Orientation of Gamma-ray Telescope

by

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Nature of the Problem

The M.I.T. Cosmic Ray Group is currently constructing a gamma-ray telescope that will be placed in orbit about the earth early in 1961. To interpret the data provided by the telescope it is necessary to know the direction of observation as a function of time. This paper will discuss the problem of determining this orientation and present several solutions that are of a provisional nature due to current uncertainties as to the exact difficulties involved. The final solution must necessarily await the successful launching of the satellite and the aquisition and processing of actual data.

The telescope being considered is essentially a collimated, unidirectional detector of gamma-rays whose energies exceed 50 Mev. It consists of Cherenkov and scintillation counters in coincidence to detect and identify high energy photons. The events so detected will be combined to provide a map of the intensity of sources of gamma radiation upon the celestial sphere. The collimation of the telescope defines a 20° field of view, so that the resolution of the telescope does not place severe demands upon the accuracy of the orientation determination. Unfortunately, the problem is not appreciably simplified by this concession.

The telescope will be placed in an orbit whose nominal parameters were chosen so that the satellite stay up one year yet remain below the Van Allen radiation belts (to avoid jamming the detectors). Thus the satellite will have: period - 90 min, perigee height - 220 km, apogee height - 1040 km. The complete satellite is essentially an elongated cylinder with the gamma-ray telescope pointing along the axis (Fig. 1). It also contains the last stage rocket and auxiliary equipment for the telescope, including:

Photoelectric sensors for orientation determination,
A tape recorder and telemetry system for data transmission,
A solar power supply.

Since the tape recorder has a capacity of 100 minutes, it will accumulate data during one revolution that will be read out at high speed on command. The transmitted data will be recorded on magnetic tape at ground stations and sent to Washington where it will be processed and relayed to M.I.T. The processing consists of a conversion of the phase-modulated signal to an amplitude-modulated visual presentation recorded on 35 mm film. The problem of determining the orientation reduces to correlating the orbit of the satellite, as determined by radio tracking, with the signals from the orientation sensors.

To provide stabilization during launch, the upper stages will be spun at 700 rpm so that upon injection into orbit, the satellite will be spinning rapidly about its longitudinal axis. The moments of inertia of the satellite about the two transverse axes differ by 5 percent, in addition being 70 times that about the longitudinal axis, so that the satellite is a tri-axial body. Rotation about the minimum moment of inertia is an unstable state that will be upset by the dissipation of energy through vibration, an effect that will be encouraged by the inclusion of flexible antennae or a liquid dash pot on the satellite. Conservation of angular momentum will result in the satellite spinning about its maximum moment of inertia at 10 rpm (Fig. 1). It is intended that this stabilization be complete in several days, but there is considerable uncertainty as to how long it will take. The state of minimum energy is poorly defined, and evidence suggests that Explorer IV required several months to stabilize, even with flexible antennae.

If there were no torques acting upon the satellite, the angular momentum vector would remain fixed in space with the (known) orientation at injection. However, aerodynamic drag and the gravitional gradient produce a net torque, which causes a precession of the angular momentum vector of 1° / Day.[(1)] As the telescope rotates it will scan a 20° band upon the celestial sphere centered about the plane perpendicular to the angular momentum vector. As this vector precesses, the entire sphere will eventually be scanned. Rapid precession will not affect the experiment, but will complicate the problem of determining the orientation. Unfortunately, evidence indicates that Explorer IV precessed 10 times as rapidly as expected,[(2)] suggesting effects whose origin Dr. C. Lundquist of the National Aeronautics and Space Administration (NASA) is investigating. With additional data from current satellites, this problem should be understood before the present satellite is launched. The only conclusion possible at present is that rapid changes in the orientation of the telescope are possible.

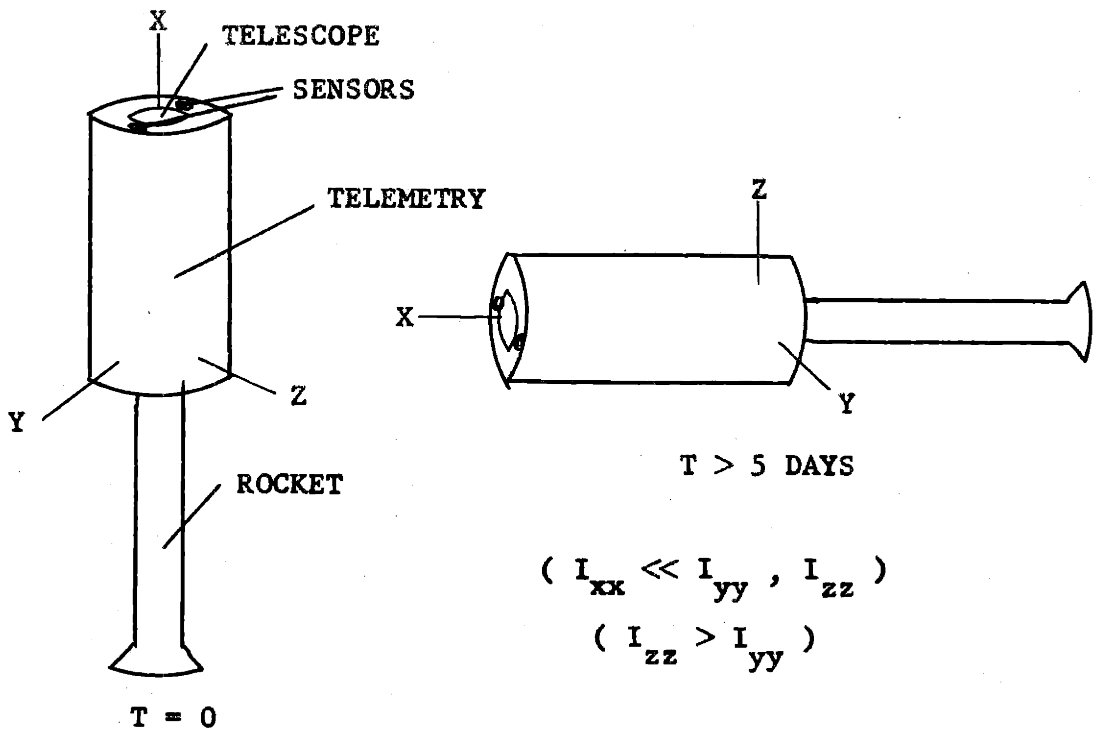


Fig. 1 · Location of principle body axes of satellite.
Initial and final modes of rotation.

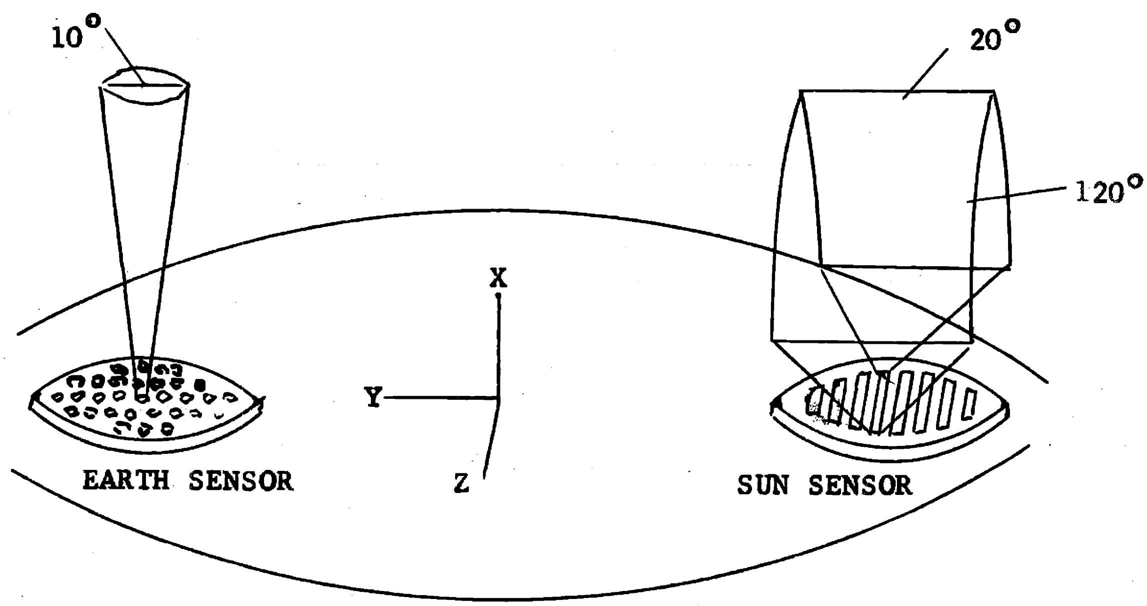


Fig. 2 Fields observed by orientation sensors:
Earth sensor: circular field - 10° dia.
Sun sensor: rectangular field - 20° wide, 120° long, parallel to angular momentum vector (Z).

The Orientation Data

The information available for determining the orientation consists of the signals of two photoelectric detectors sensitive to the visual wavelengths. They are mounted beside the telescope pointing in the same direction, so that the orientation of the sensors defines the orientation of the telescope. The system was designed by Dr. J. Kupperian of NASA to allow a determination of the orientation from the signals produced during one rotation. (3) If large scale data processing is utilized the data are redundant and the sensors unnecessarily elaborate. However, they render a solution possible with a minimum of processing.

The 'sun sensor' is designed to measure the angle between the sun and the plane of rotation of the satellite. It is collimated with a 120° field parallel to the angular momentum vector and a 10° field perpendicular to it (Fig. 2). Thus, as the satellite rotates, the sun will enter the field briefly producing a spike whose amplitude is (roughly) proportional to the cosine of the angle to the sun. This sensor is responsible for the requirement that the satellite be tri-axial: the body axis about which the satellite will rotate must be known in advance in order to position the detector properly.

The 'earth sensor' is collimated with a 10° circular field. Its sensitivity is such that it is saturated while observing the illuminated disk of the earth, so that it produces a pulse whose beginning and end mark the crossings of the earth's (illuminated) horizons. This signal is subtracted from the sun signal before phase modulation, so that both signals occupy a single recording and telemetry channel.

It is important to recognize the limitations characteristic of the particular sensing system to be used: The amplitude of the sun signal is used to determine the angle between the sun and the direction of observation. This implies that the response of the detector as a function of angle is known. This calibration will be carried out prior to launch using the sun as observed at ground level for a reference and correcting for the greater intensity above the atmosphere. The accuracy of the calibration is not so important as the fact that it will undoubtedly change with time as continued operation and the environment act upon the equipment. Hence, great reliance should not be placed upon the accuracy of this datum. Moreover, a simple interpretation of the signal from the sun sensor requires that the satellite be rotating about its stable axis, whereas the satellite may possess some residual roll about its longitudinal axis. This roll will produce a signal at a time which is not necessarily the instant at which the angle between sun and sensor is a minimum (solar transit). This condition should be easy to recognize since the amplitudes of consecutive spikes will vary; however, this may restrict the value of the datum.

The earth sensor suffers the severe restriction that it observes only the illuminated earth, and it can see an illuminated disk at most one-half of each revolution. Moreover, the distinction between a crossing of the terminator on the earth and the horizon must be made, a distinction of unknown difficulty which may limit useful observing time to one-third of each revolution. Another general assumption must be recognized. That is, that the satellite is rotating in a plane. If there is nutation there is no simple solution to the problem. In particular, the direct solutions which follow are not valid. Although the design of the satellite is intended to fulfill these conditions, it will be necessary to verify that they have been fulfilled in order to guarantee the validity of the results.

A record as received for reduction will consist of alternating positive spikes and negative pulses representing observations of the sun and earth (Fig. 3). A time signal will also be recorded so that the times of the events may be measured. The information directly available includes:

Times of two horizon crossings,
Time of solar transit,
Angle to sun at transit.

From these may be inferred:

Time of earth (center) transit,
Duration of earth pulse,
Separation of earth and sun transits.

Since the information is recorded on film, it is necessary to project the film in some manner for examination and measurement. A convenient digitization of the time measurements may be obtained by arranging a film drive and a phototransistor detector to count the time pulses. The spike amplitude may be measured with an appropriately graduated scale.

Determining the Orientation

The reduction of the data described above may be accomplished by three distinct techniques:

- The use of an analog computer,
- The computation of digital solutions employing:
  - A geometrical method,
  - An iterative, least squares method.

The essential features of each approach will be mentioned below. A detailed description of the use of each method may be found in the appendices.

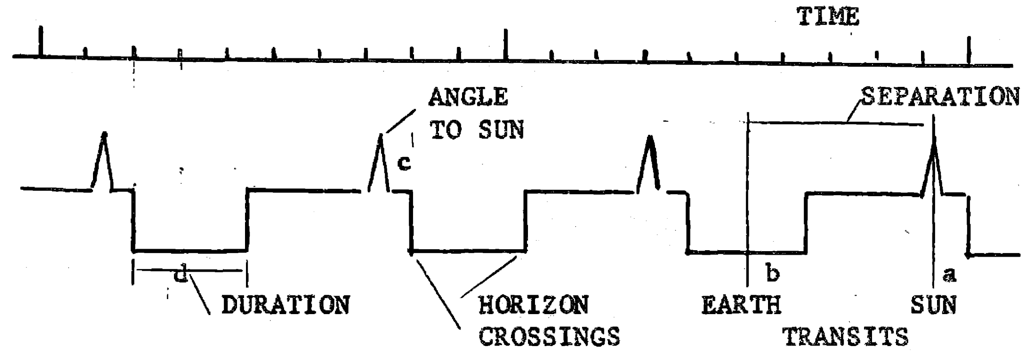


Fig. 3 Representation of time and aspect channels of telemetry record. Labeled are: crossings of earth's horizons, time of earth transit (b), duration of earth pulse (d), time of solar transit (a), separation of earth and sun transits (ab), angle to sun at transit (c).

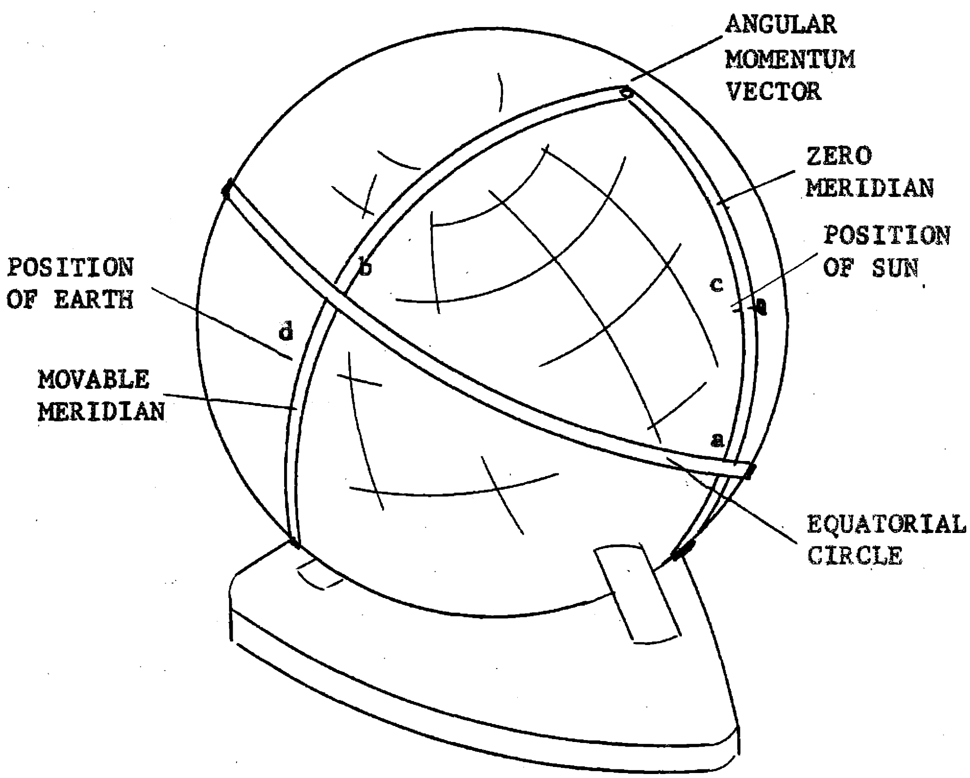


Fig. 4 Sketch of analog computer indicating relative positions of circular arcs when set from data of Fig. 3. Letters indicate the position at which each datum is entered (see Fig. 3).

A common feature of the three solutions is that all utilize the device of a local, space-fixed coordinate system with origin at the center of mass of the satellite. The body axes of the satellite at a reference time are used to define a rectangular frame in which the satellite rotates. The assumption that this frame is space-fixed is valid only so long as the angular momentum vector may be considered constant. If the satellite precesses as expected, this should be one day. As seen from the local coordinate system, the earth and sun appear to revolve about the satellite. It is useful to treat them, therefore, as if they moved in their respective orbits about the center of mass of the satellite.

The angular velocity of the satellite forms the connection between the time recorded on the aspect record and the orientation of the satellite. It is assumed known for it may be readily determined by measuring the time required for a large number of rotations, as indicated by the sun signals.

Of the three methods listed above, the analog computer provides the most direct solution by displaying a schematic model of the satellite-sun-earth configuration. The computer itself is essentially a spherical slide rule representing the celestial sphere and the local coordinate system. The celestial sphere is represented by a ball on which is engraved the sidereal coordinate grid of right ascension and declination. The local coordinate system is represented by an 'equatorial circle' for the plane of rotation of the satellite, and two meridians, one of which is movable. The fixed meridian defines the 'x axis' of the coordinate system,which is specified as the direction of observation at the time of solar transit. Thus the 'local coordinate system' forms a framework surrounding and supporting the 'celestial sphere' which is free to rotate in all directions.

The use of the computer (sketched in Fig. 4) is briefly as follows: The positions of the earth and sun at the time for which a determination of the orientation is desired are marked upon the sphere. The sphere is then rotated so that the sun lies beneath the zero meridian at the distance from the plane of rotation indicated by the amplitude of the sun signal. The movable meridian is positioned at an angle from the fixed meridian corresponding to the separation of the earth and sun transits. The sphere is adjusted so that the center of the earth lies beneath this meridian (see Appendix II). Then the direction of the angular momentum vector may be read from the sphere at the pole of the equatorial circle and the orientation of the satellite at a gamma-ray event may be read from the sphere along the equatorial circle at an angle from the zero meridian corresponding to the separation of the event and the sun signal.

One feature of the method is that since the angular momentum vector precesses slowly, once a solution has been found, small corrections to the position of the sphere are sufficient to update the solution for successive determinations.

If the procedure followed in obtaining the analog solution is expressed analytically, the geometrical method described in Appendix III may be derived. These two solutions are completely analogous, even to suffering in the same manner the limitations of the aspect sensing apparatus mentioned earlier. The analog description of the approach illustrates this most clearly:

The satellite is assumed to rotate in a plane. If it doesn't, a great circle is not traced on the celestial sphere and the entire approach breaks down. If the satellite rolls about its longitudinal axis, the zero meridian is not defined by the time of the sun signal. If the calibration of the sun sensor has changed, the angle between the sun and the plane of rotation is not known which removes an important, but not essential, datum. If the earth sensor detects twilight instead of a horizon, the center of the earth signal cannot be located and the duration is not known so that the solution fails completely. It is essential to recognize these difficulties when they occur so that invalid solutions may be avoided.

If a determination is required at a time when the earth or sun is not observed, the orientation must be determined at the closest possible time and the orientation desired assumed to be identical. Such a procedure reduces the accuracy of the results, but the error should remain within the acceptable limits.

The third approach to determining the orientation of the local reference frame employs a larger volume of input data by using measurements made over many rotations of the satellite instead of just one. It may be described as an 'iterative, least squares' method and yields values for the parameters defining the orientation which minimize the sum of the squares of the differences between the observed and predicted values of the data. It has several important advantages: The poor reliability of an individual datum is compensated for by using a large number of data. A statistical estimate may be obtained of the accuracy of the determination and the sun sensor may be re-calibrated on the basis of the redundant data. Moreover, determinations of the orientation may be made taking into account the disturbing effects of roll and nutation of the satellite as it rotates. If a situation arises in which the earth pulse may indicate a horizon crossing or a crossing of the terminator, the method is capable of deciding which and using the datum appropriately.

The iterative, least squares method assumes approximate values for the orientation parameters and uses the measured data to compute successively more nearly correct values of these parameters. The parameters used specify the orientation of the local coordinate system in terms of the matrix rotation that connects it with the sidereal coordinate system. This matrix involves three independent parameters. The frequency of rotation may be treated as an unknown parameter, but this may not be necessary since it may be determined accurately and will remain constant. In addition, the angle of nutation (if the satellite is not rotating in a plane) and the angle of roll (if present) may be defined as parameters. As noted before, an important advantage of this method is that such parameters need not be neglected. Indeed, they may be compared with the observational data to decide if they have an appreciable influence.

To actually carry out a solution by this technique requires a digital computer to perform the appreciable volume of calculations required. Appendix IV describes the method in considerable detail and illustrates how a computer program using it would operate.

The use of the methods described above to determine the orientation of the gamma-ray telescope requires that the tools they require be available. Toward this end the analog computer has been designed and is currently being constructed. Computer programs have been written to compute the position of the earth and to simulate the operation of the orientation sensors so that the data that will eventually be obtained from the satellite may be used in trial determinations before the satellite is launched. The geometrical solution has been programmed, but the program has not yet been tested. The program utilizing the iterative, least squares method is now being tested and the convergence of the iterations examined.

In summary, the orientation of the gamma-ray satellite will be determined from observations of the earth and sun. The orientation itself may be considered as a rapid rotation within a slowly moving reference frame. A straightforward geometrical solution to the problem of determining this orientation has been worked out. However, the most effective use of the information available will be made through a statistical analysis of a large body of observational data. Upon launch of the satellite and stabilization of its rotation, computer programs will provide a means for rapidly acquiring and automatically tracking changes in the orientation of the telescope as a function of time.

The author wishes to express his appreciation for the discussion and criticism offered by Dr. G. Clark in preparing this paper.

(1) V.V. Beletskii, "Motion of an Artificial Earth Satellite about its Center of Mass," in Artificial Earth Satellites, ed. L.V. Kurnosova, 2 vols. (New York, 1960), pp. 30-54.

(2) R.J. Naumann, "Recent Information Gained from Satellite Orientation Measurement," presented at the 4th Symposium on Ballistic Missiles and Space Technology, Los Angeles (1959).

(3) J. E. Kupperian, Jr., and R.W. Kreplin, "Optical Aspect System for Rockets," Review of Scientific Instruments, XXVIII (1957), pp. 14-19.

(4) F.B. Hildebrand, Introduction to Numerical Analysis, (New York, 1956), pp. 261-269.

(5) G. Veis, unpublished notes (1959).

Appendix I

Satellite Position Determination

The orbit of an earth satellite may be approximated by an ellipse whose focus is at the center of mass of the earth. The approximation results from the fact that the earth is not a sphere, but an oblate spheroid, which causes the orientation of the ellipse to change continuously. Atmospheric drag also introduces a change in the size and shape of the ellipse, the net result being that the satellite follows an open curve, however, one that may be approximated by an ellipse over a short interval of time.

The following discussion refers to a close earth satellite and represents only a first approximation to the actual position. Additional refinement introduces considerable complication and is necessary only if a positional accuracy greater than 10 km is required.

The position of the satellite is most conveniently specified in 'elliptical' coordinates which will be defined below; These may then be transformed into sidereal coordinates. 'Elliptical' coordinates specify the size, shape and orientation of the approximating ellipse and the position of the satellite along the ellipse:

The size of the ellipse is specified by the maximum radius vector, as measured from the center of the ellipse (semi-major axis - a) (Fig. 5). The inverse square gravitational field of the earth implies the 'law of areas' establishing a relationship between the frequency of revolution ($\dot{M}$) and the semi-major axis:

$$a^3 \dot{M}^2 = (274.5)^2 \text{ Mm}^3 \text{ Rev}^2/\text{Day}^2 = k$$

Thus the semi-major axis need not be explicitly stated if the frequency is known.

The shape of the sllipse is specified by its eccentricity (e):

$$e = (a^2 - b^2)^{.5} / a$$

Both semi-major axis and eccentricity decrease slowly with time (the orbit tends to become smaller and more nearly circular).

The orientation of the ellipse is specified by:

The inclination of the plane of the orbit to the equatorial plane of the earth (inclination - i).

The right ascension of the intersection of the orbit with the equatorial plane as the satellite passes from south to north (ascending node - Ω).

The angular distance traveled by the satellite from the ascending node to perigee (minimum radius vector - π) measured at the focus of the ellipse (argument of perigee - ω).

The ascending node and argument of perigee change linearly with time with the change rapid enough so that they must be constantly re-evaluated (5°/Day). Inclination is constant.

The position of the satellite along the orbit is specified by the fraction of a revolution it has traveled from perigee (mean anomaly - M). The first derivative of mean anomaly is the frequency of revolution (mean motion - $\dot{M}$). Mean anomaly is usually represented by a quadratic polynomial in time, the quadratic term representing the effect of atmospheric drag upon the velocity of the satellite.

However, mean anomaly does not directly define a position because it represents a uniform rotation while motion in an ellipse, though periodic, is not uniform. The angular distance traveled by the satellite from perigee measured at the <u>center</u> of the ellipse (eccentric anomaly - E) is an implicit function of mean anomaly:

$$M = E - e \sin E$$

The angular distance traveled by the satellite from perigee measured at the focus of the ellipse (true anomaly - v) is a function of the eccentric anomaly:

$$\sin v = \frac{(1 - e^2)^{.5}}{1 - e \cos E} \qquad \cos v = \frac{\cos E - e}{1 - e \cos E}$$

The transformation of these 'elliptical' elements into rectangular sidereal coordinates requires only the radius vector as a function of eccentric anomaly:

$$R = a\ (1 - e \cos E)$$

The complete solution may be summarized:

$$\omega = \omega_o + \omega_1\ t{-}t_o$$

$$\Omega = \Omega_o + \Omega_1\ t{-}t_o$$

$$i = i_o$$

$$e = e_o$$

$$M = M_o + M_1\ t{-}t_o + M_2\ t{-}t_o{}^2$$

$$a = (k/\dot{M})^{2/3}$$

$$E = M + e \sin E$$

$$\sin v = \frac{(1 - e^2)^{.5}}{1 - e \cos E} \qquad \cos v = \frac{\cos E - e}{1 - e \cos E}$$

$$R = a\ (1 - e \cos E)$$

$$x = R\ (\cos \omega{+}v \cos \Omega - \sin \omega{+}v \cos i \sin \Omega)$$

$$y = R\ (\cos \omega{+}v \sin \Omega + \sin \omega{+}v \cos i \cos \Omega)$$

$$z = R\ (\sin \omega{+}v \sin i)$$

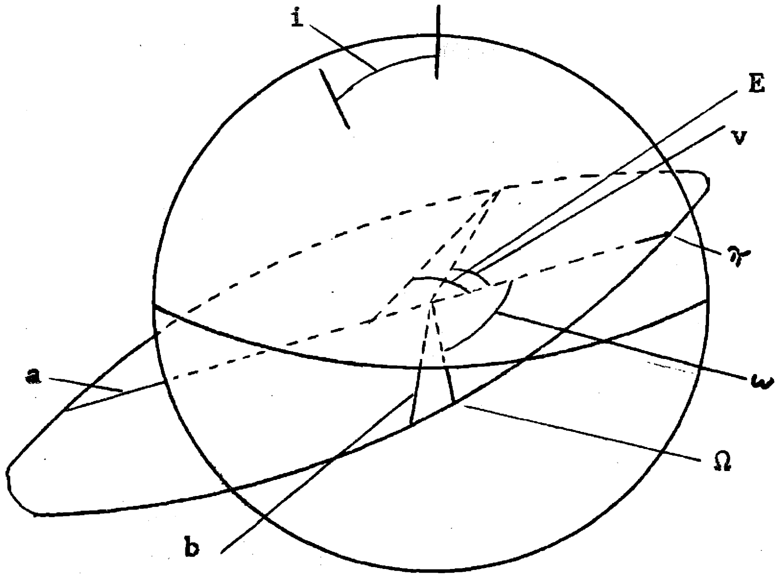


Fig. 5 The orbital elements of a satellite which have geometrical meanings are indicated by their symbols. See Appendix I for their definition.

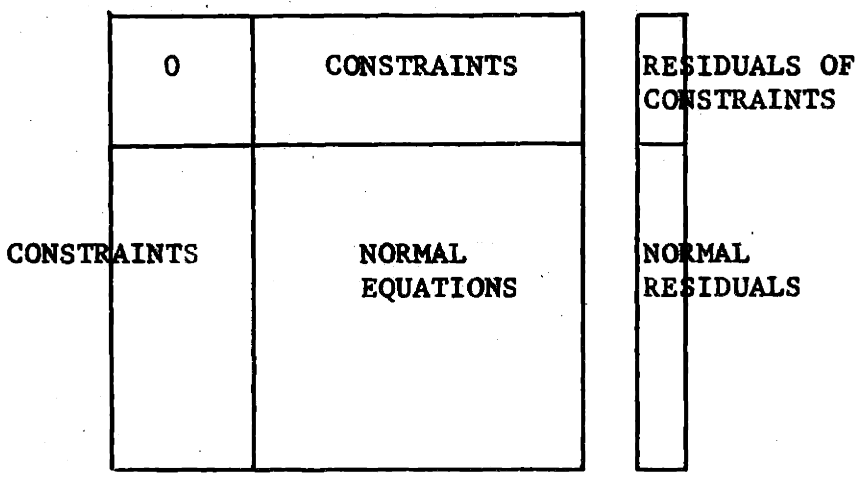


Fig. 6 Schematic of matrix of coefficients formed during the iterative, least squares solution. The normal equations are symmetric, and the constraint equations are symmetrically placed with respect to them.

Appendix II
Use of Analog Computer

This appendix is intended to provide a detailed outline of the procedure involved in obtaining the orientation of the satellite at the time of an event with the analog computer. Reference is made to the problem specified by Fig. 4.

I The position of the sun is marked on the sphere. This position may be obtained from the American Ephemeris and Nautical Almanac: 1960 which tabulates the right ascension and declination of the sun for each day

II The slide on the zero meridian is set at the angle corresponding to the distance of the sun from the plane of rotation of the satellite, as indicated by the amplitude of the sun signal (spike).

III The sphere is positioned so that the pin on this marker bears directly upon the location of the sun. This defines a pole about which the sphere may be rotated, corresponding to the fixed distance between sun and plane of rotation.

IV The position of the earth is marked on the sphere. The position of the earth as seen from the satellite is the negative of the position of the satellite as seen from the center of the earth, a position that changes rapidly. Such a rapid change renders impractical any attempt at tabulation and hand computation is rather involved (see Appendix I). To avoid these difficulties a program has been written for the 704 utilizing the orbital elements and producing the position of the earth and the height of the satellite at any specified time.

V The movable meridian semi-circle is set at the meridian corresponding to the separation of the sun spike and the center of the earth pulse. The center of the earth must lie on this meridian.

VI The sphere is rotated until the location of the center of the earth is directly beneath the movable meridian.

VII The direction of the angular momentum vector may be read from the sphere at the upper or lower pivot of the movable meridian semi-circle. Which pivot is appropriate depends upon the sense of rotation of the satellite, which may be determined from the relative positions of the earth and sun and the separation of the earth and sun signals. If the separation of the sun signal and the <u>next</u> earth signal is less than one-half revolution, then the satellite rotated from the zero meridian to the movable meridian along the shortest arc. Otherwise vice-versa.

VIII It is possible, indeed likely, that several solutions may correspond to a single set of data. Two methods are available to remove the degeneracy:

The distance of the center of the earth from the plane of rotation (δ) may be computed from the duration of the earth pulse (2σ) and the height of the satellite (H):

$$\cos \delta = \frac{(1 - 6378/6378+H)^{1/2}}{\cos \sigma}$$

That solution may be selected which most nearly conforms to this additional datum.

If another set of data are selected at a later time such that the position of the earth is appreciable different, the solution they produce should be identical to the previous. If necessary, several sets of data may be employed until a unique solution has been defined.

Appendix III
Geometrical Solution

The data obtained from the aspect record may be interpreted in a manner that permits a simple, symmetrical statement of the problem of determining the orientation. That data used are the amplitude of the sun spike at solar transit and the duration of the earth pulse. The amplitude of the sun spike is used to indicate the cosine of the angle between the sun and the sensor. The fact that this is known at transit means that at this instant, since the angular momentum vector, sun and sensor are coplanar, the angle between the sun and the angular momentum vector is known. That is:

$$\bar{s}\cdot\dot{\bar{o}}_1 = 0$$
$$\bar{s}\cdot\bar{o}_1 = (1 - a^2)^{1/2}$$
$$\bar{p}\cdot\bar{s} = \pm a$$

where $\bar{s}$ indicates the direction of the sun, $\bar{o}_1$ the orientation of the satellite and $\bar{p}$ the direction of the angular momentum vector.

The duration of the earth signal may be used to infer the same information for the relative positions of the center of the earth and the angular momentum vector. The height of the satellite determines the apparant size of the earth disk, and the duration of the signal indicates the time required for the satellite to cross the disk. From these data the minimum angle to the center of the earth (earth transit) may be computed (see Appendix II), using relationships analogous to those above:

$$\bar{e}\cdot\dot{\bar{o}}_2 = 0$$
$$\bar{e}\cdot\bar{o}_2 = (1 - b^2)^{1/2}$$
$$\bar{p}\cdot\bar{e} = \pm b$$

where $\bar{e}$ indicates the direction of earth center and $\bar{o}_2$ the orientation of the satellite at the time of earth transit.

Thus, two linear equations are obtained for the unknown components of $\bar{p}$.

The condition that $\bar{p}$ be a unit vector may be expressed in a linear fashion by expressing the projection of $\bar{p}$ on a unit vector in the direction of $\bar{s}$ $\bar{e}$:

$$\bar{d} = \frac{\bar{s} \times \bar{e}}{|\bar{s} \times \bar{e}|}$$

The identity

$$\bar{p} \cdot \bar{d} = (1 - (\bar{p}\ \bar{d})^2)^{1/2}$$

may be used to derive

$$\bar{p} \cdot \bar{d} = \pm c = \pm (1 - \frac{a^2 + b^2 - 2ab(\bar{s} \cdot \bar{e})}{(\bar{s} \times \bar{e})^2})^{1/2}$$

Thus, three linear equations are obtained for the components of $\bar{p}$:

$$\bar{p} \cdot \bar{s} = \pm a$$

$$\bar{p} \cdot \bar{e} = \pm b$$

$$\bar{p} \cdot \bar{d} = \pm c$$

The uncertainty of the signs of the right hand sides of these equations results from the fact that the aspect record indicates magnitudes, but yields no information as to the direction of the vectors. Eight solutions may be obtained, but four will differ by merely a sign change, so that four directions are specified. The decision as to which solution to choose must be made according to the criteria discussed in Appendix II.

Another independent, and evidently redundant, datum is available that has not yet been used. This datum is the separation of the earth and sun transits:

$$\bar{o}_1 \cdot \bar{o}_2 = f$$

One way to employ f is to use it with b to compute a value of $(\bar{p} \cdot \bar{s})$ as shown below. This value of $(\bar{p} \cdot \bar{s})$ (a') may then be used with b in the simultaneous equations given above. By the law of cosines applied to the spherical triangle formed by $\bar{s}$, $\bar{e}$ and $\bar{p}$:

$$\bar{e}\cdot\bar{s} = (\bar{p}\cdot\bar{e})(\bar{p}\cdot\bar{s}) + (\bar{o}_1\cdot\bar{e})(\bar{o}_2\cdot\bar{s})(\bar{o}_1\cdot\bar{o}_2)$$

$$a' = \frac{(e\cdot s)b \pm (e\cdot s^2 b^2 - (b^2 + c^2(1 - b^2))(e\cdot s^2 - c^2(1 - b^2)))}{b^2 + c^2(1 - b^2)}$$

Thus, two values of a' are obtained. Likewise, a value of $(\bar{p}\cdot\bar{e})$ may be computed from f and a and used in conjunction with a in the above equations. If all combinations of the three data are employed, 40 solutions may be obtained, however, not all need to be computed in any given case. The simultaneous equations have the same matrix of coefficients for all these solutions, so they should be solved by inverting this matrix and multiplying the inverse by the independent variables.

The solution may be systematized as follows:

1 Compute $\bar{s}$, $\bar{e}$, $\bar{d}$ and $\bar{s}\cdot\bar{e}$

2 Invert the matrix of coefficients: $\begin{pmatrix} s_x & s_y & s_z \\ e_x & e_y & e_z \\ d_x & d_y & d_z \end{pmatrix} = M$

3 Compute a, b and c

4 Obtain the eight solutions $\bar{p} = M^{-1}\begin{pmatrix} \pm a \\ \pm b \\ \pm c \end{pmatrix}$

5 Select the relevant solution

6 Replace a with a' in the solution for $\bar{p}$ above. Only that value of a' needs to be employed that agrees in magnitude and sign with the value of a leading to the solution determined to be relevant. The others solutions will be extraneous.

7 Replace b with b' and obtain another value for the relevant solution only.

8 Average the three relevant solutions to determine the final answer. The scatter in the three determinations of $\bar{p}$ will indicate the accuracy attained.

In addition to the indeterminancies mentioned, others might arise in special cases. It is the difficulty of anticipating and specifying all these decisions that makes this solution ill-suited for automatic machine computation. The program that is being prepared to carry out this solution is limited to printing the basic 20 solutions. The decision as to which three should be employed must be made separately.

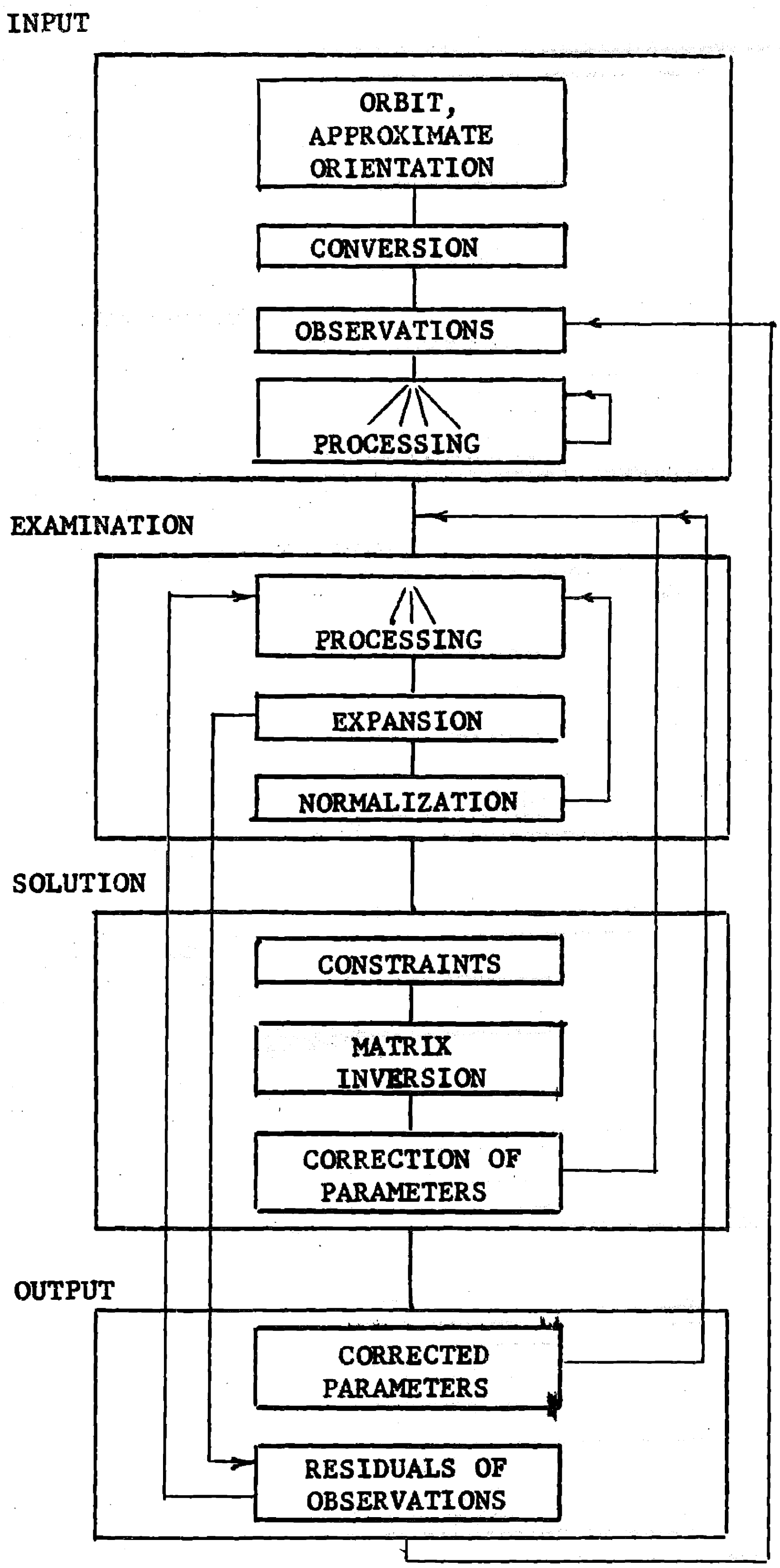


Fig. 7 Flow diagram of program determining the orientation of the satellite by the iterative, least squares technique.

Appendix IV
Iterative, Least Squares Method

The iterative least squares method for determining the orientation of the satellite will be discussed in terms of the flow diagram (Fig. 7) of the computer program that is being prepared to carry it out. This organization has the advantage of permitting direct reference to the program and presents the operations in the sequence in which they are logically encountered. The titles employed below refer to the headings of the subdivisions of the flow diagram.

Input

Orbit, Approximate Orientation

The program first reads in the orbital elements of the satellite(see Appendix I) and the approximate values of the parameters used to define the orientation. It is estimated that the approximate orientation will have to be within 30^{o} of the true orientation in order for the process to converge, but no evidence has yet been obtained to support this estimate. During routine use of the program, the orientation previously determined will be available as an approximation. An initial approximation must be obtained by the direct solutions described in Appendices II and III.

The parameters chosen to specify the orientation refer to the local coordinate system defined earlier. They are:

Frequency of rotation of the satellite (ω),
Direction of the angular momentum vector in spherical sidereal coordinates (α_3,δ_3),
Direction of observation at reference time (T_o) (α_1,δ_1).

These parameters specify the direction of the body axes of the satellite at the reference time. The reference time is chosen to be the epoch of the orbital elements since the orientation parameters are a logical extension of the orbital elements to the case of rigid body motion.

These parameters (orbit and approximate orientation) are read from BCD cards and immediately written onto the output tape.

Conversion

Once obtained, the data are converted to a form more convenient for machine processing. All angular quantities are converted to radians and the specification of the local coordinate system is changed from spherical to rectangular coordinates. The axes of the reference frame are represented by the components of three orthogonal unit vectors defined as follows:

$$\bar{x}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} \cos\delta_3 \cos\alpha_3 \\ \cos\delta_3 \sin\alpha_3 \\ \sin\delta_3 \end{pmatrix}$$

$$\bar{x}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \cos\delta_1 \cos\alpha_1 \\ \cos\delta_1 \sin\alpha_1 \\ \sin\delta_1 \end{pmatrix}$$

$$\bar{x}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \bar{x}_3 \times \bar{x}_1$$

These vectors are now used to define a rotation matrix (A) which transforms a direction in the local system ($\bar{o}_1$) to a direction in the sidereal system ($\bar{o}$).

$$\bar{o} = A\,\bar{o}_1 \qquad A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

A correction matrix (ΔA) is also defined consisting of corrections to be added to the rotation matrix specified by the approximate orientation parameters.

Observations

Next the measurements made on the aspect record are read from BCD cards. These measurements are termed 'observations' in the following discussion. Each observation is composed of an identifying alphabetic code followed by numeric information.

When the datum is a time it measures a number of oscillations of a 350 cps clock carried on the satellite. The signal produced by this clock is recorded on the time channel of the telemetry record, scaled to a convenient frequency. Time measurements thus consist of a count of the number of oscillations from the beginning of the record to the position being measured. This 'satellite time' is related to Universal Time (UT) at the end of the record, a point which corresponds to the known instant at which the satellite received the command initiating high speed read-out.

The first two measurements read in fix the relation between satellite time and UT; they are a count of the total number of oscillations recorded on the record and the universal time corresponding to the last oscillation. They are followed by the following types of observations (in any order):

- Time of horizon crossing: a single datum representing the time at which a crossing of the earth's horizon occurred,
- Time of solar transit: a single datum representing the time at which solar transit occurred,
- Amplitude of sun signal: two data: the time at which the signal was obtained and the amplitude of the signal.

This description of the measurements illustrates an important point. The sun signal normally contains two items of information: a measure of the angle to the sun and the statement that this angle is a minimum. If the satellite rolls the signal may no longer represent a solar transit, and the angle may not be the minimum angle. If calibration is lost, the amplitude may not be reliable, but the signal may still identify the time of solar transit. Thus, it is essential to recognize these data as independent and to treat them separately.

Processing

The term 'processing' refers to individual treatment of each observation. In this case the observations are identified by their code and the necessary conversions performed. This includes a translation of the satellite time to UT and the conversion of the amplitude of the sun signal to the cosine of the angle to the sun by applying a calibration curve.

Examination

Processing

Having completed the input, the program examines the observations for their contribution to the solution. Each observation is again identified . The identification is used to determine the weight of the observation and to define the nature of the following computations. The weight employed is the inverse square of the expected accuracy of the observation, allowing observations of widely differing accuracies to be properly treated.

The approximate orientation parameters are employed to determine the direction of observation at the time of observation in both the local and the sidereal systems:

$$\bar{o}_1 = \begin{pmatrix} \cos \omega t \\ \sin \omega t \\ 0 \end{pmatrix} \qquad \bar{o} = A\, \bar{o}_1$$

The time is also used to determine the direction of the center of the earth, or the direction of the sun, whichever is appropriate for the observation.

From this data the difference between the known value of the observed quantity and the value computed using the approximate orientation parameters is determined. The procedure is specified below for each type of observation.

Time of horizon crossing:

The quantity observed is the time at which an horizon was observed by the earth sensor. To specify this event analytically we note that since the apparent size of the earth disk is known from the position of the satellite the angle between the direction of the orientation vector and the center of the earth ($\bar{e}$) must equal the radius of the earth disk (ρ). ρ may be computed from the radius vector of the satellite (r) expressed in units of the earth's radius (r_o = 6378 km):

$$\cos \rho = (1 - (\frac{r_o}{r})^2)^{1/2}$$

The known value of the scalar product of $\bar{o}$ and $\bar{e}$ is cos ρ. The predicted value is ($\bar{o} \cdot \bar{e}$). The difference between the known and predicted value of an observation is termed the 'residual' of the observation, in this case:

$$\Delta(\bar{o} \cdot \bar{e}) = \cos \rho - \bar{o} \cdot \bar{e}$$

However, since the observed quantity was a time, the residual in the scalar product is used to find a corresponding residual in time. This residual may be obtained by expanding the error in the scalar product as an error in orientation caused by an error in time:

$$\Delta(\bar{o} \cdot \bar{e}) = \bar{e} \cdot \Delta\bar{o} = \bar{e} \cdot \dot{\bar{o}}\, \Delta t$$

Thus the residual in time may be determined from the error in the scalar product according to the equation:

$$\Delta t = \frac{\cos \rho - \bar{e} \cdot \bar{o}}{\bar{e} \cdot \dot{\bar{o}}}$$

Time of solar transit:

A similar technique is used to treat the observation of the time of solar transit. In order for a transit to have occurred, the scalar product of the orientation and the sun ($\bar{s}$) must have been a minimum. Thus the derivative of the scalar product must have been zero. Its predicted value ($\bar{s}\cdot\dot{\bar{o}}$) is determined from the time derivative of the orientation vector ($\dot{\bar{o}}$):

$$\dot{\bar{o}}_1 = \begin{pmatrix} -\sin \omega t \\ \cos \omega t \\ 0 \end{pmatrix} \qquad \dot{\bar{o}} = A\,\dot{\bar{o}}_1$$

The residual in time is again computed from the difference between the observed and predicted values of the scalar product by differentiating. There results:

$$\Delta t = \frac{-\bar{s}\cdot\dot{\bar{o}}}{\bar{s}\cdot\ddot{\bar{o}}}$$

Amplitude of sun signal

This observation is composed of a time at which a measurement of the scalar product of the orientation vector and the sun was made. Thus, the difference between the measured value of the product (k) and the value computed using the approximate orientation parameters ($\bar{s}\cdot\bar{o}$) defines the residual of the observation directly:

$$\Delta(\bar{s}\cdot\bar{o}) = k - \bar{s}\cdot\bar{o}$$

Expansion

The residuals thus obtained represent an error in the assumed orientation. We blame this error on the approximate orientation parameters (rotation matrix and frequency of rotation; n = 10 parameters), by expressing the error in orientation (or its derivative) as a function of errors in the parameters:

$$\bar{o} = A\,\bar{o}_1$$

$$\Delta\bar{o} = \Delta A\,\bar{o}_1 + A\Delta\bar{o}_1 \qquad A\,\Delta\bar{o}_1 = \frac{t}{\omega}\,\dot{\bar{o}}\,\Delta\omega$$

$$\Delta\dot{\bar{o}} = \Delta A\,\dot{\bar{o}}_1 + A\,\Delta\dot{\bar{o}}_1 \qquad A\,\Delta\dot{\bar{o}}_1 = \frac{t}{\omega}\,\ddot{\bar{o}}\,\Delta\omega$$

Likewise, we blame the error in a scalar product (the computed function employed by all observations) on an error in orientation (or its derivative).

Thus:

$$\Delta(\bar{s}\cdot\bar{o}) = \bar{s}\cdot\Delta\bar{o}$$

The residuals of each observation are expanded in this fashion obtaining:

Time of horizon crossing: $\Delta t = \frac{\bar{e}\cdot\Delta\bar{o}}{\bar{e}\cdot\dot{\bar{o}}}$

Time of solar transit: $\Delta t = \frac{\bar{s}\cdot\Delta\dot{\bar{o}}}{\dot{\bar{s}}\cdot\bar{o}}$

Amplitude of sun signal: $\Delta(\bar{s}\cdot\bar{o}) = \bar{s}\cdot\Delta\bar{o}$

The coefficients of the equations thus defined are evaluated by the program, so that given a particular observation, an equation is obtained relating errors in the parameters (dependent variables) to the residuals of the observation (independent variable).

Normalization

The above equation, obtained from one observation, is a member of a set of linear equations obtained from the entire set of observations. This set contains many more equations than unknowns, and it is customary to solve such equations by the method of least squares which is a technique combining the equations in such a way as to minimize the sum of the squared errors of the solutions. This method is discussed in the standard texts on numerical analysis and will not be considered here.[4]

One convenient modification to the procedure often followed is that the contribution of each equation to this solution is computed immediately, avoiding the storage of a large number of equations. Instead, only the n(n+1) coefficients employed in the least squares solution need to be stored.

Each observation is treated in this manner until all have been considered.

Solution

Constraints

The result of the examination and combination of the observations is an n×n matrix of coefficients and a vector containing n normalized residuals. These represent the set of equations which express the necessary corrections to the n orientation parameters. There is one restriction to be considered before the solution is obtained.

The nine variables which represent the corrections to the rotation matrix (ΔA) are not independent. They may not be changed arbitrarily, but must always conform to the specification that the columns of the matrix form an ortho-normal set of vectors. This condition is specified by the six scalar products that may be formed among the vectors:

$$\bar{x}_i \cdot \bar{x}_j = \delta_{ij} \quad i=1,3 \quad j=i,3$$

These six conditions state that only three of the nine variables are independent. Expanding these equations about the approximate rotation matrix, linear relations are obtained between the corrections to the components of the vectors:

$$\delta_{ij} - \bar{x}_i \cdot \bar{x}_j = \Delta\bar{x}_i \cdot \bar{x}_j + \bar{x}_i \cdot \Delta\bar{x}_j \quad i=1,3 \quad j=i,3$$

The residuals thus obtained express the possibility that the approximate coordinate axes are not precisely ortho-normal, a condition that must be corrected.

Adding the six equations to the normal equations obtained from the least squared combination of observational data expresses the conditions that must hold among the variables, but it also increases the number of variables by six. These additional variables are often referred to as Lagrange multipliers. The combination of the two sets of equations is done so that the matrix of coefficients ultimately obtained appears as in Fig. 6.[5]

Matrix Inversion

The equations thus defined are solved by inverting the matrix of coefficients (M). This procedure is readily carried out on a digital computer and has the advantage that the correlation matrix (inverse matrix) is available for obtaining statistical estimates of accuracy.

Correction of Parameters

If the complete set of equations finally obtained is represented by

$$\Delta\bar{a} = M\,\Delta\bar{b}$$

where $\Delta\bar{a}$ represents the residual vector and $\Delta\bar{b}$ the unknowns then:

$$\Delta\bar{b} = M^{-1}\,\Delta\bar{a} \qquad \bar{b}_{i+1} = \bar{b}_i + \Delta\bar{b}$$

In this manner corrected values of the orientation parameters are obtained which define successively better approximations to the orientation. Using the improved values, the observations are re-examined and the solution repeated until no further corrections are necessary. This process need not converge, but in practice a reasonable initial approximation usually leads to rapid convergence.

Output

Parameters

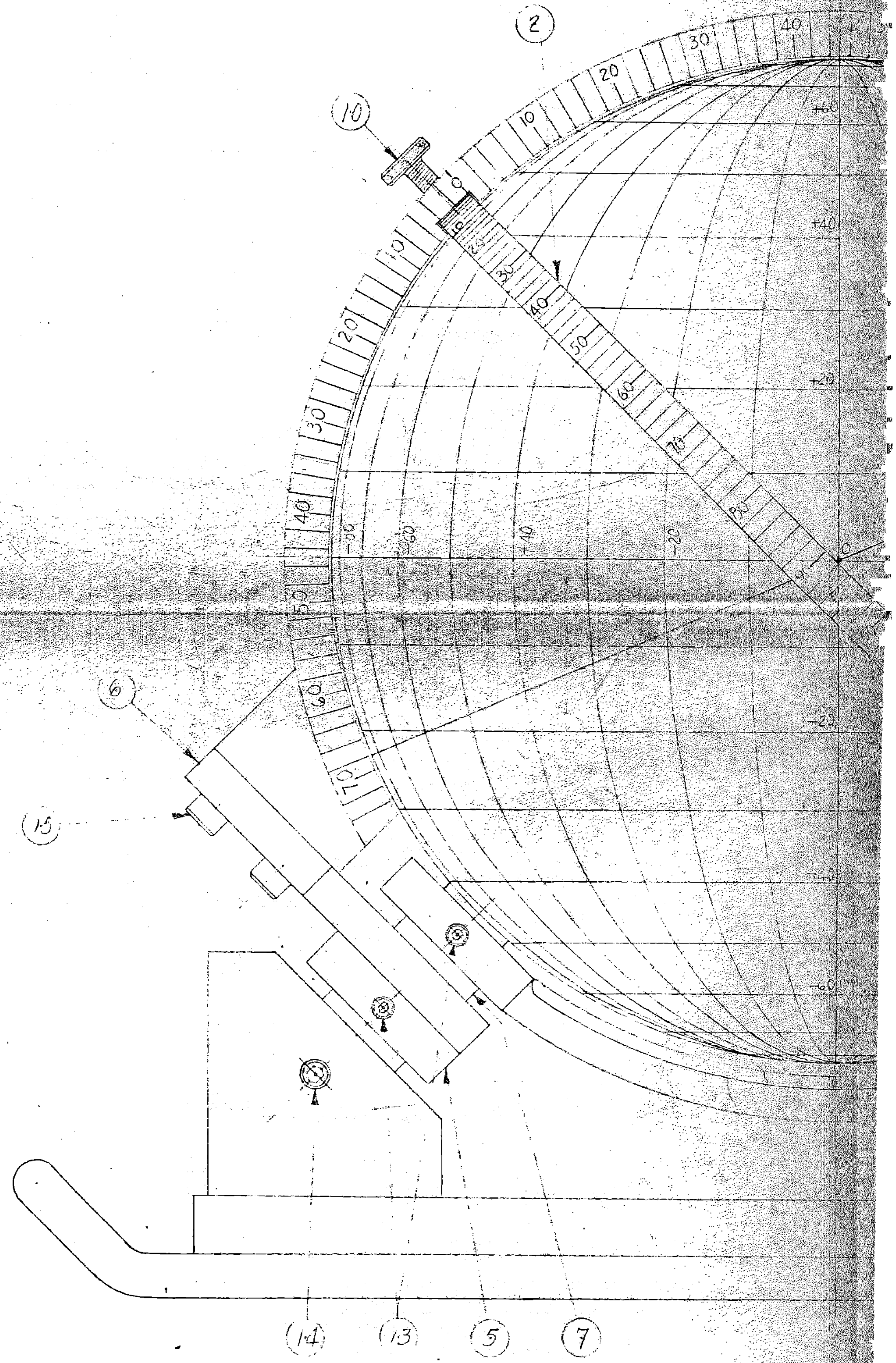
The improved values of the orientation parameters, represented by the coordinates of the axes of the local coordinate system, are converted back to spherical coordinates and written on the output tape. These coordinates refer to the same reference time as those read in initially, but they are valid over the interval of time covered by the observations.

Residuals of Observations

Each observation is re-examined with the final values of the orientation parameters and its residual written on the output tape.

The object of this output is to provide a check of the accuracy of the measurements as a feedback toward improving the techniques of measurement. It also provides an indication of the quality of the determination and information that may be used to re-calibrate the sun sensor.

The program then returns to the input mode and reads more observational data in preparation for another solution over another interval of time.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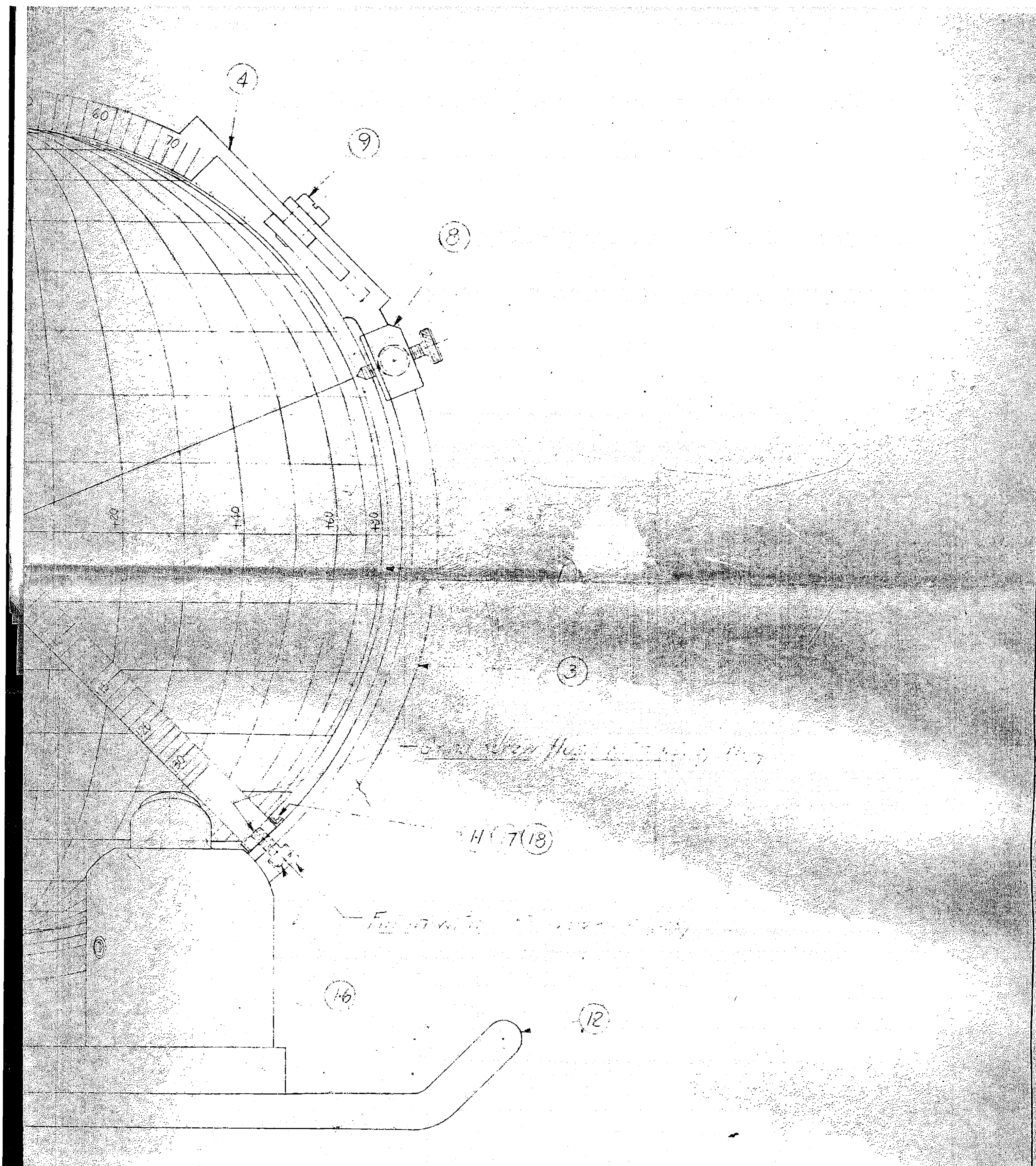

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