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Pointed Hopf actions on fields, II

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POINTED HOPF ACTIONS ON FIELDS, II

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ABSTRACT. This is a continuation of the authors' study of finite-dimensional pointed Hopf algebras H which act inner faithfully on commutative domains. As mentioned in Part I of this work, the study boils down to the case where H acts inner faithfully on a field. These Hopf algebras are referred to as Galois-theoretical.

In this work, we provide classification results for finite-dimensional pointed Galois-theoretical Hopf algebras H of finite Cartan type. Namely, we determine when such H of type $A_1^{\times r}$ and some H of rank two possess the Galois-theoretical property. Moreover, we provide necessary and sufficient conditions for Reshetikhin twists of small quantum groups to be Galois-theoretical.

1. INTRODUCTION

The goal of this paper is to continue the study of finite-dimensional pointed Hopf actions on commutative domains over an algebraically closed field $\Bbbk$ of characteristic zero, which we started in [15]. [1] By [15, Lemma 9 and Remark 3], this reduces to the study of Hopf actions on fields containing $\Bbbk$. Moreover, by passing to appropriate Hopf quotients, it suffices to consider *inner faithful* actions, i.e., those not factoring through a 'smaller' Hopf algebra (Definition 2.2), and we will do so throughout the paper. Since faithful actions of finite groups on fields are studied by Galois theory, in [15] we made the following definition.

Definition 1.1. A Hopf algebra H over $\Bbbk$ is said to be *Galois-theoretical* if it acts inner faithfully on a field containing $\Bbbk$.

Examples of Galois-theoretical Hopf algebras are provided in [15, Theorem 2]. They include Taft algebras, $u_q(\mathfrak{sl}_2)$, and twists of small quantum groups. At the same time, it is shown in [15] that many familiar examples of finite-dimensional Hopf algebras (such as $\mathrm{gr}(u_q(\mathfrak{sl}_2))$ and generalized Taft algebras) are not Galois-theoretical.

Our goal here is to classify Galois-theoretical finite-dimensional pointed Hopf algebras in as many cases as possible. We believe that the methods of this paper with some additional work can lead to a complete solution of this problem, at least in the special case when the group of grouplike elements is abelian.

Recall that an important invariant of a pointed Hopf algebra is its *rank* θ (Definition 2.4). Our first result discusses the rank one case. Namely, let H be a finite-dimensional pointed Hopf algebra of rank one. Then by [16, Theorem 1(a)], H is generated by the group of grouplike elements $G = G(H)$ and a $(g, 1)$-skew-primitive element x, for some g contained in the center $Z(G)$ of G.

Theorem 1.2 (Theorem 4.2). *Let H be a finite-dimensional pointed Hopf algebra of rank one, as above. Then, H is Galois-theoretical if and only if the Hopf subalgebra generated by $\{g, x\}$ is a Taft algebra.*

For finite-dimensional pointed Hopf algebras H of higher rank, we make the following assumptions for the rest of the paper, unless stated otherwise.

[1]The reference numbers for the published version of [15] differ from the reference numbers of the preprint version; see the Appendix of `http://arxiv.org/abs/1403.4673`(v4) for the key.

Hypothesis 1.3. Take H to be a finite-dimensional pointed Hopf algebra and assume that $G = G(H)$ is abelian. Now the associated coradically graded Hopf algebra $\operatorname{gr}(H)$ is isomorphic to the bosonization of a Nichols algebra $\mathfrak{B}(V)$ by $\Bbbk G$ (see Proposition 2.5). In other words, H is a lifting of $\mathfrak{B}(V) \# \Bbbk G$. Assume that $\mathfrak{B}(V)$ is of *finite Cartan type* (see Definition 2.6(b)). [2]

Under the hypotheses above, H is generated by G and $(g_i, 1)$-skew-primitive elements x_i for $g_i \in G$, so that $g_i x_j = \chi_j(g_i) x_j g_i$, where χ_j is a character of G. If $x_i^{n_i} = 0$ for some $n_i \geq 2$, then H is a finite-dimensional Hopf quotient of the (typically infinite-dimensional) pointed Hopf algebra $H(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$ of rank θ from Definition 3.1.

For this reason, the following theorem will play a central role in this paper. For any $g \in G$ and $\chi \in \widehat{G}$ (the character group of G), let $I_{g,\chi} := \{1 \leq i \leq \theta \mid g_i = g, \chi_i = \chi\}$.

Theorem 1.4 (Theorem 3.3). *Let L be a field containing $\Bbbk$ and equipped with a faithful action of G. Then, the extensions of the G-action on L to a (not necessarily inner faithful) $H(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$-action are defined by the formula*

$$(1.5) \qquad\qquad\qquad\qquad x_i \mapsto w_i(1 - g_i),$$

where $w_i \in L$ is such that $g \cdot w_i = \chi_i(g) w_i$ for all $g \in G$.

Moreover, for each g, χ, the skew-primitive elements $\{x_i \mid i \in I_{g,\chi}\}$ are linearly independent as $\Bbbk$-linear operators on L if and only if so are the elements w_i; this can be achieved by an appropriate choice of w_i. [3]

Using this result, we provide in Example 3.14 an illustration of how to construct an inner faithful action of a finite-dimensional pointed Hopf algebra on a field as an extension of a faithful group action.

Now we continue our study in the coradically graded case. We have the following result for type $A_1^{\times \theta}$.

Theorem 1.6 (Theorem 4.5). *Let H be a finite-dimensional pointed coradically graded Hopf algebra of finite Cartan type $A_1^{\times \theta}$. Then, H is Galois-theoretical if and only if the Hopf subalgebra H' of H generated by $\{g_1, \ldots, g_\theta, x_1, \ldots, x_\theta\}$ is the tensor product of*

- *Taft algebras $T(n, \zeta)$,*
- *Nichols Hopf algebras $E(n)$, and*
- *book algebras $\mathbf{h}(\zeta, 1)$.*

See [15, Sections 3.1, 3.2, 3.4] for the presentations of $T(n, \zeta)$ $(=:T(n))$, $E(n)$, and $\mathbf{h}(\zeta, 1)$, respectively.

The next theorem describes the Galois-theoretical properties of finite-dimensional pointed coradically graded Hopf algebras of rank two.

Theorem 1.7 (Theorem 5.7). *Let H be a finite-dimensional pointed coradically graded Hopf algebra of rank two, subject to condition (2.8) on the values of $\chi_i(g_i)$. Then, H is Galois-theoretical if and only if the Hopf subalgebra H' generated by g_1, g_2, x_1, x_2 is one of the following:*

(a) *of type $A_1 \times A_1$, namely*
  - *the tensor product of Taft algebras $T(n, \zeta) \otimes T(n', \zeta')$ for $n, n' \geq 2$, or*
  - *the 8-dimensional Nichols Hopf algebra $E(2)$, or*
  - *the book algebra $\mathbf{h}(\zeta, 1)$;*

(b) *of type A_2, B_2, or G_2 with $\chi_2(g_1) = 1$ or $\chi_1(g_2) = 1$ (here, H' is isomorphic to a twist $u_q^{\geq 0}(\mathfrak{g})^J$); or*

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[2]In fact, all finite-dimensional pointed Hopf algebras with G abelian, and with all prime divisors of $|G|$ being > 7 (subject to additional mild conditions), are liftings of bosonizations of Nichols algebras of finite Cartan type by $\Bbbk G$ [10]. See Theorem 2.9 for more details.

[3]In many cases, $|I_{g,\chi}| \leq 1$, and the condition that $\{x_i \mid i \in I_{g,\chi}\}$ are linearly independent boils down to the condition $x_i \neq 0$ as operators on L.

(c) one of the 3^4-dimensional (resp. 5^5-dimensional, 7^7-dimensional) Hopf algebras of type A_2 (resp. B_2, G_2) from [6, Theorem 1.3(ii) (resp. (iii), (iv))], where $\chi_2(g_1)$, $\chi_1(g_2) \neq 1$.

Due to the complexity of the rank two case, we leave the general study of coradically graded Galois-theoretical Hopf algebras of higher rank to future work.

Next, we discuss the Galois-theoretical properties of liftings H of coradically graded finite-dimensional pointed Hopf algebras in the cases when H is of finite Cartan type $A_1^{\times \theta}$ and of rank two. To our knowledge, liftings of Nichols algebras of type G_2 are unclassified, so we do not address this case here.

Theorem 1.8 (Theorem 6.1). *Let H be a lifting of bosonization of a Nichols algebra $\mathfrak{B}(V)$ of finite Cartan type as classified in [5] for type $A_1^{\times \theta}$. Then, H is Galois-theoretical if and only if the Hopf subalgebra of H generated by $\{g_1, \ldots, g_\theta, x_1, \ldots, x_\theta\}$ is the quotient by a group of central grouplike elements of a tensor product of:*

- *$u'_q(\mathfrak{gl}_2)$, the Hopf algebras from [15, Definition 11],*
- *Taft algebras $T(n, \zeta)$,*
- *Nichols Hopf algebras $E(n)$, and*
- *book algebras $\mathbf{h}(\zeta, 1)$.*

Theorem 1.9 (Propositions 6.2, 6.3). *Let H be a lifting of bosonization of a Nichols algebra $\mathfrak{B}(V)$ of finite Cartan type as classified in [7] for type A_2, and in [14] for type B_2. Here, we assume that the braiding parameter is a root of unity of odd order $m > 1$, with $m \geq 5$ for type A_2, and $m \neq 5$ for type B_2. In either case, if H is Galois-theoretical, then H is coradically graded.*

Finally, we address the Galois-theoretical property of twists of small quantum groups.

Theorem 1.10 (Propositions 7.1 and 7.3). *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with Cartan matrix (a_{ij}). Let $q \in \Bbbk$ be a root of unity of odd order $m \geq 3$, with $m > 3$ for type G_2. For parts (d) and (e) below, we also require that m is relatively prime to $\det(a_{ij})$, and to 3 in type G_2. Let J be a Drinfeld twist coming from the Cartan subgroup of the small quantum group (i.e., a Reshetikhin twist). Then:*

(a) *$u_q(\mathfrak{g})$ is Galois-theoretical if and only if $\mathfrak{g} = \mathfrak{sl}_2$.*

(b) *$u_q^{\geq 0}(\mathfrak{g})$ is Galois-theoretical if and only if $\mathfrak{g} = \mathfrak{sl}_2$.*

(c) *$gr(u_q(\mathfrak{g}))$ is not Galois-theoretical.*

(d) *$u_q(\mathfrak{g})^J$ can be Galois-theoretical if and only if $\mathfrak{g} = \mathfrak{sl}_n$, and in this case, there are only two of such twists J for $n \geq 3$, and one (namely, $J = 1$) for $n = 2$.*

(e) *There are precisely $2^{rank(\mathfrak{g})-1}$ twists J for which $u_q^{\geq 0}(\mathfrak{g})^J$ is Galois-theoretical.*

In the theorem above, twists are counted up to gauge transformations, as gauge equivalent twists produce isomorphic Hopf algebras.

The paper is organized as follows. Background material on Hopf algebra actions and pointed Hopf algebras of finite Cartan type is provided in Section 2. Preliminary results on the Galois-theoretical property of finite-dimensional pointed Hopf algebras of finite Cartan type are provided in Section 3. Here, we prove Theorem 1.4 and define *minimal* Hopf algebras which will be used throughout this work. Next, in Section 4, we study the Galois-theoretical property in the type $A_1^{\times \theta}$ case, which includes the rank one case; we establish Theorems 1.2 and 1.6 here. In Section 5, we restrict our attention to the coradically graded case and study Galois-theoretical H of rank two; namely, we prove Theorem 1.7. Then, in Section 6, we determine when liftings of bosonizations of certain Nichols algebras of finite Cartan type are Galois-theoretical; we verify Theorems 1.8 and 1.9 here. Finally, in Section 7, we prove Theorem 1.10 on the Galois-theoretical property of twists of small quantum groups. Suggestions for further directions of this work are presented in Section 8.

2. Background material

In this section, we provide a background discussion of Hopf algebra actions and pointed Hopf algebras of finite Cartan type. We set the following notation for the rest of the article. Unless specified otherwise:

$\Bbbk$ = an algebraically closed base field of characteristic zero; all unadorned tensor products are over $\Bbbk$

ζ, q = a primitive root of unity in $\Bbbk$ of order n and m, respectively

H = a finite-dimensional Hopf algebra with coproduct Δ, counit ε, antipode S

G = the group of grouplike elements $G(H)$ of H

$\widehat{G}$ = character group of $G = \{\alpha : G \to \Bbbk^\times\}$

$I_{g,\chi}$ = the index set $\{i \mid g_i = g,\ \chi_i = \chi\}$ for given elements $g \in G$, $\chi \in \widehat{G}$

q_{ij} = the scalar $\chi_j(g_i)$ for $\chi_j \in \widehat{G}$ and $g_i \in G$

n_i = the order of q_{ii}

(a_{ij}) = the Cartan matrix associated to a finite-dimensional semisimple Lie algebra

L = an H-module field containing $\Bbbk$

F = the subfield of invariants L^H

E = an intermediate field of the extension $L^H \subset L$

${}^G_G\mathcal{YD}$ = the category of Yetter-Drinfeld modules over $\Bbbk G$

(V, c) = finite-dimensional braided vector space in ${}^G_G\mathcal{YD}$ with basis $x_1, \ldots, x_\theta$

Since G is abelian by Hypothesis 1.3, we have that V is just a G-graded G-module.

2.1. **Hopf algebra actions.** We recall basic facts about Hopf algebra actions; refer to [17] for further details. A left H-module M has a left H-action structure map denoted by $\cdot : H \otimes M \to M$.

Definition 2.1. Given a Hopf algebra H and an algebra A, we say that H *acts on* A (from the left) if A is a left H-module, $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$, and $h \cdot 1_A = \varepsilon(h) 1_A$ for all $h \in H$, $a, b \in A$. Here, $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler notation). In the case that H acts on a field L, we refer to L as an H-*module field*.

We restrict ourselves to H-actions that do not factor through 'smaller' Hopf algebras.

Definition 2.2. Given a left H-module M, we say that M is an *inner faithful* H-module if $IM \neq 0$ for every nonzero Hopf ideal I of H. Given an action of a Hopf algebra H on an algebra A, we say that this action is *inner faithful* if the left H-module (algebra) A is inner faithful.

When given an H-action on an algebra A, one can always pass uniquely to an inner faithful $\overline{H}$-action on A, where $\overline{H}$ is some quotient Hopf algebra of H.

Next, we consider elements of a field L invariant under the H-action on L.

Definition 2.3. Let H be a Hopf algebra that acts on a field L containing $\Bbbk$ from the left. The *subfield of invariants* for this action is given by

$$L^H = \{\ell \in L \mid h \cdot \ell = \varepsilon(h)\ell \ \text{ for all } h \in H\}.$$

2.2. **Pointed Hopf algebras of finite Cartan type.** Let us begin with a discussion of pointed Hopf algebras. A nonzero element $g \in H$ is *grouplike* if $\Delta(g) = g \otimes g$, and the group of grouplike elements of H is denoted by $G = G(H)$. An element $x \in H$ is (g, g')-*skew-primitive*, if for grouplike elements g, g' of $G(H)$, we have that $\Delta(x) = g \otimes x + x \otimes g'$. The space of such elements is denoted by $P_{g,g'}(H)$. The *coradical* H_0 of a Hopf algebra H is the sum of all simple subcoalgebras of H. The *coradical filtration* $\{H_n\}_{n \geq 0}$ of H is defined inductively by

$$H_n = \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H), \quad \text{for } n \geq 1$$

where $H = \bigcup_{n \geq 0} H_n$. We say that a Hopf algebra H is *pointed* if all of its simple H-comodules (or equivalently, if all of its simple H-subcoalgebras) are 1-dimensional. When H is pointed, we have that $H_0 = \Bbbk G$ and $H_1 = \Bbbk G + \sum_{g,g' \in G} P_{g,g'}(H)$. Following [16], we measure the complexity of a pointed Hopf algebra by considering its *rank*.

Definition 2.4. Let H be a Hopf algebra with coradical filtration $\{H_n\}_{n \geq 0}$, where the coradical H_0 is a Hopf subalgebra and H is generated by H_1 as an algebra. The *rank* of H is θ if $\dim(\Bbbk \otimes_{H_0} H_1) = \theta + 1$.

To study finite-dimensional pointed Hopf algebras H, it is convenient to assume that $G = G(H)$ is abelian (as we have done in Hypothesis 1.3) since we have the following result.

Proposition 2.5. [12] *Let H be a finite-dimensional pointed Hopf algebra with an abelian group of grouplike elements G. Then, the associated coradically graded Hopf algebra $gr(H)$ is isomorphic to a bosonization $\mathfrak{B}(V) \# \Bbbk G$ of a Nichols algebra $\mathfrak{B}(V)$ by the group algebra $\Bbbk G$. In other words, H is generated by grouplike and skew-primitive elements.*

Proof. This follows from [12, Theorem 4.15]. Namely, such an H satisfies the Andruskiewitsch-Schneider conjecture [8, Conjecture 1.4]: it is generated in degree one (by grouplike and skew-primitive elements). $\square$

We refer the reader to [9, Section 2] for a review of Nichols algebras associated to braided vector spaces.

Now we consider a special subclass of the finite-dimensional pointed Hopf algebras, those of *finite Cartan type*. Refer to [9, Section 1.2] and [10] for further details.

Definition 2.6. Let (V, c) be a finite-dimensional braided vector space.
  (a) (V, c) is of *diagonal type* if there exists a basis $x_1, \ldots, x_\theta$ of V and scalars $q_{ij} \in \Bbbk^\times$ so that
  $$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$$
  for all $1 \leq i, j \leq \theta$. The matrix (q_{ij}) is called the *braiding matrix*.
  (b) (V, c) is of *finite Cartan type* if it is of diagonal type and
  (2.7) $$q_{ii} \neq 1 \quad \text{and} \quad q_{ij} q_{ji} = q_{ii}^{a_{ij}}$$
  where $(a_{ij})_{1 \leq i, j \leq \theta}$ is a Cartan matrix associated to a finite-dimensional semisimple Lie algebra.
  (c) The same terminology applies to the Nichols algebra $\mathfrak{B}(V)$ and a Hopf algebra H when $gr(H) \cong \mathfrak{B}(V) \# \Bbbk G$. In this case, H is a *lifting* of a finite-dimensional pointed Hopf algebra of finite Cartan type, and further, *trivial lifting* when H is coradically graded.

Note that since $G(H)$ is abelian here, the corresponding braided vector space (V, c) is automatically of diagonal type; indeed, it suffices to choose x_i to be eigenvectors of the G-action. See [15, Theorem 2] for examples of finite-dimensional pointed Hopf algebras of finite Cartan type.

Motivated by Proposition 2.5, we now give the precise presentation of finite-dimensional pointed Hopf algebras H where the order of $G(H)$ has large prime divisors; H is of finite Cartan type in this case.

Notation $[\Phi^+, \alpha, n_I]$. In the case where V is of finite Cartan type, let Φ be the root system of the Cartan matrix $(a_{ij})_{1 \leq i, j, \leq \theta}$, and let Φ^+ be the subset of positive roots. Let $\alpha_1, \ldots, \alpha_\theta$ be the simple roots. We write $i \sim j$ if the roots α_i and α_j are in the same connected component of the Dynkin diagram of Φ. Let $\mathcal{X}$ be the set of such connected components. Assume that

(2.8) $$q_{ii} \text{ has odd order, and is prime to 3 if } i \text{ lies in a component } G_2.$$

The order of q_{ii} and of q_{jj} are equal when $i \sim j$. So, we set n_I to be the order of any q_{ii} with $\alpha_i \in I$ of $\mathcal{X}$.

Theorem 2.9. (a) [10, Theorem 0.1(2)] *Let H be a finite-dimensional pointed Hopf algebra, where G is abelian so that the prime divisors of $|G|$ are > 7. Then, H is of finite Cartan type.*

(b) [8, Theorem 4.5] *Let H be a finite-dimensional pointed Hopf algebra of finite Cartan type, with G abelian. Let $(q_{ij} := \chi_j(g_i))$ be the braiding matrix of H and assume (2.8). Then, H is generated by G and by $(g_i, 1)$-skew-primitive elements x_i for $g_i \in G$ and $i = 1, \ldots, \theta$. Further, $gr(H) \cong \mathfrak{B}(V) \# \Bbbk G$ is subject to the relations of G along with the following relations:*

$$gx_i = \chi_i(g)x_i g \quad \text{for all } i, \text{ and all } g \in G$$
$$ad_c(x_i)^{1-a_{ij}}(x_j) = 0 \quad \text{for all } i \neq j$$
$$x_\alpha^{n_I} = 0 \quad \text{for all } \alpha \in \Phi_I^+, I \in \mathcal{X}.$$

Here, $(ad_c x_i)(y) = x_i y - (q_{ij_1} \cdots q_{ij_t})y x_i$ for $y = x_{j_1} \cdots x_{j_t}$. $\square$

3. Preliminary results on pointed Galois-theoretical Hopf algebras

We begin this section by examining actions of Hopf algebras $H(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$, which are infinite-dimensional when $\theta \geq 2$. We also define and discuss *minimal* Hopf algebras, which will be used throughout the rest of the paper.

3.1. **On the Hopf algebras** $H(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$**.** We consider the actions of the Hopf algebra $H(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$ defined below.

Definition 3.1. Let G be a finite abelian group. Let $g_i \in G$ be an element of order $n_i \geq 2$, for $i = 1, \ldots, \theta$. Fix characters $\chi_i : G \to \Bbbk^\times$, for $i = 1, \ldots, \theta$, so that $q_{ii} := \chi_i(g_i)$ has order n_i. Let $\underline{g} := (g_1, \ldots, g_\theta)$ and $\underline{\chi} := (\chi_1, \ldots, \chi_\theta)$. The Hopf algebra $H(G, \underline{g}, \underline{\chi})$ is generated by G and $(g_i, 1)$-skew-primitive elements x_i, for $i = 1, \ldots, \theta$, subject to the relations of G,

$$x_i^{n_i} = 0 \quad \text{and} \quad gx_i = \chi_i(g)x_i g,$$

for all $g \in G$.

Thus, $H(G, \underline{g}, \underline{\chi})$ is a quotient of the bosonization of $\Bbbk\langle x_1, \ldots, x_\theta \rangle$ by $\Bbbk G$, by the Hopf ideal of relations $(x_i^{n_i})_{i=1}^\theta$. If $\theta \geq 2$, then $H(G, \underline{g}, \underline{\chi})$ is infinite-dimensional. For a given i, the subalgebra of $H(G, \underline{g}, \underline{\chi})$ generated by $\{g_i, x_i\}$ is a Taft algebra.

Proposition 3.2. *Let H be a finite-dimensional pointed Hopf algebra of rank θ with $G = G(H)$ abelian, generated by G and $(g_i, 1)$-skew-primitive elements x_i for $i = 1, \ldots, \theta$. Then, any inner faithful action of H on a field $L \supset \Bbbk$ descends from an action of $H(G, \underline{g}, \underline{\chi})$ on L (via a surjective Hopf algebra homomorphism $H(G, \underline{g}, \underline{\chi}) \to H$) so that, for any g, χ, $\{x_i \mid i \in I_{g,\chi}\}$ are linearly independent as $\Bbbk$-linear operators on L.*

Proof. Suppose H is Galois-theoretical, and we are given an inner faithful action of H on a field $L \supset \Bbbk$. For each $1 \leq i \leq \theta$, consider the Hopf subalgebra $H_i \subset H$ generated by g_i and x_i. By [16, Theorem 1(a)], H_i is a generalized Taft algebra $T(n_i, m_i, \alpha_i)$, where m_i divides n_i. Since H_i is Galois-theoretical, by [15, Propositions 10(3) and 21] we must have that H_i is an ordinary Taft algebra $T(n_i, \zeta_i)$, that is, $x_i^{n_i} = 0$ and $g_i x_i = q_{ii} x_i g_i$ for some $q_{ii} \in \Bbbk^\times$ with ord$(q_{ii}) = n_i$. By taking $\chi_i \in \widehat{G}$ with $\chi_i(g_i) = q_{ii}$, this implies that the H-action on L descends from an $H(G, \underline{g}, \underline{\chi})$-action.

Moreover, since H acts inner faithfully on L, the operators defined by x_i on L must be linearly independent. Indeed, if $\{x_i \mid i \in I_{g,\chi}\}$ are linearly dependent, then there exists an element $f = \sum_i a_i x_i$, for $a_i \in \Bbbk$ not all zero, that acts on L as zero. So, $\langle f \rangle$ forms a Hopf ideal (as f is $(g, 1)$-skew-primitive), which contradicts inner faithfulness. $\square$

Proposition 3.2 shows that it is important to determine the structure of the $H(G, \underline{g}, \underline{\chi})$-module fields L. This is done in the following theorem.

Theorem 3.3. *Let L be a field containing $\Bbbk$ and equipped with a faithful action of G. Then, the extensions of the G-action on L to a (not necessarily inner faithful) $H(G, \underline{g}, \underline{\chi})$-action are defined by the formula*

$$(3.4) \qquad\qquad x_i \mapsto w_i(1 - g_i),$$

where $w_i \in L$ is such that $g \cdot w_i = \chi_i(g)w_i$ for all $g \in G$. In other words, x_i acts as $w_i(1 - g_i)$, as $\Bbbk$-linear operators on L.

In this case:

(a) *L is a Galois extension of the field of invariants $F = L^H = L^G$; and*

(b) *L is also a Galois extension of the intermediate field $E = L^{H'} = L^{G'}$, where G' is the subgroup of G generated by $g_1, \ldots, g_\theta$, and H' is the Hopf subalgebra of H generated by G' and $\{x_i\}_{i=1}^\theta$.*

Moreover, for each g, χ, the skew-primitive elements $\{x_i \mid i \in I_{g,\chi}\}$ are linearly independent as $\Bbbk$-linear operators on L if and only if so are the elements w_i; this case can be achieved by an appropriate choice of elements $w_i \in L$.

Remark 3.5. For any $\chi : G \to \Bbbk^\times$, there exists $w \in L^\times$ such that $g \cdot w = \chi(g)w$ for all $g \in G$. The element w is unique up to multiplying by an element of $F := L^G$. This follows by the Normal Basis Theorem: L is a free FG-module of rank one. As a consequence, the extension of the G-action to an $H(G, \underline{g}, \underline{\chi})$-action as in Theorem 3.3 is unique up to renormalization, $x_i \mapsto \lambda_i x_i$ for $\lambda_i \in F$.

Proof of Theorem 3.3. Take $F := L^G$. Since L is a free FG-module of rank one by the Normal Basis Theorem, for each $\alpha \in \widehat{G}$, we can choose $u_\alpha \in L^\times$ such that $g \cdot u_\alpha = \alpha(g)u_\alpha$; see Remark 3.5. Here, $u_\alpha u_\beta = \psi(\alpha, \beta)u_{\alpha\beta}$, where ψ is a 2-cocycle of $\widehat{G}$ with values in $F^\times$; that is to say, $\psi(\alpha, \beta)\psi(\alpha\beta, \gamma) = \psi(\beta, \gamma)\psi(\alpha, \beta\gamma)$ for all $\alpha, \beta, \gamma \in \widehat{G}$. This follows from the associativity of L: $(u_\alpha u_\beta)u_\gamma = u_\alpha(u_\beta u_\gamma)$.

Since $gx_i = \chi_i(g)x_i g$ for all $g \in G$, we claim for all $\alpha \in \widehat{G}$ that

$$(3.6) \qquad\qquad x_i \cdot u_\alpha = c_i(\alpha)u_{\alpha\chi_i},$$

where $c_i(\alpha) \in F$ satisfies two conditions:

$$(3.7) \qquad\qquad c_i(\alpha) = 0, \qquad \text{if } \alpha(g_i) = 1;$$

$$(3.8) \qquad\qquad \alpha(g_i)\psi(\alpha, \beta\chi_i)c_i(\beta) + \psi(\alpha\chi_i, \beta)c_i(\alpha) = \psi(\alpha, \beta)c_i(\alpha\beta).$$

To verify (3.7), note that if $\alpha(g_i) = 1$, then $u_\alpha \in L^{g_i}$. Since $x_i^{n_i} = 0$, we can employ [15, Theorem 11(i)] to get $x_i \cdot u_\alpha = 0$. So, we conclude that $c_i(\alpha) = 0$. To verify (3.8), compare the coefficients of $u_{\alpha\beta\chi_i}$ in the equation $x_i \cdot (u_\alpha u_\beta) = x_i \cdot (\psi(\alpha, \beta)u_{\alpha\beta})$ (using the coproduct of x_i on the left hand side).

Let $K_i := c_i(\chi_i)$. Then, setting $\alpha = \chi_i$ and $\beta = \chi_i^{m-1}$, we get from (3.8):

$$K_i\psi(\chi_i^2, \chi_i^{m-1}) + q_{ii}\psi(\chi_i, \chi_i^m)c_i(\chi_i^{m-1}) = \psi(\chi_i, \chi_i^{m-1})c_i(\chi_i^m).$$

Thus, setting $b_i(m) = \dfrac{c_i(\chi_i^m)}{q_{ii}^m\psi(\chi_i, \chi_i^m)}$, we get

$$\frac{K_i \; \psi(\chi_i^2, \chi_i^{m-1})}{q_{ii}^m \; \psi(\chi_i, \chi_i^m)\psi(\chi_i, \chi_i^{m-1})} + b_i(m-1) = b_i(m).$$

Using the 2-cocycle property of ψ, we get that $\dfrac{K_i}{q_{ii}^m \; \psi(\chi_i, \chi_i)} + b_i(m-1) = b_i(m)$. Since $b_i(0) = 0$ by (3.7), we get $b_i(m) = \dfrac{K_i}{\psi(\chi_i, \chi_i)}(q_{ii}^{-1} + q_{ii}^{-2} + \cdots + q_{ii}^{-m})$. Hence

$$(3.9) \qquad c_i(\chi_i^m) \;=\; K_i(1 + q_{ii} + \cdots + q_{ii}^{m-1})\frac{\psi(\chi_i, \chi_i^m)}{\psi(\chi_i, \chi_i)} \;=\; \widehat{K_i}\psi(\chi_i, \chi_i^m)(1 - q_{ii}^m),$$

where $\widehat{K_i} = \dfrac{K_i}{\psi(\chi_i, \chi_i)(1 - q_{ii})}$.

Now let $\beta \in \widehat{G}$ be a function so that $\beta(g_i) = 1$, and set $\alpha = \chi_i^m$ in (3.8) to get

(3.10)
$$\psi(\chi_i^{m+1}, \beta)c_i(\chi_i^m) \;=\; \psi(\chi_i^m, \beta)c_i(\chi_i^m\beta).$$

We get by (3.9) and (3.10) that

$$\psi(\chi_i^{m+1}, \beta)\widehat{K_i}\psi(\chi_i, \chi_i^m)(1 - q_{ii}^m) = \psi(\chi_i^m, \beta)c_i(\chi_i^m\beta).$$

Hence

$$c_i(\chi_i^m\beta) \;=\; \widehat{K_i}(1 - q_{ii}^m)\dfrac{\psi(\chi_i\chi_i^m, \beta)\psi(\chi_i, \chi_i^m)}{\psi(\chi_i^m, \beta)} \;=\; \widehat{K_i}(1 - q_{ii}^m)\psi(\chi_i, \chi_i^m\beta).$$

Since any $\alpha \in \widehat{G}$ is of the form $\chi_i^m\beta$, where $\beta(g_i) = 1$, and $0 \le m \le \operatorname{ord}(g_i) - 1$, we have that

(3.11)
$$c_i(\alpha) = \widehat{K_i}(1 - \alpha(g_i))\psi(\chi_i, \alpha).$$

By setting $w_i := \widehat{K_i}u_{\chi_i}$, we get

$$w_i(1 - g_i) \cdot u_\alpha \;=\; \widehat{K_i}u_{\chi_i}(1 - \alpha(g_i))u_\alpha \;=\; \widehat{K_i}(1 - \alpha(g_i))\psi(\chi_i, \alpha)u_{\alpha\chi_i}.$$

Thus by (3.6) and (3.11), we get that $x_i \mapsto w_i(1 - g_i)$. Moreover, $g \cdot w_i = \chi_i(g)w_i$ for all $g \in G$, as required.

Conversely, if we choose $w_i \in L$ such that $g \cdot w_i = \chi(g)w_i$ for $g \in G$, then it is easy to see that the formula $x_i \mapsto w_i(1 - g_i)$ defines an extension of the G-action to an $H(G, \underline{g}, \chi)$-action.

Next, it is clear that the elements $\{x_i \mid i \in I_{g,\chi}\}$ are linearly independent as $\Bbbk$-linear operators on L for each g, χ if and only if so are the elements w_i. The latter can be achieved since $\dim_{\Bbbk} F = \infty$.

Now we establish statements (a) and (b) as follows. By [15, Corollary 13], we have that $L^H = L^G (=: F)$ and the extension $L^G \subset L$ is Galois. For the same reasons, we also have that L is a Galois extension of $E := L^{H'} = L^{G'}$. $\qquad\square$

Remark 3.12. In (3.4), we have $w_i = \mu_i u_{\chi_i}$, where $\mu_i \in F$, $u_{\chi_i} \in L$, so that $g \cdot \mu_i = \mu_i$ and $g \cdot u_{\chi_i} = \chi_i(g)u_{\chi_i}$. In several computations below, we take $\mu_i = 1$ for all i without loss of generality, since g commutes with μ_i.

Corollary 3.13. *Let G be a finite group, not necessarily abelian, and consider the Hopf algebra $H(G, \underline{g}, \chi) := H(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$ defined as above, with g_i belonging to the center $Z(G)$ of G. Then, extensions of a faithful G-action on a field $L \supset \Bbbk$ to a (not necessarily inner faithful) $H(G, \underline{g}, \chi)$-action are given by formula (3.4) as in Theorem 3.3.*

Moreover, for each g, χ, the $(g, 1)$-skew-primitive elements $\{x_i \mid i \in I_{g,\chi}\}$ are linearly independent as $\Bbbk$-linear operators on L if and only if so are the elements w_i, and this can be achieved by an appropriate choice of the w_i.

Proof. Adapt the proof of Theorem 3.3, except that $F = L^{Z(G)}$, not L^G, and $G/Z(G)$ may act nontrivially on F. So, $\{w_i\}$ are unique not up to elements of F, but up to elements of $L^G = F^{G/Z(G)}$. $\qquad\square$

We now provide an example of how to construct an inner faithful action of a finite-dimensional pointed Hopf algebra on a field as an extension of a faithful group action.

Example 3.14. Let $m \ge 2$ and let q be a root of unity in $\Bbbk$ with $\operatorname{ord}(q^2) = m$. Consider the small quantum group $u_q(\mathfrak{sl}_2)$ generated by a grouplike element k, a $(k, 1)$-skew-primitive element e, and a $(1, k^{-1})$-skew-primitive element f, subject to relations:

$$ke = q^2ek, \quad kf = q^{-2}fk, \quad e^m = 0, \quad f^m = 0, \quad k^m = 1, \quad ef - fe = \dfrac{k - k^{-1}}{q - q^{-1}}.$$

Here, $G(u_q(\mathfrak{sl}_2)) = \mathbb{Z}_m$ and the Hopf algebra $H(G, \underline{g}, \chi)$ is generated by k, e, f, subject to the first five relations. To show that the action $H(G, \underline{g}, \chi)$ on a field descends to an action of $u_q(\mathfrak{sl}_2)$, we only need to work with the last relation of $u_q(\mathfrak{sl}_2)$.

We construct an action of $u_q(\mathfrak{sl}_2)$ on $\mathbb{k}(z)$ as follows. Let $G = \mathbb{Z}_m = \langle k \rangle$ act on $\mathbb{k}(z)$ by $k \cdot z = q^{-2}z$. Take $x_1 = e$ and $x_2 = kf$; these elements are $(k, 1)$-skew-primitive. Then $\chi_1(k) = q^2$ and $\chi_2(k) = q^{-2}$, as $ke = q^2ek$ and $k(kf) = q^{-2}(kf)k$. The last relation of $u_q(\mathfrak{sl}_2)$ then has the form

$$(3.15) \qquad q^2 x_1 x_2 - x_2 x_1 = (k^2 - 1)(q - q^{-1})^{-1}.$$

Thus, by Theorem 3.3, we get that the condition for the $H(G, \underline{g}, \chi)$-action on $\mathbb{k}(z)$ to descend to an action of $u_q(\mathfrak{sl}_2)$ is

$$q^2 w_1(1 - k) w_2(1 - k) - w_2(1 - k) w_1(1 - k) \quad = (k^2 - 1)(q - q^{-1})^{-1},$$

i.e., after simplifications on the left hand side,

$$w_1 w_2 (q^2 - 1)(1 - k^2) = (k^2 - 1)(q - q^{-1})^{-1}.$$

Thus, the condition is $w_1 w_2 = -q(q^2 - 1)^{-2}$. Now, set

$$w_1 = (1 - q^{-2})^{-1} z^{-1} \quad \text{and} \quad w_2 = -q^{-1}(q^2 - 1)^{-1} z$$

to get the action of e and f. Namely, we chose w_1 as above to get that e, which acts as $w_1(1 - k)$, satisfies $e \cdot z = 1$. On the other hand, kf, which acts as $w_2(1 - k)$, satisfies

$$kf \cdot z = -q^{-1}(q^2 - 1)^{-1}(1 - q^{-2})z^2 = -q^{-3}z^2.$$

Hence, $f \cdot z = q^4 kf \cdot z = -qz^2$. This recovers the action from [15, Proposition 25(2)].

3.2. **The Nichols algebra relations.** We now return to our main problem: determining when a finite-dimensional pointed Hopf algebra H with an abelian group of grouplike elements G is Galois-theoretical. By Proposition 2.5, H is generated by G and $(g_i, 1)$-skew-primitive elements x_i, for $i = 1, \ldots, \theta$.

Assume that H is coradically graded, i.e., H is the bosonization $\mathfrak{B}(V)\#\mathbb{k}G$. In this case, H is a quotient of $H(G, \underline{g}, \chi)$ of Definition 3.1. The kernel of the surjective homomorphism $\phi : H(G, \underline{g}, \chi) \to H$ is generated by some noncommutative polynomials P_α in the x_i, which are relations of the Nichols algebra $\mathfrak{B}(V)$. Note that since H is coradically graded, we can choose P_α to be homogeneous of some degree $d_i(\alpha)$ in each x_i.

Consider any action of $H(G, \underline{g}, \chi)$ on L such that G acts faithfully. By Theorem 3.3, this action is given by the formula $x_i \mapsto w_i(1 - g_i)$. Then, P_α acts on L by the operator $\prod_i w_i^{d_i(\alpha)} Q_\alpha$, where Q_α is an element of $\mathbb{k}G$. Note that Q_α is independent of the choice of the w_i and the choice of module field L.

Proposition 3.16. *Let H be a finite-dimensional, pointed, coradically graded Hopf algebra with an abelian group of grouplike elements G. Then, H is Galois-theoretical if and only if the elements $Q_\alpha \in \mathbb{k}G$ vanish for all α.*

Proof. To prove the forward direction, assume that H is Galois-theoretical, and acts inner faithfully on a field L. Then, we can pull back the action of H on L to an action of $H(G, \underline{g}, \chi)$, where P_α acts by zero. Since H acts on L inner faithfully, each x_i acts by nonzero. So, each w_i is nonzero. Thus, $Q_\alpha = 0$ for all α, as desired.

Conversely, fix a faithful action of G on a field L. Note that since $Q_\alpha = 0$, the formula $x_i \mapsto w_i(1 - g_i)$ defines an H-action on L. Pick $\{w_i \mid i \in I_{g, \chi}\}$ to be linearly independent; we have shown in Theorem 3.3 that this can be achieved. Then, the x_i act by linearly independent operators on L. By [17, Corollary 5.4.7], any nonzero Hopf ideal in H intersects the $\mathbb{k}$-span of $\{x_i \mid i \in I_{g, \chi}\}$ in H nontrivially. Therefore, the H-action on L is inner faithful. $\qquad\square$

This gives an effective criterion of Galois-theoreticity for coradically graded H: the relations P_α are known, which allows one to calculate explicitly the elements Q_α.

Example 3.17. In contrast with Example 3.14, take $H = \mathrm{gr}(u_q(\mathfrak{sl}_2))$. We then have a single relation $P_1 = q^2 x_1 x_2 - x_2 x_1$ of $\mathcal{B}(V)$ to consider. The corresponding element of $\Bbbk G$ is

$$Q_1 = q^2(1 - \chi_2(k)k)(1 - k) - (1 - \chi_1(k)k)(1 - k) = (q^2 - 1)(1 - k^2) \neq 0.$$

This shows that $\mathrm{gr}(u_q(\mathfrak{sl}_2))$ is not Galois-theoretical due to Proposition 3.16.

We will use the Galois-theoreticity criterion above in a number of other examples below.

Proposition 3.18. *Let H be a finite-dimensional pointed Hopf algebra with an abelian group of grouplike elements G. If H is Galois-theoretical, then a nontrivial lifting of H cannot be Galois-theoretical.*

Proof. Let $\widetilde{H}$ be a lifting of H. Assume that L is an inner faithful $\widetilde{H}$-module field. Then by Proposition 3.2, $\widetilde{H}$ is a Hopf quotient of $H(G, \underline{g}, \underline{\chi})$. Further, the action of $\widetilde{H}$ on L descends from an action of $H(G, \underline{g}, \underline{\chi})$ on L. But since H is Galois-theoretical, we have $Q_\alpha = 0$ for all α by Proposition 3.16. So, for any action of $H(G, \underline{g}, \underline{\chi})$ on L the elements P_α act by zero. In particular, the above action of $H(G, \underline{g}, \underline{\chi})$ factors through H. So, $\widetilde{H}$ is a Hopf quotient of H, of the same dimension as H. Hence, $\widetilde{H} = H$ (i.e., the action of $\widetilde{H}$ on L is, in fact, and action of H), that is, $\widetilde{H}$ is a trivial lifting of H. □

3.3. **Minimal Hopf algebras.** Let H be a pointed Hopf algebra with $G = G(H)$ abelian (so, generated by grouplike and skew-primitive elements by Proposition 2.5), and consider the following notation.

Notation $[G', H']$. Let $G' \subset G$ be the subgroup generated by all $g \in G$ for which there is a nontrivial $(g, 1)$-skew-primitive element in H. Let H' be the subalgebra of H generated by G' and all of the $(g, 1)$-skew-primitive elements for $g \in G'$.

Clearly, G' is a normal subgroup of G and $H = \Bbbk G \otimes_{\Bbbk G'} H'$.

Definition 3.19. We say that H is *minimal* if $H = H'$.

Example 3.20. We have the following examples of minimal pointed Hopf algebras from [15]:
  (a) Taft algebras $T(n, \zeta)$ with $G' = \langle g \rangle \cong \mathbb{Z}_n$;
  (b) $E(n)$ with $G' = \langle g \rangle \cong \mathbb{Z}_2$;
  (c) generalized Taft algebras $T(n, m, \alpha)$ with $G' = \langle g \rangle \cong \mathbb{Z}_n$;
  (d) book algebras $\mathbf{h}(\zeta, p)$ with $G' = \langle g \rangle = \mathbb{Z}_n$;
  (e) H_{3^4} with $G' = \langle g \rangle = \mathbb{Z}_3$; and
  (f) $u_q(\mathfrak{sl}_2)$ with $G' = \langle k \rangle = \mathbb{Z}_m$.

To study the Galois-theoretical properties of pointed coradically graded Hopf algebras with $G(H)$ abelian (as in Sections 4 and 5), we may focus on the case of H being minimal due to the following result.

Proposition 3.21. *Suppose H is a finite-dimensional, pointed, coradically graded Hopf algebra with $G = G(H)$ abelian. Then, H is Galois-theoretical if and only if so is the minimal Hopf subalgebra H'.*

Proof. Using the notation from the beginning of Subsection 3.2, we get that the elements Q_α for both H and H' are the same. So by applying Proposition 3.16 to both H and H', we get the desired result. □

4. Coradically graded Galois-theoretical Hopf algebras, type $\mathrm{A}_1^{\times \theta}$

We begin this section by classifying finite-dimensional, pointed, Galois-theoretical Hopf algebras of rank one (of type A_1). We then study the Galois-theoretical property of pointed coradically graded Hopf algebras of finite Cartan type $\mathrm{A}_1^{\times \theta}$; these are known as *bosonizations of quantum linear spaces*.

4.1. **Type A_1: Galois-theoretical Hopf algebras of rank one.** We determine precisely when a finite-dimensional pointed Hopf algebra H of rank one is Galois-theoretical. We also determine in this case the structure of the H-module fields L. The classification of finite-dimensional pointed Hopf algebras of rank one is provided in [16]; we repeat their result below.

Theorem 4.1. [16, Theorem 1(a)] *Let G be a finite group with character map $\chi : G \to \Bbbk^\times$, and take $g \in Z(G)$ and $\alpha \in \Bbbk$. Every finite-dimensional pointed Hopf algebra of rank one is generated by G and a $(g,1)$-skew-primitive element x, subject to the group relations of G and the relations*

$$x^m = \alpha(g^m - 1) \quad and \quad ax = \chi(a)xa$$

for all $a \in G$. $\square$

Note that in this situation, m is the order of the root of unity $\chi(g)$. The main result for the rank one case is as follows.

Theorem 4.2. *Let H be a finite-dimensional pointed Hopf algebra of rank one with $G = G(H)$, not necessarily abelian. Then:*
  (a) *H is Galois-theoretical if and only if the Hopf subalgebra H' generated by $\{g, x\}$ is a Taft algebra $T(n, \zeta)$. In this case, if H acts inner faithfully on a field L, then*
     (i) *L is a Galois extension of the field $F = L^G = L^H$;*
     (ii) *L is also a cyclic extension of degree n of the intermediate field $E = L^{T(n,\zeta)} = L^{\mathbb{Z}_n}$.*
  (b) *Moreover, H acts inner faithfully on any field containing $\Bbbk$ which admits a faithful action of G, so that x acts by nonzero.*

Proof. (a) First, let us assume that H is Galois-theoretical. Consider the Hopf subalgebra H' of H generated by g and x. We know by Theorem 4.1 that H' is a generalized Taft algebra $T(n, m, \alpha)$, where $n \in \mathbb{Z}_+$ so that m divides n. Now, by [15, Propositions 10(3) and 21] we must have that H' is an ordinary Taft algebra $T(n, \zeta)$.

Conversely, assume that the minimal Hopf subalgebra H' generated by g and x is a Taft algebra. Then, H' is Galois theoretical by [15, Proposition 17]. Thus, H is Galois-theoretical by Proposition 3.21.

(i,ii) Apply classical Galois theory and [15, Theorem 11].

(b) Apply Corollary 3.13. $\square$

4.2. **Type $A_1^{\times \theta}$: bosonizations of quantum linear spaces.** According to [5, Theorem 5.5], every finite-dimensional pointed coradically graded Hopf algebra of finite Cartan type $A_1^{\times \theta}$ is isomorphic to a bosonization of a *quantum linear space*. The latter is a braided Hopf algebra in ${}^G_G \mathcal{YD}$ defined in [5, Section 3]; its bosonization by $\Bbbk G$ is defined as follows.

Definition 4.3. [5] Let $\theta \geq 1$ and let G be a finite abelian group. The *bosonization of a quantum linear space* is a Hopf algebra

$$B(G, \underline{g}, \underline{\chi}) := B(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta),$$

generated by G and $(g_i, 1)$-skew-primitive x_i for $g_i \in G$, with $i = 1, \ldots, \theta$. Given characters $\chi_1, \ldots, \chi_\theta \in \widehat{G}$ with $q_{ii} := \chi_i(g_i)$ having orders $n_i \geq 2$, we have that $B(G, \underline{g}, \underline{\chi})$ is subject to the relations of G and

$$gx_i = \chi_i(g)x_i g, \qquad x_i^{n_i} = 0, \qquad x_i x_j = \chi_j(g_i)x_j x_i,$$

for all $g \in G$ and $i \neq j$. We also assume that

(4.4) $$\chi_j(g_i)\chi_i(g_j) = 1$$

for all $i \neq j$.

Note that $B(G, \underline{g}, \underline{\chi})$ is a finite-dimensional Hopf algebra quotient of the Hopf algebra $H(G, \underline{g}, \underline{\chi})$ defined in Section 3.1. Moreover, (4.4) implies that $B(G, \underline{g}, \underline{\chi})$ is of Cartan type $A_1^{\times \theta}$.

Theorem 4.5. *The Hopf algebra $B(G, \underline{g}, \underline{\chi})$ is Galois-theoretical if and only if the minimal Hopf subalgebra B' of $B(G, \underline{g}, \underline{\chi})$ generated by $\{g_1, \dots g_\theta, x_1, \dots, x_\theta\}$ is the tensor product of Taft algebras $T(n, \zeta)$, Nichols Hopf algebras $E(n)$, and book algebras $\mathbf{h}(\zeta, 1)$. In this case, if L is an inner faithful $B(G, \underline{g}, \underline{\chi})$-module field, then:*

    (i) *L is a Galois extension of the intermediate field $F = L^B = L^G$; and*

    (ii) *L is also a Galois extension of the intermediate field $E = L^{B'} = L^{G'}$, where G' is the subgroup of G generated by $g_1, \dots, g_\theta$.*

Proof. If B' is the tensor product of $T(n, \zeta)$, $E(n)$, or $\mathbf{h}(\zeta, 1)$, then B' is Galois-theoretical by [15, Propositions 10(5), 17, 19, 22]. Now, $B(G, \underline{g}, \underline{\chi})$ is Galois-theoretical by Proposition 3.21.

Conversely, if $B(G, \underline{g}, \underline{\chi})$ is Galois-theoretical with inner faithful module field L, then by Theorem 3.3 and Remark 3.12, x_i acts as $w_i(1 - g_i)$. Thus for $i \neq j$, the relations $g_i w_j = \chi_j(g_i) w_j g_i$, $x_i x_j - \chi_j(g_i) x_j x_i = 0$, and (4.4) imply that

$$
w_i(1 - g_i) w_j(1 - g_j) - \chi_j(g_i) w_j(1 - g_j) w_i(1 - g_i)
$$
$$
= w_i w_j \left[(1 - \chi_j(g_i) g_i)(1 - g_j) - \chi_j(g_i)(1 - \chi_i(g_j) g_j)(1 - g_i) \right]
$$
$$
= w_i w_j [(1 - \chi_j(g_i))(1 - g_i g_j)] = 0.
$$

Using the notation of Subsection 3.2, we have $Q_{ij} = (1 - \chi_j(g_i))(1 - g_i g_j)$. So, $Q_{ij} = 0$ by Proposition 3.16, and either $\chi_j(g_i) = 1$ or $g_i g_j = 1$ for all $i \neq j$.

To proceed, define a graph Γ whose vertices are labelled $1, \dots, \theta$, where we have an edge i—j if and only if $\chi_j(g_i) \neq 1$. In this case, $g_i g_j = 1$.

Then, any connected component C of Γ has *order* equal to the order of an element g_i for $i \in C$; this does not depend on i. (This value is also the order of $\chi_i(g_i)$ and nilpotency order of x_i, as the Hopf subalgebra generated by $\{g_i, x_i\}$ is a Taft algebra by [15, Proposition 10(3)] and Theorem 4.2(a).) We let $H(C)$ be the minimal Hopf subalgebra of $B(G, \underline{g}, \underline{\chi})$ generated by $\{g_i, x_i\}_{i \in C}$.

Sublemma 4.6. *Suppose Γ is connected and has order 2. Then, Γ is a complete graph, and $H(\Gamma) = E(\theta)$.*

Proof of Sublemma 4.6. Suppose that there are edges i—j and j—r in Γ, so we have $\chi_j(g_i) \neq 1$ and $\chi_r(g_j) \neq 1$. Then $g_i = g_j$, as Γ has order 2. Moreover, $\chi_r(g_i) = \chi_r(g_j) \neq 1$, so i—r. Thus, Γ is a complete graph. Since Γ has order 2, we have that $\text{ord}(g_i) = 2$. Now since $g_i g_j = 1$, we get that $g_i = g_j$ for all $i \neq j$. Thus, $H = E(\theta)$.

Sublemma 4.7. *Suppose that Γ is connected, has more than one vertex, and has order $n > 2$. Then, Γ has two vertices, and $H(\Gamma)$ is a book algebra $\mathbf{h}(\zeta, 1)$.*

Proof of Sublemma 4.7. Suppose that we have i—j—r in Γ. Then, let $g := g_i$, so $g_j = g^{-1}$ and $g_r = g$. Hence,

$$
(4.8) \qquad\qquad\qquad \chi_s(g_i) = \chi_s(g_j)^{-1} = \chi_s(g_r)
$$

for all s. Thus, by (4.4) and (4.8),

$$
\chi_j(g_i) = \chi_j(g_j)^{-1} = \chi_j(g_r) = \chi_r(g_j)^{-1} = \chi_r(g_i) = \chi_i(g_r)^{-1} = \chi_i(g_j) = \chi_j(g_i)^{-1},
$$

so $\chi_j(g_i)$ has order 2. This yields a contradiction, as $\chi_j(g_i) = \chi_j(g_j)^{-1}$ by (4.8), which has order $n > 2$. Thus, Γ has two vertices, say 1 and 2, connected with an edge. Set $g_1 = g$, $g_2 = g^{-1}$ and $x_1 = y$, $x_2 = g^{-1}x$, so that $\Delta(x) = 1 \otimes x + x \otimes g$, $\Delta(y) = g \otimes y + y \otimes 1$. Moreover, take $\chi_1(g_2) = \zeta^{-1}$ for ζ a primitive n-th root of unity, so that $\chi_2(g_1) = \zeta$ and $xy = yx$. Now, $H(\Gamma)$ is the book algebra $\mathbf{h}(\zeta, 1)$ generated by g, x, y.

Sublemma 4.9. *We have that $B' \cong H(C_1) \otimes \cdots \otimes H(C_m)$, where C_j are the components of Γ.*

Proof of Sublemma 4.9. We only need to show that $G(B') = G(H(C_1)) \times \cdots \times G(H(C_m))$. Suppose that we have a relation $h_1 \cdots h_m = 1$, where $h_j \in G(H(C_j))$. Our job is to show that $h_j = 1$ for all j. To do so, note that for $h_r = \prod_{t \in C_r} g_t^{p_t}$, we get $\chi_i(g_t) = 1$ for $i \in C_j$, $t \in C_r$ and $j \neq r$. Hence, $\chi_i(h_r) = 1$ for $i \in C_j$ with $j \neq r$. Since $h_1 \cdots h_m = 1$, we get that $h_j = \prod_{r \neq j} h_r^{-1}$ and $\chi_i(h_j) = 1$ for all j. This implies that $h_j = 1$, as desired.

By Sublemmas 4.6 and 4.7, $H(C_j)$ is $E(n)$, a book algebra, or a Taft algebra; the latter occurs if C_j has only one vertex. Hence, if $B(G, \underline{g}, \underline{\chi})$ is Galois-theoretical, then B' is as claimed by Sublemma 4.9.

(i,ii) Apply classical Galois theory and [15, Theorem 11]. $\qquad \square$

5. Coradically graded Galois-theoretical Hopf algebras, rank two

In this section, we study the Galois-theoretical properties of finite-dimensional pointed Hopf algebras of rank two that are coradically graded. Recall we assume that $G = G(H)$ is an abelian group, and as a result, $H \cong \mathfrak{B}(V) \# \Bbbk G$ with braiding matrix $(q_{ij} = \chi_j(g_i))$.

The finite Cartan types of rank two are $A_1 \times A_1$, A_2, B_2, or G_2, with corresponding Cartan matrices (a_{ij}):

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$
$$A_1 \times A_1 \qquad\qquad A_2 \qquad\qquad\quad B_2 \qquad\qquad\quad G_2$$

We have the following lemma.

Lemma 5.1. *Let $H \cong \mathfrak{B}(V) \# \Bbbk G$ be a finite-dimensional, pointed, coradically graded Hopf algebra of rank two, not of type $A_1 \times A_1$, subject to (2.8). Let $\{x_i\}_{i=1,2}$ be a basis of $(g_i, 1)$-skew-primitive elements for the graded braided vector space V. Let H' be the minimal Hopf subalgebra generated by g_1, g_2, x_1, x_2.*

Then, H' is a finite-dimensional Hopf algebra quotient of $H(\langle g_1, g_2 \rangle, g_1, g_2, \chi_1, \chi_2)$ from Section 3.1 subject to relations:

$$ad_c(x_i)^{1-a_{ij}}(x_j) = 0 \text{ for all } i \neq j \qquad and \qquad x_\alpha^n = 0,$$

for non-simple roots α, where x_α are the Cartan-Weyl root elements, and $n = \text{ord}(g_1) = \text{ord}(g_2)$. Here, $(ad_c x_i)(y) = x_i y - (q_{ij_1} \cdots q_{ij_t}) y x_i$ for $y = x_{j_1} \cdots x_{j_t}$.

Proof. This is a special case of Theorem 2.9. In particular, $n = \text{ord}(g_1) = \text{ord}(g_2)$ as the Hopf subalgebras generated by $\{g_i, x_i\}$, for $i = 1$ or 2, are Taft algebras. $\qquad \square$

The exact form of x_α will not be important for us, due to the following result.

Lemma 5.2. *Retain the notation from Theorem 2.9 and Subsection 3.2. Suppose that each g_i is a power of a grouplike element g of order n, and $g x_\alpha = \zeta x_\alpha g$ for a primitive n-th root of unity ζ. Then the element $Q_\alpha \in \Bbbk G$ corresponding to the relation x_α^n is equal to 0.*

Proof. Let L be a module field for $H(G, \underline{g}, \underline{\chi})$. By Theorem 3.3, x_α acts on L by $w Q(g)$, where w is a monomial in terms of the w_i, and $Q(g) \in \Bbbk[g]/(g^n - 1)$. Hence, the expression x_α^n acts on L by $(w Q(g))^n$, which is $w^n \prod_{i=0}^{n-1} Q(\zeta^i g)$. Therefore, $Q_\alpha = \prod_{i=0}^{n-1} Q(\zeta^i g)$.

By Theorem 3.3, $Q(g)$ is a multiple of $1 - g$. Hence, we have that Q_α is a multiple of $\prod_{i=0}^{n-1} (1 - \zeta^i g)$. But $\prod_{i=0}^{n-1} (1 - \zeta^i g) = 1 - g^n = 0$, so we get the desired result. $\qquad \square$

The proof of the main result, Theorem 5.7, boils down to two cases: (a) $q_{12} = 1$ or $q_{21} = 1$, and (b) $g_1^2 g_2 = 1$ or $g_1 g_2^2 = 1$. We proceed in case (b) below.

Proposition 5.3. *Retain the notation from Lemma 5.1. Assume that $q_{12}, q_{21} \neq 1$. If $g_1^2 g_2 = 1$ or $g_1 g_2^2 = 1$, then H' is Galois-theoretical if and only if either*

- *H' is one of the 3^4-dimensional Hopf algebras H_{3^4} of type A_2 from [6, Theorem 1.3(ii)];*
- *H' is one of the 5^5-dimensional Hopf algebras H_{5^5} of type B_2 from [6, Theorem 1.3(iii)]; or*
- *H' is one of the 7^7-dimensional Hopf algebras H_{7^7} of type G_2 from [6, Theorem 1.3(iv)].*

Proof. By (2.7), we have that $q_{11}^{-a_{12}} q_{12} q_{21} = 1$ and $q_{12} q_{21} q_{22}^{-a_{21}} = 1$. So there exists a unique primitive n-th root of unity ζ, and a scalar $\lambda \in \Bbbk$, such that $q_{11} = \zeta^{a_{21}}$, $q_{12} = \zeta^{a_{12}a_{21}} \lambda^{-1}$, $q_{21} = \lambda$, $q_{22} = \zeta^{a_{12}}$.

If $g_1^2 g_2 = 1$, then we can take $g := g_1$ so that $g_2 = g^{-2}$, where $\mathrm{ord}(g_1) = n \geq 3$. Now

$$gx_1 = \zeta^{a_{21}} x_1 g, \qquad g^{-2}x_1 = \lambda x_1 g^{-2}, \qquad gx_2 = \zeta^{a_{12}a_{21}} \lambda^{-1} x_2 g, \qquad g^{-2}x_2 = \zeta^{a_{12}} x_2 g^{-2}.$$

The first two equations yield $\lambda = (\zeta^{a_{21}})^{-2}$, and the last two equations yield $\zeta^{a_{12}} = (\zeta^{a_{12}a_{21}} \lambda^{-1})^{-2}$. Thus,

$$(5.4) \qquad\qquad \zeta^{a_{12}} = \zeta^{-2a_{12}a_{21} - 4a_{21}}.$$

In type A_2, we have that $a_{12} = a_{21} = -1$. So, $n = 3$ by (5.4) and we get by [6, Theorem 1.3(ii)] that H' is an 3^4-dimensional Hopf algebra H_{3^4} of type A_2. In type B_2, we have that $a_{12} = -2$ and $a_{21} = -1$. So, $n = 2$ by (5.4), which contradicts (2.8). In type G_2, we have that $a_{12} = -1$ and $a_{21} = -3$. So, $n = 7$ and H' is the bosonization of one of the Nichols algebras from [6, Theorem 1.3(iv)].

If $g_1 g_2^2 = 1$, then we can take $g := g_2$ so that $g_1 = g^{-2}$. Similarly, one gets

$$(5.5) \qquad\qquad \zeta^{a_{21}} = \zeta^{-2a_{12}a_{21} - 4a_{12}}.$$

In type A_2, we have that $n = 3$ by (5.5). So again, H' is an 3^4-dimensional Hopf algebra H_{3^4} of type A_2. In type B_2, we have that $n = 5$ by (5.5). Therefore, H' is a bosonization of one of the Nichols algebras from [6, Theorem 1.3(iii)]. In type G_2, we have that $n = 1$ by (5.5), which yields a contradiction.

Hence, if $g_1^2 g_2 = 1$ or $g_1 g_2^2 = 1$, then H' is either one of the

(i) 3^4-dimensional Hopf algebras H_{3^4} of type A_2 from [6, Theorem 1.3(ii)] where $g_1^2 g_2 = g_1 g_2^2 = 1$;

(ii) 5^5-dimensional Hopf algebras H_{5^5} of type B_2 from [6, Theorem 1.3(iii)] where $g_1 g_2^2 = 1$; or

(iii) 7^7-dimensional Hopf algebras H_{7^7} of type G_2 from [6, Theorem 1.3(iv)] where $g_1^2 g_2 = 1$.

Now let us show that each of the Hopf algebras above is Galois-theoretical.

First, the Hopf algebra in (i) has braiding matrix $q_{11} = q_{12} = q_{21} = q_{22} = $ a primitive third root of unity, and is Galois-theoretical by [15, Proposition 23].

We use Proposition 3.16 to check that the Hopf algebra in (ii) is Galois-theoretical. Namely, we need each of the elements $Q_\alpha \in \Bbbk G$ corresponding to relations of the Hopf algebra to be 0. We take $g_1 = g^3$ and $g_2 = g$, where $\mathrm{ord}(g) = \mathrm{ord}(\zeta) = 5$. Apply Lemma 5.2 to conclude that the elements Q_α corresponding to the relations $x_\alpha^5 = 0$ are 0. The remaining relations of the Hopf algebra in this case are

$$ad_c(x_1)^3(x_2) = x_1^3 x_2 - (q_{11}^2 q_{12} + q_{11}q_{12} + q_{12})x_1^2 x_2 x_1 + (q_{11}^3 q_{12}^2 + q_{11}^2 q_{12}^2 + q_{11}q_{12}^2)x_1 x_2 x_1^2 - q_{11}^3 q_{12}^3 x_2 x_1^3 = 0,$$
$$ad_c(x_2)^2(x_1) = x_2^2 x_1 - (q_{21}q_{22} + q_{21})x_2 x_1 x_2 + (q_{21}^2 q_{22})x_1 x_2^2 = 0.$$

To compute the element $Q_{12} \in \Bbbk G$ corresponding to the relation $ad_c(x_1)^3(x_2)$, we use Theorem 3.3 and Remark 3.12. For instance, $x_1^3 x_2$ acts as

$$[w_1(1 - g_1)]^3 w_2(1 - g_2) = w_1^3 w_2[(1 - q_{11}^2 q_{12}g_1)(1 - q_{11}q_{12}g_1)(1 - q_{12}g_1)(1 - g_2)],$$

so $(1 - q_{11}^2 q_{12}g_1)(1 - q_{11}q_{12}g_1)(1 - q_{12}g_1)(1 - g_2)$ is a summand of Q_{12}. Now we obtain that the element Q_{12} is 0 by the Maple code below:

```
Q12:= (1-q11^2*q12*g1)*(1-q11*q12*g1)*(1-q12*g1)*(1-g2)
      -(q11^2*q12+q11*q12+q12)*(1-q11^2*q12*g1)*(1-q11*q12*g1)*(1-q21*g2)*(1-g1)
      +(q11^3*q12^2+q11^2*q12^2+q11*q12^2)*(1-q11^2*q12*g1)*(1-q21^2*g2)*(1-q11*g1)*(1-g1)
```

```
        -q11^3*q12^3*(1-q21^3*g2)*(1-q11^2*g1)*(1-q11*g1)*(1-g1);
g1:=g^3;                g2:=g;                          zeta:=exp((2/5)*Pi*I);
q11:=zeta^(-1);         q12:=zeta^2*lambda^(-1);        q21:=lambda;            q22:=zeta^(-2);
collect(simplify(Q12),[g],'distributed');
## The only powers of g that arise are 0, 10.
simplify(coeff(Q12,g,0)+coeff(Q12,g^(10)));
>>                                              0
```

Similarly, we obtain that the element Q_{21}, corresponding to the relation $ad_c(x_2)^2(x_1)$, is 0 as follows:

```
Q21:= (1-q21*q22*g2)*(1-q21*g2)*(1-g1)
      -(q21*q22+q21)*(1-q21*q22*g2)*(1-q12*g1)*(1-g2)
      +(q21^2*q22)*(1-q12*q12*g1)*(1-q22*g2)*(1-g2);
g1:=g^3;                g2:=g;                          zeta:=exp((2/5)*Pi*I);
q11:=zeta^(-1);         q12:=zeta^2*lambda^(-1);        q21:=lambda;            q22:=zeta^(-2);
collect(simplify(Q21),[g],'distributed');
## The only powers of g that arise are 0, 5.
simplify(coeff(Q21,g,0)+coeff(Q21,g^(5)));
>>                                              0
```

Now we use Proposition 3.16 to check that the Hopf algebra in (iii) is Galois-theoretical, in the same fashion as above. We take $g_1 = g$ and $g_2 = g^5$, where $\text{ord}(g) = \text{ord}(\zeta) = 7$. Apply Lemma 5.2 to conclude that the elements Q_α corresponding to the relations $x_\alpha^7 = 0$ are 0. We obtain that the elements Q_{12} and Q_{21}, corresponding to remaining relations $ad_c(x_1)^2(x_2)$ and $ad_c(x_2)^4(x_1)$, respectively, are 0 by the following Maple code:

```
Q12:= (1-q11*q12*g1)*(1-q12*g1)*(1-g2)
      -(q11*q12+q12)*(1-q11*q12*g1)*(1-q21*g2)*(1-g1)
      +(q11*q12^2)*(1-q21*q21*g2)*(1-q11*g1)*(1-g1);

Q21:= (1-q21*q22^3*g2)*(1-q21*q22^2*g2)*(1-q21*q22*g2)*(1-q21*g2)*(1-g1)
      -(q21*q22^3+q21*q22^2+q21*q22+q21)* (1-q21*q22^3*g2)*(1-q21*q22^2*g2)*(1-q21*q22*g2)*(1-q12*g1)*(1-g2)
      +(q21^2*q22^5+q21^2*q22^4+q21^2*q22^3+q21^2*q22^3+q21^2*q22^2+q21^2*q22)
          *(1-q21*q22^3*g2)*(1-q21*q22^2*g2)*(1-q12^2*g1)*(1-q22*g2)*(1-g2)
      -(q21^3*q22^6+q21^3*q22^5+q21^3*q22^4+q21^3*q22^3)
          *(1-q21*q22^3*g2)*(1-q12^3*g1)*(1-q22^2*g2)*(1-q22*g2)*(1-g2)
      +(q21^4*q22^6)*(1-q12^4*g1)*(1-q22^3*g2)*(1-q22^2*g2)*(1-q22*g2)*(1-g2);

g1:=g;                  g2:=g^5;                        zeta:=exp((2/7)*Pi*I);
q11:=zeta^(-3);         q12:=zeta^3*lambda^(-1);        q21:=lambda;            q22:=zeta^(-1);

collect(simplify(Q12),[g],'distributed');
## The only powers of g that arise are 0, 7.
simplify(coeff(Q12,g,0)+coeff(Q12,g^(7)));
>>                                              0

collect(simplify(Q21),[g],'distributed');
## The only powers of g that arise are 0, 21.
simplify(coeff(Q21,g,0)+coeff(Q21,g^(21)));
>>                                              0
```

Thus, by Proposition 3.16, the Hopf algebras in (ii) and (iii) are Galois-theoretical as well.   □

Now let us deal with case (a) discussed before Proposition 5.3, i.e., $q_{12} = 1$ or $q_{21} = 1$. Let $\widetilde{u}_{\overline{q}}^{\geq 0}(\mathfrak{g})$ be the positive part of the small quantum group of adjoint type, see [15, Definition 12]. Note that in our situation (of rank 2), we have $\widetilde{u}_{\overline{q}}^{\geq 0}(\mathfrak{g}) = u_{\overline{q}}^{\geq 0}(\mathfrak{g})$ unless we are in type A_2 and n is divisible by 3.

Proposition 5.6. *Retain the hypotheses of Lemma 5.1. Suppose that $q_{12} = 1$ or $q_{21} = 1$. Then,*

(a) *H' is isomorphic to a Drinfeld twist $\widetilde{u}_{\overline{q}}^{\geq 0}(\mathfrak{g})^J$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra of type A_2, B_2, or G_2, described in [15, Definition 12]; and*

(b) *H' is Galois-theoretical.*

Proof. (a) It is easy to see that the subgroup generated by g_1, g_2 has to be isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$. The rest is a straightforward verification; namely, the twist J can be chosen so that the skew primitive elements of $\widetilde{u}_{\overline{q}}^{\geq 0}(\mathfrak{g})^J$ have corresponding braiding parameters where either q_{12} or q_{21} is equal to 1.

(b) This follows from [15, Proposition 38] and part (a). $\square$

Now we state and prove the main result of this subsection.

Theorem 5.7. *Let H be a finite-dimensional, pointed, coradically graded Hopf algebra of rank two, subject to (2.8). Let $\{x_i\}_{i=1,2}$ be a basis of $(g_i, 1)$-skew-primitive elements for the graded braided vector space V. Then, H is Galois-theoretical if and only if the minimal Hopf subalgebra H' of H generated by g_1, g_2, x_1, x_2 is either*

(a) *of type $A_1 \times A_1$, namely*
   - *the tensor product of Taft algebras $T(n, \zeta) \otimes T(n', \zeta')$ for $n, n' \geq 2$,*
   - *the 8-dimensional Nichols Hopf algebra $E(2)$, or*
   - *the book algebra $\mathbf{h}(\zeta, 1)$;*

(b) *of type A_2, B_2, or G_2 with $q_{12} = 1$ or $q_{21} = 1$ (here, H' is isomorphic to a twist $\widetilde{u}_{\overline{q}}^{\geq 0}(\mathfrak{g})^J$); or*

(c) *of type A_2, B_2, or G_2 with $q_{12}, q_{21} \neq 1$, where H' is one of the*
   - *3^4-dimensional Hopf algebras of type A_2 from [6, Theorem 1.3(ii)] (here, $g_1^2 g_2 = g_1 g_2^2 = 1$ and $ord(g_i) = 3$),*
   - *5^5-dimensional Hopf algebras H_{5^5} of type B_2 from [6, Theorem 1.3(iii)] (here, $g_1 g_2^2 = 1$ and $ord(g_i) = 5$), or*
   - *7^7-dimensional Hopf algebras H_{7^7} of type G_2 from [6, Theorem 1.3(iv)] (here, $g_1^2 g_2 = 1$ and $ord(g_i) = 7$).*

Proof. Both directions for part (a) hold by Theorem 4.5. So, we only need to study types A_2, B_2, G_2. Consider the Serre relation of H given below:

$$(5.8) \qquad \qquad \mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0 \quad \text{if } i \neq j.$$

Recall that $(\mathrm{ad}_c x_i)(y) = x_i y - q_{ij_1} \cdots q_{ij_t} y x_i$ for $y = x_{j_1} \cdots x_{j_t}$.

For types A_2 and G_2, we have that (5.8) yields the following Serre relation for H:

$$x_1^2 x_2 - (q_{11} q_{12} + q_{12}) x_1 x_2 x_1 + q_{12}^2 q_{11} x_2 x_1^2 = 0.$$

If H is Galois-theoretical with module field L, then apply Theorem 3.3, Remark 3.12, and (2.7) to conclude that

$$(5.9) \qquad \begin{aligned} &w_1(1-g_1)w_1(1-g_1)w_2(1-g_2) - (q_{11}q_{12} + q_{12})w_1(1-g_1)w_2(1-g_2)w_1(1-g_1) \\ &+ q_{12}^2 q_{11} w_2(1-g_2)w_1(1-g_1)w_1(1-g_1) = 0. \end{aligned}$$

Using the notation of Subsection 3.2, direct computation with $g_i x_j = \chi_j(g_i) x_j g_i$, $\chi_j(g_i) = q_{ij}$, and (5.9) yields

(5.10)
$$\begin{aligned} Q_{12} = {}& (q_{11}q_{12}q_{21} - q_{11}q_{12})(q_{12} - q_{11}q_{12}q_{21})g_1^2 g_2 && + q_{12}(q_{21} - 1)(q_{11} + 1)(q_{11}q_{12}q_{21} - 1)g_1 g_2 \\ && + (q_{11}q_{12}q_{21} - 1)(1 - q_{12}q_{21})g_2 \\ && + (q_{11}q_{12} - 1)(q_{12} - 1) && = 0. \end{aligned}$$

That is, we employed Proposition 3.16. By (2.7), we have that $q_{11}q_{12}q_{21} = 1$. So, (5.10) implies that

$$(1 - q_{11}q_{12})(q_{12} - 1)(g_1^2 g_2 - 1) = 0.$$

So again by (2.7), $q_{12} = 1$ or $q_{21} = 1$, or $g_1^2 g_2 = 1$. Now apply Propositions 5.3, 5.6, and 3.21 to yield parts (b) and (c) in the case that H is of type A_2 or G_2.

For type B_2, we have that (5.8) yields the following Serre relation for H:

$$x_2^2 x_1 - (q_{22}q_{21} + q_{21})x_2 x_1 x_2 + q_{21}^2 q_{22} x_1 x_2^2 = 0.$$

Arguing as above, with $q_{12}q_{21}q_{22} = 1$, we conclude that

$$(1 - q_{21}q_{22})(q_{21} - 1)(g_1 g_2^2 - 1) = 0.$$

So, by (2.7), $q_{12} = 1$ or $q_{21} = 1$, or $g_1 g_2^2 = 1$. Apply Propositions 5.3, 5.6, and 3.21 to yield part (b) and (c) in the case that H is of type B_2. $\qquad\square$

6. Galois-theoretical lifts of type $\mathrm{A}_1^{\times \theta}$ and of rank two types

In this section, we discuss the Galois-theoretical property of liftings of various finite-dimensional, pointed, coradically graded Hopf algebras. In particular, we discuss liftings of bosonizations of quantum linear spaces and of Nichols algebras of types A_2 and B_2, which are classified by Andruskiewitsch-Schneider [5, 7] and Beattie-Dăscălescu-Raianu [14]. To our knowledge, the liftings of Nichols algebras of type G_2 have not been classified, so this case is not addressed.

6.1. **Lifting bosonizations of quantum linear spaces** $B(G, g_1, \ldots, g_\theta, \chi_1, \ldots, \chi_\theta)$, **type $\mathrm{A}_1^{\times \theta}$.** Recall that G is a finite abelian group and take $\theta \geq 2$. The finite-dimensional, pointed Hopf algebras H so that $gr(H) \cong B(G, \underline{g}, \underline{\chi})$ (of Section 4.2) have been classified in [5, Section 5]. These Hopf algebras, denoted by $A(G, \underline{g}, \underline{\chi}, \underline{\alpha}, \underline{\lambda})$, are generated by G and $(g_i, 1)$ skew-primitive elements x_i, for $g_i \in G$ with $i = 1, \ldots, \theta$. Take character maps $\chi_i \in \widehat{G}$, for $i = 1, \ldots, \theta$, so that $q_{ii} := \chi_i(g_i)$ has order n_i. Here, we have that

$$\chi_i(g_j)\chi_j(g_i) = 1.$$

Then $A(G, \underline{g}, \underline{\chi}, \underline{\alpha}, \underline{\lambda})$ is subject to the relations of G and

$$g x_i = \chi_i(g) x_i g, \qquad x_i^{n_i} = \alpha_i(1 - g_i^{n_i}), \qquad x_i x_j = \chi_j(g_i) x_j x_i + \lambda_{ij}(1 - g_i g_j),$$

for $\alpha_i, \lambda_{ij} \in \mathbb{k}$ and $i \neq j$. We may (and will) assume that $\lambda_{ij} = 0$ if $g_i g_j = 1$.

Theorem 6.1. *Let A' be the minimal Hopf subalgebra of $A = A(G, \underline{g}, \underline{\chi}, \underline{\alpha}, \underline{\lambda})$ generated by $\{g_1, \ldots, g_\theta, x_1, \ldots, x_\theta\}$. Then, A is Galois-theoretical if and only if A' is the quotient by a group of central grouplike elements of a tensor product of*

- *Hopf algebras $u'_q(\mathfrak{gl}_2)$ from [15, Definition 11],*
- *Taft algebras $T(n, \zeta)$,*
- *Nichols Hopf algebras $E(n)$, or*
- *book algebras $\mathbf{h}(\zeta, 1)$.*

Proof. The Hopf algebras $u'_q(\mathfrak{gl}_2)$, $T(n, \zeta)$, $E(n)$, $\mathbf{h}(\zeta, 1)$ are Galois-theoretical by [15, Propositions 17, 19, 22, and 33], and by [15, Proposition 10(5)], so is any tensor product H of these Hopf algebras. Also, it is easy to check that any quotient of H by a group Z of central grouplike elements is Galois-theoretical, by looking at the action of $\overline{H} := H/(z - 1, z \in Z)$ on the field of Z-invariants L^Z in an inner faithful H-module field L. So the "if" direction follows from Proposition 3.21.

Conversely, suppose that $A(G, \underline{g}, \underline{\chi}, \underline{\alpha}, \underline{\lambda})$ is Galois-theoretical, then so is the subalgebra A^i generated by g_i and x_i by [15, Proposition 10(3)]. Since A^i is a generalized Taft algebra, then by [15, Proposition 21], we get that A^i is coradically graded. Thus, $\alpha_i = 0$ for all i. Now it suffices to show that the minimal Hopf subalgebra A' is a central quotient of the tensor product of $u'_q(\mathfrak{gl}_2)$, $T(n, \zeta)$, $E(n)$, or $\mathbf{h}(\zeta, 1)$.

Applying Theorem 3.3 and Remark 3.12 to the relation $x_i x_j = \chi_j(g_i) x_j x_i + \lambda_{ij}(1 - g_i g_j)$ yields:

$$w_i(1 - g_i)w_j(1 - g_j) - \chi_j(g_i)w_j(1 - g_j)w_i(1 - g_i) - \lambda_{ij}(1 - g_i g_j)$$
$$= [w_i w_j(1 - \chi_j(g_i)) - \lambda_{ij}](1 - g_i g_j) = 0.$$

So either $w_i w_j(1 - \chi_j(g_i)) = \lambda_{ij}$ or $g_i g_j = 1$ for all $i \neq j$.

To proceed, define a graph Γ whose vertices are labelled $1, \dots, \theta$ and which has two kinds of edges — dotted ones and solid ones. Namely,

* a solid edge i—j if and only if $g_i g_j = 1$.
* a dotted edge $i \cdots j$ if and only if $g_i g_j \neq 1$ and $w_i w_j(1 - \chi_j(g_i)) = \lambda_{ij}$, with $\lambda_{ij} \neq 0$.

Thus, i and j are not connected if and only if $g_i g_j \neq 1$, $\lambda_{ij} = 0$, and $\chi_i(g_j) = 1$. Note that two vertices cannot be connected by a solid and a dotted edge at the same time.

Note that if i—j then the order of g_i equals the order of g_j. Also, if $i \cdots j$, then $w_i w_j \in \mathbb{k}$ commutes with g_s for all s, and $\chi_i(g_s) = \chi_j(g_s)^{-1}$. So $\chi_i(g_i) = \chi_j(g_i)^{-1} = \chi_i(g_j) = \chi_j(g_j)^{-1}$. Thus if two vertices are connected by a dotted edge, then g_i and g_j also have the same order. Hence, to each component C of Γ one can attach its order, which is the order of g_i for any $i \in \Gamma$. This is also the nilpotency order of x_i for any $i \in \Gamma$, since $\alpha_i = 0$.

For a component C of Γ, let $H(C)$ be the Hopf subalgebra of $A(G, \underline{g}, \underline{\chi}, \underline{0}, \underline{\lambda})$ generated by $\{g_i, x_i\}_{i \in C}$.

Let us consider the case when Γ is connected. Assume the order of Γ is $n > 2$. Suppose we have i—j—r in Γ. Then by the proof of Sublemma 4.7, we come to a contradiction. Suppose now we have $i \cdots j \cdots r$ in Γ. Then a similar proof leads to a contradiction:

$$\chi_i(g_j) = \chi_j(g_j)^{-1} = \chi_r(g_j) = \chi_j(g_r)^{-1} = \chi_i(g_r) = \chi_r(g_i)^{-1} = \chi_j(g_i) = \chi_i(g_j)^{-1}.$$

Finally, suppose that $i \cdots j$—r. Then by the above, i is not connected to r. So, $\chi_i(g_r) = 1$, and $\chi_j(g_r) = \chi_j(g_j) = 1$, a contradiction.

This means that either Γ has one vertex (so, $H(\Gamma) = T(n, \zeta)$) or two vertices connected by a solid edge (so, $H(\Gamma) = \mathbf{h}(\zeta, 1)$) or dotted edge (so, $H(\Gamma) = u'_q(\mathfrak{gl}_2)$ or its central quotient; see, e.g. the discussion before [15, Proposition 30]).

Now assume that Γ is connected and the order of Γ is $n = 2$. Then by the proof of Sublemma 4.6, Γ is a complete graph with solid and dotted edges. Indeed, the matrix $(\chi_i(g_j))$ is symmetric (as $\chi_i(g_j) = \pm 1$), thus all its entries for $i, j \in \Gamma$ are the same, so they have to be -1.

It is clear that if i—j—r then i—r. Thus, if we only keep the solid edges, Γ will fall apart into components $C_1, \dots, C_m$, such that for any $i \in C_k$, $j \in C_l$, $k \neq l$, we have $i \cdots j$. Thus if $i, r \in C_s$ and $j \in C_l$, $s \neq l$, then w_i and w_r are both scalar multiples of w_j^{-1}, and $g_i = g_r$ (as $n = 2$), so we have a contradiction with inner faithfulness. Thus, either all edges in Γ are solid or all are dotted. If all edges are solid, then $H(C) = E(n)$ for some n. Also, we cannot have a dotted triangle with vertices i, j, r, as then w_i is proportional to w_j^{-1}, hence to w_r, hence to w_i^{-1}, so $w_i \in \mathbb{k}$, a contradiction. Thus, if all edges of Γ are dotted (and there is at least one edge), then $|\Gamma| = 2$, and we get the Hopf algebra $u'_q(\mathfrak{gl}_2)$ for $q = -1$.

Finally, let Γ be arbitrary. Then, by mimicking Sublemma 4.9, we see that our Hopf algebra A' is a quotient of the tensor product of the algebras corresponding to the connected components of Γ by a subgroup of central grouplike elements. The proposition is proved. $\square$

6.2. Lifting bosonizations $\mathfrak{B}(V)\#\Bbbk G$, for V of type A_2.

Let H be a finite-dimensional, pointed Hopf algebra so that $gr(H) \cong \mathfrak{B}(V)\#\Bbbk G$, for $\mathfrak{B}(V)$ of type A_2. Such H have been classified for braiding parameter q, a primitive m-th root of unity, with $m > 1$ an odd integer. Namely, by [7, Theorems 3.6 and 3.7] for $m > 3$ and by [14, Proposition 3.3] for $m = 3$.

We proceed in the case when $m > 3$. In this case, H is generated by G and x_1, x_2, where x_i is $(g_i, 1)$-skew-primitive and $g_i \in G$ and is subject to the relations of G,

- $gx_i = \chi_i(g)x_ig$,   for $i = 1, 2$,
- $x_i^m = \mu_i(1 - g_i^m)$,   for $i = 1, 2$,
- $[x_1x_2 - \chi_2(g_1)x_2x_1]\, x_1 = \chi_1(g_2)x_1\, [x_1x_2 - \chi_2(g_1)x_2x_1]$,

along with other relations irrelevant to the proof of the result below. Here, $\mu_1, \mu_2, \lambda \in \Bbbk$, $q = \chi_1(g_1) = \chi_2(g_2)$ and $\chi_1(g_2)\chi_2(g_1) = q^{-1}$.

Proposition 6.2. *Let H be a finite-dimensional pointed Hopf algebra so that $gr(H) \cong \mathfrak{B}(V)\#\Bbbk G$, for $\mathfrak{B}(V)$ of type A_2, as above. If H is Galois-theoretical, then H is coradically graded, in which case we are in the setting of Theorem 5.7(b).*

Proof. Suppose that H is Galois-theoretical. Then, the subalgebra H^i generated by g_i and x_i is also Galois-theoretical by [15, Proposition 10(3)], for $i = 1, 2$. Since H^i is a generalized Taft algebra, then by [15, Proposition 21], H^i is coradically graded. Thus, $\mu_1 = \mu_2 = 0$. Now, H is a quotient of $H(G, g_1, g_2, \chi_1, \chi_2)$ studied in Section 3.1.

We get by Theorem 3.3 and Remark 3.12 that x_i acts as $w_i(1 - g_i)$ for $i = 1, 2$. From the relations $g_iw_j = \chi_j(g_i)w_jg_i$ and $\chi_1(g_1)\chi_1(g_2)\chi_2(g_1) = 1$ and

$$[x_1x_2 - \chi_2(g_1)x_2x_1]\, x_1 = \chi_1(g_2)x_1\, [x_1x_2 - \chi_2(g_1)x_2x_1],$$

we have that

$$
\begin{aligned}
&q_{21}w_1(1-g_1)w_1(1-g_1)w_2(1-g_2) - (q_{12}q_{21}+1)w_1(1-g_1)w_2(1-g_2)w_1(1-g_1) \\
&\quad +q_{12}w_2(1-g_2)w_1(1-g_1)w_1(1-g_1) \\
&= w_1{}^2w_2\left[(q_{12}-1)(q_{21}-1)(g_1^2g_2-1)\right] = 0.
\end{aligned}
$$

So, $q_{21} = \chi_1(g_2) = 1$, $q_{12} = \chi_2(g_1) = 1$, or $g_1^2g_2 = 1$. By Theorem 5.7, we have that H is a lift of a finite-dimensional, pointed, coradically graded Hopf algebra of type A_2 that is Galois-theoretical. Now, H is coradically graded by Proposition 3.18. $\square$

6.3. Lifting bosonizations $\mathfrak{B}(V)\#\Bbbk G$, for V of type B_2.

Let H be a finite-dimensional pointed Hopf algebra so that $gr(H) \cong \mathfrak{B}(V)\#\Bbbk G$, for $\mathfrak{B}(V)$ of type B_2. Such H have been classified for braiding parameter q, a primitive m-th root of unity, with $m \neq 5$ an odd integer. Namely, by [14, Theorems 2.6 and 2.7], H is generated by G and x_1, x_2, where x_i is $(g_i, 1)$-skew-primitive and $g_i \in G$ and is subject to the relations of G,

- $gx_i = \chi_i(g)x_ig$ for $i = 1, 2$,
- $x_i^m = \mu_i(g_i^m - 1)$ for $i = 1, 2$;
- $x_2[x_2x_1 - \chi_1(g_2)x_1x_2] - \chi_1(g_2)\chi_2(g_2)[x_2x_1 - \chi_1(g_2)x_1x_2]x_2 = 0$,

along with other relations irrelevant to the proof of the result below. Here, $\mu_1, \mu_2, \lambda \in \Bbbk$, $q^2 = \chi_1(g_1)$, $q = \chi_2(g_2)$, and $\chi_1(g_2)\chi_2(g_1) = q^{-2}$.

Proposition 6.3. *Let H be a finite-dimensional, pointed Hopf algebra so that $gr(H) \cong \mathfrak{B}(V)\#\Bbbk G$, for $\mathfrak{B}(V)$ of type B_2, as above. If H is Galois-theoretical, then H is coradically graded, in which case we are in the setting of Theorem 5.7(b).*

Proof. Suppose that H is Galois-theoretical. Then, the subalgebra H^i generated by g_i and x_i is also Galois-theoretical by [15, Proposition 10(3)], for $i = 1, 2$. Since H^i is a generalized Taft algebra, then by [15, Proposition 21], H^i is coradically graded. Thus, $\mu_1 = \mu_2 = 0$ and $m = \text{ord}(g_1) = \text{ord}(g_2)$.

Now, H is a quotient of $H(G, g_1, g_2, \chi_1, \chi_2)$ studied in Section 3.1, and we get by Theorem 3.3 and Remark 3.12 that x_i acts as $w_i(1-g_i)$ for $i = 1, 2$. From the relations $g_i w_j = \chi_j(g_i)w_j g_i$, $\chi_1(g_2)\chi_2(g_1)\chi_2(g_2) = 1$ and

$$x_2\left[x_2 x_1 - \chi_1(g_2)x_1 x_2\right] - \chi_1(g_2)\chi_2(g_2)\left[x_2 x_1 - \chi_1(g_2)x_1 x_2\right]x_2 = 0,$$

we have that

$$w_2(1-g_2)w_2(1-g_2)w_1(1-g_1) - (q_{21} + q_{21}q_{22})w_2(1-g_2)w_1(1-g_1)w_2(1-g_2)$$
$$+q_{21}^2 q_{22}w_1(1-g_1)w_2(1-g_2)w_2(1-g_2)$$
$$= w_1{}^2 w_2\left[(q_{12}^{-1}-1)(1-q_{21})(g_1 g_2^2 - 1)\right] = 0.$$

So, $q_{12} = 1$, $q_{21} = 1$, or $g_1 g_2^2 = 1$. Since $m \neq 5$, we have by Theorem 5.7 H is a lift or a finite-dimensional, pointed, coradically graded Hopf algebra of type B_2 that is Galois-theoretical. Now H is coradically graded by Proposition 3.18.   □

7. GALOIS-THEORETICAL SMALL QUANTUM GROUPS OF HIGHER RANK

We study the Galois-theoretical property of the small quantum groups $u_q(\mathfrak{g})$, $u_q^{\geq 0}(\mathfrak{g})$, and $gr(u_q(\mathfrak{g}))$, along with their Drinfeld twists. Here, let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra of type X. Note that $u_q^{\geq 0}(\mathfrak{g})$, $u_q(\mathfrak{g})$, and $gr(u_q(\mathfrak{g}))$ (and their twists) are pointed Hopf algebras of finite Cartan type X, X × X, and X × X, respectively. We use the notation and terminology of [15, Sections 1.2, 1.4, and 3.9].

Proposition 7.1. *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, not of type $A_1^{\times r}$ for $r \geq 2$. Let $q \in \Bbbk$ be a root of unity of odd order $m \geq 3$, with $m > 3$ for $\mathfrak{g}$ containing a component of type G_2. Then, we have the statements below.*

(a) *$u_q(\mathfrak{g})$ is Galois-theoretical if and only if $\mathfrak{g} = \mathfrak{sl}_2$.*
(b) *$u_q^{\geq 0}(\mathfrak{g})$ is Galois-theoretical if and only if $\mathfrak{g} = \mathfrak{sl}_2$. In this case, $u_q^{\geq 0}(\mathfrak{sl}_2)$ is a Taft algebra of dimension m^2.*
(c) *$gr(u_q(\mathfrak{g}))$ is not Galois-theoretical.*

Proof. We have that both $u_q(\mathfrak{sl}_2)$ and $u_q^{\geq 0}(\mathfrak{sl}_2)$ are Galois-theoretical by [15, Propositions 25(2) and 17]. Moreover, $gr(u_q(\mathfrak{sl}_2))$ is not Galois-theoretical by [15, Proposition 25(1)].

On the other hand, let $\mathfrak{g}$ have rank ≥ 2. Note that there exists $i \neq j$, so that $a_{ij} = -1$ with $|i - j| = 1$, since $\mathfrak{g}$ is not of type $A_1^{\times r}$ for $r \geq 2$. The Hopf algebras $u_q(\mathfrak{g})$, $u_q^{\geq 0}(\mathfrak{g})$ and $gr(u_q(\mathfrak{g}))$ have a Serre relation given as follows:

$$(7.2) \qquad\qquad e_i^2 e_j - (q^d + q^{-d})e_i e_j e_i + e_j e_i^2 = 0,$$

where $i \neq j$ as above, and e_i is a $(k_i, 1)$-skew-primitive element for a grouplike element k_i. Here, $d = 1$ for type ADE, $d = 2$ for type BCF, and $d = 3$ for type G.

By way of contradiction, suppose that these Hopf algebras H are Galois-theoretical with module field L. These Hopf algebras are quotients of $H(G, \underline{g}, \underline{\chi})$ studied in Section 3.1, so we get by Theorem 3.3 that e_i

acts on L by formula (3.4). By Remark 3.12, e_i acts as $w_i(1 - k_i)$. Now the relations between k_i and e_j of H from [15, Definition 2] imply that

$$\chi_i(k_i) = q^{2d}, \qquad \chi_i(k_j) = q^{-d}, \qquad \chi_j(k_i) = q^{-d}, \qquad \chi_j(k_j) = q^2.$$

As $k_i w_j = \chi_j(k_i) w_j k_i$, we have from (7.2) that

$$
\begin{aligned}
0 = \ & w_i(1 - k_i)w_i(1 - k_i)w_j(1 - k_j) - (q^d + q^{-d})w_i(1 - k_i)w_j(1 - k_j)w_i(1 - k_i) \\
& + w_j(1 - k_j)w_i(1 - k_i)w_i(1 - k_i) \\
= \ & w_i{}^2 w_j[(1 - q^d k_i)(1 - q^{-d}k_i)(1 - k_j) - (q^d + q^{-d})(1 - q^d k_i)(1 - q^{-d}k_j)(1 - k_i) \\
& + (1 - q^{-2d}k_j)(1 - q^{2d}k_i)(1 - k_i)] \\
= \ & \begin{cases} w_i{}^2 w_j[q^{-1}(q - 1)^2(k_i^2 k_j - 1)] & \text{if } d = 1, \\ w_i{}^2 w_j[q^{-2}(q - 1)^2(q + 1)^2(k_i^2 k_j - 1)] & \text{if } d = 2, \\ w_i{}^2 w_j[q^{-3}(q - 1)^2(q^2 + q + 1)^2(k_i^2 k_j - 1)] & \text{if } d = 3. \end{cases}
\end{aligned}
$$

Given the conditions on the order of q and the fact that $k_i^2 k_j \neq 1$, we arrive at a contradiction. Thus, $u_q(\mathfrak{g})$, $u_q^{\geq 0}(\mathfrak{g})$, and $\mathrm{gr}(u_q(\mathfrak{g}))$ are not Galois-theoretical when $\mathfrak{g}$ is of rank ≥ 2 not of type $A_1^{\times r}$. $\qquad \square$

The following result characterizes the Galois-theoretical properties of Drinfeld twists of small quantum groups.

Proposition 7.3. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and retain the notation of Proposition 7.1. Let m be relatively prime to $\det(a_{ij})$, and to 3 in type G_2. Let J be a Drinfeld twist of $u_q(\mathfrak{g})$ induced from its Cartan subgroup. Then, we have the following statements.*

(a) *There are precisely $2^{rank(\mathfrak{g})-1}$ twists J (up to gauge transformation) so that $u_q^{\geq 0}(\mathfrak{g})^J$ is Galois-theoretical.*

(b) *The Hopf algebra $u_q(\mathfrak{g})^J$ can be Galois-theoretical if and only if $\mathfrak{g} = \mathfrak{sl}_n$. In this case, there are only two of such twists J for $n \geq 3$, and one (namely, $J = 1$) for $n = 2$, up to a gauge transformation.*

Proof. (a) [15, Proposition 37] provides $2^{\mathrm{rank}(\mathfrak{g})-1}$ twists giving rise to Galois-theoretical examples, so our job is to show that no other twist works. Let J be a twist such that $u_q^{\geq 0}(\mathfrak{g})^J$ is Galois-theoretical. Then by the classification in the rank 2 case (Theorem 5.7), for each i, j, either $q_{ij} = 1$ or $q_{ji} = 1$ (and both hold if i, j are not connected). This defines an orientation on the Dynkin diagram of $\mathfrak{g}$ (with $i \to j$ if $q_{ij} \neq 1$), and there are exactly $2^{\mathrm{rank}(\mathfrak{g})-1}$ possibilities. Thus, there are no possibilities for J beyond the above, proving (a).

(b) Let J be a twist such that $u_q(\mathfrak{g})^J$ be Galois-theoretical, and b_J be the alternating bicharacter corresponding to J as in [15, Proposition 7]. Let us consider generators e_i and f_j of $u_q(\mathfrak{g})^J$, for $j \neq i$. Together with Cartan elements, they generate a coradically graded Hopf subalgebra of type $A_1 \times A_1$. Let H_{ij} be its minimal Hopf subalgebra. Then by Theorem 5.7, H_{ij} is either a tensor product of two Taft algebras or a book algebra of type $\mathbf{h}(\zeta, 1)$.

Let $b_{ij} := b_J(k_i, k_j)$. It is easy to check that the Cartan matrix of H_{ij} has the form

$$
A_{ij} = \begin{pmatrix} q^{2d_i} & q^{-d_i a_{ij}} b_{ij} \\ q^{d_j a_{ji}} b_{ij}^{-1} & q^{-2d_j} \end{pmatrix}
$$

So, if H_{ij} is a tensor product of two Taft algebras, we must have $b_{ij} = q^{d_i a_{ij}}$. Then $b_{ji} \neq q^{d_j a_{ji}}$, as the matrix $(d_i a_{ij})$ is symmetric, while $b_{ij} = b_{ji}^{-1}$. Thus, if i—j then H_{ij} or H_{ji} is a book algebra.

If H_{ij} is a book algebra, then

(7.4) $$q^{2d_i} = q^{-d_i a_{ij}} b_{ij},$$

and if H_{ji} is a book algebra, then

(7.5) $$q^{-2d_i} = q^{d_i a_{ij}} b_{ji}^{-1}.$$

Taking the product of (7.4),(7.5), we get $b_{ij} = b_{ji}$, a contradiction. Thus, H_{ij} and H_{ji} cannot both be book algebras. So, if i—j, then exactly one of the algebras H_{ij}, H_{ji} is a tensor product and exactly one is a book algebra.

Introduce an orientation on the Dynkin diagram of $\mathfrak{g}$ by putting $i \to j$ if H_{ij} is a book algebra. If $i \to j$, then H_{ji} is a tensor product of Taft algebras, so $b_{ji} = q^{d_j a_{ji}}$ and hence $b_{ij} = q^{-d_i a_{ij}}$. Thus, by (7.4) we have $q^{2d_i} = q^{-2d_i a_{ij}}$. Also $q^{2d_i} = q^{2d_j}$. So $a_{ij} = a_{ji} = -1$, i.e., $\mathfrak{g}$ is simply laced.

Next, by (the proof of) Sublemma 4.7, we cannot have $i \to j$ and $r \to j$ for i, j, r distinct, or $j \to i$ and $j \to r$ for i, j, r distinct. Hence, the Dynkin diagram of $\mathfrak{g}$ cannot contain a triple vertex. So the diagram Q and the Lie algebra $\mathfrak{g}$ must be of type A_{n-1} for some n. Moreover, the above constraint on the orientation also implies that all the edges of the Dynkin diagram are oriented in the same direction, i.e. we are left with just two orientations (which coincide if $n = 2$, i.e., there are no edges).

Finally, the twists $J^{\pm}$ corresponding to the remaining two orientations indeed give rise to Galois-theoretical Hopf algebras by [15, Corollary 31]. The theorem is proved. $\qquad\square$

8. Further directions

In this work we studied the Galois-theoretical property of the most extensively studied class of finite-dimensional, pointed Hopf algebras: those of finite Cartan type. The study of the Galois-theoretical property of such Hopf algebras of higher rank (≥ 3) is open in general (and some cases of liftings in rank 2 have also not been treated here). We also suggest the following tasks: investigate the Galois-theoretical property of finite-dimensional, pointed Hopf algebras H with $G(H)$ abelian

- of *standard type* (which properly includes finite Cartan type) [1, 11],
- of *super type* [3], or
- of *unidentified type* [13].

Moreover, it would be interesting to continue this study for known finite-dimensional, pointed Hopf algebras over a non-abelian group of grouplike elements [4, 18], or over a cosemisimple Hopf algebra (especially when the Nichols algebra is of Cartan type A) [2].

Acknowledgments

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