot \exp\left(C\frac{(w - z/2)^2}{n}\right), \tag{4.8.16}$$

provided $n \geq N'$, $|w - z/2| \leq \epsilon_0 n$ and $|z - pn| \leq \epsilon_0 n$.

Notice that by possibly taking $\epsilon_0$ smaller we can make $\sigma(n,p) \leq \sqrt{n/4} \cdot c_p|z/n - p|$, where $c_p = \frac{2}{p(1-p)}$. Using the latter together with (4.8.16) and Jensen's inequality we have for any $k \in \mathbb{N}$ that

$$\mathbb{E}\left[e^{(1/k)|W-Z|}\middle|W = w\right] \leq \mathbb{E}\left[e^{|W-Z|}\middle|W = w\right]^{1/k} \leq (\sqrt{n}C)^{1/k} \cdot \exp\left(\frac{c_p(z - pn)^2}{nk} + C\frac{(w - z/2)^2}{nk}\right),$$

and if we further use that $(x + y)^2 \leq 2x^2 + 2y^2$ above we see that

$$\mathbb{E}\left[e^{(1/k)|W-Z|}\middle|W = w\right] \leq (\sqrt{n}C)^{1/k} \cdot \exp\left(\frac{(c_p + 1/2)(z - pn)^2}{nk} + \frac{2C(w - pm)^2}{nk}\right), \tag{4.8.17}$$

provided $n \geq N'$, $|w - z/2| \leq \epsilon_0 n$ and $|z - pn| \leq \epsilon_0 n$.

Suppose now that $b_1$ is given, and let $k$ be sufficiently large so that

$$\frac{c_p + 1/2}{k} \leq b_1 \text{ and } \frac{2C}{k} \leq b_1.$$

If $a_1 \leq 1/k$ we see from (4.8.17) that

$$\mathbb{E}\left[e^{a_1|W-Z|}\middle|W = w\right] \leq C^{1/k}\sqrt{n} \cdot \exp\left(\frac{b_1(z - pn)^2}{n} + \frac{b_1(w - pm)^2}{n}\right), \tag{4.8.18}$$

provided $n \geq N'$, $|w - z/2| \leq \epsilon_0 n$ and $|z - pn| \leq \epsilon_0 n$. If $|z - pn| > \epsilon_0 n$ or $|w - z/2| > \epsilon_0 n$ we observe that

$$\frac{b_1(z - pn)^2}{n} + \frac{b_1(w - pm)^2}{n} \geq \frac{b_1\epsilon_0^2 n}{3}.$$

One easily observes that if $a_1 \leq a_0$ with $a_0$ sufficiently small and $C \geq \tilde{c}$ with $\tilde{c}$ sufficiently large we have for any $w$ such that $\mathbb{P}(W = w) > 0$ that

$$\mathbb{E}\left[e^{a_1|W-Z|}\middle|W = w\right] \leq C^{1/k}\sqrt{n} \cdot \exp\left(\frac{b_1\epsilon_0^2 n}{3}\right).$$

The latter statements suggest that (4.8.18) holds for all $w$ such that $\mathbb{P}(W = w) > 0$ and $n \geq N'$, which concludes the proof of the lemma. □

# Bibliography


[1] A. Aggarwal, *Current fluctuations of the stationary ASEP and six-vertex model*, (2016), Preprint, arXiv:1608.04726.

[2] ———, *Current fluctuations of the stationary ASEP and six-vertex model*, (2016), Preprint, arXiv:1608.04726.

[3] ———, *Convergence of the stochastic six-vertex model to the ASEP*, Math. Phys. Anal. Geom. **20** (2017).

[4] A. Aggarwal and A. Borodin, *Phase transitions in the ASEP and stochastic six-vertex model*, (2016), Preprint, arXiv:1607.08684.

[5] T. Alberts, K. Khanin, and J. Quastel, *The intermediate disorder regime for directed polymers in dimension 1 + 1*, Ann. Probab. **42** (2014), 1212–1256.

[6] G. Amir, I. Corwin, and J. Quastel, *Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions*, Comm. Pure Appl. Math. **64** (2011), 466–537.

[7] G. Andrews, R. Askey, and R. Roy, *Special functions*, Cambridge University Press, Cambridge, 2000.

[8] Y. Baryshnikov, *Gues and queues*, Probab. Theory Relat. Fields **119** (2001), 256–274.

[9] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, reprint of the 1982 original, Academic Press, London, 1989.

[10] L. Bertini and G. Giacomin, *Stochastic Burgers and KPZ equations from particle systems*, Comm. Math. Phys. **183** (1997), 571–607.

[11] D. Betea, C. Boutillier, G. Chapuy, S. Corteel, and M. Vuletić, *Perfect sampling algorithm for Schur processes*, (2014), Preprint, arXiv:1407.3764.

[12] D. Betea and M. Wheeler, *Refined Cauchy and Littlewood identities, plane partitions and symmetry classes of alternating sign matrices*, J. Comb. Theory Ser. A. **137** (2016), 126–165.

[13] P. Billingsley, *Convergence of probability measures, Second edition*, Academic Press, New York, 1999.

[14] O. Bodini, É. Fusy, and C. Pivoteau, *Random sampling of plane partitions*, Comb., Prob. and Comp. **19** (2010), 201–226.

[15] V. Bogachev, *Measure Theory Vol. I*, Springer, Berlin, 2007.

[16] N.M. Bogoliubov, A.G. Pronko, and M.B. Zvonarev, *Boundary correlation functions of the six-vertex model*, J. Phys. A **35** (2002), 5525–5541.

[17] A. Borodin, *Schur dynamics of the Schur processes*, Adv. Math. **228** (2011), 2268–2291.

[18] ______, *CLT for spectra of submatrices of Wigner random matrices*, Mosc. Math. J. **14** (2014), 29–38.

[19] ______, *Stochastic higher spin six vertex model and Macdonald measures*, (2016), Preprint, arXiv:1608.01553.

[20] ______, *On a family of symmetric rational functions*, Adv. Math. **306** (2017), 973–1018.

[21] A. Borodin and A. Bufetov, *An irreversible local Markov chain that preserves the six vertex model on a torus*, (2015), Preprint, arXiv:1509.05070.

[22] A. Borodin, A. Bufetov, and I. Corwin, *Directed random polymers via nested contour integrals*, (2015), Preprint, arXiv:1511.07324v1.

[23] A. Borodin, A. Bufetov, and M. Wheeler, *Between the stochastic six vertex model and Hall-Littlewood processes*, (2016), Preprint, arXiv:1611.09486.

[24] A. Borodin and I. Corwin, *Macdonald processes*, Probab. Theory Related Fields **158** (2014), 225–400.

[25] ______, *Discrete time q-TASEPs*, Int. Math. Res. Notices (2015), doi:10.1093/imrn/rnt206.

[26] A. Borodin, I. Corwin, and P. L. Ferrari, *Free energy fluctuations for directed polymers in random media in 1 + 1 dimension*, Comm. Pure and Appl. Math. **67** (2014), 1129–1214.

[27] A. Borodin, I. Corwin, and V. Gorin, *Stochastic six-vertex model*, Duke Math. J. **165** (2016), 563–624.

[28] A. Borodin, I. Corwin, V. Gorin, and S. Shakirov, *Observables of Macdonald processes*, Trans. Amer. Math. Soc. to appear. (2013), Preprint, arXiv:1306.0659.

[29] A. Borodin, I. Corwin, and D. Remenik, *Log-Gamma polymer free energy fluctuations via a Fredholm determinant identity*, Commun. Math. Phys. **324** (2013), 215–232.

[30] A. Borodin and P. L. Ferrari, *Anisotropic growth of random surfaces in 2 + 1 dimensions*, Comm. Math. Phys. **325** (2014), 603–684.

[31] A. Borodin and V. Gorin, *Shuffling algorithm for boxed plane partitions*, Adv. Math. **220**, 1739–1770.

[32] ———, *Shuffling algorithm for boxed plane partitions*, Adv. Math. **220** (2009), 1739–1770.

[33] A. Borodin and L. Petrov, *Higher spin six vertex model and symmetric rational functions*, (2016), Preprint, arXiv:1601.05770.

[34] A. Bufetov and K. Matveev, *Hall-Littlewood RSK field*, (2017), Preprint, arXiv:1705.07169.

[35] P. Calabrese, P.L. Doussal, and A. Rosso, *Free-energy distribution of the directed polymer at high temperature*, Euro. Phys. Lett. **90** (2010), :20002.

[36] R. Cerf and R. Kenyon, *The low-temperature expansion of the Wulff crystal in the 3d Ising model*, Com. in Math. Phys. **222** (2001), 147–179.

[37] E. Çinlar, *Probability and stochastics*, Springer, New York, 2011.

[38] F. Colomo, A.G. Pronko, and A. Sportiello, *Generalized emptiness formation probability in the six-vertex model*, J. Phys. A: Math. Theor. **49** (2016).

[39] F. Colomo and A. Sportiello, *Arctic curves of the six-vertex model on generic domains: The tangent method*, J. Stat. Phys. **164** (2016), 1488–1523.

[40] I. Corwin, *The Kardar-Parisi-Zhang equation and universality class*, Random Matrices Theory Appl. **1** (2012).

[41] I. Corwin and E. Dimitrov, *Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood Gibbsian line ensembles*, Comm. Math. Phys., , to appear, 2017. Preprint, arXiv:1703.07180.

[42] I. Corwin and A. Hammond, *Brownian Gibbs property for Airy line ensembles*, Invent. Math. **195** (2014), 441–508.

[43] ———, *KPZ line ensemble*, (2015), Preprint, arXiv:1312.2600v2.

[44] I. Corwin and L. Petrov, *Stochastic higher spin vertex models on the line*, Comm. Math. Phys. **343** (2016), 651–700.

[45] M. Defosseux, *Orbit measures, random matrix theory and interlaced determinantal processes*, Ann. Inst. Henri Poincaré Probab. Stat. **46** (2010), 209–249.

[46] P. Diaconis and J. A. Fill, *Strong stationary times via a new form of duality*, Ann. Probab. **18** (1990), 1483–1522.

[47] E. Dimitrov, *KPZ and Airy limits of Hall-Littlewood random plane partitions*, Ann. Inst. Henri Poincaré Probab. Stat., , to appear, 2016. Preprint, arXiv:1602.00727.

[48] ———, *Six-vertex models and the GUE-corners process*, Int. Math. Res. Notices, , to appear, 2016. Preprint, arXiv:1610.06893.

[49] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, *Alternating-sign matrices and domino tilings*, **1** (1992), 111–132, 219–234, J. Algebraic Combin.

[50] P. L. Ferrari and B. Veto, *Tracy-Widom asymptotics for q-TASEP*, (2014), Preprint, arXiv:1310.2515v4.

[51] P.L. Ferrari and H. Spohn, *Step fluctuations for a faceted crystal*, J. Stat. Phys. **113** (2003), 1–46.

[52] V. Gorin, *From alternating sign matrices to the Gaussian uinitary ensemble*, Commun. Math. Phys. **332** (2014), 437–447.

[53] H.-L. Gwa and H. Spohn, *Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian*, Phys. Rev. Lett. **68** (1992), 725–728.

[54] K. Johansson and E. Nordenstam, *Eigenvalues of GUE minors*, Electron. J. Probab. **11** (2006), 1342–1371.

[55] O. Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, New York, 1997.

[56] I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*, Volume 113 of Graduate Texts in Mathematics, Springer, 1988.

[57] R. Kenyon and A. Okounkov, *Limit shapes and the complex Burgers equation*, Acta Math. **199** (2007), 263–302.

[58] V. Korepin, *Calculations of norms of Bethe wave functions*, Commun. Math. Phys. **86** (1982), 391–418.

[59] C. Korff, *Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra*, Commun. Math. Phys. **318** (2013), 173–246.

[60] G.F. Lawler and J.A. Trujillo-Ferreras, *Random walk loop-soup*, Trans. Amer. Math. Soc. **359** (2007), 767–787.

[61] P. D. Lax, *Functional analysis*, Wiley-Interscience, New York, 2002.

[62] D. Levin, Y. Peres, and E. Wilmer, *Markov chains and mixing times*, AMS, Providence, 2008.

[63] T. Liggett, *Interacting Particle Systems*, Springer, Berlin, 2005.

[64] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2 ed., Oxford University Press Inc., New York, 1995.

[65] W. H. Mills, D. P. Robbins, and H. Rumsey, *Alternating sign matrices and descending plane partitions*, J. Comb. Theory. Series A. **34** (1983), 340–359.

[66] C. Mueller, *On the support of solutions to the heat equation with noise*, Stochastics **37** (1991), 225–246.

[67] C. Mueller and R. Nualart, *Regularity of the density for the stochastic heat equation*, Electron. J. Probab. **13** (2008), 2248–2258.

[68] E. Nordenstam, *Interlaced particles in tilings and random matrices*, Doctoral Thesis, KTH (2009).

[69] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003), 581–603.

[70] ———, *The birth of a random matrix*, Mosc. Math. J. **6** (2006), 553–566.

[71] G. Olshanski and A. Vershik, *Ergodic unitarily invariant measures on the space of infinite Hermitian matrices*, Amer. Math. Soc. Transl. Ser. 2 **175** (1996), 137–175.

[72] K. R. Parthasarathy, *Probability measures on metric spaces*, Wiley, New York, 1967.

[73] M. Prähofer and H. Spohn, *Scale invariance of the PNG Droplet and the Airy process*, J. Stat. Phys. **108** (2002), 1071–1106.

[74] N. Reshetikhin, *"Lectures on the integrability of the six-vertex model"* in Exact methods in Low-dimensional Statistical Physics and Quantum Computing (Les Houches, 2008), Oxford Univ. Press, Oxford, 2010.

[75] B. Simon, *Trace ideals and their applications*, 2 ed., AMS, Providence, 2005.

[76] F. Spitzer, *Interaction of Markov processes*, Adv. Math. **5** (1970), 246–290.

[77] E. Stein and R. Shakarchi, *Complex analysis*, Princeton University Press, Princeton, 2003.

[78] C. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, Comm. Math. Phys. **159** (1994), 151–174.

[79] ———, *Asymptotics in ASEP with step initial condition*, Commun. Math. Phys. **290** (2009), 129–154.

[80] M. Vuletić, *The shifted Schur process and asymptotics of large random plane partitions*, Int. Math. Res. Not. IMRN **14** (2007), Art. ID rnm043, 53 pp. MR2349310 (2009c:60029).

[81] ———, *A generalization of MacMahon's formula*, Trans. Amer. Math. Soc. **361** (2009), 2789–2804.

[82] J. Warren, *Dyson's Brownian motions, intertwining and interlacing*, Electr. J. Probab. **7** (2007), 573–590.