Localization at b_{10} in the stable category of comodules over the Steenrod reduced powers

by

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Abstract

Chromatic localization can be seen as a way to calculate a particular infinite piece of the homotopy of a spectrum. For example, the (finite) chromatic localization of a p-local sphere is its rationalization, and the corresponding chromatic localization of its Adams E_2 page recovers just the zero-stem. We study a different localization of Adams E_2 pages for spectra, which recovers more information than the chromatic localization. This approach can be seen as the analogue of chromatic localization in a category related to the derived category of comodules over the dual Steenrod algebra, a setting in which Palmieri has developed an analogue of chromatic homotopy theory. We work at $p = 3$ and compute the E_2 page and first nontrivial differential of a spectral sequence converging to $b_{10}^{-1} \operatorname{Ext}_P^*(\mathbb{F}_3, \mathbb{F}_3)$ (where P is the Steenrod reduced powers), and give a complete calculation of other localized Ext groups, including $b_{10}^{-1} \operatorname{Ext}_P^*(\mathbb{F}_3, \mathbb{F}_3[\xi_1^3])$.

Thesis Supervisor: Haynes Miller
Title: Professor of Mathematics

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Contents

0.1 Notation

We use the following notation extensively in this thesis.

k	$\mathbb{F}_p$ (for p odd, specialized to $p=3$ in Chapter 4)
$M \Box_\Gamma N$	cotensor product of Γ-comodules M, N
$E[x_1, \dots, x_n]$	Exterior algebra $k[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$
$P[x_1, \dots, x_n]$	Polynomial algebra $k[x_1, \dots, x_n]$
$D[x_1, \dots, x_n]$	Truncated height-p polynomial algebra $k[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$
$\overline{R}$	Coaugmentation ideal $\text{coker}(k \to R)$, for a unital k-algebra R
A	mod-p Steenrod dual $P[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$
P	Steenrod reduced powers algebra $P[\xi_1, \xi_2, \dots]$
ξ_n	Antipode of the generator usually called ξ_n (see Notation 4.1.5)
Comod_Γ	Category of Γ-comodules (for a coalgebra Γ)
$\text{Stable}(\Gamma)$	See Definition 2.1.6
Ext_Γ	Derived functors of $\text{Hom}_{\text{Comod}_\Gamma}$ (see §2.1)
Sp	Any symmetric monoidal model for the category of spectra
$\pi_{**}(M)$	$\text{Hom}_{\text{Stable}(\Gamma)}(k, M)$ (see Notation 2.1.13)
D	$D[\xi_1]$
B	$P \Box_D k = k[\xi_1^3, \xi_2, \xi_3, \dots]$ (see Notation 4.1.5)
$K(\xi_1)$	$\text{colim}(B \xrightarrow{b_{10}} B \xrightarrow{b_{10}} \dots)$ as an object of $\text{Stable}(P)$
R	$b_{10}^{-1} \text{Ext}_D(k,k) = E[h_{10}] \otimes P[b_{10}^{\pm}]$
$C_\Gamma^*(M,N)$	Cobar complex $\mathcal{N}\overset{L}{C}{}_\Gamma^*(M,N)$ as defined in Definitions 3.1.5 and 3.1.11
$\overset{\Delta}{\otimes}, \overset{L}{\otimes}, \overset{R}{\otimes}$	Diagonal, left, and right comodule structures on a tensor product (see Definition 3.1.1)
MPASS	See Definition 2.2.2
$M(n)$	D-comodule isomorphic to $k[\xi_1]/\xi_1^{n+1}$

Chapter 1

Introduction

The goal of the work described in this thesis is to compute graded abelian groups of the form

$$b_{10}^{-1} \operatorname{Ext}_P^*(\mathbb{F}_3, M)$$

where:

- P is the mod-3 Steenrod reduced powers algebra $\mathbb{F}_3[\xi_1, \xi_2, \dots]$,
- $b_{10} \in \operatorname{Ext}_P^2(\mathbb{F}_3, \mathbb{F}_3)$ is the element with cobar representative $[\xi_1|\xi_1^2] + [\xi_1^2|\xi_1]$, and
- M is a P-comodule.

The main focus is the case where $M = \mathbb{F}_3$: we describe the E_2 page and first nontrivial differentials of a spectral sequence converging to $b_{10}^{-1} \operatorname{Ext}_P^*(\mathbb{F}_3, \mathbb{F}_3)$. We also have complete calculations of $b_{10}^{-1} \operatorname{Ext}_P^*(\mathbb{F}_3, M)$ for some other comodules M, and a conjecture about the general structure of these Ext groups. We will begin by explaining the motivation for this project by situating it within the larger context of chromatic homotopy theory. Later in this chapter we will give a summary of our main techniques and results.

1.1 Motivation: chromatic localizations in Stable(P)

This section is structured as follows: first we will describe a class of objects we would like to better understand (Adams E_2 pages), then we will describe an approximation technique (chromatic localization) that does not seem to immediately apply to the desired objects of study, and finally we will describe a way to apply the technique to the objects of study (by re-constructing the machinery of chromatic homotopy theory within an algebraic category related to Adams E_2 pages). None of the work described in this section is ours; the main ingredients are the nilpotence and periodicity theorems of Devinatz-Hopkins-Smith and the work by Palmieri about stable categories of comodules. Unless explicitly stated otherwise, we will work localized at an odd prime p, which will eventually be specialized to 3, and write $k = \mathbb{F}_p$.

1.1.1 Adams E_2 pages

Given a finite p-local spectrum X, there is an Adams spectral sequence

$$E_2(X) = \mathrm{Ext}^*_A(k, H_*(X)) \implies \pi_* X^\wedge_p$$

converging to the p-complete homotopy of X. Here A is the mod-p Steenrod algebra dual $k[\xi_1, \xi_2, \dots, \tau_0, \tau_1, \dots]/(\tau_i)^2$, viewed as a Hopf algebra, and (as will be the case throughout this document) Ext denotes comodule Ext. For spectra X of interest, such as the p-local sphere, the E_2 page is more computationally tractable than $\pi_* X$: the E_2 page is purely algebraic, and can be computed algorithmically in a finite range. However, for many X of interest, there is no hope of obtaining a closed-form formula for $E_2(X)$, and its structure encodes deep information about $\pi_* X$. Thus, we are interested in obtaining information about the structure of graded abelian groups of the form $\mathrm{Ext}^*_A(k, M)$ for A-comodules M.

Let $P = k[\xi_1, \xi_2, \dots]$ be the Steenrod reduced powers algebra, and let E be the quotient

Hopf algebra $k[\tau_0, \tau_1, \dots]/(\tau_i^2)$. Then there is an extension of Hopf algebras $P \to A \to E$, which gives rise to a Cartan-Eilenberg spectral sequence

$$E_2^{**} = \mathrm{Ext}_P^*(k, \mathrm{Ext}_E^*(k, M)) \implies \mathrm{Ext}_A^*(k, M).$$

There is a third grading on this spectral sequence that comes from powers of E which causes it to collapse at E_2 when M has trivial E-coaction—in particular, when $M = k$, which corresponds to the case $X = S$. This motivates us to consider the following goal:

Goal 1.1.1. Study $\mathrm{Ext}_P^*(k, N)$ where N is a P-comodule.

1.1.2 Chromatic localization

Now we will review chromatic localization from the perspective of Adams spectral sequence vanishing lines. The main idea is that, given a finite p-local spectrum X, there is a localization $v_n^{-1}\pi_*X$ of π_*X that can be seen as an approximation to π_*X in the sense that $v_n^{-1}\pi_*X$ is often easier to compute than π_*X and it agrees with π_*X in an infinite region.

Theorem 1.1.2 (Hopkins-Smith [HS98]). *There is a filtration of the category* Sp_p^{fin} *of* p*-local finite spectra*

$$\mathrm{Sp}_p^{fin} = \mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \dots$$

such that if X *is in* $\mathcal{C}_n$ *and not* $\mathcal{C}_{n+1}$*, there is a non-nilpotent self-map* $\Sigma^k X \to X$ *(for some* k*) satisfying certain nice properties, which we denote* v_n^i*. This gives rise to a non-nilpotent operator* v_n^i *on every page of the Adams spectral sequence.*

If X is in $\mathcal{C}_n$ and not $\mathcal{C}_{n+1}$, we say that X has *type* n.

Theorem 1.1.3 (Hopkins-Palmieri-Smith [HPS99]). *Suppose* X *is a finite* p*-local spectrum of type* n*. Then* $E_\infty(X)$ *vanishes above a line of slope* $\frac{1}{|\tau_n|-1}$ *(which* v_n^i *acts* *parallel to), and in the wedge between this line and a lower line of slope* $\frac{1}{|\tau_{n+1}|-1}$*, we have*

$$E_\infty(X) \cong v_n^{-1} E_\infty(X)$$

where $v_n^{-1}E_\infty(X) = \mathrm{colim}(E_\infty(X) \xrightarrow{v_n^i} E_\infty(X) \xrightarrow{v_n^i} \dots)$.

Chromatic localization is v_n-localization, and this theorem shows that if we know $v_n^{-1}\pi_* X$, we know an infinite amount of information about $\pi_* X$. (Of course, since the v_n-periodic region is defined in terms of Adams filtration, we do not learn $\pi_k X$ for any given stem k.)

Example 1.1.4. The sphere spectrum has type 0, and the first chromatic localization is $v_0^{-1}\pi_* \widehat{S_p} = p^{-1}\pi_*\widehat{S_p} = \pi_* S \otimes \mathbb{Q}$. Serre [Ser53] proved that $\pi_* S \otimes \mathbb{Q} = \pi_0 S \otimes \mathbb{Q} = \mathbb{Q}$. Theorem 1.1.3 only makes guarantees for vanishing and periodicity in the E_∞ page of the Adams spectral sequence, but in this case we can see this illustrated in E_2.

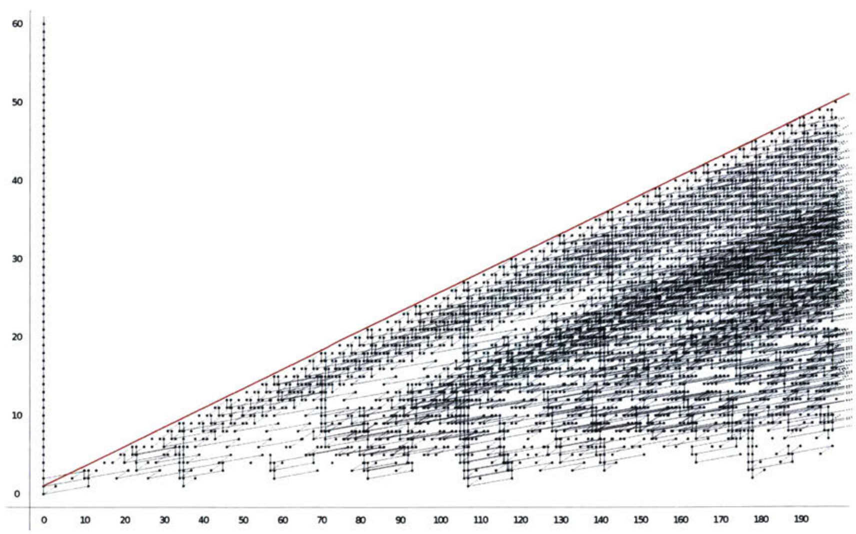


Figure 1-1: $E_2(S)$ at $p = 3$, with line above which this is p-periodic

The vanishing line has infinite slope, and the line drawn in the picture is the line above which $\pi_* S$ is p-periodic; the only elements in the p-periodic wedge are powers of the class $[\tau_0]$ representing the map $S \xrightarrow{p} S$.

Example 1.1.5. The mod-p Moore spectrum S/p has type 1. At odd primes, Miller [Mil78, Corollary 3.6] showed that

$$v_1^{-1}\pi_*(S/p) = P[q_1^{\pm 1}] \otimes E[h_{i,0} \, : \, i \geq 1] \otimes P[b_{i,0} \, : \, i \geq 1]$$

(where $q_1 = [\tau_1]$) by computing $v_1^{-1}E_2(S/p)$ and showing that the spectral sequence collapses at E_2.

These localizations give information about π_*S: given an element $x \in \pi_*(S/p)$ we can form an infinite family $S \to S/p \xrightarrow{v_1^k} S/p \xrightarrow{x} S$ (where the first map is inclusion of the bottom cell), and similarly one studies infinite v_n-periodic families in π_*S for higher n.

1.1.3 Chromatic localization in Stable(P)

Chromatic localization, as described above, gives information about homotopy groups of spectra, not Ext groups. We will describe an algebraic category Stable(P) and describe Palmieri's construction of a partial analogue of chromatic homotopy theory in this category, such that the analogue of chromatic localization gives information about $\mathrm{Ext}_P^*(k, M)$ for P-comodules M, in accordance with Goal 1.1.1.

We will give a fuller summary of the construction and properties of Stable(P) in Section 2.1, but for now define Stable(P) as the category whose objects are unbounded cochain complexes of injective P-comodules, and whose morphisms are chain complex morphisms modulo chain homotopy. The idea is that it is a modification of the derived category of P-comodules $D(P)$, in order for it to be better-behaved for localizations. There is a functor $i : \mathrm{Comod}_P \to \mathrm{Stable}(P)$ taking a P-comodule to an injective resolution, and the important property is that, similar to $D(P)$, we have

$$\mathrm{Hom}_{\mathrm{Stable}(P)}(i(M), i(N)) \cong \mathrm{Ext}_P^*(M, N)$$

for M, N in Comod_P.

This category has many structural similarities to the homotopy category of spectra Ho(Sp), and Palmieri [Pal01] proves algebraic analogues of many results in homotopy theory, including partial analogues of the nilpotence and periodicity theorems mentioned in Section 1.1.2. Some of these results are summarized in Section 2.3. The analogue of "homotopy groups"—maps in Ho(Sp) from the unit object to a certain object—is $\mathrm{Hom}_{\mathrm{Stable}(P)}(k, X)$, and if $X = i(M)$, this is $\mathrm{Ext}_P^*(k, M)$. In the analogue of chromatic homotopy theory in Stable(P), the full set of periodicity operators is difficult to enumerate explicitly, but it contains powers of the May spectral sequence elements $b_{ts} = \frac{1}{p}\sum_{0<i<p^{s+1}} \binom{p^{s+1}}{i} [\xi_t^i | \xi_t^{p^{s+1}-i}]$ for $s < t$. The corresponding chromatic localizations have the form $b_{ts}^{-1}\,\mathrm{Ext}_P^*(k, M)$, and (if M has the right analogue of "type" to have b_{ts}-periodicity) they agree with $\mathrm{Ext}_P^*(k, M)$ in a range of dimensions.

Classically, every p-local finite spectrum X has a unique type n, and only one chromatic localization $v_n^{-1}\pi_*(X)$ is defined and nonzero. An object of Stable(P) might have an action of multiple periodicity elements, and the analogue of the Nishida nilpotence theorem (which says that every element in $\pi_* S_{(p)}$ is nilpotent except for the multiplication-by-p map) is much more complicated: for example, at $p = 3$, $\mathrm{Ext}_P^*(k, k)$ has an action by at least two non-nilpotent operators, b_{10} and b_{11}.

Our main focus in this project is to study the Stable(P) analogue of Example 1.1.4—that is, to compute that localization of $\mathrm{Ext}_P^*(k, k)$ (the Stable(P) analogue of $\pi_* S$) by the first periodicity operator, namely b_{10}. Slightly more generally, we will discuss the computation of

$$b_{10}^{-1}\,\mathrm{Ext}_P^*(k, M)$$

for several P-comodules M. This can be thought of as an approximation of $\mathrm{Ext}_P^*(k, M)$

in the sense that this agrees with its localization above a line of $\frac{1}{p(p^2-1)-1}$. We end up specializing to $p = 3$; see Section 1.2.1.

1.1.4 Connection to motivic homotopy theory

Another part of the motivation for this project is its potential applications to motivic homotopy theory; this is part of planned future work.

The element $b_{10} \in \mathrm{Ext}^2_A(k, k)$ survives the Adams spectral sequence and converges to $\beta_1 \in \pi_* S$. While β_1 is nilpotent (and hence $\beta_1^{-1}\pi_* S = 0$) by the Nishida nilpotence theorem, it is non-nilpotent in $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, as well as in the homotopy of the p-complete $\mathbb{C}$-motivic sphere $(\mathbb{S}^{mot})^{\widehat{}}_p$. So studying its localization gives topological, as opposed to purely algebraic, information in the context of motivic homotopy theory.

In particular, there is an element $\tau \in \pi_{0,-1}((\mathbb{S}^{\mathrm{mot}})^{\widehat{}}_p)$ in the homotopy of the p-completed motivic sphere over $\mathbb{C}$ such that the realization map from motivic homotopy theory to classical homotopy theory corresponds to inverting τ. That is, the τ-periodic part of $\pi_{**}((\mathbb{S}^{\mathrm{mot}})^{\widehat{}}_p)$ corresponds to classical homotopy theory, and so recent work on understanding the unique properties of motivic homotopy theory centers around studying $C\tau$, the cofiber of multiplication by τ. Gheorghe, Wang, and Xu [GWX] show that $\pi_{**}C\tau \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, and the motivic Adams spectral sequence for $C\tau$ coincides with the algebraic Novikov spectral sequence

$$E_2 = \mathrm{Ext}_P(k, Q) \implies \mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$$

where $Q = \mathrm{Ext}^*_{E[\tau_0,\tau_1,\dots]}(k, k)$.

The element $b_{10} \in \mathrm{Ext}^*_P(k, Q)$ converges to β_1 in $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, which acts parallel to the vanishing line. So understanding the b_{10}-localization of the E_2 page of the algebraic Novikov spectral sequence is the first step to understanding $\beta_1^{-1}\,\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \cong \beta^{-1}\pi_{**}C\tau$ at $p = 3$. This would be the $p = 3$ analogue to Andrews and Miller's computation [AM17] of $\alpha^{-1}\operatorname{Ext}_{BP_*BP}(BP_*, BP_*)$ at $p = 2$.

1.2 Techniques

1.2.1 Margolis-Palmieri Adams spectral sequence

Our main technique is an Adams spectral sequence constructed in the category Stable(P): given a monoid object E in Stable(P) and another object X satisfying some finiteness conditions (see Proposition 2.2.5), there is a convergent spectral sequence

$$E_1 = \pi_{**}(E \otimes \overline{E}^{\otimes s} \otimes X) \implies \pi_{**}(X)$$

where $\pi_{**}(X)$ denotes $\operatorname{Hom}_{\mathsf{Stable}(P)}(k, X)$, and $\overline{E}$ is the cofiber of the unit map $k \to E$. If, in addition, $\pi_{**}(E \otimes E)$ is flat over $\pi_{**}(E)$ (an analogue of the Adams flatness condition), the E_2 page has the form $\operatorname{Ext}^*_{\pi_{**}(E\otimes E)}(\pi_{**}(E), \pi_{**}(E \otimes X))$. As this spectral sequence was first studied by Margolis [Mar83] and Palmieri [Pal01], we call it the *Margolis-Palmieri Adams spectral sequence* (abbreviated MPASS).

To study $b_{10}^{-1}\operatorname{Ext}^*_P(k, M)$, we apply the MPASS in the case where the monoid object E is

$$K(\xi_1) := \operatorname{colim}\left(i(P \,\square_{D[\xi_1]} k) \xrightarrow{b_{10}} i(P \,\square_{D[\xi_1]} k) \xrightarrow{b_{10}} \quad \dots \quad \right)$$

and $X = b_{10}^{-1} i(M)$. (In general we use the notation $D[x]$ to denote $k[x]/x^p$.) This satisfies Adams flatness at $p = 3$ but not for $p > 3$ (and the connective version $i(P \,\square_{D[\xi_1]} k)$ does not satisfy Adams flatness at any prime). One reason to expect simpler behavior at lower primes is that we have

$$\pi_{**}(K(\xi_1) \otimes K(\xi_1)) = b_{10}^{-1}\operatorname{Ext}^*_P(k, (P \,\square_{D[\xi_1]} k) \otimes (P \,\square_{D[\xi_1]} k)) \cong b_{10}^{-1}\operatorname{Ext}^*_{D[\xi_1]}(k, P \,\square_{D[\xi_1]} k)$$

by the change of rings theorem (Corollary 3.1.10), and the category $\operatorname{Comod}_{D[\xi_1]}$ is simpler at lower primes: every comodule over a height-p truncated polynomial algebra is a sum of comodules of the form $k[x]/x^i$ for $1 \leq i \leq p$, and there are fewer of these for lower primes. One illustration of the extra simplicity at $p = 3$ is that there is a Künneth isomorphism for the functor $b_{10}^{-1} \operatorname{Ext}^*_{D[\xi_1]}(k, -)$ only for $p = 3$.

1.2.2 Different forms of the MPASS

When doing computations with the MPASS as outlined above, we use the fact that this spectral sequence coincides starting at E_1 with the b_{10}-localized versions of the following two spectral sequences:

(1) the spectral sequence associated to the filtration of the cobar complex $C^*_P(k, M)$, where $F^s C^*_P(k, M)$ consists of elements $\{[a_1 | \dots | a_n]m\}$ such that at least s of the a_i's are in $\ker(P \to D[\xi_1])$;

(2) a generalized version of the Cartan-Eilenberg spectral sequence associated to the map $P \square_{D[\xi_1]} k \to P$.

For (2), recall that there is a Cartan-Eilenberg spectral sequence

$$E_2 = \operatorname{Ext}^*_B(k, \operatorname{Ext}^*_C(k, M)) \implies \operatorname{Ext}^*_A(k, M)$$

associated to an extension of Hopf algebras $B \to A \to C$. We present a similar construction that can be defined if B is only an A-comodule algebra, instead of a Hopf algebra of the form $A \square_C k$; we believe that the construction, with this level of generality, is new. In Section 3.2, we show that the Cartan-Eilenberg spectral sequence for an A-comodule algebra B agrees with the B-based MPASS in Stable(A). The filtration spectral sequence (1) is only defined in the case that B is a subalgebra of A of the form $A \square_C k$, and in Section 3.3 we show that this agrees with the Cartan-Eilenberg spectral sequence (and hence also the MPASS). This generalizes the classical fact that the filtration spectral sequence in (1) coincides at E_1 with the (classical) Cartan-Eilenberg spectral sequence.

These spectral sequences are useful at different times. Though not ideal for large-scale computation due to the lack of structure, the filtration spectral sequence is very concrete and useful for computing differentials in low degrees. The MPASS is useful largely because of the form of the E_2 term (in the case where flatness is satisfied). From the Cartan-Eilenberg spectral sequence variant we obtain structure such as power operations in some cases (see [Saw82]).

1.2.3 Twisting cochains

When computing $b_{10}^{-1}\operatorname{Ext}_P^*(k, k[\xi_1^3])$ in Section 6.3, we use a very different technique, inspired by the theory of twisting cochains, which we feel is worth pointing out here. The technique applies to computing $\operatorname{Ext}_\Gamma^*(k,k)$ for a Hopf algebra Γ, as well as localized versions of algebras of this form; it is applicable to the case at hand because $k[\xi_1^3] \cong P \,\square_{k[\xi_1,\xi_2,\dots]/(\xi_1^3)} k$ and so by the change of rings theorem, we have

$$b_{10}^{-1}\operatorname{Ext}_P^*(k, k[\xi_1^3]) \cong b_{10}^{-1}\operatorname{Ext}^*_{k[\xi_1,\xi_2,\dots]/(\xi_1^3)}(k,k).$$

Suppose we wish to show that $b_{10}^{-1}\operatorname{Ext}_\Gamma^*(k,k) \cong H^*(Q)$, where Q is a cochain complex. The idea is to explicitly construct a map from the cobar complex $C_\Gamma^*(k,k)$ to Q, and then show that the resulting map is a quasi-isomorphism after inverting b_{10}. Recall that $C_\Gamma^*(k,k)$ is a dga where the algebra structure comes from the concatenation product; thus it is multiplicatively generated by $C_\Gamma^1(k,k)$. So to construct a map $\theta' : C_\Gamma^*(k,k) \to Q^*$, it suffices to construct a map $\theta : C_\Gamma^1(k,k) \to Q^1$, and then extend the map multiplicatively to all of $C_\Gamma^*(k,k)$. However, one also needs to make sure the resulting map is a chain map, and one can show (see Proposition 6.3.2) that it suffices to check

$$d_Q(\theta(x)) = \sum \theta(x')\theta(x'')$$

for all $x \in \overline{\Gamma} = C^1_\Gamma(k,k)$, where $\sum x' \otimes x''$ is the reduced diagonal of $x \in \Gamma$.

Once the map θ has been constructed, one way to show that θ' is a quasi-isomorphism after inverting b_{10} is to define a filtration on Q^* such that θ' is a filtration-preserving map with respect to the filtration on $C^*_\Gamma(k,k)$ described in Section 1.2.2. This gives rise to a map of filtration spectral sequences, and the idea is to use knowledge of H^*Q to show the spectral sequences coincide.

1.3 Outline and main results

In Chapter 2, we first give a construction of the category Stable(A) that we are working in, and explain the properties that makes it a desirable setting. We then give details about the construction of the MPASS, and review several of Palmieri's results about the analogues of the nilpotence and periodicity theorems in Stable(A) and Stable(P). As was sketched in Section 1.1.3, we use this to motivate our quest to compute $b_{10}^{-1}\operatorname{Ext}^*_P(k,k)$, by situating it as the most basic chromatic localization in the category Stable(P)..

Chapter 3 is devoted to setting up a more general version of the Cartan-Eilenberg spectral sequence and proving the comparison results mentioned in Section 1.2.2. The key point, which we explain in depth in Section 3.1, involves two isomorphic ways to construct the cobar complex: one way produces the familiar cobar differential, and the other way arises from the cosimplicial object associated to a free-forgetful monad and has a differential $x \mapsto 1 \otimes x$. The Cartan-Eilenberg spectral sequence for the Hopf algebra extension $B \to A \to C$ arises from a double complex $C^*_A(k,A) \,\square_A C^*_B(B,k)$. Using the usual construction of the cobar complex, $C^*_B(B,k)$ only makes sense if B is a coalgebra, but if we replace these cobar complexes with the second version, this can be defined when B is an A-comodule algebra. We show that this coincides with the MPASS in Theorem 3.2.4. Section 3.3 is devoted to proving the comparison with a filtration spectral sequence on the cobar complex, following the classical proof.

In Chapters 4 and 5 we turn to the $K(\xi_1)$-based MPASS for computing $b_{10}^{-1}\operatorname{Ext}_P^*(k,k)$. In Section 4.2 we determine the structure of $P\square_{D[\xi_1]}k$ as a $D[\xi_1]$-comodule in order to obtain an expression for $K(\xi_1)_{**}K(\xi_1) = b_{10}^{-1}\operatorname{Ext}^*_{D[\xi_1]}(k, A\square_{D[\xi_1]}k)$ as a vector space. For this computation, we work at an arbitrary odd prime. We find that $K(\xi_1)_{**}K(\xi_1)$ is flat over $K(\xi_1)_{**}$ at $p=3$, and so we specialize to $p=3$ going forward. In Section 4.3 we prove:

Theorem. The Hopf algebroid $(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$ is an exterior algebra over $K(\xi_1)_{**} = E[h_{10}]\otimes P[b_{10}^{\pm 1}]$ on generators $e_2, e_3, \ldots$ where e_n is in degree $2(3^n+1)$.

Corollary. The E_2 page of the $K(\xi_1)$-based Adams spectral sequence for computing $\pi_{**}(b_{10}^{-1}k)$ is

$$\operatorname{Ext}^*_{K(\xi_1)_{**}K(\xi_1)}(K(\xi_1)_{**}, K(\xi_1)_{**}) = K(\xi_1)_{**}\otimes P[w_2, w_3, \ldots]$$

where $w_n = [e_n]$ has Adams filtration 1 and internal degree $2(3^n+1)$.

Using a degree argument, we show (Proposition 5.1.1) that $d_r(x) = 0$ unless $r\equiv 4 \pmod 9$ or $r \equiv 8 \pmod 9$. Chapter 5 is devoted to computing the first differential.

Theorem. The element w_2 is a permanent cycle, and for $n\geq 3$, there is a differential

$$d_4(w_n) = b_{10}^{-4}h_{10}w_2^2w_{n-1}^3.$$

The strategy is to use comparison with the MPASS computing $b_{10}^{-1}\operatorname{Ext}^*_{P_n}(k,k)$, where

$$P_n = k[\xi_1, \xi_2, \xi_{n-2}, \xi_{n-1}, \xi_n]/(\xi_1^9, \xi_2^3, \xi_{n-2}^{27}, \xi_{n-1}^9, \xi_n^3).$$

It is not hard to reduce to showing the differential $d_4(w_n) = b_{10}^{-4}h_{10}w_2^2w_{n-1}^3$ in this simpler MPASS. The main strategy is: (1) compute enough of the E_2 page of the simpler MPASS to identify classes of interest such as w_n and $b_{10}^{-4}h_{10}w_2^2w_{n-1}^3$; (2) show that $b_{10}^{-1}\operatorname{Ext}^*_{P_n}(k,k)$ is zero in the stem of $b_{10}^{-4}h_{10}w_2^2w_{n-1}^3$ (so it is either the source or target of a differential); (3) show that it is a permanent cycle and, for degree reasons, w_n is the only element that can hit it. For step (2), we calculate part of $b_{10}^{-1}\operatorname{Ext}^*_{P_n}(k,k)$ using the dual of the May spectral sequence.

We conjecture the following behavior for d_8 in the MPASS converging to $b_{10}^{-1}\operatorname{Ext}^*_P(k,k)$: if $d_4(x) = h_{10}y$ and $d_4(y) = h_{10}z$, then $d_8(x) = b_{10}z$. Furthermore, we conjecture that the remaining differentials in the spectral sequence are zero. In Chapter 6, we state the following more general conjecture:

Conjecture. Let $D = k[\xi_1]/(\xi_1^3)$. There is a functor $W : \mathrm{Comod}_P \to \mathrm{Comod}_D$ such that

$$b_{10}^{-1}\operatorname{Ext}^*_P(k,M) \cong b_{10}^{-1}\operatorname{Ext}^*_D(k,W(M))$$

and $W(k) = k[\widetilde{w}_2, \widetilde{w}_3, \dots]$ with D-coaction given by $\psi(\widetilde{w}_n) = 1 \otimes \widetilde{w}_n + \xi_1 \otimes \widetilde{w}_2^2\widetilde{w}_{n-1}^3$ for $n \geq 3$ and $\psi(\widetilde{w}_2) = 1 \otimes \widetilde{w}_2$.

(Here $\widetilde{w}_n = b_{10}^{-1}w_n$.) In the remainder of the chapter, we prove two results that support this conjecture.

Theorem. We have the following:

(1) $b_{10}^{-1}\operatorname{Ext}^*_P(k, k[\xi_1^9, \xi_2^3, \xi_3, \xi_4, \dots]) \cong b_{10}^{-1}\operatorname{Ext}^*_D(k, k[\widetilde{w}_2, \widetilde{b}_{20}])$ where $\psi(\widetilde{w}_2) = 1 \otimes \widetilde{w}_2$ and $\psi(\widetilde{b}_{20}) = 1 \otimes \widetilde{b}_{20} + \xi_1 \otimes \widetilde{w}_2^4$;

(2) $b_{10}^{-1}\operatorname{Ext}^*_P(k, k[\xi_1^3]) \cong b_{10}^{-1}\operatorname{Ext}^*_P(k, k[h_{20}, b_{20}, w_3, w_4, \dots]/h_{20}^2)$ where ψ acts trivially on all the generators.

Compare the conjecture above with the following analogue at $p = 2$:

Theorem 1.3.1 (Milgram-May [MM81]). *At $p = 2$, let $h_{10} = [\xi_1]$ and $P = \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots]$.*

Then for a P-comodule M we have

$$h_{10}^{-1}\operatorname{Ext}_P^*(\mathbb{F}_2, M) \cong h_{10}^{-1}\operatorname{Ext}_{E[\xi_1]}^*(\mathbb{F}_2, M).$$

Note that this theorem is much simpler than the $p = 3$ case we study; this can be seen in the fact that the $p = 2$ MPASS collapses.

Chapter 2

Homotopy theory in the stable category of comodules

Let (A, Γ) be a Hopf algebroid. This chapter will describe a program begun by Margolis [Mar83] and Palmieri [Pal01] to study the homotopy theory of Γ-comodules. The eventual goal is to set up an analogy between Ho(Sp) and a homotopically nice algebraic category Stable(Γ) into which Comod_Γ embeds, such that Ext_Γ groups correspond to homotopy groups in classical homotopy theory. This analogy can be developed to the point that classical techniques for studying homotopy groups, such as the Adams spectral sequence, can be imported into Stable(Γ) and applied for the study of Ext groups. In the first section, we will define the category Stable(Γ). In the second section, we will discuss the analogue of the Adams spectral sequence in Stable(Γ), which we call the Margolis-Palmieri Adams spectral sequence. This will be our main computational tool in the rest of this thesis. In the third section, we discuss the analogue of some features of chromatic homotopy theory in Stable(Γ), and explain how this fits our central problem of computing $b_{10} \mathrm{Ext}_P^*(k, M)$ into a larger conceptual framework.

2.1 The category Stable(Γ)

It is a general fact about abelian categories (see e.g. [Sta18, Tags 06XQ], [Sta18, Tag 06XS]) that to any abelian category $\mathcal{A}$ one can form the derived category $D(\mathcal{A})$ by inverting the homology isomorphisms in the category $K(\mathcal{A})$ of chain complexes in $\mathcal{A}$ up to chain homotopy, and there is an isomorphism

$$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) := \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) \cong H^i(\mathrm{Hom}_{K(\mathcal{A})}(X, I^\bullet_Y))$$

where $I^\bullet_Y$ is an injective resolution of Y. In particular, one can apply this to the category Comod_Γ of (A, Γ)-comodules (see [Rav86, A1.1.2] for a precise definition of this category).

Definition 2.1.1. Define $\mathrm{Ext}_\Gamma(M, N) = \mathrm{Hom}_{D(\mathrm{Comod}_\Gamma)}(M, N)$ where M and N on the right hand side are identified with their image in the derived category. If the context is clear, we abbreviate $D(\mathrm{Comod}_\Gamma)$ as $D(\Gamma)$.

Our eventual goal is to try to use homotopy-theoretic techniques to study Ext groups, and $D(\Gamma)$ is not a bad first guess as a setting for this work. A fair amount of homotopy theory only depends on the existence of a small number of formal properties of Ho(Sp), such as the existence of (co)fiber sequences, an invertible suspension functor, and a symmetric monoidal smash product. Given an arbitrary category with a symmetric monoidal product (generalizing the smash product) and triangulated structure (generalizing the (co)fiber sequences of homotopy theory), we are well on our way to at least being able to write down analogues of many of the major constructions in homotopy theory. The derived category $D(\Gamma)$ fits this criterion: the shift functor gives rise to a triangulation, and tensor product of chain complexes is symmetric monoidal.

In [HPS97], Hovey, Palmieri, and Strickland consider a set of axioms for Ho(Sp)-like categories, which they call *stable homotopy categories*, and develop analogues of classical homotopy theory in this generality.

Definition 2.1.2 ([HPS97, Definition 1.1.4]). A *stable homotopy category* is a symmetric monoidal triangulated category $\mathcal{C}$ (such that the symmetric monoidal product is compatible with the triangulation) along with a set $\mathcal{G}$ of strongly dualizable objects of $\mathcal{C}$ such that

$$\mathrm{Loc}_{\mathcal{C}}(\mathcal{G}) \simeq \mathcal{C}, \tag{2.1.1}$$

where $\mathrm{Loc}_{\mathcal{C}}(\mathcal{G})$ indicates the localizing subcategory of $\mathcal{C}$ generated by $\mathcal{G}$—that is, the smallest thick subcategory that is closed under filtered colimits in $\mathcal{C}$.

Before checking whether $D(\Gamma)$ fits this definition, however, we note that $D(\Gamma)$ already has a problem that needs to be corrected first: while the derived category seems like a good setting for studying Ext groups, it turns out it is not a good setting for studying *localized* Ext groups. In particular, we would like to study groups of the form $x^{-1}\operatorname{Ext}^*_\Gamma(A, M)$ where M is a Γ-comodule and $x \in \operatorname{Ext}^*_\Gamma(A, A)$ is non-nilpotent, and one might hope that $x^{-1}\operatorname{Ext}_\Gamma(A, M) = \operatorname{Hom}_{D(\Gamma)}(A, x^{-1}M)$; this is the same as asking for the equality

$$\begin{aligned} \operatorname{colim}\big(\operatorname{Hom}_{D(\Gamma)}(A, M) \xrightarrow{x} \operatorname{Hom}_{D(\Gamma)}(A, M) \xrightarrow{x} \operatorname{Hom}_{D(\Gamma)}(A, M) \to \dots\big) \\ = \operatorname{Hom}_{D(\Gamma)}\big(A,\ \operatorname{colim}(M \xrightarrow{x} M \xrightarrow{x} M \to \dots)\big) \end{aligned} \tag{2.1.2}$$

where in the sequence $M \xrightarrow{x} M \to \dots$ we are identifying M with its image in $D(\Gamma)$, i.e. the class in $\mathrm{Ch}(\Gamma)$ represented by a Γ-injective resolution of M. This would hold if the unit object A were compact, but that is not true in general, and in fact (2.1.2) does not hold in general, as we show with the following counterexample.

Example 2.1.3. Let $(A, \Gamma) = (k, E[t])$, the exterior Hopf algebra on one generator over the field k. Then $\operatorname{Ext}_{E[t]}(k, k) = P[\alpha]$ where α is the class in homological degree 1, and k has injective resolution $I = (E[t] \xrightarrow{\partial} E[t] \xrightarrow{\partial} E[t] \to \dots)$ where ∂ is the comodule map taking $1 \mapsto 0$ and $t \mapsto 1$. In $D(E[t])$ we have

$$\operatorname{colim}(k \xrightarrow{\alpha} k \xrightarrow{\alpha} k \to \dots) = \operatorname{colim}(I \xrightarrow{\alpha} I \xrightarrow{\alpha} I \to \dots)$$

$$= \dots \xrightarrow{\partial} E[t] \xrightarrow{\partial} E[t] \xrightarrow{\partial} E[t] \xrightarrow{\partial} \dots.$$

This is acyclic, and hence zero in $D(E[t])$. So the right hand side of (2.1.2) in this case is zero, and the left hand side is $\alpha^{-1}\operatorname{Ext}_{E[t]}(k,k) = k[\alpha^{\pm 1}]$ by definition.

We would like to fix this problem with the derived category and work in a category such that localized Ext groups can be described as Hom-sets between localized objects. More precisely, we would like to work in a category $\mathcal{A}$ such that:

(1) If M and N are Γ-comodules, then $\operatorname{Hom}_{\mathcal{A}}(M,N) = \operatorname{Ext}_{\Gamma}(M,N)$.

(2) We have $x^{-1}\operatorname{Ext}_{\Gamma}(M,N) = \operatorname{Hom}_{\mathcal{A}}(M, x^{-1}N)$.

(3) The category $\mathcal{A}$ is a stable homotopy category in the sense of Definition 2.1.2.

The correct choice of $\mathcal{A}$ is called Stable(Γ); there are three equivalent constructions. First we need some preliminaries. Given a category $\mathcal{C}$, the Ind construction Ind($\mathcal{C}$) is designed to force

$$\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(X, \operatorname*{colim}_i Y_i) = \operatorname*{colim}_i \operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(X, Y_i)$$

where $\operatorname{colim} Y_i$ is a filtered colimit. More precisely:

Definition 2.1.4. Given a category $\mathcal{C}$, let Ind($\mathcal{C}$) be the category whose objects are diagrams $F : \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ is a small filtered category, and if $F : \mathcal{D} \to \mathcal{C}$ and $F' : \mathcal{D}' \to \mathcal{C}$ are objects, then

$$\operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(F, G) = \lim_{d \in \mathcal{D}} \operatorname*{colim}_{d' \in \mathcal{D}'} \operatorname{Hom}_{\mathcal{C}}(F(d), F'(d')).$$

By design, there is a full and faithful embedding $\mathcal{C} \to \operatorname{Ind}(\mathcal{C})$ such that objects in the image are compact in Ind($\mathcal{C}$). This suggests we define Stable(Γ) $=$ Ind($D(\Gamma)$), but we still need to satisfy (2.1.1). The following lemma provides some intuition for the definition.

Lemma 2.1.5 ([BHV15, Lemma 2.15]). *If $\mathcal{G} \subset \mathcal{C}$ is a set of compact generators of $\mathcal{C}$, then*

$$\mathrm{Ind}(\mathrm{Thick}_{\mathcal{C}}(\mathcal{G})) \simeq \mathrm{Loc}_{\mathrm{Thick}_{\mathcal{C}}(\mathcal{G})}(\mathcal{G}).$$

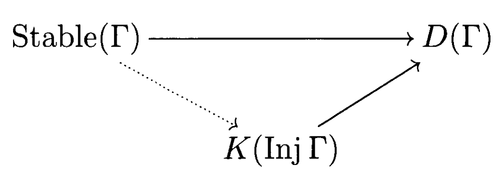
Definition 2.1.6 ([BHV15, Definition 4.8]). Let $\mathcal{G}$ denote the set of dualizable Γ-comodules, and let $\mathrm{Thick}_{D(\Gamma)}(\mathcal{G})$ denote the thick subcategory of $D(\Gamma)$ generated by the image of $\mathcal{G}$ in $D(\Gamma)$. Then define

$$\mathrm{Stable}(\Gamma) = \mathrm{Ind}(\mathrm{Thick}_{D(\Gamma)}(\mathcal{G})).$$

It turns out that this is equivalent to the following, somewhat more concrete, construction:

Definition 2.1.7. Define $K(\mathrm{Inj}\,\Gamma)$ to be the category whose objects are unbounded complexes of injective Γ-comodules, and whose morphisms are chain complex morphisms modulo chain homotopies.

Since $D(\Gamma)$ is cocomplete, the universal property of the Ind construction gives rise to a functor $\mathrm{Stable}(\Gamma) \to D(\Gamma)$, which can be regarded as a left Bousfield localization functor. The claim is that this functor factors through $K(\mathrm{Inj}\,\Gamma)$



and the functor $\mathrm{Stable}(\Gamma) \to K(\mathrm{Inj}\,\Gamma)$ induces an equivalence of categories under certain hypotheses.

Theorem 2.1.8 ([BHV15, Proposition 4.17]). *Suppose A is Noetherian, Γ is flat over A, and every compact object in the image of Comod_Γ is in $\mathrm{Thick}_{D(\Gamma)}(A)$. Then there is an equivalence of categories $\mathrm{Stable}(\Gamma) \simeq K(\mathrm{Inj}\,\Gamma)$.*

The conditions on the theorem are satisfied for the Hopf algebroid (R_*, R_*R) if R_*R is commutative and R is Landweber exact over MU or over the quotient of BP by a finite regular invariant sequence (see [BHV15, after Definition 4.14]). Examples of R satisfying this condition include $H\mathbb{F}_p$ and BP.

There is a third way of thinking of this category:

Remark 2.1.9. Under a niceness assumption on (A, Γ) satisfied by Adams Hopf algebroids, Hovey [Hov04] constructs the *homotopy model structure* on $\mathrm{Ch}(\Gamma)$ as a localization of the projective model structure and shows that its homotopy category is a stable homotopy category in the sense of Definition 2.1.2. In [BHV15, §4.5] it is shown that this homotopy category is equivalent to $\mathrm{Stable}(\Gamma)$ as defined above.

Warning 2.1.10. Let A denote the Steenrod algebra, or more generally any algebra that can be expressed as a union of Poincaré algebras. In [Mar83, Chapter 14 §1], Margolis defines an enlargement $\mathrm{StMod}(A)$ of the category of A-modules that he calls the "stable category." Its objects are left A-modules and its morphisms are A-module morphisms modulo those that factor through a projective module. One might wonder if this agrees with the dual of the definitions of stable categories of comodules above, but this is not true in general; see [BK08] for a discussion of the difference between $\mathrm{StMod}(kG)$ and the category $K(\mathrm{Inj}\, kG)$ of chain complexes of injective kG-modules up to chain homotopy. In particular, $\mathrm{StMod}(kG)$ is equivalent to the subcategory of $K(\mathrm{Inj}\, kG)$ consisting of acyclic complexes.

Theorem 2.1.11 ([HPS97, Theorem 9.5.1], [BHV15, Lemma 4.21]). *Under the hypotheses of Theorem 2.1.8,* $\mathrm{Stable}(\Gamma)$ *is a stable homotopy category in the sense of Definition 2.1.2, and if M and N are Γ-comodules, then*

$$\mathrm{Ext}_\Gamma(M, N) \cong \mathrm{Hom}_{\mathrm{Stable}(\Gamma)}(M, N).$$

Moreover, (2.1.2) is satisfied, and hence $x^{-1}\operatorname{Ext}_\Gamma(M,N) = \operatorname{Hom}_{\mathsf{Stable}(\Gamma)}(M, x^{-1}N)$, because all objects in the image of Comod_Γ are compact by definition of Ind.

Remark 2.1.12 (Symmetric monoidal structure)**.** For concreteness, suppose we are using the $K(\operatorname{Inj}\Gamma)$ model of the stable category of Γ-comodules. If X and Y are objects of Stable(Γ), it is clear that the symmetric monoidal product $X \otimes Y$ should be the tensor product of chain complexes—that is, $(X\otimes Y)_n = \bigoplus_{i+j} X_i \otimes Y_j$—but we need to give it the structure of a chain complex *of Γ-comodules.* We define the Γ-coaction on $X_i \otimes Y_j$ to be the *diagonal coaction*, namely

$$\psi(x\otimes y) = \sum x'y' \otimes x'' \otimes y''$$

where the Γ-coaction on X_i and Y_j are given by $\psi(x) = \sum x' \otimes x''$ and $\psi(y) = \sum y' \otimes y''$, respectively. We write $X_i \overset{\Delta}{\otimes} Y_j$ to denote this tensor product as an object of Comod_Γ with the diagonal coaction, and write $X \overset{\Delta}{\otimes} Y$ for the tensor product of chain complexes with the levelwise Γ-comodule structure given by the diagonal coaction.

Notation 2.1.13. The idea is that Stable(Γ) behaves enough like Ho(Sp) that we should be able to port over a large amount of classical homotopy theory for the study of Stable(Γ). To emphasize this analogy, we adopt the following notation and make the following observations:

- As k is the unit object in Stable(Γ), write

  $$\pi_{**}(X) = \operatorname{Hom}_{\mathsf{Stable}(\Gamma)}(k, X)$$

  for an object X of Stable(Γ). We assume that Γ is a graded Hopf algebra, and hence objects of Stable(Γ) are bi-graded: the first grading in π_{**} will refer to homological degree from regarding X as a chain complex in $K(\operatorname{Inj}\Gamma)$, and the second grading will refer to internal degree. If M is the Stable(Γ) representative of a Γ-comodule, then $\pi_{s,t}(M) = \operatorname{Ext}^{s,t}_\Gamma(k, M)$.

- As in topology, write $X_{**}X = \pi_{**}(X \otimes X)$.
- Notice that the analogue of the homotopy groups of spheres in this category is $\mathrm{Ext}_\Gamma(k,k)$ (self-maps of the unit object). In particular, if $\Gamma = A$, then this is the E_2 page of the Adams spectral sequence for the sphere, and if Y is a (topological) spectrum, $\pi_{**}(H_*Y) = \mathrm{Ext}_A(k, H_*Y)$ is the Adams E_2 page for Y. The idea to take tools originally designed to study (classical) homotopy groups, and construct them internally in Stable(A) so they can be used to study Adams E_2-pages $\pi_{**}(H_*Y)$.

In the next section, we will extend this analogy and define a version of the Adams spectral sequence within the category Stable(Γ). In Section 2.3 we will extend this even further and talk about analogues of chromatic homotopy theory in Stable(A).

2.2 The Margolis-Palmieri Adams spectral sequence

Let E be a (classical) ring spectrum and X a finite spectrum, and let $\overline{E}$ be the cofiber of the unit map $S \to E$. Recall that the classical Adams spectral sequence

$$E_1 = E_*(\overline{E}^{\wedge s} \wedge X) \implies \pi_* X^\wedge_E$$

is constructed by applying $\pi_*(-)$ to the tower of fiber sequences

$$\begin{array}{ccccccc}
X & \leftarrow & \overline{E} \wedge X & \leftarrow & \overline{E}^{\wedge 2} \wedge X & \leftarrow & \cdots \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & \\
E \wedge X & & E \wedge \overline{E} \wedge X & & E \wedge \overline{E}^{\wedge 2} \wedge X & &
\end{array}$$

where the fiber sequence

$$\begin{array}{ccc}
\overline{E}^{\wedge s} \wedge X & \leftarrow & \overline{E}^{\wedge s+1} \wedge X \\
\downarrow & \nearrow & \\
E \wedge \overline{E}^{\wedge s} \wedge X & &
\end{array}$$

is obtained by smashing

$$\begin{array}{ccc} S & \leftarrow\cdots & \overline{E} \\ \downarrow & \nearrow & \\ E & & \end{array}$$

with $\overline{E}^{\wedge s} \wedge X$ on the right.

Proposition 2.2.1. *If E_*E is flat as an E_*-algebra, then one can define the structure of a Hopf algebroid on $(E_*,\ E_*E)$ as follows:*

- *The left and right units come from applying π_* to the maps $E \wedge S \to E \wedge E$ and $S \wedge E \to E \wedge E$, respectively.*
- *The antipode comes from applying π_* to the swap map $E \wedge E \to E \wedge E$.*
- *The counit $\varepsilon : \pi_*(E \wedge E) \to \pi_* E$ comes from applying π_* to the multiplication map on E.*
- *For the comultiplication, note that there is a natural map $E_*E \otimes_{E_*} E_*E \to E_*(E \wedge E)$ induced by $\pi_*(E \wedge E) \otimes \pi_*(E \wedge E) \to \pi_*(E \wedge E \wedge E \wedge E) \overset{-\wedge\mu\wedge-}{\longrightarrow} \pi_*(E \wedge E \wedge E)$. The flatness condition implies that this map is an isomorphism. Then the coaction on E_*E comes from the composition*

$$\pi_*(E \wedge E) \overset{\alpha}{\to} \pi_*(E \wedge E \wedge E) \overset{\cong}{\leftarrow} E_*E \otimes_{E_*} E_*E$$

where α is induced by the ring spectrum map $E \wedge E \to E \wedge S \to E \to E \wedge E \wedge E$.

In this flat case, we have

$$E_2 = \mathrm{Ext}_{E_*E}(E_*, E_*X).$$

Definition 2.2.2 ([Pal01]). Given a monoid object ("ring spectrum") E in Stable(Γ) and another object X in Stable(Γ), we can define an analogous tower of fibrations in Stable(Γ) and apply the functor $\mathrm{Hom}_{\mathrm{Stable}(\Gamma)}(A, -) = \pi_{**}(-)$, obtaining a spectral sequence with $E_1 = E_{**}(\overline{E}^{\otimes s} \otimes X)$ abutting to $\pi_{**}X$. We call this the E-based *Margolis-Palmieri Adams spectral sequence* for computing $\pi_{**}X$, henceforth abbreviated as

MPASS.

Remark 2.2.3. The unit map $X \to E \otimes X$ is given by $m \mapsto 1 \otimes m$. One can check that this respects the Γ-coaction since $E \otimes X$ is endowed with the diagonal Γ-coaction (see Remark 2.1.12).

As in the classical case, if $E_{**}E$ is flat over E_{**}, then we can define a Hopf algebroid structure on the pair $(E_{**}, E_{**}E)$ analogously to Proposition 2.2.1. In this case, the MPASS has E_2 term

$$E_2 = \mathrm{Ext}_{E_{**}E}(E_{**}, E_{**}(X)).$$

This Adams flatness condition is satisfied in the following common situation.

Proposition 2.2.4 ([Pal01, Proposition 1.4.6]). *Suppose $f : (A, \Gamma) \to (A, \Sigma)$ is a map of Hopf algebroids such that $\Phi := \Gamma \square_\Sigma A$ is a subalgebra of Γ, and such that the Σ-coaction on Φ (defined by composing the Γ-coaction on Φ with f) is trivial. Then $(\Phi_{**}, \Phi_{**}\Phi)$ is flat.*

Proof. We have

$$\Phi_{**}\Phi = \mathrm{Ext}^*_\Gamma(A, (\Gamma \square_\Sigma A) \otimes (\Gamma \square_\Sigma A)) \cong \mathrm{Ext}^*_\Sigma(A, \Gamma \square_\Sigma A)$$

by the variant of the change of rings theorem in Corollary 3.1.10, and by the hypothesis about the coaction, this is $\mathrm{Ext}^*_\Sigma(A, A) \otimes \Phi \cong \Phi_{**} \otimes \Phi$. □

We will eventually use this machinery in the special case where $(A, \Gamma) = (\mathbb{F}_3, P)$, where $P = \mathbb{F}_3[\xi_1, \xi_2, \dots]$ is the reduced powers; our ring spectrum E will be $b_{10}^{-1}(P \square_{\mathbb{F}_3[\xi_1]/\xi_1^3} \mathbb{F}_3)$, which does not satisfy the hypotheses of the above proposition, but will end up satisfying flatness regardless due to special properties of working in characteristic 3.

In the world of classical homotopy theory, in general the Adams spectral sequence converges not to π_*X but to the E-completion $\pi_*(X)^{\widehat{}}_E$. However, in Stable(Γ), a connectivity argument shows that the inverse limit of the Adams tower is contractible in most of the cases we care about, and so in these cases the spectral sequence converges to $\pi_{**}(X)$. More precisely:

Proposition 2.2.5 ([Pal01, Proposition 1.4.3]). *Suppose (A, Γ) is a Hopf algebra where A is a field. Let E be a ring spectrum satisfying the following conditions:*

- *$\pi_{ij}(E) = 0$ if $i < 0$ or $j - i < 0$;*
- *the unit map η induces an isomorphism on $\pi_{0,0}$;*
- *$\mu_* : \pi_{0,0}E \otimes \pi_{0,0}E \longrightarrow \pi_{0,0}$ induced by the multiplication map μ is an isomorphism;*
- *the homology of the chain complex E is a finite-dimensional k-vector space in each bi-degree.*

Also suppose X is weakly connective*: that is, there exists i_0 and j_0 such that $\pi_{ij}X = 0$ if $i < i_0$ or $j < j_0$. Then the MPASS converges to $\pi_{**}(X)$.*

Remark 2.2.6. There is an alternate construction of the Adams spectral sequence as the spectral sequence associated to the augmented cosimplicial spectrum

$$X \longrightarrow E \wedge X \underset{\eta_R}{\overset{\eta_L}{\underset{\longrightarrow}{\overset{\longrightarrow}{\leftarrow \mu}}}} E \wedge E \wedge X \overset{\longrightarrow}{\underset{\longrightarrow}{\Delta}} E \wedge E \wedge E \wedge X \qquad \ldots \tag{2.2.1}$$

(For more about this approach, see [Lur10, Lecture 8] or [Pet16, §3.1].) This is the cosimplicial spectrum associated to the monad arising from the free-forgetful adjunction

$$F : \mathrm{Sp} \rightleftarrows \mathrm{Mod}_E : U$$
$$X \overset{F}{\longmapsto} E \wedge X.$$

One can obtain a spectral sequence in Stable(Γ) analogously: let E be a monoid object in Stable(Γ). Then there is a free-forgetful adjunction Stable(Γ) $\rightleftarrows \mathrm{Mod}_E$ as above, where Mod_E denotes the category of E-modules in Stable(Γ), and the free functor sends $X \mapsto E \overset{\Delta}{\otimes} X$. The MPASS is the spectral sequence arising from the resulting augmented cosimplicial object.

2.3 Nilpotence and periodicity in Stable(A) and Stable(P)

In this section we continue our review of homotopy theory constructions that can be performed internally in Stable(Γ), focusing on analogues of the nilpotence and periodicity theorems of Devinatz, Hopkins, and Smith, which form an important part of the foundations of chromatic homotopy theory. First we state the original theorems, whose setting is the category $\mathrm{Sp}_p^{\mathrm{fin}}$ of finite p-local spectra for a fixed prime p, and then discuss partial analogues in Stable(Γ), focusing on the cases where Γ is the dual Steenrod algebra A or the Steenrod reduced powers P. In the last subsection, we discuss a relationship between some periodicity operators over Stable(A) and the classical theory of E_2 vanishing lines, and show how our project of computing $b_{10}^{-1}\operatorname{Ext}_P(k, M)$ for P-comodules M can be viewed in a chromatic framework as the first chromatic localization in Stable(P).

Fix a prime p and let $\mathrm{Sp}_p^{\mathrm{fin}}$ denote the category of p-local finite (classical) spectra. The nilpotence and periodicity theorems are about a collection of ring spectra $K(n)$ for $n \geq 0$ with $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ which detect nilpotent maps, parametrize thick subcategories of $\mathrm{Sp}_p^{\mathrm{fin}}$, and describe vanishing lines in Adams spectral sequences.

Theorem 2.3.1 (Nilpotence theorem, [DHS88, Theorem 1], [HS98, Theorem 3]). *The collection $\{K(n)\}_{n\geq 0}$ detects nilpotence:*

(1) Given a p-local ring spectrum R, an element $\alpha \in \pi_* R$ is nilpotent if and only if for all $0 \leq n \leq \infty$, $K(n)_*(\alpha)$ is nilpotent.

(2) A self-map $f : \Sigma^k X \to X$ (for X in $\mathrm{Sp}_p^{\mathit{fin}}$) is nilpotent if and only if $K(n)_* f$ is nilpotent for all $0 \leq n < \infty$.

(3) A map $f : F \to X$ from a finite spectrum to a p-local spectrum is smash nilpotent

if and only if $K(n)_*f = 0$ *for all* $0 \leq n \leq \infty$.

There is also a single spectrum BP *that detects nilpotence in an analogous sense.*

Theorem 2.3.2 (Periodicity theorem, [HS98, Theorem 9]). *Given a finite p-local spectrum* X, *if* $K(n)_*(X) \neq 0$ *and* $K(n-1)_*(X) = 0$ *then there is an essentially unique self map* $\Sigma^{|v_n^{p^i}|} : X \to X$ *(for some* i*) such that the induced map* $K(n)_*(X) \to K(n)_*(X)$ *is multiplication by* $v_n^{p^i}$ *(or, in the case* $n = 0$, *multiplication by a rational number), and the induced map* $K(m)_*(X) \to K(m)_*(X)$ *is zero for* $m > n$. *We call this a* v_n*-map.*

Theorem 2.3.3 (Thick subcategory theorem, [HS98, Theorem 7]). *The poset of thick subcategories of* Sp_p^{fin} *is the system*

$$\mathrm{Sp}_p^{fin} = \mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \dots$$

where $\mathcal{C}_n$ *is the subcategory of* Sp_p^{fin} *generated by the spectra* X *such that* $K(n-1)_*X = 0$.

We say that X has *type* n if it is contained in $\mathcal{C}_n$ and not $\mathcal{C}_{n+1}$. This filtration gives information about the Adams spectral sequence:

Theorem 2.3.4 ([HPS99]). *If* X *has type* n, *then the* E_∞ *page of the Adams spectral spectral sequence* $E_r^{s,t}(X) \implies \pi_*(X)$ *has a vanishing line of slope*

$$\frac{1}{|\tau_n| - 1} = \frac{1}{2p^n - 2}.$$

These theorems touch on deep structure in $\mathrm{Sp}_p^{\mathrm{fin}}$, and so one does not expect them to generalize easily to an arbitrary stable homotopy category $\mathcal{C}$ in the sense of Definition 2.1.2. In the classical setting, most of the work is in proving the nilpotence theorem, and the thick subcategory theorem and periodicity theorem follow from it with a somewhat more formal argument. A version of the thick subcategory theorem in the setting of stable homotopy categories, assuming the existence of a nilpotence theorem, can be found in [HPS97, Corollary 5.2.3], but the hypotheses on the nilpotence-detecting family are not satisfied in Stable(A); indeed, we will see that the thick subcategory poset is more complicated than the nilpotence-detecting family.

It is, however, a general fact about stable categories that Adams E_∞ vanishing lines are related to a thick subcategory classification: an object X has an Adams E_∞ vanishing line of a given slope if and only if every object in Thick(X) has a vanishing line of that slope. (In fact, the proof of Theorem 2.3.4 amounts to proving this assertion, and using the fact from [HS98, §4] that for every n, the finite type n spectrum constructed by Smith in [Smi92] has the indicated vanishing line on its Adams E_2 page.)

There is a large body of work (see [Mar83], [Pal01], [Pal94], [BH17], [Kra18]) focused on finding analogues of Theorems 2.3.1–2.3.4 in Stable(Γ) for various Hopf algebroids (A, Γ) of interest.

2.3.1 Nilpotence, periodicity, and thick subcategory theorems for Stable(A) and Stable(P)

Recall that the mod-2 Steenrod dual has the form $A = \mathbb{F}_2[\xi_1, \xi_2, \dots]$, and for $p > 2$ the mod-p Steenrod dual has the form $A = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$; for $p > 2$ recall the dual Steenrod reduced powers algebra is $P = \mathbb{F}_p[\xi_1, \xi_2, \dots]$. In this subsection we will state theorems and conjectures by Palmieri on the structure of the thick subcategory poset and a nilpotence-detecting family, and describe how a subset of that family relates to vanishing lines in Adams E_2 pages.

For A at $p = 2$ and P for $p > 2$, we have both an analogue of BP and of the collection of $K(n)$'s. Recall that an elementary Hopf algebra is a tensor product of Hopf algebras of the form $\mathbb{F}_p[x]/x^2$ and $\mathbb{F}_p[x]/x^{p^n}$ for primitive generator x.

Theorem 2.3.5 (Palmieri, [Pal01, 2.1.7, 5.1.5, 5.1.6, 5.1.7(f)], [Pal96a, 4.2, 4.3], [Pal96b,

§5]). *Let* $p = 2$ *and let*

$$E(r) = \mathbb{F}_2[\xi_r, \xi_{r+1}, \xi_{r+2}, \dots]/(\xi_r^{p^r}, \xi_{r+1}^{p^r}, \xi_{r+2}^{p^r}, \dots).$$

(These are the maximal elementary Hopf algebra quotients of A*.) The collection of ring spectra* $\{A \,\square_{E(r)} \mathbb{F}_2\}$ *detects nilpotence in the same sense as Theorem 2.3.1, except (1) is an "if" instead of an "if and only if." Furthermore, let* $C = A/(\xi_1^2, \xi_2^4, \xi_3^8, \dots)$*; then* $A \,\square_C \mathbb{F}_2$ *detects nilpotence in* Stable(A).

At $p > 2$*, let*

$$Q(r) = \mathbb{F}_p[\xi_r, \xi_{r+1}, \xi_{r+2}, \dots]/(\xi_r^p, \xi_{r+i}^{p^{r+1}}).$$

Then the collection $\{P \,\square_{Q(r)} \mathbb{F}_p\}$ *detects nilpotence in a sense made precise in [Pal96a, Theorem 4.3]. Let* $C' = P/(\xi_1^p, \xi_2^{p^2}, \xi_3^{p^3}, \dots)$*. Then* $P \,\square_{C'} \mathbb{F}_p$ *detects nilpotence over* P *in a sense made precise in [Pal96a, Theorem 4.2].*

(The issue with the other direction of (1) is a finiteness issue—there might be infinitely many elementary quotients.)

Conjecture 2.3.6 (Palmieri, [Pal01, Conjecture 5.4.1]). Let $p > 2$. Let $\mathcal{Q}$ be the collection of *quasi-elementary* quotient Hopf algebras of A (see [Pal01, Definition 2.1.10]), which includes the maximal elementary quotients

$$E(-1) = E[\tau_0, \tau_1, \dots]$$
$$E(r) = A/(\xi_1, \dots, \xi_r, \xi_{r+1}^{p^{r+1}}, \xi_{r+2}^{p^{r+1}}, \xi_{r+3}^{p^{r+1}}, \dots; \tau_0, \dots, \tau_r).$$

Then the collection $\{A \,\square_E \mathbb{F}_p \,:\, E \in \mathcal{Q}\}$ detects nilpotence in Stable(A). Furthermore, if we write $C = A/(\xi_1^p, \xi_2^{p^2}, \xi_3^{p^3}, \dots)$ then $A \,\square_C \mathbb{F}_p$ detects nilpotence in Stable(A).

The thick subcategory conjecture below is reminiscent of the classical theorem that the thick subcategories $\mathcal{C}_n$ of $\mathrm{Sp}_p^{\mathrm{fin}}$ are in bijection with the invariant ideals of $\pi_* BP$.

Conjecture 2.3.7 (Palmieri, [Pal99, Conjecture 1.4], [Pal01, 6.7.3]). The thick subcategories of finite A-modules are in one-to-one correspondence with radical ideals of $\pi_{**}(A\,\square_C \mathbb{F}_p) = \mathrm{Ext}^*_C(\mathbb{F}_p, \mathbb{F}_p)$ satisfying a finiteness condition that are invariant under the coaction of $A\,\square_C \mathbb{F}_p$. In particular, an invariant ideal I gets sent to the full subcategory generated by finite objects X such that $I(X) \supset I$, where $I(X)$ is the radical of the ideal

$$\left\{y \in \pi_{**}(A\,\square_C \mathbb{F}_p) \,:\, X \overset{y\otimes -}{\to} (A\,\square_C \mathbb{F}_p) \otimes X \text{ is null}\right\}.$$

For $y \in \pi_{**}(A\,\square_C \mathbb{F}_p)$, there is a notion of a "y-map" similar to the classical v_n-maps, though with some technical differences (see [Pal01, Definition 6.2.1, Remark 6.2.2, Definition 6.2.5, Lemma 6.2.6] for details). The analogue of the periodicity theorem is:

Theorem 2.3.8 (Palmieri, [Pal01, Theorem 6.1.3, Theorem 6.2.4]). *Let $p = 2$ and let X be a finite object in* Stable(A). *For every $y \in \pi_{**}(A\,\square_C \mathbb{F}_2)$ that maps to an A-invariant element of $\pi_{**}(A\,\square_C \mathbb{F}_2)/I(X)$, X has a y-map that is central in the ring $[X, X]_{**}$. Furthermore, the collection of objects having a y-map (for fixed y) forms a thick subcategory of* Stable(A).

The theorem is only proved at $p = 2$, in part because there is no known classification of quasi-elementary Hopf algebras, but one can conjecture analogous behavior for A at $p > 2$ and for P.

2.3.2 Vanishing lines

Unlike BP_*, the ring $\pi_{**}(A\,\square_C \mathbb{F}_p)$ is very complicated, which is an impediment to studying periodicity operators y. The Morava K-theory analogues $A\,\square_{Q(r)} \mathbb{F}_p$ and $A\,\square_{E(r)} \mathbb{F}_p$ of Theorem 2.3.5 and Conjecture 2.3.6 are much more tractable, though not as simple as classical Morava K-theories. In particular, for $p > 2$ every generator in

$E(r)$ is primitive in the cobar complex $C_{E(r)}(\mathbb{F}_p, \mathbb{F}_p)$, and so we have

$$\begin{aligned}\pi_{**}(A \,\square_{E(r)} \mathbb{F}_p) &= \mathrm{Ext}_{E(r)}(\mathbb{F}_p, \mathbb{F}_p) \\ &= E[h_{r+i,j} \,:\, 1 \leq i,\ 0 \leq j \leq r] \otimes P[b_{r+i,j}, v_{r+i} \,:\, 1 \leq i,\ 0 \leq j \leq r].\end{aligned}$$

Moreover, one can show (see [Pal01, Proof of Proposition 5.3.4]) that for every v_n and $b_{t,s}$ with $s < t$, there are powers $v_n^{i(n)}$, $b_{t,s}^{j(s,t)}$ that lift to $\pi_{**}(A \,\square_C \mathbb{F}_p)$. In this subsection, we will discuss spectra that have b_{ts}- and v_n-maps, and show how this relates to the classical theory of vanishing lines in (topological) Adams spectral sequence E_2 pages.

For $s < t$, let $K(\xi_t^{p^s}) := b_{ts}^{-1}(A \,\square_{D[\xi_t^{p^s}]} \mathbb{F}_p)$ denote the colimit

$$\mathrm{colim}\left(A \,\square_{D[\xi_t^{p^s}]} \mathbb{F}_p \xrightarrow{b_{ts}} A \,\square_{D[\xi_t^{p^s}]} \mathbb{F}_p \xrightarrow{b_{ts}} \dots\right)$$

and similarly for $K(\tau_n) := v_n^{-1}(A \,\square_{E[\tau_n]} \mathbb{F}_p)$, where $\mathrm{Ext}_{A \,\square_{E[\tau_n]} \mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p) = \mathrm{Ext}_{E[\tau_n]}(\mathbb{F}_p, \mathbb{F}_p) = P(v_n)$. Define the indexing set

$$\mathscr{X} = \{\xi_t^{p^s}\}_{s<t} \cup \{\tau_n\}_{n \geq 0}$$

and order them by degree s, defined by

$$\begin{aligned}s(\xi_t^{p^s}) &= \frac{p|\xi_t^{p^s}|}{2} = p^{s+1}(p^t - 1) \\ s(\tau_n) &= |\tau_n| = 2p^n - 1.\end{aligned}$$

While the objects $K(v)$ for $v \in \mathscr{X}$ do not belong to the nilpotence-detecting family of Conjecture 2.3.6, they look like Morava K-theories in the sense that they are non-connective spectra with simple coefficient rings: by the change of rings theorem we have $\pi_{**}(K(\xi_t^{p^s})) = b_{ts}^{-1}\,\mathrm{Ext}_{D[\xi_t^{p^s}]}(\mathbb{F}_p, \mathbb{F}_p) = E[h_{ts}] \otimes P[b_{ts}^{\pm 1}]$ and $\pi_{**}(K(\tau_n)) = v_n^{-1}\,\mathrm{Ext}_{E[\tau_n]}(\mathbb{F}_p, \mathbb{F}_p) = P(v_n^{\pm 1})$. More importantly, we will see that they detect spectra with b_{ts}- and v_n-maps. Say that X is *type* d if $K(v)_*X = 0$ for all $v \in \mathscr{X}$ with $s(v) < d$ and $K(v)_*X \neq 0$ for (the unique) $v \in \mathscr{X}$ such that $s(v) = d$.

Theorem 2.3.9 (Palmieri, [Pal01, Theorem 2.4.3]). *Let X be a finite object in* Stable(A) *of type d. If $s(\xi_t^{p^s}) = d$ then X has a non-nilpotent b_{ts}-map; if $s(\tau_n) = d$ then X has a non-nilpotent v_n-map.*

By Theorem 2.3.8, the collection of all such X forms a thick subcategory, and by [HPS99], given a vanishing plane in the MPASS E_∞ page for one such X, any other X has a vanishing plane parallel to the first. For every n, Palmieri constructs an object of Stable(A) with a v_n-self map, and shows it has an E_2 page vanishing plane $s \geq -|\tau_n|(s+t) + u$ ([Pal01, Theorem 4.4.1]).[1] Since Adams filtrations are non-negative and $E_r^{s,t,u}$ converges to $\pi_{s+t,u}(X)$, this shows that $\pi_{s+t,u}(X) = 0$ when $0 \geq -|\tau_n|(s+t) + u$. This recovers the following classical theorem, which classifies vanishing lines in (topological) Adams spectral sequence E_2 pages, in the case where $d = s(\tau_n)$ for some n; the construction and analysis of an (algebraic) object of type $s(\xi_t^{p^s})$ would give a proof of the other case.

Theorem 2.3.10 (Miller-Wilkerson, [MW81]). *Let M be an A-comodule of type d. Then* $\mathrm{Ext}_A^{s,t}(\mathbb{F}_p, M) = 0$ *for $s > \frac{1}{d-1}(t-s) + c$ for some intercept c.*

Remark 2.3.11. In topology, a spectrum has a v_n-map for only one n. Here, this is not the case: for example, at $p > 3$ the Smith complex $V(1)$ has an E_∞ page vanishing line of slope $\dfrac{1}{|\tau_2| - 1} = \dfrac{1}{2p^2 - 2}$ by Theorem 2.3.4, but has a higher slope vanishing line $\dfrac{1}{s(\xi_1) - 1} = \dfrac{1}{p^2 - p}$ in its E_2 page, parallel to which b_{10} acts non-nilpotently. This illustrates the fact that $H_*V(1)$ has not only a v_2-map inherited from topology, but also a b_{10}-map. One can see this phenomenon even with the unit object $\mathbb{F}_p$ in Stable(A): it has a v_0-map inherited from topology, but also a b_{10}-map (see Proposition 2.3.12).

Working over P instead of A, we also have a similar family of "easy" periodicity operators that come from the nilpotence-detecting families. To compute $\pi_{**}(P \,\square_{Q(r)} \mathbb{F}_p) =$

[1]Note that our grading convention for Adams spectral sequences is different from Palmieri's: we are using s to denote Adams filtration, t to denote internal homological degree in Stable(A), and u to denote internal topological degree. If we write Palmieri's grading as (s_P, t_P, u_P) and ours as (s,t,u) then $(s_P, t_P, u_P) = (s, u-t, t)$.

$\mathrm{Ext}_{Q(r)}(\mathbb{F}_p, \mathbb{F}_p)$ we note that every generator $\xi_{r+i}^{p^j}$ of $Q(r)$ is primitive in the cobar complex except $\overline{\Delta}(\xi_{2r+i}) = \xi_r \otimes \xi_{r+i}^{p^r}$ for $i \geq 1$. So we have

$$\pi_{**}(P \,\square_{Q(r)} \mathbb{F}_p) = \mathbb{F}_p[h_{r+i,0}, h_{r+j,k}, b_{r+i,0}, b_{r+j,k} \,:\, 1 \leq i \leq r,\, 1 \leq j,\, 1 \leq k \leq r-1]/(h_{r+i,0}^2, h_{r+j,k}^2)$$
$$\otimes \mathrm{Ext}_{(D[\xi_r, \xi_{2r+i}, \xi_{r+i}^{p^r} \,:\, 1 \leq i])}(k,k).$$

Furthermore, I claim b_{r0} is non-nilpotent in $\pi_{**}(P \,\square_{Q(r)} k) = \mathrm{Ext}^*_{Q(r)}(k,k)$ using the same argument as Proposition 2.3.12, using the comparison $Q(r) \to D[\xi_r]$. One can show, as for A, that powers of the periodicity operators b_{ts} lift to $\pi_{**}(P \,\square_{C'} \mathbb{F}_p)$. In particular, b_{10} is the operator with the lowest degree, and in the next proposition we show that $\mathbb{F}_p$ has type $s(\xi_1)$ as an object of Stable(P).

Proposition 2.3.12. *Let $p > 2$. The element $b_{10} = \frac{1}{p} \sum_{0<i<p} \binom{p}{i} [\xi_1^i | \xi_1^{p-i}]$ is non-nilpotent in $\mathrm{Ext}_P^{**}(\mathbb{F}_p, \mathbb{F}_p)$ and in $\mathrm{Ext}_A^{**}(\mathbb{F}_p, \mathbb{F}_p)$.*

Proof. The Hopf algebra maps $A \to D$ and $P \to D$ give rise to graded ring maps $\mathrm{Ext}^*_A(k,k) \to \mathrm{Ext}^*_D(k,k)$ and $\mathrm{Ext}^*_P(k,k) \to \mathrm{Ext}^*_D(k,k)$. Since b_{10} has cobar formula $\frac{1}{p} \sum_{0<i<p} \binom{p}{i} \xi_1^i \otimes \xi_1^{p-i}$ in $C_A(k,k)$, $C_P(k,k)$, and $C_D(k,k)$, these maps take b_{10} to b_{10}, and hence take b_{10}^n to b_{10}^n for any n. Since $b_{10} \in \mathrm{Ext}^*_D(k,k) = E[h_{10}] \otimes P[b_{10}]$ is non-nilpotent, so are $b_{10} \in \mathrm{Ext}^2_A(k,k)$ and $b_{10} \in \mathrm{Ext}^2_P(k,k)$. □

Remark 2.3.13. This shows that the unit object $\mathbb{F}_p$ of Stable(P) has a b_{10}-map. The main goal of Chapters 4 and 5 is to study $b_{10}^{-1}\,\mathrm{Ext}_P(\mathbb{F}_3, \mathbb{F}_3)$, and by the discussion above, we can think of this as a chromatic localization of the unit object $\mathbb{F}_3$ in the category Stable(P) with respect to b_{10}, the first periodicity operator acting on $\mathbb{F}_3$.

Chapter 3

Spectral sequence comparisons

In this chapter we discuss the relationship between three spectral sequences for computing Ext groups over a Hopf algebra Γ: the MPASS introduced in the previous chapter, a particular filtration spectral sequence on the cobar complex of Γ, and the Cartan-Eilenberg spectral sequence. In order to define the third spectral sequence, one needs to start with the data of an extension of Hopf algebras $\Phi \to \Gamma \to \Sigma$; this then produces a spectral sequence

$$E_2^{**} = \mathrm{Ext}^*_\Phi(k, \mathrm{Ext}^*_\Sigma(k,k)) \implies \mathrm{Ext}^*_\Gamma(k,k).$$

If Σ is a conormal quotient of Γ, then Palmieri [Pal01, Proposition 1.4.9] shows that the Cartan-Eilenberg spectral sequence agrees with the Φ-based MPASS. However, the MPASS is more general than this: given any Γ-comodule-algebra Φ—not necessarily a Hopf algebra—one can study the Φ-based MPASS computing $\mathrm{Ext}_\Gamma(k,k)$. In Section 3.2 we discuss a modification of the construction of the Cartan-Eilenberg spectral sequence that permits it to be defined in this setting, though (as in the case of the MPASS) more conditions are necessary to show it has the desired E_2 term. We show that this more general Cartan-Eilenberg spectral sequence coincides with the MPASS at E_1. This involves some nuances of the cobar resolution, so we give a careful account of its construction in Section 3.1.

Furthermore, it is known [Ada60, §2.3] that the usual Cartan-Eilenberg spectral sequence coincides with a certain filtration of the Γ-cobar complex that depends on Φ. This filtration can be defined when Φ is a sub-Γ-comodule-algebra of Γ, and in Section 3.3 we show that the filtration spectral sequence coincides with the Cartan-Eilenberg spectral sequence defined in Section 3.2.

Our main interest in this setting comes from our desire to study the $b_{10}^{-1}B$-based MPASS for computing $b_{10}^{-1}\operatorname{Ext}_P(k,k) = \operatorname{Hom}_{b_{10}^{-1}\operatorname{Stable}(P)}(k,k)$, where $B = P\square_{D[\xi_1]}k$. In Chapters 5 and 6, we will find each of these three computational tools convenient at different points, and will make use of their equivalence.

Notation 3.0.1. Given a Hopf algebra Γ and a left Γ-comodule M, we will write $\sum m' \otimes m'' := \psi(m)$ and $\sum \gamma' \otimes \gamma'' := \Delta(\gamma)$ for $m \in M$ and $\gamma \in \Gamma$ when there is no ambiguity which coaction is in play.

We also will need notation for the iterated coproduct $\Gamma \xrightarrow{\Delta^n} \Gamma^{\otimes n+1}$ and coaction $M \xrightarrow{\psi^n} \Gamma^{\otimes n} \otimes M$; we will write $\sum m_{(1)}|\dots|m_{(n+1)} := \psi^n(m)$ and $\sum \gamma_{(1)}|\dots|\gamma_{(n+1)} := \Delta^n(\gamma)$. (Note that this notation is well-defined because of coassociativity.)

For example, $\Delta(\gamma) = \sum \gamma'|\gamma'' = \sum \gamma_{(1)}|\gamma_{(2)}$, and $\sum \Delta(\gamma_{(1)})|\gamma_{(2)} = \sum \gamma_{(1)}|\gamma_{(2)}|\gamma_{(3)}$.

We will make extensive use of the following identities, which are part of the definition of a Hopf algebra.

Fact 3.0.2. *Let Γ be a Hopf algebra with antipode c, comultiplication Δ, unit η, and coaugmentation ε.*

(1) (Coassociativity) $\sum (x')' \otimes (x')'' \otimes x'' = \sum x' \otimes (x'')' \otimes (x'')''$
(This fact is used to make the notation above well-defined; in that language, this just reads $\sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$.*)*

(2) $\sum c(x')x'' = \varepsilon(x) = \sum x'c(x'')$

(3) $\sum \varepsilon(x') \otimes x'' = 1 \otimes x$

(4) $\sum c(x)'|c(x)'' = \sum c(x'')|c(x')$

3.1 The cobar complex and the shear isomorphism

3.1.1 Constructing the cobar complex

Let Γ be a commutative Hopf algebra over k, N a left Γ-comodule, and M a right Γ-comodule. The cobar *resolution* $D_\Gamma(N)$ is a particularly nice Γ-injective resolution for N; the cobar *complex* $C_\Gamma(M, N)$ is the complex obtained by applying $M \square_\Gamma -$ to the cobar resolution. The cohomology of the cobar resolution is $\mathrm{Ext}_\Gamma(M, N)$. We will discuss two isomorphic constructions of the cobar resolution (denoted $\overset{\Delta}{D}{}^*_\Gamma(N)$ and $\overset{L}{D}{}^*_\Gamma(N)$), as they are both common in the literature; the isomorphism connecting them is the *shear isomorphism*, which we discuss first.

Definition 3.1.1. Let M and N be left Γ-comodules, with coaction denoted by $\psi(m) = \sum m' \otimes m''$ and $\psi(n) = \sum n' \otimes n''$. There are two natural ways to put a Γ-comodule structure on their tensor product $M \otimes N$: the *left coaction* $M \otimes N \to \Gamma \otimes (M \otimes N)$ is given by $m \otimes n \mapsto \sum m' \otimes m'' \otimes n$, and the *diagonal coaction* is given by $m \otimes n \mapsto \sum m'n' \otimes m'' \otimes n''$. To distinguish these, we write $M \overset{L}{\otimes} N$ for the tensor product $M \otimes N$ endowed with the left Γ-coaction, and $M \overset{\Delta}{\otimes} N$ for the diagonal coaction.

For a pair of right Γ-comodules one can analogously define the right and diagonal coactions, denoted $\overset{R}{\otimes}$ and $\overset{\Delta}{\otimes}$, respectively.

These constructions agree in the following special case:

Lemma 3.1.2 (Shear isomorphism). *If M is a left Γ-comodule, there is an isomorphism*

$S : \Gamma \overset{\triangle}{\otimes} M \to \Gamma \overset{L}{\otimes} M$ *given by:*

$$S : a \otimes m \mapsto \sum am' \otimes m''$$
$$S^{-1} : a \otimes m \mapsto \sum ac(m') \otimes m''$$

where c *is the antipode on* Γ. *Analogously, if* M *is a right* Γ*-comodule, there is an isomorphism* $S_c : M \overset{\triangle}{\otimes} \Gamma \to M \overset{R}{\otimes} \Gamma$ *given by:*

$$S_c : m \otimes a \mapsto \sum m' \otimes m''a$$
$$S_c^{-1} : m \otimes a \mapsto \sum m' \otimes c(m'')a.$$

Proof. We prove just that the pair (S, S^{-1}) are actually inverses; the statement for S_c is analogous. First we prove that $S \circ S^{-1} = \mathbb{1}$. We have

$$\begin{aligned}
S(S^{-1}(a \otimes m)) &= S(\textstyle\sum ac(m') \otimes m'') \\
&= \sum ac(m')(m'')' \otimes (m'')'' \\
&= \sum ac(m_{(1)})m_{(2)} \otimes m_{(3)} \\
&= \sum a\varepsilon(m_{(1)}) \otimes m_{(2)} \\
&= \sum a \otimes m
\end{aligned}$$

where the fourth equality is by Fact 3.0.2(2) and the last equality is by Fact 3.0.2(3).

In the other direction, analogous application of Hopf algebra properties yields:

$$\begin{aligned}
S^{-1}(S(a \otimes m)) &= S^{-1}(\textstyle\sum am' \otimes m'') \\
&= \sum am'c((m'')') \otimes (m'')'' \\
&= \sum am_{(1)}c(m_{(2)}) \otimes m_{(3)} \\
&= \sum a\varepsilon(m_{(1)}) \otimes m_{(2)} \\
&= \sum a \otimes m.
\end{aligned}$$

□

To define the first version $\overset{\triangle}{D}{}^*_\Gamma(N)$ of the cobar resolution of N, observe there is a free-forgetful adjunction

$$U : \mathrm{Comod}_\Gamma \rightleftarrows \mathrm{Mod}_k : F.$$

The free functor F sends $N \mapsto \Gamma \overset{L}{\otimes} N$ with unit $n \mapsto \sum n' \otimes n''$ where the Γ-coaction on N is written $\psi(n) = \sum n' \otimes n''$. We can form an augmented cosimplicial object from the monad FU:

$$\begin{array}{l} N \\ \downarrow \simeq \\ \overset{\triangle}{D}{}^\bullet_\Gamma(N) = \Big(\Gamma \overset{\triangle}{\otimes} N \underset{\eta_2}{\overset{\eta_1}{\underset{\longleftarrow \mu \longrightarrow}{\longrightarrow}}} \Gamma \overset{\triangle}{\otimes} \Gamma \overset{\triangle}{\otimes} N \underset{\eta_3}{\overset{\eta_1}{\longrightarrow}} \!\!\begin{smallmatrix} \longleftarrow \mu_1 \longrightarrow \\ \longleftarrow \mu_2 \longrightarrow \end{smallmatrix}\!\! \Gamma \overset{\triangle}{\otimes} \Gamma \overset{\triangle}{\otimes} \Gamma \overset{\triangle}{\otimes} N \qquad \cdots \qquad \Big) \end{array} \tag{3.1.1}$$

The codegeneracies μ_i are multiplication of the i^{th} and $(i+1)^{st}$ copies of Γ, and the coface maps η_i are given by insertion of 1 into the i^{th} spot. The second version of the cobar resolution arises from a second augmented cosimplicial object:

$$\begin{array}{l} N \\ \downarrow \simeq \\ \overset{L}{D}{}^\bullet_\Gamma(N) = \Big(\Gamma \overset{L}{\otimes} N \underset{\psi}{\overset{\Delta}{\underset{\longleftarrow \varepsilon \longrightarrow}{\longrightarrow}}} \Gamma \overset{L}{\otimes} \Gamma \overset{L}{\otimes} N \underset{\psi}{\overset{\Delta_1}{\longrightarrow}} \!\!\begin{smallmatrix} \longleftarrow \varepsilon_2 \longrightarrow \\ \longleftarrow \varepsilon_1 \longrightarrow \end{smallmatrix}\!\! \Gamma \overset{L}{\otimes} \Gamma \overset{L}{\otimes} \Gamma \overset{L}{\otimes} N \qquad \cdots \qquad \Big) \end{array} \tag{3.1.2}$$

Here the codegeneracies ε_i come from applying the coaugmentation ε to the i^{th} spot, and the coface maps $\Delta_i : \Gamma^{\otimes n} \otimes N \to \Gamma^{\otimes n+1} \otimes N$ for $1 \leq i \leq n$ come from applying Δ to the i^{th} slot; the last coface map comes from the coaction $\psi : N \to \Gamma \otimes N$.

Remark 3.1.3. If N were a right Γ-comodule, we could have built analogous cosimplicial objects $N \overset{\triangle}{\otimes} \Gamma^{\overset{\triangle}{\otimes} \bullet+1}$ and $N \otimes \Gamma^{\otimes \bullet} \overset{R}{\otimes} \Gamma$. To avoid too much notational clutter, we will use the same notation in these cases: if N is being thought of as a right Γ-comodule, the symbol $\overset{\triangle}{D}{}^\bullet_\Gamma(N)$ will mean the aforementioned cosimplicial object, and if N is a left Γ-comodule, $\overset{\triangle}{D}{}^\bullet_\Gamma(N)$ will mean the cosimplicial object in (3.1.1).

Definition 3.1.4. The *(non-normalized) cobar resolution* $\overset{\triangle}{D}{}^*_\Gamma(N)$ is the associated chain complex of $\overset{\triangle}{D}{}^\bullet_\Gamma(N)$ (that is, the complex $\Gamma^{\overset{\triangle}{\otimes} *+1} \overset{\triangle}{\otimes} N$ whose differentials are an alternating sum of coface maps in $\overset{\triangle}{D}{}^\bullet_\Gamma(N)$). Similarly, define $\overset{L}{D}{}^*_\Gamma(N)$ to be the associated chain complex of $\overset{L}{D}{}^\bullet_\Gamma(N)$.

Definition 3.1.5. The *(non-normalized) cobar complex* $\overset{\triangle}{C}{}^*_\Gamma(M,N)$ is the complex $M \,\square_\Gamma \overset{\triangle}{D}{}^*_\Gamma(N)$. Similarly, define $\overset{L}{C}{}^*_\Gamma(M,N) = M \,\square_\Gamma \overset{L}{D}{}^*_\Gamma(N)$.

Remark 3.1.6. The cobar differential most commonly used in cobar computations, e.g. as in [Rav86, A1.2.11], is $\overset{L}{C}{}^*_\Gamma(M,N)$, not $\overset{\triangle}{C}{}^*_\Gamma(M,N)$ (and Ravenel's $C_\Gamma(M,N)$ there refers to (a normalized version of) $\overset{L}{C}{}^*_\Gamma(M,N)$). Since the differential in $\overset{\triangle}{C}{}^*_\Gamma(M,N)$ looks simpler than the one in $\overset{L}{C}{}^*_\Gamma(M,N)$, one might wonder why we don't use the former in computations instead; one issue is that the complication resurfaces when trying to write down an individual term $\overset{\triangle}{C}{}^n_\Gamma(M,N)$ explicitly; by contrast, $\overset{L}{C}{}^n_\Gamma(M,N) \cong M \otimes \Gamma^{\otimes n} \otimes N$ is easy to work with. Another reason the $\overset{L}{C}{}^n_\Gamma(M,N)$ version is preferred is that it only uses the coalgebra structure of Γ.

3.1.2 More on the shear isomorphism

The isomorphism $\overset{\triangle}{D}{}^n_\Gamma(N) \xrightarrow{\cong} \overset{L}{D}{}^n_\Gamma(N)$ is given by the iterated shear isomorphism:

$$S^n : \Gamma^{\overset{\triangle}{\otimes} n} \overset{\triangle}{\otimes} M = \Gamma^{\overset{\triangle}{\otimes} n-1} \overset{\triangle}{\otimes} (\Gamma \overset{\triangle}{\otimes} M) \xrightarrow{S} \Gamma^{\overset{\triangle}{\otimes} n-1} \overset{\triangle}{\otimes} (\Gamma \overset{L}{\otimes} M) \xrightarrow{S} \Gamma^{\overset{\triangle}{\otimes} n-2} \overset{\triangle}{\otimes} (\Gamma \overset{L}{\otimes} \Gamma \overset{L}{\otimes} M) \xrightarrow{S} \dots \xrightarrow{S} \Gamma^{\overset{L}{\otimes} n} \overset{\triangle}{\otimes} M.$$

We will need an explicit formula for this.

Lemma 3.1.7. *The iterated shear isomorphism* $S^n : \Gamma^{\overset{\triangle}{\otimes} n} \overset{\triangle}{\otimes} M \to \Gamma \overset{L}{\otimes} \Gamma^{\otimes n-1} \otimes M$ *is given by*

$$S^n : x_1|\dots|x_n|m \mapsto \sum x_{1(1)}x_{2(1)}\dots x_{n(1)}m_{(1)}|x_{2(2)}\dots x_{n(2)}m_{(2)}|x_{3(3)}\dots x_{n(3)}m_{(3)}|\dots|m_{(n+1)}.$$

The iterated shear isomorphism $S_c^n : M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} n} \to M \otimes \Gamma^{\otimes n-1} \overset{R}{\otimes} \Gamma$ *is given by*

$$S_c^n : m|x_n|\dots|x_1 \mapsto \sum m_{(1)}|m_{(2)}x_{n(1)}|m_{(3)}x_{n(2)}x_{n-1(1)}|\dots|m_{(n+1)}x_{n(n)}x_{n-1(n-1)}\cdots x_{2(2)}x_1.$$

Proof. We prove just the first statement, as the second is analogous. Use induction on n. If $n = 2$ this is true by definition of S. Now suppose S^{n-1} is given by the formula above. We can write S^n as the composition

$$\Gamma^{\overset{\vartriangle}{\otimes} n} \overset{\vartriangle}{\otimes} M \xrightarrow{S^{n-1}} \Gamma \overset{\vartriangle}{\otimes} (\Gamma^{\overset{L}{\otimes} n-1} \otimes M) \xrightarrow{S} \Gamma \overset{L}{\otimes} (\Gamma^{\overset{L}{\otimes} n-1} \otimes M)$$

and by the inductive hypothesis the first map sends

$$x_1|x_2|\dots|x_n|m \mapsto \sum x_1|x_{2(1)}x_{3(1)}\dots x_{n(1)}m_{(1)}|x_{3(2)}\dots x_{n(2)}m_{(2)}|\dots|m_{(n)}.$$

If we write this as $x_1|y$, then the second map sends this to $\sum x_1 y_{(1)}|y_{(2)}$; remembering that the coaction on y just comes from the first component, this is:

$$\sum x_1x_{2(1)}x_{3(1)}\dots x_{n(1)}m_{(1)}|x_{2(2)}x_{3(2)}\dots x_{n(2)}m_{(2)}|x_{3(3)}\dots x_{n(3)}m_{(3)}|\dots|m_{(n+1)}. \qquad \square$$

Lemma 3.1.8. *The iterated inverse shear isomorphism* $S^{-n} : \Gamma \overset{L}{\otimes} \Gamma^{\otimes n-1} \otimes M \to \Gamma^{\overset{\vartriangle}{\otimes} n} \overset{\vartriangle}{\otimes} M$ *is given by*

$$S^{-n} : x_1|\dots|x_n|m \mapsto \sum x_1c(x_2')|x_2''c(x_3')|x_3''c(x_4')|\dots|x_n''c(m')|m''.$$

The iterated inverse shear isomorphism $S_c^{-n} : M \otimes \Gamma^{\otimes n-1} \overset{R}{\otimes} \Gamma \to M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} n}$ *is given by*

$$S_c^{-n} : m|x_n|\dots|x_1 \mapsto \sum m'|c(m'')x_n'|c(x_n')x_{n-1}'|\dots|c(x_2')x_1.$$

Proof. Again we only prove the first statement, and again this is by induction on n. If $n = 1$, this is the definition of S^{-1} in Lemma 3.1.2.

Assume the formula holds for $n-1$. Write S^{-n} as the composition

$$\Gamma \overset{L}{\otimes} (\Gamma^{\overset{L}{\otimes} n-1} \otimes M) \xrightarrow{S^{-(n-1)}} \Gamma \overset{L}{\otimes} (\Gamma^{\overset{\Delta}{\otimes} n} \overset{\Delta}{\otimes} M) \xrightarrow{S^{-1}} \Gamma \overset{\Delta}{\otimes} (\Gamma^{\overset{\Delta}{\otimes} n} \overset{\Delta}{\otimes} M)$$

and by the inductive hypothesis the first map sends

$$x_1|x_2|\dots|x_n|m \mapsto \sum x_1|x_2c(x_3')|x_3''c(x_4')|\dots|x_n''c(m')|m''.$$

If we write this as $x_1|y$, then the second map sends this to $\sum x_1 c(y_{(1)})|y_{(2)}$, which is

$$\begin{aligned}
&\sum x_1 c\big((x_2c(x_3')x_3''c(x_4')\dots x_n''c(m')m'')'\big)\,|x_2''c(x_3')''|(x_3'')''c(x_4')''|\dots|(x_n'')''c(m')''|(m'')''\\
&\quad= \sum x_1 c\big(x_{2(1)}c(x_{3(2)})x_{3(3)}c(x_{4(2)})x_{4(3)}\dots c(m_{(2)})m_{(3)}\big)\,|x_{2(2)}c(x_{3(1)})|x_{3(4)}c(x_{4(1)})|\\
&\qquad\qquad \dots|x_{n(4)}c(m_{(1)})|m_{(4)}\\
&\quad= \sum x_1 c\big(x_{2(1)}\varepsilon(x_{3(2)}\dots x_{n(2)}m_{(2)})\big)|x_{2(2)}c(x_{3(1)})|x_{3(3)}c(x_{4(1)})|\\
&\qquad\qquad \dots|x_{n(3)}c(m_{(1)})|m_{(3)}\\
&\quad= \sum x_1 c(x_{2(1)})|x_{2(2)}c(x_{3(1)})|x_{3(2)}c(x_{4(1)})|\dots|x_{n(2)}c(m_{(1)})|m_{(2)}.
\end{aligned}$$

Here the first equality uses the fact that $\sum c(x')|c(x'') = \sum c(x)''|c(x)'$, the second uses the fact that $c(x')x'' = \varepsilon(x)$, and the third uses the fact that $\sum \varepsilon(x')|x'' = \sum 1|x$. □

Eventually, we will work in a setting where $q : \Gamma \to \Sigma$ is a map of Hopf algebras, and $\Phi = \Gamma \Box_\Sigma k$. In this situation, we will make extensive use of the following lemma.

Lemma 3.1.9. *Let M be a Γ-comodule. Then $\Gamma \Box_\Sigma M \subset \Gamma \overset{L}{\otimes} M$ inherits a left Γ-comodule structure, and the shear isomorphism $S : \Gamma \overset{\Delta}{\otimes} M \to \Gamma \overset{L}{\otimes} M$ restricts to an isomorphism*

$$\Phi \overset{\Delta}{\otimes} M \xrightarrow{\cong} \Gamma \Box_\Sigma M.$$

The shear isomorphism $S_c : M \overset{\Delta}{\otimes} \Gamma \to M \overset{R}{\otimes} \Gamma$ restricts to an isomorphism $M \overset{\Delta}{\otimes} \Phi \xrightarrow{\cong} M \Box_\Sigma \Gamma$.

Proof. First we check that the left comodule structure on $\Gamma \overset{L}{\otimes} M$ restricts to a comodule structure on $\Gamma \,\square_\Sigma M$. I claim that both squares below commute:

$$\begin{array}{ccc}
\Gamma \overset{L}{\otimes} M & \xrightarrow{\psi\otimes\mathbb{1}} & \Gamma\otimes\Gamma\otimes M \\
{\scriptstyle \psi\otimes\mathbb{1}}\Big\downarrow\Big\downarrow{\scriptstyle \mathbb{1}\otimes\psi} & & {\scriptstyle \mathbb{1}\otimes\psi\otimes\mathbb{1}}\Big\downarrow\Big\downarrow{\scriptstyle \mathbb{1}\otimes\mathbb{1}\otimes\psi} \\
\Gamma\otimes\Sigma\otimes M & \xrightarrow{\Delta\otimes\mathbb{1}\otimes\mathbb{1}} & \Gamma\otimes\Gamma\otimes\Sigma\otimes M
\end{array}$$

This comes from coassociativity of Γ, plus the fact that the coaction $\Gamma \to \Sigma \otimes \Gamma$ comes from composing the comultiplication on Γ with the given Hopf algebra map $q : \Gamma \to \Sigma$. An element is an element of $\Gamma \overset{L}{\otimes} M$ that equalizes the left vertical maps. Given an element $a \otimes m \in \Gamma \,\square_\Sigma M$ (i.e. an element that equalizes the left vertical maps), we need to show that $\psi(a \otimes m)$ lands in $\Gamma \otimes (\Gamma \,\square_\Sigma M)$ (i.e. that this element is in the equalizer of the right vertical maps). This is given by the commutativity of the diagram.

Write M_η for M with the trivial Σ-coaction. Then we have

$$\Phi \overset{\triangle}{\otimes} M \cong (\Gamma \,\square_\Sigma k) \overset{\triangle}{\otimes} M \cong \Gamma \,\square_\Sigma (k \overset{\triangle}{\otimes} M) \cong \Gamma \,\square_\Sigma M_\eta.$$

To show S restricts to a map $\Phi \overset{\triangle}{\otimes} M \cong \Gamma \,\square_\Sigma M_\eta \to \Gamma \,\square_\Sigma M$, using the same argument as above it suffices to find a map f such that both of the squares in the diagram below commute:

$$\begin{array}{ccc}
\Gamma \overset{\triangle}{\otimes} M & \xrightarrow{S} & \Gamma \overset{L}{\otimes} M \\
{\scriptstyle \psi\otimes\mathbb{1}}\Big\downarrow\Big\downarrow{\scriptstyle \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}} & & {\scriptstyle \psi\otimes\mathbb{1}}\Big\downarrow\Big\downarrow{\scriptstyle \mathbb{1}\otimes\psi} \\
\Gamma\otimes\Sigma\otimes M & \xrightarrow{f} & \Gamma\otimes\Sigma\otimes M
\end{array}$$

Define $f : x \otimes y \otimes z \mapsto \sum x(z')' \otimes y(z')'' \otimes z''$. Checking commutativity of the diagram uses the fact (from coassociativity of $\Delta : \Gamma \to \Gamma \otimes \Gamma$) that $\sum (z')' \otimes (z')'' \otimes z'' = \sum z' \otimes (z'')' \otimes (z'')''$.

Finally, we show that $S^{-1} : \Gamma \overset{L}{\otimes} M \to \Gamma \overset{\triangle}{\otimes} M$ restricts to a morphism $\Gamma \,\square_\Sigma M \to \Gamma \,\square_\Sigma M$.

As before, it suffices to show commutativity of

$$
\begin{array}{ccc}
\Gamma \overset{L}{\otimes} M & \xrightarrow{\quad S^{-1} \quad} & \Gamma \overset{\Delta}{\otimes} M \\
{\scriptstyle \psi\otimes\mathbb{1}} \Big\downarrow \Big\downarrow {\scriptstyle \mathbb{1}\otimes\psi} & & {\scriptstyle \psi\otimes\mathbb{1}} \Big\downarrow \Big\downarrow {\scriptstyle \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}} \\
\Gamma \otimes \Sigma \otimes M & \xrightarrow{\quad g \quad} & \Gamma \otimes \Sigma \otimes M
\end{array}
$$

for some g. Take g to be the morphism $x \otimes y \otimes z \mapsto \sum xc(z')' \otimes yc(z')'' \otimes z''$. It is obvious that the square obtained by taking the leftmost of each pair of vertical arrows is commutative, but the other square needs to be checked. The bottom left composition applied to $a \otimes m$ is $C_B := \sum ac((m'')')' \otimes m'c((m'')')'' \otimes (m'')''$ and the top right composition is $C_T := \sum ac(m') \otimes 1 \otimes m''$. We have

$$
\begin{aligned}
C_B &= \sum ac(m_{(2)})'|m_{(1)}c(m_{(2)})''|m_{(3)} & \\
&= \sum ac(m_{(3)})|m_{(1)}c(m_{(2)})|m_{(4)} & \text{Fact 3.0.2(4)} \\
&= \sum c(m_{(2)})|\varepsilon(m_{(1)})|m_{(3)} & \text{Fact 3.0.2(2)} \\
&= \sum c(m_{(1)})|1|m_{(2)} = C_T. &
\end{aligned}
$$

□

The change of rings theorem

$$\mathrm{Ext}^*_\Gamma(M, \Gamma \,\square_\Sigma N) \cong \mathrm{Ext}^*_\Sigma(M, N)$$

is a standard result in homological algebra (see, e.g., [CE99, §VI.4]). For future reference we record the following variant, obtained using Lemma 3.1.9.

Corollary 3.1.10 (Change of rings theorem). *Let M be a right Γ-comodule and N a left Γ-comodule, and let $\Phi = \Gamma \,\square_\Sigma N$. Then there is an isomorphism*

$$\mathrm{Ext}^*_\Gamma(M, \Phi \overset{\Delta}{\otimes} N) \cong \mathrm{Ext}^*_\Sigma(M, N).$$

3.1.3 The normalized cobar complex

Finally, we discuss two useful smaller versions of the cobar complex that turn out to be chain-homotopic to the cobar complex; they are isomorphic to each other, and are both referred to as the *normalized complex.*

Definition 3.1.11. Let $A^\bullet$ be a cosimplicial object in an abelian category, with associated complex A^*. Define the subcomplex $\mathcal{N}A^*$ of A^* and the quotient complex $\mathcal{Q}A^*$ of A^* as follows:

$$\mathcal{N}A^n = \bigcap_{i=0}^{n-1} \ker(s^i : A^n \to A^{n-1})$$
$$\mathcal{Q}A^n = A^n / \sum_{i=1}^{n} \operatorname{im}(d^i : A^{n-1} \to A^n).$$

Theorem 3.1.12. *There are chain homotopy equivalences $\mathcal{N}A^* \simeq A^* \simeq \mathcal{Q}A^*$, and there is an isomorphism of chain complexes $\mathcal{N}A^* \cong \mathcal{Q}A^*$. (In particular, we can write $A^* = \mathcal{N}A^* \oplus \mathcal{D}A^*$ for a contractible complex D^*, such that $\mathcal{Q}A^* = A^*/\mathcal{D}A^*$.)*

For a proof of this theorem in the dual (simplicial) case, see [GJ09, Theorem III.2.1 and Theorem III.2.4].

Remark 3.1.13. Note that $\mathcal{N}\overset{L}{C}{}^*_\Gamma(M, N)$ is just usual normalized cobar complex $M \otimes \overline{\Gamma}^{\otimes n} \otimes N \cong M \,\square_\Gamma (\Gamma \overset{L}{\otimes} \overline{\Gamma}^{\otimes n} \otimes N)$. There is a resolution $\Gamma \overset{\vartriangle}{\otimes} \overline{\Gamma}^{\overset{\vartriangle}{\otimes} *} \overset{\vartriangle}{\otimes} N$ of N, but this is $\mathcal{Q}(\overset{\vartriangle}{D}{}^\bullet_\Gamma(N))$, not $\mathcal{N}(\overset{\vartriangle}{D}{}^\bullet_\Gamma(N))$. Instead, by definition we have

$$\mathcal{N}\overset{\vartriangle}{D}{}^n_\Gamma(N) = \bigcap_{i=0}^{n-1} \ker(\mu_i : \Gamma^{\overset{\vartriangle}{\otimes} n+1} \to \Gamma^{\overset{\vartriangle}{\otimes} n}) \overset{\vartriangle}{\otimes} N \subset \overset{\vartriangle}{D}{}^n_\Gamma(N)$$

where μ_i multiplies the i^{th} and $(i+1)^{st}$ factors of Γ.

Since $\overset{\vartriangle}{D}{}^\bullet_\Gamma(N)$ is defined to be zero in degrees < 0, we have $\mathcal{N}\overset{\vartriangle}{D}{}^0_\Gamma(N) = \Gamma \overset{\vartriangle}{\otimes} N$.

3.2 The Cartan-Eilenberg spectral sequence

3.2.1 Classical Cartan-Eilenberg spectral sequence

Let Γ be a Hopf algebra. Given an extension of Hopf algebras

$$\Phi \longrightarrow \Gamma \longrightarrow \Sigma$$

(so in particular $\Phi = \Gamma \,\square_\Sigma k$), a right Γ-comodule M, and a left Φ-comodule N, the Cartan-Eilenberg spectral sequence for computing $\mathrm{Cotor}_\Gamma(M, N)$ arises from the double complex (Γ-resolution of M) $\square_\Gamma$ (Φ-resolution of N). If we use the usual normalized cobar resolutions $\mathcal{N}\overset{L}{D}{}^*_\Gamma(M)$ and $\mathcal{N}\overset{L}{D}{}^*_\Phi(N)$, the double complex is

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & (M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s} \otimes N) & \xrightarrow{(-1)^t \mathbb{1} \otimes d_\Phi} & (M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s+1} \otimes N) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle d_\Gamma \otimes \mathbb{1}} & & \downarrow{\scriptstyle d_\Gamma \otimes \mathbb{1}} & \\
\cdots \longrightarrow & (M \otimes \overline{\Gamma}^{\otimes t+1} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s} \otimes N) & \xrightarrow{(-1)^{t+1} \mathbb{1} \otimes d_\Phi} & (M \otimes \overline{\Gamma}^{\otimes t+1} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s+1} \otimes N) & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & \\
 & \vdots & & \vdots &
\end{array} \tag{3.2.1}$$

The signs come from the usual formula for the tensor product of chain complexes, and satisfy $d_{\mathrm{vert}} d_{\mathrm{horiz}} + d_{\mathrm{horiz}} d_{\mathrm{vert}} = 0$. The spectral sequence that starts by taking homology in the vertical direction first has

$$\begin{aligned}
E_1^{s,t} &= \mathrm{Cotor}^t_\Gamma(M, \, \Phi \otimes \overline{\Phi}^{\otimes s} \otimes N) \\
&\cong \mathrm{Cotor}^t_\Gamma(M, \, (\Gamma \,\square_\Sigma k) \otimes \overline{\Phi}^{\otimes s} \otimes N) \\
&\cong \mathrm{Cotor}^t_\Sigma(M, \, \overline{\Phi}^{\otimes s} \otimes N).
\end{aligned}$$

where the last isomorphism is by the change of rings theorem. For the spectral sequence that starts by taking homology in the horizontal direction first, exactness of the functor

$(M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_\Gamma -$ gives

$$E_1^{*,t} \cong H^*((M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes *} \otimes N)) \cong (M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_\Gamma H^*(\Phi \otimes \overline{\Phi}^{\otimes *} \otimes N)$$

and by the exactness of the resolution $\Phi \otimes \overline{\Phi}^{\otimes *} \otimes N$ of N, this is concentrated in degree zero as $(M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_\Gamma N$. The E_2 page then takes cohomology in the t direction, obtaining $E_2 \cong E_\infty \cong \mathrm{Cotor}_\Gamma(M, N)$. This implies that the spectral sequence that starts by taking homology in the vertical direction first also converges to $\mathrm{Cotor}_\Gamma(M, N)$. The Cartan-Eilenberg spectral sequence is the vertical-first spectral sequence, and we have just shown that it has

$$E_1^{s,t} = \mathrm{Cotor}^t_\Sigma(M, \overline{\Phi}^{\otimes s} \otimes N) \implies \mathrm{Cotor}^{s+t}_\Gamma(M, N).$$

If Φ has trivial Σ-coaction, then we have $E_1^{s,t} \cong \mathrm{Cotor}^t_\Sigma(M, N) \otimes \overline{\Phi}^{\otimes s}$, whose cohomology is:

$$E_2 = \mathrm{Cotor}^s_\Phi(k, \mathrm{Cotor}^t_\Sigma(M, N)).$$

The spectral sequence converges because it is a first-quadrant double complex spectral sequence (see e.g. [McC01, Theorem 2.15]).

Remark 3.2.1. The E_2 page is independent of the Φ-resolution of N and the Γ-resolution of M, but the E_1 page does depend on the Φ-resolution of N.

3.2.2 Weakening the hypotheses

The goal of this section is to remove the requirement that Φ be a coalgebra. More precisely, let Γ be a Hopf algebra and Φ be any Γ-comodule-algebra. The first issue with defining an analogue of (3.2.1) is that it is unclear what category N should be in, seeing as there is no such thing as a Φ-comodule. Furthermore, the cosimplicial object $\overset{L}{D}{}^\bullet_\Phi(N)$ can't be defined, not just for the aforementioned reason but also because the coface maps are defined in terms of the coproduct on Φ. To remedy this, let N be a

Γ-comodule, and—because we assumed that Φ is an algebra—we can write down the analogue $\overset{\vartriangle}{D}{}^{\bullet}_{\Phi}(N)$ of $\overset{\vartriangle}{D}{}^{\bullet}_{\Gamma}(N)$:

$$\begin{array}{l} N \\ \downarrow \simeq \\ \overset{\vartriangle}{D}{}^{\bullet}_{\Phi}(N) = \Big(\Phi \overset{\vartriangle}{\otimes} N \underset{\eta_2}{\overset{\eta_1}{\underset{\longleftarrow \mu \longrightarrow}{\longrightarrow}}} \Phi \overset{\vartriangle}{\otimes} \Phi \overset{\vartriangle}{\otimes} N \underset{\eta_3}{\overset{\eta_1}{\underset{\longleftarrow \mu_2 \longrightarrow}{\overset{\longleftarrow \mu_1 \longrightarrow}{\longrightarrow}}}} \Phi \overset{\vartriangle}{\otimes} \Phi \overset{\vartriangle}{\otimes} \Phi \overset{\vartriangle}{\otimes} N \qquad \cdots \qquad \Big). \end{array} \tag{3.2.2}$$

This is a cosimplicial object in Γ-comodules which is quasi-isomorphic to N.

It can also be described in a more natural way. Since Φ is a monoid object in Comod_Γ, we can define the category Mod_Φ of Φ-modules in Comod_Γ. There is a free-forgetful adjunction

$$F_\Phi : \mathrm{Comod}_\Gamma \rightleftarrows \mathrm{Mod}_\Phi : U$$

where $F_\Phi(N) = \Phi \overset{\vartriangle}{\otimes} N$. Then (3.2.2) is the cosimplicial object associated to the monad UF_Φ.

Definition 3.2.2. In this context, define the Cartan-Eilenberg spectral sequence to be the spectral sequence associated to the double complex

$$(\mathcal{N}\overset{\vartriangle}{D}{}^{*}_{\Gamma}(M)) \,\square_\Gamma (\mathcal{N}\overset{\vartriangle}{D}{}^{*}_{\Phi}(N)).$$

The spectral sequence is unchanged starting at E_1 if we replace the right-most complex by a chain-homotopic one, and in Section 3.3 we will find it more convenient to use the complex

$$\overset{\vartriangle}{D}{}^{*}_{\Gamma}(M) \,\square_\Gamma\, (\mathcal{N}\overset{\vartriangle}{D}{}^{*}_{\Phi}(N)). \tag{3.2.3}$$

By definition, we have the E_1 term

$$E_1^{s,t} = \mathrm{Cotor}^t_\Gamma(M, \mathcal{N}\overset{\vartriangle}{D}{}^{*}_{\Phi}(N))$$

and it converges to $\mathrm{Cotor}_\Gamma(M, N)$ as with the usual construction of the Cartan-Eilenberg spectral sequence.

Remark 3.2.3. If Φ did have a coalgebra structure, we can also define the spectral sequence in Section 3.2.1, and it is clear that these two spectral sequences are isomorphic via the shear isomorphism.

3.2.3 Comparison with MPASS

Theorem 3.2.4. *Given a left Γ-comodule-algebra Φ and a left Γ-comodule N, the Cartan-Eilenberg spectral sequence*

$$\overset{\vartriangle}{E}{}_1^{s,*} = H^*\big(\overset{\vartriangle}{D}{}^*_\Gamma(k)\,\square_\Gamma(\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N))\big) \implies \mathrm{Cotor}^*_\Gamma(k, N) \cong \mathrm{Ext}^*_\Gamma(k, N)$$

coincides starting at E_1 with the Φ-based MPASS

$$E_1^{s,*} = \mathrm{Ext}^*_\Gamma(k, \Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N) \implies \mathrm{Ext}^*_\Gamma(k, N).$$

Proof. By Theorem 3.1.12, there is an isomorphism of chain complexes $\mathcal{N}\overset{\vartriangle}{D}{}^*_\Phi(N) \cong \mathcal{Q}\overset{\vartriangle}{D}{}^*_\Phi(N)$, so instead of the double complex $\overset{\vartriangle}{D}{}^*_\Gamma(k)\,\square_\Gamma(\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N))$ we may use

$$\overset{\vartriangle}{D}{}^*_\Gamma(k)\,\square_\Gamma(\mathcal{Q}\overset{\vartriangle}{D}{}^\bullet_\Phi(N)) = \Gamma^{\overset{\vartriangle}{\otimes} t+1}\,\square_\Gamma(\Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N).$$

Let T^* be the total complex, defined by $T^n = \bigoplus_{s+t=n} \Gamma^{\overset{\vartriangle}{\otimes} t+1}\,\square_\Gamma(\Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N)$. Define a filtration F^s on this total complex as follows:

$$F^{s_0}T^n = \bigoplus_{\substack{s+t=n \\ s \geq s_0}} \Gamma^{\overset{\vartriangle}{\otimes} t+1}\,\square_\Gamma(\Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N).$$

For the associated graded we have:

$$F^{s_0}/F^{s_0+1}T^n = \Gamma^{\overset{\triangle}{\otimes} n-s_0+1} \Box_\Gamma (\Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s_0} \overset{\triangle}{\otimes} N)$$

$$H^*(F^{s_0}/F^{s_0+1}T^*) = \mathrm{Cotor}^*_\Gamma(k, \Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N).$$

By definition, the Cartan-Eilenberg spectral sequence arises from the exact couple

$$\begin{array}{ccc} H^*(F^sT^*) & \longleftarrow & H^*(F^{s+1}T^*) \\ & \searrow \quad \quad \nearrow & \\ & H^*(F^s/F^{s+1}T^*). & \end{array} \qquad (3.2.4)$$

Let X^s denote the image of the complex

$$\Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N \xrightarrow{\eta \otimes \mathbb{1}} \Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N \xrightarrow{\eta \otimes \mathbb{1}} \Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s+2} \overset{\triangle}{\otimes} N \xrightarrow{\eta \otimes \mathbb{1}} \dots$$

in Stable(Γ) (that is, a complex of injective comodules quasi-isomorphic to the above complex). Note that the complex represented by X^0 is a Γ-comodule resolution for N, and hence is quasi-isomorphic to N; in general, there is a quasi-isomorphism

$$X^s \overset{\mathrm{qis}}{\simeq} \ker(\Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N \to \Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N) \cong \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N. \qquad (3.2.5)$$

We can express (3.2.4) as the exact couple arising from applying $\mathrm{Cotor}_\Gamma(k, -)$ to the cofiber sequence

$$X^{s+1} \to X^s \to \Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N. \qquad (3.2.6)$$

On the other hand, the MPASS comes from the exact couple obtained by applying the functor $\mathrm{Ext}_\Gamma(k, -)$ to the cofiber sequence

$$\overline{\Phi}^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N \to \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N \to \overline{\Phi}^{\otimes s} \overset{\triangle}{\otimes} \Phi \overset{\triangle}{\otimes} N. \qquad (3.2.7)$$

in Stable(Γ). There is an isomorphism $\mathrm{Ext}^*_\Gamma(k, M) \cong \mathrm{Cotor}^*_\Gamma(k, M)$ for all M, so we are applying the same functor to the two cofiber sequences. Moreover, there is a map of cofiber sequences from (3.2.7) to (3.2.6), and by (3.2.5) this is a quasi-isomorphism; in particular, the induced map of exact couples is an isomorphism. □

3.3 Cartan-Eilenberg vs. filtration spectral sequence

It is a classical fact [Ada60, §2.3] that the Cartan-Eilenberg spectral sequence associated to the Hopf extension $\Phi \to \Gamma \to \Sigma$ computing $\mathrm{Cotor}_\Gamma(M, N)$ coincides with a filtration spectral sequence on the cobar complex $C_\Gamma(M, N)$ defined by

$$F^s C^n_\Gamma(M, N) = \{m[a_1|\dots|a_n]\nu \in C^n_\Gamma(M, N) \ : \ \#(\{a_1, \dots, a_n\} \cap G) \geq s\}$$

where

$$G := \ker(\Gamma \to \Sigma).$$

As G is an ideal in Γ and the cobar complex $C^*_\Gamma(k, k)$ is a ring under the concatenation product, one can say this filtration of $C^*_\Gamma(M, N) = M \otimes C^*_\Gamma(k, k) \otimes N$ comes from the G-adic filtration of $C^*_\Gamma(k, k)$. In the previous section, we defined a variant of the Cartan-Eilenberg spectral sequence that makes sense when Φ is an arbitrary Γ-comodule-algebra. In this section, we will impose the additional condition that there is an inclusion $\Phi \hookrightarrow \Gamma$ preserving the relevant structure, and that we can write $\Phi = \Gamma \square_\Sigma k$ where $\Gamma \to \Sigma$ is a map of Hopf algebras.

Let E^{**}_r denote this filtration spectral sequence, and let $\overset{\vartriangle}{E}{}^{**}_r$ denote the generalized Cartan-Eilenberg spectral sequence. We will show that these agree starting at $r = 1$. As a double complex spectral sequence can be viewed as a filtration spectral sequence on the total complex, it suffices to show the following:

Theorem 3.3.1. *There is a filtration-preserving chain map*

$$\theta : \bigoplus_{s+t=n} (M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\mathcal{N} \overset{\vartriangle}{D}{}^s_\Phi(N)) \longrightarrow C^n_\Gamma(M, N)$$

whose induced map of spectral sequences $\overset{\vartriangle}{E}{}_r^{**} \to E_r^{**}$ is an isomorphism on E_1.

Corollary 3.3.2. *We have an isomorphism $\overset{\vartriangle}{E}{}_r^{**} \to E_r^{**}$ for $r \geq 1$.*

Definition 3.3.3. Define $\widetilde{\theta}$ as the composition

$$\widetilde{\theta} : (M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\Phi^{\overset{\vartriangle}{\otimes} s+1} \overset{\vartriangle}{\otimes} N) \xrightarrow{S_c \otimes S} (M \otimes \Gamma^{\otimes t} \overset{R}{\otimes} \Gamma) \square_\Gamma (\Gamma \overset{L}{\otimes} \Gamma^{\otimes s} \otimes N)$$
$$\xrightarrow{e} M \otimes \Gamma^{\otimes s+t} \otimes N$$

where the second map e is defined to be

$$(m|a_1|\dots|a_t|a) \otimes (b|b_1|\dots|b_s|n) \mapsto \varepsilon(ab)m|a_1|\dots|a_t|b_1|\dots|b_s|n.$$

Define θ to be the restriction of $\widetilde{\theta}$ to $(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\mathcal{N} \overset{\vartriangle}{D}{}^s_\Phi(N))$.

In Lemma 3.3.6, we will show that this restriction lands in $(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\Gamma \square_\Sigma G(s) \square_\Sigma N)$, where

$$G(s) := \underbrace{G \square_\Sigma \dots \square_\Sigma G}_{s}.$$

We will see that $E_0^{0,*}(M,N)$ is easy to describe (and in particular it is easy to show that θ induces an isomorphism $\overset{\vartriangle}{E}{}_0^{0,*}(M,N) \cong E_0^{0,*}(M,N)$), and most of the work involves identifying $E_0^{s,*}(M,N)$ (for $s > 0$) with $E_0^{0,*}(M,N')$ for a different comodule N', in a way that is compatible with a similar identification for $\overset{\vartriangle}{E}{}_0^{s,*}$. More precisely, we will show that there is a map β of chain complexes making the following diagram commute.

$$\begin{array}{ccccc}
(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes}*}) \square_\Gamma \mathcal{N} \overset{\vartriangle}{D}{}^0_\Phi(G(s) \square_\Sigma N) & = & \overset{\vartriangle}{E}{}_0^{0,*}(M, G(s) \square_\Sigma N) & \xrightarrow[\simeq]{\theta} & E_0^{0,*}(M, G(s) \square_\Sigma N) \\
{\scriptstyle 1 \otimes S^{-1}} \downarrow \cong & & & & \simeq \downarrow \beta \\
(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes}*}) \square_\Gamma \mathcal{N} \overset{\vartriangle}{D}{}^s_\Phi(N) & = & \overset{\vartriangle}{E}{}_0^{s,*}(M,N) & \xrightarrow{\theta} & E_0^{s,*}(M,N)
\end{array} \tag{3.3.1}$$

It suffices to show the following:

(1) θ is a filtration-preserving chain map;

(2) S^{-1} gives rise to an isomorphism $\mathcal{N}\overset{\vartriangle}{D}{}^0_\Phi(G(s)\,\square_\Sigma N) \to \mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N)$;

(3) there exists a chain equivalence β making the diagram commute;

(4) θ is a chain equivalence for $s = 0$.

(1) says we have written down a filtration-preserving map between total complexes, and (2)–(4) allow us to use the diagram to show that θ is a chain equivalence for all $s \geq 0$. We prove (1) in Lemma 3.3.4 and Corollary 3.3.7, (2) in Corollary 3.3.8, (3) in Corollary/ Definition 3.3.11, and (4) in Proposition 3.3.13.

Both the structure of the proof and the entirety of (2), the hardest part, are taken from an argument attributed to Ossa appearing as [Rav86, A1.3.16], showing that the classical Cartan-Eilenberg spectral sequence coincides with the filtration spectral sequence under discussion. The only new part we offer is the translation, via iterated shear isomorphisms, to the generalized Cartan-Eilenberg spectral sequence as defined in Section 3.2.

Lemma 3.3.4. *$\widetilde{\theta}$ is a chain map $\bigoplus_{s+t=n}(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1})\,\square_\Gamma \overset{\vartriangle}{D}{}^s_\Phi(N) \to C^n_\Gamma(M, N)$.*

Proof. Since S and S_c are maps of chain complexes of Γ-comodules, there is an induced map on the tensor product of chain complexes

$$(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} *+1}) \otimes (\Phi^{\overset{\vartriangle}{\otimes} *+1} \overset{\vartriangle}{\otimes} N) \to (M \otimes \Gamma^{*+1}) \otimes (\Gamma^{\otimes *+1} \otimes N)$$

and since these are maps of chain complexes of Γ-comodules, this passes to a map on the cotensor product

$$(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} *+1})\,\square_\Gamma(\Phi^{\overset{\vartriangle}{\otimes} *+1} \overset{\vartriangle}{\otimes} N) \to (M \otimes \Gamma^{*+1})\,\square_\Gamma(\Gamma^{\otimes *+1} \otimes N).$$

Then $\widetilde{\theta}$ is formed by post-composing with the map

$$e : (M \otimes \Gamma^{\otimes t+1}) \,\square_\Gamma (\Gamma^{\otimes s+1} \otimes N) \to M \otimes \Gamma^{t+s} \otimes N$$

which takes $m[a_1| \dots |a_t]a_{t+1} \otimes b_0[b_1| \dots |b_s]n \mapsto \varepsilon(a_{t+1}b_0)m[a_1| \dots |a_t|b_1| \dots |b_s]n$. To see this is a chain map, it suffices to check the following diagram commutes.

$$\begin{array}{ccc}
(M \otimes \Gamma^{\otimes t+1}) \,\square_\Gamma (\Gamma^{\otimes s+1} \otimes N) & \xrightarrow{\mathbb{1}\otimes\varepsilon\otimes\mathbb{1}} & M \otimes \Gamma^{\otimes t} \otimes \Gamma^{\otimes s} \otimes N \\
\Big\downarrow {\scriptstyle d\ \text{double complex}} & & \Big\downarrow {\scriptstyle d_{\text{cobar}}} \\
\begin{array}{c}(M \otimes \Gamma^{\otimes t+1}) \,\square_\Gamma (\Gamma^{\otimes s+2} \otimes N) \\ \oplus\ (M \otimes \Gamma^{\otimes t+2}) \,\square_\Gamma (\Gamma^{\otimes s+1} \otimes N)\end{array} & \xrightarrow{\mathbb{1}\otimes\varepsilon\otimes\mathbb{1}} & M \otimes \Gamma^{\otimes t+s+1} \otimes N
\end{array}$$

This requires keeping track of signs: the double complex differential is $d_\Gamma \otimes \mathbb{1} + (-1)^t \mathbb{1} \otimes d_\Phi$, or more explicitly:

$$\begin{aligned}
a_0[a_1| \dots |a_t]a_{t+1} \otimes b_0[b_1| \dots |b_s]b_{s+1} \mapsto & \sum_i (-1)^i a_0[\dots |a_i'|a_i''| \dots]a_{t+1} \otimes b_0[b_1| \dots |b_s]b_{s+1} \\
& + \sum_i (-1)^{i+t} a_0[a_1| \dots |a_t]a_{t+1} \otimes b_0[\dots |b_i'|b_i''| \dots]b_{s+1}
\end{aligned}$$

and the cobar differential is

$$\begin{aligned}
a_0[a_1| \dots |a_t|b_1| \dots |b_s]b_{s+1} \mapsto & \sum_i (-1)^i a_0[a_1| \dots |a_i'|a_i''| \dots |b_1| \dots |b_s]b_{s+1} \\
& + \sum_i (-1)^{t+i} a_0[a_1| \dots |a_t|b_1| \dots |b_i'|b_i''| \dots |b_s]b_{s+1}.
\end{aligned}$$

In particular, notice that, on the bottom left composition, the terms corresponding to $a_0[\dots|a_{t+1}']a_{t+1}'' \otimes b_0[\dots]b_{s+1}$ cancel in $M \otimes \Gamma^{\otimes t+s+1} \otimes N$ with the terms corresponding to $a_0[\dots]a_{t+1} \otimes b_0'[b_0''| \dots]b_{s+1}$. □

While $\widetilde{\theta}$ is not filtration-preserving, we will show that its restriction to $(M \overset{\triangle}{\otimes} \Gamma^{\overset{\triangle}{\otimes} t+1}) \,\square_\Gamma \mathcal{N} \overset{\triangle}{D}{}^s_\Phi$ is.

Lemma 3.3.5. *The iterated shear isomorphism* $S : \Gamma^{\overset{\triangle}{\otimes} *+1} \overset{\triangle}{\otimes} N \to \Gamma^{\overset{L}{\otimes} *+1} \otimes N$ *restricts* *to an isomorphism of chain complexes*

$$S : \Phi^{\overset{\triangle}{\otimes} *+1} \overset{\triangle}{\otimes} N \to \underbrace{\Gamma \square_\Sigma \dots \square_\Sigma \Gamma}_{*+1} \square_\Sigma N. \tag{3.3.2}$$

Proof. For any Γ-comodule M, by Lemma 3.1.9 the shear isomorphism gives an isomorphism $\Phi \overset{\triangle}{\otimes} N \xrightarrow{\cong} \Gamma \square_\Sigma N$, and iterating the shear map gives an isomorphism $\Phi^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N \xrightarrow{\cong} \underbrace{\Gamma \square_\Sigma \dots \square_\Sigma \Gamma}_{s+1} \square_\Sigma N$. □

Lemma 3.3.6. *The iterated shear map* $S : \Gamma^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N \to \Gamma^{\overset{L}{\otimes} s+1} \otimes N$ *restricts to an isomorphism* $\mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N) \to \Gamma \square_\Sigma G(s) \square_\Sigma N$.

Proof. It suffices to check the inclusions $S^{-1}(\Gamma \square_\Sigma G(s) \square_\Sigma N) \subset \mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N)$ and $S(\mathcal{N}\overset{\triangle}{D}{}^s_\Phi(M)) \subset \Gamma \square_\Sigma G(s) \square_\Sigma N$. For the first inclusion, use Lemma 3.1.8 to observe that

$$S^{-1}(a|g_1|\dots|g_s|n) = \sum ac(g_1')|g_1''c(g_2')|g_2''c(g_3')|\dots|g_s''c(n')|n'' \tag{3.3.3}$$

and for $1 \le i \le s$ we have

$$\begin{aligned} \mu_i(\textstyle\sum ac(g_1')|g_1''c(g_2')|g_2''c(g_3')|\dots|g_s''c(n')|n'') &= \textstyle\sum ac(g_1')|g_1''c(g_2')|\dots|g_{i-1}''c(g_i')g_i''c(g_{i+1}')|\dots|n'' \\ &= \textstyle\sum ac(g_1')|g_1''c(g_2')|\dots|g_{i-1}''\varepsilon(g_i)c(g_{i+1}')|\dots|n'' \end{aligned}$$

which is zero since $g_i \in G$ (and so $g_i \notin k$). This shows (3.3.3) is in $\mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N)$.

For the other direction, let $x_0|\dots|x_s|n \in \mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N) \subset \Phi^{\overset{\triangle}{\otimes} s+1} \otimes N$. By Lemma 3.1.7, we have

$$S(x_0|\dots|x_s|n) = \sum x_{0(1)}x_{1(1)}\dots x_{s(1)}n_{(1)}|x_{1(2)}\dots x_{s(2)}n_{(2)}|x_{2(3)}\dots n_{(3)}|\dots|n_{(s+2)}. \tag{3.3.4}$$

The goal is to show that each component $x_{k(k+1)}x_{k+1(k+1)}\cdots x_{s(k+1)}n_{(k+1)}$ is in G for $1 \le k \le s$. Since Φ is a left Γ-comodule, if $x \in \Phi$ then $\Delta^j(x) = x_{(1)}|\dots|x_{(j)}$ and so $x_{(j)} \in \Phi$. By assumption, all of the x_i's are in Φ, and since (3.3.4) involves the iterated coproduct $\Delta^{i+1}(x_i) = x_{i(1)}|\dots|x_{i(i+1)}$ for every i, we have $x_{i(i+1)} \in \Phi$. If we could guarantee $x_{k(k+1)}$ were in $\overline{\Phi}$, then we would be done (since $G = \overline{\Phi}\Gamma$). Instead, we show that the terms where $x_{k(k+1)} = 1$ sum to zero.

The terms where $x_{k(k+1)} = 1$ are:

$$\sum x_{0(1)}x_{1(1)}\dots x_{k-1(1)}x_{k(1)}\dots x_{s(1)}n_{(1)}|\dots|x_{k-2(k-1)}x_{k-1(k-1)}x_{k(k-1)}\dots \\ |x_{k-1(k)}x_{k(k)}x_{k+1(k)}\dots|x_{k+1(k+1)}x_{k+2(k+1)}\dots|\dots|n_{(s+2)}. \tag{3.3.5}$$

The assumption that $x_0|\dots|x_s$ is in $\mathcal{N}\overset{\triangle}{D}{}^s_\Phi(M)$ implies that $x_{k-1}x_k = 0$ (this is where we use the fact that $k \geq 1$), and hence

$$0 = \Delta^k(x_{k-1}x_k) = \sum x_{k-1(1)}x_{k(1)}|\dots|x_{k-1(k-1)}x_{k(k-1)}|x_{k-1(k)}x_{k(k)}.$$

Observing how $\Delta(x_{k-1}x_k)$ is embedded in (3.3.5), we have (3.3.5) $= 0$. □

Corollary 3.3.7. *θ is filtration-preserving.*

Proof. This is a direct consequence of Lemma 3.3.6. □

Corollary 3.3.8. *There are isomorphisms*

$$\mathcal{N}\overset{\triangle}{D}{}^0_\Phi(G(s)\,\square_\Sigma N) = \Phi \overset{\triangle}{\otimes} (G(s)\,\square_\Sigma N) \xrightarrow{S\otimes\mathbb{1}} \Gamma\,\square_\Sigma G(s)\,\square_\Sigma N \xrightarrow{S^{-1}} \mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N).$$

This gives the left vertical isomorphism in (3.3.1).

Our next task is to define the map β in (3.3.1) and show it is a chain equivalence. Most of the work for that is done in Lemma 3.3.10; the next lemma is helpful for that, and the result is summarized in Corollary/ Definition 3.3.11.

Lemma 3.3.9. *For fixed s, there is an isomorphism of complexes $F^s/F^{s+1}C_\Gamma(M,N) = E_0^{s,*}(M,N) \cong M\,\square_\Sigma E_0^{s,*}(M,\Sigma)\,\square_\Sigma N$.*

In particular, $E_0^{s,*}(M,N)$ only depends on the Σ-coaction on N, not the full Γ-coaction. We will abuse notation by writing $E_0^{s,*}(M,N)$ where N has a Σ-coaction and not a Γ-coaction (specifically, we do this for $N = G$).

Proof. We begin by showing that $F^s/F^{s+1}C_\Gamma(M,N)$ only depends on the Σ-coaction on N: given $x = m[\gamma_1|\dots|\gamma_n]\nu$ in $F^sC_\Gamma(M,N)$, the term $m[\gamma_1|\dots|\gamma_n|\nu']\nu''$ in $d(x)$ is in F^{s+1} if $\nu' \in G$. So, if we write $\psi(\nu) = \sum \underline{\nu}'|\underline{\nu}''$ for the coaction $\psi : N \to \Sigma \otimes N$, we can say that $d(x) \equiv \sum m[\gamma_1|\dots|\gamma_n|\underline{\nu}']\underline{\nu}''$ in $F^s/F^{s+1}C_\Gamma^{n+1}(M,N)$.

We have an isomorphism $\psi : N \xrightarrow{\cong} \Sigma \,\square_\Sigma N$ of Σ-comodules, where the coaction on the right hand side is $\sigma \otimes \nu \mapsto \sigma' \otimes \sigma'' \otimes \nu$. This shows that the following diagram commutes

$$\begin{array}{ccc} E_0^{s,t}(M,N) & \xrightarrow{\psi} & E_0^{s,t}(M,\Sigma)\,\square_\Sigma N \\ \downarrow d & & \downarrow d \\ E_0^{s,t+1}(M,N) & \xrightarrow{\psi} & E_0^{s,t+1}(M,\Sigma)\,\square_\Sigma N \end{array}$$

and so there is chain complex isomorphism $E_0^{s,*}(M,N) \cong E_0^{s,*}(M,\Sigma)\,\square_\Sigma N$ for every s. □

Lemma 3.3.10 ([Rav86, A1.3.16]). *The map*

$$\begin{aligned} \delta : E_0^{s-1,*}(M,G) &\longrightarrow E_0^{s,*}(M,\Sigma) \\ m[a_1|\dots|a_{s-1}]g &\longmapsto m[a_1|\dots|a_{s-1}|g']g''. \end{aligned}$$

is a chain equivalence, where $\sum g' \otimes g''$ is the image of $g \in G$ along the map $\Gamma \xrightarrow{\Delta} \Gamma \otimes \Gamma \to \Gamma \otimes \Sigma$.

Proof. We introduce a second filtration $\widetilde{F}^s$ which is defined only on $C_\Gamma(M,\Gamma)$:

$$\widetilde{F}^s C_\Gamma^n(M,\Gamma) = \{m[\gamma_1|\dots|\gamma_n]\gamma : \text{ at least } s \text{ of } \{\gamma,\gamma_1,\dots,\gamma_n\} \text{ are in } G\}\ [1].$$

[1] This is off by one from the grading convention used in [Rav86, A1.3.16].

There is a short exact sequence of complexes

$$0 \to F^s/\widetilde{F}^{s+1}C^*_\Gamma(M,\Gamma) \to \widetilde{F}^s/\widetilde{F}^{s+1}C^*_\Gamma(M,\Gamma) \to \widetilde{F}^s/F^sC^*_\Gamma(M,\Gamma) \to 0. \tag{3.3.6}$$

Unlike F, the new filtration $\widetilde{F}$ preserves the contracting homotopy on $C^*_\Gamma(M,\Gamma)$ given by $m[\gamma_1|\ldots|\gamma_n]\gamma \mapsto \varepsilon(\gamma)m[\gamma_1|\ldots|\gamma_{n-1}]\gamma_n$. So $\widetilde{F}^*C_\Gamma(M,\Gamma)$ is contractible, and so is the quotient complex $\widetilde{F}^*/\widetilde{F}^{*+1}C_\Gamma(M,\Gamma)$. The short exact sequence (3.3.6) gives rise to a long exact sequence in cohomology, and contractibility of the middle complex means that the boundary map

$$\delta : H^*(\widetilde{F}^s/F^sC^*_\Gamma(M,\Gamma)) \to H^*(F^s/\widetilde{F}^{s+1}C^{*+1}_\Gamma(M,\Gamma)) \tag{3.3.7}$$

is an isomorphism. We will identify $\widetilde{F}^s/F^sC^*_\Gamma(M,\Gamma)$ and $F^s/\widetilde{F}^{s+1}C^{*+1}_\Gamma(M,\Gamma)$ with the source and target of the desired map in the lemma statement, and show that δ can be lifted to a map on chains.

Levelwise, we can write

$$\widetilde{F}^{s+1}C^n_\Gamma(M,\Gamma) = F^{s+1}C^n_\Gamma(M,\Gamma) + F^sC^n_\Gamma(M,G) \tag{3.3.8}$$

but this is an abuse of notation—as G is not a Γ-comodule, $C^*_\Gamma(M,G)$ is not a complex (but we can still talk about $C^n_\Gamma(M,G) \subset C^n_\Gamma(M,\Gamma)$ as a sub-module). We will see that this will cease to be a problem upon passing to the associated graded E_0.

For each n, we have

$$\begin{aligned} \widetilde{F}^s/F^sC^*_\Gamma(M,\Gamma) &\cong \left(F^sC^n_\Gamma(M,\Gamma) + F^{s-1}C^n_\Gamma(M,G)\right)/F^sC^n_\Gamma(M,\Gamma) && (3.3.9)\\ &\cong F^{s-1}/F^sC^n_\Gamma(M,G) \\ F^s/\widetilde{F}^{s+1}C^{*+1}_\Gamma(M,\Gamma) &\cong F^sC^n_\Gamma(M,\Gamma)/\left(F^{s+1}C^n_\Gamma(M,\Gamma) + F^sC^n_\Gamma(M,G)\right) && (3.3.10)\\ &= \left(F^sC^n_\Gamma(M,\Gamma)/F^{s+1}C^n_\Gamma(M,\Gamma)\right)/F^sC^n_\Gamma(M,G) \\ &\cong F^s/F^{s+1}C^n_\Gamma(M,\Sigma). \end{aligned}$$

While $F^sC^*_\Gamma(M,G)$ is not a complex, Lemma 3.3.10 shows that $F^{s-1}/F^sC^*_\Gamma(M,G)$ is a complex, and the isomorphisms $\widetilde{F}^s/F^sC^n_\Gamma(M,\Gamma) \cong F^{s-1}/F^sC^n_\Gamma(M,G)$ and $F^s/\widetilde{F}^{s+1}C^{*+1}_\Gamma(M,\Gamma) \cong F^s/F^{s+1}C^n_\Gamma(M,\Sigma)$ extend to isomorphisms of complexes. I claim the boundary map (3.3.7) can be identified as the map

$$\begin{aligned} H^*(F^{s-1}/F^sC^*_\Gamma(M,G)) &\xrightarrow{\delta} H^*(F^s/F^{s+1}C^*_\Gamma(\Sigma,N)) \\ m[a_1|\dots|a_n]g &\longmapsto \sum m[a_1|\dots|a_n|\underline{\underline{g'}}]\underline{\underline{g''}} \end{aligned}$$

where $\sum \underline{\underline{g'}}|\underline{\underline{g''}}$ is the image of g under the right Σ-coaction. As the boundary map, this is just given by the cobar differential, but in order for $m[a_1|\dots|a_n]g$ to be a cycle, the sum of all the terms except the one in the formula for δ is in $F^sC^{n+1}_\Gamma(M,\Gamma)$. Furthermore, I claim this can be extended to a map on chains:

$$\begin{aligned} \delta : F^{s-1}/F^sC^*_\Gamma(M,G) &\longrightarrow F^s/F^{s+1}C^*_\Gamma(\Sigma,N) \\ m[a_1|\dots|a_n]g &\longmapsto \sum m[a_1|\dots|a_n|\underline{\underline{g'}}]\underline{\underline{g''}}. \end{aligned}$$

It suffices to show that the image of $m[a_1|\dots|a_n]g \in F^sC^*_\Gamma(M,G)$ lies in $F^{s+1}C^*_\Gamma(M,\Sigma)$, and this holds because $\underline{\underline{g''}}$ is the $(s+1)^{st}$ term in G. □

Using Lemma 3.3.9, we can write this as a map

$$\begin{array}{c} E_0^{s-1,*}(M, G(s)\,\square_\Sigma N) \\ = E_0^{s-1,*}(M,G)\,\square_\Sigma N \end{array} \xrightarrow{\delta} E_0^{s,*}(M,\Sigma)\,\square_\Sigma N = E_0^{s,*}(M,\Sigma\,\square_\Sigma N) \xrightarrow{\cong} E_0^{s,*}(M,N)$$

$$m[a_1|\dots|a_n]g|\nu \longmapsto \sum m[a_1|\dots|a_n\underline{\underline{g'}}]\underline{\underline{g''}}\nu \longmapsto \sum m[a_1|\dots|a_n|g]\nu.$$

Corollary/ Definition 3.3.11. Iterating δ gives rise to a chain equivalence

$$E_0^{0,*}(M, G(s)\,\square_\Sigma N) \xrightarrow{\delta} E_0^{1,*}(M, G(s-1)\,\square_\Sigma N) \xrightarrow{\delta} \dots \xrightarrow{\delta} E_0^{s,*}(M,N)$$

sending

$$m[a_1|\dots|a_n]g_1|\dots|g_s|\nu \longmapsto m[a_1|\dots|a_n|g_1|\dots|g_s]\nu.$$

Let β denote this composition.

It is now easy to see that (3.3.1) commutes. Our final task is to show (4) after (3.3.1); first we need an easy lemma.

Lemma 3.3.12. *Let Γ be a Hopf algebra and M be an Γ-comodule. Then the coaction $\psi : M \to \Gamma \square_\Gamma M$ is an isomorphism with inverse $T : \Gamma \square_\Gamma M \to M$ sending $a \otimes m \mapsto \varepsilon(a)m$.*

Proof. First we check that the coaction ψ lands in the cotensor product $\Gamma \square_\Gamma M$: we need to check that $\psi(m) = \sum m' \otimes m''$ lands in the kernel of $\Delta \otimes \mathbb{1} - \mathbb{1} \otimes \psi : \Gamma \otimes M \to \Gamma \otimes \Gamma \otimes M$. But $\sum (m')' \otimes (m')'' \otimes m'' - \sum m' \otimes (m'')' \otimes (m'')'' = 0$ by coassociativity.

Next, we check that T is an inverse. We have $T\psi(m) = T(\sum m' \otimes m'') = \sum \varepsilon(m')m''$. This is equal to m by Fact 3.0.2(3). For the other composition, we have $\psi T(a \otimes m) = \sum \varepsilon(a)m' \otimes m''$. Since $a \otimes m$ is in $\Gamma \square_\Gamma M$, we have $\sum a \otimes m' \otimes m'' = \sum a' \otimes a'' \otimes m$. Applying $\varepsilon \cdot \mathbb{1} \otimes \mathbb{1}$ to this, we have $\sum \varepsilon(a)m' \otimes m'' = \sum \varepsilon(a')a'' \otimes m = \sum a \otimes m$. So $\psi \circ T = \mathbb{1}$. $\square$

Proposition 3.3.13. *θ induces an isomorphism $\overset{\vartriangle}{E}{}_1^{0,*} \to E_1^{0,*}$.*

Proof. First notice that we have an isomorphism

$$F^0/F^1(M \otimes \Gamma^{\otimes t} \otimes N) \cong M \otimes \Sigma^{\otimes t} \otimes N$$

since $m[\gamma_1|\dots|\gamma_s]\nu$ is in F^1 if any of the γ_i's are in G. On the other hand, we have

$$H^*(\overset{\vartriangle}{E}{}_1^{0,*}) = H^*((M \mathbin{\overset{\vartriangle}{\otimes}} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\Phi \mathbin{\overset{\vartriangle}{\otimes}} N)) = \mathrm{Cotor}^*_\Gamma(M, \Phi \mathbin{\overset{\vartriangle}{\otimes}} N) \cong \mathrm{Cotor}^*_\Sigma(M, N)$$

by the change of rings isomorphism. In the rest of this proof we make this isomorphism more explicit, enough to see that the isomorphism $\overset{\vartriangle}{E}{}_1^{0,*} \to E_1^{0,1}$ is induced by θ.

Since the shear map $\Gamma \overset{\vartriangle}{\otimes} \Gamma \to \Gamma \overset{L}{\otimes} \Gamma$ commutes with the map $\Gamma \otimes \Gamma \xrightarrow{q \otimes q} \Sigma \otimes \Sigma$, we have a commutative diagram

$$\begin{array}{ccccc}
(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\Phi \overset{\vartriangle}{\otimes} N) & & & & \\
\downarrow {\scriptstyle \mathbb{1}^{t+2} \otimes S} & & & & \\
(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\Gamma \square_\Sigma N) & \xrightarrow{\mathbb{1} \otimes q^{t+1} \otimes \mathbb{1}^2} & (M \overset{\vartriangle}{\otimes} \Sigma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Gamma (\Gamma \square_\Sigma N) & \xrightarrow{\cong} & (M \overset{\vartriangle}{\otimes} \Sigma^{\overset{\vartriangle}{\otimes} t+1}) \square_\Sigma N \\
\downarrow {\scriptstyle S_c^t \otimes \varepsilon \cdot \varepsilon \cdot \mathbb{1}} & & & & \downarrow {\scriptstyle S_c^{t+1} \otimes \varepsilon \cdot \mathbb{1}} \\
F^0/F^1(M \otimes \Gamma^t \otimes N) & & \xrightarrow{\cong} & & M \otimes \Sigma^t \otimes N
\end{array}$$

Note that the left vertical composition is θ, by definition. The middle horizontal composition is the chain equivalence inducing the change of rings isomorphism $\mathrm{Cotor}^*_\Gamma(M, \Gamma \square_\Sigma N) \cong \mathrm{Cotor}_\Sigma(M, N)$. By Lemma 3.3.12, the right vertical map is $S_c^{t+1} \otimes T$, an isomorphism. So the bottom left vertical map is a chain equivalence. The top left vertical map is an isomorphism, so θ is a chain equivalence. $\square$

Chapter 4

The E_2 page of the $K(\xi_1)$-based MPASS

Unless otherwise indicated, henceforth we will work at $p = 3$, and let $k = \mathbb{F}_3$. We also let $D = k[\xi_1]/\xi_1^3$ throughout.

4.1 Overview of the $K(\xi_1)$-based MPASS

Our goal is to compute $\pi_{**}(b_{10}^{-1}k) = b_{10}^{-1}\operatorname{Ext}_P(k,k)$ using a MPASS based at

$$K(\xi_1) := b_{10}^{-1}B \qquad \text{where } B := P\,\square_D k.$$

Since B is an algebra, $K(\xi_1)$ is a ring object in Stable(P). At $p = 3$ we will show that $K(\xi_1)_{**}K(\xi_1)$ is flat over $K(\xi_1)_{**}$, and so the E_2 term is:

$$E_2 = \operatorname{Ext}_{K(\xi_1)_{**}K(\xi_1)}(K(\xi_1)_{**}, K(\xi_1)_{**}) \implies \pi_{**}(b_{10}^{-1}k) = b_{10}^{-1}\operatorname{Ext}^*_P(k,k).$$

This flatness property does not hold at higher primes; this is the main reason this problem is significantly more tractable at the prime 3.

In Section 4.4 we will show that this spectral sequence converges, and in Section 4.5 we will show that $\mathrm{Ext}^*_P(k,k)$ agrees with its b_{10}-localization above a line of slope $\frac{1}{23}$. The bulk of the chapter, in sections 4.2 and 4.3, is devoted to determining the structure of the Hopf algebroid

$$\begin{aligned}\big(K(\xi_1)_{**},\ K(\xi_1)_{**}K(\xi_1)\big) &= \big(b_{10}^{-1}\,\mathrm{Ext}^*_P(k,B),\ b_{10}^{-1}\,\mathrm{Ext}^*_P(k,B\otimes B)\big)\\ &\cong \big(b_{10}^{-1}\,\mathrm{Ext}^*_D(k,k),\ b_{10}^{-1}\,\mathrm{Ext}^*_D(k,B)\big)\end{aligned}$$

(where the last isomorphism is by the change of rings theorem) in order to determine the structure of the E_2 page. The coefficient ring $K(\xi_1)_{**}$ is easy to compute using the change of rings theorem:

$$\begin{aligned}K(\xi_1)_{**} &= b_{10}^{-1}\,\mathrm{Ext}^*_P(k,B) = b_{10}^{-1}\,\mathrm{Ext}^*_P(k,P\,\Box_D k)\\ &= b_{10}^{-1}\,\mathrm{Ext}^*_D(k,k) = E[h_{10}]\otimes P[b_{10}^{\pm 1}]\end{aligned}$$

where h_{10} is in homological degree 1 and b_{10} is in homological degree 2. It will be useful to have notation for this coefficient ring:

$$R := E[h_{10}]\otimes P[b_{10}^{\pm}]. \tag{4.1.1}$$

Our goal is to show the following:

Theorem 4.1.1. *The ring of co-operations $K(\xi_1)_{**}K(\xi_1)$ is flat over $K(\xi_1)_{**}$, and moreover there is an isomorphism of Hopf algebras*

$$K(\xi_1)_{**}K(\xi_1) = K(\xi_1)_{**}\otimes E[e_2,e_3,\dots]$$

for generators e_n in homological degree 1 and internal degree $2(3^n+1)$. That is, e_n is primitive, and $K(\xi_1)_{**}K(\xi_1)$ is exterior as a Hopf algebra over $K(\xi_1)_{**}$.

Corollary 4.1.2. *The E_2 page of the $K(\xi_1)$-based Adams spectral sequence for com-*

puting $\pi_{**}(b_{10}^{-1}k)$ *is*

$$E_2^{**} \cong R \otimes P[w_2, w_3, \dots]$$

where w_n *has Adams filtration 1 and internal degree* $2(3^n+1)$.

Remark 4.1.3. The generator w_2 is a permanent cycle, and converges to $g_0 = \langle h_{10}, h_{10}, h_{11}\rangle \in \mathrm{Ext}_P^*(k,k)$. We will see in Chapter 5 that the other w_n's support differentials, so it is less easy to see how these generators connect to familiar elements in the Adams E_2 page. One useful heuristic is that $w_n = \langle h_{10}, h_{10}, h_{n-1,1}\rangle$ over $P/(\xi_1^3, \xi_2^9, \xi_3^9, \dots)$.

Remark 4.1.4. As B is a P-comodule algebra, there is a Hopf algebroid $(B, B \overset{\vartriangle}{\otimes} B)$ in Stable(P), where the comultiplication is given by

$$B \otimes B \overset{-\otimes\eta\otimes-}{\longrightarrow} B \otimes B \otimes B \cong (B \otimes B) \otimes_B (B \otimes B).$$

The Hopf algebroid above is given by applying $b_{10}^{-1}\pi_{**}(-) = b_{10}^{-1}\,\mathrm{Ext}_P^*(k,-)$ to this one.

Notation 4.1.5. We have chosen to define B as a left P-comodule. It can be written explicitly as $\mathbb{F}_3[\overline{\xi}_1^3, \overline{\xi}_2, \overline{\xi}_3, \dots]$. To simplify the notation, everywhere in the remaining chapters of this thesis we will redefine the symbol ξ_n to mean the antipode of the usual ξ_n. Thus, going forward, we will have $\Delta(\xi_n) = \sum_{i+j=n} \xi_i \otimes \xi_j^{p^i}$, and $B = \mathbb{F}_3[\xi_1^3, \xi_2, \xi_3, \dots]$.

4.2 D-comodule structure of B

In this section we work at an arbitrary prime p. We will write $k = \mathbb{F}_p$, $D = \mathbb{F}_p[\xi_1]/\xi_1^p$, and $B = P \square_D k = \mathbb{F}_p[\xi_1^p, \xi_2, \xi_3, \dots]$ (using the convention of Notation 4.1.5). Note that B is an algebra and a P-comodule, but not a coalgebra. Let ψ denote the D-coaction $B \to D \otimes B$ that comes from composing the P-coaction $B \to P \otimes B$ with the surjection $P \to D$.

Definition 4.2.1. If we write

$$\psi(x) = 1 \otimes x + \xi_1 \otimes a_1 + \xi_1^2 \otimes a_2 + \cdots + \xi_1^{p-1} \otimes a_{p-1}$$

for some a_i's, define

$$\partial(x) := a_1.$$

For example, since $\Delta(\xi_n) = 1 \otimes \xi_n + \xi_1 \otimes \xi_{n-1}^p + \dots$ we have $\partial(\xi_n) = \xi_{n-1}^p$, and $\partial(\xi_{n-1}^p) = 0$. One can show using coassociativity that $a_k = \frac{1}{k!}\partial^{k-1}a_1$. As ξ_1 is dual to P_1^0 in the Steenrod algebra, the operator $\partial : P \to P$ is dual to the operator $P^\vee \to P^\vee$ given by left P_1^0-multiplication. In particular, $(P_1^0)^p = 0$ implies $\partial^p = 0$.

Lemma 4.2.2. *We have* $\partial(xy) = \partial(x)y + x\partial(y)$.

Proof. We have

$$\begin{aligned}\Delta(xy) = \Delta(x)\Delta(y) &= (1 \otimes x + \xi_1 \otimes \partial x + \dots)(1 \otimes y + \xi_1 \otimes \partial y + \dots) \\ &= 1 \otimes xy + \xi_1 \otimes (y\partial x + x\partial y) + \dots.\end{aligned}$$

□

The structure theorem for modules over a PID says that modules over $D^\vee \cong D$ decompose as sums of modules isomorphic to $\mathbb{F}_p[\xi_1]/\xi_1^i$ for $1 \le i \le p$. Dually, we have the following:

Lemma 4.2.3. *Let* $M(n)$ *denote the* D*-comodule* $\mathbb{F}_p[\xi_1]/\xi_1^{n+1}$*. Then every* D*-comodule splits uniquely as a direct sum of* D*-comodules isomorphic to* $M(n)$ *for* $n \le p-1$.

Note that $M(0) \cong \mathbb{F}_p$ and $M(p-1) \cong D$.

The goal of this section is to prove the following proposition.

Proposition 4.2.4. *Define the indexing set* $\mathscr{B}$ *to be the set of monomials of the form*

$\prod_{j=1}^{n} \xi_{i_j}^{e_j}$ *such that* $1 \leq e_j \leq p-2$, *and for* $X \in \mathscr{B}$, *write* $x_j(X) := \xi_{i_j}^{e_j}$ *and* $e_j(X) := e_j$. *Then there is a* D*-comodule isomorphism*

$$B \cong \bigoplus_{X \in \mathscr{B}} \bigotimes_{j=1}^{n} M(e_j(X))_{x_j(X)} \quad \oplus \quad F$$

where F *is a free* D*-comodule and* $M(e)_{\xi_i^e} := \mathbb{F}_p\{\xi_i^e, \partial \xi_i^e, \dots, \partial^e \xi_i^e\} \cong M(e)$.

If $e \leq p-1$ then $M(e)_{\xi_n^e}$ is a sub-D-comodule of B with dimension $e+1$. By the Leibniz rule (Lemma 4.2.2) we have

$$M(e+pf)_{\xi_n^{e+pf}} = \mathbb{F}_p\{\xi_n^e \xi_n^{pf}, \partial(\xi_n^e)\xi_n^{pf}, \dots, \partial^e(\xi_n^e)\xi_n^{pf}\} = M(e)_{\xi_n^e} \otimes \mathbb{F}_p\{\xi_n^{pf}\}$$

for $e \leq p-1$. For any collection of $e_i \in \mathbb{N}$, define

$$T(\xi_{n_1}^{e_1} \dots \xi_{n_d}^{e_d}) := M(e_1)_{\xi_{n_1}^{e_1}} \overset{\vartriangle}{\otimes} \dots \overset{\vartriangle}{\otimes} M(e_d)_{\xi_{n_d}^{e_d}}. \tag{4.2.1}$$

This is a sub-D-comodule spanned (as a vector space) by monomials of the form $\partial^{k_1}(\xi_{n_1}^{e_1}) \dots \partial^{k_d}(\xi_{n_d}^{e_d})$. Clearly, $B = \sum_{\substack{\text{monomials} \\ \prod \xi_{n_i}^{e_i} \in B}} T(\xi_{n_1}^{e_1} \dots \xi_{n_d}^{e_d})$, but this is not a direct sum decomposition—any given monomial appears in many different summands. To fix this, we will study the poset of $T(X)$'s, and find that B is a direct sum of the maximal elements of that poset.

Notation 4.2.5. Define the notation

$$\left\langle \prod_{i \geq 1} \xi_i^{e_i} \ ; \ \prod_{i \geq 2} \xi_i^{f_i} \right\rangle := \prod \xi_i^{e_i} \ \prod \xi_{i-1}^{pf_i}.$$

(These are not formal products; they only make sense if $e_i = 0 = f_i$ for all but finitely many i.) For example, we have $\langle X \ ; \ 1 \rangle = X$ for any monomial X, and $\langle 1 \ ; \ \xi_n \rangle = \xi_{n-1}^p = \partial(\xi_n)$. Expressions $\left\langle \prod_{i \geq 2} \xi_i^{e_i} \ ; \ \prod_{i \geq 2} \xi_i^{f_i} \right\rangle$ represent elements of $B \subset P$, and conversely every element of B has a representation of this form (note that

$\xi_1^p = \langle 1 \ ; \ \xi_2 \rangle)$. Monomials in B do not have unique expressions of the form $\langle X \ ; \ Y \rangle$: for example, $\left\langle \xi_{n-1}^p \ ; \ 1 \right\rangle = \langle 1 \ ; \ \xi_n \rangle$.

Lemma 4.2.6. *There is a bijection*

$$\{\textit{monomials in } B\} \longleftrightarrow \left\{ \left\langle \textstyle\prod_{i\geq 2} \xi_i^{e_i} \ ; \ \prod_{i\geq 2} \xi_i^{f_i} \right\rangle \ : \ e_i \leq p-1 \right\}. \tag{4.2.2}$$

Say that a bracket expression is *admissible* if it is of the form on the right hand side.

Proof. Given a monomial, the admissible bracket expression is the one with the greatest number of terms on the right-hand side. □

Lemma 4.2.7. *If X is a monomial with admissible bracket expression $\left\langle \prod \xi_i^{e_i} \ ; \ \prod \xi_i^{f_i} \right\rangle$ and Y is a monomial in $T(X)$, then Y (up to invertible scalar) has admissible expression $\left\langle \prod \xi_i^{e_i - c_i} \ ; \ \prod \xi_i^{f_i + c_i} \right\rangle$ for a set of $c_i \geq 0$ that are zero for all but finitely many i.*

The idea is that Y is obtained from X by moving terms from the left to the right.

Proof. If $e \leq p-1$ then we have

$$\partial^i(\xi_n^e) = \frac{e!}{(e-i)!} \xi_n^{e-i} \xi_{n-1}^{pi}.$$

By definition, $X = \prod_{i\geq 1} \xi_i^{e_i + pf_{i+1}}$ where $e_1 = 0$, and

$$\begin{aligned} Y &= \prod \partial^{k_i} \xi_i^{e_i + pf_{i+1}} = \prod (\partial^{k_i} \xi_i^{e_i}) \xi_i^{pf_{i+1}} = \prod \frac{e_i!}{(e_i - k_i)!} \xi_i^{e_i - k_i + pk_{i+1}} \xi_i^{pf_{i+1}} \\ &= \left\langle \prod \frac{e_i!}{(e_i - k_i)!} \xi_i^{e_i - k_i} \ ; \ \prod \xi_i^{k_i + f_i} \right\rangle \end{aligned}$$

using the fact that $\partial \xi_i^p = 0$. So we can take $c_i = k_i$ in the lemma statement. □

Definition 4.2.8. For monomials X and Y, write $X \geq Y$ if $Y \in T(X)$.

It is easy to check that this makes the set of monomials into a poset, and that $X \geq Y$ if and only if $T(X) \supset T(Y)$.

Lemma 4.2.9. *Suppose W is a monomial with admissible bracket expression $\langle \prod \xi_i^{e_i} \ ; \ \prod \xi_i^{f_i} \rangle$. Let $\widetilde{W} = \langle \prod \xi_i^{c_i} \ ; \ \prod \xi_i^{d_i} \rangle$ where $c_i = \min\{e_i + f_i, p-1\}$ and $d_i = f_i - (c_i - e_i)$. Then $\widetilde{W}$ is the maximal object $\geq W$.*

Proof. Let X be an arbitrary monomial, written in its unique admissible bracket expression. Then $X \geq W$ if and only X can be obtained from W by moving terms in W from the right to the left side of the bracket expression. Note that $\widetilde{W}$ is the bracket expression obtained by moving as many terms to the left as possible while still keeping the resulting expression admissible. This implies $\widetilde{W}$ is maximal. □

Define an equivalence relation on monomials where $X \sim Y$ if $\widetilde{X} = \widetilde{Y}$.

Lemma 4.2.10. *There is a direct sum decomposition $B \cong \bigoplus_{\substack{\text{eq. class} \\ \text{reps. } X}} T(\widetilde{X})$.*

Proof. I claim that $T(\widetilde{X}) = \mathbb{F}_p\{Y : X \sim Y\}$; this follows from the fact that, by definition, $T(\widetilde{X})$ is generated by Y such that $Y \leq \widetilde{X}$. So the direct sum decomposition comes from partitioning monomials into their equivalence classes. □

Let $\mathscr{I}$ be the set of admissible bracket expressions X such that $\widetilde{X} = X$. By Lemma 4.2.9 we have the following.

Lemma 4.2.11. *$\mathscr{I}$ is the set of admissible bracket expressions $\left\langle \prod \xi_i^{e_i} \ ; \ \prod \xi_i^{f_i} \right\rangle$ such that $e_i \leq p-1$ and if $e_i < p-1$ then $f_i = 0$.*

Lemma 4.2.12. *If $X = \langle \prod \xi_i^{e_i} \ ; \ \prod \xi_i^{f_i} \rangle$ is an admissible expression, there is an isomorphism of D-comodules $T(\langle \prod \xi_i^{e_i} \ ; \ 1 \rangle) \cong T(X)$.*

Proof. By Lemma 4.2.7, every Y in $T(X)$ has a bracket expression obtained from X by moving terms from the left to the right, so the right hand side of the bracket expression for Y is divisible by $\prod \xi_i^{f_i}$, and so Y is divisible by $u := \langle 1\ ;\ \prod \xi_i^{f_i} \rangle = \prod \xi_{i-1}^{pf_i}$. So multiplication by u gives a map $T(\langle \prod \xi_i^{e_i}\ ;\ 1\rangle) \to T(X)$, and moreover from the above description of $Y \in T(X)$ it is easy to see that this is a bijection. Finally, since $\partial(u) = 0$, this is an isomorphism of D-comodules. ☐

Lemma 4.2.13. *If $X = \langle \prod \xi_i^{e_i}\ ;\ \prod \xi_i^{f_i} \rangle$ is an admissible expression such that $e_k = p-1$ for some k then $T(X)$ is a free D-comodule.*

Proof. By definition, we have $T(X) = \overset{\Delta}{\bigotimes} M(e_i)_{\xi_{n_i}^{e_i}}$, and $M(e_k)_{\xi_{n_k}^{e_k}} \cong M(p-1) \cong D$ by assumption. Rearranging terms and using the shear isomorphism, we have $T(X) \cong D \overset{\Delta}{\otimes} \overset{\Delta}{\bigotimes}_{i \neq k} M(e_i)_{\xi_{n_i}^{e_i}} \cong D \overset{L}{\otimes} \overset{\Delta}{\bigotimes}_{i \neq k} M(e_i)_{\xi_{n_i}^{e_i}}$, which is free. ☐

By Lemmas 4.2.12 and 4.2.13, we have:

Corollary 4.2.14. *If $X = \langle \prod \xi_i^{e_i}\ ;\ \prod \xi_i^{f_i} \rangle$ is an admissible bracket expression in $\mathscr{I}$ such that $f_i \neq 0$ for any i, then $T(X)$ is free as a D-comodule.*

Proof of Proposition 4.2.4. From Lemma 4.2.10 we have $B \cong \bigoplus_{X \in \mathscr{I}} T(X)$, and by Corollary 4.2.14 there are free D-comodules F and F' such that

$$\begin{aligned} B \ &\cong \bigoplus_{\langle X\ ;\ 1\rangle \in \mathscr{I}} T(\langle X\ ;\ 1\rangle) \oplus F \ = \bigoplus_{\langle X\ ;\ 1\rangle \in \mathscr{I}} T(X) \oplus F \\ &\cong \bigoplus_{\substack{\langle X\ ;\ 1\rangle \text{ s.t.} \\ e_i(X) \leq p-2}} T(X) \oplus F' \\ &= \bigoplus_{X \in \mathscr{B}} T(X) \oplus F' \\ &\cong \bigoplus_{X \in \mathscr{B}} \overset{\Delta}{\bigotimes_i} M(e_i(X))_{x_i(X)} \oplus F'. \end{aligned}$$ ☐

We conclude with a useful lemma that simplifies checking relations in certain b_{10}-local Ext groups of interest.

Lemma 4.2.15. *Let $I(n) = (\xi_1^{pn}, \xi_2^{pn}, \ldots)B$. Then $I(p-1)$ is contained in the free part of B according to the decomposition in Proposition 4.2.4. In particular, if $x \in \operatorname{Ext}_P^*(k, P \square_D I(p-1))$ then $x = 0$ in $b_{10}^{-1} \operatorname{Ext}_P^*(k, P \square_D B)$.*

Proof. Consider an arbitrary monomial $q = \xi_n^{(p-1)p} X$ in $I(p-1)$. If X has an admissible expression $\left\langle \prod \xi_i^{e_i} \ ; \ \prod \xi_i^{f_i} \right\rangle$ then q has an admissible expression $\left\langle \prod \xi_i^{e_i} \ ; \ \xi_{n+1}^{p-1} \prod \xi_i^{f_i} \right\rangle$. By Lemmas 4.2.10 and 4.2.13, it suffices to show that $\widetilde{q} = \langle \prod \xi_i^{c_i} \ ; \ \prod \xi_i^{d_i} \rangle$ satisfies $c_k = p-1$ for some k. Using the formula for $\widetilde{q}$ in Lemma 4.2.9, we have $c_{n+1} = p-1$. □

Corollary 4.2.16. *Let $I(n)$ be as in Lemma 4.2.15. If $x \in \operatorname{Ext}_P^*(k, P \square_D (P \square_D I(p-1)))$, then x is zero in $b_{10}^{-1} \operatorname{Ext}_P^*(k, P \square_D (P \square_D I(p-1)))$.*

4.3 Hopf algebra structure

Convention 4.3.1. Unless indicated otherwise, we will work at $p = 3$ in this section (and everywhere hereafter in this thesis). The reason for making this simplification is the simplicity of the structure of Comod_D and the Künneth formula (Lemma 4.3.5), which imply that $K(\xi_1)_{**}K(\xi_1) = b_{10}^{-1} \operatorname{Ext}_D(k, B)$ is flat (in fact, free) over $K(\xi_1)_{**} = b_{10}^{-1} \operatorname{Ext}_D^*(k, k)$. All of these points are discussed in Section 4.3.1.

4.3.1 Vector space structure of $K(\xi_1)_{**}K(\xi_1)$ at $p = 3$

Using the shear isomorphism (Corollary 3.1.10), we have

$$K(\xi_1)_{**}K(\xi_1) := \operatorname{Ext}_P(k, K(\xi_1)_{**}K(\xi_1)) \cong b_{10}^{-1} \operatorname{Ext}_P(k, B \mathbin{\hat{\otimes}} B)$$

$$\cong b_{10}^{-1}\operatorname{Ext}_P(k, P\square_D B) \cong b_{10}^{-1}\operatorname{Ext}_D(k,B).$$

The main result of the previous section allows us to write

$$\begin{aligned} b_{10}^{-1}\operatorname{Ext}_D^*(k,B) &\cong b_{10}^{-1}\operatorname{Ext}_D^*(k, \bigoplus_{\substack{\xi_{n_1}^{e_1}\cdots\xi_{n_d}^{e_d}\\ e_i\le p-2}} \bigotimes_{i=1}^{d} M(e_i)_{\xi_{n_i}^{e_i}} \oplus F) \\ &\cong \bigoplus_{\substack{\xi_{n_1}^{e_1}\cdots\xi_{n_d}^{e_d}\\ e_i\le p-2}} b_{10}^{-1}\operatorname{Ext}_D^*(k, \bigotimes_{i=1}^{d} M(e_i)_{\xi_{n_i}^{e_i}}) \end{aligned} \tag{4.3.1}$$

at all primes.

There is a formula that allows us to decompose the tensor products $\bigotimes M(e_i)$ into a sum of the basic comodules $M(n)$, but in general it is rather complicated:

Theorem 4.3.2 (Renaud, [Ren79, Theorem 1]). *At all primes,*

$$M(r)\otimes M(s) \cong (r-c)M(p-1) + \sum_{i=1}^{c} M(s-r+2i-2) \quad \textit{for } c = \begin{cases} r & \textit{if } r+s\le p \\ p-s & \textit{otherwise.} \end{cases}$$

At $p=3$, however, the only D-comodules are $M(0)=k$, $M(1)$, and $M(2)=D$, and it is easy to see directly that $M(1)\otimes M(1)\cong D\oplus\Sigma^{0,|\xi_1|}k$. (Here we use bigraded notation for the shift for consistency with viewing these objects in Stable(D), so $\Sigma^{0,|\xi_1|}$ denotes a shift of 0 in the homological dimension and $|\xi_1|$ in internal degree). In particular,

$$k\{x,\partial x\}\otimes k\{y,\partial y\} \;\cong\; k\{xy,\ \partial(x)y + x\partial(y),\ \partial(x)\partial(y)\}\oplus k\{\partial(x)y - x\partial(y)\}.$$

After inverting b_{10}, free comodules become zero, and the only basic types of comodules are $M(0)=k$ and $M(1)$.

Remark 4.3.3. We will repeatedly use the fact that $\operatorname{Ext}_D^*(k,D)$ is a 1-dimensional

k-vector space in homological degree 0 and zero otherwise, and for $i \in \{0,1\}$, $\operatorname{Ext}^*_D(k, M(i))$ is 1-dimensional in homological degree ≥ 0. As b_{10} is the generator of $\operatorname{Ext}^2_D(k,k)$, we have $b_{10}^{-1}\operatorname{Ext}^*_D(k,D) = 0$, and $b_{10}^{-1}\operatorname{Ext}^*_D(k, M(i))$ is a 1-dimensional k-vector space in every dimension. Furthermore, for any D-comodule M, the localization map $\operatorname{Ext}^*_D(k,M) \to b_{10}^{-1}\operatorname{Ext}^*_D(k,M)$ is an isomorphism in homological degree > 0.

Lemma 4.3.4. *In* Stable(D), *we have an isomorphism*

$$b_{10}^{-1}M(1) \cong \Sigma^{-1,2|\xi_1|}b_{10}^{-1}M(0).$$

Proof. A representative for $M(1)$ in Stable(D) (i.e., an injective resolution for it) is $0 \to D \xrightarrow{\partial^2} \Sigma^{0,2|\xi_1|}D \xrightarrow{\partial} \Sigma^{0,3|\xi_1|}D \xrightarrow{\partial^2} \Sigma^{0,5|\xi_1|}D \to \dots$, and so $b_{10}^{-1}M(1) := \operatorname{colim}(M(1) \xrightarrow{b_{10}} \Sigma^{2,-|b_{10}|}M(1) \to \dots)$ is represented by

$$\dots \to \Sigma^{0,-|\xi_1|}D \xrightarrow{\partial} \underbrace{D}_{\text{hom.deg.0}} \xrightarrow{\partial^2} \Sigma^{0,2|\xi_1|}D \xrightarrow{\partial} \Sigma^{0,3|\xi_1|}D \to \dots.$$

Similarly, $b_{10}^{-1}M(0)$ is represented by

$$\dots \to \Sigma^{0,-2|\xi_1|}D \xrightarrow{\partial^2} \underbrace{D}_{\text{hom.deg.0}} \xrightarrow{\partial} \Sigma^{0,|\xi_1|}D \xrightarrow{\partial^2} \Sigma^{0,3|\xi_1|}D \to \dots.$$

and so there is a degree-preserving isomorphism $b_{10}^{-1}M(1) \to \Sigma^{-1,2|\xi_1|}b_{10}^{-1}M(0)$. ☐

(At arbitrary primes, the formula $b_{10}^{-1}M(n) \cong \Sigma^{-1,(p-1)|\xi_1|}b_{10}^{-1}M(p-2-n)$ holds for the same reason.) Therefore, if M is a D-comodule, then $b_{10}^{-1}M \in \text{Stable}(D)$ is a sum of shifts of the unit object $k \cong M(0)$. Remembering that Stable(D) was constructed so that $\operatorname{Hom}_{\text{Stable}(D)}(k, b_{10}^{-1}M) = b_{10}^{-1}\operatorname{Ext}_D(k,M)$, we obtain the following Künneth isomorphism:

Lemma 4.3.5. *If M and N are D-comodules, then*

$$b_{10}^{-1}\operatorname{Ext}^*_D(k, M \otimes N) \cong b_{10}^{-1}\operatorname{Ext}^*_D(k,M) \otimes b_{10}^{-1}\operatorname{Ext}^*_D(k,N).$$

This only works at $p = 3$, and is the essential reason we have made the simplification of working at $p = 3$.

Applying this to (4.3.1) we have the following.

Corollary 4.3.6. *We have an isomorphism*

$$b_{10}^{-1} \operatorname{Ext}_D^*(k, B) \cong \bigoplus_{\substack{monomials \\ \xi_{n_1} \cdots \xi_{n_d}}} b_{10}^{-1} \operatorname{Ext}_D^*(k, \Sigma^{-d, 2|\xi_1|} k_{\xi_{n_1} \cdots \xi_{n_d}})$$

where $\Sigma^{-d,2d|\xi_1|} k_{\xi_{n_1} \cdots \xi_{n_d}}$ is the copy of $\Sigma^{-d,2d|\xi_1|} k$ isomorphic to $\bigotimes_{i=1}^d M(1)_{\xi_{n_i}}$ under Lemma 4.3.4. In particular, $K(\xi_1)_{**} K(\xi_1) = b_{10}^{-1} \operatorname{Ext}_D^*(k, B)$ is free over $K(\xi_1)_{**} = b_{10}^{-1} \operatorname{Ext}_D(k, k)$.

So $b_{10}^{-1} \operatorname{Ext}_D(k, B)$ has R-module generators in bijection with monomials of the form $\xi_{n_1} \ldots \xi_{n_d}$ (where $n_i \neq n_j$ if $i \neq j$). Now we will be more precise in choosing these generators.

Lemma 4.3.7. *Suppose N is a D-comodule algebra with sub-D-comodules $k\{x, \partial x\} \cong M(1)$ and $k\{y, \partial y\} \cong M(1)$.*

(1) The image of $\operatorname{Ext}_D^1(k, k\{x, \partial x\})$ in $\operatorname{Ext}_D^1(k, N)$ is generated by $e(x) = [\xi_1]x - [\xi_1^2]\partial x$.

(2) We have

$$e(x) \cdot e(y) = b_{10}(y \partial x - x \partial y)$$

in the multiplication $\operatorname{Ext}_D^*(k, N) \otimes \operatorname{Ext}_D^*(k, N) \to \operatorname{Ext}_D^*(k, N)$ induced by the product structure on N. In particular, $e(x)^2 = 0$.

(3) If the multiplication map embeds $k\{x, \partial x\} \otimes k\{y, \partial y\}$ in N injectively, then $b_{10}^{-1} \operatorname{Ext}_D^2(k, k\{x, \partial x\} \otimes k\{y, \partial y\}) \subset b_{10}^{-1} \operatorname{Ext}_D^2(k, N)$ is a 1-dimensional vector space with generator $e(x) \cdot e(y)$.

Since $\operatorname{Ext}^i_D(k,M) = b_{10}^{-1}\operatorname{Ext}^i_D(k,M)$ for $i>0$, note that this also gives a generator of $b_{10}^{-1}\operatorname{Ext}^1_D(k,N)$.

Proof. Since $\operatorname{Ext}^1_D(k, M(1))$ is a 1-dimensional k-vector space, for (1) it suffices to show that $e(x)$ is a cycle that is not a boundary. Indeed, since $dx = [\xi_1]\partial x$ and $d(\partial x) = 0$, we have $d(e(x)) = -[\xi_1|\xi_1]\partial x + [\xi_1|\xi_1]\partial x = 0$, and $e(x)$ is not in $d(C^0_D(k, k\{x, \partial x\})) = d(k\{x, \partial x\})$.

For (2), we use a special case of the cobar complex multiplication formula in [Mil78, Proposition 1.2]:

Fact 4.3.8. *The multiplication $C^1_D(k,M)\otimes C^1_D(k,N) \to C^2_D(k, M\otimes N)$ is given by*

$$[\xi]m \otimes [\omega]n \mapsto \sum [\xi \otimes m'\omega](m''\otimes n).$$

Thus the product $C^1_D(k,N)\otimes C^1_D(k,N) \to C^2_D(k,N\otimes N) \xrightarrow{\mu} C^2_D(k,N)$ takes $[\xi]m \otimes [\omega]n \mapsto \sum[\xi\otimes m'\omega]m''n$. Using this formula, we have:

$$\begin{aligned}
e(x)\cdot e(y) &= [\xi_1|x]\cdot[\xi_1|y] - [\xi_1|x]\cdot[\xi_1^2|\partial y] \\
&\qquad - [\xi_1^2|\partial x]\cdot[\xi_1|y] + [\xi_1^2|\partial x]\cdot[\xi_1^2|\partial y] \\
[\xi_1|x]\cdot[\xi_1|y] &= \sum [\xi_1|x'\xi_1]x''y = [\xi_1|\xi_1]xy + [\xi_1|\xi_1^2](\partial x)y \\
[\xi_1|x]\cdot[\xi_1^2|\partial y] &= \sum [\xi_1|x'\xi_1^2]x''\partial y = [\xi_1|\xi_1^2]x\partial y \\
[\xi_1^2|\partial x]\cdot[\xi_1|y] &= \sum [\xi_1^2|(\partial x)'\xi_1](\partial x)''y = [\xi_1^2|\xi_1](\partial x)y \\
[\xi_1^2|\partial x]\cdot[\xi_1^2|\partial y] &= \sum [\xi_1^2|\xi_1^2(\partial x)'](\partial x)''\partial y = [\xi_1^2|\xi_1^2]\partial x\partial y \\
d([\xi_1^2]xy) &= 2[\xi_1|\xi_1]xy - [\xi_1^2|\xi_1](\partial x)y - [\xi_1^2|\xi_1]x\partial y - [\xi_1^2|\xi_1^2]\partial x\partial y \\
e(x)\cdot e(y) + d([\xi_1^2]xy) &= [\xi_1|\xi_1^2](\partial x)y + [\xi_1^2|\xi_1](\partial x)y - [\xi_1|\xi_1^2]x\partial y - [\xi_1^2|\xi_1]x\partial y \\
&= b_{10}((\partial x)y - x\partial y)
\end{aligned}$$

For (3), note that there is a decomposition of D-comodules

$$\begin{aligned} k\{x, \partial x\} \otimes k\{y, \partial y\} &\xrightarrow[\cong]{\mu} k\{xy, x\partial y, (\partial x)y, (\partial x)(\partial y)\} \\ &= k\{xy, (\partial x)y + x\partial y, (\partial x)(\partial y)\} \oplus k\{(\partial x)y - x(\partial y)\} \end{aligned}$$

and since $\mathrm{Ext}_D^{*>0}(k, D) = 0$, the quotient map

$$b_{10}^{-1} \mathrm{Ext}_D^2(k, k\{x, \partial x\} \otimes k\{y, \partial x\}) \cong b_{10}^{-1} \mathrm{Ext}_D^2(k, k\{x\partial y - (\partial x)y\})$$

is an isomorphism. By (2), $e(x) \cdot e(y)$ is a generator of the latter Ext group. □

Lemma 4.3.9. *Suppose N is a D-comodule algebra with sub-D-comodules $k\{x, \partial x\} \cong M(1)$ and $k\{y\} \cong k$.*

(1) The image of $\mathrm{Ext}_D^0(k, k\{y\})$ in $\mathrm{Ext}_D^0(k, N)$ is generated by y.

(2) We have

$$e(x) \cdot y = [\xi_1]xy - [\xi_1^2](\partial x)y = y \cdot e(x).$$

(3) If the multiplication map embeds $k\{x, \partial x\} \otimes k\{y\}$ in N injectively, then $e(x) \cdot y$ is a generator of the 1-dimensional vector space $b_{10}^{-1} \mathrm{Ext}_D^1(k, k\{x, \partial x\} \otimes k\{y\})$.

Proof. (1) is clear. (2) follows from the cobar complex multiplication formulas

$$\begin{aligned} C_D^0(k, M) \otimes C_D^1(k, N) &\to C_D^1(k, M \otimes N) & m \otimes [\xi]n &\mapsto [\xi](m \otimes n) \\ C_D^1(k, M) \otimes C_D^0(k, N) &\to C_D^1(k, M \otimes N) & [\xi]n \otimes m &\mapsto [\xi](n \otimes m). \end{aligned}$$

For (3), note that $k\{x, \partial x\} \otimes k\{y\} = k\{xy, (\partial x)y\}$. Note that $(\partial x)y = \partial(xy)$. From Lemma 4.3.7, $b_{10}^{-1} \mathrm{Ext}_D^1(k, k\{xy, \partial(xy)\})$ is generated by $e(xy) = [\xi_1]xy - [\xi_1^2]\partial(xy) = e(x) \cdot y$. □

Definition 4.3.10. Define $e_n := e(\xi_n)$ as the chosen generator of $b_{10}^{-1}\operatorname{Ext}^1_D(k, M(1)_{\xi_n})$.

Lemma 4.3.11. *Under the change of rings isomorphism*

$$b_{10}^{-1}\operatorname{Ext}_D(k,B) \cong b_{10}^{-1}\operatorname{Ext}_P(k, P\,\square_D B)$$

the image of $e(x)$ *in* $\operatorname{Ext}^1_P(k, P\,\square_D B)$ *has cobar representative*

$$[\xi_1](1|x) - [\xi_1^2](1|\partial x) + [\xi_1](\xi_1|\partial x) \in \overline{P} \otimes (P\,\square_D B).$$

Proof. The change of rings isomorphism $\operatorname{Ext}_D(k,M) \cong \operatorname{Ext}_P(k, P\,\square_D M)$ works as follows: since P is free over D, the functor $P\,\square_D -$ is exact, and so given an injective D-resolution $M \to X^\bullet$ for M, the complex $P\,\square_D M \to P\,\square_D X^\bullet$ is an injective P-resolution. So we have $\operatorname{Ext}^i_D(k,M) \cong \operatorname{Cotor}^i_D(k,M) = H^i(k\,\square_D X^\bullet)$, which agrees with $\operatorname{Ext}^i_P(k, P\,\square_D M) \cong \operatorname{Cotor}^i_P(k, P\,\square_D M) = H^i(k\,\square_P(P\,\square_D X^\bullet)) \cong H^i(k\,\square_D X^\bullet)$.

In particular, $\operatorname{Ext}_P(k, P\,\square_D B)$ can be computed by applying $k\,\square_P -$ to the resolution

$$P\,\square_D C_D(k,B) = \big(P\,\square_D B \to P\,\square_D(D\otimes B) \to P\,\square_D(D\otimes \overline{D}\otimes B) \to \dots\big). \tag{4.3.2}$$

By Lemma 4.3.7, $e(x)$ has representative $[1|\xi_1]x - [1|\xi_1^2]\partial x \in D\otimes\overline{D}\otimes B$ in the D-cobar *resolution* for B, and so its representative in (4.3.2) is $1|1|\xi_1|x - \cdot 1|1|\xi_1^2|\partial x$.

But we wanted a representative in the cobar complex $C_P(k, P\,\square_D B)$, so we will write down part of an explicit map from the P-cobar resolution for $P\,\square_D B$ to (4.3.2):

$$
\begin{array}{ccc}
P\,\square_D B & = & P\,\square_D B \\
\downarrow & & \downarrow \\
P\otimes(P\,\square_D B) & \xrightarrow{f^0} & P\otimes B \\
\downarrow & & \downarrow \\
P\otimes\overline{P}\otimes(P\,\square_D B) & \xrightarrow{f^1} & P\otimes\overline{D}\otimes B \\
\downarrow & & \downarrow \\
P\otimes\overline{P}^{\otimes 2}\otimes(P\,\square_D B) & \longrightarrow & P\otimes\overline{D}^{\otimes 2}\otimes B \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

By basic homological algebra, the map f^* exists and is unique, so to find f^0 and f^1 it suffices to find P-comodule maps that make the first two squares commute. In particular, one can check that the maps

$$f^0(a|b|c) = \varepsilon(b)a|c$$
$$f^1(a|b|c|d) = \varepsilon(c)a|b|d$$

make the diagram commute, and $z := [1|\xi_1](1|x) + [1|\xi_1](\xi_1|\partial x) - [1|\xi_1^2](1|\partial x)$ is a cycle in $P\otimes\overline{P}\otimes(P\,\square_D B)$ such that $(k\,\square_P f)(z) = e(x)$. $\square$

4.3.2 Multiplicative structure

Proposition 4.3.12. *The summand*

$$b_{10}^{-1}\operatorname{Ext}_D^d(k, M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_d}}) \subset b_{10}^{-1}\operatorname{Ext}_D^d(k, B)$$

is generated by the product $e_{n_1}\ldots e_{n_d}$.

Proof. Since

$$b_{10}^{-1}\operatorname{Ext}_D^d(k,\bigotimes M(1)_{\xi_{n_i}}) = \begin{cases} \Sigma^{d,0}\, b_{10}^{-1}\operatorname{Ext}_D^0(k,\bigotimes M(1)_{\xi_{n_i}}) & d \text{ is even} \\ \Sigma^{d-1,0}\, b_{10}^{-1}\operatorname{Ext}_D^1(k,\bigotimes M(1)_{\xi_{n_i}}) & d \text{ is odd,} \end{cases}$$

it suffices to show that $b_{10}^{-1}\operatorname{Ext}_D^0(k, M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_d}})$ is generated by $b_{10}^{-d/2}e_{n_1}\ldots e_{n_d}$ when d is even, and $b_{10}^{-1}\operatorname{Ext}_D^1(k, M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_d}})$ is generated by $b_{10}^{-(d-1)/2}e_{n_1}\ldots e_{n_d}$ when d is odd. We proceed by induction on d. The base case $d=1$ is by definition.

Case 1: d is even. The tensor product $M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_{d-1}}}$ is isomorphic to $M(1)\oplus F$ for a free summand F. By Lemma 4.3.7, $b_{10}^{-1}\operatorname{Ext}_D^2(k,(M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_{d-1}}})\otimes M(1)_{\xi_{n_d}})$ is generated by $e(x)\cdot e_{n_d}$ where $e(x)$ is a generator of $b_{10}^{-1}\operatorname{Ext}_D^1(k, M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_{d-1}}})$. By the inductive hypothesis, we can take $e(x) = b_{10}^{-(d-2)/2}e_{n_1}\ldots e_{n_{d-1}}$. So then $b_{10}^{-1}e(x)e_{n_d} = b_{10}^{-d/2}e_{n_1}\ldots e_{n_d}$ is a generator for $b_{10}^{-1}\operatorname{Ext}_D^0(k, M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_d}})$.

Case 2: d is odd. In this case, $M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_{d-1}}}$ is isomorphic to $k\oplus F$ for a free summand F. By Lemma 4.3.9, $b_{10}^{-1}\operatorname{Ext}_D^1(k,(M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_{d-1}}})\otimes M(1)_{\xi_{n_d}})$ is generated by $y\cdot e_{n_d}$ where y is a generator of $b_{10}^{-1}\operatorname{Ext}_D^0(k, M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_{d-1}}})$. By the inductive hypothesis, we can take $y = b_{10}^{-(d-1)/2}e_{n_1}\ldots e_{n_{d-1}}$. ☐

Recall we defined $R = b_{10}^{-1}\operatorname{Ext}_D(k,k) = E[h_{10}]\otimes P[b_{10}^{\pm1}]$.

Corollary 4.3.13. *There is an R-module isomorphism* $b_{10}^{-1}\operatorname{Ext}_D^*(k, M(1)_{\xi_{n_1}}\otimes\ldots\otimes M(1)_{\xi_{n_d}}) \cong R\{e_{n_1}\ldots e_{n_d}\}$ *where the generator* $e_{n_1}\ldots e_{n_d}$ *is in degree d.*

Corollary 4.3.14. *The map* $R\otimes E[e_2,e_3,\ldots]\to b_{10}^{-1}\operatorname{Ext}_D^*(k,B)$ *is an isomorphism of R-algebras.*

4.3.3 Antipode

The antipode is the map induced on Ext by the swap map $\tau : B \overset{\vartriangle}{\otimes} B \to B \overset{\vartriangle}{\otimes} B$.

Recall (see Lemma 3.1.9) there is a shear isomorphism $S_M : B \overset{\vartriangle}{\otimes} M \to P \square_D M$ sending $a \otimes m \mapsto \sum am' \otimes m''$. It has an inverse $S_M^{-1} : a \otimes m \mapsto \sum ac(m') \otimes m''$. In order to be able to apply Lemma 4.2.15, we now obtain an explicit formula for the induced map $\tau' := S_B \circ \tau \circ S_B^{-1} : P \square_D B \to P \square_D B$. This map is:

$$\begin{array}{ccc} B \overset{\vartriangle}{\otimes} B & \xrightarrow{\ \tau\ } & B \overset{\vartriangle}{\otimes} B \\ {\scriptstyle S_B^{-1}} \uparrow & & \downarrow {\scriptstyle S_B} \\ P \square_D B & \overset{\tau'}{\dashrightarrow} & P \square_D B \end{array} \qquad \begin{array}{ccc} \sum xc(y')|y'' & \longmapsto & \sum y''|xc(y') \\ \uparrow & & \downarrow \\ x|y & & \sum y'' \cdot x'c(y')'|x''c(y')'' \end{array}$$

Using coassociativity we have

$$\begin{aligned} \tau'(x \otimes y) &= \sum y'' \cdot x'c(y')'|x''c(y')'' \\ &= \sum x'y''c((y')'')|x''c((y')') && \text{Fact 3.0.2(4)} \\ &= \sum x'y_{(3)}c(y_{(2)})|x''c(y_{(1)}) \\ &= \sum x'\varepsilon(y_{(2)})|x''c(y_{(1)}) && \text{Fact 3.0.2(2)} \\ &= \sum x'|x''c(y). && \text{Fact 3.0.2(3)} \end{aligned}$$

Since $(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$ is a Hopf algebroid, the antipode is multiplicative, so to determine it, it suffices to show:

Proposition 4.3.15. *We have:*

(1) $c(h) = h$

(2) $c(e_n) = -e_n$.

Proof. The antipode is given by the map $\tau'_* : \mathrm{Ext}^*_P(k, P \square_D B) \to \mathrm{Ext}^*_P(k, P \square_D B)$

induced by τ', defined so that $\tau'_*([x_1|\dots|x_s]m) = [\xi_1|\dots|x_s]\tau'(m)$. Since $h = [\xi_1](1|1) \in \mathrm{Ext}^1_P(k, P\,\square_D B)$, we have $c(h) = \tau'_*(h) = h$. For (2), we need an explicit formula for the antipode in the dual Steenrod algebra:

Fact 4.3.16 ([Mil58, Lemma 10]). *asdfasdf Let* $\mathrm{Part}(n)$ *be the set of ordered partitions of* n, $\ell(\alpha)$ *the length of the partition* α, *and* $\sigma_i(\alpha) = \sum_{j=1}^{i} \alpha_j$ *be the partial sum. Then we have:*

$$c(\xi_n) = \sum_{\alpha \in \mathrm{Part}(n)} (-1)^{\ell(\alpha)} \prod_{i=1}^{\ell(\alpha)} \xi_{\alpha_i}^{p^{\sigma_{i-1}(\alpha)}}.$$

In particular, if $n \geq 2$ *then* $c(\xi_n) \equiv -\xi_n + \xi_1\xi_{n-1}^p \pmod{\overline{P}^{p^2} P}$ *and* $c(\xi_{n-1}^p) \equiv -\xi_{n-1}^p \pmod{\overline{P}^{p^2} P}$.

Recall (Notation 4.1.5) that we have defined ξ_n to be the antipode of its usual definition, so here we have $\Delta(\xi_n) = \sum_{i+j=n} \xi_i \otimes \xi_j^{p^i}$. (Since the antipode is a ring homomorphism, the formula in Fact 4.3.16 is the same in either case.)

Combining this antipode formula with the formula for e_n in Lemma 4.3.11 we have:

$$\begin{aligned}
\tau'_*(e_n) &= \tau'_*([\xi_1](1|\xi_n) - [\xi_1^2](1|\xi_{n-1}^3) + [\xi_1](\xi_1|\xi_{n-1}^3)) \\
&= [\xi_1](1|c(\xi_n)) - [\xi_1^2](1|c(\xi_{n-1}^3)) + [\xi_1](\xi_1|c(\xi_{n-1}^3) + 1|\xi_1 c(\xi_{n-1}^3)) \\
&= [\xi_1](-1|\xi_n + 1|\xi_1\xi_{n-1}^3 + 1|A) - [\xi_1^2](-1|\xi_{n-1}^3 + 1|B) \\
&\qquad + [\xi_1](-\xi_1|\xi_{n-1}^3 + \xi_1|C - 1|\xi_1\xi_{n-1}^3 + 1|D) \\
&= -e_n + [\xi_1](1|A + \xi_1|C + 1|D) - [\xi_1^2](1|B)
\end{aligned}$$

for A, B, C, and D in $\overline{P}^9 P = I(3)$. By Lemma 4.2.15 these terms are zero in b_{10}-local cohomology, and $c(e_n) = \tau'_*(e_n) = -e_n$. □

Corollary 4.3.17. *We have* $\eta_L = \eta_R$; *that is, the Hopf algebroid* $(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$ *is, in fact, a Hopf algebra.*

Proof. One of the axioms of a Hopf algebroid is $c \circ \eta_R = \eta_L$. Since η_L is just the inclusion of R into $b_{10}^{-1} \operatorname{Ext}_D^*(k, B)$, its image is invariant under the antipode c. □

4.3.4 Comultiplication

To define the comultiplication map $b_{10}^{-1} \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B) \to b_{10}^{-1} \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B)^{\otimes 2}$, first consider the maps

$$\operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B) \xrightarrow{\alpha_*} \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B \overset{\triangle}{\otimes} B) \xleftarrow{\beta} \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B) \otimes \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B)$$

where α_* is the map on Ext induced by $\alpha : B^{\otimes 2} \to B^{\otimes 3}$ with $\alpha : a \otimes b \mapsto a \otimes 1 \otimes b$, and β is defined as the map in the factorization

$$\begin{array}{ccccc} \operatorname{Ext}_P(k, B^{\overset{\triangle}{\otimes} 2}) \otimes \operatorname{Ext}_P(k, B^{\overset{\triangle}{\otimes} 2}) & \xrightarrow{\text{Künneth}} & \operatorname{Ext}_P(k, B^{\overset{\triangle}{\otimes} 2} \overset{\triangle}{\otimes} B^{\overset{\triangle}{\otimes} 2}) & \xrightarrow{-\otimes\mu\otimes-} & \operatorname{Ext}_P(k, B^{\overset{\triangle}{\otimes} 3}) \\ & \searrow & & \overset{\beta}{\nearrow} & \\ & & \operatorname{Ext}_P(k, B^{\overset{\triangle}{\otimes} 2}) \otimes_{\operatorname{Ext}_P(k,B)} \operatorname{Ext}_P(k, B^{\overset{\triangle}{\otimes} 2}) & & \end{array} \tag{4.3.3}$$

It follows from the shear isomorphism $\operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} M) \cong \operatorname{Ext}_D(k, M)$ and the Künneth isomorphism for b_{10}-local cohomology over D (Lemma 4.3.5) that β is an isomorphism after inverting b_{10}, and we define the comultiplication map on $b_{10}^{-1} \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B)$ by $\Delta := \beta^{-1} \circ \alpha_*$.

In particular, flatness of $K(\xi_1)_{**}K(\xi_1)$ over $K(\xi_1)_{**}$ implies that $(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$ is a Hopf algebroid using the definitions of comultiplication, antipode, counit, and unit above. In a Hopf algebroid, the comultiplication is a homomorphism, and so to determine Δ explicitly it suffices to determine $\Delta(e_n)$. We prove this in Proposition 4.3.19. Lemma 4.3.11 gives an expression for e_n in $\operatorname{Ext}_P^1(k, P \square_D B)$, so we prefer to calculate $\Delta : b_{10}^{-1} \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B) \to b_{10}^{-1} \operatorname{Ext}_P(k, B \overset{\triangle}{\otimes} B)^{\otimes 2}$ after composing with the shear isomorphism; that is, there is a commutative diagram

$$\begin{array}{ccccc} b_{10}^{-1}\operatorname{Ext}_P(k, B\hat{\otimes} B) & \xrightarrow{\alpha_*} & b_{10}^{-1}\operatorname{Ext}_P(k, B\hat{\otimes} B\hat{\otimes} B) & \xleftarrow{\beta} & b_{10}^{-1}\operatorname{Ext}_P(k, B\hat{\otimes} B)^{\otimes 2} \\ \downarrow{\scriptstyle (S_B)_*} & & \downarrow{\scriptstyle ((\mathbb{1}\otimes S_B)\circ S_{B\otimes B})_*} & & \downarrow{\scriptstyle S_B\otimes S_B} \\ b_{10}^{-1}\operatorname{Ext}_P(k, P\square_D B) & \xrightarrow{\alpha'_*} & b_{10}^{-1}\operatorname{Ext}_P(k, P\square_D(P\square_D B)) & \xleftarrow{\beta'} & b_{10}^{-1}\operatorname{Ext}_P(k, P\square_D B)^{\otimes 2} \end{array}$$

and we will show that $\alpha'_*(e_n) = \beta'(1\otimes e_n + e_n \otimes 1)$ in $b_{10}^{-1}\operatorname{Ext}_P(k, P\square_D(P\square_D B))$. (We have chosen to use an extra application of the shear isomorphism on the middle term in order to apply Corollary 4.2.16.)

Lemma 4.3.18. *If* $a \in \operatorname{Ext}_P(k, P\square_D B)$ *has cobar representative* $[a_1|\dots|a_s](p|q)$, *we have*

$$\alpha'_*(a) = \sum [a_1|\dots|a_s](p|q'|q'')$$
$$\beta'(1\otimes a + a\otimes 1) = [a_1|\dots|a_s](\sum p'|p''|q + p|q|1)$$

in $\operatorname{Ext}_P(k, P\square_D(P\square_D B))$.

So to check that a is primitive after inverting b_{10}, it suffices to check

$$\sum [a_1|\dots|a_s](p|q'|q'') - [a_1|\dots|a_s](\sum p'|p''|q + p|q|1) = 0 \qquad (4.3.4)$$

in $b_{10}^{-1}\operatorname{Ext}_P(k, P\square_D(P\square_D B))$.

Proof. By definition, α' is the map induced on Ext by the composition

$$P\square_D B \xrightarrow{S_B^{-1}} B\hat{\otimes} B \xrightarrow{-\otimes\eta\otimes-} B\hat{\otimes} B\hat{\otimes} B \xrightarrow{S_{B\otimes B}} P\square_D(B\hat{\otimes} B) \xrightarrow{P\square_D S_B} P\square_D(P\square_D B).$$

On elements, we have:

$$x|y \longmapsto \sum xc(y')|y'' \longmapsto \sum xc(y')|1|y'' \longmapsto \sum xc(y')(y'')'|1|(y'')''$$
$$\longmapsto \sum xc(y')(y'')'|((y'')'')'|((y'')'')'' = \sum x|y'|y''$$

where the last equality is a coassociativity argument similar to the one at the beginning of Section 4.3.3. That is, we have $\alpha'(x \otimes y) = \sum x \otimes y' \otimes y''$, which implies

$$\alpha'_*([a_1| \dots |a_s](p|q)) = \sum [a_1| \dots |a_s](p|q'|q'').$$

The map β' comes from the bottom composition in

$$\begin{array}{ccccc}
\mathrm{Ext}_P(k, B^{\overset{\vartriangle}{\otimes}2})^{\otimes 2} & \xrightarrow{\text{Künneth}} & \mathrm{Ext}_P(k, B^{\overset{\vartriangle}{\otimes}2} \overset{\vartriangle}{\otimes} B^{\overset{\vartriangle}{\otimes}2}) & \xrightarrow{(-\otimes\mu\otimes-)_*} & \mathrm{Ext}_P(k, B^{\overset{\vartriangle}{\otimes}3}) \\
\downarrow {\scriptstyle (S_B)_*\otimes(S_B)_*} & & \downarrow {\scriptstyle (S_B\otimes S_B)_*} & & \downarrow {\scriptstyle (S_{B\otimes B})_*} \\
\mathrm{Ext}_P(k, P \,\square_D B)^{\otimes 2} & \xrightarrow{\text{Künneth}} & \mathrm{Ext}_P(k, (P \,\square_D B) \overset{\vartriangle}{\otimes} (P \,\square_D B)) & \xrightarrow{\gamma_*} & \mathrm{Ext}_P(k, P \,\square_D (P \,\square_D B)).
\end{array}$$

We will only give an explicit expression for β' on elements of the form $1 \otimes a$ and $a \otimes 1$, where 1 denotes the unit $1 \otimes 1 \in \mathrm{Ext}^0_P(k, P \,\square_D B)$ and $a = [a_1| \dots |a_s](p \otimes q) \in \mathrm{Ext}^s_P(k, P \,\square_D B)$. In [Mil78], there is a full description of the Künneth map K on the level of cochains, but here all we need are the maps $K : C^0_P(k, M) \otimes C^s_P(k, N) \to C^s_P(k, M \otimes N)$ and $K : C^s_P(k, N) \otimes C^0_P(k, M) \to C^s_P(k, M \otimes N)$. The former sends $m \otimes [a_1| \dots |a_s]n \mapsto [a_1| \dots |a_s](m \otimes n)$ and the latter sends $[a_1| \dots |a_s]n \otimes m \mapsto [a_1| \dots |a_s](n \otimes m)$. In particular, $K(1 \otimes a) = [a_1| \dots |a_s](1|1|p|q)$ and $K(a \otimes 1) = [a_1| \dots |a_s](p|q|1|1)$ in $\mathrm{Ext}^s_P(k, (P \,\square_D B) \otimes (P \,\square_D B))$.

To determine β', it remains to determine the map $\gamma : (P \,\square_D B) \otimes (P \,\square_D B) \to P \,\square_D (P \,\square_D B)$ induced by $- \otimes \mu \otimes -$. This is accomplished by calculating the effect of shear isomorphisms as follows:

$$\begin{array}{ccccc}
(B \overset{\vartriangle}{\otimes} B) \otimes (B \overset{\vartriangle}{\otimes} B) & \xrightarrow{-\otimes\mu\otimes-} & B^{\overset{\vartriangle}{\otimes}3} & & \\
\uparrow {\scriptstyle S_B^{-1}\otimes S_B^{-1}} & & \downarrow {\scriptstyle S_{B\otimes B}} & & \\
(P \,\square_D B) \otimes (P \,\square_D B) & & P \,\square_D (B \overset{\vartriangle}{\otimes} B) & \xrightarrow{P \,\square_D S_B} & P \,\square_D (P \,\square_D B)
\end{array}$$

$$
\begin{array}{ccc}
\sum xc(y')|y'' \otimes zc(w')|w'' & \longmapsto & \sum xc(y')|y''zc(w')|w'' \\
\Big\uparrow & & \Big\downarrow \\
x|y \otimes z|w & & \begin{array}{l} \sum xc(y')(y'')'z'c(w')'(w'')' \\ \quad \otimes (y'')''z''c(w')'' \otimes (w'')'' \longmapsto \sum xz'|yz''|w. \\ \quad = \sum xz'|yz''c(w')|w'' \end{array}
\end{array}
$$

That is, $\gamma(x|y \otimes z|w) = \sum xz'|yz''|w$, which implies

$$
\begin{aligned}
\beta'(1 \otimes a + a \otimes 1) &= \gamma_* K(1 \otimes a + a \otimes 1) \\
&= \gamma_*([a_1|\dots|a_s](1|1|p|q + p|q|1|1)) \\
&= [a_1|\dots|a_s]\gamma(1|1|p|q + p|q|1|1) \\
&= [a_1|\dots|a_s](\textstyle\sum p'|p''|q + p|q|1).
\end{aligned}
$$

□

Proposition 4.3.19. *The element e_n is primitive.*

Proof. We need to check the criterion (4.3.4) for $a = e_n$. Recall we had the formula

$$e_n = [\xi_1](1|\xi_n) - [\xi_1^2](1|\xi_{n-1}^3) + [\xi_1](\xi_1|\xi_{n-1}^3) \in C_P^1(P \,\square_D B)$$

from Lemma 4.3.11. It suffices to check that $\alpha'_*(e_n) - \beta'_*(1 \otimes e_n + e_n \otimes 1)$ is zero in $b_{10}^{-1} \operatorname{Ext}_P(k, P \,\square_D (P \,\square_D B))$. Using Lemma 4.3.18 we have:

$$
\begin{aligned}
\alpha'_*(e_n) - \beta'_*(1 \otimes e_n + e_n \otimes 1) &= \big([\xi_1](1|\Delta\xi_n) - [\xi_1^2](1|\Delta\xi_{n-1}^3) + [\xi_1](\xi_1|\Delta\xi_{n-1}^3)\big) \\
&\quad - \big([\xi_1](1|1|\xi_n + 1|\xi_n|1) - [\xi_1^2](1|1|\xi_{n-1}^3 + 1|\xi_{n-1}^3|1) \\
&\quad + [\xi_1](1|\xi_1|\xi_{n-1}^3 + \xi_1|1|\xi_{n-1}^3 + \xi_1|\xi_{n-1}^3|1)\big) \\
&= [\xi_1] \sum_{\substack{i+j=n \\ 2 \le i \le n-1}} 1|\xi_i|\xi_j^{3^i} - [\xi_1^2] \sum_{\substack{i+j=n-1 \\ 1 \le i \le n-2}} 1|\xi_i^3|\xi_j^{3^{i+1}} + [\xi_1] \sum_{\substack{i+j=n-1 \\ 1 \le i \le n-2}} \xi_1|\xi_i^3|\xi_j^{3^{i+1}}
\end{aligned}
$$

But all the remaining terms in the difference are in $C_P(P \,\square_D (P \,\square_D I(3)))$ so by Corollary 4.2.16 they are zero in b_{10}-local cohomology. □

Putting together Lemma 4.3.14, Proposition 4.3.15, Corollary 4.3.17, and Proposition

4.3.19, we have the following:

Theorem 4.3.20. *The map* $R \otimes E[e_2, e_3, \dots] \to b_{10}^{-1}\operatorname{Ext}^*_D(k, B)$ *is an isomorphism of Hopf algebras. That is, the Hopf algebroid* $(K(\xi_1)_{**}, K(\xi_1)_{**}K(\xi_1))$ *is an exterior Hopf algebra over* R *on the generators* $e_2, e_3, \dots$ *where* e_n *has internal degree* $2(3^n + 1)$.

4.4 Convergence

The convergence argument will only rely on the form of the E_1 page of our spectral sequence. Recall $B = P\square_D k$ and $K(\xi_1) = b_{10}^{-1}B$. By the definition of the MPASS (Definition 2.2.2), we have $E_1^{s,*} = b_{10}^{-1}K(\xi_1)_{**}(\overline{K(\xi_1)}^{\otimes s}) = b_{10}^{-1}\operatorname{Ext}_P(k, B \otimes \overline{B}^{\otimes s})$. By the change of rings theorem, this is $b_{10}^{-1}\operatorname{Ext}_D(k, \overline{B}^{\otimes s})$.

Proposition 4.4.1. *The* b_{10}*-localized* $K(\xi_1)$*-based MPASS*

$$E_1^{s,*} = b_{10}^{-1}\operatorname{Ext}_D(k, \overline{B}^{\otimes s}) \implies b_{10}^{-1}\operatorname{Ext}_P(k, k)$$

converges.

The proof is a slight modification of [Pal01, Proposition 4.4.1, Proposition 4.2.6].

We use the following grading convention: $x \in E_1^{s,t,u}$ is an element in $\operatorname{Ext}^t_P(k, B \otimes \overline{B}^{\otimes s})$ with internal degree u. Note $b_{10} \in E_1^{0,2,12}$.

Lemma 4.4.2. *Let* M *be a bounded-below graded* D*-module and suppose* $u_M = \min\{u(x) : x \in M\}$. *If* $x \in \operatorname{Ext}^*_D(k, M)$ *is a nonzero element of degree* (s, t, u) *and* $x \neq 0$, *then* $u \geq u_M + 6t - 2$.

Proof. First we check the cases when $M \cong k, M(1)$, or D.

Case 1: $M \cong k$. Let y be the generator of M, in degree $(t, u) = (0, u(y))$. We have $\mathrm{Ext}_D^*(k, k\{y\}) = E[h_{10}] \otimes P[b_{10}] \otimes k\{y\}$ where h_{10} is in degree $(t, u) = (1, |\xi_1|) = (1, 4)$ and b_{10} is in degree $(t, u) = (2, 12)$. The minimum degree element is y, so $u_M = u(y)$. Every element has the form $h_{10}b_{10}^n y$ or $b_{10}y$ for $n \geq 0$, and both of these satisfy $u \geq u_M + 6t - 2$.

Case 2: $M \cong M(1)$. Write $M = k\{y, \partial y\}$, where ∂y is in degree $(0, u(\partial y))$ and ∂y is in degree $(0, u(\partial y) + 4)$. By Lemma 4.3.7(1), $\mathrm{Ext}_D^*(k, M) = \mathbb{F}_p[b_{10}] \otimes k\{\partial y, e(y)\}$ where $e(y)$ is in degree $(t, u) = (1, u(\partial y) + 8)$. The minimum degree element is ∂y, and all the elements satisfy $u \geq u_M + 6t$.

Case 3: $M \cong D$. Here, $\mathrm{Ext}_D^0(k, M) \cong k$ has degree $(t, u) = (0, u_M)$ and $\mathrm{Ext}_D^t(k, M) = 0$ for $t > 0$.

In general, a homogeneous element $x \in M$ is a sum $\sum x_i$ for $x_i \in M_i$ where M_i is a summand of the above type, and by definition, $u_{M_i} \geq u_M$. So $u(x) = u(x_i) \geq u_{M_i} + 6t - 2 \geq u_M + 6t - 2$. □

Proposition 4.4.3. *There is a vanishing plane in the E_1 page of our spectral sequence: $E_1^{s,t,u} = 0$ if $u < 12s + 6t - 2$.*

Proof. Recall $E_1^{s,t,*} = \mathrm{Ext}_P^t(k, P \,\square_D\, \overline{B}^{\otimes s}) \cong \mathrm{Ext}_D(k, \overline{B}^{\otimes s})$. The element in $\overline{B}$ of smallest internal degree is ξ_1^3, which has $u = 12$. Therefore $x \in \overline{B}^{\otimes s}$ has $u \geq 12s$. By Lemma 4.4.2, if $x \in E_1^{s,t,u}$ has degree (s, t, u), then $u \geq 12s + 6t - 2$. □

Corollary 4.4.4. *The differential $d_r : E_r^{s,t,u} \to E_r^{s+r,t-r+1,u}$ is zero if $r > \frac{1}{6}(u - 12s - 6t - 4)$.*

Proof. Given $x \in E_r^{s,t,u}$, $d_r(x) \in E_r^{s',t',u'} = E_r^{s+r,t-r+1,u}$ will be zero because of the vanishing plane if $12s' + 6t' - 2 - u' > 0$. But

$$12s' + 6t' - 2 - u' = 12(s + r) + 6(t - r + 1) - 2 - u = (12s + 6t + 4 - u) + 6r$$

which is > 0 for r as indicated. □

Corollary 4.4.5. *There is a vanishing line in* $\mathrm{Ext}^*_P(k, k)$*: if* $x \in \mathrm{Ext}^{t',u}_P(k, k)$ *and* $u - 6t' + 2 < 0$ *then* $x = 0$.

Proof. Permanent cycles in $E_1^{s,t,u}$ converge to elements in $\mathrm{Ext}^{s+t,u}_P(k, k)$. Any such x would then be represented by a permanent cycle in $E_1^{s,t,u}$ with $u - 6(s + t) + 2 < 0 \leq 6s$ (since Adams filtrations are non-negative), which falls in the vanishing region of Proposition 4.4.3. □

Note that $b_{10} \in \mathrm{Ext}^{2,12}_P(k, k)$ acts parallel to this vanishing line; this is an illustration of the Stable(P) version of Theorem 1.1.3.

Proof of Proposition 4.4.1. The non-localized spectral sequence converges by Proposition 2.2.5. There are two things that can go wrong with convergence of a localized spectral sequence: (1) a b_{10}-tower of permanent cycles is not in $b_{10}^{-1}E_\infty$ because the tower is split into infinitely many pieces in the spectral sequence, connected by hidden multiplications; (2) a b_{10}-periodic tower supports a differential to an infinite sequence of torsion elements, and hence this differential is not recorded in $b_{10}^{-1}E_r$. (The reverse of (2), where a sequence of torsion elements supports a differential that hits a b_{10}-tower, cannot happen: if $d_r(x) = y$ and $b_{10}^n x = 0$ in E_r, then $0 = d_r(b_{10}^n x) = b_{10}^n d_r(x) = b_{10}^n y$.)

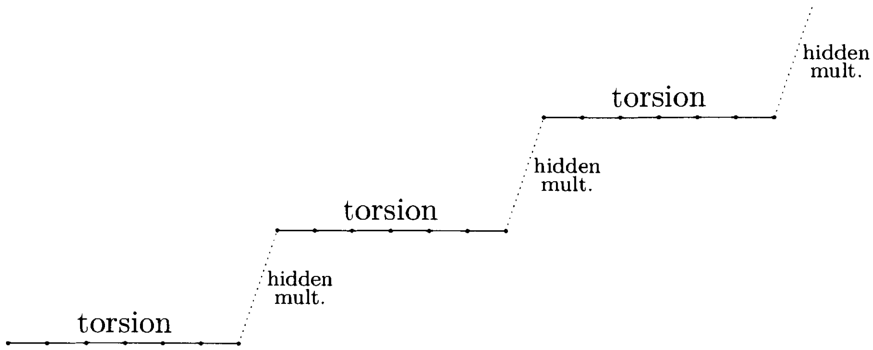


Figure 4-1: Illustration of (1): this represents a b_{10}-tower in $\pi_* X$

For (1), suppose x has degree (s_x, t_x, u_x). If there were no multiplicative extensions, then $b_{10}^i x$ would have degree $(s_x, t_x + 2i, u_x + 12i)$. But multiplicative extensions cause it to have the expected internal degree u and stem $s + t$, but higher s. That is, $b_{10}^i x$ has degree $(s_x + n_i, t_x + 2i - n_i, u_x + 12i)$ for some $n_i > 0$, and because this scenario involves the existence of infinitely many multiplicative extensions, the sequence $(n_i)_i$ is increasing and unbounded above. This causes us to run afoul of the vanishing plane (Proposition 4.4.3) for sufficiently large i:

$$\begin{aligned} 12s + 6t - 2 - u &= 12(s_x + n_i) + 6(t_x + 2i - n_i) - 2 - (u_x + 12i) \\ &= 12s_x + 6t_x - 2 - u_x + 6n_i \end{aligned}$$

which is > 0 for $i \gg 0$.

For (2), the scenario is, more precisely, as follows: we have a b_{10}-periodic element $x \in \mathrm{Ext}_P^*(k, k)$, and a sequence of differentials $d_{r_i}(b_{10}^i x) = y_i \neq 0$, where every y_i is b_{10}-torsion. The sequence $(r_i)_i$ must be increasing and bounded above: if $b_{10}^{n_i} y_i = 0$ then $d_{r_i}(b_{10}^{n_i} x) = b_{10}^{n_i} y_i = 0$, and so if $b_{10}^{n_i} x$ is to support a differential $d_{r_{n_i}}$, we must have $r_{n_i} > r_i$. Note that the condition on r in Corollary 4.4.4 is the same for all $b_{10}^i x$. So some of the r_i's will be greater than this bound, contradicting the assumption that $d_{r_i}(b_{10}^i x) \neq 0$. □

4.5 Identifying the b_{10}-periodic region

In this section, we determine a line of slope $\frac{1}{23}$ above which $\mathrm{Ext}^*_P(k,k)$ is b_{10}-periodic. Our main input is the following theorem, which Palmieri states for the Steenrod dual A instead of P, as we do below. The only difference is that, over A, one must also take into account the objects $K(\tau_n)$, which do not come into play over P. (This reasoning follows from the discussion in [Pal01, §2.3.2].)

Recall in Section 2.3.2, we defined $s(\xi_t^{p^s}) = \frac{1}{2}p|\xi_t^{p^s}| = p^{s+1}(p^t-1)$ and discussed how this related to vanishing lines on Adams E_2 pages.

Theorem 4.5.1 ([Pal01, Theorem 2.3.1]). *Suppose X is a spectrum in* Stable(P) *satisfying the following conditions:*

(1) There exists an integer i_0 such that $\pi_{i,*}X = 0$ if $i < i_0$,

(2) There exists an integer j_0 such that $\pi_{i,j}X = 0$ if $j - i < j_0$,

(3) There exists an integer i_1 such that the homology of the cochain complex X vanishes in homological degree $> i_1$. (In particular, this is satisfied if X is the resolution of a bounded-below comodule.)

Suppose $d = s(\xi_{t_0}^{p^{s_0}})$ (with $s_0 < t_0$) has the property that $K(\xi_t^{p^s})_{**}(X) = 0$ for all (s,t) with $s < t$ and $s(\xi_t^{p^s}) < d$. Then $\pi_{**}X$ has a vanishing line of slope d: for some c, $\pi_{i,j}X = 0$ when $j < di - c$.

As elsewhere in this thesis, we abuse notation by identifying a P-comodule with its image in Stable(P), and here we take that one step further by writing k/b_{10} for the cofiber in Stable(P) of $b_{10} \in \mathrm{Ext}^2_P(k,k)$, thought of as a map $k \to k$ in Stable(P). We will make use of the cofiber sequence

$$k \xrightarrow{b_{10}} k[2] \to k/b_{10}[2] \tag{4.5.1}$$

and its induced long exact sequence

$$\cdots \to \mathrm{Ext}_P^{s,t}(k,k) \to \underbrace{\mathrm{Ext}_P^{s+2,t+12}(k,k)}_{\mathrm{Ext}_P^{s,t+12}(k,k[2])} \to \mathrm{Ext}_P^{s+2,t+12}(k,k/b_{10}) \to \mathrm{Ext}_P^{s+1,t}(k,k) \to \dots. \tag{4.5.2}$$

(Here we use $\mathrm{Ext}_P^{**}(k,k/b_{10})$ to denote $\mathrm{Hom}_{\mathrm{Stable}(P)}(k,k/b_{10})$.)

Claim 4.5.2. *The object k/b_{10} satisfies the conditions of Theorem 4.5.1 for $d = 24$.*

Proof. First we check the three homotopy boundedness conditions.

(1) k satisfies the condition for $i_0 = 0$, so (4.5.2) shows that k/b_{10} satisfies the condition for $i_0 = 0$.

(2) k satisfies this condition for $j_0 = 0$, so by (4.5.2), k/b_{10} satisfies the condition for $j_0 = 0$.

(3) k/b_{10} is (the resolution of) the 2-cell complex $k \stackrel{b_{10}}{\to} k$.

Now we check the main condition in Theorem 4.5.1 with $d = 24 = s(\xi_2)$. Since ξ_1 is the first $\xi_t^{p^s}$ with $s < t$ and ξ_2 is the second, we just have to check $K(\xi_1)_{**}(k/b_{10}) = 0$. This is essentially by construction: consider the long exact sequence of (4.5.1) in $K(\xi_t^{p^s})_{**}$:

$$K(\xi_t^{p^s})_{*,*} \xrightarrow{b_{10}} K(\xi_t^{p^s})_{*+2,*+12} \longrightarrow K(\xi_t^{p^s})_{*+2,*+12}(k/b_{10}).$$

Since $(P\,\square_{D[\xi_1]}k)_{**} = B_{**} \cong E[h_{10}] \otimes P[b_{10}]$, the non-connective version $K(\xi_1) = b_{10}^{-1}(P\,\square_{D[\xi_1]}k)$ has $K(\xi_1)_{**} \cong E[h_{10}] \otimes P[b_{10}^{\pm}]$, i.e. multiplication by b_{10} is an isomorphism $K(\xi_1)_{s,*} \to K(\xi_1)_{s+2,*+12}$ for all s, and so the LES shows $K(\xi_1)_{**}(k/b_{10}) = 0$. □

Thus, we can use Palmieri's theorem to conclude that there exists some c such that $\pi_{s,t}(k/b_{10}) = 0$ when $t < 24s - c$. Going back to (4.5.2), we see that multiplication by

b_{10} is an isomorphism in this range; more specifically, from the exact sequence

$$\underbrace{\operatorname{Ext}_P^{s+1,t+12}(k,k/b_{10})}_{0 \text{ if } t+12<24(s+1)-c} \to \operatorname{Ext}_P^{s,t}(k,k) \to \operatorname{Ext}_P^{s+2,t+12}(k,k) \to \underbrace{\operatorname{Ext}_P^{s+2,t+12}(k,k/b_{10})}_{0 \text{ if } t+12<24(s+2)-c}$$

we see that $\operatorname{Ext}_P^{s,t}(k,k) \overset{b_{10}}{\to} \operatorname{Ext}_P^{s+2,t+12}(k,k)$ is an isomorphism if $t < 24s + 12 - c$, or equivalently $\frac{1}{23}(t-s) + \frac{1}{23}(c-12) < s$. If $x \in \operatorname{Ext}_P^{s,t}(k,k)$ is nonzero with s,t satisfying this condition, then so is $b_{10}^k x$ for every k. Therefore:

Proposition 4.5.3. *The localization map* $\operatorname{Ext}_P^{s,t}(k,k) \to b_{10}^{-1}\operatorname{Ext}_P^{s,t}(k,k)$ *is an isomorphism in the range* $s > \frac{1}{23}(t-s) + c'$ *for some constant* c'.

In [Pal01, 2.3.5(c)], Palmieri gives an explicit expression for the constant c, which allows us to calculate the y-intercept in the above line to be $c' \approx 6.39$.

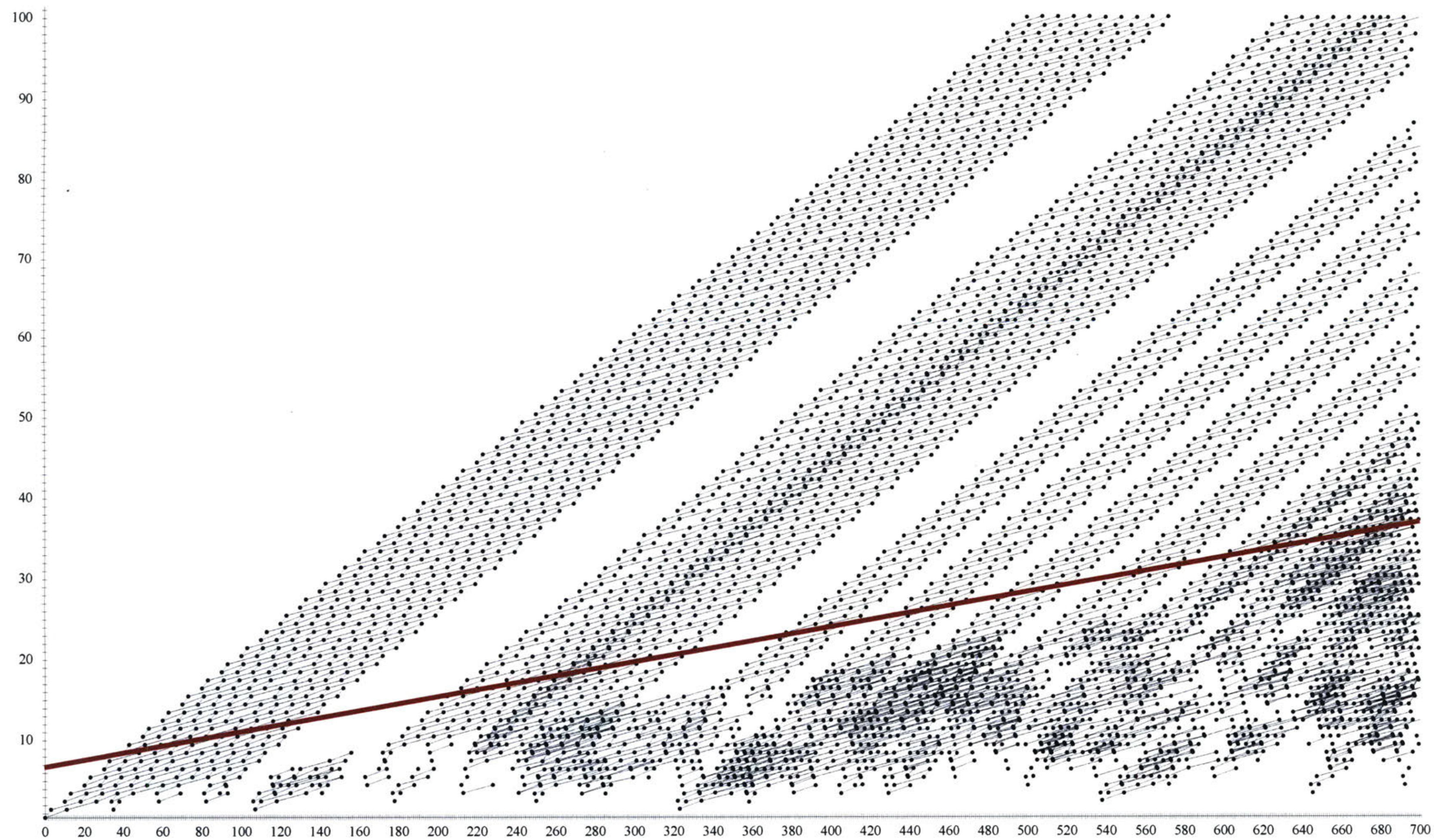


Figure 4-2: Chart of $\mathrm{Ext}^*_P(k,k)$ with the line of Proposition 4.5.3 drawn in red: classes above the line are b_{10}-periodic.

Chapter 5

Computation of d_4

Notation 5.0.1. As in all of the computational parts of this thesis, we are working at $p = 3$. Recall we have set

- $D = k[\xi_1]/\xi_1^3$,
- $K(\xi_1) = b_{10}^{-1}B$ where $B = P\,\square_D k$, and
- $R = K(\xi_1)_{**} = E[h_{10}] \otimes P[b_{10}^{\pm 1}]$.

Finally, recall from 4.1.5 that we have established the convention that the symbol ξ_n means what is usually called $\overline{\xi}_n$: that is, we have

$$\Delta(\xi_n) = \sum_{i+j=n} \xi_i \otimes \xi_j^{p^i}.$$

This makes it easier to talk about $B = P\,\square_D k$, which can be written $k[\xi_1^3, \xi_2, \xi_3, \dots]$ with the above unconventional notation.

5.1 Overview of the computation

In the previous chapter, we've shown that the $K(\xi_1)$-based MPASS computing $b_{10}^{-1}\operatorname{Ext}_P(k,k)$ has the form

$$E_2^{**} = E[h_{10}] \otimes P[b_{10}^{\pm 1}, w_2, w_3, \dots] \implies b_{10}^{-1}\operatorname{Ext}_P^*(k,k)$$

where $s(w_n) = 1$, $t(w_n) = 1$, and $u(w_n) = 2(3^n+1)$. (Recall s is Adams filtration, t is internal homological degree, and u is internal topological degree, so $E_1^{s,t,u} = \operatorname{Ext}_P^{t,u}(k, B \otimes \overline{B}^{\otimes s})$.)

Proposition 5.1.1. *If $r \geq 2$ and $r \not\equiv 4 \pmod 9$ or $r \not\equiv 8 \pmod 9$, then $d_r = 0$. Furthermore, if we let $W^+ = P[b_{10}^{\pm 1}, w_2, w_3, \dots]$ and $W^- = W^+\{h_{10}\}$, then*

$$d_{4+9n}(W^+) \subset W^- \qquad d_{4+9n}(W^-) = 0$$

$$d_{8+9n}(W^+) = 0 \qquad d_{8+9n}(W^-) \subset W^+.$$

Proof. This is a degree argument, so we simplify to considering $d_r(x)$ where x is a monomial.

Case 1: $x = w_{k_1} \dots w_{k_d}$ and $d_r(x) = b_{10}^N w_{n_1} \dots w_{n_{d+r}}$. I claim this is not possible because $t(d_r(x))$ has the wrong parity. Recall that $t(w_n) = 1$, $t(b_{10}) = 2$, and $t(d_r(x)) = 1 - r + t(x)$. But here we have $t(d_r(x)) + r - t(x) = (2N + d + r) + r - d = 2N + 2r \not\equiv 1 \pmod 2$.

Case 2: $x = w_{k_1} \dots w_{k_d}$ and $d_r(x) = b_{10}^N h_{10} w_{n_1} \dots w_{n_{d+r}}$. We will measure degree using $u' := u - 6(s+t)$; this has the property that $u'(b_{10}) = 0$, $u'(h_{10}) = -2$, and $u'(w_k) = 2(3^k - 5)$ for all k. Furthermore $u'(d_r(x)) = u'(x) - 6$. Using the fact that

$u'(w_n) = 2(3^n - 5)$, $u'(h_{10}) = -2$, and $u'(b_{10}) = 0$, this equality becomes

$$\sum_{i=1}^{d+r} 2(3^{n_i} - 5) - 2 = \sum_{j=1}^{d} 2(3^{k_j} - 5) - 6$$
$$\sum_{i=1}^{d+r} 3^{n_i} - \sum_{j=1}^{d} 3^{k_j} = 5r - 2$$

Since $n_i, k_j \geq 2$ for all i, j, taking this mod 9 gives $0 \equiv 5r-2 \pmod 9$, so $r \equiv 4 \pmod 9$.

Case 3: $x = h_{10}w_{k_1}\dots w_{k_d}$ and $d_r(x) = b_{10}^N w_{n_1}\dots w_{n_{d+r}}$. As in Case 2, $u'(d_r(x)) = u'(x) - 6$ becomes:

$$\sum_{i=1}^{d+r} 2(3^{n_i} - 5) = \sum_{j=1}^{d} 2(3^{k_j} - 5) - 2 - 6$$
$$\sum_{i=1}^{d+r} 3^{n_i} - \sum_{j=1}^{d} 3^{k_j} = 5r - 4$$

and taking this mod 9 yields $r \equiv 8 \pmod 9$.

Case 4: $x = h_{10}w_{k_1}\dots w_{k_d}$ and $d_r(x) = b_{10}^N h_{10} w_{n_1}\dots w_{n_{d+r}}$. This can't happen for the same reason as Case 1. □

So the next possibly nontrivial differential is d_4.

Proposition 5.1.2. *We have the following:*

$$d_r(h_{10}) = 0 \text{ for } r \geq 2$$
$$d_r(w_2) = 0 \text{ for } r \geq 2$$
$$d_4(w_3) = \pm b_{10}^{-1} h_{10} w_2^5$$
$$d_4(w_4) = \pm b_{10}^{-1} h_{10} w_2^2 w_3^3.$$

Proof. The first two facts can be seen directly in the cobar complex $C_P(k, k)$, using the cobar representatives $h_{10} = [\xi_1]$ and $w_2 = [\xi_1|\xi_2] - [\xi_1^2|\xi_1^3]$, which are permanent cycles.

The differentials on w_3 and w_4 were deduced from the chart of $\operatorname{Ext}_P^*(k,k)$ up to the 700 stem that appears as Figure 4-2 (generated by the software [Nas]). In Proposition 4.5.3, we showed that $\operatorname{Ext}_P^*(k,k)$ agreed with $b_{10}^{-1}\operatorname{Ext}_P^*(k,k)$ in the range of dimensions depicted in the chart. Thus we know which classes in $E_2 = R \otimes P[w_2, w_3, \dots]$ in this range of dimensions die in the spectral sequence, and, using multiplicativity of the spectral sequence, this forces the differentials above. □

The goal of this chapter is to prove the following:

Theorem 5.1.3. *For $n \geq 5$, there is a differential in the MPASS*

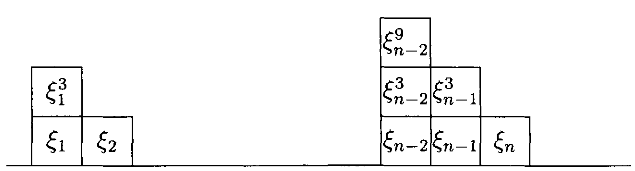
$$d_4(w_n) = \pm b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3.$$

Since the spectral sequence is multiplicative and $d_4(h_{10}) = 0$, this determines d_4.

The main idea is to use comparison with the MPASS computing $b_{10}^{-1}\operatorname{Ext}_{P_n}(k,k)$, where

$$P_n = k[\xi_1, \xi_2, \xi_{n-2}, \xi_{n-1}, \xi_n]/(\xi_1^9, \xi_2^3, \xi_{n-2}^{27}, \xi_{n-1}^9, \xi_n^3).$$

(The idea is that this is the smallest algebra in which the desired differential can be seen.) This is a quotient Hopf algebra of P by the classification of such (see [Pal01, Theorem 2.1.1.(a)]). Here's a picture:



Let $B = P \square_D k$ and $B_n = P_n \square_D k$. There is a $b_{10}^{-1}B_n$-based MPASS computing $b_{10}^{-1}\operatorname{Ext}_{P_n}(k,k)$, which we will denote $E_r(k, B_n)$. Let $E_r(k, B)$ denote the $b_{10}^{-1}B$-based

MPASS for $b_{10}^{-1}\operatorname{Ext}_P(k,k)$ discussed above. Then the diagram

$$\begin{array}{ccccc} B & \longrightarrow & P & \longrightarrow & D \\ \downarrow & & \downarrow & & \| \\ B_n & \longrightarrow & P_n & \longrightarrow & D \end{array}$$

shows there is a map of spectral sequences $E_r(k,B) \to E_r(k,B_n)$.

Lemma 5.1.4. *It suffices to show that $d_4(w_n) \neq 0$ in $E_4(k,B)$.*

Proof. Since $s(d_4(w_n)) = 4 + s(w_n) = 5$, we know that $d_4(w_n)$ is a linear combination of terms of the form $b_{10}^N h_{10} w_{k_1} \dots w_{k_5}$. Using the grading $u' := u - 6(s+t)$, we have

$$\begin{aligned} u'(w_n) &= u'(b_{10}^N h_{10} w_{k_1} \dots w_{k_5}) + 6 \\ 2(3^n - 5) &= -2 + \sum_{i=1}^{5} 2(3^{k_i} - 5) + 6 \\ 3^n + 18 &= \sum_{i=1}^{5} 3^{k_i} \end{aligned}$$

Note that $k_i \geq 2$. Looking at this mod 9, we see that two of the k_i's have to be $= 2$, say k_1 and k_2. Then we have $3^n = 3^{k_3} + 3^{k_4} + 3^{k_5}$. The only possibility is $n - 1 = k_3 = k_4 = k_5$. So if $d_4(w_n) \neq 0$ then $d_4(w_n) = b_{10}^N h_{10} w_2^2 w_{n-1}^3$, and checking internal degrees shows $N = -4$. □

When we discuss $E_r(k,B_n)$ it will be easy to see that there is a class $w_n \in E_2(k,B_n)$ which is the target of $w_n \in E_2(k,B)$ along the quotient map.

$$\begin{array}{ccc} E_4(k,B) & \xrightarrow{d_4} & E_4(k,B) \\ \downarrow & & \downarrow \\ E_4(k,B_n) & \xrightarrow{d_4} & E_4(k,B_n) \end{array}$$

Lemma 5.1.4 says that it suffices to show $d_4(w_n) \neq 0$ in $E_4(k,B_n)$, but it turns out to be the same amount of work to show the following more attractive statement.

Goal 5.1.5. There is a differential $d_4(w_n) = \pm b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ in $E_r(k, B_n)$.

Using the same argument as Proposition 5.1.1, we know that $d_2 = 0 = d_3$ in $E_r(k, B_n)$, so $h_{10} w_2^2 w_{n-1}^3$ is not the target of an earlier differential.

We will use the following strategy to show the desired differential in $E_r(k, B_n)$:

(1) Calculate $E_2(k, B_n)$ in a region and identify classes w_2, w_{n-1}, w_n that are the targets of their namesake classes under the quotient map $E_2(k, B) \to E_2(k, B_n)$.

(2) Show that $b_{10}^{-1} \operatorname{Ext}_{P_n}^*(k, k)$ is zero in the stem of $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$. This implies that $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ either supports a differential or is the target of a differential.

(3) Show that $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ is a permanent cycle in the MPASS (so it must be the target of a differential) and show that, for degree reasons, w_n is the only element that can hit it. By looking at filtrations, we see this differential is a d_4.

In order to show (2), we introduce another spectral sequence for calculating $b_{10}^{-1} \operatorname{Ext}_{P_n}^*(k, k)$, the Ivanovskii spectral sequence (ISS). This is the (b_{10}-localized version of the) dual of the May spectral sequence; that is, it is the spectral sequence obtained by filtering the cobar complex on P_n by powers of the augmentation ideal.

In Section 5.2 we will introduce notation and record facts about gradings. In Section 5.3 we will compute $E_1(k, B_n)$ and the relevant part of $E_2(k, B_n)$, and show (1) and (3) assuming (2). In Section 5.4 we will calculate the relevant part of the ISS and show (2). Convergence of the localized ISS is discussed in Section 5.5.

5.2 Notation and gradings

Since most of the work in this chapter consists of degree-counting arguments, we will now record how differentials and convergence affect the various gradings at play. We also introduce a change of coordinates on degrees that simplifies degree arguments by putting b_{10} in degree zero.

MPASS gradings

As mentioned above, when working with MPASS's we use the grading (s, t, u) where s is MPASS filtration, t is internal homological degree, and u is internal topological degree, such that $E_1^{s,t,u} = \mathrm{Ext}_P^{t,u}(k, B \otimes \overline{B}^{\otimes s})$. This has the property that the differential has the form

$$d_r : E_r^{s,t,u} \longrightarrow E_r^{s+r,t-r+1,u}$$

and a permanent cycle in $E_r^{s,t,u}$ converges to an element in $b_{10}^{-1}\,\mathrm{Ext}_P^{s+t,u}(k,k)$.

Instead of working with the grading (s, t, u), we perform a change of coordinates by setting

$$u' := u - 6(s + t)$$

and track (u', s) instead. This is more convenient because $u'(b_{10}) = 0 = s(b_{10})$, so all classes in a b_{10}-tower have the same (u', s)-degree. The differential under the change of coordinates has the form

$$d_r : E_r^{u',s} \longrightarrow E_r^{u'-6,s+r}$$

and a permanent cycle in $E_r^{u',s}$ converges to an element in $b_{10}^{-1}\,\mathrm{Ext}_P^{a,b}(k,k)$ (where b is internal topological degree and a is homological degree) with $b - 6a = u'$.

Definition 5.2.1. Let *stem* in $b_{10}^{-1}\,\mathrm{Ext}_P^{a,b}(k,k)$ denote the quantity $b - 6a$. Then a permanent cycle in $E_r^{u',s}$ converges to an element in the u' stem.

Finally, define

$$u'' := u - 6t.$$

This is only useful for looking at the E_1 page of the MPASS, as d_1 fixes u''.

ISS gradings

The *Ivanovskii spectral sequence* computing $b_{10}^{-1}\operatorname{Ext}_{P_n}(k,k)$ is the spectral sequence obtained by filtering the cobar complex on P_n by powers of the augmentation ideal: for example, $[\xi_1\xi_2|\xi_{n-1}^3]$ has filtration $2+3=5$. Let E_r^{ISS} denote the E_r page of the Ivanovskii spectral sequence.

We use slightly different grading conventions: classes have degree (s,t,u) where s is ISS filtration, t denotes degree in the cobar complex, and u denotes internal topological degree (as in the MPASS). The differential has the form

$$d_r^{ISS} : E_r^{s,t,u} \longrightarrow E_r^{s+r,t+1,u}$$

and a permanent cycle in $E_r^{s,t,u}$ converges to an element in $b_{10}^{-1}\operatorname{Ext}_P^{t,u}(k,k)$.

We will use the change of coordinates

$$u' := u - 6t$$

which is designed so that $u'(b_{10}) = 0$. (This has a different formula from the MPASS change of coordinates simply because (s,t,u) correspond to different parameters here.) The differential has the form

$$d_r^{ISS} : E_r^{u',s} \longrightarrow E_r^{u'-6,s+r}$$

and a permanent cycle in $E_r^{u',s}$ converges to an element in $b_{10}^{-1}\operatorname{Ext}_P^{a,b}(k,k)$ with $u' = b-6a$, i.e. an element in the u' stem.

Note that u' has different formulas for the MPASS and ISS, but in both spectral sequences u' corresponds to stem, with the definition given above. Now we will introduce another grading on P_n (for $n \geq 5$) preserved by the comultiplication.

Extra grading on P_n

Let $P'_n = k[\xi_1, \xi_2, \xi_{n-2}^3, \xi_{n-1}, \xi_n]/(\xi_1^9, \xi_2^3, \xi_{n-2}^{27}, \xi_{n-1}^9, \xi_n^3)$. Note that every monomial in P_n can be written $\xi_{n-2}^e x$ where $e \in \{0, 1, 2\}$ and $x \in P'_n$.

Lemma 5.2.2. *For $n \geq 5$, P'_n is a sub-coalgebra of P_n.*

Proof. This is clear from the comultiplication formulas

$$\begin{aligned} \Delta(\xi_n) &= 1 \otimes \xi_n + \xi_1 \otimes \xi_{n-1}^3 + \xi_2 \otimes \xi_{n-2}^9 \\ \Delta(\xi_{n-1}) &= 1 \otimes \xi_{n-1} + \xi_1 \otimes \xi_{n-2}^3 + \xi_{n-1} \otimes 1 \\ \Delta(\xi_{n-2}^3) &= 1 \otimes \xi_{n-2}^3 + \xi_{n-2}^3 \otimes 1 \end{aligned} \tag{5.2.1}$$

and the assumption $n \geq 5$ guarantees that $\xi_1, \xi_2 \neq \xi_{n-2}$. □

Proposition 5.2.3. *Let $n \geq 3$. There is an extra grading α on P_n that respects the comultiplication, defined by the property that it is multiplicative on P'_n, and*

$$\begin{aligned} \alpha(\xi_1) = \alpha(\xi_2) &= \alpha(\xi_{n-2}) = 0, \\ \alpha(\xi_{n-2}^3) &= \alpha(\xi_{n-1}) = 3, \\ \alpha(\xi_n) &= 9, \\ \alpha(\xi_{n-1}^e x) &= \alpha(x) \text{ for } e \in \{0,1,2\} \text{ and } x \in P'_n. \end{aligned}$$

Proof. First we check that α respects the comultiplication when restricted to P'_n. Since it is defined to be multiplicative on P'_n, it suffices to check that $\alpha(y) = \alpha(\Delta y)$ for y as each of the multiplicative generators. This is clear from the comultiplication formulas (5.2.1).

Now suppose $y = \xi_{n-2}x$ where $x \in P'_n$. We have

$$\Delta(\xi_{n-2}x) = (1 \otimes \xi_{n-2} + \xi_{n-2} \otimes 1)\Delta x = \sum \left(x' \otimes x''\xi_{n-2} + x'\xi_{n-2} \otimes x''\right)$$

and the α degrees of both sides agree since P'_n is a coalgebra. Similarly, if $y = \xi_{n-2}^2 x$ for $x \in P'_n$, we have

$$\begin{aligned}\alpha(\Delta y) &= \alpha((1 \otimes \xi_{n-2}^3 + 2\xi_{n-2} \otimes \xi_{n-2} + \xi_{n-2}^2 \otimes 1)(\Delta x)) \\ &= \alpha\left(\sum x' \otimes \xi_{n-2}^2 x'' + 2\xi_{n-2}x' \otimes \xi_{n-2}x'' + \xi_{n-2}^2 x' \otimes x''\right) = \alpha(\Delta x).\end{aligned}$$

□

5.3 The E_2 page of the $b_{10}^{-1}B_n$-based MPASS

The goal of this section is to prove the following:

Proposition 5.3.1. *If $b_{10}^{-4}h_{10}w_2^2w_{n-1}^3$ is the target of a differential in the $b_{10}^{-1}B_n$-based MPASS calculating $b_{10}^{-1}\operatorname{Ext}^*_{P_n}(k,k)$, that differential must be*

$$d_4(w_n) = \pm b_{10}^{-4}h_{10}w_2^2w_{n-1}^3.$$

The main task is to calculate enough of $E_2(k, B_n)$ to do a degree-counting argument (Proposition 5.3.9), where

$$B_n = P_n \square_D k = k[\xi_1^3, \xi_2, \xi_{n-2}, \xi_{n-1}, \xi_n]/(\xi_1^9, \xi_2^3, \xi_{n-2}^{27}, \xi_{n-1}^9, \xi_n^3).$$

As in the calculation of the E_2 page of the $b_{10}^{-1}B$-based MPASS (Section 4.3), the

Künneth formula for the functor $b_{10}^{-1}\operatorname{Ext}_D^*(k,-)$ (Lemma 4.3.5) guarantees flatness of $(b_{10}^{-1}B_n)_{**}(b_{10}^{-1}B_n)$ over $(b_{10}^{-1}B_n)_{**}$. So we can use the formula

$$E_2 \cong \operatorname{Ext}^*_{(b_{10}^{-1}B_n)_{**}b_{10}^{-1}B_n}((b_{10}^{-1}B_n)_{**},(b_{10}^{-1}B_n)_{**}) \tag{5.3.1}$$

where $(b_{10}^{-1}B_n)_{**} = b_{10}^{-1}\operatorname{Ext}^*_{P_n}(k,B_n) = R$ and $(b_{10}^{-1}B_n)_{**}(b_{10}^{-1}B_n) = b_{10}^{-1}\operatorname{Ext}^*_{P_n}(R,B_n^{\otimes 2}) \cong b_{10}^{-1}\operatorname{Ext}^*_D(k,B_n)$ by the change of rings theorem. We will simultaneously determine the vector space structure and the comultiplication on $b_{10}^{-1}\operatorname{Ext}_D(k,B_n)$.

Remark 5.3.2. By the definition of the MPASS (Definition 2.2.2), the change of rings theorem, and the Künneth formula mentioned above, we have

$$E_1^{s,*}(k,B_n) \cong b_{10}^{-1}\operatorname{Ext}^*_P(k,B_n\otimes \overline{B}_n^{\otimes s}) \cong b_{10}^{-1}\operatorname{Ext}^*_D(k,\overline{B}_n^{\otimes s}) \cong b_{10}^{-1}\operatorname{Ext}_D(k,\overline{B}_n)^{\otimes s}$$

and so the coproduct on $b_{10}^{-1}\operatorname{Ext}^*_D(k,B_n)$ coincides with d_1 on $E_1^{1,*}$.

We can write B_n as a tensor product

$$B_n = k[\xi_2,\xi_1^3]/(\xi_2^3,\xi_1^9)\otimes k[\xi_{n-2}]/\xi_{n-2}^3\otimes k[\xi_{n-1},\xi_{n-2}^3]/(\xi_{n-1}^3,\xi_{n-2}^{27})\otimes k[\xi_n,\xi_{n-1}^3]/(\xi_n^3,\xi_{n-1}^9)$$

illustrated in Figure 5-1.

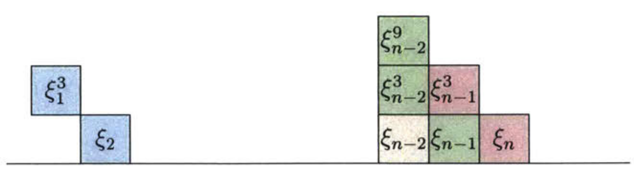


Figure 5-1: Illustration of the decomposition of B_n into tensor factors

Since we have a Künneth formula for $b_{10}^{-1}\operatorname{Ext}_D^*(k,-)$, it suffices to apply this functor to each of the four factors of B_n above.

Factor 1: $k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9)$

We can explicitly see that this breaks up as a D-comodule as

$$k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9) \cong \underbrace{k\{1\}}_{\cong k} \oplus \underbrace{k\{\xi_2, \xi_1^3\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_2^2, \xi_1^3\xi_2, \xi_1^6\}}_{\cong D} \oplus \underbrace{k\{\xi_1^3\xi_2^2, \xi_1^6\xi_2\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_1^6\xi_2^2\}}_{\cong k}. \tag{5.3.2}$$

(Recall $M(1)$ was defined to be the D-comodule $k[\xi_1]/\xi_1^2$, and every D-comodule is a sum of copies of k, $M(1)$, and D.) As a module over $R := E[h_{10}] \otimes P[b_{10}^{\pm 1}]$, this is generated by a class $e_2 = e(\xi_2)$ in $b_{10}^{-1} \operatorname{Ext}_D^1(k, k\{\xi_2, \xi_1^3\})$, a class $f_{20} = e(\xi_1^3\xi_2^2)$ in $b_{10}^{-1} \operatorname{Ext}_D^1(k, k\{\xi_1^3\xi_2^2, \xi_1^6\xi_2\})$, and a class c_2 in $b_{10}^{-1} \operatorname{Ext}_D^0(k, k\{\xi_1^6\xi_2^2\})$. As $b_{10}^{-1} \operatorname{Ext}_D^*(k, D) = 0$, we may ignore the free summands.

Using Lemma 4.3.7, we can give explicit representatives for the classes in $b_{10}^{-1} \operatorname{Ext}_D^*(k, k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9))$ coming from the decomposition (5.3.2):

$$\begin{aligned} e_2 &:= e(\xi_2) = [\xi_1]\xi_2 - [\xi_1^2]\xi_1^3 \in \operatorname{Ext}_D^*(k, k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9)) \\ f_{20} &:= e(\xi_1^3\xi_2^2) = [\xi_1]\xi_1^3\xi_2^2 + [\xi_1^2]\xi_1^6\xi_2 \\ c_2 &= \xi_1^6\xi_2^2 \end{aligned}$$

satisfying relations $e_2^2 = 0 = f_{20}^2$ and $b_{10}c_2 = e_2 f_{20}$.

Lemma 5.3.3. *The classes e_2 and f_{20} are primitive in the coalgebra $b_{10}^{-1} \operatorname{Ext}_D^*(k, B_n)$.*

Proof. By the results of Section 3.3, we can interpret the MASS as a filtration spectral sequence on the cobar complex $C_{P_n}(k, k)$, where $[a_1|\dots|a_s]$ is in filtration n if $\geq n$ a_i's are in $\overline{B}_n P_n$. The elements e_2 and f_{20} correspond to elements in $F^1/F^2 C_{P_n}^2(k, k)$ with the same formulas, and by Remark 5.3.2 it suffices to show that $d_1(e_2) = 0 = d_1(f_{20})$ in the filtration spectral sequence. One checks explicitly that $d_{\text{cobar}}(e_2) = 0$, so it is a permanent cycle. This is not true of f_{20}, but we can write down explicit correcting terms in higher filtration:

$$f_{20} \equiv \widetilde{f}_{20} := [\xi_2|\xi_2^2] + [\xi_2^2|\xi_2] - [\xi_1\xi_2|\xi_2\xi_1^3] + [\xi_1\xi_2^2|\xi_1^3] + [\xi_1^2\xi_2|\xi_1^6] + [\xi_1^2|\xi_2\xi_1^6] + [\xi_1|\xi_2^2\xi_1^3]$$

and then check that $d_{\text{cobar}}(\widetilde{f}_{20}) = [\xi_1^3|\xi_1^6|\xi_1^3] + [\xi_1^3|\xi_1^3|\xi_1^6]$. This has filtration 3, and so $d_1(f_{20}) = 0$. □

So we've proved:

Proposition 5.3.4. *There is an isomorphism of Hopf algebras*

$$b_{10}^{-1}\operatorname{Ext}_D(k, k[\xi_2, \xi_1^3]/(\xi_2^3, \xi_1^9)) \cong R \otimes E[e_2, f_{20}]$$

where e_2 and f_{20} are primitive.

We can summarize the degree information as follows:

element	s	t	u	$u'' = u - 6t$	α
1	0	0	0	0	0
h_{10}	0	1	4	-2	0
b_{10}	0	2	12	0	0
$e_2 = [\xi_1]\xi_2 - [\xi_1^2]\xi_1^3$	1	1	20	14	0
$f_{20} = [\xi_1]\xi_1^3\xi_2^2 + [\xi_1^2]\xi_1^6\xi_2$	1	1	48	42	0
$c_2 = \xi_1^6\xi_2^2$	1	0	56	56	0

Factor 2: $k[\xi_{n-2}]/\xi_{n-2}^3$

This decomposes as $k\{1\} \oplus k\{\xi_{n-2}\} \oplus k\{\xi_{n-2}^2\}$ so we have three R-module generators:

element	s	t	u	$u'' = u - 6t$	α
1	0	0	0	0	0
ξ_{n-2}	1	0	$2(3^{n-2}-1)$	$2(3^{n-2}-1)$	0
ξ_{n-2}^2	1	0	$2 \cdot 2(3^{n-2}-1)$	$2 \cdot 2(3^{n-2}-1)$	0

As a Hopf algebra we have

$$b_{10}^{-1} \operatorname{Ext}_D(k, k[\xi_{n-2}]/\xi_{n-2}^3) \cong R \otimes D[\xi_{n-2}].$$

Factor 3: $k[\xi_{n-1}, \xi_{n-2}^3]/(\xi_{n-1}^3, \xi_{n-2}^{27})$

Similarly to (5.3.2), for the third factor of B_n we have a D-comodule decomposition

$$k[\xi_{n-1}, \xi_{n-2}^3]/(\xi_{n-1}^3, \xi_{n-2}^{27}) \cong \underbrace{k\{1\}}_{\cong k} \oplus \underbrace{k\{\xi_{n-1}, \xi_{n-2}^3\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_{n-1}^2\xi_{n-2}^{21}, \xi_{n-1}\xi_{n-2}^{24}\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_{n-1}^2\xi_{n-2}^{24}\}}_{\cong k} \oplus F$$

where F is a free D-comodule, which gives the following R-module generators of $b_{10}^{-1} \operatorname{Ext}_D^*(k, k[\xi_{n-1}, \xi_{n-2}^3]/(\xi_{n-1}^3, \xi_{n-2}^{27}))$:

element	s	t	u	$u'' = u - 6t$	α
1	0	0	0	0	0
$e_{n-1} := [\xi_1]\xi_{n-1} - [\xi_1^2]\xi_{n-2}^3$	1	1	$2(3^{n-1}+1)$	$2(3^{n-1}-2)$	3
$y_{n-1} := [\xi_1]\xi_{n-1}^2\xi_{n-2}^{21} + [\xi_1^2]\xi_{n-1}\xi_{n-2}^{24}$	1	1	$2(3^{n+1}-21)$	$2(3^{n+1}-24)$	27
$z_{n-1} := \xi_{n-1}^2\xi_{n-2}^{24}$	1	0	$2(3^{n+1}+3^{n-1}-26)$	$2(3^{n+1}+3^{n-1}-26)$	30

Lemma 5.3.5. *e_{n-1} is a permanent cycle in $E_r(k, B_n)$. In particular, $d_1(e_{n-1}) = 0$.*

Proof. Use the filtration spectral sequence interpretation of the MPASS described in the proof of Lemma 5.3.3, where e_{n-1} has representative

$$[\xi_1|\xi_{n-1}] - [\xi_1^2|\xi_{n-2}^3]$$

in $C_{P_n}(k,k)$. It is clear that this is a cycle in $C_{P_n}(k,k)$, hence a permanent cycle in the spectral sequence. □

Factor 4: $k[\xi_n, \xi_{n-1}^3]/(\xi_n^3, \xi_{n-1}^9)$

There is a D-comodule decomposition

$$k[\xi_n, \xi_{n-1}^3]/(\xi_n^3, \xi_{n-1}^9) \cong \underbrace{k\{1\}}_{\cong k} \oplus \underbrace{k\{\xi_n, \xi_{n-1}^3\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_n^2, \xi_{n-1}^3\xi_n, \xi_{n-1}^6\}}_{\cong D} \oplus \underbrace{k\{\xi_{n-1}^3\xi_n^2, \xi_{n-1}^6\xi_n\}}_{\cong M(1)} \oplus \underbrace{k\{\xi_{n-1}^6\xi_n^2\}}_{\cong k}.$$

The non-free summands lead to R-module generators of $b_{10}^{-1}\operatorname{Ext}_D^*(k, k[\xi_n, \xi_{n-1}^3]/(\xi_n^3, \xi_{n-1}^9))$ which have representatives (in order):

element	s	t	u	$u'' = u - 6t$	α
1	0	0	0	0	0
$e_n := [\xi_1]\xi_n - [\xi_1^2]\xi_{n-1}^3$	1	1	$2(3^n+1)$	$2(3^n-2)$	9
$f_{n0} := [\xi_1]\xi_{n-1}^3\xi_n^2 - [\xi_1^2]\xi_{n-1}^6\xi_n$	1	1	$2(3^{n+1}-3)$	$2(3^{n+1}-6)$	27
$c_n := \xi_{n-1}^6\xi_n^2$	1	0	$2(3^{n+1}+3^n-8)$	$2(3^{n+1}+3^n-8)$	36

Corollary 5.3.6. *There is an isomorphism of R-modules*

$$b_{10}^{-1}\operatorname{Ext}_D(k, B_n) \cong R\{1, e_2, f_{20}, c_2\} \otimes R\{1, \xi_{n-2}, \xi_{n-2}^2\}$$
$$\otimes R\{1, e_{n-1}, y_{n-1}, z_{n-1}\} \otimes R\{1, e_n, f_{n,0}, c_n\}.$$

We have already computed part of the Hopf algebra structure on $b_{10}^{-1}\operatorname{Ext}_D(k, B_n) =$

$E_1^{1,*}(k, B_n)$ but do not need to finish this; we just need one more piece of information.

Lemma 5.3.7. *e_n is primitive in $b_{10}^{-1}\operatorname{Ext}_D(k, B_n)$*

Proof. Write $\psi(e_n) = \sum_i c[x_i|y_i]$, where $c \in R$ and $x_i, y_i \in b_{10}^{-1}\operatorname{Ext}_D(k, B_n)$. As the cobar differential preserves the grading α (see Proposition 5.2.3) and ψ can be given in terms of the cobar differential (see e.g. Remark 5.3.2), ψ also preserves α. Since $\alpha(e_n) = 9$, in order for $d_1(e_n)$ to have $\alpha = 9$, we need $\alpha(x_i) + \alpha(y_i) = 9$. Looking at α degrees in the above charts of R-module generators in $b_{10}^{-1}\operatorname{Ext}_D(k, B_n)$, the only options are for $e_n \mid x_i$ or y_i, or for $e_{n-1}^2 \mid x_i$ or y_i. But $e_{n-1}^2 = 0$ by Lemma 4.3.7, and so the only option is for e_n to be primitive. □

Combining Lemmas 5.3.3, 5.3.5, and 5.3.7 we have:

Corollary 5.3.8. *In $b_{10}^{-1}\operatorname{Ext}_D(k, B_n)$, the elements e_2, f_{20}, e_{n-1}, and e_n are exterior generators in the Hopf algebra sense—they are primitive and square to zero.*

Now we have computed enough of $E_2(k, B_n)$ to show Proposition 5.3.1. If $b_{10}^{-4}h_{10}w_2^2w_{n-1}^3$ (which is in degree $\alpha = 9$, $u' = 2(3^n - 8)$, and $u = 2(3^n + 1)$) is the target of a differential, it must be a d_r for $r \leq 4$ (since the target is in filtration 5), and the source of that differential must have degree $\alpha = 9$, $u' = 2(3^n - 5)$, and $u = 2(3^n + 1)$. Thus it suffices to prove Proposition 5.3.9.

Proposition 5.3.9. *The only element in $E_2(k, B_n)$ with $s \leq 4$, $\alpha = 9$, $u' = 2(3^n - 5)$, and $u = 2(3^n + 1)$ is $\pm w_n$.*

Proof. There is a map $R \otimes E[e_2, f_{20}, e_{n-1}, e_n] \otimes D[\xi_{n-2}] \to b_{10}^{-1}\operatorname{Ext}_D(k, B_n)$ that is an isomorphism on degree $u'' < 2(3^{n+1} - 24)$ and induces a map on cobar complexes

$$C^s_{R\otimes E[e_2,f_{20},e_{n-1},e_n]\otimes D[\xi_{n-2}]}(R, R) \to C^s_{b_{10}^{-1}\operatorname{Ext}_D(k,B_n)}(R, R).$$

I claim the map of cobar complexes is an isomorphism in degree $u'' < -2 + 2(3^{n+1} - 24) + 14(s-1)$. One can see this by noting that a minimal-degree element in $C^s_{b_{10}^{-1}\operatorname{Ext}_D(k,B_n)}(R,R)$ not in the image is $h_{10}[y_{n-1}|e_2|\dots|e_2]$, in degree $-2+2(3^{n+1} - 24) + 14(s-1)$. (We use u'' degree here because it is additive with respect to multiplication within $b_{10}^{-1}\operatorname{Ext}_D(k,B_n) = E_1^{1,*}$, whereas u' degree is additive with respect to multiplication of cohomology classes in $H^*E_1 = E_2$.) Note that the desired degrees $u'' = u' + 6s = 2(3^n - 5) + 6s$ fall into the region described here for every s.

Now we look at the map induced on Ext in this region. Since d_r differentials increase u'' degree by $6(r-1)$ (they preserve u and decrease t by $r-1$) and increase s by r, differentials originating in the region $u'' < -2 + 2(3^{n+1} - 24) + 14(s-1)$ stay in the region, but there might be differentials originating outside the region hitting elements in the region. Instead of showing that the map on Ext is an isomorphism in a smaller region, note that this is already enough for our purposes: we want to check that $\operatorname{Ext}_{b_{10}^{-1}\operatorname{Ext}_D(k,B_n)}(R,R)$ is zero in particular dimensions, and it suffices to check that in $\operatorname{Ext}_{R\otimes E[e_2,f_{20},e_{n-1},e_n]\otimes D[\xi_{n-2}]}(R,R)$.

We have

$$\operatorname{Ext}_{R\otimes E[e_2,f_{20},e_{n-1},e_n]\otimes D[\xi_{n-2}]}(R,R) \cong R\otimes P[w_2,b_{20},b_{n-2,0},w_{n-1},w_n]\otimes E[h_{n-2,0}]$$

where $w_i = [e_i]$, $b_{20} = [f_{20}]$, and $\operatorname{Ext}_{D[\xi_{n-2}]}(R,R) = R\otimes E[h_{n-2,0}]\otimes P[b_{n-2,0}]$. Degree information is as follows:

element	s	t	u	u'	α
w_2	1	1	20	8	0
b_{20}	1	1	48	36	0
$h_{n-2,0}$	1	0	$2(3^{n-2}-1)$	$2(3^{n-2}-1)$	0
$b_{n-2,0}$	2	0	$2(3^{n-1}-3)$	$2(3^{n-1}-3)$	0
w_{n-1}	1	1	$2(3^{n-1}+1)$	$2(3^{n-1}-5)$	3
w_n	1	1	$2(3^n+1)$	$2(3^n-5)$	9
h_{10}	0	1	4	-2	0
b_{10}	0	2	12	0	0

Of course, w_n has the right degree. Any other monomial with the right degree must be in $R \otimes P[w_2, b_{20}, b_{n-2,0}, w_{n-1}] \otimes E[h_{n-2,0}]$, and it is clear from looking at α degree above that it must have the form $w_{n-1}^3 x$ (where $x \in R \otimes P[w_2, b_{20}, b_{n-2,0}] \otimes E[h_{n-2,0}]$). Since $u'(w_{n-1}^3) = 2(3^n - 15)$, we need $u'(x) = 20$, which is not possible using w_2 in degree 8, b_{20} in degree 36, h_{10} in degree -2 (where $h_{10}^2 = 0$), and $h_{n-2,0}$ and $b_{n-2,0}$ in higher degree.

So the element must be $\pm b_{10}^N w_n$, and by checking u degree we see that the power N has to be zero. □

5.4 Degree-counting in the ISS

Recall that $b_{10}^{-4} h_{10} w_2^2 w_{n-1}^3$ has $\alpha = 9$ and $u' = 2(3^n - 8)$; if it were a permanent cycle, it would converge to an element of $b_{10}^{-1} \operatorname{Ext}_{P^n}^{a,b}(k,k)$ with stem $b - 6a = 2(3^n - 8)$ (see Definition 5.2.1) and $\alpha = 9$. The goal of this section is to prove:

Proposition 5.4.1. *The sub-vector space of $b_{10}^{-1} \operatorname{Ext}_{P_n}^{*}(k,k)$ consisting of elements in stem $2(3^n - 8)$ and $\alpha = 9$ is zero.*

We will prove this using a (localized) Ivanovskii spectral sequence (ISS) computing $b_{10}^{-1}\operatorname{Ext}_{P_n}(k,k)$. In our case, the ISS is constructed by filtering the cobar complex for P_n by powers of the augmentation ideal. For example, $[\xi_n]$ is in filtration 1, and in the Milnor diagonal

$$d_{\text{cobar}}([\xi_n]) = [\xi_1|\xi_{n-1}^3] + [\xi_2|\xi_{n-2}^9],$$

$[\xi_1|\xi_{n-1}^3]$ is in filtration 4 (since $[\xi_1]$ is in filtration 1 and $[\xi_{n-1}^3]$ is in filtration 3), and $[\xi_2|\xi_{n-2}^9]$ is in filtration 10. In general, all of the multiplicative generators $\xi_1, \xi_2, \xi_{n-2}, \xi_{n-1}, \xi_n$ are primitive in the associated graded, i.e. they are in $\ker d_0$. To form the b_{10}-localized spectral sequence, take the colimit of multiplication by b_{10}. In Section 5.5 we show that the (localized and un-localized) ISS converges in our case.

So we have $E_0 \cong D[\xi_1, \xi_1^3, \xi_2, \xi_{n-2}, \xi_{n-2}^3, \xi_{n-2}^9, \xi_{n-1}, \xi_{n-1}^3, \xi_n]$ and

$$E_1^{ISS} = E[h_{1i}, h_{20}, h_{n-2,j}, h_{n-1,i}, h_{n0}]_{\substack{i\in\{0,1\}\\ j\in\{0,1,2\}}} \otimes P[b_{10}^{\pm1}, b_{11}, b_{20}, b_{n-2,j}, b_{n-1,i}, b_{n,0}]_{\substack{i\in\{0,1\}\\ j\in\{0,1,2\}}}.$$

Here $h_{ij} = [\xi_i^{3^j}]$ has filtration 3^j and b_{ij} has filtration 3^{j+1}. To help with the degree-counting argument in Proposition 5.4.1, here is a table of the degrees of the multiplicative generators of the E_1 page.

element	s	t	u	$u' = u - 6t$	α
h_{10}	1	1	4	-2	0
b_{10}	3	2	12	0	0
h_{11}	3	1	12	6	0
b_{11}	9	2	36	24	0
h_{20}	1	1	16	10	0
b_{20}	3	2	48	36	0
$h_{n-2,0}$	1	1	$2(3^{n-2}-1)$	$2(3^{n-2}-4)$	0
$b_{n-2,0}$	3	2	$2(3^{n-1}-3)$	$2(3^{n-1}-9)$	0
$h_{n-2,1}$	3	1	$2(3^{n-1}-3)$	$2(3^{n-1}-6)$	3
$b_{n-2,1}$	9	2	$2(3^n-9)$	$2(3^n-15)$	9

$h_{n-2,2}$	9	1	$2(3^n - 9)$	$2(3^n - 12)$	9
$b_{n-2,2}$	27	2	$2(3^{n+1} - 27)$	$2(3^{n+1} - 33)$	27
$h_{n-1,0}$	1	1	$2(3^{n-1} - 1)$	$2(3^{n-1} - 4)$	3
$b_{n-1,0}$	3	2	$2(3^n - 3)$	$2(3^n - 9)$	9
$h_{n-1,1}$	3	1	$2(3^n - 3)$	$2(3^n - 6)$	9
$b_{n-1,1}$	9	2	$2(3^{n+1} - 9)$	$2(3^{n+1} - 15)$	27
$h_{n,0}$	1	1	$2(3^n - 1)$	$2(3^n - 4)$	9
$b_{n,0}$	3	2	$2(3^{n+1} - 3)$	$2(3^{n+1} - 9)$	27

Proof of Proposition 5.4.1. The argument has two parts:

(1) show that (up to powers of b_{10}) the only generators in E_1^{ISS} in degree $(u' = 2(3^n - 8), \alpha = 9)$ are $h_{10}h_{20}h_{n-2,2}$ and $h_{10}h_{11}h_{20}b_{n-2,1}$;

(2) show that those elements are targets of higher differentials in the b_{10}-local ISS.

From looking α degrees we see that no monomial in E_1 in degree $(u' = 2(3^n - 8), \alpha = 9)$ can be divisible by $b_{n-2,2}$, $b_{n-1,1}$, or $b_{n,0}$, and moreover by looking at u' degree we see it is not possible for $b_{n-1,0}$, $h_{n-1,1}$, or $h_{n,0}$ to be a factor of such a monomial. The only monomial of the right degree divisible by $h_{n-2,2}$ is $b_{10}^N h_{10}h_{20}h_{n-2,2}$. Any remaining elements of the right degree are in

$$E[h_{10}, h_{11}, h_{20}, h_{n-2,0}, h_{n-2,1}, h_{n-1,0}] \otimes P[b_{10}^{\pm 1}, b_{11}, b_{20}, b_{n-2,0}, b_{n-2,1}].$$

Of these generators, only $h_{n-2,1}$, $h_{n-1,0}$, and $b_{n-2,1}$ have $\alpha > 0$. Since $h_{n-2,1}^2 = 0 = h_{n-1,0}^2$, a monomial with $\alpha = 9$ needs to be divisible by $b_{n-2,1}$. If $u'(b_{n-2,1}x) = 2(3^n - 8)$ then $u'(x) = 14$, and the only possibility is $x = b_{10}^N h_{10}h_{11}h_{20}$. (Here we are using the assumption $n \geq 5$ to determine that $u'(h_{n-2,0}) = 2(3^{n-2} - 4) \geq 46$, and the elements following it in the chart have greater degree).

This concludes part (1) of the argument; for (2) it suffices to show

$$d_9(h_{10}h_{20}b_{n-1,0}) = h_{10}h_{20}h_{11}b_{n-2,1} - b_{10}h_{10}h_{20}h_{n-2,2} \tag{5.4.1}$$
$$d_9(b_{10}h_{10}h_{n0}) = -b_{10}h_{10}h_{20}h_{n-2,2}. \tag{5.4.2}$$

First, I claim that $h_{10}h_{20}$ is a permanent cycle; it is represented by $[\xi_1|\xi_2] - [\xi_1^2|\xi_1^3] = w_2$, which we've seen is a permanent cycle in the cobar complex. The class $b_{n-1,0}$ has cobar representative $[\xi_{n-1}|\xi_{n-1}^2] + [\xi_{n-1}^2|\xi_{n-1}]$ and

$$\begin{aligned} b_{n-1,0} \equiv [\xi_{n-1}|\xi_{n-1}^2] + [\xi_{n-1}^2|\xi_{n-1}] - [\xi_1\xi_{n-1}|\xi_{n-1}\xi_{n-2}^3] + [\xi_1\xi_{n-1}^2|\xi_{n-2}^3] \\ + [\xi_1^2\xi_{n-1}|\xi_{n-2}^6] + [\xi_1^2|\xi_{n-1}\xi_{n-2}^6] + [\xi_1|\xi_{n-1}^2\xi_{n-2}^3] \in (F^3/F^4)C_{P_n}^2(k,k). \end{aligned}$$

Computing the cobar differential on this class (and remembering that $\xi_{n-3}^9 = 0$ in P_n), we see that $d_9(b_{n-1,0}) = h_{11}b_{n-2,1} - b_{10}h_{n-2,2}$. So

$$d_9(h_{10}h_{20}b_{n-1,0}) = h_{10}h_{20}\, d_9(b_{n-1,0}) = h_{10}h_{20}(h_{11}b_{n-1,1} - b_{10}h_{n-2,2}).$$

We have $h_{10}h_{n0} \equiv [\xi_1|\xi_n] - [\xi_1^2|\xi_{n-1}^3] = w_n \in F^2/F^3$ and there is a cobar differential

$$d_{\text{cobar}}([\xi_1|\xi_n] - [\xi_1^1|\xi_{n-1}^3]) = -[\xi_1|\xi_2|\xi_{n-2}^9] + [\xi_1^2|\xi_1^3|\xi_{n-2}^9].$$

This implies (5.4.2). (We didn't check that $h_{10}h_{20}h_{11}b_{n-2,1}$ and $h_{10}h_{20}b_{10}h_{n-2,2}$ survive to the E_9 page, because that is not necessary: we only have to check that these elements die somehow in the spectral sequence, and if they have already died before the E_9 page, then that is good enough for this argument.) □

5.5 ISS convergence

It is easy to see that the (unlocalized) ISS converges: it is based on a decreasing filtration of the cobar complex that clearly satisfies both $\bigcap_s F^sC_{P_n}(k,k) = \{0\}$

and $\bigcup_s F^s C_{P_n}(k,k) = C_{P_n}(k,k)$. In the rest of this section, we will check that the b_{10}-localized ISS converges; this boils down to the fact that it has a vanishing line parallel to b_{10}. Let E_r^{ISS} denote the E_r page of the unlocalized ISS and $b_{10}^{-1}E_r^{ISS}$ denote the E_r page of the localized ISS.

Lemma 5.5.1. *There is a slope $\frac{1}{4}$ vanishing line in E_1^{ISS} in (u,s) coordinates. That is, if $x \in E_1^{ISS}$ has $s(x) > \frac{1}{4}u(x)$ then $x = 0$.*

Proof. In Section 5.4 we computed the E_1 page:

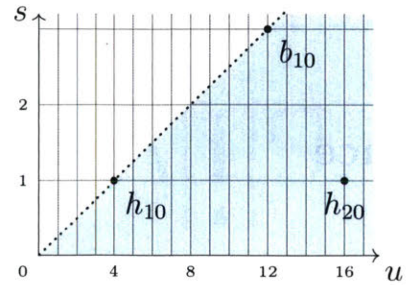
$$E_1^{ISS} = \bigotimes_{(i,j)\in I} E[h_{ij}] \otimes P[b_{ij}]$$

where $I = \{(1,0),(1,1),(2,0),(n-2,0),(n-2,1),(n-2,2),(n-1,0),(n-1,1),(n,0)\}$. These generators occur in the following degrees:

element	u	s	u/s
h_{ij}	$2(3^i-1)3^j$	3^j	$2(3^i-1)$
b_{ij}	$2(3^i-1)3^{j+1}$	3^{j+1}	$2(3^i-1)$

So we have $\frac{u}{s} \geq 2(3^1-1) = 4$, which proves the lemma. Note that b_{10}, in degree $(u=12, s=3)$, acts parallel to the vanishing line. □

Here is a picture:

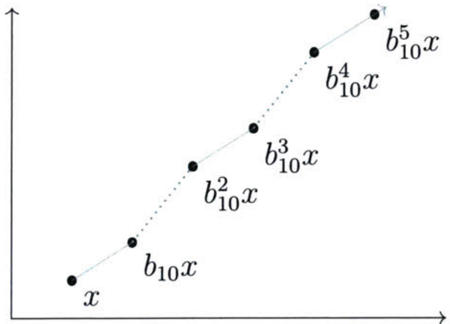


Differentials are vertical: d_r takes elements in degree (u, s) to degree $(u, s + r)$.

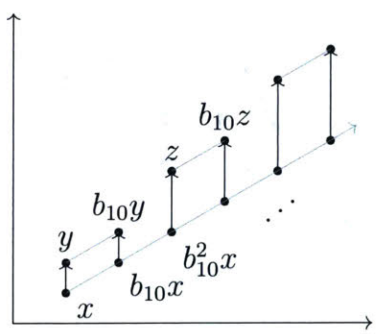
Proposition 5.5.2. *The b_{10}-localized ISS converges.*

Proof. There are two ways convergence could fail:

(1) There could be a b_{10}-tower x in E_∞^{ISS} that does not appear in $b_{10}^{-1}E_\infty$ because it is broken into a series of b_{10}-torsion towers connected by hidden multiplications.



(2) There could be a b_{10}-tower x in $b_{10}^{-1}E_\infty^{ISS}$ that is not a permanent cycle in E_∞^{ISS} because there it supports a series of increasing-length differentials to b_{10}-torsion elements (so these differentials would be zero in $b_{10}^{-1}E_r^{ISS}$).



In both of these cases, it is clear from the pictures that these cannot happen if there is a vanishing line of slope equal to the degree of b_{10}.

(Notice that the reverse of (2) can't happen—the b_{10} tower x can't be hit by a differential originating at a b_{10}-torsion element y, because $d_r(b_{10}^N y) = b_{10}^N d_r(y) = b_{10}^N x \neq 0$ which implies $b_{10}^N y \neq 0$ for all N.) ☐

Remark 5.5.3. The same proof shows that the ISS for $b_{10}^{-1}\operatorname{Ext}_P(k,k)$ converges; in particular, the vanishing line in Lemma 5.5.1 goes through even with more h_{ij}'s and b_{ij}'s in the E_1 page.

Chapter 6

Conjectures and Examples

Recall we have been using the following notation: $B = P\Box_D k$, $K(\xi_1) = b_{10}^{-1}B$, and $R = b_{10}^{-1}\operatorname{Ext}_D^*(k,k) = P[b_{10}^{\pm 1}] \otimes E[h_{10}]$. Furthermore, we maintain the convention, as used in the last two chapters, of silently applying the antipode to the generators ξ_n of P, so $\Delta(\xi_n) = \sum_{i+j=n} \xi_i \otimes \xi_j^{p^i}$ instead of $\sum_{i+j=n} \xi_j^{p^i} \otimes \xi_i$ as is the customary definition of the symbol ξ_n.

6.1 Conjectures

In the previous two chapters, we studied the $K(\xi_1)$-based Adams spectral sequence

$$E_2 = \operatorname{Ext}^*_{K(\xi_1)_{**}K(\xi_1)}(K(\xi_1)_{**}, K(\xi_1)_{**}) \implies b_{10}^{-1}\operatorname{Ext}_P^*(k,k).$$

We showed that the E_2 page is isomorphic to $R \otimes P[w_2, w_3, \dots]$ and showed that the first nonzero differential is $d_r(w_n) = b_{10}^{-1}h_{10}w_2^2w_{n-1}^3$. By a degree argument (Proposition 5.1.1) we know that $d_r = 0$ (for $r \geq 2$) unless $r \equiv 4 \pmod 9$ and $r \equiv 8 \pmod 9$. We conjecture that d_8 is the only other nonzero differential. More precisely:

Conjecture 6.1.1.

(1) If $d_4(x) = h_{10}y$ and $d_4(y) = h_{10}z$, then $d_8(h_{10}x) = h_{10}z$.

(2) $d_r = 0$ for $r \geq 8$.

Evidence for this conjecture includes the fact that it agrees with b_{10}-periodic range of the computer calculation of $\mathrm{Ext}^*_P(k,k)$ up to the 700 stem (see Figure 4-2). See Figures 6-3–6-5 for charts depicting these differentials. In the range of the pictures, the differentials are known, not conjectured, because they can be deduced from the aforementioned chart of $\mathrm{Ext}^*_P(k,k)$. Conjecture 6.1.1, together with information about multiplicative extensions, allows one to conclude that $b_{10}^{-1}\mathrm{Ext}_P(k,k)$ has a particularly attractive form.

Proposition 6.1.2. *Suppose Conjecture 6.1.1 holds. Then there is an isomorphism on the level of vector spaces*

$$b_{10}^{-1}\mathrm{Ext}^*_P(k,k) \cong b_{10}^{-1}\mathrm{Ext}^*_D(k, k[\widetilde{w}_2, \widetilde{w}_3, \dots])$$

where the D-coaction on $\widetilde{w}_n$ *is given by* $\psi(\widetilde{w}_n) = 1 \otimes \widetilde{w}_n + \xi_1 \otimes \widetilde{w}_2^2\widetilde{w}_{n-1}^3$ *for* $n \geq 3$ *and* $\psi(\widetilde{w}_2) = 1 \otimes \widetilde{w}_2$. *This is an isomorphism of R-modules if, for every differential* $d_4(x) = h_{10}y$ *such that* y *is a permanent cycle, there is a hidden multiplication* $h_{10}\cdot(h_{10}x) = b_{10}y$ *in* $b_{10}^{-1}\mathrm{Ext}^*_P(k,k)$.

Proof. We begin by determining the isomorphism on the level of vector spaces. Given any D-comodule M with coaction $\psi : M \to D \otimes M$, let $\partial : M \to M$ denote the operator defined by $\psi(m) = 1 \otimes m + \xi_1 \otimes \partial(m) - \xi_1^2 \otimes \partial^2(m)$ (see Definition 4.2.1). Then there is a resolution $D \xrightarrow{\partial} D \xrightarrow{\partial^2} D \xrightarrow{\partial} \dots$ for k, and applying $-\square_D M$ gives rise to a complex $M \xrightarrow{\partial} M \xrightarrow{\partial^2} M \xrightarrow{\partial} \dots$ whose cohomology is $\mathrm{Cotor}_D(k,M) \cong \mathrm{Ext}^*_D(k,M)$, and $b_{10}^{-1}\mathrm{Ext}^*_D(k,M)$ is the cohomology of the periodic complex $\cdots \to M \xrightarrow{\partial} M \xrightarrow{\partial^2} M \to \dots$. Let $\widetilde{w}_n = b_{10}^{-1}w_n$ and let $W = k[\widetilde{w}_2, \widetilde{w}_3, \dots]$. Note that $d_4(\widetilde{w}_n) = h_{10}\widetilde{w}_2^2\widetilde{w}_{n-1}^3$. We will show that the E_∞ term in the MPASS is isomorphic to the cohomology of $W^* : \cdots \to W \xrightarrow{\partial} W \xrightarrow{\partial^2} W \to \cdots$.

We have $E_2 = R \otimes P[w_2, w_3, \dots] \cong R \otimes W$. Write $E_2 = W^+ \oplus W^-$, where $W^+ = W \otimes P[b_{10}^{\pm 1}]$ and $W^- = W^+\{h_{10}\}$. By Proposition 5.1.1, we know that elements in W^+ could be the source of a d_4 differential or the target of a d_8 differential, and vice versa for W^-. Using Conjecture 6.1.1(2), the E_∞ page of the MPASS is obtained by taking the cohomology of E_2 by d_4 and d_8, and in fact this is

$$E_\infty \cong \ker(d_4|_{W^+})/\operatorname{im}(d_8|_{W^-}) \oplus \ker(d_8|_{W^-})/\operatorname{im}(d_4|_{W^+}). \tag{6.1.1}$$

By Conjecture 6.1.1(1), there is a map f of chain complexes

$$\begin{array}{ccccccccc} \cdots \longrightarrow & W & \xrightarrow{\partial} & W & \xrightarrow{\partial^2} & W & \xrightarrow{\partial} & \cdots \\ & \downarrow{\scriptstyle f^{2n}} & & \downarrow{\scriptstyle f^{2n+1}} & & \downarrow{\scriptstyle f^{2n+2}} & & \\ \cdots \longrightarrow & W\{b_{10}^n\} & \xrightarrow{d_4} & W\{h_{10}b_{10}^n\} & \xrightarrow{d_8} & W\{b_{10}^{n+1}\} & \xrightarrow{d_4} & \cdots \end{array}$$

where the vertical maps are the obvious isomorphisms. By construction the cohomology of the top complex is $b_{10}^{-1}\operatorname{Ext}_D(k, W)$, and by (6.1.1), the cohomology of the bottom complex is E_∞.

Now we check that this respects the R-module structure, assuming the extra hypothesis. We will just check that it commutes with multiplication by h_{10}. Note that the powers of b_{10} and h_{10} on the bottom row refer to names in the MPASS E_2 page. If $\omega = [x] \in W^{2n}$ is a cycle then $h_{10}\omega$ is represented by $[x] \in W^{2n+1}$. So $f_*^{2n+1}(h_{10}\omega) = [h_{10}b_{10}^n x] = h_{10}[b_{10}^n x] = h_{10}f_*^{2n}(\omega)$. The other case is a bit more complicated. If $\upsilon = [y] \in W^{2n+1}$ is a cycle then $h_{10}\upsilon$ is represented by $[\partial y] \in W^{2n}$. We need to show that $f_*^{2n+2}(h_{10}\upsilon) = [b_{10}^{n+1}(\partial y)]$ can be represented as $h_{10} \cdot [h_{10}b_{10}^n y] = h_{10} \cdot f_*^{2n+1}(\upsilon)$. From the commutativity of the diagram we have $d_4([b_{10}^n y]) = [h_{10}b_{10}^n \partial y] = h_{10}[b_{10}^n \partial y]$, and $[b_{10}^n \partial y]$ is a permanent cycle because $\partial^2 y = 0$ by assumption. From the assumption about hidden multiplications, we have $h_{10} \cdot [h_{10}b_{10}^n y] = b_{10}[b_{10}^n \partial y]$ as desired. □

Remark 6.1.3. One can try to show the hidden multiplication by use of Massey products. First, one would like to use the Massey product convergence theorem (see

[Rav86, A1.4.10]) to show that the E_5 Massey product $\langle h_{10}, h_{10}, [b_{10}^n \partial y]\rangle$ converges to a Massey product in $b_{10}^{-1}\operatorname{Ext}_P(k,k)$. The crossing differentials hypothesis is automatically satisfied (assuming Conjecture 6.1.1): this says that there can be no nontrivial differentials d_r with $r > 4$ hitting classes in the same stem as $h_{10}[b_{10}^n \partial y]$, and this is true because any such differential would be a d_8, which only hits classes with $u' \equiv 0 \pmod 4$ (that is, classes whose E_2 representatives are in $k[w_2, w_3, \dots]$, as opposed to $k[w_2, w_3, \dots]\{h_{10}\}$). However, to use the Massey product convergence theorem, we also need to show that $\langle h_{10}, h_{10}, [b_{10}^n \partial y]\rangle$ is strictly defined in $b_{10}^{-1}\operatorname{Ext}_P^*(k,k)$, in particular that there is no nonzero hidden multiplication $h_{10} \cdot [b_{10}^n \partial y]$.

If this can be shown, then the Massey product shuffling relations

$$h_{10} \cdot [h_{10} b_{10}^n y] = h_{10}\,\langle h_{10}, h_{10}, [b_{10}^n \partial y]\rangle = \langle h_{10}, h_{10}, h_{10}\rangle\,[b_{10}^n \partial y]$$

in $b_{10}^{-1}\operatorname{Ext}_P^*(k,k)$ give rise to the desired hidden multiplication.

The expression in Proposition 6.1.2 is the $k = M$ case of the following more general conjecture.

Conjecture 6.1.4. There is a functor $W : \mathrm{Comod}_P \to \mathrm{Comod}_D$ such that for any P-comodule M, we have

$$b_{10}^{-1}\operatorname{Ext}_P^*(k, M) \cong b_{10}^{-1}\operatorname{Ext}_D^*(k, W(M)).$$

We do not have a conjecture for the form of $W(M)$ in general, though we believe it to be related to the MPASS E_2 page.

Remark 6.1.5. Since $b_{10}^{-1}\operatorname{Ext}_D^*(k, W(M)) \cong b_{10}^{-1}\operatorname{Ext}_P^*(k, P \,\square_D W(M))$, it is tempting to guess that the isomorphism in Conjecture 6.1.4 comes from a map $M \to P\,\square_D W(M)$.

However, I claim that this cannot be true for $M = k$. There is a free-forgetful adjunction

$$U : \mathrm{Comod}_P \leftrightarrows \mathrm{Comod}_D : F$$

where the free functor F takes $M \mapsto P \square_D M$. Given a D-comodule $W(k)$, this shows that every P-comodule map $k \to P \square_D W(k)$ factors through the adjunction unit $k \to P \square_D k$. So the supposed isomorphism $b_{10}^{-1} \operatorname{Ext}_P^*(k, k) \to b_{10}^{-1} \operatorname{Ext}_P^*(k, P \square_D W(M))$ factors through $b_{10}^{-1} \operatorname{Ext}_P^*(k, P \square_D k) \cong b_{10}^{-1} \operatorname{Ext}_D^*(k, k) = E[h_{10}] \otimes P[b_{10}^{\pm 1}]$, which is clearly false in light of what we know about $b_{10}^{-1} \operatorname{Ext}_P^*(k, k)$.

In the remainder of this chapter, we present two simpler, but complete, calculations which provide evidence for 6.1.4. In Section 6.2, we show that

$$b_{10}^{-1} \operatorname{Ext}_P^*(k, k[\xi_1^9, \xi_2^3, \xi_3, \xi_4, \ldots]) \cong b_{10}^{-1} \operatorname{Ext}_D^*(k, k[\widetilde{w}_2, \widetilde{b}_{20}])$$

where $\psi(\widetilde{w}_2) = 1 \otimes \widetilde{w}_2$ and $\psi(\widetilde{b}_{20}) = 1 \otimes \widetilde{b}_{20} + \xi_1 \otimes \widetilde{w}_2^4$. (We actually compute $b_{10}^{-1} \operatorname{Ext}^*_{k[\xi_1, \xi_2]/(\xi_1^9, \xi_2^3)}(k, k)$, which is isomorphic to the left hand side due to the change of rings theorem corresponding to the fact that $P \square_{k[\xi_1,\xi_2]/(\xi_1^9,\xi_2^3)} k \cong k[\xi_1^9, \xi_2^3, \xi_3, \xi_4, \ldots]$.) In Section 6.3, we compute

$$b_{10}^{-1} \operatorname{Ext}_P^*(k, k[\xi_1^3]) = b_{10}^{-1} \operatorname{Ext}_D^*(k, k[\widetilde{h}_{20}, \widetilde{b}_{20}, \widetilde{w}_3, \widetilde{w}_4, \ldots]/\widetilde{h}_{20}^2)$$

where the generators $\widetilde{h}_{20}$, $\widetilde{b}_{20}$, and $\widetilde{w}_n$ have trivial D-coaction. To summarize, our conjectural functor W should satisfy:

M	$W(M)$/free D-comodule summands
k	$k[\widetilde{w}_2, \widetilde{w}_3, \dots]$ (conjectural) $\psi(\widetilde{w}_n) = 1 \otimes \widetilde{w}_n + \xi_1 \otimes \widetilde{w}_2^2 \widetilde{w}_{n-1}^3$ $(n \geq 3)$
$k[\xi_1^3]$	$k[\widetilde{h}_{20}, \widetilde{b}_{20}, \widetilde{w}_3, \widetilde{w}_4, \dots]/(\widetilde{h}_{20}^2)$ trivial D-coaction
$k[\xi_1^9, \xi_2^3, \xi_3, \xi_4, \dots]$	$k[\widetilde{w}_2, \widetilde{b}_{20}]$ $\psi(\widetilde{b}_{20}) = 1 \otimes \widetilde{b}_{20} + \xi_1 \otimes \widetilde{w}_2^4$
$B = k[\xi_1^3, \xi_2, \xi_3, \dots]$	k
P	0

6.2 Localized cohomology of $P(1)$

Let $P(1) = k[\xi_1, \xi_2]/(\xi_1^{p^2}, \xi_2^p)$. Henderson [Hen97], building on work of Liulevicius [Liu62], computes $\operatorname{Ext}^*_{P(1)}(k,k)$ at all odd primes. In this section we will compute $b_{10}^{-1} \operatorname{Ext}^*_{P(1)}(k,k)$ at $p=3$; as $\operatorname{Ext}^*_{P(1)}(k,k)$ at $p=3$ was already b_{10}-periodic, we recover Henderson's result on the vector space level, but the multiplicative structure is much simpler after inverting b_{10}.

The main goal of this section is to prove the following.

Proposition 6.2.1. *There are classes $\widetilde{w}_2$ in internal degree 8 and $\widetilde{b}_{20}$ in internal degree 36 such that there is an isomorphism*

$$b_{10}^{-1} \operatorname{Ext}^*_{P(1)}(k,k) \cong b_{10}^{-1} \operatorname{Ext}^*_{D}(k, k[\widetilde{w}_2, \widetilde{b}_{20}])$$

where $\psi(\widetilde{b}_{20}) = 1 \otimes \widetilde{b}_{20} + \xi_1 \otimes \widetilde{w}_2^4$ and $\psi(\widetilde{w}_2) = 1 \otimes \widetilde{w}_2$.

Since $P \Box_{P(1)} k = k[\xi_1^9, \xi_2^3, \xi_3, \xi_4, \dots]$, by the change of rings theorem we have the following.

Corollary 6.2.2. *There is an isomorphism*

$$b_{10}^{-1}\operatorname{Ext}_P(k, k[\xi_1^9, \xi_2^3, \xi_3, \xi_4, \dots]) \cong b_{10}^{-1}\operatorname{Ext}_D(k, k[\widetilde{w}_2, \widetilde{b}_{20}])$$

where $\psi(\widetilde{b}_{20}) = 1 \otimes \widetilde{b}_{20} + \xi_1 \otimes \widetilde{w}_2^4$ *and* $\psi(\widetilde{w}_2) = 1 \otimes \widetilde{w}_2$.

We approach this computation the way we approached the computation of $b_{10}\operatorname{Ext}_P^*(k,k)$ in previous chapters. That is, we set

$$B_1 = P(1) \square_D k = k[\xi_1^3, \xi_2]/(\xi_1^9, \xi_2^3)$$

and begin by computing the E_2 page of the $b_{10}^{-1}B_1$-based MPASS. First, note that $(b_{10}B_1)_{**} = b_{10}^{-1}\operatorname{Ext}_{P(1)}^*(k, B_1) = b_{10}^{-1}\operatorname{Ext}_D^*(k,k) = R$.

Lemma 6.2.3. *The E_2 page of the $b_{10}^{-1}B_1$-based MPASS is*

$$E_2 = R \otimes k[w_2, b_{20}]$$

where w_2 has degree $(s,t,u) = (1,1,20)$ and b_{20} has degree $(s,t,u) = (1,1,48)$.

These generators relate to those in Proposition 6.2.1 by $\widetilde{w}_2 = b_{10}^{-1}w_2$ and $\widetilde{b}_{20} = b_{10}^{-1}b_{20}$; for most of the computation we find it easier to work with classes with actual representatives in the non-localized cobar complex. Recall that s is Adams filtration, t is internal homological degree, and u is internal topological degree, so $E_1^{s,t,u} = b_{10}^{-1}\operatorname{Ext}_{P(1)}^{t,u}(k, B_1 \otimes \overline{B}_1^{\otimes s})$.

Proof. This is the same calculation as Proposition 5.3.4. □

In Section 3.3 we showed that the MPASS coincides with a filtration spectral sequence on the cobar complex, which is in this case given by

$$F^s C_{P(1)}^*(k,k) = \{[a_1|\dots|a_n] \,:\, \#\big(\{a_1,\dots,a_n\} \cap \overline{B}_1 P(1)\big) \geq s\}.$$

We can pick out some obvious permanent cycles in this spectral sequence:

$$\underline{h_{10}} = [\xi_1] \qquad \underline{h_{11}} := [\xi_1^3] \qquad \underline{b_{11}} := [\xi_1^3|\xi_1^6] + [\xi_1^6|\xi_1^3]$$

We use the underlined versions as above to refer to explicit classes in the cobar complex, while the non-underlined versions, e.g. h_{10}, refer to their cohomology classes. By the spectral sequence comparison result, the permanent cycles above have to correspond to classes in the MPASS; we clarify this relationship in the next lemma. It is clear that $\underline{h_{10}}$ here represents the same class as h_{10} in the MPASS coefficient ring $(B_1)_{**}$. We have the same formula for w_2 as in the $b_{10}^{-1}\operatorname{Ext}_P(k,k)$ case, namely

$$w_2 = [\xi_1|\xi_2] - [\xi_1^2|\xi_1^3]$$

and by observing their cobar representatives, it is clear that both w_2 and h_{10} are permanent cycles.

It is clear from its cobar representative that w_2 is a permanent cycle, as is h_{10}.

Lemma 6.2.4. *There are relations in* $\operatorname{Ext}^*_{P(1)}(k,k)$*:*

$$b_{10}^2[\underline{b_{11}}] = \pm w_2^3.$$
$$b_{10}[\underline{h_{11}}] = \pm h_{10}w_2$$

Proof. These are Massey product relations in $\operatorname{Ext}^*_{P(1)}(k,k)$. First observe that our formula $w_2 = [\xi_1|\xi_2] - [\xi_1^2|\xi_1^3]$ implies that $w_2 = \langle h_{10}, h_{10}, h_{11}\rangle$. Using Massey product shuffling relations we have:

$$\begin{aligned} w_2^2 = \langle h_{10}, h_{10}, h_{11}\rangle^2 &= \pm\langle h_{10}, h_{10}, \langle h_{10}, h_{10}, h_{11}\rangle h_{11}\rangle \\ &= \pm\langle h_{10}, h_{10}, h_{10}\langle h_{10}, h_{11}, h_{11}\rangle\rangle \\ &= \pm\langle h_{10}, h_{10}, h_{10}\rangle\langle h_{10}, h_{11}, h_{11}\rangle \\ h_{10}w_2 = h_{10}\langle h_{10}, h_{10}, h_{11}\rangle &= \pm\langle h_{10}, h_{10}, h_{10}\rangle h_{11} = \pm b_{10}h_{11}. \end{aligned}$$

$$
\begin{aligned}
w_2^3 = w_2 \cdot w_2^2 &= \pm \langle h_{10}, h_{10}, h_{11} \rangle \cdot b_{10} \langle h_{10}, h_{11}, h_{11} \rangle \\
&= \pm b_{10} \langle h_{10} \langle h_{10}, h_{10}, h_{11} \rangle, h_{11}, h_{11} \rangle \\
&= \pm b_{10} \langle \langle h_{10}, h_{10}, h_{10} \rangle h_{11}, h_{11}, h_{11} \rangle \\
&= \pm b_{10}^2 \langle h_{11}, h_{11}, h_{11} \rangle = \pm b_{10}^2 b_{11}
\end{aligned}
$$

We need to check Massey product indeterminacy: we need to show that all the Massey products above are strictly defined. For $\langle h_{10}, h_{10}, h_{11} \rangle$, we need to show there are no cycles in the degree of $[\xi_1^2]$ (which hits h_{10}^2) and $[\xi_2]$ (which hits $h_{10}h_{11}$). It suffices to check using Lemma 6.2.3 that there are no elements in E_2 that could converge to these elements—that is, that there are no elements in E_2 with the given stem $u' = u - 6(s+t)$. For $\langle h_{10}, h_{10}, \langle h_{10}, h_{10}, h_{11} \rangle h_{11} \rangle$ we need to check the degree of $[\xi_1^2]$ and $w_2[\xi_2]$; for $\langle h_{10}, h_{11}, h_{11} \rangle$ we need to check the degree of $[\xi_2]$ and $[\xi_1^6]$; for $\langle h_{10}, h_{10}, h_{10} \langle h_{10}, h_{11}, h_{11} \rangle \rangle$ we need to check the degree of $[\xi_1^2]$ and $b_{10}^{-1} w_2^2 [\xi_1^2]$. All of these can easily be seen as there are not very many classes in these low degrees.

For the second relation, use Massey products similarly:

$$
b_{10}[\underline{h_{11}}] = \langle h_{10}, h_{10}, h_{10} \rangle h_{11} = \pm h_{10} \langle h_{10}, h_{10}, h_{11} \rangle = \pm h_{10} w_2. \qquad \square
$$

Lemma 6.2.5. *There are differentials:*

$$
\begin{aligned}
d_3(b_{20}) &= \pm b_{10}^{-3} h_{10} w_2^4 \\
d_3(b_{20}^2) &= \mp b_{10}^{-3} w_2^4 b_{20} \\
d_3(b_{20}^3) &= 0.
\end{aligned}
$$

Proof. We will use the following cobar representative for b_{20}:

$$
\underline{b_{20}} = [\xi_2|\xi_2^2] + [\xi_2^2|\xi_2] - [\xi_1\xi_2|\xi_2\xi_1^3] + [\xi_1\xi_2^2|\xi_1^3] + [\xi_1^2\xi_2|\xi_1^6] + [\xi_1^2|\xi_2\xi_1^6] + [\xi_1|\xi_2^2\xi_1^3] \quad (6.2.1)
$$

in $F^1/F^2C_{P(1)}(k,k)$. One can check directly that

$$d_{\text{cobar}}(\underline{b_{20}}) = -\underline{h_{11}}|\underline{b_{11}}. \tag{6.2.2}$$

The first differential then follows from Lemma 6.2.4, and the second and third follow from the first by multiplicativity. □

Lemma 6.2.6. *b_{20}^3 is a permanent cycle.*

Proof. May [May70] constructed Steenrod operations on the cohomology of a Hopf algebra; this is functorial, and the Steenrod operations on $\text{Ext}^*_{P(1)}(k,k)$ are the image of the operations on $\text{Ext}^*_A(k,k)$ along the quotient map $\text{Ext}^*_A(k,k) \to \text{Ext}^*_{P(1)}(k,k)$. Sawka [Saw82] shows that double complex spectral sequences (such as the Cartan-Eilenberg spectral sequence) commute with Steenrod operations. In particular, using [Saw82, Proposition 2.5(3)] we have

$$\begin{aligned} d_7(b_{20}^3) &= d_7(P^1 b_{20}) = P^1 d_3(b_{20}) = P^1(-h_{11}b_{11}) \\ &= P^0(-h_{11})P^1(b_{11}) = -h_{12}b_{11}^3 \end{aligned}$$

which is zero since $h_{12} = [\xi_1^9]$ is zero in $C_{P(1)}(k,k)$. This shows that $d_7(b_{20}^3) = 0$, which is all that we need for now; by the time we get to the E_7 page, it will be easy to check (see e.g. Figure 6-2) that there is no room for higher differentials. □

So

$$E_4 = k[b_{10}^{\pm}, w_2, b_{20}^3]\{1, h_{10}, h_{10}b_{20}, h_{10}b_{20}^2\}/(hw_2^4, hb_{20}w_2^4).$$

Furthermore, $h_{10}b_{20}$ has a cobar representative

$$\xi_1|\underline{b_{20}} - \xi_2|\underline{b_{11}}$$

which is a permanent cycle.

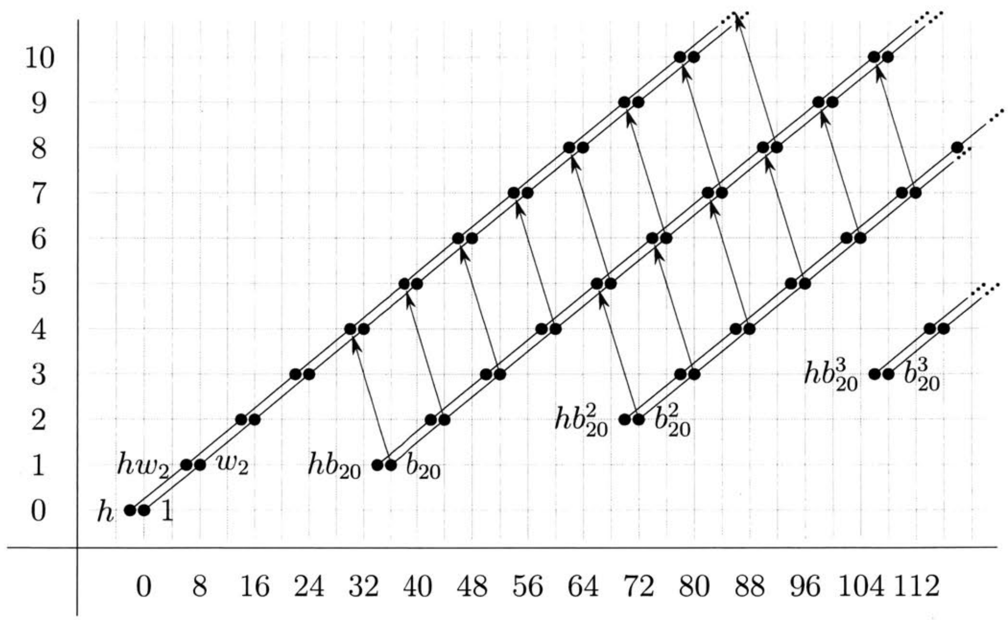


Figure 6-1: E_3 page, with coordinates (u', s) (note that b_{10} is at $(0,0)$)

Lemma 6.2.7. $d_6(h_{10}b_{20}^2) = \pm b_{10}^{-5} w_2^8$

Proof. Let $\underline{b_{20}}$ be as in (6.2.1) and let $y = [\xi_1^6|\xi_1^6]$. One can compute that

$$d_{cobar}(\underline{b_{20}}) = -\underline{h_{11}}|\underline{b_{11}}$$
$$d_{cobar}(\underline{b_{20}} - y) = -\underline{b_{11}}|\underline{h_{11}}.$$

Since $y \in F^2$ and $\underline{b_{20}} \in F^1$, we see that $(\underline{b_{20}} - y)|\underline{h_{10}}|\underline{b_{20}} \in F^2/F^3$ is a representative for b_{20}. Then we have:

$$\begin{aligned} d\big((\underline{b_{20}} - y)|\underline{h_{10}}|\underline{b_{20}} - \underline{b_{11}}|c(\xi_2)|\underline{b_{20}} - (\underline{b_{20}} - y)|\xi_2|\underline{b_{11}}\big) \\ = -\underline{b_{11}}|\underline{h_{11}}|\underline{h_{10}}|\underline{b_{20}} - (\underline{b_{20}} - y)|\underline{h_{10}}|\underline{h_{11}}|\underline{b_{11}} \\ + \underline{b_{11}}|\underline{h_{11}}|\underline{h_{10}}|\underline{b_{20}} + \underline{b_{11}}|c(\xi_2)|\underline{h_{11}}|\underline{b_{11}} \\ + \underline{b_{11}}|\underline{h_{11}}|\xi_2|\underline{b_{11}} + (\underline{b_{20}} - y)|\underline{h_{10}}|\underline{h_{11}}|\underline{b_{11}} \\ = \underline{b_{11}}|\underline{h_{11}}|\xi_2|\underline{b_{11}} + \underline{b_{11}}|c(\xi_2)|\underline{h_{11}}|\underline{b_{11}} \end{aligned}$$

and this is a representative for $b_{11}^2 \langle h_{11}, h_{10}, h_{11} \rangle$ which can be written $\pm b_{10}^{-5} w_2^8$ by Lemma 6.2.4. □

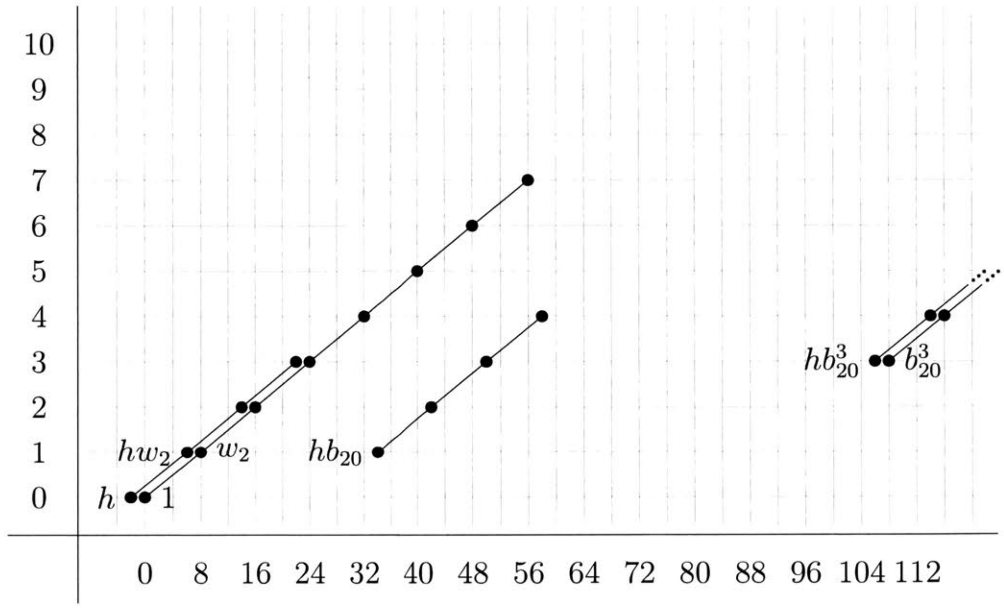


Figure 6-2: E_7 page

So we have

$$E_7 = \left(k[w_2]/w_2^8 \oplus (k[w_2]/w_2^4)\{h_{10}\} \oplus (k[w_2]/w_2^4)\{h_{10}b_{20}\}\right) \otimes k[b_{20}^3]$$

and we have seen that all these classes are permanent cycles.

Proof of Proposition 6.2.1. Set $\widetilde{w}_2 = b_{10}^{-1} w_2$ and $\widetilde{b}_{20} = b_{10}^{-1} b_{20}$. Using Lemmas 6.2.5 and 6.2.7, there is an obvious analogue of Proposition 6.1.2 with the (d_3, d_6) pair here in place of the (d_4, d_8) pair there, and it suffices to show the condition about hidden multiplications. By Remark 6.1.5, it suffices to show that $\langle h_{10}, h_{10}, [b_{10}^n \partial y] \rangle$ is strictly defined in $b_{10}^{-1} \operatorname{Ext}_{P(1)}^*(k,k)$ whenever $d_4(x) = h_{10}(\partial y)$. Looking at Figure 6-1, we see there is no room for such hidden multiplications. □

6.3 Localized cohomology of a large quotient of P

In this section we will show:

Theorem 6.3.1. *Let $D_{1,\infty} = k[\xi_1, \xi_2, \ldots]/(\xi_1^3)$. Then*

$$b_{10}^{-1} \operatorname{Ext}^*_{D_{1,\infty}}(k, k) \cong E[h_{10}, h_{20}] \otimes P[b_{10}^{\pm 1}, b_{20}, w_3, w_4, \ldots].$$

In particular, one can write

$$b_{10}^{-1} \operatorname{Ext}^*_{D_{1,\infty}}(k, k) \cong b_{10}^{-1} \operatorname{Ext}^*_D(k, k[h_{20}, b_{20}, w_3, w_4, \ldots]/(h_{20}^2)$$

where all the generators h_{20}, b_{20}, w_n are D-primitive.

It is interesting that $D_{1,\infty}$ seems reasonably close to P in size, and yet the computation of its b_{10}-local cohomology is much simpler. In particular, attempting to apply the methods in this section (especially the explicit construction in Lemma 6.3.7) to computing $b_{10}^{-1} \operatorname{Ext}^*_P(k, k)$ quickly results in an intractable mess.

The strategy is to explicitly construct a map from the cobar complex $C_{D_{1,\infty}}(k, k)$ to another complex which is designed to have the right cohomology, and then show the map is a quasi-isomorphism. Note that the cobar complex is a dga under the concatenation product, so every element is a product of elements in degree 1. Thus if our target complex is a dga, it suffices to construct a map out of $C^1_{D_{1,\infty}}(k, k) = \overline{D_{1,\infty}}$, and then extend the map to all of $C^*_{D_{1,\infty}}(k, k)$ by multiplicativity. In order to ensure the resulting map is a map of complexes, there is a criterion that the map on degree 1 needs to satisfy:

Proposition 6.3.2. *Let Γ be a Hopf algebra over k, Q^* be a dga with augmentation*

$k \to Q^*$, and $\theta : \overline{\Gamma} \to Q^1$ be a k-linear map such that

$$d_Q(\theta(x)) = \sum \theta(x')\theta(x'') \tag{6.3.1}$$

for all $x \in \overline{\Gamma}$, where $\sum x' \otimes x''$ is the reduced diagonal $\overline{\Delta}(x)$. Then there is a map of dga's $f : C^*_{\Gamma}(k,k) \to Q^*$ sending $[a_1|\dots|a_n]$ to $\prod \theta(a_i)$.

Proof. We just need to check that f commutes with the differential; that is, we have to check the following diagram commutes:

$$\begin{array}{ccc} C^n_{\Gamma}(k,k) & \xrightarrow{f} & Q^n \\ \downarrow d_{cobar} & & \downarrow d_Q \\ C^{n+1}_{\Gamma}(k,k) & \xrightarrow{f} & Q^{n+1} \end{array}$$

For $n = 1$, this is precisely what the condition (6.3.1) guarantees. Commutativity for $n > 1$ follows from the Leibniz rule. The map on $n = 0$ is the augmentation. □

Remark 6.3.3. This is an example of the more general construction of *twisting cochains*; see [HMS74, §II.1]. A morphism θ satisfying (6.3.1) will be called a *twisting morphism.*

The target of our desired twisting morphism will be the complex $b_{10}^{-1}\widetilde{U}^* \otimes W'$, where

- $W' = k[w_3, w_4, \dots]$, with $u(w_n) = 2(3^n - 1)$, is in homological degree zero with zero differential, and
- $\widetilde{U}^* := UL^*(\xi_1) \otimes UL^*(\xi_2) \subset C^*_{D[\xi_1,\xi_2]}(k,k)$ where the sub-dga $UL^*(x) \subset C^*_{D[x]}(k,k)$ is defined below.

Definition 6.3.4. Given a height-3 truncated polynomial algebra $D[x]$, let $UL^*(x)$ be the sub-dga of $C^*_{D[x]}(k,k)$ multiplicatively generated by the elements $\alpha = [x]$, $\beta = [x^2]$, and $\gamma = [x|x^2] + [x^2|x]$. This inherits from $C^*_{D[x]}(k,k)$ the differentials $d(\alpha) = 0$, $d(\beta) = -\alpha^2$, and $d(\gamma) = 0$, along with the relations $\alpha\beta + \beta\alpha = \gamma$, $\alpha^3 = 0$, and $\beta^2 = 0$.

Remark 6.3.5. This is (up to signs) the $p = 3$ case of a construction due to Moore: let UL^* be the dga which has multiplicative generators $a_1, \dots, a_{p-1}$ in degree 1 and $t_2, \dots, t_p$ in degree 2 with $d(a_i) = t_i$, subject to

$$a_1^2 = t_2 \qquad a_i^2 = 0 \text{ for } i \neq 1 \qquad a_1^p = 0 \qquad a_i a_j = -a_j a_i \text{ for } i, j \neq 1$$
$$a_j a_1 = -a_1 a_j + t_{j+1} \qquad a_i t_j = t_j a_i \qquad t_i t_j = t_j t_i.$$

This is a dga quasi-isomorphic to, and much smaller than, $C_{k[x]/x^p}(k,k)$. It also has the nice property that t_p (which, in the case $x = \xi_1$, represents b_{10}) is central.

Notation 6.3.6. Denote the generators of $UL^*(\xi_1)$ by $a_1 = [\xi_1]$, $a_2 = [\xi_1^2]$, and $b_{10} = [\xi_1|\xi_1^2] + [\xi_1^2|\xi_1]$, and the generators of $UL^*(\xi_2)$ by $q_1 = [\xi_2]$, $q_2 = [\xi_2^2]$, and $b_{20} = [\xi_2|\xi_2^2] + [\xi_2^2|\xi_2]$. (This definition of b_{10} and b_{20} does, of course, match up with the image of b_{10} and b_{20} along $\mathrm{Ext}^*_P(k,k) \to \mathrm{Ext}^*_{D[\xi_1,\xi_2](k,k)}$, and even $\mathrm{Ext}^*_P(k,k) \to \mathrm{Ext}^*_{D_{1,\infty}}(k,k)$.) Note that

$$H^*(\widetilde{U}) = H^*(C_{D[\xi_1,\xi_2]}(k,k)) = E[h_{10}, h_{20}] \otimes P[b_{10}, b_{20}].$$

So our target complex $b_{10}^{-1}\widetilde{U} \otimes W'$ has cohomology

$$H^*(b_{10}^{-1}\widetilde{U} \otimes W') = H^*(b_{10}^{-1}\widetilde{U}) \otimes W' = E[h_{10}, h_{20}] \otimes P[b_{10}^{\pm 1}, b_{20}] \otimes W'.$$

6.3.1 Defining $\theta : \overline{D_{1,\infty}} \to b_{10}^{-1}\widetilde{U} \otimes W'$

The definition of the map $\theta : \overline{D_{1,\infty}} \to b_{10}^{-1}\widetilde{U}^* \otimes W'$ is quite ad hoc, and will be done in several stages. The map will arise as a composition $D_{1,\infty} \to D' \to \widetilde{U}^* \otimes W' \to b_{10}^{-1}\widetilde{U}^* \otimes W'$, where the first map is the natural surjection to

$$D' := k[\xi_1, \xi_2, \dots]/(\xi_1^3, \xi_2^9, \xi_3^9, \dots)$$

and the last map is the natural localization map; the main goal is to construct a map $D' \to \widetilde{U}^* \otimes W'$ satisfying the twisting morphism condition, and we begin by constructing a map out of a slightly smaller coalgebra.

Lemma 6.3.7. *Let*

$$C = k[\xi_1, \xi_2^3, \xi_3, \xi_4, \dots]/(\xi_1^3, \xi_2^9, \xi_3^9, \dots).$$

There is a twisting morphism $\theta : \overline{C} \to UL^1(\xi_1) \otimes W'$.

Proof. For $n, m, k \geq 3$, make the following definitions:

$$\begin{aligned}
\theta(\xi_1) &= a_1 \\
\theta(\xi_1^2) &= a_2 \\
\theta(\xi_{n-1}^3) &= -a_1 w_n \\
\theta(\xi_n) &= a_2 w_n \\
\theta(\xi_1 \xi_{n-1}^3) &= -a_2 w_n \\
\theta(\xi_1 \xi_n) &= 0 \\
\theta(\xi_{n-1}^3 \xi_{m-1}^3) &= a_2 w_n w_m \\
\theta(\xi_n \xi_{m-1}^3) &= 0 \\
\theta(\xi_n \xi_m) &= 0 \\
\theta(\xi_1^2 \xi_{n-1}^3) &= 0 \\
\theta(\xi_1 \xi_{n-1}^3 \xi_{m-1}^3) &= 0 \\
\theta(\xi_{n-1}^3 \xi_{m-1}^3 \xi_{k-1}^3) &= 0
\end{aligned}$$

It is a straightforward computation with the cobar differential to check that each of these does not violate the twisting morphism condition

$$d(\theta(x)) = \sum \theta(x') \cdot \theta(x'') \tag{6.3.2}$$

where $\overline{\Delta}(x) = \sum x' \otimes x''$. (Note that, in C, we have $\overline{\Delta}(\xi_{n-1}^3) = 0$ and $\overline{\Delta}(\xi_n) = \xi_1 | \xi_{n-1}^3$.)

Now it suffices to prove the following.

Claim 6.3.8. *Defining $\theta(X) = 0$ for all monomials X except the ones listed above defines a twisting morphism.*

Define a (non-multiplicative) grading ρ on C where

$$\rho(1) = 0 \quad \rho(\xi_1) = 1 \quad \rho(\xi_1^2) = 2 \quad \rho(\xi_{n-1}^3) = 1 \quad \rho(\xi_{n-1}^6) = 2 \quad \rho(\xi_n) = 2 \quad \rho(\xi_n^2) = 4$$

for $n \geq 3$, and $\rho(\prod_i \xi_i^{a_i+3b_i}) = \sum \rho(\xi_i^{a_i}) + \rho(\xi_i^{3b_i})$ (where $a_i, b_i \in \{0,1,2\}$). The reason for considering this grading is the following:

Claim 6.3.9. *Writing $\Delta(x) = \sum x' \otimes x''$, we have $\rho(x') + \rho(x'') \leq \rho(x)$.*

Proof of Claim 6.3.9. If $X = \prod \xi_i^{a_i+3b_i}$ for $a_i, b_i \in \{0,1,2\}$, consider the collection $\mathscr{T}_X = \{\xi_i^{a_i} : a_i \neq 0\} \cup \{\xi_i^{3b_i} : b_i \neq 0\}$. Use induction on $n := \#\mathscr{T}_X$. If $n = 1$, then it suffices to check explicitly the Milnor diagonal of each of the terms $\{\xi_1, \xi_1^2, \xi_{i-1}^3, \xi_{i-1}^6, \xi_i, \xi_i^2\}$. (In fact, we find $\rho(x) = \rho(x') + \rho(x'')$ for each of these terms.)

For general monomials a, b, we have

$$\rho(ab) \leq \rho(a) + \rho(b). \tag{6.3.3}$$

By definition, if x and y are products of non-overlapping subsets of $\mathscr{T}_X$, then

$$\rho(xy) = \rho(x) + \rho(y). \tag{6.3.4}$$

Write $X = xy$ where $x \in \mathscr{T}_X$ and y is a product of terms in $\mathscr{T}_X$ (different from x).

Since $\Delta(xy) = \sum x'y'|x''y''$ it suffices to prove $\rho(x'y') + \rho(x''y'') \leq \rho(xy)$. We have

$$\begin{aligned}\rho(x'y') + \rho(x''y'') &\leq \rho(x') + \rho(y') + \rho(x'') + \rho(y'') \\ &\leq \rho(x) + \rho(y) \\ &= \rho(xy)\end{aligned}$$

where the first inequality is by (6.3.3), the second inequality is by the inductive hypothesis, and the last equality is by (6.3.4). □

So the monomials in C with degree 1 are ξ_1 and ξ_{n-1}^3 for $n \geq 3$, the monomials with ρ-degree 2 are ξ_1^2, ξ_n, $\xi_{n-1}^3\xi_{m-1}^3$, and $\xi_1\xi_{n-1}^3$ for $n, m \geq 3$, and the monomials with degree 3 are $\xi_1^2\xi_{n-1}^3$, $\xi_1\xi_{n-1}^3\xi_{m-1}^3$, $\xi_{n-1}^3\xi_{m-1}^3\xi_{k-1}^3$, $\xi_1\xi_n$, and $\xi_{n-1}^3\xi_m$ for $n, m \geq 3$. Notice that θ has already been defined for these monomials above. So it remains to show that θ can be defined consistently for monomials with $\rho \geq 4$. In particular, we will show using induction on ρ degree that we can define $\theta(x) = 0$ if $\rho(x) \geq 3$ while preserving the twisting morphism condition (6.3.1).

Since we have already checked above that we can define $\theta(x) = 0$ on the monomials x with $\rho(x) = 3$, let $\rho(x) = n > 3$ and assume inductively that we have already defined $\theta(y) = 0$ if $3 \leq \rho(y) \leq n-1$. Any monomial y with $\rho(y) = 0$ is in k (and hence $\theta(y) = 0$), so we can assume that $\rho(x') < \rho(x)$ and $\rho(x'') < \rho(x)$. So by the inductive hypothesis we have $\sum \theta(x') \cdot \theta(x'') = 0$, and so we can set $\theta(x) = 0$ without violating (6.3.1). □

Lemma 6.3.10. *One may extend θ constructed in Lemma 6.3.7 to a twisting morphism $\overline{D'} \to \widetilde{U}^1 \otimes W'$ by defining:*

$$\begin{aligned}\theta(\xi_2) &= q_1 \\ \theta(\xi_2^2) &= q_2 \\ \theta(\xi_2 x) &= 0 \qquad \textit{for } x \in \overline{C}\end{aligned}$$

$$\theta(\xi_2^2 x) = 0 \qquad \textit{for } x \in \overline{C}$$

where $\overline{C}$ *is the cokernel of the unit map* $k \to C$.

Proof. Note that ξ_2 is primitive in D', and C is a sub-coalgebra of D', so we need to define θ on $\xi_2 C$ and $\xi_2^2 C$. It is straightforward to check that $\theta(\xi_2) = q_1$ and $\theta(\xi_2^2) = q_2$ is consistent with (6.3.1).

If $x = \xi_2 y$ for $y \in \overline{C}$ then every y', y'' in Δy is in C, and

$$\begin{aligned}
\sum \theta(x') \cdot \theta(x'') &= \sum \left(\theta(\xi_2 y') \cdot \theta(y'') + \theta(y') \cdot \theta(\xi_2 y'')\right) \\
&= \theta(\xi_2)\theta(y) + \theta(y)\theta(\xi_2) + \sum_{y', y'' \notin k} \left(\theta(\xi_2 y') \cdot \theta(y'') + \theta(y') \cdot \theta(\xi_2 y'')\right) \\
&= q_1\theta(y) + \theta(y)q_1 + \sum_{y', y'' \notin k} \left(\theta(\xi_2 y') \cdot \theta(y'') + \theta(y') \cdot \theta(\xi_2 y'')\right).
\end{aligned}$$

Since $\theta(y) \in UL^1(\xi_1) \otimes W'$ and q_1 anti-commutes with the generators a_1 and a_2 of $UL^1(\xi_1)$, we have $q_1\theta(y) + \theta(y)q_1 = 0$. Thus defining $\theta(\xi_2 y) = 0$ does not violate (6.3.1).

Similarly, if $x = \xi_2^2 y$ for $y \in \overline{C}$, then

$$\begin{aligned}
\sum \theta(x') \cdot \theta(x'') &= \sum \left(\theta(\xi_2^2 y') \cdot \theta(y'') + 2\theta(\xi_2 y') \cdot \theta(\xi_2 y'') + \theta(y') \cdot \theta(\xi_2^2 y'')\right) \\
&= \theta(\xi_2^2)\theta(y) + 2\theta(\xi_2)\theta(\xi_2 y) + 2\theta(\xi_2 y)\theta(\xi_2) + \theta(y)\theta(\xi_2^2) \\
&\qquad + \sum_{y', y'' \notin k} \left(\theta(\xi_2^2 y') \cdot \theta(y'') + 2\theta(\xi_2 y') \cdot \theta(\xi_2 y'') + \theta(y') \cdot \theta(\xi_2^2 y'')\right) \\
&= \theta(\xi_2^2)\theta(y) + \theta(y)\theta(\xi_2^2) + \sum_{y', y'' \notin k} \left(\theta(\xi_2^2 y')\theta(y'') + \theta(y')\theta(\xi_2^2 y'')\right)
\end{aligned}$$

where in the third equality we use the fact that $0 = \theta(\xi_2 y) = \theta(\xi_2 y') = \theta(\xi_2 y'')$ (for $y', y'' \notin k$). Again, $\theta(\xi_2^2)\theta(y) + \theta(y)\theta(\xi_2^2) = q_2\theta(y) + \theta(y)q_2$ which is zero since $\theta(y)$ is in $UL^1(\xi_1) \otimes W'$ and q_2 anti-commutes with the generators a_1 and a_2 of $UL^1(\xi_1)$. So it is consistent with (6.3.1) to define $\theta(\xi_2^2 y) = 0$. □

Now precompose with the surjection $q : D_{1,\infty} \to D'$ to obtain a twisting morphism

$$\theta : D_{1,\infty} \to D' \to \widetilde{U}^1 \otimes W'.$$

This remains a twisting morphism because it is a coalgebra map—in particular, q commutes with the coproduct—and so $d(\theta(q(x))) = \sum \theta(q(x)')\theta(q(x)'') = \sum \theta(q(x'))\theta(q(x''))$. So by Proposition 6.3.2 we get an induced map

$$\theta' : C^*_{D_{1,\infty}}(k,k) \to \widetilde{U}^* \otimes W'$$

by extending θ multiplicatively using the concatenation product on the cobar complex.

6.3.2 Showing θ is a quasi-isomorphism via spectral sequence comparison

Our goal is to show:

Theorem 6.3.11. *The map $\theta' : C^*_{D_{1,\infty}}(k,k) \to \widetilde{U}^* \otimes W'$ induces an isomorphism in cohomology after inverting b_{10}. In particular, there is an isomorphism*

$$b_{10}^{-1} \operatorname{Ext}^*_{D_{1,\infty}}(k,k) \cong E[h_{10}, h_{20}] \otimes P[b_{10}^{\pm 1}, b_{20}] \otimes W'.$$

To show this, we define filtrations on $C^*_{D_{1,\infty}}(k,k)$ and on $\widetilde{U}^* \otimes W'$ in a way that makes θ' a filtration-preserving map; this induces a map of filtration spectral sequences. We compute the E_2 pages of both sides and show that θ' induces an isomorphism of E_2 pages, hence an isomorphism of E_∞ pages.

Let $B_{1,\infty} := k[\xi_2, \xi_3, \dots] = D_{1,\infty} \square_D k$. Define a decreasing filtration on $C^*_{D_{1,\infty}}(k,k)$ where $[a_1|\dots|a_n]$ is in $F^s C^*_{D_{1,\infty}}(k,k)$ if at least s of the a_i's are in $\ker(D_{1,\infty} \to D) = \overline{B}_{1,\infty} D_{1,\infty}$. Define a decreasing filtration on $\widetilde{U}^* \otimes W'$ by the following multiplicative grading:

- $|a_1| = |a_2| = |b_{10}| = 0$
- $|q_1| = |q_2| = 1$
- $|b_{20}| = 2$
- $|w_n| = 1$.

Looking at the definition of θ in Lemma 6.3.7 and Lemma 6.3.10, it is clear that θ is filtration-preserving, and hence so is θ'.

It is a consequence of the work in Section 3.3 that the $b_{10}^{-1}B_{1,\infty}$-based MPASS for computing $b_{10}^{-1}\operatorname{Ext}^*_{D_{1,\infty}}(k,k)$ coincides with the b_{10}-localized version of this filtration spectral sequence on $C^*_{D_{1,\infty}}(k,k)$. Our next goal is to calculate the E_2 page of (the b_{10}-localized version of) the filtration spectral sequence on $C^*_{D_{1,\infty}}(k,k)$, and using this correspondence we may instead calculate the MPASS E_2 term

$$E_2^{s,*} = b_{10}^{-1}\operatorname{Ext}^s_{b_{10}^{-1}\operatorname{Ext}^*_D(k,B_{1,\infty})}(b_{10}^{-1}\operatorname{Ext}^*_D(k,k),\ b_{10}^{-1}\operatorname{Ext}^*_D(k,k)). \tag{6.3.5}$$

So we need to compute $b_{10}^{-1}\operatorname{Ext}^*_D(k,B_{1,\infty})$ and its coalgebra structure. The correspondence of spectral sequences further gives that

$$E_1^{1,*} = b_{10}^{-1}\operatorname{Ext}^*_D(k,\overline{B}_{1,\infty}) \cong b_{10}^{-1}H^*(F^1/F^2C^*_{D_{1,\infty}}(k,k)) \tag{6.3.6}$$

and the reduced diagonal on $b_{10}^{-1}\operatorname{Ext}^*_D(k,B_{1,\infty})$ coincides with d_1 in the filtration spectral sequence.

Proposition 6.3.12. *As coalgebras, we have*

$$b_{10}^{-1}\operatorname{Ext}^*_D(k,B_{1,\infty}) \cong b_{10}^{-1}E[e_3,e_4,\dots]\otimes D[\xi_2]$$

i.e. e_n *and* ξ_2 *are primitive and* $\overline{\Delta}(\xi_2^2) = 2\xi_2\otimes\xi_2$.

Proof. The first task is to determine the D-comodule structure on $B_{1,\infty}$. Let ψ denote the D-coaction induced by the D-coaction on P, and $\partial : B_{1,\infty} \to B_{1,\infty}$ denote the operator defined by $\psi(x) = 1 \otimes x + \xi_1 \otimes \partial x - \xi_1^2 \otimes \partial^2 x$ (see Definition 4.2.1). For example, $\partial(\xi_n) = \xi_{n-1}^3$, $\partial(\xi_{n-1}^3) = 0$, and ∂ satisfies the Leibniz rule.

We have a coalgebra isomorphism $B_{1,\infty} \cong D[\xi_2] \otimes k[\xi_2^3, \xi_3, \xi_4, \dots]$. Since 1, ξ_2, and ξ_2^2 are all primitive, $D[\xi_2]$ splits as D-comodule into three trivial D-comodules, generated by 1, ξ_2, and ξ_2^2 respectively. So it suffices to determine the D-comodule structure of $k[\xi_2^3, \xi_3, \xi_4, \dots]$.

As part of the determination of the structure of $b_{10}^{-1} \operatorname{Ext}_D^*(k, B)$ in Section 4.2, we showed that there is a D-comodule decomposition

$$B \cong \bigoplus_{\substack{\xi_{n_1} \cdots \xi_{n_d} \\ n_i \geq 2 \text{ distinct}}} T(\langle \xi_{n_1} \dots \xi_{n_d} \ ; \ 1 \rangle) \ \oplus \ F$$

where F is a free D-comodule and $T(\langle \xi_{n_1} \dots \xi_{n_d} \ ; \ 1 \rangle)$ is generated as a vector space by monomials of the form $\partial^{\varepsilon_1}(\xi_{n_1}) \dots \partial^{\varepsilon_d}(\xi_{n_d})$ for $\varepsilon_i \in \{0, 1\}$. I claim the surjection $f : B \to k[\xi_2^3, \xi_3, \xi_4, \dots]$ takes F to another free summand: this map preserves the direct sum decomposition into summands of the form D, $M(1)$, and k, and the image of a free summand D must be either 0 or another free summand (just as there are no D-*module* maps $k = k[x]/(x) \to D$ or $M(1) = k[x]/(x^2) \to D$, there are no D-comodule maps $D \to k$ or $D \to M(1)$).

Furthermore, I claim that f acts as zero on summands $T(\langle \xi_{n_1} \dots \xi_{n_d} \ ; \ 1 \rangle)$ where some $n_i = 2$, and is the identity otherwise. In the first case, every basis element $\partial^{\varepsilon_i}(\xi_2) \prod_{j \neq i} \partial^{\varepsilon_j}(\xi_{n_j})$ in $T(\langle \xi_{n_1} \dots \xi_{n_d} \ ; \ 1 \rangle)$ has the form $\xi_2 \prod_{j \neq i} \partial^{\varepsilon_j}(\xi_{n_j}) \in \xi_1^3 \cdot k[\xi_2^3, \xi_3, \xi_4, \dots]$ or $\xi_1^3 \prod_{j \neq i} \partial^{\varepsilon_j}(\xi_{n_j}) \in \xi_1^3 \cdot k[\xi_2^3, \xi_3, \xi_4, \dots]$, and these are sent to zero under f. If instead $n_i > 2$ for every i, then every term $\partial^{\varepsilon_1}(\xi_{n_1}) \dots \partial^{\varepsilon_d}(\xi_{n_d})$ is in $k[\xi_2^3, \xi_3, \xi_4, \dots]$ and so f

acts as the identity. So we have shown that there is a D-comodule isomorphism

$$B_{1,\infty} = \Big(\bigoplus_{\substack{\xi_{n_1}\cdots\xi_{n_d} \\ n_i \geq 3 \text{ distinct}}} T(\langle \xi_{n_1} \ldots \xi_{n_d} \ ; \ 1 \rangle) \ \oplus \ F' \Big) \ \otimes \ (k_1 \oplus k_{\xi_2} \oplus k_{\xi_2^2})$$

where F' is a free D-comodule. So we have

$$\begin{aligned} b_{10}^{-1} \operatorname{Ext}_D^*(k, B_{1,\infty}) &\cong \bigoplus_{\substack{\xi_{n_1}\cdots\xi_{n_d} \\ n_i \geq 3 \text{ distinct}}} b_{10}^{-1} \operatorname{Ext}_D^* \left(k, T(\langle \xi_{n_1} \ldots \xi_{n_d} \ ; \ 1 \rangle) \otimes k\{1, \xi_2, \xi_2^2\}\right) \\ &\cong \bigoplus_{\substack{\xi_{n_1}\cdots\xi_{n_d} \\ n_i \geq 3 \text{ distinct}}} b_{10}^{-1} \operatorname{Ext}_D^* \left(k, T(\langle \xi_{n_1} \ldots \xi_{n_d} \ ; \ 1 \rangle)\right) \otimes k\{1, \xi_2, \xi_2^2\}. \end{aligned}$$

By Proposition 4.3.7, $b_{10}^{-1} \operatorname{Ext}_D^d(k, T(\langle \xi_{n_1} \ldots \xi_{n_d} \ ; \ 1 \rangle))$ is generated by $e_{n_1} \ldots e_{n_d}$, where

$$e_n = [\xi_1]\xi_n - [\xi_1^2]\xi_{n-1}^3 \in b_{10}^{-1} \operatorname{Ext}_D^1(k, T(\langle \xi_n \ ; \ 1 \rangle))$$

is primitive. The map $B \to B_{1,\infty}$ gives rise to a map of MPASS's, and in particular a map $b_{10}^{-1} \operatorname{Ext}_D^*(k, B) \to b_{10}^{-1} \operatorname{Ext}_D^*(k, B_{1,\infty})$ of Hopf algebras over $b_{10}^{-1} \operatorname{Ext}_D^*(k, k)$ sending $e_n \mapsto e_n$ for $n \geq 3$, and $e_2 \mapsto h_{10} \cdot \xi_2$. In particular, we have

$$b_{10}^{-1} \operatorname{Ext}_D^*(k, B_{1,\infty}) \cong E[h_{10}, e_3, e_4, \ldots] \otimes P[b_{10}^{\pm 1}] \otimes k\{1, \xi_2, \xi_2^2\} \tag{6.3.7}$$

and $e_n \in b_{10}^{-1} \operatorname{Ext}_D^*(k, B_{1,\infty})$ is primitive. To find the coproduct on the elements ξ_2 and ξ_2^2, use (6.3.6), in particular the fact that the (reduced) Hopf algebra diagonal corresponds to d_1 in the filtration spectral sequence. In particular, $\xi_2 \in b_{10}^{-1} \operatorname{Ext}_D^*(k, B_{1,\infty})$ corresponds to the element $[\xi_2] \in F^1/F^2 C^1_{D_{1,\infty}}(k, k)$, and we have $\overline{d}_{\text{cobar}}([\xi_2]) = [\xi_1|\xi_1^3]$ which is zero in $C^*_{D_{1,\infty}}(k, k)$, so ξ_2 is primitive. Similarly, the cobar differential on $C^*_{D_{1,\infty}}(k, k)$ shows $\overline{\Delta}(\xi_2^2) = 2\xi_2 \otimes \xi_2$. Thus the tensor factor $k\{1, \xi_2, \xi_2^2\}$ is, as a coalgebra, a truncated polynomial algebra. This finishes the determination of the coalgebra structure of $b_{10}^{-1} \operatorname{Ext}_D^*(k, B_{1,\infty})$ in (6.3.7). □

The E_2 page (6.3.5) of the MPASS is the cohomology of the Hopf algebroid

$$(b_{10}^{-1}\operatorname{Ext}_D^*(k,k),\ b_{10}^{-1}\operatorname{Ext}_D^*(k,B_{1,\infty})) = (E[h_{10}]\otimes P[b_{10}^{\pm 1}],\ E[h_{10},e_3,e_4,\dots]\otimes P[b_{10}^{\pm 1}]\otimes D[\xi_2])$$

so we have:

Corollary 6.3.13. *The MPASS E_2 page is:*

$$E_2^{**} \cong E[h_{10},h_{20}]\otimes P[b_{10}^{\pm 1},b_{20},w_3,w_4,\dots].$$

Proposition 6.3.14. *The map θ' induces an isomorphism of E_2 pages.*

Proof. We first calculate the E_2 page of the filtration spectral sequence on $C^*_{D_{1,\infty}}(k,k)$, and observe it is isomorphic to the E_2 page of the MPASS we calculated in Corollary 6.3.13. Then we show that the map θ' induces this isomorphism.

In the associated graded, there is a differential $d_0(a_2) = -a_1^2$, but the corresponding differential on q_2 is a d_1. So the filtration spectral sequence ${}^UE_r$ computing $H^*(b_{10}^{-1}\widetilde{U}^*\otimes W')$ has E_0 page

$${}^UE_0 \cong b_{10}^{-1}UL^*(\xi_1)\otimes UL^*(\xi_2)\otimes W'$$

with differential $d_0(u_1\otimes u_2\otimes w) = d(u_1)\otimes u_2\otimes w$. So

$${}^UE_1 \cong H^*(b_{10}^{-1}UL^*(\xi_1))\otimes UL^*(\xi_2)\otimes W' \cong E[h_{10}]\otimes P[b_{10}^{\pm 1}]\otimes UL^*(\xi_2)\otimes W'$$

and the only remaining differential is generated by $d_1(q_2) = -q_1^2$, so

$${}^UE_2 \cong E[h_{10}]\otimes P[b_{10}^{\pm}]\otimes H^*(UL^*(\xi_2))\otimes W' = E[h_{10},h_{20}]\otimes P[b_{10}^{\pm 1},b_{20}]\otimes W'.$$

Then $E_r \cong E_2$ for $r \geq 2$.

To show that θ' is an isomorphism, it suffices to show that $\theta'(h_{10}) = h_{10}$, $\theta'(b_{10}) = b_{10}$,

$\theta'(h_{20}) = h_{20}$, $\theta'(b_{20}) = b_{20}$, and $\theta'(w_n) = b_{10}w_n$ for $n \geq 3$. We use the fact that θ' extends θ multiplicatively using the concatenation product in the cobar complex. So $\theta'([a_1|\dots|a_n]) = \prod \theta(a_i)$, and we have:

$$\theta'(h_{10}) = \theta'([\xi_1]) = \theta(\xi_1) = a_1$$

$$\theta'(b_{10}) = \theta'([\xi_1|\xi_1^2] + [\xi_1^2|\xi_1]) = \theta(\xi_1)\theta(\xi_1^2) + \theta(\xi_1^2)\theta(\xi_1) = a_1a_2 + a_2a_1 = b_{10}$$

$$\theta'(h_{20}) = \theta'([\xi_2]) = \theta(\xi_2) = q_1$$

$$\theta'(b_{20}) = \theta'([\xi_2|\xi_2^2] + [\xi_2^2|\xi_2]) = \theta(\xi_2)\theta(\xi_2^2) + \theta(\xi_2^2)\theta(\xi_2) = q_1q_2 + q_2q_1 = b_{20}$$

$$\theta'(w_n) = \theta'([\xi_1|\xi_n] - [\xi_1^2|\xi_{n-1}^3]) = a_1a_2w_n + a_2a_1w_n = b_{10}w_n. \qquad \square$$

Since $\theta' : C^*_{D_{1,\infty}}(k,k) \to \widetilde{U}^* \otimes W'$ induces an isomorphism of spectral sequences, it induces an isomorphism in cohomology, completing the proof of Theorem 6.3.11. isomorphism

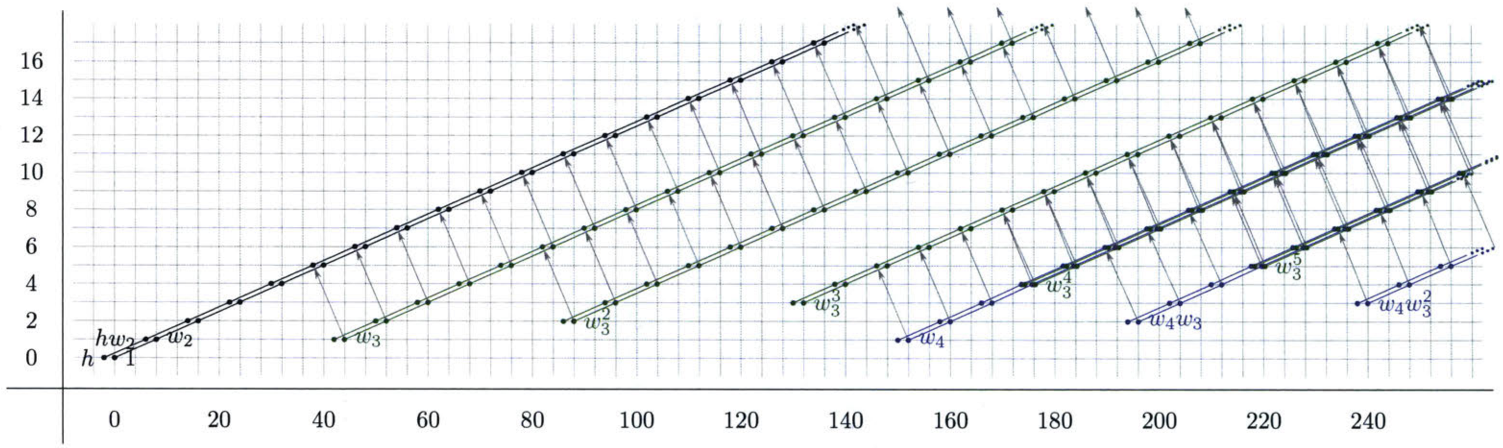


Figure 6-3: E_4 page of the $K(\xi_1)$-based MPASS, with d_4 differentials shown

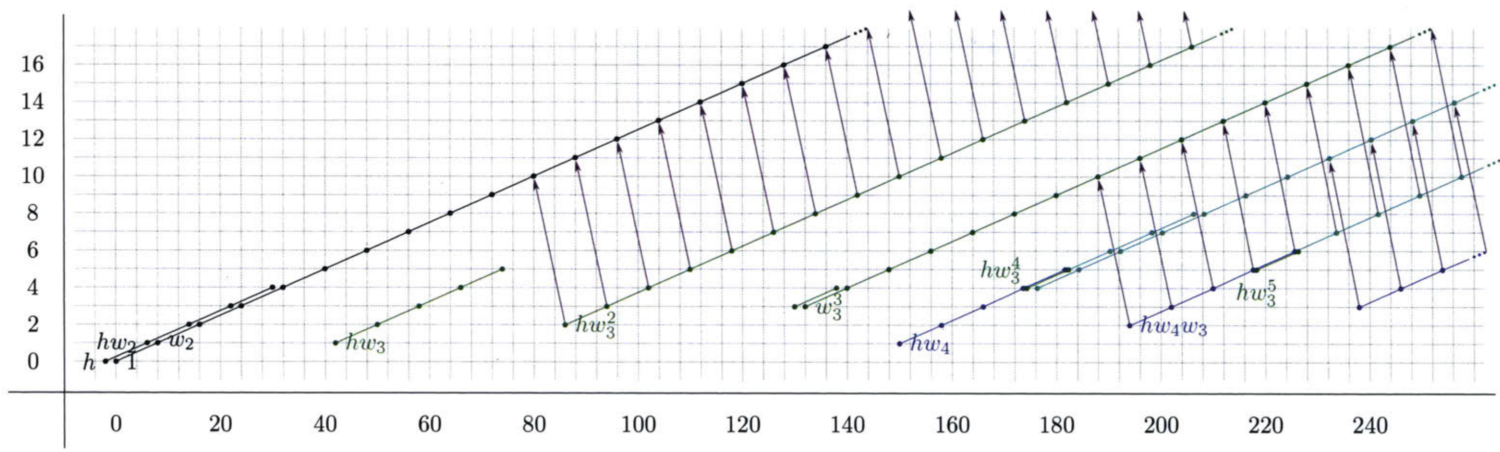


Figure 6-4: E_8 page of the $K(\xi_1)$-based MPASS

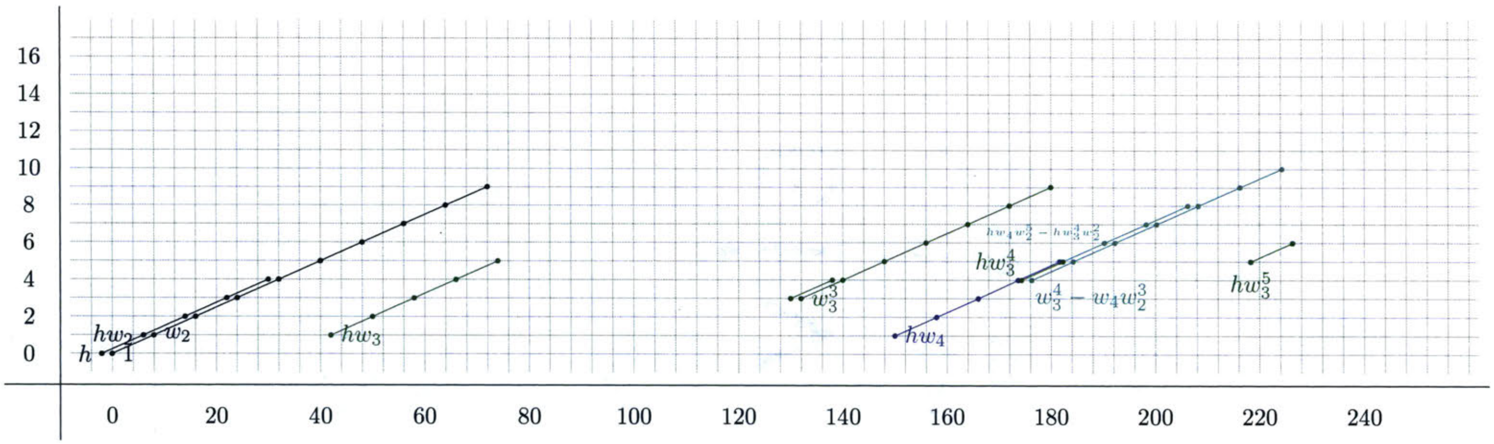


Figure 6-5: E_∞ page of the $K(\xi_1)$-based MPASS

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