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Hydrodynamics and critical slowing down

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Abstract

We introduce an effective theory which extends hydrodynamics into a regime where the critical slowing down would otherwise make hydrodynamics inapplicable.

1. Introduction

Hydrodynamics [1] is an extremely versatile theory with a wide range of applications. Its recent developments have largely concentrated on applications to relativistic heavy-ion collisions where it can describe bulk evolution of the QCD matter as well as the evolution of fluctuations and transport of charge, including anomalous chiral transport. One would like to apply hydrodynamics to describe the QCD matter evolution near the QCD critical point [2], which would greatly facilitate the analysis necessary for the discovery of this point in the beam energy scan experiments. However, hydrodynamics notoriously fails close to the critical point due to the critical slowing down. The purpose of this work is to address this shortcoming and propose a solution [3].

Hydrodynamics is an effective theory describing the dynamical space-time evolution of the densities of *conserved* quantities – energy, momentum and charge (or charges). The possibility of such a description is predicated on the separation of time scales. In linearized regime, in terms of the wave vector k, the relaxation rates of hydrodynamic modes is typically proportional to k^2 due to the conservation. All other modes are typically evolving on much faster scales set by microscopic dynamics, e.g., by collision rates or temperature.

However, there are important situations in which there are some modes in the theory which are not associated with conserved charges, but are nevertheless *parametrically* slow. That means that their evolution rate Γ can be made arbitrarily small by tuning a parameter, even though, for a fixed value of the parameter, Γ remains *finite* in the hydrodynamic limit $k \to 0$.

In this situation, hydrodynamics is not applicable in the regime where hydrodynamic modes become almost as fast as the non-hydrodynamic mode. The domain of applicability of hydrodynamics shrinks to zero as $\Gamma \to 0$. Our goal is to extend the applicability of hydrodynamics in this context. To achieve this one needs to add an additional non-hydrodynamic mode associated with the *parametrically* slow relaxation rate Γ. We shall refer to such an extension of hydrodynamics as "Hydro+".

Extending hydrodynamics by additional modes is not a new idea. The most well-known example is the Israel-Stewart theory. However, the relaxation rate of the additional modes in the Israel-Stewart extension

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of the hydrodynamics is on the order of the microscopic scale and is *not* parametrically small. As a result, there is no justification to include these microscopic modes and leave out others [4]. We wish to explore the situations in which such a parametric separation of scales between the additional slow mode and all the other microscopic fast modes makes Hydro+ a well-defined effective theory.

The two important examples of such additional parametric slowing down are: 1) The critical slowing down at the critical point; 2) The hydrodynamics of an axial or chiral charge whose conservation is violated explicitly by a small parameter, e.g., small quark mass. Both examples are of direct relevance to the study of the QCD phase diagram using heavy-ion collisions. In the case of the critical point dynamics, the parameter controlling the slow-down is the correlation length ξ: the characteristic rate of the relaxation to equilibrium is proportional to ξ^{-z}, with $z \approx 3$ [5], and can be arbitrarily small as ξ diverges near the critical point.

A direct indication that hydrodynamic approach breaks down near the critical point is the divergence of the bulk viscosity ζ, as $\xi^{z-\alpha/\nu}$, or, in the approximation sufficient for our discussion, ξ^3. The gradient expansion can be trusted for $c_s k \ll \zeta k^2/w$, which translates to the domain of applicability of hydrodynamics $k \ll c_s w/\zeta \sim \xi^{-3}$. Near the critical point this is much less than the inverse correlation length ξ^{-1}, let alone the microscopic scale such as $1/T$.

Such a situation is familiar in effective field theory. Integrating fields which are lighter than the scales we are interested in would lead to the breakdown of locality. To make effective theory *local* one needs to include all the light fields in the effective description. This is the essence of the Wilsonian paradigm.

Similarly, we can remove the divergent bulk viscosity coefficient and extend the applicability of hydrodynamics by augmenting it with an additional mode (or modes) whose relaxation rate vanishes as ξ^{-3}.

2. Hydro+

Now, to summarize, we are motivated to consider the following effective theory describing the evolution of conserved densities plus a non-hydrodynamic mode we shall call ϕ. The most important ingredient of the theory is the non-equilibrium, or more precisely, quasi-equilibrium entropy $s(\varepsilon, n, \phi)$. The microscopic meaning of this quantity is, as usual, the logarithm of the number of the quantum states of the system with given values of ε, n as well as ϕ. The derivatives of the entropy define thermodynamically conjugate quantities β, α and π:

$$ds = \beta d\varepsilon - \alpha dn - \pi d\phi. \tag{1}$$

While, due to conservation of energy and charge, $\beta = 1/T$ and $\alpha = \mu/T$ can take arbitrary values in equilibrium, the equilibrium value of π must be zero, since ϕ is not a conserved quantity and will relax to its equilibrium value $\phi_{\text{eq}}(\varepsilon, n)$, which maximizes the entropy $s(\varepsilon, n, \phi)$. This relaxation is, however, parametrically slow and allows us, in a range of time scales, to consider quasi-equilibrium states characterized by $\phi \neq \phi_{\text{eq}}$, or $\pi \neq 0$.

The equations of motion in hydrodynamics, as usual, are the conservation equations for the stress-energy tensor and the 4-current:

$$\partial_\mu T^{\mu\nu} = 0; \qquad \partial_\mu J^\mu = 0. \tag{2}$$

Also as usual, to close the system we need to supplement these equations of motion with the constitutive equations

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + p g_\perp^{\mu\nu} + \Delta T^{\mu\nu}; \qquad J^\mu = n u^\mu + \Delta J^\mu; \tag{3}$$

(where $g_\perp^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$). The pressure p and the kinetic coefficients which appear in the gradient corrections $\Delta T^{\mu\nu}$ and ΔJ^μ are functions of the variables ε, n and ϕ.

Finally, Hydro+ must include an additional equation of motion for the additional mode ϕ:

$$(u \cdot \partial)\phi = -F_\phi - G_\phi(\partial \cdot u), \tag{4}$$

where the thermodynamic restoring force F_ϕ and the coefficient G_ϕ are given by corresponding constitutive equations in terms of ε, n and ϕ and their gradients.

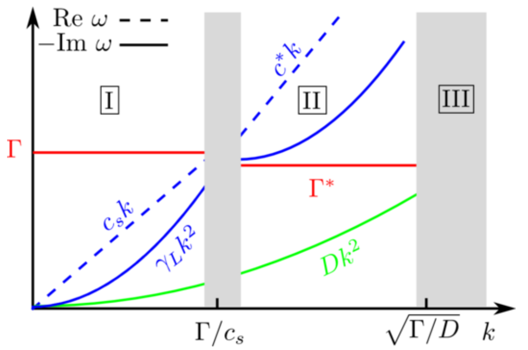

Fig. 1. The spectrum of linearized Hydro+. The conventional hydrodynamics is valid in Regime I, for as long as the relaxation rate Γ is faster than the sound oscillation rate. The sound attenuation coefficient $\gamma_L \sim \zeta/w \sim c_s^2/\Gamma$ diverges as $\Gamma \to 0$. In Regime II the sound is the fastest mode and its attenuation rate is slower.

The second law of thermodynamics $\partial \cdot s \geq 0$ imposes constraints on the constitutive equations. At the leading order (ideal hydrodynamics) it relates the pressure to the entropy and its derivatives:

$$\beta p = s - \beta\varepsilon + \alpha n + \pi G_\phi. \tag{5}$$

At the lowest nontrivial order in gradients one finds that constitutive equations must have the form

$$\Delta J_\mu = -(\lambda \partial_\mu \alpha + \lambda_{\alpha\pi} \partial_\mu \pi); \tag{6}$$

$$F_\phi = \gamma_\pi \pi - \partial_\perp \cdot (\lambda_{\pi\pi} \partial\pi + \lambda_{\alpha\pi} \partial\alpha); \tag{7}$$

$$\Delta T^{\mu\nu} = -\eta\sigma^{\mu\nu} + \zeta^* g_\perp^{\mu\nu} (\partial \cdot u) \tag{8}$$

where the coefficients γ, λ and $\lambda_{\pi\pi}$ as well as the determinant $\lambda\lambda_{\pi\pi} - \lambda_{\alpha\pi}^2$ must be non-negative, and so must the viscosities η and ζ^*.

Further constraints on the parameters can be obtained using microscopic derivation of this effective theory, and depend on the underlying theory that Hydro+ is describing. For example, if ϕ is the axial charge density, the parameter G_ϕ equals ϕ and in Eq. (5) we recover a familiar expression for the pressure as a Legendre transform of the entropy. Parity may also forbid mixing of axial and vector charges, thus, e.g., $\lambda_{\alpha\pi} = 0$. On the other hand, in the case of the critical point, we find $G_\phi = 0$. We also find that the coefficients λ_{ij} are subject to additional constraints.

The advantage of Hydro+ is that there are no anomalously large kinetic coefficients which could lead to the breakdown of the gradient expansion at the scale below $k \sim \xi^{-1}$. Most importantly, the "bare" bulk viscosity coefficient ζ^* in Eq. (8) does not diverge as $\Gamma = \gamma_\pi (\partial\pi/\partial\phi)_{\varepsilon,n} \to 0$.

3. Linearized Hydro+

To understand the dynamics and interplay of scales in Hydro+ better it is useful to consider the spectrum of linearized perturbations around equilibrium. Figure 1 summarizes the results.

As a function of the wave number k one can identify several regimes:

Regime I, or proper hydrodynamic regime. The relaxation rate of the slow mode ϕ is parametrically faster than that of all hydrodynamic modes (sound and diffusion). The mode effectively decouples, but as a consequence, the damping coefficient of the sound mode $\gamma_L \approx \zeta/(2w) \sim c_s^2/\Gamma$ is divergent when $\Gamma \to 0$. The ordinary hydrodynamic regime breaks down at $k \sim \Gamma/c_s \sim \xi^{-3}$ when the damping rate of the sound is of the order of the sound frequency and of the order of the relaxation rate Γ (see Fig. 1).

Regime II, or Hydro+ regime. In this regime the sound oscillation rate is faster than the relaxation rate of the additional slow mode. The sound speed changes by $\Delta c_s^2 = c_s^{*2} - c_s^2$ and so does the rate Γ, but there is a simple relation between their values in regimes I and II:

$$\Gamma c_s^2 = \Gamma^* c_s^{*2}. \tag{9}$$

The viscosity coefficients in regimes I and II are related by

$$\Delta\zeta \equiv \zeta - \zeta^* = \frac{w\Delta c_s^2}{\Gamma} \tag{10}$$

which is the relativistic generalization of Landau-Khalatnikov formula [1].

Note that the "bare" bulk viscosity ζ^* does not diverge, while the divergence of the true hydrodynamic bulk viscosity ζ in Regime I is due to the slow mode rate $\Gamma \to 0$.

Finally, at around $k \sim \sqrt{\Gamma/D} \sim \xi^{-1}$ the relaxation rate of the slow mode, Γ^* and the diffusion rate Dk^2 reach the same order of magnitude (note that $D \sim \xi^{-1}$). The dynamics of these slowest modes should be matched by the mode-coupling dynamics of model-H [6] in the regime III.

4. Microscopic origins of Hydro+

The fluctuations around equilibrium are controlled by the entropy via Einstein's formula $P \sim e^S$. Near the critical point the entropy as a function(al) of the hydrodynamic variables has a "flat direction". The direction in which the curvature of the entropy is the smallest near the critical point is the direction along which the fluctuations are the largest. It is convenient to rotate the basis of fluctuations from ε, n to their linear combinations $\mathcal{E}$ and $\mathcal{N}$, where the flattest direction is $\delta\mathcal{E} = 0$. The large fluctuations of the variable $\mathcal{N}$ can be described by a two-point function, to which we can apply Wigner transformation:

$$f_{\mathcal{N}}(t, \boldsymbol{x}, \boldsymbol{Q}) = \int_{\boldsymbol{y}} \langle\, \delta\mathcal{N}(t, \boldsymbol{x} + \boldsymbol{y}/2)\, \delta\mathcal{N}(t, \boldsymbol{x} - \boldsymbol{y}/2)\,\rangle e^{-i\boldsymbol{Q}\cdot\boldsymbol{y}}. \tag{11}$$

We can view $f_{\mathcal{N}}$ as a mode distribution function, similar to the particle distribution function in a kinetic theory. The variable $\boldsymbol{Q}$ can be viewed as an index of the mode. The characteristic scale for $\boldsymbol{Q}$ is $Q \sim \xi^{-1}$, thus ensuring the separation of scales between the typical gradients in Hydro+ and the underlying microscopic theory (model H). The mode distribution function obeys relaxation equation

$$(u \cdot \partial) f_{\mathcal{N}} = -\gamma(\boldsymbol{Q})\, \pi_{\mathcal{N}}(\boldsymbol{Q}), \tag{12}$$

where the relaxation coefficient $\gamma(\boldsymbol{Q})$ is proportional to the Kawasaki function known from model-coupling theory [5] and $\pi_{\mathcal{N}} = \partial s / \partial f_{\mathcal{N}}$. The equations of hydrodynamics close once the entropy functional $s(\varepsilon, n, f_{\mathcal{N}})$ is given [3]. The resulting theory can be shown to reproduce the expressions for the frequency-dependent bulk viscosity and the shift of the sound speed Δc_s^2.

Hydro+ — the hydrodynamic theory augmented by a slow mode ϕ or a family of modes $f_{\mathcal{N}}$ — should allow efficient modeling of the evolution of the fireball created in relativistic heavy-ion collisions, including the regimes where the critical slowing down and divergent bulk viscosity would otherwise render conventional hydrodynamic description inapplicable.

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