STABILITY AND TRANSIENT ANALYSIS OF CONTROLLED LONGITUDINAL MOTION OF AIRCRAFT WITH NONIDEAL AUTOMATIC CONTROLS

by

Yee Jing Liu

B.S., Electrical Engineering
Chiao Tung University, 1930

M.S., Aeronautical Engineering
Massachusetts Institute of Technology, 1938

Submitted in partial fulfillment of the requirement

for the degree of

Doctor of Science

from the

Massachusetts Institute of Technology

1941

Signature of Author Signature redacted
Department of Aeronautical Engineering

Professor in charge of research Signature redacted

Chairman of Department ____________

Committee on Graduate Students Signature redacted

Head of Department ____________

May 15, 1941

Professor George W. Swett
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dear Sir:

I hereby submit a thesis entitled "Stability and Transient Analysis of the Longitudinal Motion of Aircraft with Nonideal Automatic Controls" in partial fulfillment of the requirement for the degree of Doctor of Science.

Very respectfully yours,

Signature redacted

Yee Jing Liu

Massachusetts Institute of Technology
Cambridge, Massachusetts

A C K N O W L E D G M E N T

The author wishes to express his deepest appreciation to Professor C. S. Draper for his continued inspiration and encouragement as well as financial support in preparing this thesis.

He also wishes to thank Professor G. S. Brown for his advice in preparing the Appendix B of this thesis.

T A B L E   O F   C O N T E N T S

TABLE OF CONTENTS CONT'D

LIST OF TABLES

L I S T   O F   F I G U R E S

L I S T   O F   F I G U R E S   C O N T ' D

Figure No. | Page

A B S T R A C T

Automatic control systems have long been in practice with considerable success based upon years of cut and try experiences. The present type of automatic pilot used in aircraft has followed the same track of development. Problems dealing with automatic control of aircraft are too complicated for solution by ordinary mathematical methods because of the number of freedom involved. The equations of motion, when reduced to one dependent variable, give a linear differential equation of the sixth or higher order. However, for longitudinal stability, the pitching response can be represented by a fourth order differential equation if the control is properly designed. The theory involved is presented in Part I and its justification is carried out in Part V.

Many problems in the field of automatic control, both in aircraft and other systems, lead to the differential equation of the fourth order. For this reason it is important to develop a means for solving these quartic equations which will be useful in engineering practice. In the present thesis systematic and convenient methods for solving quartic equations are developed and presented in the form of curves and charts based on nondimensional variables. Simplified methods of determining stability criteria are also discussed

The physical significance of the nondimensional resolvent cubic derived from the nondimensionalized quartic equation is demonstrated. It is shown that at least one of the three roots of the resolvent cubic equation is the sum of the ratio of the natural frequencies of the two components and its reciprocal. A quartic chart is developed from the modified resolvent cubic equation. This quartic chart makes possible the practical solution of the quartic equation in terms of nondimensional physical constants. The scheme used gives results which are in error by less than $\pm 2\%$.

Possibilities for the improvement of stability with controls of high frequency and different coupling coefficients are investigated and the results are presented in charts. These results are expressed in terms of figures of merit which are called advantages. A table of such advantages is given which is very very useful for finding the compounding effect of controls with exciting forces proportional to different order of time derivatives of error. The theory of compounding effects is investigated and a generalized rule for the expression of the control advantage is presented. Special compounding controls are investigated with the purpose of improving the damping ratio of the system to be controlled, and the results are presented in a series of charts.

Particular relationships between the damping ratio and natural frequency of the control and the damping ratio and natural frequency of the member to be controlled should be maintained so that the controlled results may have unique frequency and unique damping ratio. The use of a properly selected frequency and damping ratio for the control is called tuning. The requirements for proper tuning are presented in the form of charts.

A knowledge of the stability of a system is not sufficient to give the whole picture of the response to a forcing function. A method suitable for a generalized transient analysis of automatic control problems is given in detail as a result of the application of the Heaviside Expansion Theorem. For aircraft applications particular attention is paid to the response of pitching and vertical motion of the airplane when a vertical surging gust is encountered. Both controlled and uncontrolled results are given in the form of plots. The control system considered in the analysis is based upon an assumed specification of keeping the frequency of the oscillations of the control and the aircraft unchanged. Any other suitable specification can be used and the control and coupling factor can be easily evaluated with the aid of the previously developed study of stability.

Four appendices are presented in this thesis.

Appendix A is the development of the cubic chart upon the established nondimensional form used by Weiss. Its improvement for use in evaluation is that only one chart is sufficient for the evaluation of physical constants (nondimensional) of the cubic equation which is often met in constant speed control systems.

Appendix B is a concise set of directions for using the quartic chart.

Appendix C is the derivation of the response when a system (with no repeating roots in its stability equation) is encountered with a surging disturbance similar to that often met in bumpy air.

Appendix D is a semi-graphical application of De Moivre's Theorem to the problem of evaluating polynomial functions involving complex numbers.

The material presented in this thesis should be useful as a common tool for the automatic control designer when the control system involves linear differential equations of the fourth order. The longitudinal control of an airplane is used as a particular illustration of the methods developed in this thesis.

INTRODUCTION

For the modern aircraft, whether commercial or military, a dependable and efficient piloting system is of growing importance. A dependable and efficient piloting system may be defined as one which consists of a combination of groups of dependable pilot mechanisms operated by a certain efficient agency. The proper coordination must be determined according to the condition of flight as well as the condition of the environment. However, during long range flight the aircraft must be kept constantly in course whether the course is straight, follows the path of a great circle, or even follows a number of connected broken lines determined by the convenience of available radio beacons. Keeping in course is the only way for the airplane, which is already available in service, to save the total fuel consumption, to reduce wearing of the engines, to save time for the passengers of the commercial aircraft or increase the swiftness of military operation. In rough weather, not only may the course be subject to drift, but annoying oscillations are always associated with it if the aircraft is not properly piloted or controlled. It is advisable to eliminate as far as possible any annoying oscillations due to such disturbances because it assures the comfort of passengers and diminishes the fatigue of the crew. Briefly, we obtain greater efficiency in our flying activity with a proper control system.

In the early stages of aviation, the pilot was the sole factor carrying the responsibility of control. Due to the lack of sensitivity to certain modes of motion and the sluggishness of reaction of the human pilot, the aircraft could never be expected to achieve its best performance. Moreover, the constant vigilance of the human pilot led him to a state of extreme fatigue, and erroneous controlling of the plane was frequently the result.

Since World War I the automatic pilot has been introduced to aircraft for normal flight. The result has been very promising. As one of its leading industries, the Sperry Gyroscope Company[1] has developed an automatic pilot system to perfection entirely through a series of elaborate experiments. The success of the present Sperry system depends upon its logical procedure of development, but strict mathematical investigation has never been bothered with until recently when Weiss[2] and Lin[3] made a mathematical study of the controlled motion of airplanes. Haus[4] has tabulated every possible means of detection of deviation from normal quantities which is available as control source. However, the Sperry Company has so far only made an exhaustive development of simple deviation control (simple displacement control, using θ, the absolute inclination of the airplane as the control source). This is true also of the British Smith automatic pilot system[5]. Incidentally, the fact that Sperry and Smith have followed the same track might be due to a lack of theoretical light for other possible or even better control such as the compounding control used aboard the S. S. New Mexico reported by Minorsky[6]. Of course, due to the wide difference of mediums of support, and henceforth the difference in construction of steamships and aircraft, entirely different types of control may be needed, respectively, to produce successfully controlled motion. However, the principle of continuity and the advantage of higher derivative control has been explained very clearly by Minorsky in his paper[6] based upon the expansion theory of Taylor's series for a continuous function. In a simple word, "the higher time derivatives are capable of giving us the necessary warning as to what is going to happen a few instants ahead of the present time". It is logical to predict that with proper coordination of the compounding of derivative and displacement controls, the controlled result may be better than that obtainable from a simple control. It is the purpose of this thesis to throw considerable light on the exploration of simple controls other than the displacement type, as well as compounding controls which are hitherto hidden from aeronautical engineers.

From the general law of motion of a system composed of many degrees of freedom which are intercoupled by the nature of its construction, as many simultaneous linear differential equations can be written if the coefficients of each variable and its time derivatives are constant; that is, if each of the component forces varies linearly with its corresponding variable or time derivative of the variable. Due to physical properties such strict linear variation could not exist for certain variables, especially when their variations are of large range. However, if we are dealing with small variations, the assumption of linearity is acceptable especially when the motion is reduced to small magnitudes by well designed controls. The solution of the general simultaneous linear differential equations can be written using the principle of determinants[7] and the principle of operational calculus[8]. However, the task is not a pleasant one when the number of degree of freedom gets higher. Even a two-degrees-of-freedom system, which usually involves a solution of fourth order linear differential equation, becomes unmanageable as far as the general characteristic of the system is concerned.

Uncontrolled longitudinal motion[9,10,11] of aircraft already yields a linear differential equation of the fourth order. The addition of nonideal control[4,5] raises the equation from fourth to sixth order in which it is unmanageable to investigate systematically the possibility of better control than the conventionally successful simple displacement type. Probably, being handicapped by the unmanageability of the higher order differential equation, both Weiss and Lin confined their investigation to the simple displacement control. With utmost effort, a better displacement control may be discovered, but the improvement shall be predictably limited as the manufacturers have definitely fought their way with long experience to the present degree of success on their simple displacement control.

It is well known that the uncontrolled longitudinal motion after disturbance consists of two oscillatory motions of different frequency with different damping. The so-called short oscillation is usually associated with considerable damping which damps the motion very quickly. The long oscillation contains only a little damping so that it dies away in a few oscillations. Sometimes the damping is slightly negative, so the long oscillation will accordingly increase in magnitude after a disturbance. The application of control aims to redistribute the original poor distribution of damping among the two oscillations. In fact, as we shall see from the following parts of this thesis, a sole change of coefficient or coefficients of the fourth order differential equation, which represents the stability of the uncontrolled longitudinal motion, will attain the end of redistribution of damping. The nature really leaves plenty of flexibility for us to play these coefficients of the fourth order differential equation. The raising of the order of the differential equation by the addition of control is inevitable. But could we manage the additional degree of freedom in such a way that it remains its characteristic (damping and frequency) no matter what the coupling may be, and thus let it only affect the coefficient or coefficients of the original uncontrolled fourth order differential equation? The answer to this question is "yes, we can." From the analysis of the equation for controlled longitudinal motion which appears in Part I, Chapter II of this thesis, we shall see the mathematical proof of this answer together with a logical argument of good reasons for the introduction of such kind of control for the airplane.

For lateral disturbed motion the lack of course stability of an uncontrolled airplane can be overcome by the addition of control of rudder movement. The order of differential equation shall be raised from fifth to ninth if aileron control is provided together. The solution for a best control is far more difficult than in the case of longitudinal motion.

However, the thesis is carried on to insure full understanding of fourth order differential equations with which the design of a most desired longitudinal control is made possible. As the quartic equation has been a stumbling block to the understanding of higher degree equations, its full understanding shall improve the ladder for us to attack the still higher degree equation as that which we shall face on the lateral controls.

Maxwell[12] has done a good deal of work on servos involving the two-degree-of-freedom equation, yet due to lack of systematization his results are not readily applicable.

Besides the automatic control of airplanes, the growing importance of other control problems encourages the writer to systematize the mathematical presentation in a much wider range than it should be if only applicable to aircraft control engineers, in the hope that it will minimize the effort for every control engineer in his particular design work.

P A R T I

GENERAL THEORY OF AUTOMATIC CONTROL ON LONGITUDINAL MOTION OF AIRCRAFT

CHAPTER ONE

CONTROLLED LONGITUDINAL MOTION OF AIRCRAFT WITH CONVENTIONAL θ CONTROL

1. Separability of Longitudinal and Lateral Motions of the Airplane

A rigid body moving in free space possesses six degrees of freedom, three translational ones along three mutually perpendicular axes and three rotational ones about the same three axes. If the motion away from its steady state is small in magnitude, or all the physical coefficients are of "linearity", the unsteady components of motion of such a body would comprise six simultaneous oscillations having different frequencies and different dampings (or their equivalent in subsidences) which can be solved from the six simultaneous equations of motion -- each for one degree of freedom -- or their resultant represented by a linear differential equation of the twelfth order. The stability equation of such motion is, therefore, of the twelfth order.

However, the forces and moments on the airplane are indifferent to displacements along each of its three axes, so three roots of the resultant equation are zero. In addition to this, the uncontrolled airplane lacks the sense in azimuth (about vertical axis) so another root is also zero. The stability equation is therefore reduced to the eighth degree.

Due to the plane-symmetrical construction of the airplane, the six degrees of freedom can be separated into two groups of three each. One group involves only motion in the plane of symmetry and the solution of this group of equations gives the longitudinal motion of the airplane. The second group, if considered for small displacement alone, involves only asymmetric or lateral motion of the airplane. The separation of longitudinal motion from the lateral one greatly simplifies the mathematical mess otherwise involved.

2. Uncontrolled Longitudinal Motion

For the static and dynamic stabilities of longitudinal motion the aircraft engineer has designed a horizontal tail for his airplane. The tail is again split into a stabilizer and an elevator immediately behind the stabilizer for the convenience of controlling the airplane when it is disturbed by gusts of wind. However, when the elevator is locked, the airplane is said to be uncontrolled and the longitudinal motion is then defined as uncontrolled longitudinal motion.

For convenience of study, equations of uncontrolled longitudinal motion may be written, with reference to wind axes,[14] as follows:

$$X = m(\dot{u} + Wq)^{*} \qquad (1.01)$$

---

*The notation adopted in this thesis will be the same as that used by Metcalf[9] in his Resume on Airplane Longitudinal Stability. A table of symbols and definitions is prepared in the beginning of this part of the thesis.

$$Z = m(\dot{w} - Uq) \qquad (1.02)$$

$$M = B\dot{q} \qquad (1.03)$$

where the left sides of the above equations represent aerodynamic forces and moments while the right sides represent the inertial forces and moments necessary for the balance of the state of motion (small oscillation).

On expanding,

$$X = \frac{\partial X}{\partial u} u + \frac{\partial X}{\partial w} w + \frac{\partial X}{\partial q} q + \frac{\partial X}{\partial \theta} \theta \quad \cdots\cdot \qquad (1.04)$$

$$Z = \frac{\partial Z}{\partial u} u + \frac{\partial Z}{\partial w} w + \frac{\partial Z}{\partial q} q + \frac{\partial Z}{\partial \theta} \theta \quad \cdots\cdot \qquad (1.05)$$

$$M = \frac{\partial M}{\partial u} u + \frac{\partial M}{\partial w} w + \frac{\partial M}{\partial q} q + \frac{\partial M}{\partial \theta} \theta \quad \cdots\cdot \qquad (1.06)$$

Let $X_u = \frac{1}{m} \frac{\partial X}{\partial u}$ (1.07)

$$Z_w = \frac{1}{m} \frac{\partial Z}{\partial w} \qquad (1.08)$$

$$M_q = \frac{1}{B} \frac{\partial M}{\partial q} \qquad (1.09)$$

Then Eqs. (1.01), (1.02) and (1.03) can be written in the following form:

$$\dot{u} + W_q = X_u u + X_w w + X_q q + g\cos\textcircled{H}\,\theta \qquad (1.10)$$

$$\dot{w} - U_q = Z_u u + Z_w w + Z_q q + g\sin\textcircled{H}\,\theta \qquad (1.11)$$

$$\ddot{\theta} = M_u u + M_w w + M_q q \qquad (1.12)$$

Rearrange the terms and replace the time operator $\frac{d}{dt}$ by the symbol D. The above equations take the following form on neglecting terms with negligible coefficients:

$$(D - X_u)u - X_w w - g\cos\textcircled{H}\,\theta = 0 \qquad (1.13)$$

$$- Z_u u + (D - Z_w)w - (DU_o + g\textcircled{H})\,\theta = 0 \qquad (1.14)$$

$$- M_u u + M_w w + (D^2 - DM_q)\,\theta = 0 \qquad (1.15)$$

Equations (1.13), (1.14) and (1.15) represent the motion when the airplane is disturbed, but the disturbing force has already ceased. However, when the airplane encounters a gust such as the vertical one $w_o 1$* of magnitude w_o beginning from a state of equilibrium, equations (1.13), (1.14) and (1.15) become:

$$(D - X_u)u - X_w w - g\cos Ⓗ\,\theta = -X_w w_o 1 \quad (1.16)$$

$$- Z_u u + (D - Z_w)w - (DU_o + gⒽ)\,\theta = -Z_w w_o 1 \quad (1.17)$$

$$- M_u u - M_w w + (D^2 - DM_q)\,\theta = -M_w w_o 1 \quad (1.18)$$

By the principle of determinants[7] the solution can be expressed in operational form:

$$u = \frac{\Delta' u}{\Delta' o} w_o 1 \quad (1.19)$$

$$\theta = \frac{\Delta' \theta}{\Delta' o} w_o 1 \quad (1.20)$$

$$w = \frac{\Delta' w}{\Delta' o} w_o 1 \quad (1.21)$$

where Δ'_o is the stability determinant of the system

$$\Delta'_o = \begin{vmatrix} D - X_u & -X_w & -g\cos Ⓗ \\ - Z_u & D-Z_w & -DU_o-gⒽ \\ - M_u & -M_w & D^2-DM_q \end{vmatrix} \quad (1.22)$$

and Δ'_u, Δ'_θ and Δ'_w are the quality determinants of forward velocity, inclination angle and vertical velocity.

$$\Delta'_u = \begin{vmatrix} -X_w & -X_w & -g\cos H \\ -Z_w & D-Z_w & -DU_o-g H \\ -M_w & -M_w & D^2-DM_q \end{vmatrix} = D \begin{vmatrix} -X_w & -g\cos Ⓗ \\ & \\ -M_w & D^2-DM_q \end{vmatrix} \quad (1.23)$$

*The symbol 1 means unit step function; that is, when $t < 0$ the function is zero, when $t \geq 0$, the function is unity.

$$\Delta'_{\theta} = \begin{vmatrix} D-X_u & -X_w & -X_w \\ -Z_u & D-Z_w & -Z_w \\ -M_u & -M_w & -M_w \end{vmatrix} = D \begin{vmatrix} D-X_u & -X_w \\ -M_u & -M_w \end{vmatrix} \qquad (1.24)$$

$$\Delta'_{w} = \begin{vmatrix} D-X_u & -X_w & -g\cos\Theta \\ -Z_u & -Z_w & -DU_o - g\Theta \\ -M_u & -M_w & D^2-DM_q \end{vmatrix} = \Delta'_{o} - D \begin{vmatrix} D-X_u & -g\cos\Theta \\ -M_u & D^2-DM_q \end{vmatrix} \qquad (1.25)$$

Eq. (1.25) can also be written as

$$\Delta'_{w} = \Delta'_{o} - (\Delta'_{u} + \Delta'_{\theta} + \tilde{\Delta}') \qquad (1.25)a$$

where

$$\tilde{\Delta}' = \begin{vmatrix} D - X_u & -X_w & -g\cos\Theta \\ 1 & 1 & 1 \\ - M_u & -M_w & D^2-DM_q \end{vmatrix} \qquad (1.26)$$

The numerical solution of Equations (1.19), (1.20), and (1.21) is deferred until all the determinants are nondimensionalized and the control theory is well established.

When Eq. (1.22) is developed and equated to zero, an equation of fourth degree in algebraic form is obtained. It is this equation from which stability criteria of the disturbed motion can be evaluated. The extensive study of stability criteria of such equations is deferred to Part II of this thesis.

From the previous work of other investigators, it is understood that the longitudinal motion of the airplane after being disturbed consists of two components:

(a) A heavily damped oscillation of short period (of the order of a few seconds). This component disappears almost at once and in most airplanes is not noticeable.

(b) A lightly damped slow oscillation, during which the airplane produces noticeable changes of forward speed, altitude, and attitude.

It is customary to neglect the quick oscillation because of its almost immediate disappearance and no report is available from the pilots as they are never bothered by such rapid oscillation. Some writers even imply denunciation of the actual presence of such rapid yet fast dying oscillation. However, Jones[15] points out that although the heavy damping of this mode of motion insures its rapid subsidence in calm air, it imposes an effective restraint against movements of the airplane relative to the air, which results in violent movements of the airplane in gusts. This conclusion has been paid attention by Weiss[16] when he started his theory of automatic control in an attempt to reduce the sharp response of the airplane to the gusts by introducing the automatic control.

3. The Conventional θ Control and the Controlled Longitudinal Motion

The most convenient way to control the disturbed longitudinal motion is to operate the elevator manually or automatically. The elevator, when operated by automatic control, follows definite law in accordance with a certain disturbance detector. Haus[4] gives the following table of disturbance detectors and the quantity to which each is sensitive:

T A B L E I

LONGITUDINAL DISTURBANCE DETECTORS AVAILABLE FOR AUTOMATIC CONTROLS

Instrument	Recording Quantity	Symbol
a. Airspeed Indicator	Relative Speed	U
b. Wind Vane	Incidence	$\alpha = -\frac{w}{U}$
c. Free Gyro suspended at its center of gravity	Absolute Inclination	θ
d. Motor-driven Gyro with Precessional Moment	Angular Velocity	$q = \dot{\theta}$
e. Accelerometer along X axis	Direction of Apparent Gravity	$\frac{du}{dt} - g\sin\theta$
f. Accelerometer along Z axis	Magnitude of Apparent Gravity	$\frac{dw}{dt} + g\cos\theta$
g. Lift Indicator	Magnitude of Lift	wU
h. Rate of Climb Meter	Vertical Airspeed	w or $U\sin\theta$
i. Torsional About Y Axis	Angular Acceleration	$\ddot{\theta}$

Lin[17] gives the generalized equation of motion due to several controls combined together, but no conclusions are drawn. Both Weiss and Lin eventually resigned their work within the scope of conventional θ control (type c. according to the above table) as manufactured by Sperry[1] and Smith[5].

In order to discuss briefly the advantages and disadvantages of this type of control and henceforth lead to new control theory, it is necessary to start with the equation of motion of the system including the control.

The fundamental equation takes the following form:

$$X = m(\dot{u} + W_q) \quad (1.27)$$

$$Z = m(w - U_q) \quad (1.28)$$

$$+M' + M = B_{\dot{q}} \quad (1.29)$$

$$-K_\theta\theta = m_c\ddot{\sigma} + F_c\dot{\sigma} + K_c\sigma \quad (1.30)$$

where M' = moment exerted by the elevator which is controlled by the θ control

σ = control movement with respect to position when control is locked

m_c = equivalent mass of the control system

F_c = equivalent damping coefficient of the control system

K_c = equivalent spring constant of the control system

K_θ = equivalent exciting force coefficient for θ control

If Eq. (1.30) is divided throughout by m_c, we have

$$- F_\theta\theta = \ddot{\sigma} + 2\zeta_c\omega_{nc}\dot{\sigma} + \omega^2_{nc}\sigma \quad (1.31)$$

where $-F_\theta = -\frac{K_\theta}{m_c}$ = equivalent exciting coefficient per unit equivalent mass of the θ control. (1.32)

$\zeta_c^* = \frac{F_c}{2\sqrt{m_cK_c}}$ = damping ratio[18,19] of the equivalent control system (1.33)

$\omega_{nc} = \sqrt{\frac{K_c}{m_c}}$ = undamped angular natural frequency[18,19] of the equivalent control system. (1.34)

The value of natural frequency and damping ratio of the entire control system can be obtained from free vibration experiment when the exciting force is kept zero. The detail of such technique is referred to General Principles of Instru-

---

*The subscript c here is used to indicate the belonging of control. It does not mean critical as those used in Draper's paper.

ment Analysis[18] by Draper and Schliestett.

If Eqs. (1.27) and (1.28) are divided throughout by m and Eq. (1.29) divided by B and developed as we did for Eqs. (1.01), (1.02) and (1.03), the equations will appear in the following form:

$$\dot{u} + W_q = X_u u + X_w w + X_q q + g\cos \textcircled{H}\, \theta \qquad (1.35)$$

$$\dot{w} - U_q = Z_u u + Z_w w + Z_q q + g\sin \textcircled{H}\, \theta \qquad (1.36)$$

$$\ddot{\theta} = M_u u + M_w w + M_q q \qquad + M_\sigma \sigma \qquad (1.37)$$

$$\ddot{\sigma} = \qquad - F_\theta \theta \qquad -2\zeta_c \omega_{nc} \dot{\sigma} - \omega_{nc}^2 \sigma \qquad (1.38)$$

where $M_\sigma = \frac{1}{B}\frac{\partial M}{\partial \sigma}$ (1.39)

Use D for $\frac{d}{dt}$, rearrange the terms and neglect the insignificantly small terms, the above group of equations appear as follows:

$$(D-X_u)u \quad -X_w w \quad -g\cos \textcircled{H}\, \theta \qquad + 0 = 0 \qquad (1.40)$$

$$-Z_u u + (D-Z_w)w - (DU_o + g\sin \textcircled{H}\,) \quad 0 = 0 \qquad (1.41)$$

$$-M_u u \quad -M_w w \; + (D^2 - DM_q)\theta - M_\sigma \sigma = 0 \qquad (1.42)$$

$$0 \;+\; 0 \qquad F_\theta \theta + (D^2 + 2\zeta_c \omega_{nc} D + \omega_{nc}^2)\sigma = 0 \qquad (1.43)$$

When the controlled airplane encounters a vertical gust of $w_o l$ from equilibrium, the above equations are modified on their right sides.

$$(D-X_u)u \quad -X_w w \quad -g\cos \textcircled{H}\, \theta \quad + \quad 0 = -X_w w_o l \qquad (1.44)$$

$$-Z_u u \quad (D-Z_w)w - (DU_o + g\sin \textcircled{H}\,)\theta \; 0 = -Z_w w_o l \qquad (1.45)$$

$$-M_u u \quad -M_w w \quad (D^2 - DM_q)\,\theta - M_\sigma \sigma = -M_w w_o l \qquad (1.46)$$

$$0 \;+\; 0 \;+\; F_\theta \theta + (D^2 + 2\zeta_c \omega_{nc} D + \omega_{nc}^2)\sigma = 0 \qquad (1.47)$$

The solution of each of the variables u, w, θ, and σ can be expressed in the following form:

$$u = \frac{\Delta'_{uc}}{\Delta'_c} w_o l \qquad (1.48)$$

$$w = \frac{\Delta'_{wc}}{\Delta'_c} w_o l \qquad (1.49)$$

$$\theta = \frac{\Delta'_{\theta c}}{\Delta'_c} w_o l \qquad (1.50)$$

$$\sigma = \frac{\Delta'_{\sigma c}}{\Delta'_c} w_o l \qquad (1.51)$$

where

$$\Delta'_c = \begin{vmatrix} D-X_u & -X_w & -g\cos \Theta & 0 \\ -Z_u & D-Z_w & -DU_o-g\sin \Theta & 0 \\ -M_u & -M_w & D^2-DM_q & -M_\sigma \\ 0 & 0 & F_\theta & D^2+2\zeta_c\omega_{nc}D+\omega^2_{nc} \end{vmatrix} \qquad (1.52)$$

or

$$\Delta'_c = (D^2+2\zeta_c\omega_{nc}D+\omega^2_{nc})\Delta'_o + F_\theta M_\sigma \begin{vmatrix} D-X_u & -X_w \\ -Z_u & D-Z_w \end{vmatrix} \qquad (1.52)a$$

$$\Delta'_{uc} = \begin{vmatrix} -X_w & -X_w & -g\cos \Theta & 0 \\ -Z_w & D-Z_w & -DU_o g\sin \Theta & 0 \\ -M_w & -M_w & D^2-DM_q & -M_\sigma \\ 0 & 0 & F_\theta & D^2+2\zeta_c\omega_{nc}D+\omega^2_{nc} \end{vmatrix} \qquad (1.53)$$

or

$$\Delta'_{uc} = (D^2+2\zeta_c\omega_{nc}D+\omega^2_{nc})\Delta'_u - F_\theta M_\sigma DX_w \qquad (1.53)a$$

$$\Delta'_{wc} \begin{vmatrix} D-X_u & -X_w & -g\cos \Theta & 0 \\ -Z_u & -Z_w & -DU_o-g\sin \Theta & 0 \\ -M_u & -M_w & D^2-DM_q & -M_\sigma \\ 0 & 0 & F_\theta & D^2+2\zeta_c\omega_{nc}D+\omega^2_{nc} \end{vmatrix} \qquad (1.54)$$

or

$$\Delta'_{wc} = \Delta'_c - D(D+2\zeta_c\omega_{nc}D+\omega^2_{nc}) \begin{vmatrix} D-X_u & -g\cos \Theta \\ -M_u & D^2-DM_q \end{vmatrix} - F_\theta M_\sigma D(D-X_u) \qquad (1.54)a$$

$$\Delta'_{\theta c} = \begin{vmatrix} D-X_u & -X_w & -X_w & 0 \\ -Z_u & D-Z_w & -Z_w & 0 \\ -M_u & -M_w & -M_w & -M_\sigma \\ 0 & 0 & 0 & D+2\zeta_c\omega_{nc}D+\omega^2_{nc} \end{vmatrix} \qquad (1.55)$$

or $\Delta'_{\theta c} = (D^2 + 2\zeta_c \omega_{nc} D + \omega_{nc}^2)\, \Delta'_\theta$ (1.55)a

$$\Delta'_{\sigma c} = \begin{vmatrix} D-X_u & -X_w & -g\cos H & -X_w \\ -Z_u & D-Z_w & -DU_o - g\sin H & -Z_w \\ -M_u & -M_w & D^2 - DM_q & -M_w \\ 0 & 0 & F_\theta & 0 \end{vmatrix} \quad (1.56)$$

$$= -F_\theta D \begin{vmatrix} D-X_u & -X_w \\ -M_u & -M_w \end{vmatrix} = -F_\theta \Delta'_\theta \quad (1.56)a$$

Equation (1.52) or (1.52)a gives sixth degree algebraic equation when developed and equated to zero.

Or $\Delta'_c = 0$ will specify the stability of the disturbed motion.

Eqs. (1.48) to (1.51) can be solved, but the result will be more advantageous if it is converted into nondimensional form.[9,11]

In terms of its dimensions, Δ'_c becomes

$$\Delta'_c = \begin{vmatrix} T^{-1} & T^{-1} & LT^{-2} & X \\ T^{-1} & T^{-1} & LT^{-2} & X \\ L^{-1}T^{-1} & L^{-1}T^{-1} & T^{-2} & L^{-1}T^{-2} \\ X & X & LT^{-2} & T^{-2} \end{vmatrix} \quad (1.57)$$

The procedure of nondimensionalization can be performed as follows:

<u>First step</u> -- multiply the coefficients in the third column by unit length L. (1.57)a

<u>Second step</u> -- multiply the coefficients in the first column by unit time T.

<u>Third step</u> -- multiply the coefficient in the second column by unit time T.

Fourth step -- multiply the coefficient in the third column by $\frac{T}{V}(= L^{-1}T^{2})$. (1.57)a

Fifth step -- multiply the coefficient in the last column by T^2.

$$\therefore \Delta c = (L)(T)(T)(T^2L^{-1})(T^2)\,\Delta' c$$

$$\Delta c = T^6 \Delta' c \tag{1.58}$$

where Δ_c is defined as nondinensional stability determinant of the controlled longitudinal motion.

Likewise,

$$\Delta uc = T^6\, \Delta' uc \tag{1.59}$$

$$\Delta wc = T^6\, \Delta' wc \tag{1.60}$$

$$\Delta \theta c = LT^5\, \Delta' \theta c \tag{1.61}$$

$$\Delta \sigma c = T^5 \Delta' \sigma c \tag{1.62}$$

where Δuc, Δwc, $\Delta \theta c$ and $\Delta \sigma c$ are defined as nondimensional quality determinants of forward speed, vertical speed, inclination and control movement of the controlled longitudinal motion.

Substitute Eqs. (1.58) to (1.62) into equations (1.48) to (1.51). The expressions of u, w, θ and σ will be in terms of nondimensional determinants:

$$u = \frac{\Delta uc}{\Delta c} w_o l \tag{1.63}$$

$$w = \frac{\Delta wc}{\Delta c} w_o l \tag{1.64}$$

$$\theta = \frac{\Delta \theta c}{\Delta_c}\left[\frac{T}{L}\right] w_o l \tag{1.65}$$

$$\sigma = \frac{\Delta \sigma c}{\Delta c}\left[T\right] w_o l \tag{1.66}$$

or $$\frac{u}{w_o} = \frac{\Delta uc}{\Delta c} 1 \tag{1.63)a}$$

$$\frac{u}{w_o} = \frac{\Delta wc}{\Delta c} 1 \tag{1.64)a}$$

$$\frac{\theta}{w_o} = \frac{1}{\left[\frac{L}{T}\right]} \frac{\Delta \theta c}{\Delta c} 1 \tag{1.65)a}$$

$$\frac{\sigma}{w_o} = [T] \frac{\Delta \sigma c}{\Delta c} 1 \tag{1.66)a}$$

Eqs. (1.63)a, (1.64)a, (1.65)a and (1.66)a are defined as unit response of forward speed, vertical speed, inclination and control movement of the airplane to vertical gusts of the shape of the step function.

Unit Time T, Unit Length L and Compact Ratio μ

In order to change the dimensional coefficients such as X_w, Z_u, etc., into nondimensional coefficients x_w, z_u, etc., expressible in terms of those established fundamental aerodynamic coefficients, the length of the tail moment arm (from tail post to center of gravity of the airplane) is taken as the unit length (or characteristic length as defined elsewhere[20]). The unit time of the nondimensional system is defined by the following equation:

$$T = \frac{m}{\frac{\rho}{2} SU} \tag{1.67}$$

On this basis, the unit velocity of the nondimensional system should be

$$V = \frac{L}{T} = \frac{L}{\frac{m}{\frac{\rho}{2} SU}} \quad \text{or} \quad \frac{U}{\frac{m}{\frac{\rho}{2} SL}} \tag{1.68}$$

where $\frac{m}{\frac{\rho}{2}SL}$ receives the symbol μ

$$\text{or} \quad \mu = \frac{m}{\frac{\rho}{2}SL} \qquad (1.69)$$

which is defined as compact ratio (or relative density as defined elsewhere[20]).

The general procedure to reduce the dimensional coefficients to nondimensional one is omitted here (as it can be found elsewhere[11]) with the exception of those which have relations with the control. However a complete table is given on the next page for all the nondimensional coefficients. These coefficients correspond to those dimensional ones in Eq. (1.52) for the convenience of application.

TABLE II

NONDIMENSIONAL COEFFICIENTS FOR LONGITUDINAL STABILITY OF THE AIRPLANE

x_u	$2C_D$	$2C_D$
z_u	$2C_L$	$2C_L$
m_u	0	$\left[-C_{mw}\frac{c}{l^*}\frac{Q}{1+Q(r-1)}u\frac{dr}{du} + 0.62C_{D1}\sqrt{\frac{C_L}{C_{L_1}}}\,x\frac{h}{l^*}\right]\frac{1}{b}$
x_w	$\frac{dC_x}{d\alpha}$	$\frac{dC_x}{d\alpha}$
z_w	$\frac{dC_L}{d\alpha}$	$\frac{dC_L}{d\alpha}$
m_w	$\frac{1}{b}\frac{dC_m}{d\alpha}$	$\frac{1}{b}\frac{dC_m}{d\alpha}$
x_θ	$-\mu C_L$	$-\mu C_L$
z_θ	$-\mu C_L\theta_o$	$-\mu C_L\theta_o$
m_q	$\frac{5}{4}e\left(\frac{dC_L}{d\alpha}\right)'\frac{S'}{S}\frac{1}{b}$	$\frac{5}{4}e\left(\frac{dC_L}{d\alpha}\right)'\frac{S'}{S}\left[1+Q(r-1)\right]\frac{1}{b}$
f_θ	$\frac{1}{\theta_{w_c}}\mu C_L$	$\frac{1}{\theta_{w_c}}\mu C_L$
f_σ	$\omega^2_{nnc} = \frac{1}{\sigma_w}\frac{C_L}{L}$	$\omega^2_{nnc} = \frac{1}{\sigma_w}\frac{C_L}{L}$
f_σ	$2\zeta_c\omega_{nnc} = 2\zeta_c\sqrt{\frac{1}{\sigma_{w_c}}\frac{\mu C_L}{L}}$	$2\zeta_c\omega_{nnc} = 2\zeta_c\sqrt{\frac{1}{\sigma_{w_c}}\frac{\mu C_L}{L}}$
f_σ	1	1
m_σ	$\mu e\left(\frac{\partial C_L}{\partial\alpha}\right)'\frac{S'}{S}\frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta}{\partial\sigma}\frac{L}{b}$	$\mu\left(\frac{\partial C_L}{\partial\alpha}\right)'\frac{S'}{S}\frac{\partial\alpha'}{\partial\beta}\frac{\partial\beta}{\partial\sigma}\frac{L}{b}\left[1+Q(r-1)\right]e$

---

* Lower case L. Throughout this thesis lower case L will be so designated by the addition of an asterisk (1*) to differentiate from the figure one (1).

Control Coefficients F_θ, ω_{nc}, M_σ and Their Corresponding Nondimensional Coefficients f_θ, ω_{nnc} and m_σ

(a) $$F_\theta = \frac{1}{mc}\frac{\partial F}{\partial \theta} = \frac{K_\theta}{mc} \qquad (1.70)$$

$$f_\theta = \frac{K_\theta}{mc} X L^{-1} T^2 = \frac{K_\theta}{W_c} g X L^{-1} T^2 \qquad (1.71)$$

where W_c is the weight equivalent of the control system. Substitute Eq. (1.67) into Eq. (1.71) and assume level flight so that

$$W = C_L \tfrac{1}{2}\rho U^2 S \qquad (1.80)$$

then

$$f_\theta = \frac{1}{\theta_{W_c}} \mu C_L \qquad (1.81)$$

and $$\theta_{W_c} = \frac{W_c}{K_\theta} \qquad (1.82)$$

where θ_{W_c} is the magnitude of disturbed inclination of the airplane with which the force exerted on the control shall equal the equivalent weight of the control system.

(b) $$\omega^2_{nc} = \frac{K_c}{mc} = \frac{K_c}{W_c} g = \frac{g}{\sigma_W} \qquad (1.83)$$

$$\sigma_W = \frac{W_c}{K_c} \qquad (1.84)$$

where σ_W is defined as the static deflection of the control system due to its equivalent weight. Since

$$\omega^2_{nnc} = \omega^2_{nc} T^2 = \frac{1}{\sigma_{W_c}} g T^2 \qquad (1.85)$$

substitute Eq. (1.67) into Eq. (1.85) and assume the level flight condition. We then have

$$\omega^2_{nnc} = \frac{1}{\sigma_W}\frac{\mu C_L}{L} = f_\sigma \qquad (1.86)$$

Likewise

$$2\zeta_c \omega_{nnc} = 2\zeta_c \sqrt{\frac{1}{\sigma_{W_c}}\frac{\mu C_L}{L}} = f_{\dot\sigma} \qquad (1.87)$$

(c) $$M_\sigma = \frac{1}{B}\frac{\delta M'}{\delta \sigma} \tag{1.88}$$

$$M' = C_L \frac{\rho}{2} S' U'^2 L \tag{1.89}$$

where M' = tail moment

C_L' = tail lift coefficient

S' = tail area

U' = air stream velocity at tail plane which equals U if slip stream effect is neglected.

e = tail plane efficiency

$$\frac{\partial M'}{\partial \sigma} = e\left(\frac{\partial C_L}{\partial \alpha}\right)\left(\frac{\partial \alpha'}{\partial \beta}\right)\left(\frac{\partial \beta}{\partial \sigma}\right)\left(\frac{\rho}{2}\right) S U^2 L \tag{1.90}$$

where $\left(\frac{\partial C_L}{\partial \alpha}\right)'$ = static lift coefficient slope

β = elevator deflection from balanced position

σ = control movement

$$M_\sigma = \frac{1}{bmL^2} e\left(\frac{\partial C_L}{\partial \alpha}\right)' \frac{\partial \alpha'}{\partial \beta} \frac{\delta \beta}{\delta \sigma} \frac{S'}{S} U^2 L \tag{1.91}$$

where b = distribution factor of longitudinal moment of inertia.

Now, $$m_\sigma = \frac{1}{B}\frac{\partial M'}{\partial \sigma} L T^2 \tag{1.92}$$

Substitute equations (1.90) and (1.67) into equation (1.92) and simplify the expression,

$$m_\sigma = \mu\, e\left(\frac{\partial C_L}{d\alpha}\right)'\left(\frac{S'}{S}\right)\left(\frac{\partial \alpha'}{\partial \beta}\right)\left(\frac{\partial \beta}{\partial \sigma}\right)\left(\frac{L}{b}\right) \tag{1.93}$$

where $\frac{\partial \alpha'}{\partial \beta}$ depends upon the design of tail plane and

$\frac{\partial \beta}{\partial \sigma}$ depends upon the design of coupling of the control movement to the elevator deflection.

Stability Determinant in Nondimensional Form

Now Eq. (1.52) can be changed into nondimensional form:

$$\Delta_c = \begin{vmatrix} d-x_u & -x_w & -\mu C_L & 0 \\ -z_u & d-x_w & -d\mu-\mu C_{L\Theta} & 0 \\ -m_u & -m_w & d^2-dm_q & -m_\sigma \\ 0 & 0 & f_\theta & d^2+df_{\dot\sigma}+f_\sigma \end{vmatrix} \qquad (1.94)$$

where f_σ may be written as ω^2_{nnc} and $f_{\dot\sigma}$ as $2\zeta_c\omega_{nnc}$.

Likewise all the nondimensional quality determinants shall retain the same form as the dimensional ones [(1.53)a, (1.54)a, (1.55)a and (1.56)a] with the capital letters replaced by the small letters; also ω_{nc} replaced by ω_{nnc}.

Equation (1.94) can be partially developed into the following form:

$$\Delta_c = (d^2+2\zeta_c\omega_{nnc}d+\omega^2_{nnc})\Delta_o + f_\theta m_\sigma \begin{vmatrix} d-x_u & -x_w \\ -z_u & d-z_w \end{vmatrix} \qquad (1.94)a$$

It is very advantageous in one way to have the minor $\begin{vmatrix} d-x_u & -x_w \\ -z_u & d-z_w \end{vmatrix}$, especially when the control approaches ideal condition, because it raises the coefficient of the first degree operator of the developed stability equation with great predominance.

However, it is disadvantageous in the other way to have the same minor even when the control is ideal because it limits our freedom to adjust the coefficients of the stability equation in the most desirable way. (A detailed explanation will be given in the next chapter.)

For a nonideal control, the stability equation becomes a sixth degree equation from which three quadratic factors can be abstracted. The disturbed motion therefore comprises three oscillatory components (or the equivalent subsidence) among which one is due to the additional degree of freedom of the control, but it may be entirely different from the isolated control response. There is no literature available to draw conclusions as to the exact effect of the control characteristic upon the controlled motion and the reaction of the controlled motion to the control characteristic. The result of introducing θ control is considered successful as to easing the motion in pitch, but not at all as regards the vertical motion. Motion of pitch is eased because the control seeks to equalize the damping of the slow and fast oscillations of the uncontrolled motion so that both of them may disappear much sooner after being disturbed. For a detailed discussion the reader is referred to Lin's work.[17]

Weiss[16] carefully examines the vertical motion (due to vertical gust) assuming a full restraint in pitch (possible if the control is very fast and powerful). He points out that the sharp response (quick following-up characteristic) is almost entirely contributed by the coefficient z_w which is the slope of the lift coefficient curve and depends upon the aspect ratio of the wing. For the aerodynamic efficiency, larger aspect ratio is required, but such an airplane will give sharp response to vertical gust; i.e., the airplane will experience a large vertical acceleration during the gust-picking-up period. The vertical acceleration is painful to unaccustomed occupants. Any means of reducing the sharp response to vertical gust is therefore worth while investigating. However, the airplane's efficiency must not be violated.

Suggestions in the aim of reducing the sharpness of vertical response will be given and discussed in Part V of this thesis.

CHAPTER TWO

THEORY OF DEPARASITIZATION

4. The Parasite Minor of the Conventional θ Control

Going back to equation (1.94)a, let the control be very fast as approaching ideal condition; then:

$$\Delta_{ci} \cong \omega_{nnc}^2 \left\{ \Delta_o + \frac{f_\theta m_\sigma}{\omega_{nnc}^2} \begin{vmatrix} d - x_u & -x_w \\ -z_u & d - z_w \end{vmatrix} \right\} \quad (1.95)$$

where Δ_{ci} is the non-dimensional stability determinant of the controlled motion with an ideal control.

Develop Δ_o , the uncontrolled non-dimensional stability determinant, into the quartic form as:

$$\Delta_o = b_4 d^4 + b_3 d^3 + b_2 d^2 + b_1 d + b_o \quad (1.96)$$

where
$$b_4 = 1$$
$$b_3 = -(x_u + z_w + m_q)$$
$$b_2 = (x_u z_w - x_w z_u) + m_q (x_u + z_w) - \mu m_w$$
$$b_1 = -m_q (x_u z_w - x_w z_w) + \mu m_w (x_u - C_L \Theta) - \mu m_u (x_w + C_L)$$
$$b_o = -C_L \mu m_w (z_u - x_u \Theta) - C_L \mu m_u (x_w \Theta - z_w) \quad (1.96)\text{a}$$

Let
$$\begin{vmatrix} d - x_u & -x_w \\ -z_u & d - z_w \end{vmatrix} = c_2 d^2 + c_1 d + c_o \quad (1.97)$$

where
$$c_2 = 1$$
$$c_1 = -(x_u + z_w)$$
$$c_o = x_u z_w - x_w z_u \quad (1.98)$$

It is seen that the coefficients b_0, b_1 and b_2 are affected by c_1, c_2 and c_0 respectively with definite relative magnitudes. The common multiplier or coupling factor $\frac{f_\theta m_\sigma}{\omega_{nnc}^2}$ cannot alter these relative magnitudes at all. From the standpoint of controlling, we have surrendered our liberty of adjusting the uncontrolled coefficients b_4, b_3, b_2, b_1 and b_0 by the application of control to this determinant:

$$\begin{vmatrix} d - x_u & -x_w \\ -z_u & d - z_w \end{vmatrix}$$

which is entirely fixed by the design of the airplane and by the flying attitude. It is therefore justifiably defined as the parasite determinant of the conventional θ control, and shall bear the notation $\Delta_{p\theta}$

$$\therefore \Delta_{p\theta} = \begin{vmatrix} d - x_u & -x_w \\ -z_u & d - z_w \end{vmatrix} \tag{1.99}$$

If the control is non-ideal, the parasite determinant still holds its characteristic as to affect the coefficients of the stability equation with a definite relative magnitude.

5. Theory of Deparasitization

From the discussion of the previous chapter, it is seen that:

(a) The uncontrolled disturbed motion is comprised of a heavily damped fast oscillation and a lightly damped slow oscillation.

(b) The purpose of control (so far as θ control is concerned) is to equalize the damping of these two components and to ease the pitching oscillation in magnitude.

With ideal control, it is possible to achieve the purpose to a certain extent without introducing the complication of additional oscillatory component. But when the control is non-ideal, the presence of additional component of motion in pitch is inevitable.

Equation (1.65)a can be developed completely into operational form:

$$\frac{\theta}{W_o} = \frac{1}{\left[\frac{L}{T}\right]} \frac{(d^2+2\zeta_c \omega_{nnc} d + \omega_{nnc}^2)\, d\,(-d m_w + x_u m_w - m_u x_w)}{(d^2+2\zeta_c \omega_{nnc} d + \omega_{nnc}^2)\Delta_o + f_\theta m_\sigma \Delta_{p\theta}} \mathit{1} \qquad (1.100)$$

As the denominator is a sixth degree equation, three oscillatory components of motion should be expected.

There is no special advantage to have this complication. In fact, we have plenty to do with those two original components. By proper adjustment of coefficient, optimum distribution of damping between the two components is obtainable. The addition of third component is really unnecessary.

Fortunately, the parasite determinant itself represents a one-degree-of-freedom system. Its undamped natural frequency lies in between those of the slow and the fast oscillation and is nearer to the slow one. The damping ratio of the parasite determinant is in the vicinity of two for the average airplane. Now, if we allow the following condition:

$$\Delta_{p\theta} = (d^2+2\zeta_c \omega_{nnc} d + \omega_{nnc}^2)_p = d^2 + 2\zeta_p \omega_{nnp} d + \omega_{nnp}^2 \qquad (1.101)$$

that is, let the equivalent control system be designed according to the characteristic of parasite determinant, such non-ideal control will not introduce third oscillatory component as others. In a strict sense, the third component (pitch) has a magnitude of zero, as can be seen from the following equation:

$$\frac{\theta}{W_o} = \frac{1}{\left[\frac{L}{T}\right]} \frac{(d^2 + 2\zeta_c \omega_{nnp} d + \omega_{nnp}^2) d(-d m_w + x_u m_w - m_u x_w)}{(d^2 + 2\zeta_c \omega_{nnp} d + \omega_{nnp}^2)(\Delta_o + f_\theta m_\sigma)} 1$$

or

$$\frac{\theta}{W_o} = \frac{1}{\left[\frac{L}{T}\right]} \left\{ \frac{0}{d^2 + 2\zeta_c \omega_{nnp} d + \omega_{nnp}^2} 1 + \frac{d(-d m_w + x_u m_w - m_u x_w)}{\Delta_o + f_\theta m_\sigma} 1 \right\} \qquad (1.102)$$

Therefore, when the control is designed with the identical dynamic characteristic of the parasite determinant, it is defined as <u>deparasitized non-ideal control.</u>

It is easy to get confused by equation (1.102) where the coupling factor $f_\theta m_\sigma$ can only affect the constant term of Δ_o. There is no particular advantage gained by such control. However, equation (1.102) is only responsible to a θ control; that is, the control movement is only excited by force which is proportional θ away from the equilibrium value. If mechanical complication is allowed so that the control system is simultaneously excited by forces which are proportional to pitching velocity $\dot{\theta}$, pitching acceleration $\ddot{\theta}$, etc., the denominator of the second fraction of equation (1.102) will have the following form:

$$\Delta_o + \left(d^n f_{\overset{n}{\theta}} \cdots\cdots + d^2 f_{\ddot{\theta}} + d f_{\dot{\theta}} + f_\theta\right) m_\sigma \qquad (1.103)$$

where $f_{\overset{n}{\theta}}$, $f_{\ddot{\theta}}$, $f_{\dot{\theta}}$ and f_θ are entirely independent constants, the choice of which is entirely up to the control designer.

In practice, one or two exciting forces are needed to obtain desirable results, so the mechanical complication is not as bad as one would imagine from expression (1.103).

It should be noted, although the magnitude of the third component or the control component is zero in pitch, it is not so for forward speed nor for vertical speed. Equation (1.64)a can be fully developed as follows:

$$\frac{w}{W_0} = \frac{(d^2 + 2\zeta_p \omega_{np} d + \omega_{nnp}^2)(\Delta_0 + f_\theta m_\sigma) - d(d^2 + 2\zeta_p \omega_{nnp} d + \omega_{nnp}^2)\left[d^3 - (x_u + m_q)d^2 + x_u m_q d - g m_u C_L - f_\theta m_\sigma d(d - x_u)\right]}{(d^2 + 2\zeta_p \omega_{nnp} d + \omega_{nnp}^2)(\Delta_0 + f_\theta m_\sigma)} 1 \tag{1.104}$$

or

$$\frac{w}{W_0} = \frac{h_5 d^5 + h_4 d^4 + h_3 d^3 + h_2 d^2 + h_1 d + b_0 \omega_{nnp}^2 + \omega_{nnp}^2 f_\theta m_\sigma}{(d^2 + 2\zeta_p \omega_{nnp} d + \omega_{nnp}^2)(\Delta_0 + f_\theta m_\sigma)} 1 \qquad (1.104)a$$

The above equation gives the evidence of the presence of the control component in vertical speed. It may be seen more clearly if the reader is referred to Part V of this thesis.

Objection might be raised from the standpoint of fast control. Control lag is indeed troublesome when slow control is used as the deparasitized control, but it can be overcome by using higher derivative force or moment to excite the control. The physical significance of this overcoming property is evidently due to that higher derivative excited control controls earlier than the deviation or error; it controls the tendency of being disturbed.

The much overdamping characteristic of the deparasitized control is doubtful in its advantage. But due to the slow natural frequency of the control, the absolute damping force is not as tremendous as one might think it would be. However, even if it needs more energy to operate this type of control, it pays to do so if the controlled motion is in the most desired mode.

As far as pitching motion is concerned, the deparasitized non-ideal control may be considered as an ideal control. For one-degree-of-freedom system, the application of ideal controls 6, 12, 21 of the first class does not increase the degree of freedom. The (θ) deparasitized non-ideal control holds the same principle as far as the pitching motion is concerned. The θ control is primarily designed for pitching motion. It is for this reason that the writer feels the promise of this type of control.

It should be noted that the theory of deparasitization can be applied to controls other than the θ- elevator coupled type.

6. <u>The Stability Determinant of the θ Deparasitized Non-Ideal Control</u>

From equations (1.102) and (1.104) and expression (1.103) the stability determinant of the deparasitized control can be factored into one quadratic factor, which is actually the parasite minor, and one quartic factor. Let $\Delta_{cp\theta}$ represent the stability determinant of the deparasitized controlled motion, then:

$$\Delta_{cp\theta} = \Delta_{p\theta}\left[\Delta_o + m_\sigma(d^n f_{\overset{n}{\theta}} + \cdots d^2 f_{\ddot{\theta}} + d f_{\dot{\theta}} + f_\theta)\right] \tag{1.105}$$

Assume the highest derivative exciting force is $f_{\ddot{\theta}}$, then we have:

$$\Delta_{cp\theta} = (d^2 + 2\zeta_p \omega_{nnp} d + \omega_{nnp}^2)\left[d^4 + b_3 d^3 + (b_2 + m_\sigma f_{\ddot{\theta}})d^2 + (b_1 + m_\sigma f_{\dot{\theta}})d + b_o + m_\sigma f_\theta\right] \qquad (1.105)a$$

The quadratic factor depends upon the design of the airplane and flying attitude. In general, it gives two (real) negative roots for d when equated to zero, which indicates the presence of one subsiding pair in motion; in other words, an overdamped stable component. The quartic factor will give additional criteria of the stability of the motion.

In order to handle the distribution of damping of the slow and fast oscillation in an optimum way, thorough knowledge about the quartic equation is necessary. Part II of this thesis will be devoted to this purpose.

It should be noted when the control is isolated or the elevator is locked, m_σ is zero, the quartic factor returns to the uncontrolled form. The presence of the quadratic factor might lead to some misunderstanding. In fact, it should not be present. With close examination on equation (1.104) we may see that the expression can be reduced to the following uncontrolled form:

$$\frac{W}{W_0} = \frac{\Delta_0 - d\left[d^3 - (x_u + m_q)d^2 + x_u m_q d - \mu m_u C_L\right]}{\Delta_0} 1 \qquad (1.106)$$

when $m_\sigma = 0$; thus the stability of the disturbed motion is only determined by Δ_0, the uncontrolled stability determinant.

P A R T II

PROPERTIES OF THE QUARTIC EQUATION

CHAPTER THREE

THE IMPORTANCE OF THE QUARTIC EQUATION TO THE GENERALIZED AUTOMATIC CONTROL PROBLEM

7. Self-excited Vibrations[22]

Den Hartog, in treating the "hunting of steam-engine governors", points out that when the engine is rigidly coupled to an electric generator feeding a large network, the presence of "engine spring" causing the stability equation of the system goes up to a quartic one. Many other problems such as "Axial Oscillation of Turbines Caused by Steam Leakage", "Airplane Wing Flutter", etc., involve quartic equations.

To realize the importance of the quartic equation in the so-called "self-excited" problem, the steam engine governor system is quoted here with a few changes of notation adapted to the text of this thesis.

Let I = moment of inertia of the rotor

φ = angular displacement of the rotor from the equilibrium position

c_e = coefficient of damping torque resulting from the damper winding

k_e = coefficient of restoring torque or the magnetic spring constant in the air gap of the generator.

m = equivalent mass of the governor

x = displacement of the governor

c_g = damping coefficient of the governor system

k_g = spring constant of the governor system

C_1 = coefficient of velocity exciting force on the governor

C_2 = coefficient of displacement controlling torque

Then the two simultaneous differential equations of the problem are:

$$(mD^2 + C_gD + k_g)x = C_1 D\varphi \qquad (2.01)$$

$$(ID^2 + C_eD + k_e)\varphi = -C_2 x \qquad (2.02)$$

where $D = \frac{d}{dt}$

With some algebraic manipulation, the stability equation is established as follows:

$$D^4 + D^3\left(\frac{c_e}{I} + \frac{c_g}{m}\right) + D^2\left(\frac{k_e}{I} + \frac{k_g}{m} + \frac{c_e \cdot c_g}{Im}\right) + D\left(\frac{c_g}{m}\frac{k_e}{I} + \frac{c_e}{I}\frac{k_g}{m} + \frac{C_1C_2}{Im}\right) + \frac{k_ek_g}{Im} = 0 \qquad (2.04)$$

in which all coefficients are seen to be positive. The criterion of stability of Eq. (2.04) by the application of the Routh's discriminant[23] becomes

$$\left(\frac{c_e}{I} + \frac{c_g}{m}\right)\left(\frac{k_e}{I} + \frac{k_g}{m} + \frac{c_ec_g}{Im}\right)\left(\frac{c_gk_e}{mI} + \frac{c_e}{I}\frac{k_e}{m} + \frac{C_1C_2}{Im}\right) > \left(\frac{c_g}{m}\frac{k_e}{I} + \frac{c_e}{I}\frac{k_g}{m} + \frac{C_1C_2}{Im}\right)^2 + \frac{k_ek_g}{Im}\left(\frac{c_e}{I} + \frac{c_g}{m}\right)^2 \qquad (2.05)$$

A generalized conclusion cannot be drawn from Eq. (2.05). Den Hartog emphasizes its physical meaning only by assuming some special cases. However, in general, quantitative, not qualitative, criteria will be more desirable.

8. Constant Azimuth or Displacement Follow-up Control System

Ships as well as airplanes usually do not possess sensitivity of direction. Constant steering by means of a rudder is necessary to keep them in course or in constant azimuth. Minorsky[6] treats automatically steered bodies (using a ship as the primary subject) with ideal controls which are classified into three groups and considers the control lag due to mass and inertia of the transmission mechanism by introducing constant time lag.

In fact, control lag due to mass and damping cannot be exactly replaced by a constant time lag, as Lin[24] has verified the invalidity of this replacement when the coupling factor becomes large.

Displacement follow-up control has been in practice for some years. Hazen[21] defines this type of control as servomechanism, and treats them only with ideal controls. The automatic direction finder, manufactured by different companies, used in airplanes and the acoustic detector used in anti-aircraft artilleries belong to this type of control.

Now, let us take the automatic direction finder as an example leading to the important quartic equation. In its usual construction[25] the enclosed antenna loop is geared to a motor which supplies the following-up torque controlled by the error and error derivative signals from the antenna through a nonideal control system. The equation of motion for the rotor can be written as:

$$I\ddot{\varphi}_d + c_e\dot{\varphi}_d + k_e\varphi_d = -c\sigma \qquad (2.06)$$

and the equation of motion for the control can be written as

$$m_c\ddot{\sigma} + c_c\dot{\sigma} + k_c\sigma = C_2\ddot{E} + C_1\dot{E} + C_0E \qquad (2.07)$$

where I = equivalent moment of inertia of the rotating part.

φ_d = displacement of the rotor from zero position.

c_e = equivalent damping coefficient of the rotating part.

k_e = spring constant of the rotating part

m_c = equivalent mass of the control

c_e = equivalent damping coefficient of the control

C = coefficient of displacement controlling torque

E = error between the driving and driven angular displacement.

where $E = \varphi_d - \varphi_i l$ (2.08)

$\varphi_i l$ = the angle to be followed

C_0 = coefficient of displacement error exciting force

C_1 = coefficient of first error derivative exciting force

C_2 = coefficient of second error derivative exciting force

Substitute $D = \frac{d}{dt}$ for the overhead dot; also, substitute equation (2.08) into (2.07); then,

$$(ID^2 + C_eD + k_e)\varphi_d = -c\sigma \quad (2.09)$$

$$(m_cD^2 + c_cD + k_c)\sigma = (C_2D^2 + C_1D + C_0)\varphi_d - (C_2D^2 + C_1D + C_0)\varphi_i l \quad (2.10)$$

By canceling σ between equations (2.09) and (2.10), φ_d can be solved in terms of $\varphi_i l$.

$$\varphi_d = \frac{C(C_2D^2 + C_1D + C_0)}{(ID^2 + c_eD + k_e)(m_cD^2 + c_cD + k_c) + C(C_2D^2 + C_1D + C_0)}\varphi_i l \quad (2.11)$$

The solution can be expressed in terms of E and $\varphi_i l$

where

$$E = \frac{(ID^2 + c_eD + k_e)(m_cD^2 + c_cD + k_c)}{(ID^2 + c_eD + k_e)(m_cD^2 + c_cD + k_c) + C(C_2D^2 + C_1D + C_0)}\varphi_i l \quad (2.12)$$

In practice φ_i may be any function of time, but it is fair enough to assume a step function for the automatic direction finder when the loop is suddenly called into operation.

Eq. (2.11) can also be written in the developed form:

$$\varphi_d = \frac{B_2D^2 + B_1D + B_0}{A_4D^4 + A_3D^3 + A_2D^2 + A_1D + A_0}\varphi_i l \quad (2.13)$$

Where $A_4 = Im_c$

$A_3 = Ic_c + m_c c_e$

$A_2 = Ik_c + m_c k_e + c_e c_c + B_2$

$A_1 = k_e c_c + k_c c_e + B_1$

$A_0 = k_e k_c + B_0$

$B_2 = CC_2$

$B_1 = CC_1$

$$B_0 = CC_0 \qquad (2.13)a$$

It can be seen from Eqs. (2.13) and (2.13)a that the steady state value of φ_d cannot be equal to φ_i unless either k_e or k_c is zero or at least the product of $k_e k_c$ is very very small compared to B_0 . In most follow-up systems, the driven part usually possesses no stiffness; that is, with zero k_c so that the steady state reading is accurate or the following-up characteristic is perfect as far as the steady state is concerned.

For mathematical analysis, we may allow a very weak stiffness in the member to be controlled, which will simplify the analysis considerably.

Again the denominator of Eq. (2.13) is defined as the stability function of the controlled system, the numerator the quality function of φ_d. Divide both denominator and numerator of Eq. (2.13) by Im_c

$$\varphi_d = \frac{B_2' D^2 + B_1' D + B_0'}{D^4 + A_3' D^3 + A_2' D^2 + A_1' D + A_0'} \varphi_{i1} \qquad (2.13)b$$

where $A_3' = \frac{c_c}{m_c} + \frac{c_e}{I}$

$$A'_2 = \frac{k_c}{m_c} + \frac{k_e}{I} + \frac{c_c}{m_c} \cdot \frac{c_e}{I} + B'_2$$

$$A'_1 = \frac{c_c}{m_c}\frac{k_e}{I} + \frac{c_e}{I}\frac{k_c}{m_c} + B'_1$$

$$A'_0 = \frac{k_e}{I}\frac{k_c}{m} + B'_0$$

$$B'_2 = \frac{CC_2}{Im_c}$$

$$B'_1 = \frac{CC_1}{Im_c}$$

$$B'_0 = \frac{CC_0}{Im_c} \qquad (2.13)c$$

where B'_0 is defined as coupling factor of error-sensitive coupling,

B'_1 is defined as coupling factor or error-velocity coupling,

B'_2 is defined as coupling factor of error-acceleration coupling.

It can be seen that the stability function of Eq. (2.13)b (with the substitution of Eq. (2.13)c) takes the same form as Eq. (2.04), the stability equation of heavily laden generator controlled by ordinary flyball governor.

Now let $\frac{c_c}{2\sqrt{k_c m_c}} = \zeta_c$ [26] = damping ratio of control

$\sqrt{\frac{k_c}{m_c}} = \omega_{nc}$ = undamped angular natural frequency of the control

$\frac{c_e}{2\sqrt{k_e I}} = \zeta_o$ = damping ratio of the member to be controlled

$\sqrt{\frac{k_e}{I}} = \omega_{no}$ = undamped angular natural frequency of the member to be controlled

Then, $A'_3 = 2\zeta_c \omega_{nc} + 2\zeta_o \omega_{no}$

$A'_2 = \omega^2_{nc} + \omega^2_{no} + 4\zeta_o\zeta_c \omega_{no}\omega_{nc} + B'_2$

$$A'_1 = 2\zeta_c \omega_{nc} \omega^2_{no} + 2\zeta_o \omega_{no} \omega^2_{nc} + B'_1$$

$$A_0 = \omega^2_{no} \omega^2_{nc} + B'_0 \qquad (2.13)d$$

From the above examples, the importance of quartic equations has been established. It is believed that the better controlled result can only be obtained with a thorough knowledge of the quartic equation.

CHAPTER FOUR

STABILITY TRANSITION CURVE OF QUARTIC EQUATION WITH STANDARDIZED NONDIMENSIONAL COEFFICIENTS

9. Nondimensionalization of Stability Function

The stability function of nonideally controlled motion has been shown to be in quartic form in many cases. The general form of such function is

$$S(D) = D^4 + A_3'D^3 + A_2'D^2 + A_1'D + A_0' = 0 \qquad (2.14)$$

where A_3' , A_2' , A_1' and A_0' are physical constants with dimensions of T^{-1}, T^{-2}, T^{-3} and T^{-4}. (For longitudinal stability function of the airplane, these coefficients are nondimensional as in Eq. (1.105).

The first requirement for stability is that all coefficients must have the same sign.

Eq. (2.14) can be nondimensionalized by introducing a nondimensional operator λ which is equal to $\frac{D}{A_0'^{1/4}}$

$$\text{or} \quad D = A_0'^{1/4}\lambda \qquad (2.15)$$

Substitute Eq. (2.15) into Eq. (2.14)

$$S(D) = A_0'(\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1) = 0 \qquad (2.16)$$

$$\text{where} \quad \alpha_3 = \frac{A_3'}{A_0'^{1/4}}, \quad \alpha_2 = \frac{A_2'}{A_0'^{1/2}}, \quad \alpha_1 = \frac{A_1'}{A_0'^{3/4}} \qquad (2.17)$$

$$\text{or} \quad \psi(\lambda) = \frac{S(D)}{A_0'} = (\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1) = 0 \qquad (2.18)$$

By the process of nondimensionalization the stability of the equation is not affected, which will be proven a little later. The sole effect is that every root of the old equation is equivalent to $A_0'^{1/4}$ times the corresponding one of the new equation which is nondimensional.

10. Factorization of the Nondimensional Quartic Equation

Physically, a differential equation of fourth order with constant coefficients represents a motion or its equivalent, such as current, which consists of two vibratory components. The components may be really vibratory, divergent or convergent, or subsiding, but they are in general of different natural frequency and of different damping.

When the operator is used for the differential, the root of the equation indicates the frequencies and dampings. The same is true for nondimensionalized equations.

The factorization of the fourth degree equation or the root-finding process of the same is not an easy task, especially when it should give two pair of complex roots. Several outstanding methods have been found by different investigators.

Graeffe[28] uses the root-squaring process to insure the wide separation of roots. Lyon[29] considers the complex root as a two-dimensional vector and solves its magnitude and direction from which the roots are finally computed. Woodruff[30] extends Lyon's principle to sixth and higher order equations in which the calculating machine must be used. Ku[30] first gives the evidence of two quadratic factors for a quartic equation and finds the coefficient of the quadratics analytically from the general resolvant cubic equation[32] with the coefficients of the original quartic and solves the roots thereon from the quadratics. Recently Lin[33] discovered a method of successive approximation to factor quartic and higher degree equations. The method is simpler than any mentioned above, but analytic proof is still being sought.

None of the methods mentioned above is started from the physical significance of the fourth degree linear differential equation. The present method is based on the physical significance of the two components. Let us take the nondimensional quartic equation as our starting point and factor it into the following form:

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = (\lambda^2 + 2\zeta_1\omega_{r_1}\lambda + \omega_{r_1}^2)(\lambda^2 + 2\zeta_2\omega_{r_2}\lambda + \omega_{r_2}^2) \tag{2.19}$$

where ω_{r_1} = dimensionless undamped natural angular frequency of component 1,

ω_{r_2} = dimensionless undamped natural angular frequency of component 2,

ζ_1 = damping ratio of component 1,

ζ_2 = damping ratio of component 2.

As the components are of different frequencies and different dampings, it is convenient to take one component as reference component, and express other quantities in ratios with the reference quantity. Suppose that component 1 is of lower frequency (from now on we shall call it the low frequency component and call component 2 the high frequency component) and is taken as the reference component. This is purely arbitrary.

Now let $\omega_r = \omega_{r1}$ (arbitrarily), dimensionless undamped angu- frequency of reference component,

$\zeta_r = \zeta_1$ (arbitrarily), damping ratio of reference component,

$\rho_\omega = \frac{\omega_{r2}}{\omega_{r1}} = \frac{\omega_{n2}}{\omega_{n1}}$ ratio of undamped natural frequency

where ω_{n1} = undamped natural angular frequency of component 1

ω_{n2} = undamped natural angular frequency of component 2

$\rho_\zeta = \frac{\zeta_2}{\zeta_1}$, ratio of damping ratio

Eq. (2.19) becomes

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = \left(\lambda^2 + 2\zeta_r\omega_r\lambda + \omega_r^2\right) \times \left(\lambda^2 + 2\zeta_r\rho_\zeta\omega_r\rho_\omega\lambda + \omega_r^2\rho_\omega^2\right) \tag{2.20}$$

On developing,

$$\alpha_3 = 2\zeta_r\omega_r + 2\zeta_r\rho_\zeta\omega_r\rho_\omega \tag{2.21}$$

$$\alpha_2 = \omega_r^2 + \omega_r^2\rho_\omega^2 + 4\zeta_r^2\rho_\zeta\omega_r^2\rho_\omega \tag{2.22}$$

$$\alpha_1 = 2\zeta_r\rho_\zeta\omega_r^3\rho_\omega + 2\zeta_r\omega_r^3\rho_\omega^2 \tag{2.23}$$

$$1 = \omega_r^4\rho_\omega^2 \tag{2.24}$$

Eq. (2.24) gives us a real advantage, that since

$$\omega_r^4\rho_\omega^2 = 1$$

so $$\omega_r\rho_\omega^{1/2} = 1 \tag{2.24a}$$

$$\omega_r^2\rho_\omega = 1 \tag{2.24b}$$

and $$\omega_r^3\rho_\omega^{3/2} = 1 \tag{2.24c}$$

Substitute Eqs. (2.24)a, (2.24)b and (2.24)c into Eqs. (2.21), (2.22) and (2.23). The following equations are established.

$$\alpha_3 = 2\zeta_r\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega}\right) \tag{2.25}$$

$$\alpha_2 = \frac{1}{\rho_\omega} + \rho_\omega + 4\zeta_r^2\rho_\zeta \tag{2.26}$$

$$\alpha_1 = 2\zeta_r\left(\frac{\rho_\zeta}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) \tag{2.27}$$

The above three simultaneous equations will give solutions to the three important physical unknown quantities.

(1) ρ_ω, the ratio of undamped natural frequencies with respect to reference component,

(2) ζ_r, the damping ratio of the reference component,

(3) ρ_ζ, the ratio of damping ratios referring to the reference component.

In addition to the above three quantities, the fourth one, ω_r, can be solved from Eq. (2.24) or

$$\omega_r = \rho_\omega^{-1/2} \tag{2.28}$$

The actual process of solving these equations is deferred to the next chapter in which the Quartic Chart is designed to render the practical convenience.

11. Stability Transition Curve (Fig. 1)

It is understood that when the damping ratio of either component is zero, the system must be at a state of unending oscillation. If either damping is slightly negative, the oscillation is unstable. The loci of such transition plotted in terms of the nondimensional coefficients will be defined as stability transition curve.

Obtain the ratio $\frac{\alpha_3}{\alpha_1}$ from Eqs. (2.25) and (2.27)

$$\frac{\alpha_3}{\alpha_1} = \frac{1 + \rho_\zeta \rho_\omega}{\rho_\zeta + \rho_\omega} \tag{2.29}$$

when $\zeta_r = 0$ and the other component does possess some damping, ρ_ζ must be infinity, so at that condition

$$\frac{\alpha_3}{\alpha_1} = \rho_\omega \tag{2.30}$$

when $\zeta_r = 0$, or $\rho_\zeta = 0$, Eq. (2.26) is reduced to

$$\alpha_2 = \frac{1}{\rho_\omega} + \rho_\omega \tag{2.31}$$

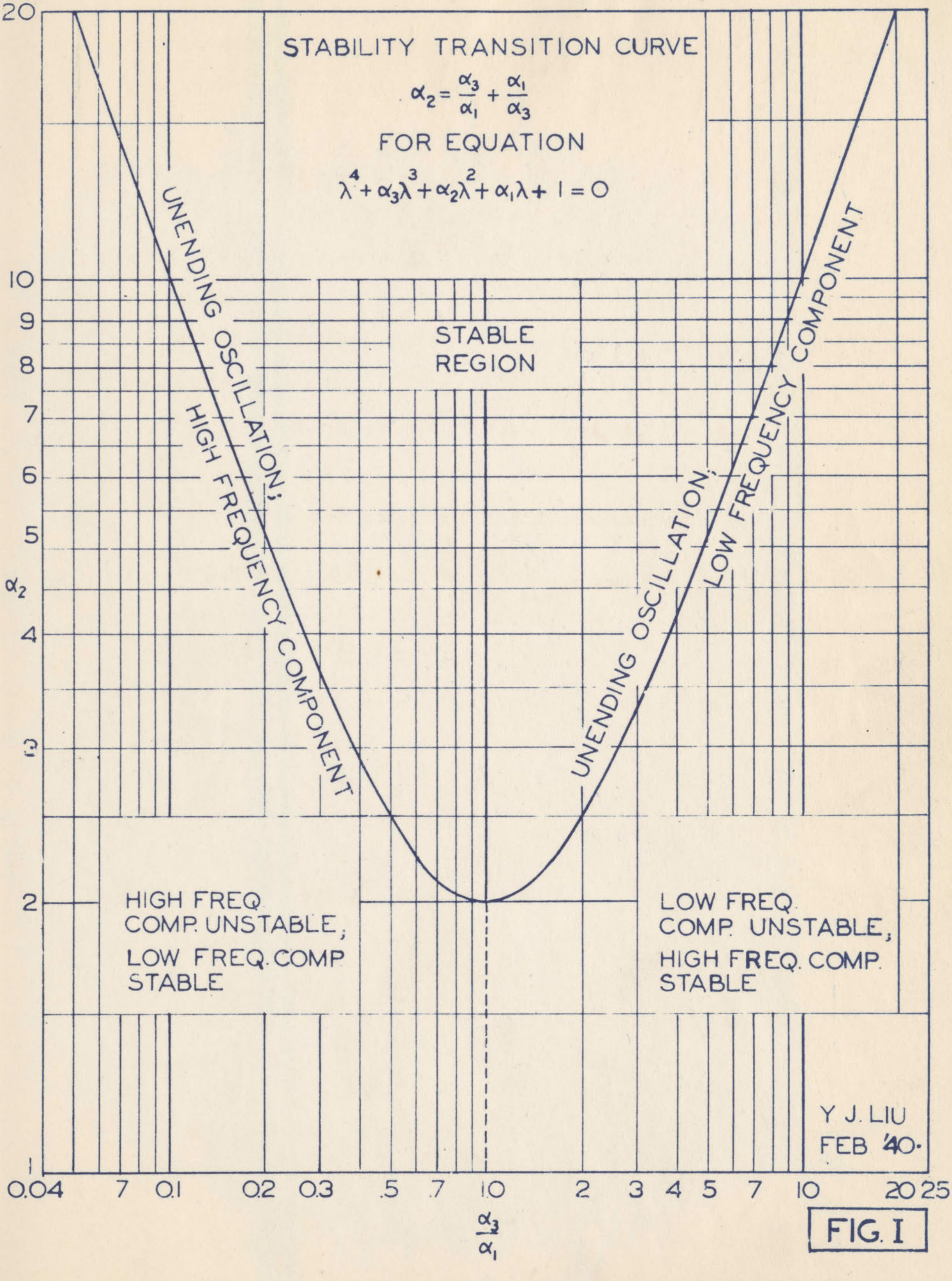

STABILITY TRANSITION CURVE
$\alpha_2 = \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}$
FOR EQUATION
$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$
STABLE REGION
UNENDING OSCILLATION; HIGH FREQUENCY COMPONENT
UNENDING OSCILLATION; LOW FREQUENCY COMPONENT
HIGH FREQ. COMP. UNSTABLE; LOW FREQ. COMP STABLE
LOW FREQ. COMP. UNSTABLE; HIGH FREQ. COMP. STABLE
Y. J. LIU
FEB '40.
α_2
20
10
9
8
7
6
5
4
3
2
1
0.04
7
0.1
0.2
0.3
.5
.7
1.0
2
3
4
5
7
10
20
25
$\frac{\alpha_3}{\alpha_1}$

FIG. I

It is evident when $\zeta_r = 0$ that the stability transition curve may be represented by the following equation

$$\alpha_2 = \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \tag{2.32}$$

Moreover, if $\zeta_r \neq 0$, but $\rho_\xi = 0$, Eq. (2.29) is reduced to

$$\frac{\alpha_3}{\alpha_1} = \frac{1}{\rho_\omega} \tag{2.32a}$$

So the stability transition curve can still be represented by Eq. (2.32) for the case.

We shall proceed to find stability criterion from the transition curve.

Take the unrestricted value of α_1 α_2 and α_3 from Eqs. (2.25), (2.26) and (2.27) and find the value of $\alpha_2 - \left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right)$ It can be shown

$$\alpha_2 - \left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right) = \frac{\frac{\rho_\xi}{\rho_\omega}\left(\rho_\omega^2 - 1\right)^2}{\rho_\xi\left(1+\rho_\omega^2\right) + \rho_\omega\left(1+\rho_\xi^2\right)} + 4\zeta_r^2\rho_\xi \tag{2.33}$$

When both ζ_r and ρ_ξ are positive, the system is stable and the right side of Eq. (2.33) is positive or greater than zero. That means

$$\alpha_2 > \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \tag{2.34}$$

is necessary for stability. The inequality (2.34) is therefore defined as the stability criterion of the standardized nondimensional equation (2.18).

The reverse condition for instability needs no further proof, but to show the rigorousness of Eq. (2.33), it is advisable to do so. Eq. (2.33) can be changed into the following form

$$\alpha_2 - \left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right) = \frac{\frac{1}{\rho_\omega}\left(\rho_\omega^2 - 1\right)^2}{\left(1+\rho_\omega^2\right) + \rho_\omega\left(\frac{1}{\rho_\xi} + \rho_\xi\right)} + 4\zeta_r^2\rho_\xi \tag{2.33a}$$

If the reference component possesses positive damping, the other component just possesses an infinitesimal negative damping; then ρ_ζ is negative, but is of infinitesimal magnitude. That is, the system is just off the transition condition and shows instability.

Let $\rho_\zeta = -\epsilon$ (2.35)

where ϵ is an infinitesimal quantity. Substitute the above equation into Eq. (2.33)a

$$\alpha_2 - \left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right) \cong -\left[\left(\rho_\omega - \frac{1}{\rho_\omega}\right)^2 + 4\zeta_r^{\,2}\right]\epsilon \qquad (2.36)$$

Eq. (2.36) shows that as soon as the motion departs from the transition condition, $\alpha_2 - \left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right)$ becomes negative, or

$$\alpha_2 < \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3} \qquad (2.37)$$

If the reference component possesses an infinitesimal negative damping while the other component possesses certain positive damping ζ_2 so that $\rho_\zeta = -\frac{\zeta_2}{\epsilon}$ which approaches infinity and has a negative sign

$$\zeta_r = -\epsilon\ , \quad \rho_\zeta = -\frac{\zeta_2}{\epsilon} \cong -\infty \qquad (2.35)\text{a}$$

Then Eq. (2.33)a becomes

$$\alpha_2 - \left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right) \cong -\left[\frac{1}{\zeta_2}\left(\rho_\omega - \frac{1}{\rho_\omega}\right)^2 + 4\zeta_2\right]\epsilon \qquad (2.36)\text{a}$$

The right side of Eq. (2.36)a is negative no matter how small ϵ is. Therefore the inequality (2.37) holds true for instability in any case.

12. The Damping Distributing Parameter $\frac{\alpha_3}{\alpha_1}$

Since the low frequency component has arbitrarily assigned as reference component, ρ_ω has to be greater than one.

Let (2.38)

$$\rho_\omega = 1 + \rho'$$

where ρ' is any positive number.

Substitute this value of ρ_ω into Eq. (2.29). The expression of $\frac{\alpha_3}{\alpha_1}$ becomes:

$$\frac{\alpha_3}{\alpha_1} = 1 + \frac{\rho'(\rho_\zeta - 1)}{1 + \rho + \rho_\zeta} \tag{2.39}$$

For a stable system (the condition is $\alpha_2 > (\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3})$ ζ_r and ρ_ζ are positive. Remember ρ_ζ is the ratio of damping ratios referred to the low frequency component. If ρ_ζ is greater than one, it means the high frequency component has a greater damping ratio. If ρ_ζ is less than one, it means that the high frequency component has a smaller damping ratio.

Now examine Eq. (2.39). If

$$\rho_\zeta > 1 \tag{2.40}$$

$$\frac{\alpha_3}{\alpha_1} = 1 + \rho'' \quad \text{where } \rho'' \text{ is a positive number}$$

or

$$\frac{\alpha_3}{\alpha_1} > 1 \tag{2.40a}$$

If

$$\rho_\zeta < 1 \tag{2.41}$$

$$\frac{\alpha_3}{\alpha_1} = 1 - \rho''' \quad \text{where } \rho''' \text{ is a positive number and less than one}$$

or

$$\frac{\alpha_3}{\alpha_1} < 1 \tag{2.41a}$$

The above condition can be extended even if ρ_ζ is negative; that is, the system is dynamically unstable.

It can therefore be safely concluded that:

(a) When $\frac{\alpha_3}{\alpha_1} > 1$, $\rho_\zeta > 1$; that is, the high frequency component possesses the greater damping ratio.

(b) When $\frac{\alpha_3}{\alpha_1} < 1$, $\rho_\zeta < 1$; that is, the high frequency component possesses the smaller damping ratio.

Apparently we have forgotten the condition when $\frac{\alpha_3}{\alpha_1} = 1$ By reexamining Eq. (2.29)

$$\frac{\alpha_3}{\alpha_1} = \frac{1 + \rho_\zeta \rho_\omega}{\rho_\zeta + \rho_\omega}$$

two parallel conclusions can be made immediately:

(c-1) When $\frac{\alpha_3}{\alpha_1} = 1$, $\rho_\zeta = 1$ at any value of ρ_ω (2.42)

That means that the two components possess the same dynamic behavior except that they are of different undamped natural frequencies. Or

(c-2) When $\frac{\alpha_3}{\alpha_1} = 1$, $\rho_\omega = 1$ at any value of ρ_ζ (2.42)a

That means that the two components are only of the same frequency, but their dynamic behaviors are different.

(c-3) There is also a possible case that both ρ_ω and ρ_ζ are unity. Mathematically it means repetition of the quadratic factor. Physically it means the system is critically damped* quadratically.

The case of $\rho_\zeta = \rho_\omega = 1$ for $\frac{\alpha_3}{\alpha_1} = 1$ is merely a special case which is common to both (c-1) and (c-2). It is the coefficient α_2 which will decide the fate of $\frac{\alpha_3}{\alpha_1} = 1$. Further detailed discussion is deferred until the development of the Quartic Chart and again when the theory of tuning is presented

---

*Critical damping in simple degree of freedom means repetition of the binomial factor mathematically, so for two degrees of freedom the repetition of the quadratic factor is also a kind of critical damping physically.

CHAPTER FIVE

THE DEVELOPMENT OF QUARTIC CHART AND STABILITY CRITERIA

13. The Dimensional Resolvent Cubic Equation

The resolvent cubic equation may be defined as one derived from an ordinary quartic equation and serves the latter as a tool to evaluate its roots (roots of the quartic). The mathematical approach of the resolvent cubic equation varies as various mathematical attacks. With purely algebraic manipulation Ferrari[32] reaches the form:

$$y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0 \qquad (2.43)$$

from the quartic equation:

$$x^4 + bx^3 + cx^2 + dx + e = 0 \qquad (2.44)$$

Lyon[29] starting with vector conception reaches the same form only in different notations. Ku[31] with biquadratic manipulation also obtains the same equation as Lyon does. The physical meaning of such resolvent cubic equation is so far hidden from the engineers' retina. The only mathematical interpretation is the relation between the roots of the quartic and of its resolvent. Such relation is expressed by the following equations:

$$y_1 = x_1x_2 + x_3x_4 \quad {}^{34}$$
$$y_2 = x_1x_3 + x_2x_4$$
$$y_3 = x_1x_4 + x_2x_3 \qquad (2.45)$$

However, equation (2.45) does not furnish any light for engineers' understanding.

14. The Non-dimensional Resolvent Cubic Equation and Its Physical Significance

In paragraph 10 of the last chapter we have established equations (2.25), (2.26) and (2.27), which we shall rewrite here for the starting point of the non-dimensional resolvent cubic equation:

$$\sigma_3 = 2\varphi_r\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\varphi\sqrt{\rho_\omega}\right) \tag{2.25}$$

$$\sigma_2 = \frac{1}{\rho_\omega} + \rho_\omega + 4\varphi_r^2\,\rho_\varphi \tag{2.26}$$

$$\sigma_1 = 2\varphi_r\left(\frac{\rho_\varphi}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) \tag{2.27}$$

From equations (2.25) and (2.26) we have the product:

$$\sigma_1\sigma_3 = 4\varphi_r^2\,\rho_\varphi\left[\rho_\omega + \frac{1}{\rho_\omega} + \rho_\varphi + \frac{1}{\rho_\varphi}\right]$$

or

$$4\varphi_r^2\,\rho_\varphi = \frac{\sigma_1\sigma_3}{\rho_\omega + \frac{1}{\rho_\omega} + \rho_\varphi + \frac{1}{\rho_\varphi}} \tag{2.46}$$

The ratio of σ_3 to σ_1 has been obtained as equation (2.29).

$$\frac{\sigma_3}{\sigma_1} = \frac{1 + \rho_\varphi\rho_\omega}{\rho_\varphi + \rho_\omega}$$

or solve for ρ_φ:

$$\rho_\varphi = \frac{\rho_\omega\left(\frac{\sigma_3}{\sigma_1}\right) - 1}{\rho_\omega - \frac{\sigma_3}{\sigma_1}} \tag{2.47}$$

With purely algebraic manipulation on the sum of equation (2.47) and its reciprocal expression, we get:

$$\rho_\varphi + \frac{1}{\rho_\varphi} = \frac{\left(\rho_\omega + \frac{1}{\rho_\omega}\right)\left(\frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3}\right) - 4}{\left(\rho_\omega + \frac{1}{\rho_\omega}\right) - \left(\frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3}\right)} \tag{2.48}$$

Now, define: $\rho_\omega + \frac{1}{\rho_\omega} = \rho_m$ (2.49)

where ρ_m is called mutual frequency ratio.

Substitute the notation ℓ_m into equations (2.26), (2.46) and (2.48); and substitute equation (2.48) into equation (2.46), and finally substitute the expression thus obtained for $4\varphi_r^2 \ell_p$ into equation (2.26), and we will get the following expression:

$$\alpha_2 = \ell_m + \cfrac{\alpha_3 \alpha_1}{\ell_m + \cfrac{\ell_m\left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right) - 4}{\ell_m - \left(\frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}\right)}} \qquad (2.50)\text{a}$$

Clear up the fraction of the above equation and the non-dimensional resolvent cubic equation is obtained:

$$\ell_m^3 - \alpha_2 \ell_m^2 + (\alpha_3 \alpha_1 - 4)\ell_m - \left[(\alpha_3^2 + \alpha_1^2) - 4\alpha_2\right] = 0 \qquad (2.50)$$

One can immediately observe the similarity between the dimensional and non-dimensional resolvent cubic equation. The equivalent e term in the non-dimensional equation is unity, so the non-dimensional resolvent cubic equation is much simplified. Yet the physical significance is evident in the non-dimensional resolvent cubic equation whose root (or roots) is the mutual frequency ratio, or:

$$\ell_\omega + \frac{1}{\ell_\omega}$$

Three roots are obtainable from the non-dimensional resolvent cubic equation. At least one of them is real and greater than the other two to furnish real value of ℓ_ω; the other two may be real or complex. What do these mean? We shall discuss them when the quartic chart is constructed.

15. The Modified Non-dimensional Resolvent Cubic and Stability Criteria

Let $\ell_m = \alpha_2 \ell_\alpha$ [27] and substitute into equation (2.50); the nondimensional resolvent is modified to the following form:

$$\rho_\sigma^3 - \rho_\sigma^2 + \left(\frac{\sigma_3\sigma_1 - 4}{\sigma_2^2}\right)\rho_\sigma - \left[\frac{(\sigma_3^2+\sigma_1^2) - 4\sigma_2}{\sigma_2^3}\right] = 0 \qquad (2.51)$$

where σ_2 is the middle coefficient of the original non-dimensional quartic equation and ρ_σ is defined as modified mutual frequency ratio, or:

$$\rho_\sigma = \frac{1}{\sigma_2}\rho_m = \frac{1}{\sigma_2}\left(\rho_\omega + \frac{1}{\rho_\omega}\right) \qquad (2.52)$$

Equation (2.51) will look much simpler if we define the lumped coefficients:

$$\frac{\sigma_3\sigma_1 - 4}{\sigma_2^2} = M \qquad (2.53)$$

and

$$\frac{\sigma_3^2 + \sigma_1^2 - 4\sigma_2}{\sigma_2^3} = N \qquad (2.54)$$

So equation (2.51) assumes the following form:

$$\rho_\sigma^3 - \rho_\sigma^2 + M\rho_\sigma - N = 0 \qquad (2.55)$$

It is seen from equation (2.55) that M and N are the constants which will give solution to ρ_σ, and then the process can be traced back until all the non-dimensional constants with physical significance are determined. It is therefore believed that certain special combination of M and N will mark:

(a) the boundary line between stably and unstably oscillatory regions

(b) the boundary line between stably oscillatory and stably non-oscillatory regions

Therefore, M and N are defined as stability criteria of the non-dimensional quartic equation.

(a) The Boundary Line Between Stably and Unstably Oscillatory Region

Substitute the equation of stability transition curve:

$$\sigma_2 = \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3}$$

into equations (2.53) and (2.54):

$$\frac{M}{N} = \frac{(\sigma_3\sigma_1 - 4)\,\sigma_2}{(\sigma_3^2+\sigma_1^2) - 4\sigma_2} = \frac{\sigma_3\sigma_1\sigma_2 - 4\sigma_2}{\sigma_3^2+\sigma_1^2 - 4\sigma_2} = 1 \qquad (2.56)$$

or when $M=N$, the system is in the state of unending oscillation. When φ_r or $\rho_\varphi=0, \sigma_3, \sigma_1$ are positive, it is possible $M = N =$ positive value or negative value depending upon their relative magnitude. But when $\varphi_r = \rho_\varphi = 0, \sigma_3 = \sigma_1 = 0$; therefore, it is only possible that $M=N=$ negative value; it can never be positive.

$\therefore$ $M=N=$ positive	One component is at unending oscillation, and the ratio $\frac{\sigma_3}{\sigma_1}$ will tell whether this component is a high frequency one or low frequency one.
$M=N=$ negative	One or both components are at unending oscillation.

When $\sigma_3=\sigma_1=0$, and $\sigma_2=2$ (or $\rho_\omega=1$), then $M=N=-1$. This is a special condition; when the equation has two identical quadratic factors, both of them miss their damping term.

The inequality $\sigma_2 > \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3}$ must be maintained if both components are to be stable, or:

$$\sigma_2\sigma_1\sigma_3 > \sigma_3^2 + \sigma_1^2 \qquad (2.57)$$

When this is substituted into equation (2.56), it becomes the following inequality:

$$\frac{\mathcal{M}}{\mathcal{N}} > 1 \tag{2.58}$$

or $\mathcal{M} > \mathcal{N}$ is required to have stable operation.

If the inequality $\sigma_2 < \frac{\sigma_3}{\sigma_1} + \frac{\sigma_1}{\sigma_3}$ is substituted for the unstable region, equation (2.56) becomes the following inequality:

$$\frac{\mathcal{M}}{\mathcal{N}} < 1 \tag{2.59}$$

or $\mathcal{N} > \mathcal{M}$ is the region of unstable operation.

Therefore, when $\mathcal{N}$ is plotted as ordinate against $\mathcal{M}$ as abscissa, this boundary condition $\mathcal{M} = \mathcal{N}$ is a straight line passing through origin. The region above this line is unstable and the region below it stable.

(b) The Boundary Line Between Stably Oscillatory and Stably Non-oscillatory Regions, Or the Locus of Critical Damping

When the two components are both critically damped,

$$\zeta_r = \rho_\zeta \zeta_r = 1$$

that makes:

$$\sigma_3 = \sigma_1 = 2\left(\frac{1}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) \tag{2.60}$$

and

$$\sigma_2 = \rho_\omega + \frac{1}{\rho_\omega} + 4 \tag{2.61}$$

This condition will mark the oscillatory and non-oscillatory region, because if both ζ_r and $\rho_\zeta \zeta_r$ are greater than unity, both components are overdamped, and four distinct real roots shall be observed from the quartic equation.

Substitute equations (2.60) and (2.61) into the general expressions of $\mathcal{M}$ and $\mathcal{N}$ (equations (2.53) and (2.54)); $\mathcal{M}$ and $\mathcal{N}$ can be evaluated as:

$$\mathcal{M} = \frac{4\left(\rho_\omega + \frac{1}{\rho_\omega} + 1\right)}{\left(\rho_\omega + \frac{1}{\rho_\omega} + 4\right)^2} = \frac{4(\rho_m + 1)}{(\rho_m + 4)^2} \tag{2.62}$$

and
$$\mathcal{N} = \frac{4\left(\rho_\omega + \frac{1}{\rho_\omega}\right)}{\left(\rho_\omega + \frac{1}{\rho_\omega} + 4\right)^3} = \frac{4\,\rho_m}{(\rho_m + 4)^3} \tag{2.63}$$

The useful range of ρ_m can be extended from $+\infty$ to -2. When -2 $< \rho_m <$ 2, ρ_ω becomes a complex number. Mathematically it is correct, but it is hard to be interpreted physically. (The confusion will be cleared up when the Quartic Chart is completed.)

Another form of expression of $\mathcal{M}$ and $\mathcal{N}$ for this oscillatory and non-oscillatory boundary can be obtained by considering starting with both components critically damped so that their factorized expression of the quartic equation can be written as:

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = \left(\lambda + \frac{1}{\rho_\omega}\right)\left(\lambda + \frac{1}{\rho_\omega}\right)(\lambda + \rho_\omega)(\lambda + \rho_\omega) \tag{2.64}$$

$$= \left[\left(\lambda + \frac{1}{\rho_\omega}\right)(\lambda + \rho_\omega)\right]^2$$

$$= \left[\lambda^2 + \left(\frac{1}{\rho_\omega} + \rho_\omega\right)\lambda + 1\right]^2$$

$$= \left[\lambda^2 + 2\varphi_r\lambda + 1\right]^2 \tag{2.64a}$$

The physical meaning of equation (2.64)a is that the two vibratory components are of the same frequency and same damping ratio; in fact, the system is critically damped quadratically. In symbol, they are:

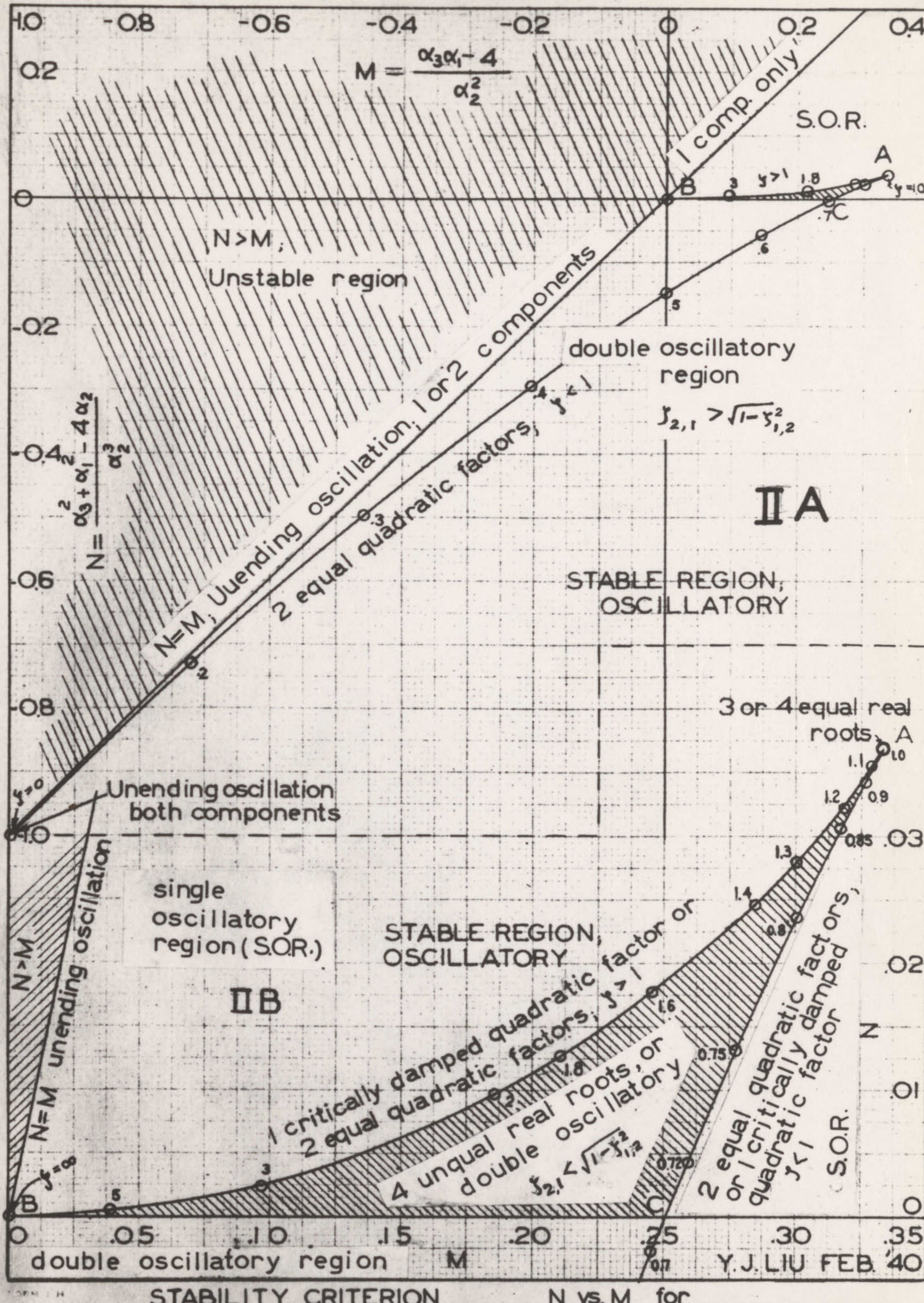
$$\rho_\omega = 1, \quad \rho_\varphi = 1 \tag{2.65}$$

That makes:

$$\alpha_3 = \alpha_1 = 4\varphi_r \tag{2.66}$$

and

$$\alpha_2 = 2(1 + 2\varphi_r^2) \tag{2.67}$$



STABILITY CRITERION N vs. M for

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$$

FIG. IIA & IIB

STABILTY CRITERION N vs. M for

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$$

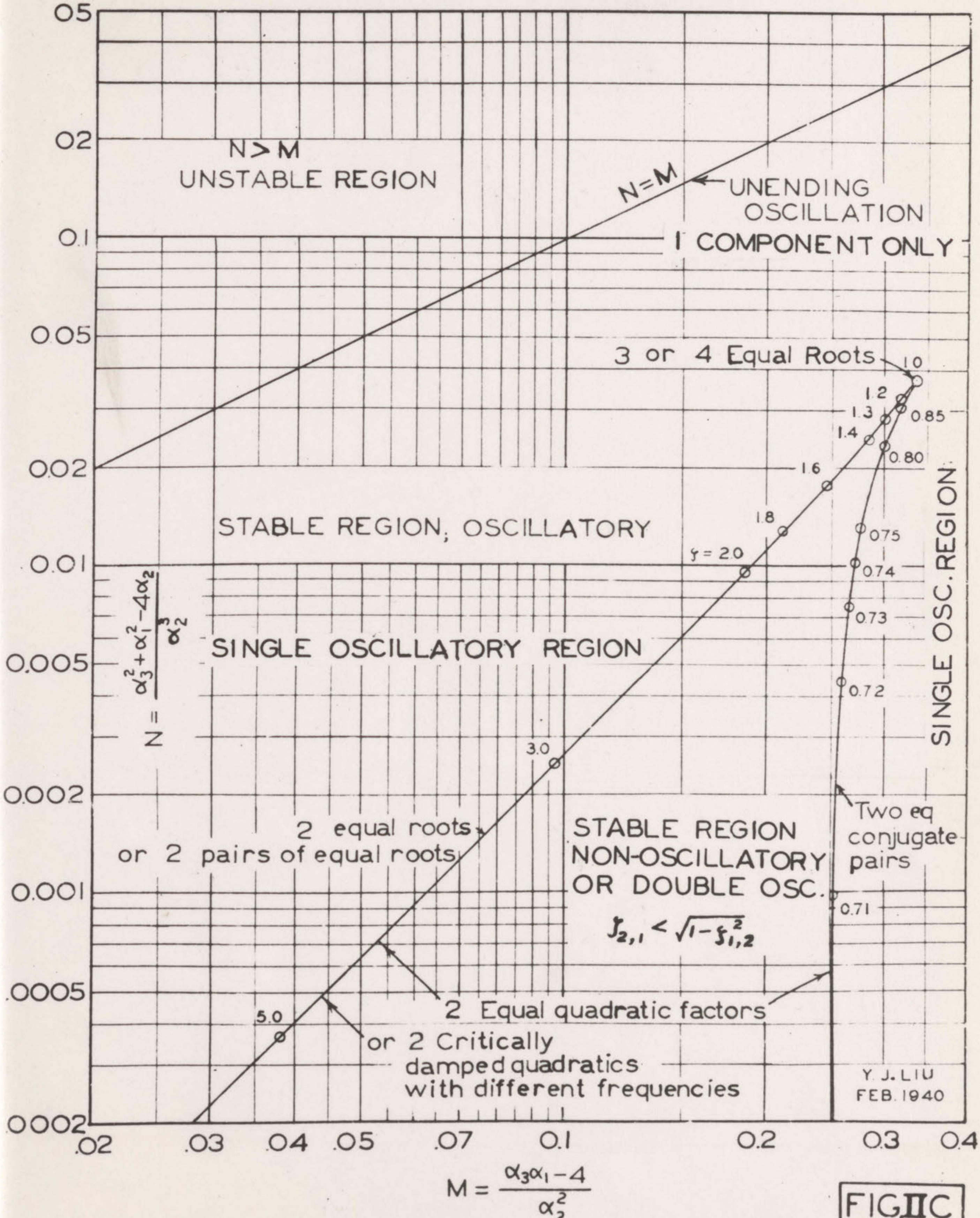


FIG. IIC

When equations (2.66) and (2.67) are substituted into the general form of $\mathcal{M}$ and $\mathcal{N}$, they appear as:

$$\mathcal{M} = \frac{4\varphi_r^2 - 1}{(1 + 2\varphi_r^2)^2} \qquad (2.68)$$

and

$$\mathcal{N} = \frac{2\varphi_r^2 - 1}{(1 + 2\varphi_r^2)^3} \qquad (2.69)$$

Here the usable range of φ_r extends from $+\infty$ to 0.

Figures IIA, B and C show the plot of equations (2.68), (2.69) and (2.58). Figure IIA covers a wide range of $\mathcal{M}$ and $\mathcal{N}$. Stability can be verified with practically every possible combination of $\mathcal{M}$ and $\mathcal{N}$. The non-oscillatory region is bounded by the curve BA, AC and BC, and is shaded. Figure IIB is an enlargement of the non-oscillatory region where four unequal real roots are present. Figure IIC is plotted in logarithmic scales to render better the visualization of very small quantities of $\mathcal{M}$ and $\mathcal{N}$ forming part of the boundary between the oscillatory and non-oscillatory regions. Figure I can be referred to as an indication of relative damping between the two components. Such indication is not available in Figure II.

The shaded area, being the non-oscillatory region, is not so evident; the pure mathematical proof which is tedious is excluded here. However, a simple logic proof will be given when the Quartic Chart is completed.

Equations (2.62) and (2.68), and equations (2.63) and (2.69) are mutually transferable with the following relation:

$$\rho_m + 4 = 2(1 + 2\varphi_r^2) \qquad (2.70)$$

When $\varphi_r = 1$ for equations (2.68) and (2.69), it means that the two quadratic factors are identically the same, and both are critically damped, and

$$\therefore (M, N) = (\tfrac{1}{3}, \tfrac{1}{27}) \qquad (2.71)$$

is the point of cusp where four equal real roots are possibly obtainable from the quartic equation.

16. The Development of the Quartic Chart for the Non-dimensional Quartic Equation $\lambda^4 + \sigma_3\lambda^3 + \sigma_2\lambda^2 + \sigma_1\lambda + 1 = 0$

When the stability criteria M and N is obtained from the coefficients σ_3, σ_2 and σ_1, the modified resolvent cubic equation is fixed:

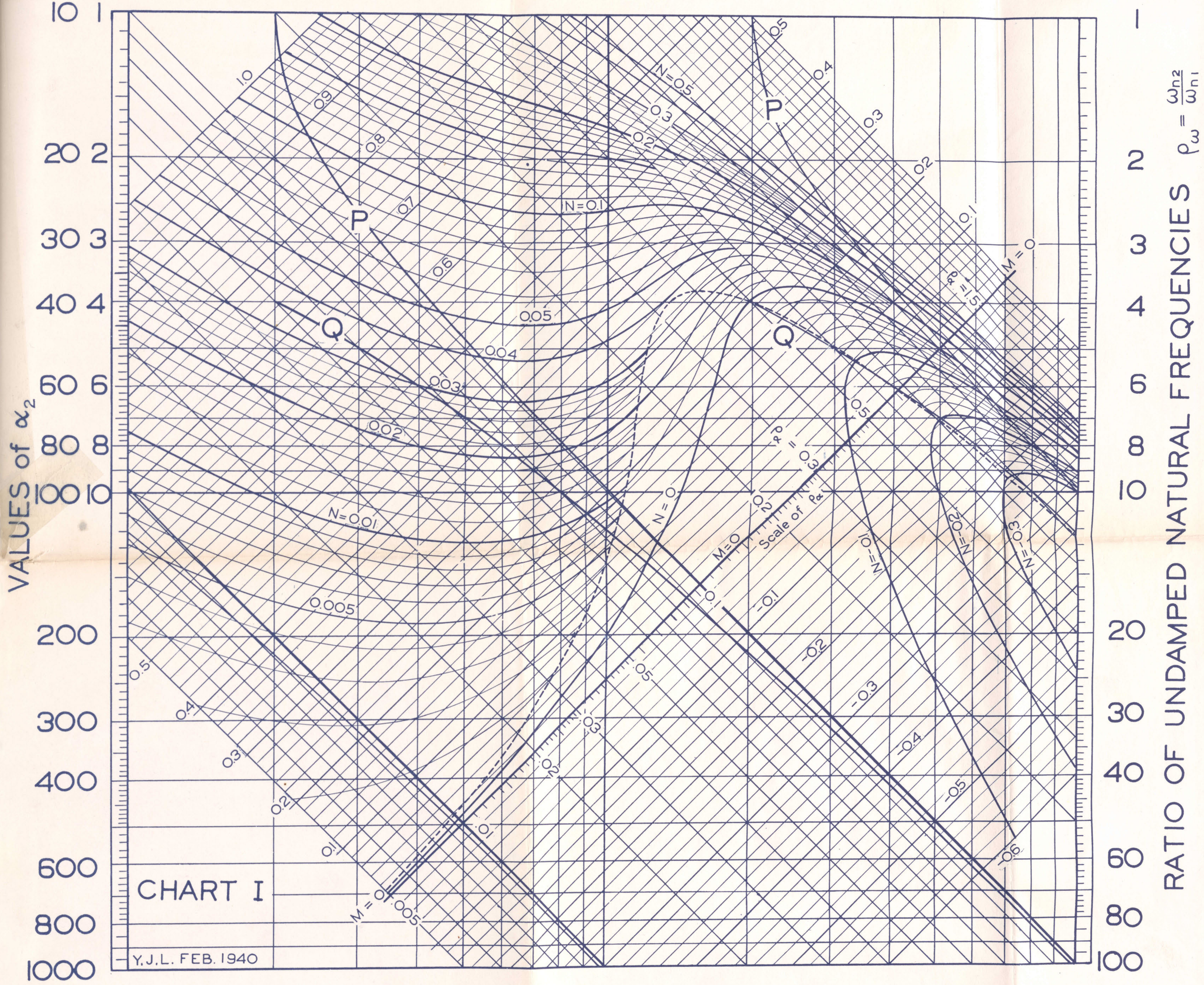
$$\rho_\sigma^3 - \rho_\sigma^2 + M\rho_\sigma - N = 0 \qquad (2.55)$$

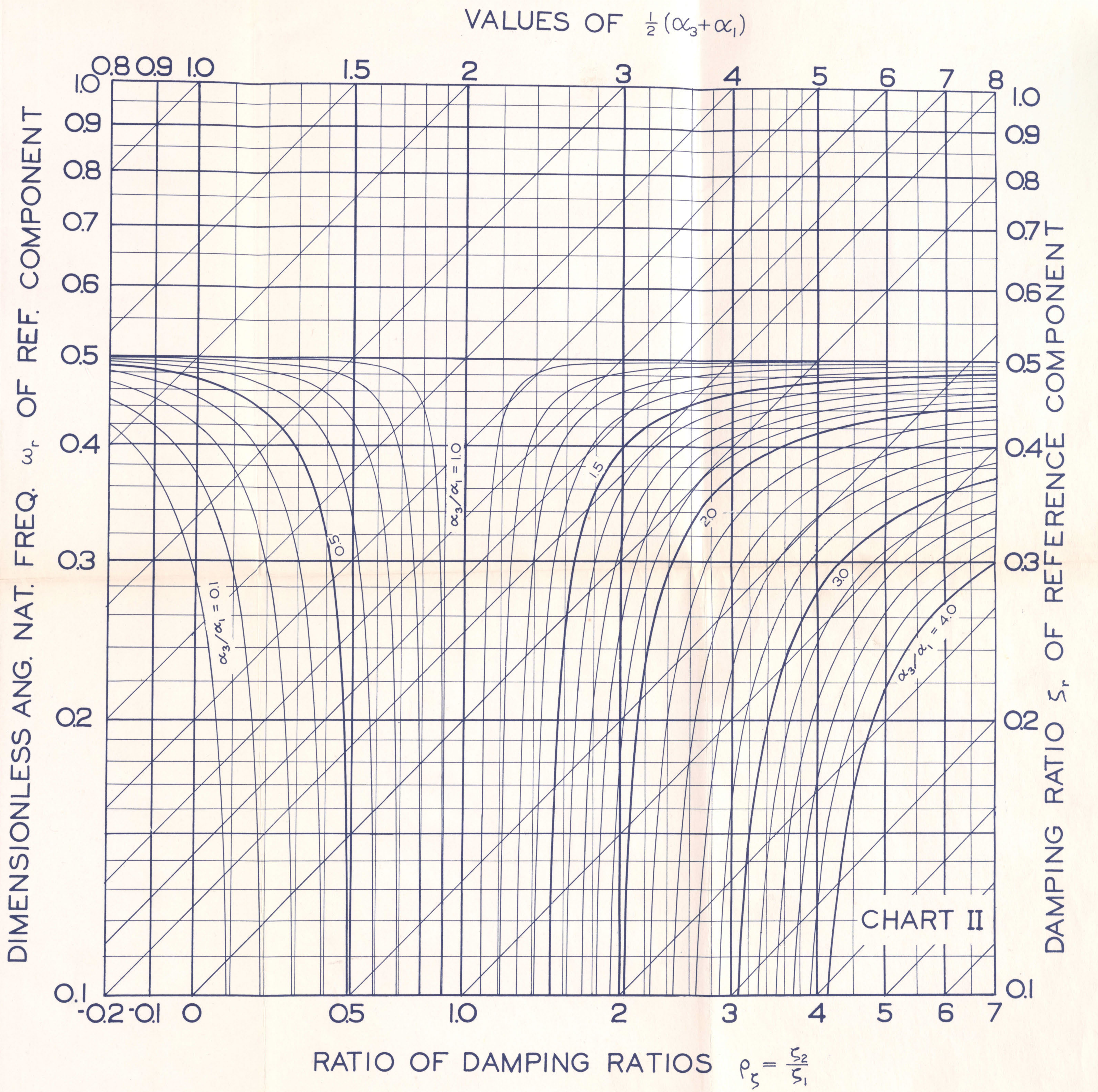
It looks like a simple matter to solve ρ_σ from the above equation, but solving a cubic equation analytically is usually tedious. Had the equation been transferred to Weiss'[35*] form, much effort could be saved. However, to preserve a simpler form of stability criteron and to simplify the further direct graphical solution, the writer decided to take the form as obtained above. Neverthe-less, Weiss' chart does not extend to regions of three unequal real roots, while at the present study of quartic equations, the region of four unequal real roots cannot be logically proven with-out the help of the region of three unequal real roots of its resolvent cubic equation.

* The writer has revised Weiss' cubic charts to a single chart for the cubic equation in Weiss' form. It is presented in Appendix A for the interest of readers.

THE QUARTIC CHART

FOR EQUATION $\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$ AND EQUATION $a_4\frac{d^4x}{dt^4} + a_3\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = 0$





Graphical solution of equation (2.55) is constructed as Chart I of the Quartic Chart. The curves are plotted for constant values of N with M as ordinate and ρ_α as abscissa. For the convenience of further application of the graphical result, the scale of M is linear, and that of ρ_α logarithmic, and they are both rotated 45 degrees from ordinary horizontal and vertical axes. It is noticed, when $N=0$, equation (2.55) becomes:

$$\rho_\alpha(\rho_\alpha^2 - \rho_\alpha + M) = 0 \qquad (2.72)$$

one of the root becomes zero, and the other roots are:

$$\rho_\alpha = \frac{1 \pm \sqrt{1-4M}}{2} \qquad (2.73)$$

When M is negative, one of the roots is negative, which is of no interest in our problem; therefore it is excluded. The root zero implies pure imaginary ρ_ω which is also of no interest. Therefore, only the root $\rho_\alpha = \frac{1+\sqrt{1-4M}}{2}$ is shown on the chart. (That is, the only intersection of constant $N(=0)$ and constant M appears on the chart.)

When M is greater than zero, ρ_α has two values until $M = \frac{1}{4}$ where the curve $N=0$ reaches its maximum point. Beyond $M=\frac{1}{4}$, there will be no intersection of $N=0$ with any M; that is, there is no real value for ρ_α.

For $N<0$, the curves have simpler shape with one maximum, because negative roots are excluded. When $\frac{1}{27} > N > 0$ the curves give one maximum at greater ρ_α and a minimum at smaller ρ_α. Again, when N exceeds $\frac{1}{27}$, neither maximum nor minimum can be seen for positive values of ρ_α.

It is very interesting to notice the relation between the locus of maxima and minima of the constant N curves of the Quartic Chart with the boundary line that separates the oscillatory and non-oscillatory regions on the stability criteria plot (Figure II).

Write equation (2.55) in the following form:

$$M = \frac{N - \rho_\alpha^3 + \rho_\alpha^2}{\rho_\alpha} \qquad (2.74)$$

Let $\frac{\partial M}{\partial \rho_\alpha} = 0$, $N = -2\rho_\alpha^3 + \rho_\alpha^2$ for M_{max} or M_{min} (2.75)a

$$M_{min} \text{ or } M_{max} = 3\rho_\alpha^2 + 2\rho_\alpha \qquad (2.75)$$

Assume the locus of M_{max} and M_{min} represents the repetition of quadratic factor; that is, at $\rho_\omega = 1$ and $\rho_\varphi = 1$. Then,

$$\rho_\alpha = \frac{\rho_\omega + \frac{1}{\rho_\omega}}{\alpha_2} = \frac{\rho_\omega + \frac{1}{\rho_\omega}}{\rho_\omega + \frac{1}{\rho_\omega} + 4\varphi_r^2} = \frac{1}{1 + 2\varphi_r^2} \qquad (2.76)$$

Substitute equation (2.76) into equations (2.75)a and (2.75) and simplify the expressions; we have:

$$M_{min} \text{ or } M_{max} = \frac{4\varphi_r^2 - 1}{(1 + 2\varphi_r^2)^2} \qquad (2.77)$$

$$N = \frac{2\varphi_r^2 - 1}{(1 + 2\varphi_r^2)^3} \text{ for } M_{min} \text{ or } M_{max} \qquad (2.78)$$

Equations (2.77) and (2.78) are exactly the same as equations (2.68) and (2.69) which are derived directly from the condition of repetition of quadratic factor. Therefore, the assumption just made is correct.

This locus M_{max} and M_{min} (or $\frac{\partial M}{\partial \rho_\alpha} = 0$) is plotted on Chart I as the dotted curve. The part of the dotted curve to the left of its vertex (or passing through $\mathcal{N}_{min}$) corresponds to the boundary line BA of Figure II. (Both of them start from $\mathcal{M},\mathcal{N}=0,0$ and end at $\mathcal{M},\mathcal{N}=\frac{1}{3},\frac{1}{27}$).

The part of the dotted curve to the right of its vertex (or passing through $\mathcal{M}_{max}$) corresponds to the boundary line ACD. (Both start from $\mathcal{M},\mathcal{N}=\frac{1}{3},\frac{1}{27}$ and end at $\mathcal{M},\mathcal{N}=-1,-1$).

The scale of ρ_α is marked along the line $\mathcal{M}=0$ whenever the intersection is fixed by a particular pair of $\mathcal{M}$ and $\mathcal{N}$; its projection onto the line $\mathcal{M}=0$ will give the value of ρ_α. (Precaution: Take the rightmost intersection if more than one intersection are observed for a particular pair of $\mathcal{M}$ and $\mathcal{N}$ to avoid the trouble in getting complex quantity in ratio of undamped natural frequencies.)

The Scale of α_2, The Concealed Scale of ρ_m* and Scale of ρ_ω, Ratio of Undamped Natural Frequencies

Lines (which are perpendicular to $\mathcal{M}=$ constant) of 135 degrees to horizontal lines represent constant ρ_α. The vertical scale on the left is provided for the middle constant α_2 in logarithmic scale. The intersection of the horizontal line of constant α_2 and the 145 degree line of constant ρ_α gives the product $\alpha_2\rho_\alpha$ or ρ_m when it is projected vertically down to the horizontal base of which the scale should be provided for ρ_m, but it is concealed. In other words, the vertical line represents constant ρ_m.

The Curve P

The curve P is the plotted result of $\rho_\omega+\frac{1}{\rho_\omega}=\rho_m$. When constant

* The detailed explanation of charting is omitted. If the reader is interested in it, please read "Graphical and Mechanical Computation", Chapter II, by J. Lipka, John Wiley and Sons, Inc., 1918

ℓ_m line (vertical) falls on P curve and deflects horizontally to the right until it reaches the scale on the right ordinate of Chart I, the reading thus obtained is ℓ_ω.

There are two curves of P, one appearing at the center part of the curve, another in the crowded zone. The former should be matched with the left scale of α_2 (range 10 - 1000); the latter with the right scale of α_2 (range 1 - 10). (The matching of scales is exactly the same for Q curves).

The Scale of ω_r, the Dimensionless Angular Natural Frequency Of Reference Component

The relation between ℓ_ω and ω_r ($\omega_r = \ell_\omega^{-1/2}$, equation (2.28)) offers a simple nomogramic solution. Prolong the horizontal line just obtained from the deflection on P curve until it meets the left ordinate scale on Chart II. This intersection gives the value of ω_r.

The Curve Q and Chart II

With much juggling for further practical convenience of charting, a new variable ℓ_q is introduced with the following relation:

$$\ell_q = (\ell_m + 2) \tag{2.79}$$

or

$$\ell_q = \left(\frac{1}{\sqrt{\ell_\omega}} + \sqrt{\ell_\omega}\right)^2 \tag{2.79a}$$

Equation (2.79) is plotted on Chart I as the Q curve with right ordinate scale of Chart I as Q's ordinate, and the horizontal concealed scale for ℓ_m.

As the left ordinate scale of Chart II bears a nomogramic relation ($\omega_r = \rho_\omega^{-1/2}$) to the right ordinate scale of Chart I, therefore when vertical ρ_m is deflected on Q horizontally toward the right and continued until it hits the left ordinate scale of Chart II, the reading on this scale will be $\rho_q^{-1/2}$;

$$\text{or} \qquad q = \rho_q^{-1/2} = \frac{1}{\frac{1}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}} \qquad (2.80)$$

We have solved ρ_ζ from equations (2.25) and (2.27) that:

$$\rho_\zeta = \frac{\rho_\omega\left(\frac{\alpha_3}{\alpha_1}\right) - 1}{\rho_\omega - \left(\frac{\alpha_3}{\alpha_1}\right)} \qquad (2.47)$$

$q, \rho_\zeta(\rho_\omega, \frac{\alpha_3}{\alpha_1})$ are plotted on Chart II with ρ_ζ as abscissa and q as ordinate (left). <u>So when certain deflected horizontal line from Q curve on Chart I cuts the constant $\frac{\alpha_3}{\alpha_1}$ curve, the intersection projected onto the bottom scale of Chart II</u> gives ρ_ζ, the ratio of damping ratios with respect to the reference frequency.

It is seen from Chart II that $\frac{\alpha_3}{\alpha_1}$ actually does the function of damping distribution as has been discussed in Paragraph 12, Chapter Four.

<u>The Effective Damping Parameter 1/2 ($\alpha_3 + \alpha_1$)</u>

The addition of equations (2.25) and (2.27) gives:

$$\alpha_3 + \alpha_1 = 2\zeta_r(1+\rho_\zeta)\left(\frac{1}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) = 2(\zeta_1 + \zeta_2)\left(\frac{1}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) \qquad (2.81)\text{a}$$

$$\text{or} \qquad \frac{\alpha_3 + \alpha_1}{2} = \zeta_r(1+\rho_\zeta)\left(\frac{1}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) = \zeta_r(1+\rho_\zeta)/q \qquad (2.81)$$

The definition of effective damping parameter is evident from equation (2.81)a, because it is proportional to the sum of damping ratios of the two components of the system.

The Scales of q, e_y, and γ_r, and The Constant $\frac{q}{1+e_y}$ Lines

The vertical scale of q $\left(= \frac{1}{\sqrt{e_w}+\frac{1}{\sqrt{e_w}}}\right)$ is drawn logarithmically upward on the left side of Chart II. It is exactly the same as that for ω_r but the denomination of the scale is omitted.

The scale of e_y is drawn on the bottom of Chart II with values of $1+e_y$ logarithmically rightward, but the scale is numbered according to values of e_y itself.

With the above arrangement of scales of q and e_y, each line 45 degrees inclined to horizontal one represents a constant value of $\frac{1+e_y}{q}$.

Transfer equation (2.81) into the following form:

$$\frac{\frac{1}{2}(\alpha_3+\alpha_1)}{\gamma_r} = \frac{1+e_y}{q} \qquad (2.82)$$

which offers a simple way to obtain γ_r from the chart by scaling 1/2 $(\alpha_3+\alpha_1)$ horizontally rightward and γ_r vertically upward, both logarithmically identical with those for $1+e_y$ and q. To avoid confusion, the scale of 1/2 $(\alpha_3+\alpha_1)$ is laid on top of Chart II and scale of γ_r along the right ordinate.

For any horizontal line deflected from Q curve on Chart I which meets a certain $\frac{\alpha_3}{\alpha_1}$ curve, the projection of the intersection onto the bottom of Chart II gives e_y. Start from the same intersection on the particular $\frac{\alpha_3}{\alpha_1}$; draw a 45 degree line until it hits a particular 1/2 $(\alpha_3+\alpha_1)$ vertical line. The intersection on such vertical line will give the value of γ_r on the rightmost ordinate scale.

17. Cyclic Shifting of Logarithmic Scales

In case no intersection on the particular 135 degree inclined line (for constant ρ_α) and the horizontal line of particular value of α_2 can be found within Chart I, we may shift the proper 135 degree inclined line one logarithmic cycle left (or right), but the scale of α_2 should never be changed. By this process the matched P and Q curves are automatically shifted one logarithmic cycle left (or right) with the shifted 135 degree inclined line. So the local P and Q curves are available.

Shifting of the 45 degree lines on Chart II one logarithmic cycle up (or down) is also permissible. However, the decimal points of the ordinates scale for φ_r must be shifted one figure left (or right). Moreover, the scales of 1/2 $(\alpha_3+\alpha_1)$ and of φ_r can be multiplied by a common factor, for instance, 10 simultaneously. In the latter case the constancy of the 45 degree lines are not affected.

18. Option in Reference Component

Sometimes the horizontal line from Q curve does not intersect the particular $\frac{\alpha_3}{\alpha_1}$ curve within the range of the chart (Chart II). In that case the ratio $\frac{\alpha_1}{\alpha_3}$ may be used instead of $\frac{\alpha_3}{\alpha_1}$ (or considering that all the numbers marked on $\frac{\alpha_3}{\alpha_1}$ curve are now for $\frac{\alpha_1}{\alpha_3}$). The rest of the procedure is exactly the same as before, but the data obtained is referred to high frequency as reference component; that is, φ_r is the damping ratio of high frequency component, and ρ_γ the ratio of damping ratio of low frequency component to that of the high frequency component.

Mathematical proof can be given for such transformation.

Let: $\gamma_r = \gamma_1 \quad , \quad \gamma_r' = \gamma_2$

$$\rho_y = \frac{\gamma_2}{\gamma_1} \quad , \quad \rho_y' = \frac{\gamma_1}{\gamma_2} = \frac{1}{\rho_y}$$

Starting from equation (2.47):

$$\rho_y = \frac{\rho_w\left(\frac{\alpha_3}{\alpha_1}\right) - 1}{\rho_w - \frac{\alpha_3}{\alpha_1}} \tag{2.47}$$

take the reciprocal expression of the above equation and multiply both numerator and denominator by $\frac{\alpha_1}{\alpha_3}$. We have:

$$\rho_y' = \frac{1}{\rho_y} = \frac{\rho_w\left(\frac{\alpha_1}{\alpha_3}\right) - 1}{\rho_w - \frac{\alpha_1}{\alpha_3}} \tag{2.83}$$

Equation (2.83) is of the same form as equation (2.47), so the transformation of $\frac{\alpha_3}{\alpha_1}$ to $\frac{\alpha_1}{\alpha_3}$ is to change ρ_y to ρ_y'.

Next, take equation (2.82) and substitute $\rho_y = \frac{1}{\rho_y'}$. There we have:

$$\frac{\frac{1}{2}(\alpha_3 + \alpha_1)}{\frac{\gamma_r}{\rho_y'}} = \frac{\rho_y' + 1}{q} \tag{2.84}$$

What is $\frac{\gamma_r}{\rho_y'}$? As $\gamma_r = \gamma_1 \quad , \quad \rho_y' = \frac{\gamma_1}{\gamma_2}$

Therefore,

$$\frac{\gamma_r}{\rho_y'} = \gamma_2 = \gamma_r'$$

or

$$\frac{\frac{1}{2}(\alpha_3 + \alpha_1)}{\gamma_r'} = \frac{\rho_y' + 1}{q} \tag{2.84a}$$

which is of the same form as equation (2.82); therefore, the graphical procedure is entirely the same if $\frac{\alpha_1}{\alpha_3}$ is used instead of $\frac{\alpha_3}{\alpha_1}$.

19. Factorization of Quartic Equation by Means of the Quartic Chart

With the understanding of the development of the Quartic Chart, one should be able to find* the four non-dimensional physical constants ρ_ω, ω_r, ρ_ζ and ζ_r. As soon as they are obtained, the factorized quartic equation can be written in the form of equation (2.19) or (2.20).

Returning to the dimensional quartic equation, with

$$D = A_0'^{1/4} \lambda \qquad \text{or} \qquad \lambda = \frac{D}{A_0'^{1/4}} \tag{2.85a}$$

$$\omega_{n1} = \omega_{r1} A_0'^{1/4} \quad \text{(undamped angular natural frequency of component 1)} \tag{2.85b}$$

and $$\omega_{n2} = \omega_{r2} A_0'^{1/4} \quad \text{(undamped angular natural frequency of component 2)} \tag{2.85c}$$

Substitution of equations (2.85)a, b and c into equation (2.14) will give:

$$D^4 + A_3' D^3 + A_2' D^2 + A_1' D + A_0' = (D^2 + 2\zeta_1 \omega_{n1} D + \omega_{n1}^2)(D^2 + 2\zeta_2 \omega_{n2} D + \omega_{n2}^2) \tag{2.86}$$

20. Graphical and Analytic Solutions

ρ_α is a principal datum to the four dimensionless physical quantities ρ_ω, ω_r, ρ_ζ and ζ_r. Chart I serves the most practical and convenient way to find the value of ρ_α which could be obtained analytically only after elaborate formulation and substitution. However, when ρ_α is obtained from Chart I, the following formulae can be used for the evaluation of ρ_ω, ω_r, ρ_ζ and ζ_r:

* Complete directions for the Quartic Chart are presented as Appendix B, which has been issued to the Class of Servo-mechanism at the Institute, and which proved to be practical. A limited number of mimeographed prints is available at the Institute Instrumentation Laboratory.

$$\frac{1}{\alpha_2}\left(\rho_\omega + \frac{1}{\rho_\omega}\right) = \rho_\alpha \qquad \text{or} \qquad \rho_\omega = \frac{1}{2}\left[\alpha_2 \rho_\alpha + \sqrt{(\alpha_2 \rho_\alpha)^2 - 4}\right] \tag{2.87}$$

$$\omega_r^2 \rho_\omega = 1 \qquad \text{or} \qquad \omega_r = \frac{1}{\sqrt{\rho_\omega}} \tag{2.88}$$

$$\frac{\alpha_3}{\alpha_1} = \frac{1 + \rho_\zeta \rho_\omega}{\rho_\zeta + \rho_\omega} \qquad \text{or} \qquad \rho_\zeta = \frac{\rho_\omega \left(\frac{\alpha_3}{\alpha_1}\right) - 1}{\rho_\omega - \frac{\alpha_3}{\alpha_1}} \tag{2.89}$$

$$\frac{\alpha_3 + \alpha_1}{2} = \zeta_r (1 + \rho_\zeta)\left(\sqrt{\rho_\omega} + \frac{1}{\sqrt{\rho_\omega}}\right) \qquad \text{or} \qquad \zeta_r = \frac{\frac{1}{2}(\alpha_3 + \alpha_1)}{(1 + \rho_\zeta)\left(\sqrt{\rho_\omega} + \frac{1}{\sqrt{\rho_\omega}}\right)} \tag{2.90}$$

$$\alpha_2 = \frac{1}{\rho_\omega} + \rho_\omega + 4\zeta_r^2 \rho_\zeta \qquad \text{or} \qquad \zeta_r = \frac{1}{2}\sqrt{\frac{\alpha_2 (1 - \rho_\alpha)}{\rho_\zeta}} \tag{2.90a}$$

$$\alpha_3 = 2\zeta_r \left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta \sqrt{\rho_\omega}\right) \qquad \text{or} \qquad \zeta_r = \frac{\alpha_3}{2\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta \sqrt{\rho_\omega}\right)} \tag{2.90b}$$

$$\alpha_1 = 2\zeta_r \left(\frac{\rho_\zeta}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) \qquad \text{or} \qquad \zeta_r = \frac{\alpha_1}{2\left(\frac{\rho_\zeta}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right)} \tag{2.90c}$$

CHAPTER SIX

DETAILED ANALYSIS OF STABILITY CRITERIA M AND N

21. The Rigorous Proof of The Nonoscillatory Region Bounded By BACB

As we have proven the correspondence of the boundary line ABC to the dotted curve on the quartic chart, the bottom line CB on Fig. 2 is N = 0, so it corresponds to the curve N = 0 on the quartic chart. Therefore, the shaded area BCAB on Fig. 2 may be considered as the area bounded by the curves $\frac{\partial M}{\partial \rho_\alpha} = 0$, and N = 0 on the quartic chart.

It is easily seen that any curve for $0 < N < \frac{1}{27}$ enters the area (bounded by $\frac{\partial M}{\partial \rho_\alpha} = 0$ and N = 0) at M = max, and leaves the same at M = min. A particular M (which must be greater than zero, but less than 1/3) which intersects one particular N curve inside this area gives two additional intersections outside the area -- one to its left and another to its right. That is, for the particular M and N which do intersect inside the shaded area BACB, three intersections are obtainable to give three distinct ρ_α. That means that this particular pair, M and N, will give three values of ρ_m because α_2 is constant for the particular problem. This in turn gives three values of ρ_ω.

By common sense, if a quartic equation can be factored into four real and distinct binomial factors as

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)=0 \qquad (2.91)$$

there are three ways to combine them into two distinct quadratics which are both critically damped from physical point of view, and accordingly three real distinct ρ_ω's can be observed.

It can also be observed that:

(a) The constant M line which is minimum to a particular N curve has another intersection with the same constant M to the right of the dotted curve. This corresponds to $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$ so that two different ways can be at our liberty to combine the four factors into two distinct quadratics. Thus we have two real distinct ρ_ω's. The intersection at the minimum point evidently will give $\rho_\omega = \rho_\xi = 1$ and $\zeta_r > 1$, while the right intersection will give $\rho_\xi = \zeta_r = 1$ and $\rho_\omega > 1$.*

Hold the particular N curve. The slightest increase of M immediately gives three intersections with this N curve. However, increasing M from its minimum value actually corresponds to the rightward increment of M from the boundary line BA of Fig. 2. This means entering the shaded area on Fig. 2 which corresponds to giving 4 distinct real roots for the quartic equation.

With the slightest decrease of M only one intersection with the same N is observed which means that two of the roots are a conjugate pair. The natural way to factor such a quartic into two simple quadratics (with real coefficients) is restricted to one. Physically it means to the left of the boundary line BA (Fig. 2) the system is oscillatory.

---

* This conclusion is only valid for $\frac{\alpha_3}{\alpha_1} = 1$. Further discussion for $\frac{\alpha_3}{\alpha_1} \neq 1$ appears in section 22.

(b) The constant M line which is maximum to a particular N curve has another intersection with the same N curve to the left of the dotted curve. Along $\frac{dM}{d\rho_a} = 0$ and M = max. the quartic equation has been shown to have two identical quadratic factors with ζ_r less than one (at the vertex of $\frac{dM}{d\rho_a} = 0$, or at the cusp of M vs. N curve, $\zeta_r = 1$). Mathematically,

For $\zeta_r < 1$ $\quad (\lambda^2+2\zeta_r\lambda+1)^2 = [\lambda+(\zeta_r+i\sqrt{1-\zeta_r^2})]^2[\lambda+(\zeta_r-i\sqrt{1-\zeta_r^2})]^2$ (2.92)

From the right side of Eq. (2.80) it is easily seen that

$$\rho_\omega^2 = \frac{(\zeta_r+i\sqrt{1-\zeta_r^2})^2}{(\zeta_r-i\sqrt{1-\zeta_r^2})^2} \quad \text{or} \quad \rho_\omega = \frac{\zeta_r+i\sqrt{1-\zeta_r^2}}{\zeta_r-i\sqrt{1-\zeta_r^2}} \tag{2.93}$$

Rationalizing equation (2.93) we get

$$\rho_\omega = 2\zeta_r^2-1+2i\zeta_r\sqrt{1-\zeta_r^2} \tag{2.94)a}$$

and

$$\rho_m = \rho_\omega + \frac{1}{\rho_\omega} = 2(2\zeta_r^2-1) \tag{2.94}$$

which is the same thing as $\quad \rho_m + 4 = 2(2\zeta_r^2+1)$ (2.70)

That makes the expressions of M and also of N mutually transferable either in terms of ρ_m or ζ_r.

From the above mathematical analysis we can understand that the left intersection of M and N (the same M is tangent to the same N at the right) is mathematically correct giving real ρ_a thence real ρ_m, but complex ρ_ω which is not interested physically.

It is clear that the right intersection gives larger ρ_a than the left one for the same pair of M and N. In other words, right $\rho_m >$ left ρ_m (if α_2 is kept constant, and it is so because we are dealing with one and the same quartic equation).

And because

$$\rho_m = \frac{1}{\rho_\omega} + \rho_\omega \qquad \text{or} \qquad \rho_\omega = \frac{\rho_m \pm \sqrt{\rho_m^2 - 4}}{2} \qquad (2.95)$$

Therefore, when $\rho_m = 2$, $\rho_\omega = 1$

$-2 < \rho_m < 2$, ρ_ω = complex

ρ_m for $\rho_\omega = 1$ is larger than ρ_m for ρ_ω = complex. There is no further confusion to take the tangential point instead of the left intersection whose physical significance is hard to interpret, although it is mathematically correct.

Hold the particular N curve. The slightest decrease of M immediately gives three intersections with this N curve. However, decreasing in M from its maximum value actually corresponds to leftward increment of M from the boundary line AC of Fig. 2. This means that the entering of the shaded area on Fig. 2, which corresponds to giving four distinct real roots for the quartic equation. With the slightest increase in M, only one intersection with the same N is observed which means that two of the roots are a conjugate pair. The natural way to factor such a quartic equation into two simple quadratics with real coefficients is restricted to one. Physically it means that to the right of the boundary line AC (Fig. 2) the system is oscillatory.

(c) The boundary curve N = 0 shows that when N is less than zero, there is no chance to have three intersections with any M. But slightly above the boundary of $N > 0$ (but less than $\frac{1}{27}$) three intersections are obtainable with suitable M's. This boundary curve N = 0 corresponds to the base line N = 0 of the shaded area BACB on Fig. 2. It is therefore safe to say that the shaded area of Fig. 2 bounded by $M = \frac{4\zeta_r^2 - 1}{(1-2\zeta_r^2)^2}$,

$N = \frac{2\zeta_r^2 - 1}{(1+2\zeta_r^2)^3}$ and N = 0 is the nonoscillatory region where four distinct real roots are present.

It is possible to show that for N = 0 the two components fall on either one of the following conditions:

1. $\rho_\omega > 1, \quad \zeta_r = .707 = \zeta_2$ (2.96)
2. $\rho_\omega = 1, \quad \zeta_2 = \sqrt{1-\zeta_r^2}$ (2.97)

22. The Most Lenient Behavior Affixing to The Boundary Lines That Separate The Oscillatory and Nonoscillatory Region

All the above conditions for the boundary lines between the oscillatory and nonoscillatory are apparently derived from the least condition that $\frac{\alpha_3}{\alpha_1} = 1$* which forms the following possible combinations:

(a) $\rho_\omega = 1 \quad \rho_\zeta \neq 1$ Boundary lines BA and CB
(b) $\rho_\zeta = 1 \quad \rho_\omega > 1$ Boundary lines BA and CB
(c) $\rho_\omega = 1 \quad \rho_\zeta = 1$ Boundary lines BA and AC

However, with simple and logical reasoning, if one component is being critically damped, but another is being overdamped, the factored form of the quartic equation can be arranged in two and only two combinations.

$$[(\lambda + \lambda_{12})(\lambda + \lambda_{12})][(\lambda + \lambda_3)(\lambda + \lambda_4)] = 0 \quad (2.98)$$

or

$$[(\lambda + \lambda_{12})(\lambda + \lambda_3)][(\lambda + \lambda_{12})(\lambda + \lambda_4)] = 0 \quad (2.98)a$$

In this case two and only two different real frequency ratios are obtainable. On the quartic chart two and only two ρ_α's should be present. Therefore, one of the two M - N inter-

*See Section 12, Chapter II of this thesis sections must be at $\frac{dM}{d\rho_\alpha} = 0$. With this example, it is evident that λ_3 may not be equal to λ_4, yet their stability criteria M and N fall on the boundary line between oscillatory and nonoscillatory regions as $\lambda_3 = \lambda_4$ does.

Apparently the lenient condition is $\zeta_r = 1$, $\rho_\omega > 1$ and $\rho_\xi \neq 1$ (ρ_ξ may be $\gtrless 1$, the condition $\rho_\xi = 1$ has been treated). From that condition

$$\alpha_3 = 2\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\xi\sqrt{\rho_\omega}\right) \tag{2.99}$$

$$\alpha_2 = \frac{1}{\rho_\omega} + \rho_\omega + 4\rho_\xi \tag{2.100}$$

$$\alpha_1 = 2\left(\frac{\rho_\xi}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega}\right) \tag{2.101}$$

The first evidence of the above condition is

$$\frac{\alpha_3}{\alpha_1} = \frac{1 + \rho_\xi \rho_\omega}{\rho_\xi + \rho_\omega} \neq 1 \tag{2.102}$$

With the above condition substitute Eqs. (2.99) to (2.101) into Eqs. (2.53) and (2.54) and simplify with $\rho_m = \rho_\omega + \frac{1}{\rho_\omega}$

$$M = \frac{4\,\rho_\xi\,(\rho_m + \rho_\xi)}{(\rho_m + 4\rho_\xi)^2} \tag{2.103}$$

and $$N = \frac{4\,\rho_\xi^2\,\rho_m}{(\rho_m + 4\rho_\xi)^3} \tag{2.104}$$

Eqs. (2.103) and (2.104) cannot be directly identified as anything along the boundary line, but if both denominator and numerator of Eq. (2.103) are divided by ρ_ξ^2 and those of Eq. (2.104) by ρ_ξ^3, and let $\rho_m' = \frac{\rho_m}{\rho_\xi}$ for both equations.

Then $$M = \frac{4(\rho_m' + 1)}{(\rho_m' + 4)^2} \tag{2.103a}$$

and $$N = \frac{4\,\rho_m'}{(\rho_m' + 4)^3} \tag{2.104a}$$

Eqs. (2.103)a and (2.104)a take the same form as Eqs. (2.62) and (2.63), only with ρ_m changed to ρ_m' . It is true that with one critically damped component (or two equal roots), the stability criteria M and N fall also on the boundary line that separates the oscillatory and nonoscillatory regions. However, the point is shifted from $M,N(\rho_m)$ to $M.N(\rho_m')$. Such shifting sometimes means shifting from branch BA to branch ACD. An example will show this statement clearly. Suppose that the original conditions are:

$$\rho_\omega = 5.0, \qquad \rho_\xi = 5.2$$

Then $\rho_m = 5.2$ or $M,N(\rho_m) = 0.293,\ 0.0266$ (On BA)

but $\rho_m' = \dfrac{5.2}{5.2} = 1.0$ or $M,N(\rho_m') = 0.32,\ 0.032$ (On AC)

For the bottom line, or N = 0, the lenient condition will reduce to M = 0; that is, one component is at unending oscillation. Apparently this is only one point of N = 0 (at M = 0) so it is not applicable to the whole range of N = 0. In other words, such lenient condition is not applicable to the boundary line N = 0.

23. The Lenient Behavior Affixing To The Cusp A of The Shaded Area

Logically speaking, at the cusp A (Fig. 2) or the corresponding vertex of $\frac{\partial M}{\partial \rho_\alpha} = 0$, only one way of factoring the quartic equation is possible to give real ρ_α . The condition of four equal real roots is too strict. However, only three equal real roots are sufficient to reach the cusp of the shaded area.

This can be proved mathematically by the following considerations. Let the quartic equation be factored as: :

$$\lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+1=(\lambda+\lambda_1)^2\left[(\lambda+\lambda_1)(\lambda+\frac{1}{\lambda_1^3})\right] \quad (2.105)$$

where $\lambda_1 \neq 1$ (2.105)a

$$\alpha_3 = 3\lambda_1 + \frac{1}{\lambda_1^3} \quad (2.105)b$$

$$\alpha_2 = 3(\lambda_1^2 + \frac{1}{\lambda_1^2}) \quad (2.105)c$$

and $\alpha_1 = \frac{3}{\lambda_1} + \lambda_1^3$ (2.105)d

from which $\frac{\alpha_3}{\alpha_1} = \frac{3\lambda_1^4+1}{(3+\lambda_1^4)\lambda_1^2} \neq 1$ (This is an essential indication)

$$M = \frac{\alpha_1\alpha_3-4}{\alpha_2^2} = \frac{1}{3}\ , \quad N = \frac{\alpha_3^2+\alpha_1^2-4\alpha_2}{\alpha_2^3} = \frac{1}{27}$$

Therefore the logical prediction is correct.

24. Double and Single Oscillatory Regions

(A) Double Oscillatory Regions

From the quartic chart (Chart I) it is seen that all curves of $N < 0$ give two intersections with a constant M line which is less than M_{max} for that N curve. By logical reasoning for such condition of M and N, two ways of factoring may yield two different ρ_m's. We know that when $N < 0$, the system is oscillatory. At least one of the two components is oscillatory; the other component may or may not be oscillatory. However, the two-intersection behavior will help us to clear up any such uncertainty.

Assume both components to be oscillatory --

$$\lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+1=(\lambda^2+2\zeta_1\omega_{nn1}\lambda+\omega_{nn1}^2)(\lambda^2+2\zeta_2\omega_{nn2}\lambda+\omega_{nn2}^2) \quad (2.106)$$

where $\omega_{nn1}\omega_{nn2} = 1$ (2.106)a

$\zeta_1 < 1, \quad \zeta_2 < 1 \quad \& \quad \zeta_1 \neq \zeta_2$ (2.106)b

The right hand side of Eq. (2.106) can also be written as

$$\left[\lambda + (\zeta_1 + i\sqrt{1-\zeta_1^2})\omega_{nn1}\right]\left[\lambda + (\zeta_1 - i\sqrt{1-\zeta_1^2})\omega_{nn1}\right]\left[\lambda + (\zeta_2 + i\sqrt{1-\zeta_2^2})\omega_{nn2}\right]\left[\lambda + (\zeta_2 - i\sqrt{1-\zeta_2^2})\omega_{nn2}\right] \quad (2.107)$$

From expression (2.107) we have three ways to express ρ_ω:

$$\rho_{\omega_1} = \frac{\omega_{nn2}}{\omega_{nn1}} \quad (2.107)a$$

$$\rho_{\omega_2} = \left[\frac{(\zeta_1 + i\sqrt{1-\zeta_1^2})(\zeta_2 + i\sqrt{1-\zeta_2^2})}{(\zeta_1 - i\sqrt{1-\zeta_1^2})(\zeta_2 - i\sqrt{1-\zeta_2^2})}\right]^{\frac{1}{2}} \quad (2.107)b$$

$$\rho_{\omega_3} = \left[\frac{(\zeta_1 + i\sqrt{1-\zeta_1^2})(\zeta_2 - i\sqrt{1-\zeta_2^2})}{(\zeta_1 - i\sqrt{1-\zeta_1^2})(\zeta_2 + i\sqrt{1-\zeta_2^2})}\right]^{\frac{1}{2}} \quad (2.107)c$$

From the above expressions of , expressions of can be deduced:

$$\rho_{m_1} = \frac{\omega_{nn2}}{\omega_{nn1}} + \frac{\omega_{nn1}}{\omega_{nn2}} \qquad \text{real}, \quad > 2 \quad (2.108)a$$

$$\rho_{m2} = 2\left[\zeta_1\zeta_2 - \sqrt{(1-\zeta_1^2)(1-\zeta_2^2)}\right] \qquad \text{real}, \quad < 2, \ \gtreqless 0 \quad (2.108)b$$

$$\rho_{m_3} = 2\left[\zeta_1\zeta_2 + \sqrt{(1-\zeta_1^2)(1-\zeta_2^2)}\right] \qquad \text{real}, \quad < 2, \ > 0 \quad (2.108)c$$

It is necessary to prove $\frac{\rho_{m_2}}{2}$ or $\frac{\rho_{m_3}}{2} < 1$

Let us assume

$$\zeta_1\zeta_2 \mp \sqrt{(1-\zeta_1^2)(1-\zeta_2^2)} < 1 \quad (2.109)$$

add $-\zeta_1\zeta_2$ to both sides of the inequality and square both sides

$$1 - \zeta_1^2 - \zeta_2^2 + \zeta_1^2\zeta_2^2 < 1 - 2\zeta_1\zeta_2 + \zeta_1^2\zeta_2^2$$

Canceling and transferring the terms we have

$$0 < (\zeta_1 - \zeta_2)^2 \qquad (2.109)\text{a}$$

Equation (2.109)a is a true fact except $\zeta_1 = \zeta_2$ which has been excluded.

ρ_{m_3} is always positive because ζ_1 and ζ_2 are both less than unity. ρ_{m_2} may be positive, negative, or zero. The condition for the transition of ρ_{m_2} from negative to positive can be found in the following treatment:

$$\zeta_1\zeta_2 - \sqrt{(1-\zeta_1^2)(1-\zeta_2^2)} > 0$$

$$(1-\zeta_1^2)(1-\zeta_2^2) > \zeta_1^2\zeta_2^2$$

$$1 - \zeta_1^2 - \zeta_2^2 > 0$$

$$\zeta_2 < \sqrt{1-\zeta_1^2} \qquad (2.110)$$

or

$$\zeta_1 < \sqrt{1-\zeta_2^2}$$

With the help of Chart I and the above analysis, it can be seen that

(a) if $\zeta_1 \neq \zeta_2$, $\zeta_1 < 1$, $\zeta_2 < 1$ and $\zeta_{2,1} > \sqrt{1-\zeta_{1,2}^2}$ only two intersections at positive ρ_α can be obtained from the stability criteria M and N of such a system. All the curves for N < 0 satisfy this condition, therefore the region below N = 0 is a double oscillatory region restricted to the condition $\zeta_{2,1} > \sqrt{1-\zeta_{1,2}^2}$

(b) If $\zeta_1 \neq \zeta_2$, $\zeta_1 < 1$, $\zeta_2 < 1$ and $\zeta_{2,1} < \sqrt{1-\zeta_{1,2}^2}$ three intersections at positive ρ_α should be observed from the stability criteria M and N of such a system. The shaded region of Fig. 2, which so far has been claimed to be nonoscillatory, satisfies this condition. Therefore this shaded region serves twofold to indicate (1) nonoscillatory, and (2) double oscillatory, restricted to the condition $\zeta_{2,1} < \sqrt{1-\zeta_{1,2}^2}$ The decision between the two states of motion merely depends upon the magnitude of the middle constant α_2 of the quartic equation.

From the above analysis it is understood that if two or three intersections are observed for certain pairs of M and N of a quartic equation, the rightmost intersection will always give results according to physical ways of decomponentization (that is, give real ρ_ω and real ζ_r etc.). The other intersection or intersections may or may not lead to complex ρ_ω, ζ_r etc. according to whether this system is doubly oscillatory or nonoscillatory.

(B) Single Oscillatory Region

Going back to Eq. (2.146) and letting one of the ζ's be greater than one, the other less than one, (for convenience let $\zeta_2 > 1$, $\zeta_1 < 1$), the four factors of equation (2.106) become:

$$\left[\lambda+(\zeta_1+i\sqrt{1-\zeta_1^2})\omega_{nn1}\right]\left[\lambda+(\zeta_1-i\sqrt{1-\zeta_1^2})\omega_{nn1}\right]\left[\lambda+(\zeta_2+\sqrt{\zeta_2^2-1})\omega_{nn2}\right]\left[\lambda+(\zeta_2-\sqrt{\zeta_2^2-1})\omega_{nn2}\right] \quad (2.111)$$

For expression (2.111) we also have three ways in which to express ρ_ω:

$$\rho_{\omega_1} = \frac{\omega_{nn2}}{\omega_{nn1}} \quad (2.111)a$$

$$\rho_{\omega_2} = \left[\frac{(\zeta_1+i\sqrt{1-\zeta_1^2})(\zeta_2+\sqrt{\zeta_2^2-1})}{(\zeta_1-i\sqrt{1-\zeta_1^2})(\zeta_2-\sqrt{\zeta_2^2-1})}\right]^{\frac{1}{2}} \quad (2.111)b$$

$$\rho_{\omega_3} = \left[\frac{(\zeta_1+i\sqrt{1-\zeta_1^2})(\zeta_2-\sqrt{\zeta_2^2-1})}{(\zeta_1-i\sqrt{1-\zeta_1^2})(\zeta_2+\sqrt{\zeta_2^2-1})}\right]^{\frac{1}{2}} \quad (2.111)c$$

From the above expressions for ρ_ω , expressions for ρ_m can be deduced:

$$\rho_{m_1} = \frac{\omega_{nn2}}{\omega_{nn1}} + \frac{\omega_{nn1}}{\omega_{nn2}} \qquad (2.112)a$$

$$\rho_{m_2} = 2\left[\zeta_1\zeta_2 - i\sqrt{(1-\zeta_1^2)(\zeta_2^2-1)}\right] \qquad (2.112)b$$

$$\rho_{m_3} = 2\left[\zeta_1\zeta_2 + i\sqrt{(1-\zeta_1^2)(\zeta_2^2-1)}\right] \qquad (2.112)c$$

The presence of i in the expression of ρ_m can be interpreted as meaning that no more real intersections can be observed for the M-N pair obtained from a single oscillatory system other than the one which possesses physical significance; that is, giving solution to real ρ_ω , real ζ_r , etc. Accordingly the region above N = 0 and outside the nonoscillatory (or double oscillatory) region satisfies the above condition. Therefore, the region above N = 0 and outside the nonoscillatory region is defined as the single oscillatory region.

25. Summary of Stability Analysis

The following table (Table III) may serve as a good summary of stability analysis for the quartic equation.

TABLE III

STABILITY BEHAVIOR OF THE QUARTIC EQUATION IN NONDIMENSIONAL FORM

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$$

$$\text{with } M = \frac{\alpha_3\alpha_1 - 4}{\alpha_2^2}, \quad N = \frac{\alpha_3^2 + \alpha_1^2 - 4\alpha_2}{\alpha_2^3}$$

$N > M$	Unstable				
$N = M$	Unending oscillation		$N > 0$		One component only
			$N < 0$		One or both components
$N=M=-1$	Unending oscillation of both components				
$N < M$	Outside region BABC	At least one component is oscillatory	$N>0$ $N=0$ $N<0$		One component oscillatory Nonphysical existence Both components oscillatory
Stable	Along boundary BABC $N = \frac{2\zeta^2-1}{(1+2\zeta^2)^3}$ $M = \frac{4\zeta^2-1}{(1+2\zeta^2)^2}$ ζ may be any real number	Two equal quadratic factors or one critically damped quadratic factor accompanying another of any damping ratio	Along AB $\zeta > 1$	$\alpha_3 = \alpha_1$	Two pair of equal roots Two equally overdamped quadratics
				$\alpha_3 \neq \alpha_1$	Two equal roots. Another component may be overdamped.
			Along AC	$\alpha_3 = \alpha_1$	Two equal conjugate-paired roots
			Also CD $\zeta < 1$	$\alpha_3 \neq \alpha_1$	Two equal roots. Another component overdamped.
			At Vertex A $\zeta = 1$ $M = \frac{1}{3}$ $N = \frac{1}{27}$	$\alpha_3 = \alpha_1$	Four equal roots
				$\alpha_3 \neq \alpha_1$	Three equal roots
			Along CB	$\zeta_1 = \zeta_2 = .707$ at $\rho_\omega > 1$ or $\rho_\omega = 1$ $\quad \zeta_2 = \sqrt{1-\zeta_1^2}$	
	Inside region ABC	Four distinct real roots or both components oscillatory limited by $\zeta_{2,1} < \sqrt{1-\zeta_{1,2}^2}$			
$\alpha_3 > \alpha_1$		High frequency component has greater damping ratio.			
$\alpha_3 < \alpha_1$		High frequency component has lesser damping ratio			
$\alpha_3 = \alpha_1$		Two components of same frequency, but with different damping Two components of different frequency, but with same damping or two components of same frequency and same damping			

P A R T III

STABILITY IMPROVEMENT WITH DIFFERENT CONTROLS

CHAPTER SEVEN

STABILITY TRANSITION CURVE WITH DIFFERENT COUPLING COEFFICIENTS

26. Coupling Factor and Coupling Coefficient

In reviewing Section 8, Chapter Three, the coupling factors of a control system are recollected. The coefficients of the quartic equation of the controlled motion can be broken into two parts; one is due to the idly dynamic combination of the control and controlled member, and the other due to the coupling effect.

The idly dynamic combination may be defined as one in which the coupling factors of the system are all zero (physically the control is locked) and the coefficients of the quartic behave in such a way that the quartic equation may be resolved into two factors, one identified as the identical characteristic of the member to be controlled and the other the identical characteristic of the control.

In symbols, Eq. (2.13)c -- or (2.13)d -- can be written as follows:

$$
\begin{aligned}
A'_3 &= A'_{30} \\
A'_2 &= A'_{20} + B'_2 \\
A'_1 &= A'_{10} + B'_1 \\
A'_0 &= A'_{00} + B'_0
\end{aligned}
\qquad (3.01)
$$

where $A'_{30} = \frac{C_1}{m_c} + \frac{c_e}{I} = 2\zeta_c\omega_{nc} + 2\zeta_o\omega_{no}$

$$A'_{20} = \frac{k_c}{m_c} + \frac{k_e}{I} + \frac{c_c\ c_e}{m_c\ I} = \omega_{nc}^2 + \omega_{no}^2 + 4\zeta_o\zeta_c\omega_{no}\omega_{nc}$$

$$A'_{10} = \frac{c_c\ k_e}{m_c\ I} + \frac{c_e\ k_c}{I\ m_c} = 2\zeta_c\omega_{nc}\omega_{no}^2 + 2\zeta_o\omega_{no}\omega_{nc}^2$$

and $A'_{00} = \frac{k_e\ k_c}{I\ \ m} = \omega_{no}^2\omega_{nc}^2$ (3.01)a

are defined as idle coefficients.

Eq. (3.01) can be written in the following form:

$$A'_3 = A'_{30}$$

$$A'_2 = A'_{20}(1 + \gamma_2)$$

$$A'_1 = A'_{10}(1 + \gamma_1)$$

$$A'_0 = A'_{00}(1 + \gamma_0) \qquad (3.02)$$

where $\gamma_2 = \frac{B'_2}{A'_{20}}$, defined as second derivative coupling coefficient

$\gamma_1 = \frac{B'_1}{A'_{10}}$, defined as first derivative coupling coefficient

$\gamma_0 = \frac{B'_0}{A'_{00}}$, defined as error sensitive coupling coefficient

The stability function S(D) in the form of Eq. (2.14) can be expressed in the following way:

$$S(D) = D^4 + A'_{30}D^3 + A'_{20}(1 + \gamma_2)D^2 + A'_{10}(1 + \gamma_1)D + A'_{00}(1 + \gamma_0) \qquad (3.03)$$

By introducing

$$D = A'^{1/4}_{00}\lambda \qquad (3.04)$$

We may write the stability function in nondimensional form:

$$\psi_0(\lambda) = \frac{S(D)}{A'_{00}} = \lambda^4 + \alpha_{30}\lambda^3 + \alpha_{20}(1+\gamma_2)\lambda^2 + \alpha_{10}(1+\gamma_1)\lambda + \alpha_{00}(1+\gamma_0) \qquad (3.05)$$

where $\alpha_{30} = \frac{A'_{30}}{A'^{1/4}_{00}} = 2\zeta_o\left(\frac{1}{\sqrt{\rho_{\omega o}}} + \rho_{\zeta o}\sqrt{\rho_{\omega o}}\right)$ (3.06)

$$\alpha_{20} = \frac{A'_{20}}{A'^{1/2}_{00}} = \rho_{\omega o} + \frac{1}{\rho_{\omega o}} + 4\zeta_o^2\rho_{\zeta o} \qquad (3.07)$$

$$\alpha_{10} = \frac{A'_{10}}{A'^{\,3/4}_{00}} = 2\zeta_0\left(\frac{\rho_{\zeta_0}}{\sqrt{\rho_{\omega_0}}} + \sqrt{\rho_{\omega_0}}\right) \qquad (3.08)$$

$$\alpha_{00} = \frac{A'_{00}}{A'_{00}} = 1 \qquad (3.09)$$

with $\rho_{\omega_0} = \frac{\omega_{nc}}{\omega_{no}} = \frac{\text{Undamped natural frequency of control}}{\text{Undamped natural frequency of member to be controlled}}$ (3.10)

$\rho_{\zeta_0} = \frac{\zeta_c}{\zeta_o} = \frac{\text{Damping ratio of the control}}{\text{Damping ratio of the member to be controlled}}$ (3.11)

and α_{30}, α_{20}, α_{10}, and α_{00} are defined as nondimensional idle coefficients.

27. Effect of Coupling Coefficients on Stability Transition Curve

It is understood that a mechanical system of one degree of freedom cannot be unstable. When such member is controlled by a mechanical non-ideal control at idle condition, the resultant quartic equation must show the stable behavior. Graphically, the point α_{20} vs. $\frac{\alpha_{30}}{\alpha_{10}}$ must lie above the stability transition curve.

$$\alpha_{20} = \frac{\alpha_{30}}{\alpha_{10}} + \frac{\alpha_{10}}{\alpha_{30}} \qquad (3.12)$$

However, in a two-degree-of-freedom system such as the uncontrolled longitudinal stability of an airplane, it may be stable or unstable; that is α_{20} vs. $\frac{\alpha_{30}}{\alpha_{10}}$ of such a system may lie above or below the transition curve

When the control is put into action, the same point α_{20} vs. $\frac{\alpha_{30}}{\alpha_{10}}$ will change its relative position with the new transition curve which is affected by the coupling coefficient. The effect of different coupling coefficients can be analyzed separately and their resultant effect can be easily interpreted.

(A) Effect of Error Sensitive Coupling Coefficient

With only error sensitive coupling coefficient, Eq. (3.05) appears in the following form:

$$\lambda^4 + \alpha_{30}\lambda^3 + \alpha_{20}\lambda^2 + \alpha_{10}\lambda + (1+\gamma_0) = 0 \qquad (3.13)$$

which can be transformed into:

$$\lambda'^4 + \alpha_3\lambda'^3 + \alpha_2\lambda'^2 + \alpha_1\lambda' + 1 = 0 \qquad (3.14)$$

where

$$\alpha_3 = \frac{\alpha_{30}}{(1+\gamma_0)^{1/4}}, \qquad \alpha_2 = \frac{\alpha_{20}}{(1+\gamma_0)^{1/2}},$$

$$\alpha_1 = \frac{\alpha_{10}}{(1+\gamma_0)^{3/4}} \quad \text{and} \quad \lambda = (1+\gamma_0)^{1/4}\lambda' \qquad (3.14)a$$

The new stability transition curve evidently is

or

$$\alpha_2 = \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}, \quad \text{or} \quad \alpha_{20} = \frac{\alpha_{30}}{\alpha_{10}}(1+\gamma_0) + \frac{\alpha_{10}}{\alpha_{30}} \qquad (3.15)$$

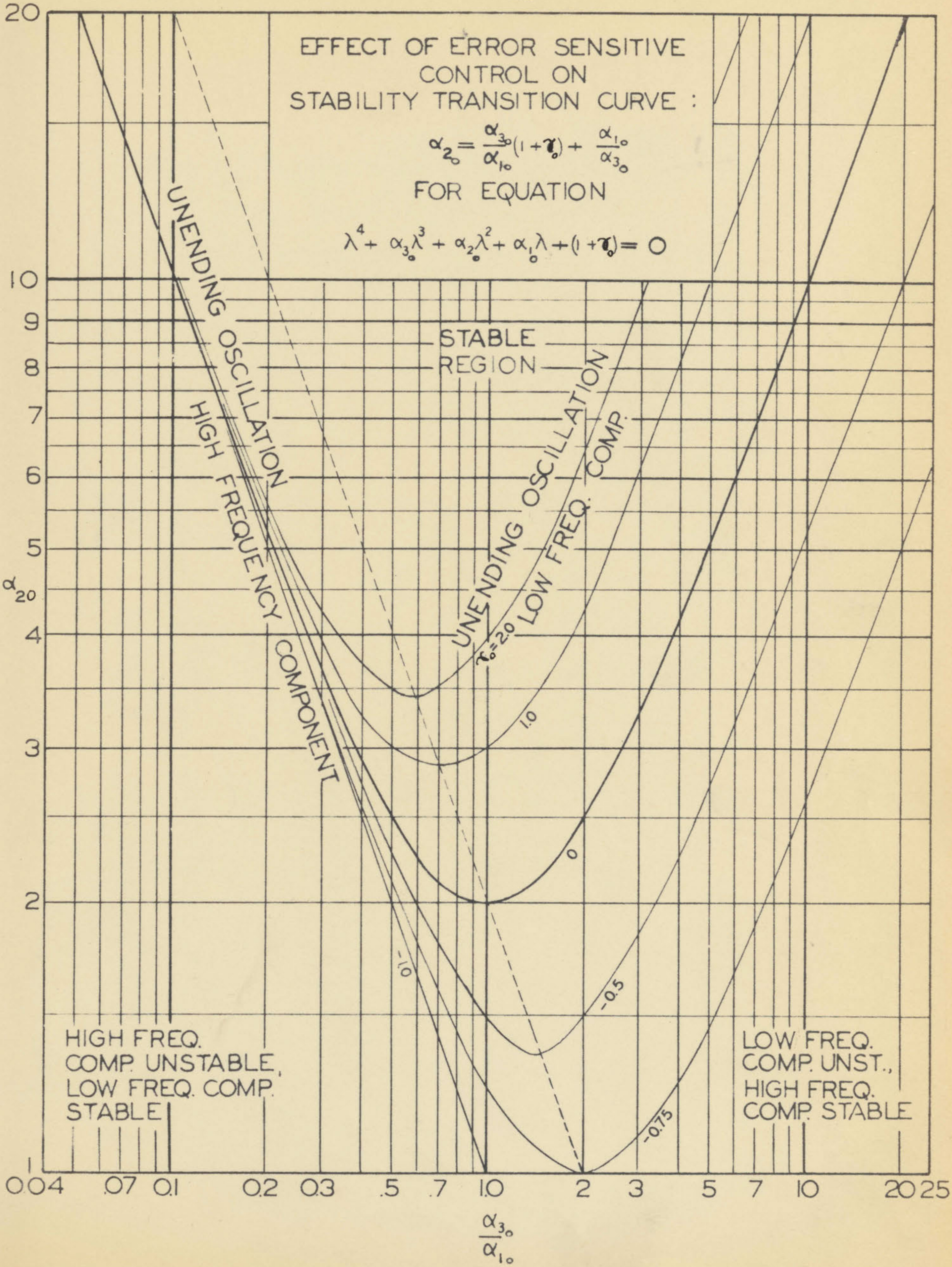
or in symmetric form

$$\frac{\alpha_{20}}{(1+\gamma_0)^{1/2}} = \frac{\alpha_{30}(1+\gamma_0)^{1/2}}{\alpha_{10}} + \frac{\alpha_{10}}{\alpha_{30}(1+\gamma_0)^{1/2}} \qquad (3.15)a$$

Eq. (3.15) is plotted as Fig. 3 with α_{20} as ordinate against $\frac{\alpha_{30}}{\alpha_{10}}$ as abscissa with γ_0 as varying parameter.

In the high performance control (which will be discussed later in Chapter Eight) the control frequency is usually higher than that of the controlled member, and the damping ratio of control is usually higher than that of the controlled member. Symbolically the high performance controlled system possesses

FIG. III


EFFECT OF ERROR SENSITIVE CONTROL ON STABILITY TRANSITION CURVE :
$\alpha_{2_0} = \frac{\alpha_{3_0}}{\alpha_{1_0}}(1+\gamma_0) + \frac{\alpha_{1_0}}{\alpha_{3_0}}$
FOR EQUATION
$\lambda^4 + \alpha_{3_0}\lambda^3 + \alpha_{2_0}\lambda^2 + \alpha_{1_0}\lambda + (1+\gamma_0) = 0$
STABLE REGION
UNENDING OSCILLATION HIGH FREQUENCY COMPONENT
UNENDING OSCILLATION LOW FREQ. COMP.
$\gamma_0 = 2.0$
1.0
0
-0.5
-0.75
-1.0
HIGH FREQ. COMP. UNSTABLE, LOW FREQ. COMP. STABLE
LOW FREQ. COMP. UNST., HIGH FREQ. COMP. STABLE
α_{20}
20
10
9
8
7
6
5
4
3
2
1
0.04
.07
0.1
0.2
0.3
.5
.7
1.0
2
3
5
7
10
20
25
$\frac{\alpha_{3_0}}{\alpha_{1_0}}$

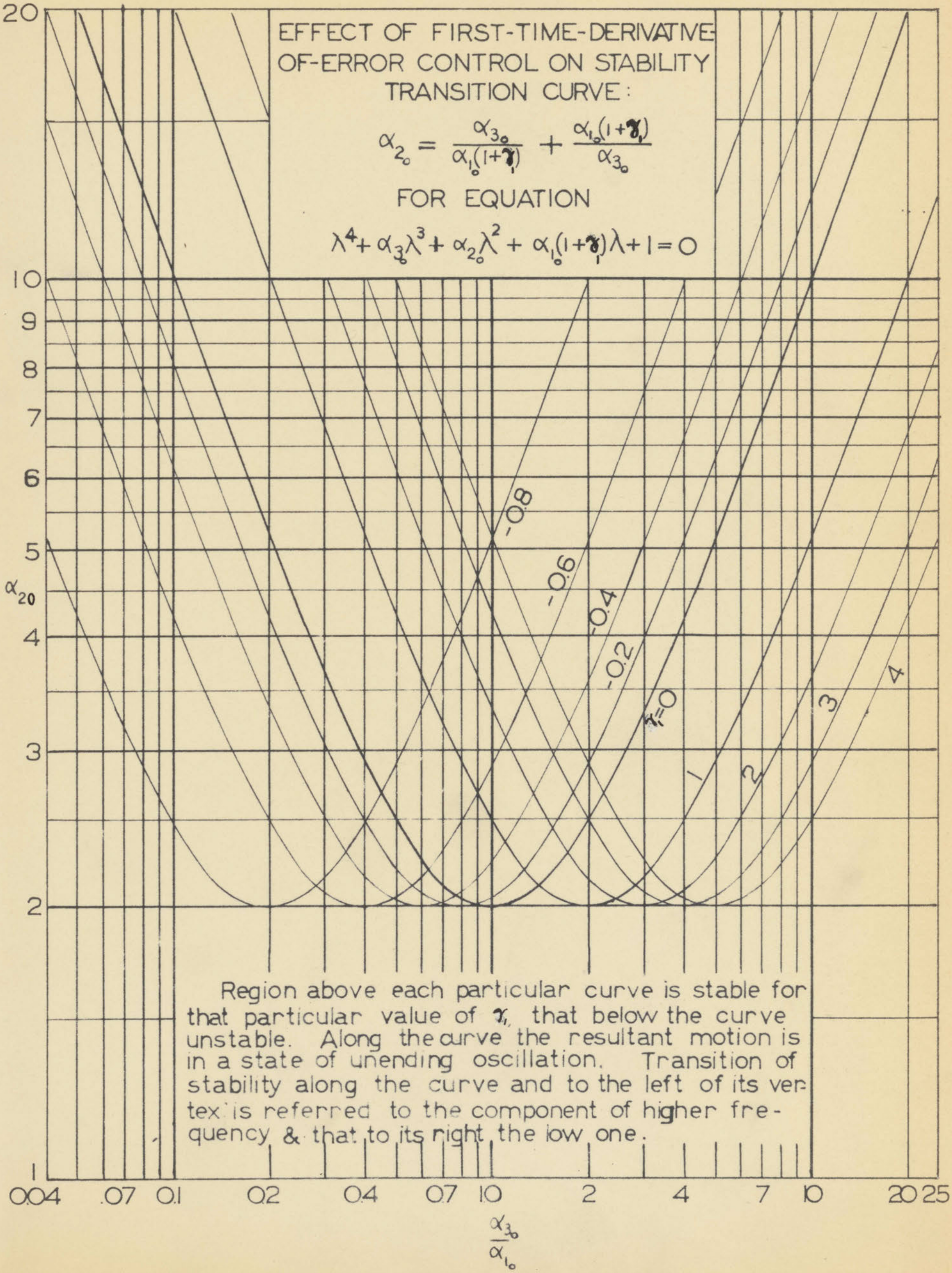
the two conditions $\rho_{\omega 0} >> 1$ and $\rho_{\zeta 0} > 1$. With control at idle condition, the point α_{20} vs. $\frac{\alpha_{30}}{\alpha_{10}}$ is slightly above the transition curve of $\gamma_0 = 0$ at some values of $\frac{\alpha_{30}}{\alpha_{10}} > 1$ (because $\rho_{\zeta 0} > 1$). It is therefore seen that with a positive coupling coefficient, the system will soon reach unending oscillation when the transition curve, raised by the coupling coefficient, passes through the point α_{20} vs. $\frac{\alpha_{30}}{\alpha_{10}}$ This sets a limit to $+\gamma_0$ beyond which the system will be unstable. However, if negative coupling coefficient is used, there is no such limit, and the system is always stable until $\gamma_0 = -1$ when the quartic equation is reduced to cubic one with one root equal to zero. At $\gamma_0 = -1$ the transition curve becomes a straight line* ($\alpha_{20} = \frac{\alpha_{10}}{\alpha_{30}}$) in a log-log plot and there the meaning of high frequency or low frequency becomes obscure.

It is also interesting to notice the shifting of the vertices of the stability transition curves along a straight line $\left[\alpha_{20} = 2\frac{\alpha_{10}}{\alpha_{30}}\right]$ in the log-log plot. It is therefore possible to make the high frequency component possess the smaller damping ratio by introducing negative γ_0 (to make the specified point α_{20} vs. $\frac{\alpha_{30}}{\alpha_{10}}$ appear above the left branch of the transition curve). For a good follow-up control, the least requirement is to have steady state reading equal to quantity which is to be followed. Going back to Eq. (2.13)b the above requirement cannot be fulfilled unless $B'_0 = A'_0$.

---

*Compare this with the stability criteria chart of the cubic equation in Appendix A.

FIG. IV


EFFECT OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL ON STABILITY TRANSITION CURVE:
$$\alpha_{2_0} = \frac{\alpha_{3_0}}{\alpha_{1_0}(1+\gamma_1)} + \frac{\alpha_{1_0}(1+\gamma_1)}{\alpha_{3_0}}$$
FOR EQUATION
$$\lambda^4 + \alpha_{3_0}\lambda^3 + \alpha_{2_0}\lambda^2 + \alpha_{1_0}(1+\gamma_1)\lambda + 1 = 0$$
Region above each particular curve is stable for that particular value of γ_1, that below the curve unstable. Along the curve the resultant motion is in a state of unending oscillation. Transition of stability along the curve and to the left of its vertex is referred to the component of higher frequency & that to its right the low one.
-0.8
-0.6
-0.4
-0.2
$\gamma_1=0$
1
2
3
4
α_{20}
20
10
9
8
7
6
5
4
3
2
1
0.04
.07
0.1
0.2
0.4
0.7
1.0
2
4
7
10
20
25
$\frac{\alpha_{3_0}}{\alpha_{1_0}}$

But from Eq. (2.13)c it is seen that $A_o' = \frac{k_e k_c}{Im} + B_o'$, so A_o' cannot be equal to B_o' unless k_e (or k_c) is zero. Approximate truth can be reached if k_e is very small, so that $\frac{k_e k_c}{Im}$ is negligible to B_o'. In such type control it is necessary to use large positive γ_o to satisfy the requirement of true steady state reading. However, as positive γ_o easily leads the system into an unstable condition, (such as is shown in Fig. 3), means of improving damping is therefore of equal importance after the introduction of overcontrolled positive γ_o.

If the system is not aimed at the following-up characteristic like the controlled longitudinal motion of the airplane, only moderate positive γ_o may be needed.

(B) Effect of First Derivative Coupling Coefficient

With only error velocity coupling, Eq. (3.05) appears in the following form:

$$\lambda^4 + \alpha_{30}\lambda^3 + \alpha_{20}\lambda^2 + \alpha_{10}(1+\gamma_1)\lambda + 1 = 0 \qquad (3.16)$$

of which the stability transition curve becomes

$$\alpha_{20} = \frac{\alpha_{30}}{\alpha_{10}(1+\gamma_1)} + \frac{\alpha_{10}(1+\gamma_1)}{\alpha_{30}} \qquad (3.17)$$

Eq. (3.17) is plotted as Fig. 4 with α_{20} as ordinate against $\frac{\alpha_{30}}{\alpha_{10}}$ as abscissa with γ_1 as varying parameter. Positive γ_1 shifts the curve to the right of the original one while negative γ_1 moves it to the left.

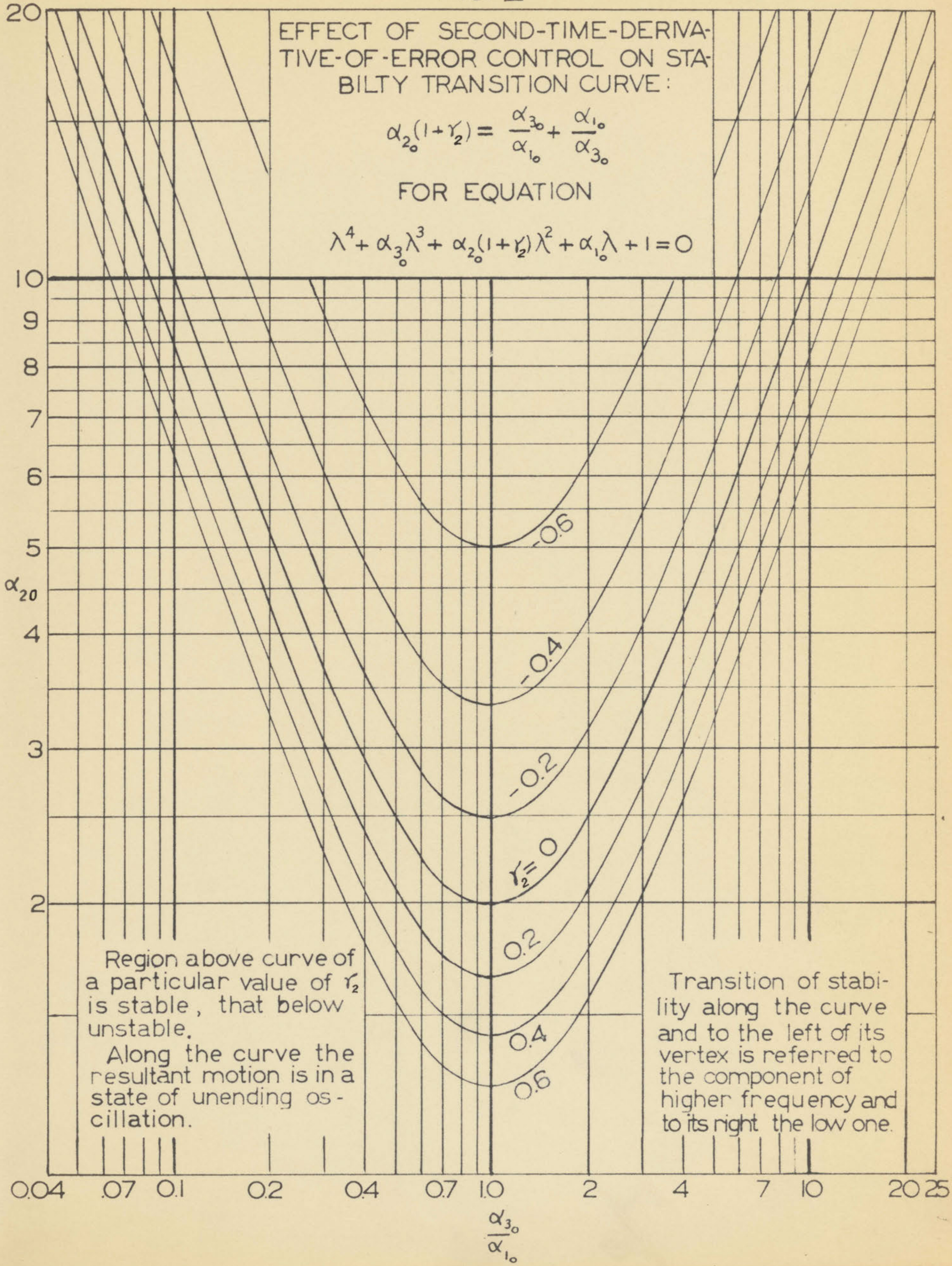
With particular control and controlled member the point α_{20} vs. $\frac{\alpha_{30}}{\alpha_{10}}$ may be located slightly above the right branch of $\gamma_1 = 0$. Too big a positive γ_1 will lead the system to instability of which the high frequency component will first undergo unending oscillation. A slight negative γ_1 will also lead the system to instability of which the low frequency component will first pass through a stable oscillation to an unstable one.

Introducing positive γ_1 decreases the ratio $\frac{\alpha_{30}}{\alpha_{10}(1+\gamma_1)}$ which adjusts the distribution of damping ratio between the fast and the slow components. In fact, the damping ratio of the controlled member (low frequency component) is noticeably improved with a slight positive γ_1 while the damping ratio of the control (high frequency component) is only slightly decreased. It is possible to adjust the coupling coefficient so that $\frac{\alpha_{30}}{\alpha_{10}(1+\gamma_1)} = 1$. In most cases of high performance control, such a condition gives two components of motion with the same frequency, but with different damping ratio. When γ_1 exceeds such a limit, the high frequency component will have less damping ratio than the low frequency component. Physically, the change in damping ratio of either component should be a continuous variation. However, Maxwell[36] finds that there is an abrupt change of damping distribution when coupling coefficient (defined in a different way with the symbols used in this thesis) reaches a certain value. The abrupt change is only due to the exchange of title of the high and the low frequency components. Detailed application on the condition $\frac{\alpha_{30}}{\alpha_{10}(1+\gamma_1)} = 1$ will be developed into another chapter.*

---

*Chapter Ten, Tuning Controls.

FIG. V


EFFECT OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL ON STABILTY TRANSITION CURVE:
$\alpha_{2_0}(1+\gamma_2) = \frac{\alpha_{3_0}}{\alpha_{1_0}} + \frac{\alpha_{1_0}}{\alpha_{3_0}}$
FOR EQUATION
$\lambda^4 + \alpha_{3_0}\lambda^3 + \alpha_{2_0}(1+\gamma_2)\lambda^2 + \alpha_{1_0}\lambda + 1 = 0$
20
10
9
8
7
6
5
4
3
2
α_{20}
-0.6
-0.4
-0.2
$\gamma_2 = 0$
0.2
0.4
0.6
Region above curve of a particular value of γ_2 is stable, that below unstable.
Along the curve the resultant motion is in a state of unending oscillation.
Transition of stability along the curve and to the left of its vertex is referred to the component of higher frequency and to its right the low one.
0.04
.07
0.1
0.2
0.4
0.7
1.0
2
4
7
10
20
25
$\frac{\alpha_{3_0}}{\alpha_{1_0}}$

(C) Effect of Second Derivative Coupling

With only error-acceleration coupling, Eq. (3.05) appears in the following form:

$$\lambda^4 + \alpha_{30}\lambda^3 + \alpha_{20}(1+\gamma_2)\lambda^2 + \alpha_{10}\lambda + 1 = 0 \qquad (3.18)$$

of which the stability transition curve becomes

$$\alpha_{20}(1+\gamma_2) = \frac{\alpha_{30}}{\alpha_{10}} + \frac{\alpha_{10}}{\alpha_{30}} \qquad (3.19)$$

Eq. (3.19) is plotted as Fig. 5 with α_{20} as ordinate against $\frac{\alpha_{30}}{\alpha_{10}}$ as abscissa with γ_2 as varying parameter. The transition curve is shifted upward with negative coupling coefficient; that means that if overcontrolled with negative coupling, the system might be led into instability. However, no such instability would occur if positive coupling is used.

(D) Effect of Higher Derivative Coupling

As detective instruments are limited to error sensitive, velocity sensitive, and acceleration sensitive types, higher derivative instruments are not yet available. Therefore, no complication is needed to explore their effect upon the transition curve. Moreover, the combination of γ_0, γ_1 and γ_2 is widely open to yield desirable results. Therefore the analysis of transition of stability is confined to the three types of coupling.

CHAPTER EIGHT

CONTROLS WITH HIGH NATURAL FREQUENCY

28. High Performance Controls and Definition of Advantages

It is desirable to have least control lag in a follow-up control. Controls with high natural frequency can achieve this object with ideal control. A simple figure of merit[37] can be derived.

With nonideal control, it is impossible to use a single expression to sum up all the relative merits. However, to simplify the effort of design the following definitions and notations are introduced in a tabular form.

(See Table IV next page)

TABLE IV

Results with Control idle	Results with control in action	Ratio and Notation	Definition
α_{30} α_{20} α_{10}	α_3 α_2 α_1		
ω_0 *	$\omega_1 = (\omega_r)$	$\omega_1/\omega_0 = \eta_\omega$	Advantage of undamped natural frequency
ω_c	ω_2	$\omega_2/\omega_c = \eta'_\omega$	Advantage of undamped natural frequency (control component)
$\rho_{\omega_0} = \frac{\omega_c}{\omega_0}$	$\rho_\omega = \frac{\omega_2}{\omega_1}$	$\frac{\rho_\omega}{\rho_{\omega_0}} = \eta_{\rho\omega}$	Advantage of ratio of undamped natural frequencies
ζ_0	$\zeta_1 = (\zeta_r)$	$\frac{\zeta_1}{\zeta_0} = \eta_\zeta$	Advantage of damping ratio
ζ_c	ζ_2	$\frac{\zeta_2}{\zeta_c} = \eta'_\zeta$	Advantage of damping ratio (control component)
$\rho_{\zeta_0} = \frac{\zeta_c}{\zeta_0}$	$\rho_\zeta = \frac{\zeta_2}{\zeta_1}$	$\frac{\rho_\zeta}{\rho_{\zeta_0}} = \eta_{\rho\zeta}$	Advantage of ratio of damping ratios
$\omega_0 \zeta_0$	$\omega_1 \zeta_1$	$\frac{\omega_1 \zeta_1}{\omega_0 \zeta_0} = \eta_{\omega\zeta}$	Advantage of damping
$\omega_c \zeta_c$	$\omega_2 \zeta_2$	$\frac{\omega_2 \zeta_2}{\omega_c \zeta_c} = \eta'_{\omega\zeta}$	Advantage of damping (control component)

---

* All ω's in this table and henceforth in Chapters 8, 9, 10 and 10 are referred to undamped natural angular frequency.

It should be noted here that although every ratio referred to control at idle condition is defined as advantage, it is merely a mathematical symbol because a control cannot be expected to influence every quantity in the advantageous sense. Some of the "advantages" may actually work on the disadvantageous side. The designer has to use his own judgment to make a satisfactory compromise.

29. Error Sensitive Control with High Natural Frequency

Going back to equation (3.14) and (3.14)a and with substitution of Eqs. (2.25) to (2.27) and (3.06) to (3.08) we have

$$2\zeta_1\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega}\right) = \frac{2\zeta_0\left(\frac{1}{\sqrt{\rho_{\omega 0}}} + \rho_{\zeta 0}\sqrt{\rho_{\omega 0}}\right)}{(1 + r_0)^{1/4}} \tag{3.20}$$

$$\rho_\omega + \frac{1}{\rho_\omega} + 4\zeta_1^2\rho_\zeta = \frac{\rho_{\omega 0} + \frac{1}{\rho_{\omega 0}} + 4\zeta_0^2\rho_{\zeta 0}}{(1 + r_0)^{1/2}} \tag{3.21}$$

$$2\zeta_1\left(\sqrt{\rho_\omega} + \frac{\rho_\zeta}{\sqrt{\rho_\omega}}\right) = \frac{2\zeta_0\left(\sqrt{\rho_{\omega 0}} + \frac{\rho_{\zeta 0}}{\sqrt{\rho_{\omega 0}}}\right)}{(1 + r_0)^{3/4}} \tag{3.22}$$

$$\rho_\omega^2\,\omega_1^4 = \rho_{\omega 0}^2\,\omega_0^4\,(1 + r_0) \tag{3.23}$$

If $\rho_\omega >> 1$, approximation can be safely made to obtain those advantages as defined in Table IV in terms of control specification and coupling coefficient explicitly.

From Eq. (3.21) it is seen that $\frac{1}{\rho_\omega}$ and $\frac{1}{\rho_{\omega 0}}$ are definitely neglegible when $\rho_\omega >> 1$. Since $\zeta_0^2\rho_{\zeta 0} = \zeta_0\zeta_c$ and in general ζ_0 is small, (may be in the order of .1), ζ_c is in the order of 1.0, so $4\,\zeta_0^2\rho_{\zeta 0}$ is also negligible to $\rho_{\omega 0}$. It will be seen soon that the product $\zeta_1^2\rho_\zeta$ or $\zeta_1\zeta_2$ is of the same magnitude as $\zeta_o\zeta_c$. So $4\zeta_1^2\rho_\zeta$ is also negligible to ρ_ω . Therefore Eq. (3.21) is simplified to

$$\eta_{\rho\omega} = \frac{\rho_\omega}{\rho_{\omega o}} = \frac{1}{(1+r_o)^{1/2}} \tag{3.24}$$

From Eq. (3.23) expression of η_ω can be obtained as

$$\eta_\omega = \frac{\omega_1}{\omega_o} = \left(\frac{\rho_{\omega o}}{\rho_\omega}\right)^{1/2}(1+r_o)^{1/4} = (1+r_o)^{1/2} \tag{3.25}$$

In Eq. (3.20) it is safe to neglect $\frac{1}{\sqrt{\rho_\omega}}$ against $\rho_\zeta\sqrt{\rho_\omega}$ and $\frac{1}{\sqrt{\rho_{\omega o}}}$ against $\rho_{\zeta o}\sqrt{\rho_{\omega o}}$ because $\rho_\omega, \rho_{\omega o} \gg 1$ and $\rho_{\zeta_1}, \rho_{\zeta_o}$ also $\gg 1$ in general. Therefore we get

$$\frac{\zeta_1}{\zeta_o}(1+r_o)^{1/4} = \frac{\rho_{\zeta o}\sqrt{\rho_{\omega o}}}{\rho_\zeta\sqrt{\rho_\omega}} \tag{3.26)a$$

Substituting Eq. (3.24) into Eq. (3.26)a we get

$$\frac{\zeta_1}{\zeta_o} = \frac{\rho_{\zeta o}}{\rho_\zeta} \tag{3.26)b$$

or

$$\eta'_\zeta = \frac{\zeta_2}{\zeta_c} = \frac{\zeta_1\rho_\zeta}{\zeta_o\rho_{\zeta o}} = 1 \tag{3.26}$$

which means the damping ratio of the control component is not (essentially) changed.

Equation (3.26)b can also be written in the following form

$$\rho_\zeta = \frac{\rho_{\zeta o}}{\eta_\zeta} \tag{3.26)c$$

Substitute the value of ρ_ζ into Eq. (3.22) and leave every term in because $\frac{\rho_{\zeta o}}{\sqrt{\rho_{\omega o}}}$ is not negligible to $\sqrt{\rho_{\omega o}}$ and $\frac{\rho_\zeta}{\sqrt{\rho_\omega}}$ is not negligible to $\sqrt{\rho_\omega}$. We have

$$\eta_\zeta = \frac{\sqrt{\rho_{\omega o}} + \frac{\rho_{\zeta o}}{\sqrt{\rho_{\omega o}}}}{(1+r_o)^{3/4}\left(\sqrt{\rho_\omega} + \frac{\rho_{\zeta o}}{\eta_\zeta\sqrt{\rho_\omega}}\right)} \tag{3.27)a$$

Multiply both the numerator and denominator by $\frac{1}{\sqrt{\rho_{\omega o}}}$ and substitute $\rho_{\omega o} = \rho_{\omega o}\frac{\rho_{\omega}}{\rho_{\omega o}}$, Eq. (3.27)a appears in the following form:

$$\eta_{\zeta} = \frac{1 + \frac{\rho_{\zeta o}}{\rho_{\omega o}}}{(1+r_o)^{1/2}\left[1 + \frac{\rho_{\zeta o}(1+r_o)^{1/2}}{\rho_{\omega o}\ \eta_{\zeta}}\right]} \qquad (3.27)b$$

From Eq. (3.27)b η_{ζ} can be solved thus

$$\eta_{\zeta} = \frac{1}{(1+r_o)^{1/2}}\left[1 - \frac{r_o}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right] \qquad (3.27)$$

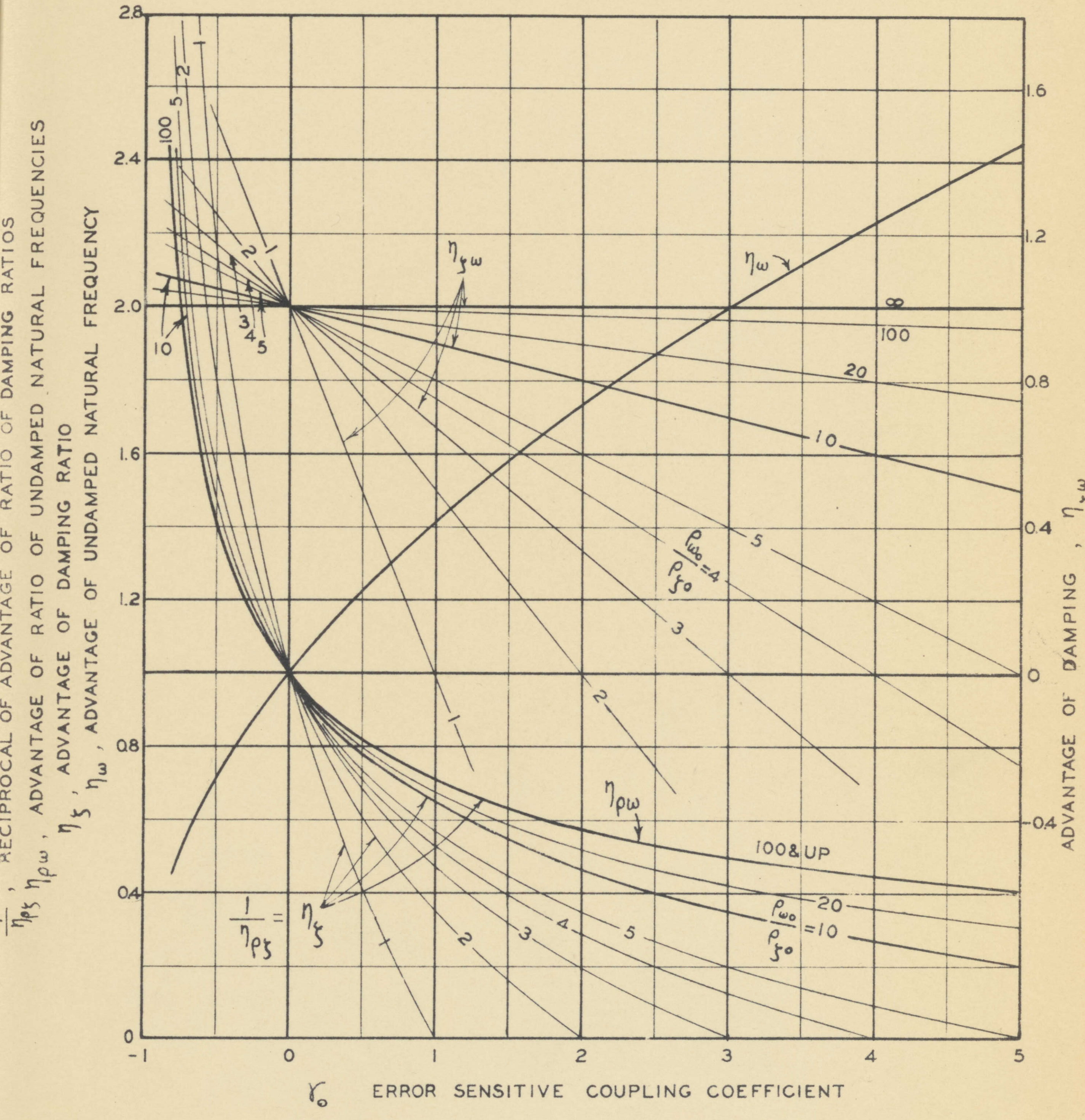
Eq. (3.27) is rather an interesting result in which the advantage of damping ratio is a function of coupling coefficient r_o and the control specification $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$ which relates the ratio of natural frequencies with the ratio of damping ratios between the controlled member and the control in a form of simple ratio. Multiply Eq. (3.27) by Eq. (3.25). $\eta_{\zeta\omega}$ is obtained.

$$\eta_{\zeta\omega} = 1 - \frac{r_o}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}} \qquad (3.28)$$

From the expressions for η_{ζ} and $\eta_{\zeta\omega}$ it is seen that when $r_o = \frac{\rho_{\omega o}}{\rho_{\zeta o}}$ (positive). Both η_{ζ} and $\eta_{\zeta\omega}$ become zero which means that at that value of r_o the system becomes unendingly oscillatory and the unendingly oscillatory component is the low frequency component. But when r_o is negative, but less than 1.0, the system can never be unstable. When r_o slightly exceeds -1.0, η_{ζ} becomes negative while $\eta_{\zeta\omega}$ is still positive.. This contradiction should be considered as a result of negligence of certain terms during the course of derivation unjustifiable at the region where r_o exceeds -1.0. Fortunately too great a coupling coefficient is not used in practice.

By definition, the following advantages can be found:

FIG. VI CHARACTERISTICS OF ERROR SENSITIVE CONTROL AT LARGE $\rho_\omega S$



$$\eta'_\omega = \frac{\omega_2}{\omega_c} = \frac{\rho_\omega \omega_1}{\rho_{\omega_0}\omega_0} = \frac{1}{(1+\gamma_0)^{1/2}}(1+\gamma_0)^{1/2} = 1 \tag{3.29}$$

$$\eta'_{\zeta\omega} = \eta'_\zeta \eta'_\omega = 1 \tag{3.30}$$

$$\eta_{\rho\zeta} = \frac{\zeta_2}{\zeta_1} \Big/ \frac{\zeta_c}{\zeta_0} = \frac{\eta'_\zeta}{\eta_\zeta} = \frac{1}{\eta_\zeta} = \frac{(1+\gamma_0)^{1/2}}{1 - \frac{\gamma_0}{\frac{\rho_{\omega_0}}{\rho_{\zeta_0}}}} \tag{3.40}$$

Eqs. (3.24), (3.25), (3.27), (3.40) and (3.28) are plotted as Fig. 6 with η_ζ, $\eta_{\zeta\omega}$, η_ω, $\eta_{\rho\omega}$, $\frac{1}{\eta_{\rho\zeta}}$, as ordinates against the coupling coefficient γ_0 as abscissa with the control specification $\frac{\rho_{\omega_0}}{\rho_{\zeta_0}}$ as varying parameter.

All other quantities -- η'_ζ, η'_ω, $\eta'_{\zeta\omega}$ -- which are approximately equal to unity are not plotted.

With Fig. 6 the stability improvement of a controlled system by a particular control with particular coupling coefficient can be picked up without any effort. When the member to be controlled possesses excess damping ratio, a control with high frequency of the error sensitive type will be satisfactory with positive coupling coefficient so long as the damping is concerned.

In case the member to be controlled possesses sufficient stiffness, but not sufficient damping, the error sensitive control will be satisfactory with slightly negative coupling coefficient.

However, the error sensitive control is primarily designed to supply the azimuthal or following-up characteristic (positively coupled). The real advantage is to increase the natural frequency of the controlled member. However, the damping is inherently spoiled. Therefore, damping improving coupling is of great necessity to compensate the spoiled damping.

30. Error-Acceleration Control with High Natural Frequency

Because of simplicity in analysis of this type of control, it is taken up ahead of the error-velocity type.

In reviewing Eqs. (3.18), (3.06) to (3.08) and (2.25) to (2.27), the following relations are obtained for the error-acceleration control.

$$\alpha_3 = \alpha_{30}$$
$$\alpha_2 = \alpha_{20}(1+\gamma_2)$$
$$\alpha_1 = \alpha_{10}$$

or

$$2\zeta_1\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega}\right) = 2\zeta_0\left(\frac{1}{\sqrt{\rho_{\omega 0}}} + \rho_{\zeta 0}\sqrt{\rho_{\omega 0}}\right) \qquad (3.41)$$

$$\rho_\omega + \frac{1}{\rho_\omega} + 4\zeta_1^2\rho_\zeta = \left(\rho_{\omega 0} + \frac{1}{\rho_{\omega 0}} + 4\zeta_0^2\rho_{\zeta 0}\right)(1+\gamma_2) \qquad (3.42)$$

$$2\zeta_1\left(\sqrt{\rho_\omega} + \frac{\rho_\zeta}{\sqrt{\rho_\omega}}\right) = 2\zeta_0\left(\sqrt{\rho_{\omega 0}} + \frac{\rho_{\zeta 0}}{\sqrt{\rho_{\omega 0}}}\right) \qquad (3.43)$$

Eq. (3.42) can be simplified, if $\rho_\omega >> 1$, with neglect of $\frac{1}{\rho_\omega}$, $4\zeta_1^2\rho_\zeta$, $\frac{1}{\rho_{\omega 0}}$ and $4\zeta_0^2\rho_{\zeta 0}$ so that

$$\eta_{\rho\omega} = \frac{\rho_\omega}{\rho_{\omega 0}} = 1 + \gamma_2 \qquad (3.44)$$

$$\therefore \quad \frac{\omega_1^4\rho_\omega^2}{\omega_0^4\rho_{\omega 0}^2} = 1$$

$$\therefore \quad \eta_\omega = \frac{\omega_1}{\omega_0} = \frac{1}{(1+\gamma_2)^{1/2}} \qquad (3.45)$$

By neglecting $\frac{1}{\sqrt{\rho_\omega}}$ against $\rho_\zeta\sqrt{\rho_\omega}$ and $\frac{1}{\sqrt{\rho_{\omega 0}}}$ against $\rho_{\zeta_0}\sqrt{\rho_{\omega 0}}$ Eq. (3.41) can be put into the following form:

$$\eta_\zeta = \frac{\zeta_1}{\zeta_0} = \eta_\omega \frac{\rho_{\zeta 0}}{\rho_\zeta} \tag{3.47a}$$

or
$$\rho_\zeta = \frac{\eta_\omega \rho_{\zeta 0}}{\eta_\zeta} \tag{3.47}$$

Nothing can be allowed to be neglected in equation (3.43). It can be written as

$$\eta_\zeta = \frac{1 + \frac{\rho_{\zeta 0}}{\rho_{\omega 0}}}{\sqrt{\frac{\rho_\omega}{\rho_{\omega 0}}}\left(1 + \frac{\rho_\zeta}{\rho_\omega}\right)} \tag{3.48}$$

With the substitution of $\rho_\omega = \rho_{\omega 0}\frac{\rho_\omega}{\rho_{\omega 0}} = \frac{\rho_{\omega 0}}{\eta_\omega^2}$ and Eqs. (3.47) and (3.44), Eq. (3.48) gives the following solution

$$\eta_\zeta = \eta_\omega \left[1 + (1-\eta_\omega^2)\frac{\rho_{\zeta 0}}{\rho_{\omega 0}}\right] \tag{3.49a}$$

or
$$\eta_\zeta = \frac{1}{(1+\gamma_2)^{1/2}}\left[1 + \frac{\gamma_2}{1+\gamma_2}\cdot\frac{1}{\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}}\right] \tag{3.49}$$

Hence
$$\eta_{\zeta\omega} = \eta_\zeta \eta_\omega = \frac{1}{1+\gamma_2}\left[1 + \frac{\gamma_2}{1+\gamma_2}\cdot\frac{1}{\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}}\right] \tag{3.50}$$

$$\eta_{\rho\zeta} = \frac{\eta_\omega}{\eta_\zeta} = \frac{1}{1 + \frac{\gamma_2}{1+\gamma_2}\cdot\frac{1}{\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}}} \tag{3.51}$$

$$\eta'_\omega = \frac{\omega_2}{\omega_c} = \frac{\rho_\omega \omega_1}{\rho_{\omega 0}\omega_0} = \eta_{\rho\omega}\eta_\omega = (1+\gamma_2)^{1/2} \tag{3.52}$$

$$\eta'_\zeta = \frac{\zeta_2}{\zeta_c} = \frac{\zeta_1 \rho_\zeta}{\zeta_0 \rho_{\zeta 0}} = \eta_\zeta \eta_{\rho\zeta} = \frac{1}{(1+\gamma_2)^{1/2}} \tag{3.53}$$

$$\eta'_{\zeta\omega} = \eta'_\zeta \eta'_\omega = \frac{1}{(1+\gamma_2)^{1/2}}(1+\gamma_2)^{1/2} = 1 \tag{3.54}$$

Eqs. (3.49) and (3.50) are separately plotted as Fig. 7A and Fig. 7B with η_ζ and $\eta_{\zeta\omega}$ as ordinates against the coupling coefficient as abscissa with $\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}$ as varying parameter. With

FIG. VII A

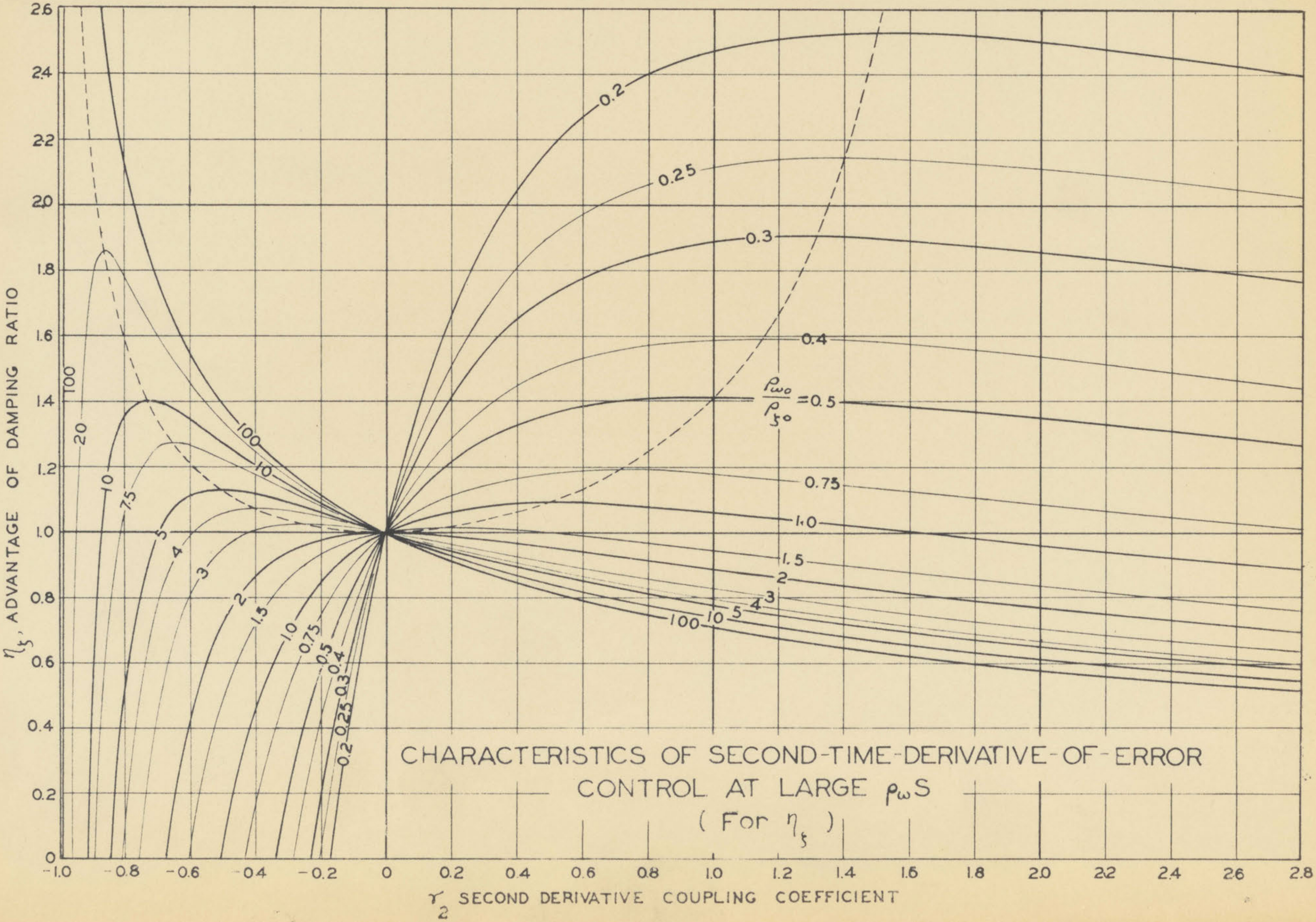

CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR
CONTROL AT LARGE $\rho_\omega S$
(For η_ζ)
η_ζ, ADVANTAGE OF DAMPING RATIO
r_2 SECOND DERIVATIVE COUPLING COEFFICIENT
$\frac{\rho_{\omega o}}{\rho_{\zeta o}} = 0.5$
0.2
0.25
0.3
0.4
0.75
1.0
1.5
2
3
4
5
7.5
10
20
100
0
0.2
0.4
0.6
0.8
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
-1.0
-0.8
-0.6
-0.4
-0.2

FIG. VII B

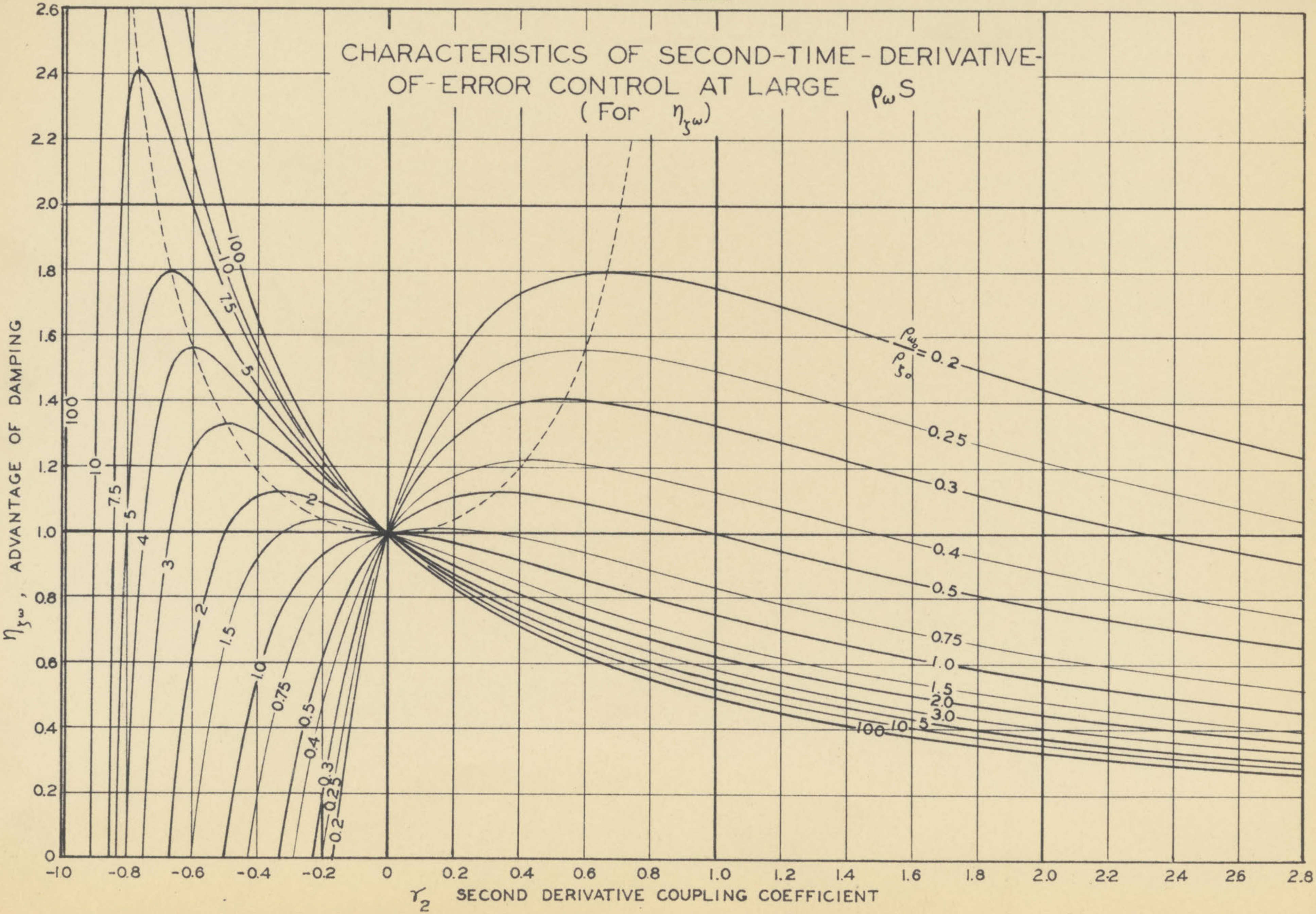

CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega S$
(For $\eta_{\zeta\omega}$)
$\eta_{\zeta\omega}$, ADVANTAGE OF DAMPING
r_2 SECOND DERIVATIVE COUPLING COEFFICIENT
$\rho_{\omega_0}/\rho_{\zeta_0} = 0.2$
0.25
0.3
0.4
0.5
0.75
1.0
1.5
2.0
3.0
5
10
100
7.5
4
2
0.2
0.25
0.3
0.4
0.5
0.75
1.0
1.5
2
3
4
5
7.5
10
100
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8

FIG. VII C

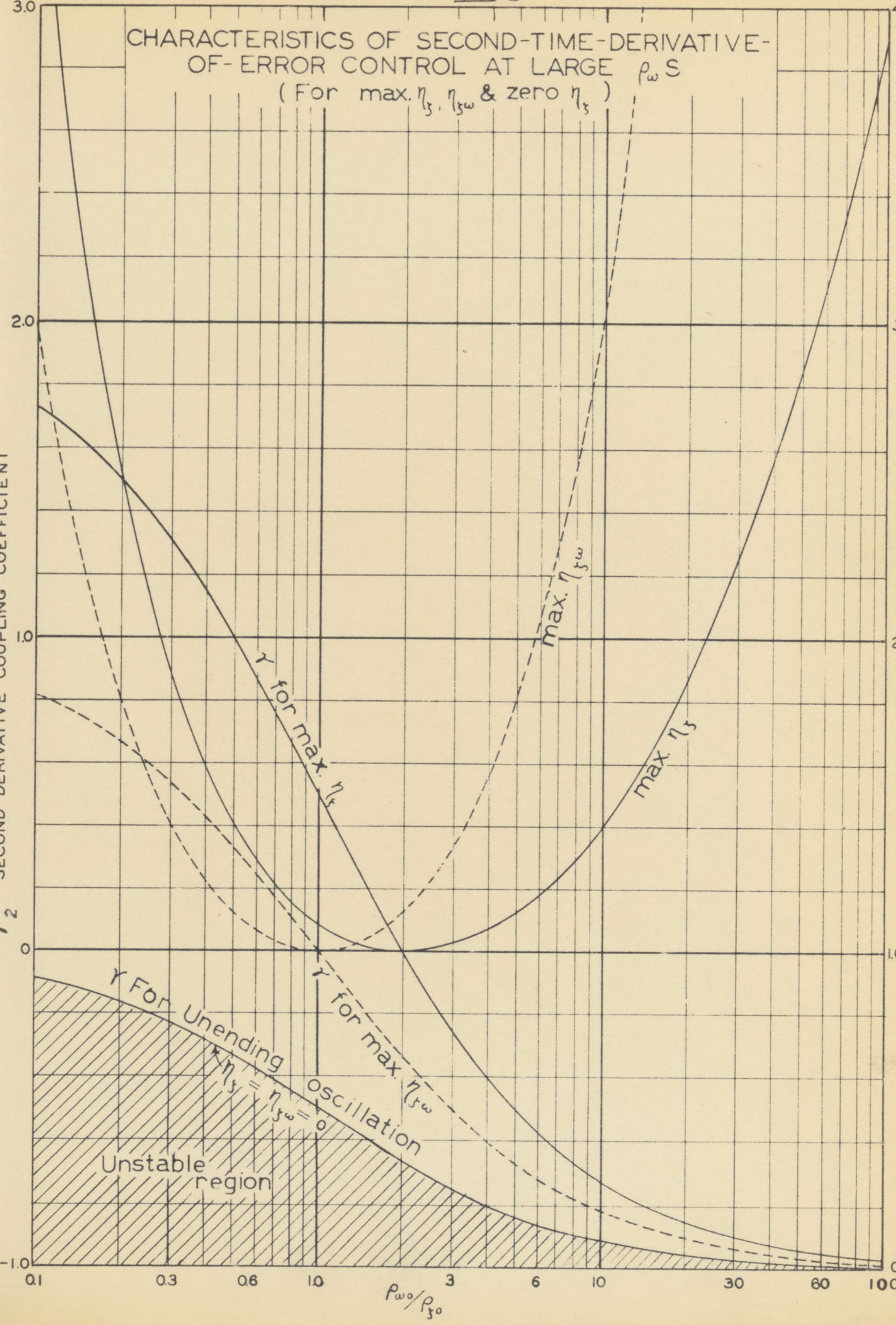
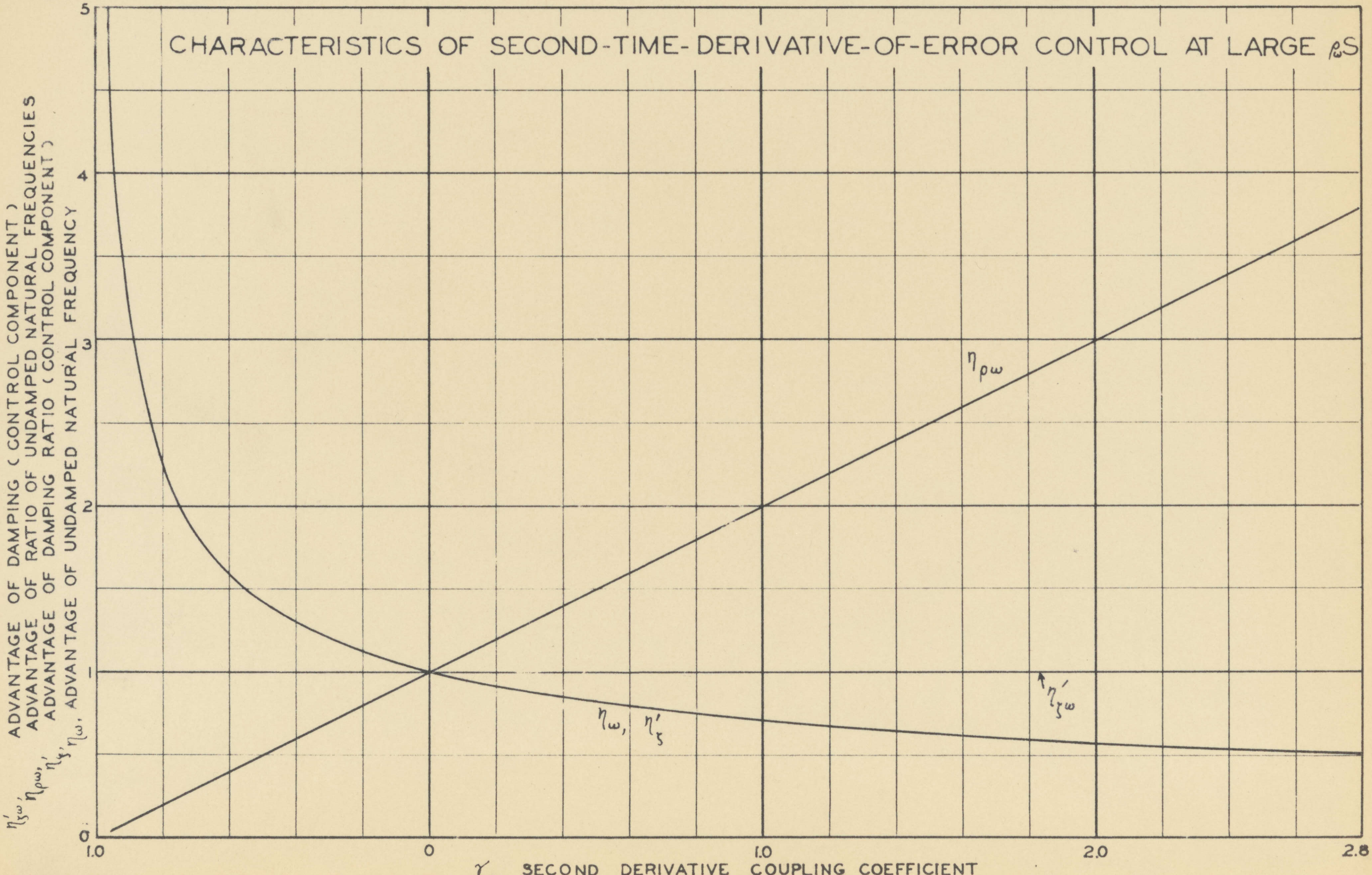

CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega S$
(For max. η_ζ, $\eta_{\zeta\omega}$ & zero η_ζ)
SECOND DERIVATIVE COUPLING COEFFICIENT
γ_2
MAXIMUM ADVANTAGE
γ for max. η_ζ
max. η_ζ
max. $\eta_{\zeta\omega}$
γ for max. $\eta_{\zeta\omega}$
γ For Unending oscillation
$\eta_\zeta = \eta_{\zeta\omega} = 0$
Unstable region
3.0
2.0
1.0
0
-1.0
4.0
3.0
2.0
1.0
0
0.1
0.3
0.6
1.0
3
6
10
30
60
100
$\rho_{\omega 0}/\rho_{\zeta 0}$

FIG. VII D


CHARACTERISTICS OF SECOND-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega S$
$\eta'_{\zeta\omega}$, ADVANTAGE OF DAMPING (CONTROL COMPONENT)
$\eta_{\rho\omega}$, ADVANTAGE OF RATIO OF UNDAMPED NATURAL FREQUENCIES
η'_ζ, ADVANTAGE OF DAMPING RATIO (CONTROL COMPONENT)
η_ω, ADVANTAGE OF UNDAMPED NATURAL FREQUENCY
5
4
3
2
1
0
1.0
0
1.0
2.0
2.8
$\eta_{\rho\omega}$
$\eta'_{\zeta\omega}$
η_ω, η'_ζ
r_2 SECOND DERIVATIVE COUPLING COEFFICIENT

117

positive coupling the system is always stable as η_ξ and $\eta_{\xi\omega}$ approach a certain asymptotic value no matter what $\frac{\rho_{\omega o}}{\rho_{\xi o}}$ is used. But with negative coupling, unending oscillation occurs when it exceeds a certain amount (because at that time, η_ξ and $\eta_{\xi\omega}$ become zero).

For a constant value of $\frac{\rho_{\omega o}}{\rho_{\xi o}}$, there is one coupling coefficient that will give maximum η_ξ and another γ_2 that will give maximum $\eta_{\xi\omega}$. Such loci are plotted as dotted curves in Figs. 7A and 7B.

In terms of this control specification $\frac{\rho_{\omega o}}{\rho_{\xi o}}$, maximum η_ξ, $\eta_{\xi\omega}$ and their corresponding γ_2 can be expressed:

$$\text{max. } \eta_\xi = \frac{2\left(1+\frac{\rho_{\omega o}}{\rho_{\xi o}}\right)^{3/2}}{3^{2/3}\frac{\rho_{\omega o}}{\rho_{\xi o}}} = \frac{0.384\left(1+\frac{\rho_{\omega o}}{\rho_{\xi o}}\right)^{3/2}}{\frac{\rho_{\omega o}}{\rho_{\xi o}}} \tag{3.55}$$

$$\text{for max. } \eta_\xi, \quad \gamma_2 = \frac{2-\frac{\rho_{\omega o}}{\rho_{\xi o}}}{1+\frac{\rho_{\omega o}}{\rho_{\xi o}}} \tag{3.55a}$$

$$\text{max. } \eta_{\xi\omega} = \frac{\left(1+\frac{\rho_{\omega o}}{\rho_{\xi o}}\right)^2}{2\frac{\rho_{\omega o}}{\rho_{\xi o}}} \tag{3.56}$$

$$\text{for max. } \eta_{\xi\omega} \quad \gamma_2 = \frac{1-\frac{\rho_{\omega o}}{\rho_{\xi o}}}{1+\frac{\rho_{\omega o}}{\rho_{\xi o}}} \tag{3.56a}$$

The limiting value of γ_2 at which the system becomes unendingly oscillatory can also be expressed in terms of $\frac{\rho_{\omega o}}{\rho_{\xi o}}$,

$$\text{for zero } \eta_\xi, \quad \gamma_2 = -\frac{\frac{\rho_{\omega o}}{\rho_{\xi o}}}{1+\frac{\rho_{\omega o}}{\rho_{\xi o}}} \tag{3.57}$$

The above results -- Eqs. (3.55) to (3.57) -- are plotted as Fig. 7C with r_2's and η's as ordinates against $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$ as abscissae. It can be seen that for the same $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$, max. $\eta_{\zeta\omega}$ obtainable is greater than max. η_ζ obtainable when $\frac{\rho_{\omega o}}{\rho_{\zeta o}} > \sqrt{2}$, and the condition is reversed when $\frac{\rho_{\omega o}}{\rho_{\zeta o}} < \sqrt{2}$. To obtain max. η_ζ, r_2 cannot be greater than 2. To obtain max. $\eta_{\zeta\omega}$, r_2 cannot be greater than 1.0.

Fig. 7D are plotted for η_ω, $\eta_{\rho\omega}$, η_ω', η_ζ' and $\eta_{\zeta\omega}'$ vs. r_2. Fig. 7D supplies the information that the damping of the control (high frequency) component is not essentially changed ($\eta_{\zeta\omega}' = 1$) because the reduction of damping ratio ($\eta_\zeta' < 1$ for $r_2 > 1$) is compensated by increase in its natural frequency ($\eta_\omega' > 1$ for $r_2 > 1$). The compensation in the region for $r_2 < 1$ is in the opposite way.

For the principal (or low frequency) component, natural frequency is decreased with positive coupling and increased with negative coupling. Because of the complication of η_ζ, which is not only a function of r_2 but also a function of $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$, the effect of the control upon the principal natural frequency therefore cannot compensate the effect upon ζ. Real advantage is then taken from negatively coupled control of the error-acceleration type with proper magnitude of the coupling coefficient so that both η_ζ and η_ω are greater than 1.0 ($\frac{\rho_{\omega o}}{\rho_{\zeta o}} > 2$) which greatly improves the stability of the principal component, and yet does not substantially affect the stability of the control component as far as damping ($\zeta_2 \omega_2$) is concerned. Such advantage is available only at the sacrifice of decreasing the inertia relating to the principal component. When the system encounters a prolonged disturbance, the diminished inertia due to negative γ_2 coupling will throw the system immediately into the disturbance. However, if such negative coupling is only called into action when the disturbance has ceased, it will definitely "quench" the disturbed motion.

Error-acceleration type control can be considered as inertia improving control. Therefore the improvement in damping, if there is any, is only a secondary effect. Hence, a system which only possesses a negligible damping cannot be improved to a satisfactory degree by the error-acceleration type control.

31. Error-Velocity Control with High Natural Frequency

In reviewing equations (3.17), (3.06) to (3.08) and (2.25) to (2.27) the following relations are obtained for the error-velocity control.

$$\alpha_3 = \alpha_{30}$$

$$\alpha_2 = \alpha_{20}$$

$$\alpha_1 = \alpha_{10}(1+\gamma_1)$$

or

$$2\zeta_1\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega}\right) = 2\zeta_0\left(\frac{1}{\sqrt{\rho_{\omega0}}} + \rho_{\zeta0}\sqrt{\rho_{\omega0}}\right) \tag{3.58}$$

$$\rho_\omega + \frac{1}{\rho_\omega} + 4\zeta_1^2\rho_\zeta = \rho_{\omega0} + \frac{1}{\rho_{\omega0}} + 4\zeta_0^2\rho_{\zeta0} \tag{3.59}$$

$$2\zeta_1\left(\sqrt{\rho_\omega} + \frac{\rho_\zeta}{\sqrt{\rho_\omega}}\right) = 2\zeta_0\left(\sqrt{\rho_{\omega0}} + \frac{\rho_{\zeta0}}{\sqrt{\rho_{\omega0}}}\right)(1+\gamma_1) \tag{3.60}$$

Write Eq. (3.58) as

$$\eta_\zeta = \frac{\frac{1}{\sqrt{\rho_{\omega0}}} + \rho_{\zeta0}\sqrt{\rho_{\omega0}}}{\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega}} = \frac{1+\rho_{\zeta0}\rho_{\omega0}}{\frac{\sqrt{\rho_{\omega0}}}{\sqrt{\rho_\omega}}\left(1+\rho_\zeta\rho_\omega\right)} \tag{3.61}$$

Since such error-velocity control is usually called for improving damping, $\rho_{\zeta 0} >> 1$, and $\rho_{\zeta} > 1$. With high natural frequency control, $\rho_{\zeta 0}\rho_{\omega 0} >> 1$, and $\rho_{\zeta}\rho_{\omega} >> 1$; therefore Eq. (3.61) may be simplified as

$$\eta_{\zeta} = \eta_{\omega} \frac{\rho_{\zeta 0}}{\rho_{\zeta}} \tag{3.62}$$

where $\eta_{\omega} = \frac{\omega_1}{\omega_0} = \frac{\sqrt{\rho_{\omega 0}}}{\sqrt{\rho_{\omega}}}$ because $\rho_{\omega 0}^2 \omega_0^4 = \rho_{\omega}^2 \omega_1^4$

To allow frequency change, Eq. (3.59) can be simplified to the following form by neglecting $\frac{1}{\rho_{\omega}}$ and $\frac{1}{\rho_{\omega 0}}$ terms only.

$$\rho_{\omega} + 4\zeta_1^2 \rho_{\zeta} = \rho_{\omega 0} + 4\zeta_0^2 \rho_{\zeta 0} \tag{3.63}$$

Substitute Eq. (3.62) into the above one , and the following relation can be obtained:

$$\eta_{\zeta} = \frac{\frac{\rho_{\omega 0}}{4\zeta_0^2 \rho_{\zeta 0}}\left(1 - \frac{1}{\eta_{\omega}^2}\right) + 1}{\eta_{\omega}} \tag{3.64a}$$

or

$$\eta_{\zeta} = \frac{\frac{\rho_{\omega 0}}{4\zeta_0 \zeta_c}\left(1 - \frac{1}{\eta_{\omega}^2}\right) + 1}{\eta_{\omega}} \tag{3.64}$$

Write Eq. (3.60) as:

$$\eta_{\zeta} = \frac{(1+\gamma_1)\left(1 + \frac{\rho_{\zeta 0}}{\rho_{\omega 0}}\right)}{\frac{\sqrt{\rho_{\omega}}}{\sqrt{\rho_{\omega 0}}}\left(1 + \frac{\rho_{\zeta}}{\rho_{\omega}}\right)}$$

and equate to Eq. (3.62). The following relation is obtained:

$$\frac{\rho_{\zeta 0}}{\rho_{\zeta}} = \frac{\eta_{\zeta}}{\eta_{\omega}} = (1+\gamma_1)\left(1 + \frac{1}{\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}}\right) - \frac{\eta_{\omega}^2}{\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}} \tag{3.65}$$

Eq. (3.64) is plotted as Fig. 8A with η_{ζ} as ordinate against $\frac{\rho_{\omega 0}}{\zeta_c \zeta_0}$ as abscissa with η_{ω} as varying parameter. It is seen, although the variation of ω is small, that it is very sensitive to η_{ζ} . If η_{ω} is allowed to be unity, η_{ζ} would be unity

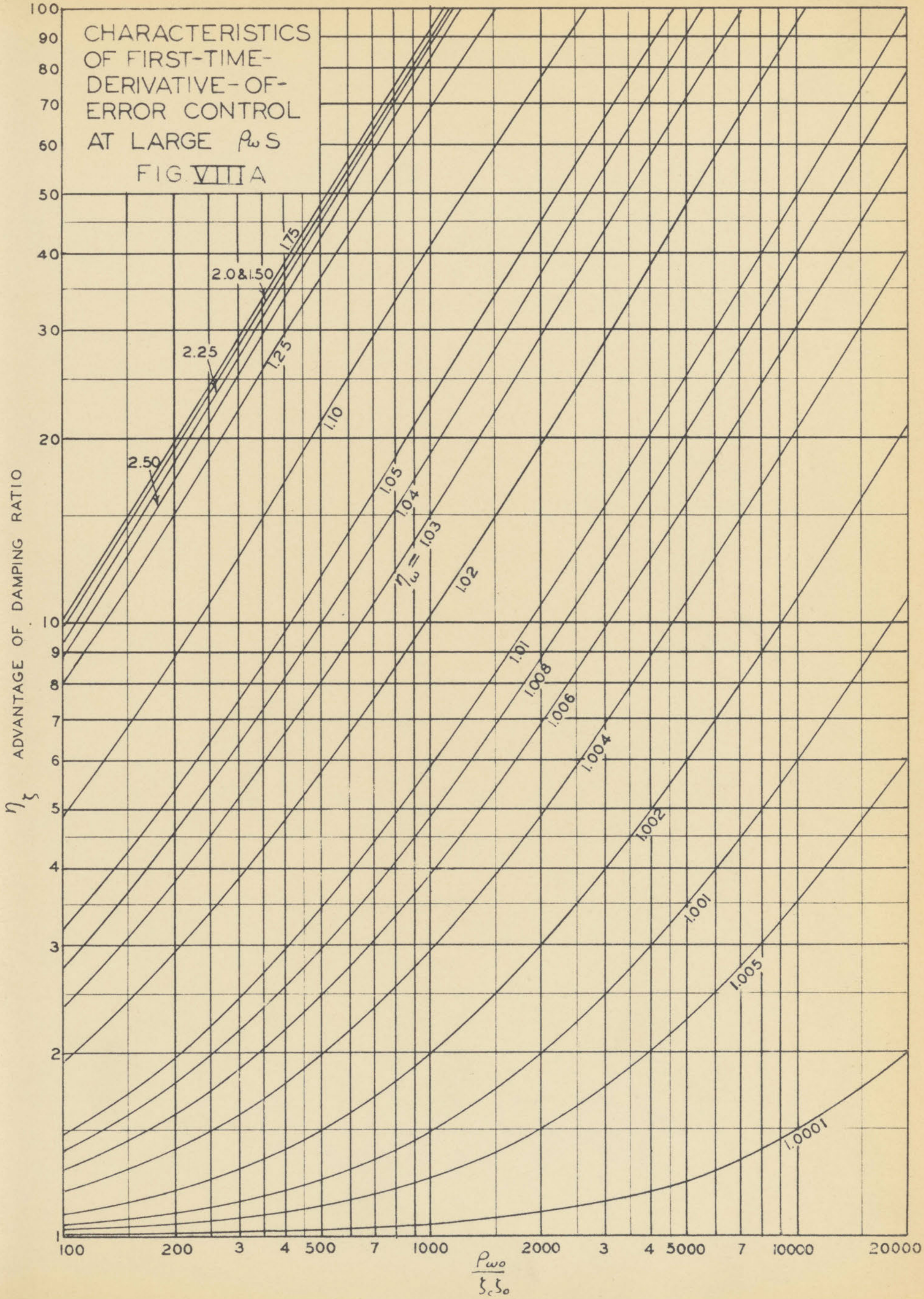
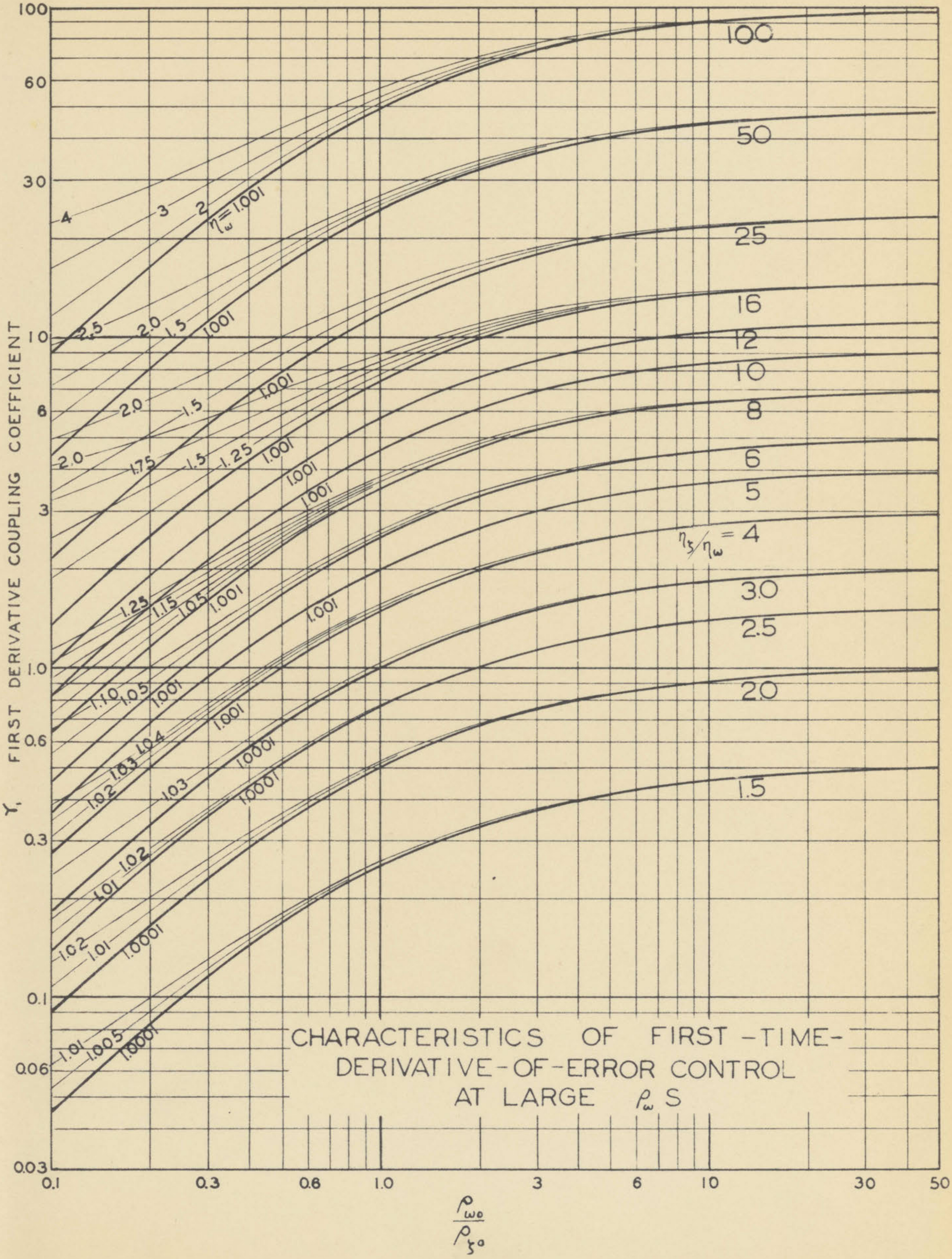

CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega S$
FIG. VIII A
ADVANTAGE OF DAMPING RATIO η_ζ
$\frac{\rho_{\omega o}}{\zeta_c \zeta_o}$
η_ω = 1.03
2.50
2.25
2.0 & 1.50
1.75
1.25
1.10
1.05
1.04
1.02
1.01
1.008
1.006
1.004
1.002
1.001
1.005
1.0001
100
200
3
4
500
7
1000
2000
3
4
5000
7
10000
20000
1
2
3
4
5
6
7
8
9
10
20
30
40
50
60
70
80
90
100

FIG. VIII B



CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE ρ_ω S

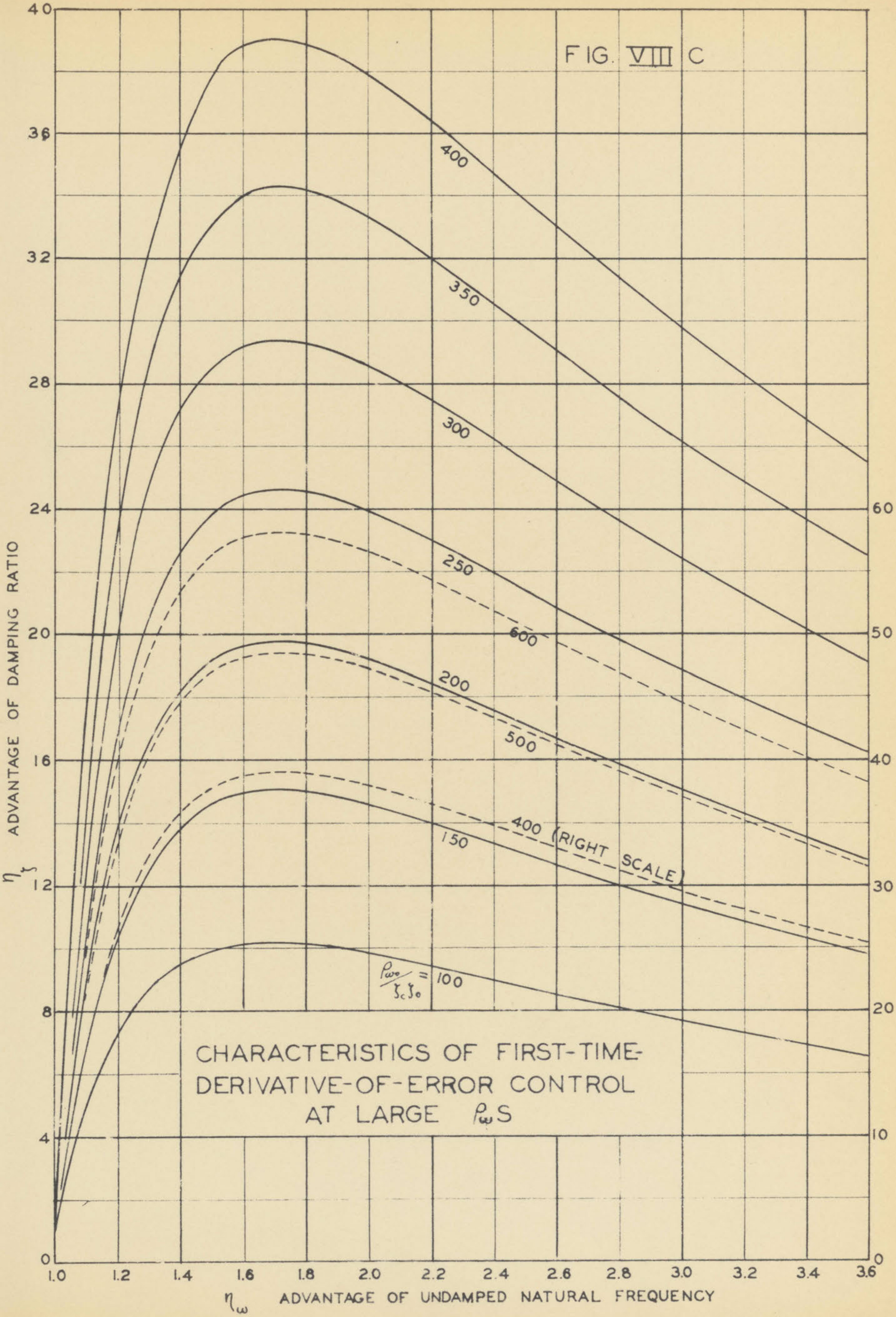


FIG. VIII C
400
350
300
250
600
200
500
400 (RIGHT SCALE)
150
$\frac{\rho_{\omega o}}{\zeta_c \zeta_o} = 100$
CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE $\rho_\omega S$
η_ζ ADVANTAGE OF DAMPING RATIO
η_ω ADVANTAGE OF UNDAMPED NATURAL FREQUENCY
0
4
8
12
16
20
24
28
32
36
40
10
20
30
40
50
60
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
3.2
3.4
3.6

at any γ_1 which is apparently not true. When η_ω reaches 1.75, η_ζ becomes maximum, so the curves retreat as η_ω further increases. However, with ordinary high frequency control, such optimum operation cannot be attained unless the natural frequency of the control is only moderately high (say ρ_ω around 10) and the damping ratio of control is well above 1.0.

To use Fig. 8A, the designer is supposed to know how many times the damping ratio of the controlled member is to be raised; that is, he must know what η_ζ he wants. He has also to know what frequency variation to allow for; that is, η_ω. Then the abscissa value or $\frac{\rho_{\omega 0}}{\zeta_c \zeta_0}$ is fixed from which $\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}$ can be easily found because ζ_0 is known from the problem. ζ_c is somewhere between 1 and 2 for conservative design.

Eq. (3.65) is plotted as Fig. 8B with γ_1 as ordinate against $\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}$ as abscissa with $\frac{\eta_\zeta}{\eta_\omega}$ (or $\frac{\rho_{\zeta 0}}{\rho_\zeta}$) as principal varying parameter and η_ω as the secondary one.

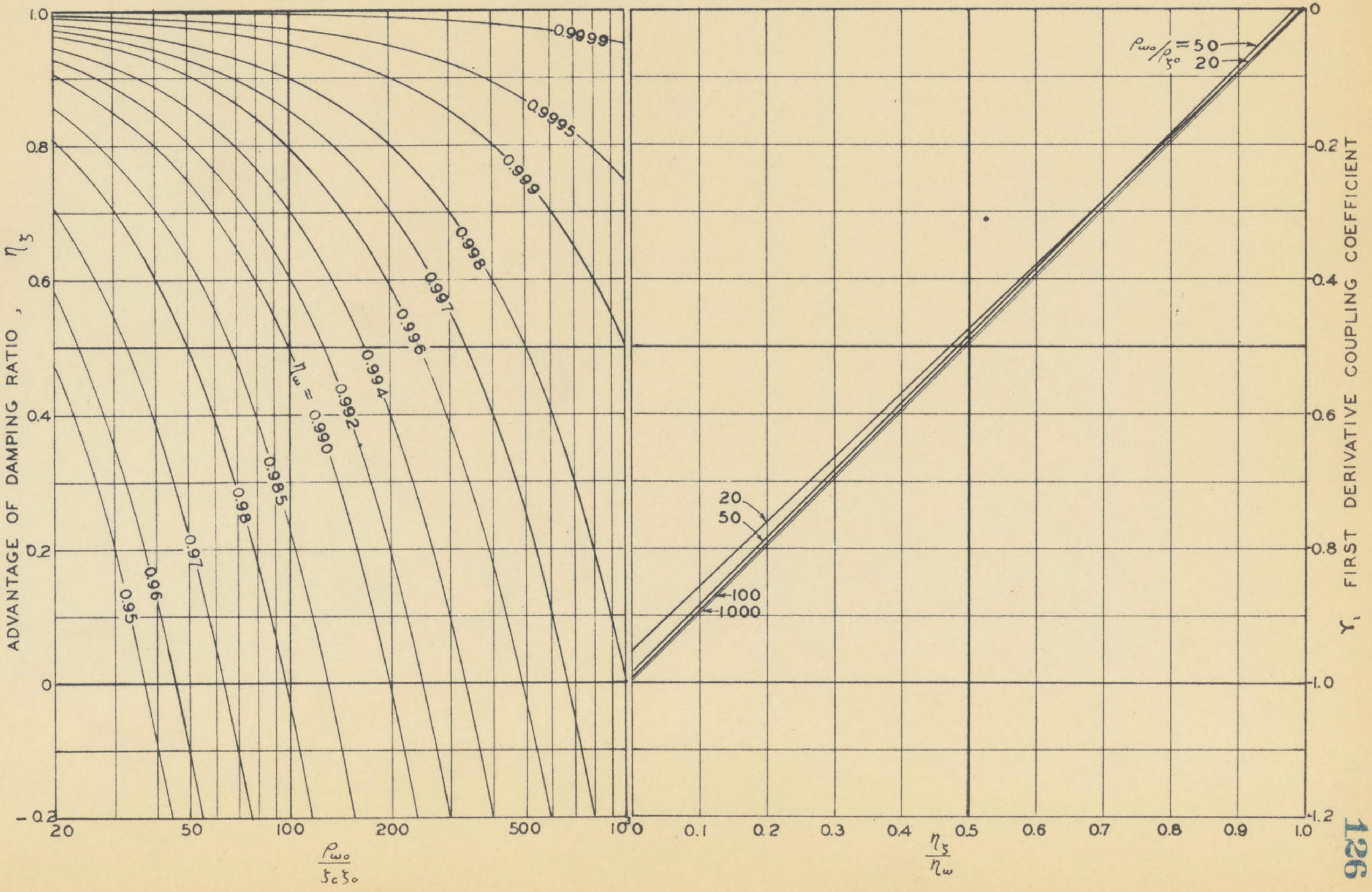
When $\frac{\rho_{\omega 0}}{\zeta_0 \zeta_c}$ is obtained from Fig. 8A, $\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}$ can be figured out. Therefore, γ_1 can be found with the known values $\frac{\eta_\zeta}{\eta_\omega}$, η_ω, and $\frac{\rho_{\omega 0}}{\rho_{\zeta 0}}$.

Any attempt to apply the Figs. (8A and 8B) in reverse order is easily confusing because the conditions are not well defined when γ_1 is chosen as starting datum.

For academic interest or rather, prospective design, optimum operation curves are plotted as Fig. 8C, with η_ζ as ordinate against η_ω as abscissa, with $\frac{\rho_{\omega 0}}{\zeta_c \zeta_0}$ as varying parameter. The curves show a flat top at maximum η_ζ. With values of η_ω lying between 1.6 and 1.8 the purpose of maximum η_ζ may be

FIG. IX

CHARACTERISTICS OF FIRST-TIME-DERIVATIVE-OF-ERROR CONTROL AT LARGE ρ_ωS



satisfactorily attained. However, the lower the value of is, the less the coupling coefficient will be required.

When η_ω is slightly less than one, η_ζ is less than one and negative coupling coefficient is required. Eqs. (3.64) and (3.65) are plotted side by side as Fig. 9 for $\eta_\omega < 1$. The left part of Fig. 9 should be applied first, leaving the coupling coefficient γ_1 to be found last from the right part of the figure.

When $\rho_{\omega o} >> 1$, ought to be only slightly away from 1.0. Eq. (3.65) can be simplified further as

$$\eta_\zeta \cong 1 + \gamma_1 \left(1 + \frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right) \tag{3.66}$$

$$\eta_\omega \cong 1.0$$

This simplified equation serves as a quick estimate of the coupling coefficient when $\rho_\omega >> 1$ and required η_ζ is known.

CHAPTER NINE

COMPOUNDING CONTROLS

32. Single Coupling and Compound Coupling

In Chapter Eight three types of control have been discussed. They are only excited by a single force derived from the error or error derivative. Such controls may be defined as single coupling control. When a control is excited by two or more forces simultaneously, it is defined as compound control. From the analysis in Chapter Eight it is understood that neither one of the three types of single coupling will give a result which may be considered as "all round". Therefore proper "compounding" should be studied.

When a control is excited by two or more forces, the result obtained is neither the sum nor the product of the separately excited controlled system. Mechanically the system adjusts itself to give a compounding result without complication. But in analysis, step-by-step consideration has to be followed.

A control started with specification $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$ will not keep on as such when the system is improved by the application of a certain coupling. Therefore the control specification $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$ has to be modified in order to consider the improvement of stability of the system with the other coupling (which actually acts simultaneously with the first one).

It is advisable to have the results obtained in the last chapter in the tabulated form in order to minimize the effort

TABLE V

APPROXIMATE STABILITY IMPROVEMENT DUE TO VARIOUS CONTROLS OF HIGH NATURAL FREQUENCY AT CERTAIN $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$

COUPLING COEFF. / ADVANTAGES	γ_0	γ_1	γ_2	γ_3	γ_4
η_ω	$(1+\gamma_0)^{1/2}$	1	$\frac{1}{(1+\gamma_0)^{1/2}}$	1	1
η_ζ	$\frac{1}{(1+\gamma_0)^{1/2}}\left[1-\gamma_0\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	$(1+\gamma_1)\left[1+\frac{\gamma_1}{1+\gamma_1}\cdot\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$ $*_1$	$\frac{1}{(1+\gamma_2)^{1/2}}\left[1+\frac{\gamma_2}{1+\gamma_2}\cdot\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	$1-\gamma_3\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}$ $*_2$	1
$\eta_{\zeta\omega}$	$1-\gamma\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}$	$1+\gamma_1\left[1+\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	$\frac{1}{1+\gamma_2}\left[1+\frac{\gamma_2}{1+\gamma_2}\cdot\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	$1-\gamma_3\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}$	1
η'_ω	1	1	$(1+\gamma)^{1/2}$	1	$\frac{1}{(1+\gamma_4)^{1/2}}$
η'_ζ	1	1	$\frac{1}{(1+\gamma)^{1/2}}$	$1+\gamma_3$	$\frac{1}{(1+\gamma_4)^{1/2}}$
$\eta'_{\zeta\omega}$	1	1	1	$1+\gamma_3$	1
$\eta_{\rho\omega}$	$\frac{1}{(1+\gamma_0)^{1/2}}$	1	$1+\gamma_2$	1	$\frac{1}{(1+\gamma_4)^{1/2}}$
$\eta_{\rho\zeta}$	$(1+\gamma_0)^{1/2}\left[1-\gamma_0\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]^{-1}$	$\left[1+\gamma_1\left(1+\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right)\right]^{-1}$	$\left[1+\frac{\gamma_2}{1+\gamma_2}\cdot\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]^{-1}$	$(1+\gamma_3)\left[1-\gamma_3\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]^{-1}$	$\frac{1}{(1+\gamma_4)^{1/2}}$
$\eta_{\frac{\rho\omega}{\rho\zeta}}$	$\frac{1}{1+\gamma_0}\left[1-\gamma_0\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	$(1+\gamma_1)\left[1+\frac{\gamma_1}{1+\gamma_1}\cdot\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	$(1+\gamma_2)\left[1+\frac{\gamma_2}{1+\gamma_2}\cdot\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	$\frac{1}{1+\gamma_3}\left[1-\gamma_3\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right]$	1
$\frac{\rho_{\omega 1}}{\rho_{\zeta 1}}$	$\frac{1}{1+\gamma_0}\left[\frac{\rho_{\omega o}}{\rho_{\zeta o}}-\gamma_0\right]$	$\gamma_1+(1+\gamma_1)\frac{\rho_{\omega o}}{\rho_{\zeta o}}$	$\gamma_2+(1+\gamma_2)\frac{\rho_{\omega o}}{\rho_{\zeta o}}$	$\frac{1}{1+\gamma_3}\left[\frac{\rho_{\omega o}}{\rho_{\zeta o}}-\gamma_3\right]$	$\frac{\rho_{\omega o}}{\rho_{\zeta o}}$

* EXPRESSION IS TRUE FOR VERY SMALL ζ_o AND VERY HIGH $\rho_{\omega o}$ WITH CONSERVATIVE ζ_o (AROUND ONE). FOR MORE ACCURATE RESULTS, CONSULT FIG. VIII A, B. IF MUCH OVERDAMPING IS ALLOWED IN THE CONTROL, η_ζ CAN BE DERIVED BY BAIRSTOW'S APPROXIMATION AS: $*_1$ $1+\gamma_1\left[1+\left\{\frac{\rho_{\omega o}}{\rho_{\zeta o}}+4\zeta_o^2\left(1+\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right)\right\}^{-1}\right]$ $*_2$ $1-\gamma_3\left[\frac{\rho_{\omega o}}{\rho_{\zeta o}}+4\zeta_o^2\left(1+\frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right)\right]^{-1}$

in an attempt to consider the compounding effect of two and more couplings. Table V not only serves as a summary of the results in the last chapter, but also extends to higher derivative coupling. (The derivations are omitted as they can be done by carefully neglecting terms which are negligible). The inclusion of the higher terms would be found useful should the time lag of the detecting instrument be considered.

33. Theory of Compounding for Controls with High Natural Frequency

It is seen from Table V that some expressions of η only contain the coupling coefficient itself. The specification of control has no influence upon it. For such a η simple multiplication of the individual expression is sufficient to give the resultant η. For instance,

$$\eta_{\omega_{012}} = \eta_{\omega_0}\eta_{\omega_1}\eta_{\omega_2} = \left(\frac{1+\gamma_0}{1+\gamma_2}\right)^{1/2} \tag{3.67}$$

where the subscripts after the η's and γ's are referred to degrees of derivative. When they appear together, it means that the system is being compoundly coupled.

But some η's are functions of γ as well as $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$. In such a case the expression can be, in general, expressed by

$$\eta_q = f_q(\gamma_q)^* \left[1 + h_q(\gamma_q)^* \frac{1}{\frac{\rho_{\omega o}}{\rho_{\zeta o}}}\right] \tag{3.68}$$

---

* For convenience in typing, $f_q(\gamma_q)$, $h_q(\gamma_q)$ and $k_q(\gamma_q)$ shall be abbreviated as f_q, h_q and k_q respectively throughout this section, and $\frac{\rho_{\omega o}}{\rho_{\zeta o}}$ abbreviated as s_c signifying specification of control.

where the subscript q is referred to the order of derivative and f and h represent two different functions. The control specification should be multiplied by the factor

$$\eta_{\frac{\rho\omega}{\rho\xi}\, q} = k_q(\gamma_q)\left[1 + h_q(\gamma_q)\frac{1}{\frac{\rho_{\omega o}}{\rho_{\xi o}}}\right] \qquad (3.69)$$

When the compounding is made between the q^{th} and s^{th} derivative, the expression can be written

$$qs = f_q\left(1 + \frac{h_q}{s_c}\right)\left\{f_s{}^{*}\left[1 + \frac{h_s}{s_c k_q\left(1 + \frac{h_q}{s_c}\right)}\right]\right\}$$

$$= f_q\left(1 + \frac{h_q}{s_c}\right)\left\{f_s\left[1 + \frac{h_s}{s_c}\left(1 - \frac{1 - \frac{1}{k_q} + \frac{h_q}{s_c}}{1 + \frac{h_q}{s_c}}\right)\right]\right\}$$

$$= f_q\left(1 + \frac{h_q}{s_c}\right)f_s\left(1 + \frac{h_s}{s_c}\right) - f_q f_s h_s\left(1 - \frac{1}{k_q} + \frac{h_q}{s_c}\right)/s_c$$

$$= \eta_q\,\eta_s - f_q f_s h_s h_q\left(\frac{1}{h_q} - \frac{1}{h_q k_q} + \frac{1}{s_c}\right)/s_c \qquad (3.70)$$

It can be shown that

$$(1/h_q)(1-1/k_q) = 1$$

or $\quad 1 - 1/k_q = h_q \qquad (3.71)$

For instance, take $q = 2$

$$k_2 = k_2(\gamma_2) = 1 + \gamma_2$$

and $1 - \frac{1}{1+\gamma_2} = \frac{\gamma_2}{1+\gamma_2} = h_2(\gamma_2) = h_2$

or take $q = 0$

$$1 - 1/k_0(\gamma_0) = 1 - (1 + \gamma_0) = -\gamma_0 = h_0(\gamma_0) = h_0$$

*Where h_s, f_s, etc. stand for $h_s(\gamma_s)$, $f_s(\gamma_s)$, etc.

Therefore

$$\eta_{qs} = \eta_q \eta_s - f_q f_s h_q h_s (1 + 1/s_c)/s_c \qquad (3.72)$$

On developing, and canceling the terms containing $1/s_c$ Eq.(3.72) becomes

$$\eta_{qs} = f_q f_s \left[1 + (h_q + h_s - h_q h_s)/s_c\right]$$
$$= f_{q,s}(1 + h_{q,s}/s_c) \qquad (3.73)$$

where

$$f_{q,s} = f_{q,s}(\gamma_q, \gamma_s) = f_q(\gamma_q) f_s(\gamma_s) = f_q f_s \qquad (3.73)a$$

$$h_{q,s} = h_{q,s}(\gamma_q, \gamma_s) = h_q + h_s - h_q h_s \qquad (3.73)b$$

It should be noted that the order of substitution is indifferent to the result. In the bracket of Eq. (3.70) it would look like $(1/h_s - 1/h_s k_s + 1/s_c)$ if subscript s is substituted for subscript q, but the simplified form takes the same expression $1 + 1/s_c$.

By the same reasoning and by following the same procedure as that given above, it can be shown that

$$\frac{\eta_{\rho\omega}}{\rho_f}{}_{qs} = k_{q,s}(1 + h_{q,s}/s_c) \qquad (3.74)$$

again, $k_{q,s} = k_q k_s$ (3.74)a

and $h_{q,s} = h_q + h_s - h_q h_s$ (3.73)b

It can also be proven that

$1 - 1/k_{q,s} = h_{q,s}$ by the same method in proving Eq. (3.71).

Therefore

$$\eta_{qsp} = \eta_{qs} \eta_p - f_{q,s} f_p h_{q,s} h_p (1 + 1/s_c)/s_c \qquad (3.76)$$

On developing

$$\eta_{qsp} = f_{q,s,p}(1 + h_{q,s,p}/s_c) \qquad (3.77)$$

where

$$f_{q,s,p} = f_q f_s f_p = f_q(\gamma_q) f_s(\gamma_s) f_p(\gamma_p) \qquad (3.77)a$$

and $$h_{q,s,p} = h_q + h_s + h_p + h_q h_s h_p - h_q h_s - h_s h_p - h_p h_q \qquad (3.77)b$$

From the above analysis several general rules can be deduced.

(1) When the advantage of a singly coupled control is a function of coupling coefficient as well as control specification, it can be expressed by the product of two factors, of which one is a simple function of the coupling coefficient; that is, $f_q(\gamma_q)$, and the other is the sum of unity and the quotient of another simple function of the coupling coefficient; that is, $h_q(\gamma_q)$, divided by the control specification.

(2) When the control is compoundly coupled, such "advantage" takes the same form although the two functions are modified according to the following rules:

(A) The function that has nothing to do with the control specification becomes the product of the individual function, or in symbol,

$$f_{q,s,\cdots\cdots p} = f_q f_s \cdots\cdots f_p \qquad (3.78)a$$

(B) The function that divided by the control specification takes the form

$$\begin{aligned} h_{q,s,\cdots\cdots p} = {} & h_q + h_s \;\cdots\cdots\; + h_p \\ & - h_q h_s - h_s h_r - \cdots\cdots - h_q h_p \\ & + h_q h_s h_{s+1} \;\cdots\cdots\; + h_q h_s h_p \qquad (3.78)b \\ & - h_q h_s h_{s+1} h_p - \cdots\cdots \\ & \pm \cdots \end{aligned}$$

where h_q stands for $h_q(\gamma_q)$, etc.

It is beneficial to master the compoundly coupled control, as the "advantages" can be actually attained to a better degree by compromising the coupling coefficients. The practical example will be left to the practical designers.

34. Special Compounding Controls

In some cases where only the damping ratio is required to be improved, the natural frequency of the member to be controlled is satisfactory and required to be kept the same no matter how much improvement has been made on the damping ratio. The problem may therefore be specified as follows:

(a) What ρ_ω is convenient to design for a specific ω_1 ?

(b) What is the initial damping present in the member to be controlled and how big is the advantage of the damping ratio expected to be obtained from the control?

(c) What is the relative damping ratio ρ_ζ between the control component and the principal component?

(d) Based upon the validity of the detecting instrument, what compounding couplings are to be used?

From the above specification, the following data can be determined:

(e) The relative damping ratio $\rho_{\zeta 0}$ between the control and the member to be controlled. This datum virtually fixes the damping ratio of the control.

(f) The coupling coefficients.

35. Constant ω_1 , Velocity-Acceleration Compounding Controls

For such a controlled system the nondimensional coefficients of the standardized quartic equation are

$$\alpha_3 = \alpha_{30}$$
$$\alpha_2 = \alpha_{20}(1+\gamma_2) \qquad (3.79)$$
$$\alpha_1 = \alpha_{10}(1+\gamma_1)$$

Because nothing has been added to the constant term (or α_{oo}), $\rho_\omega^2\omega_1^4 = \rho_{\omega o}^2\omega_o^4$, but $\omega_o = \omega_1$; therefore $\rho_{\omega_o} = \rho_\omega$.

Therefore,

$$2\zeta_1\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega}\right) = \alpha_3 = \alpha_{3o} = 2\zeta_o\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_{\zeta o}\sqrt{\rho_\omega}\right) \quad (3.80)\text{a}$$

or
$$\frac{\zeta_1}{\zeta_o} = \frac{1+\rho_{\zeta o}\rho_\omega}{1+\rho_\zeta\rho_\omega} \quad (3.80)\text{b}$$

or
$$\underline{\rho_{\zeta o} = \frac{1}{\rho_\omega}(\eta_\zeta - 1) + \rho_\zeta\eta_\zeta} \quad (3.80)$$

$$\gamma_1 = \frac{\alpha_1 - \alpha_{1o}}{\alpha_{1o}} = \frac{\alpha_1}{\alpha_{1o}} - 1$$

$$= \frac{\zeta_1\left(\sqrt{\rho_\omega} + \frac{\rho_\zeta}{\sqrt{\rho_\omega}}\right)}{\zeta_o\left(\sqrt{\rho_\omega} + \frac{\rho_{\zeta o}}{\sqrt{\rho_\omega}}\right)}$$

or
$$\underline{\gamma_1 = \eta_\zeta\left(\frac{\rho_\omega + \rho_\zeta}{\rho_\omega + \rho_{\zeta o}}\right) - 1} \quad (3.81)$$

$$\gamma_2 = \frac{\alpha_2 - \alpha_{2o}}{\alpha_{2o}} = \frac{4\left(\zeta_1^2\rho_\zeta - \zeta_o^2\rho_{\zeta o}\right)}{\rho_\omega + \frac{1}{\rho_\omega} + 4\zeta_o^2\rho_{\zeta o}}$$

or
$$\gamma_2 = \frac{\eta_\zeta^2\rho_\zeta - \rho_{\zeta o}}{\frac{1}{4\zeta_o^2}\left(\rho_\omega + \frac{1}{\rho_\omega}\right) + \rho_{\zeta o}} \quad (3.82)\text{a}$$

or
$$\underline{\gamma_2 = \frac{\rho_\zeta - \frac{1}{\eta_\zeta^2}\rho_{\zeta o}}{\frac{1}{4\zeta_1^2}\left(\rho_\omega + \frac{1}{\rho_\omega}\right) + \frac{1}{\eta_\zeta^2}\rho_{\zeta o}}} \quad (3.82)$$

Eqs. (3.80), (3.81) and (3.82) are plotted as series of charts keeping ρ_ζ as the leading parameter which only varies from one sheet to another.

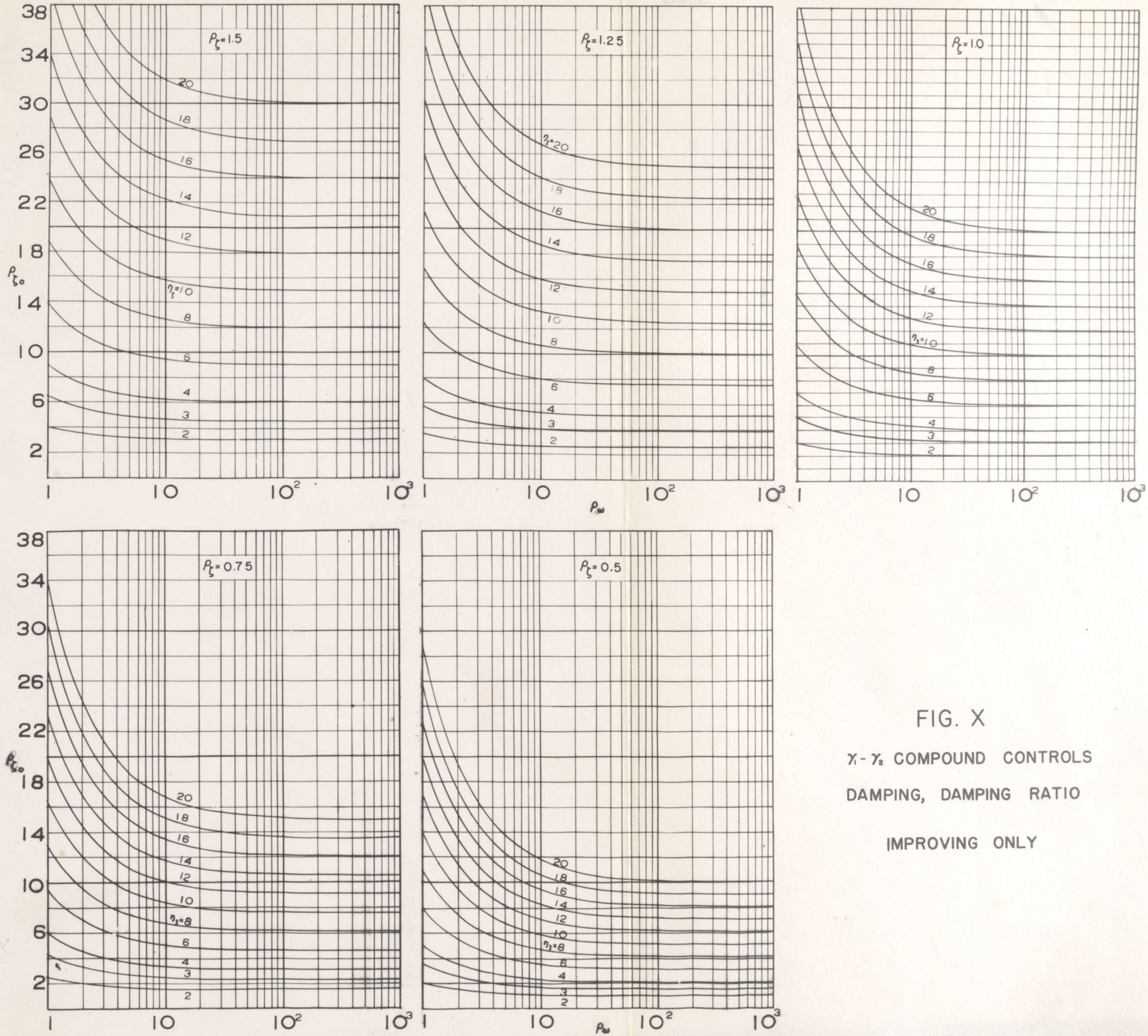


FIG. X

γ_1 - γ_2 COMPOUND CONTROLS

DAMPING, DAMPING RATIO

IMPROVING ONLY

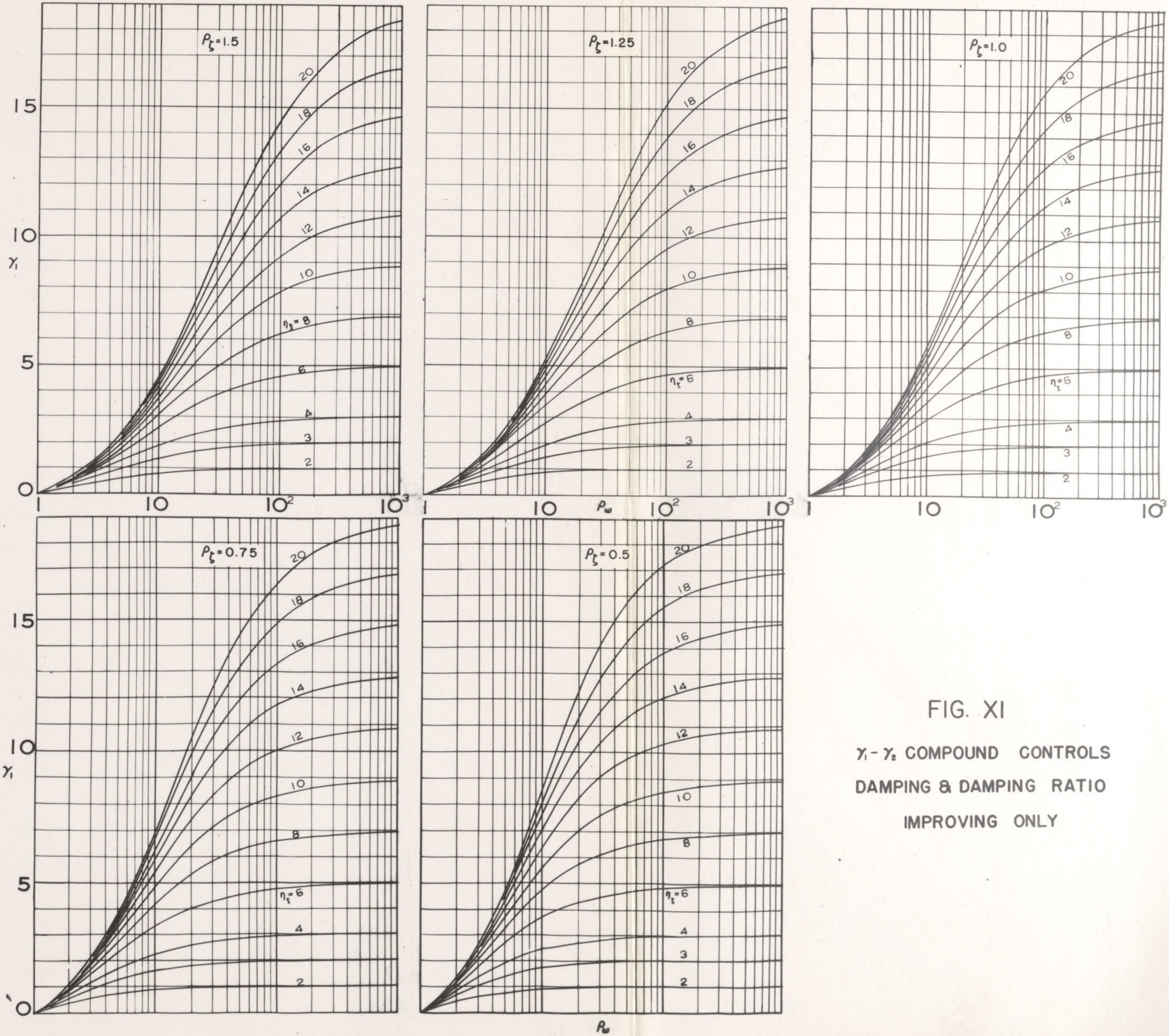


FIG. XI

$\gamma_1 - \gamma_2$ COMPOUND CONTROLS

DAMPING & DAMPING RATIO

IMPROVING ONLY

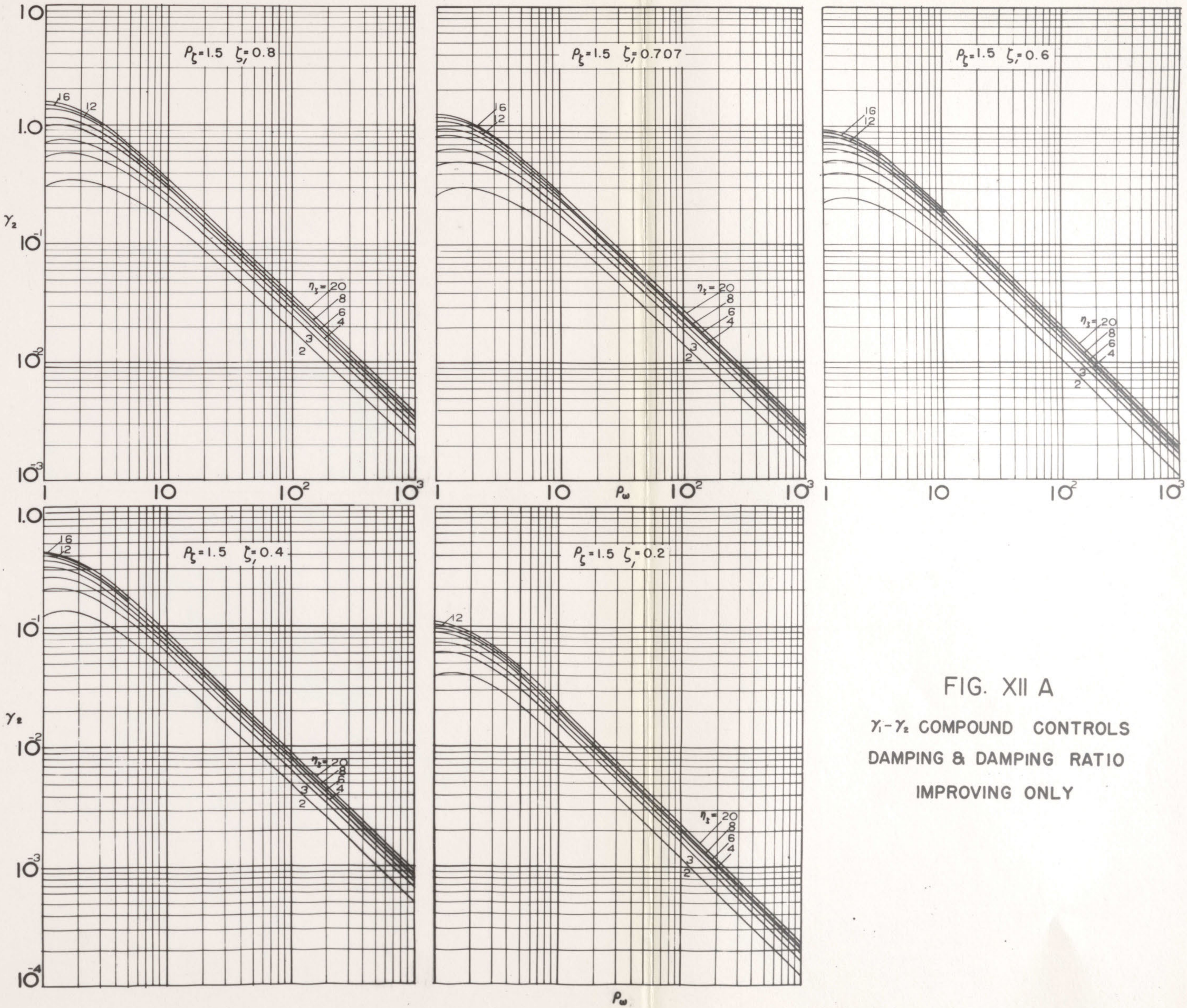


FIG. XII A

$\gamma_1 - \gamma_2$ COMPOUND CONTROLS

DAMPING & DAMPING RATIO

IMPROVING ONLY

FIG. XII B $\gamma_1-\gamma_2$ COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY

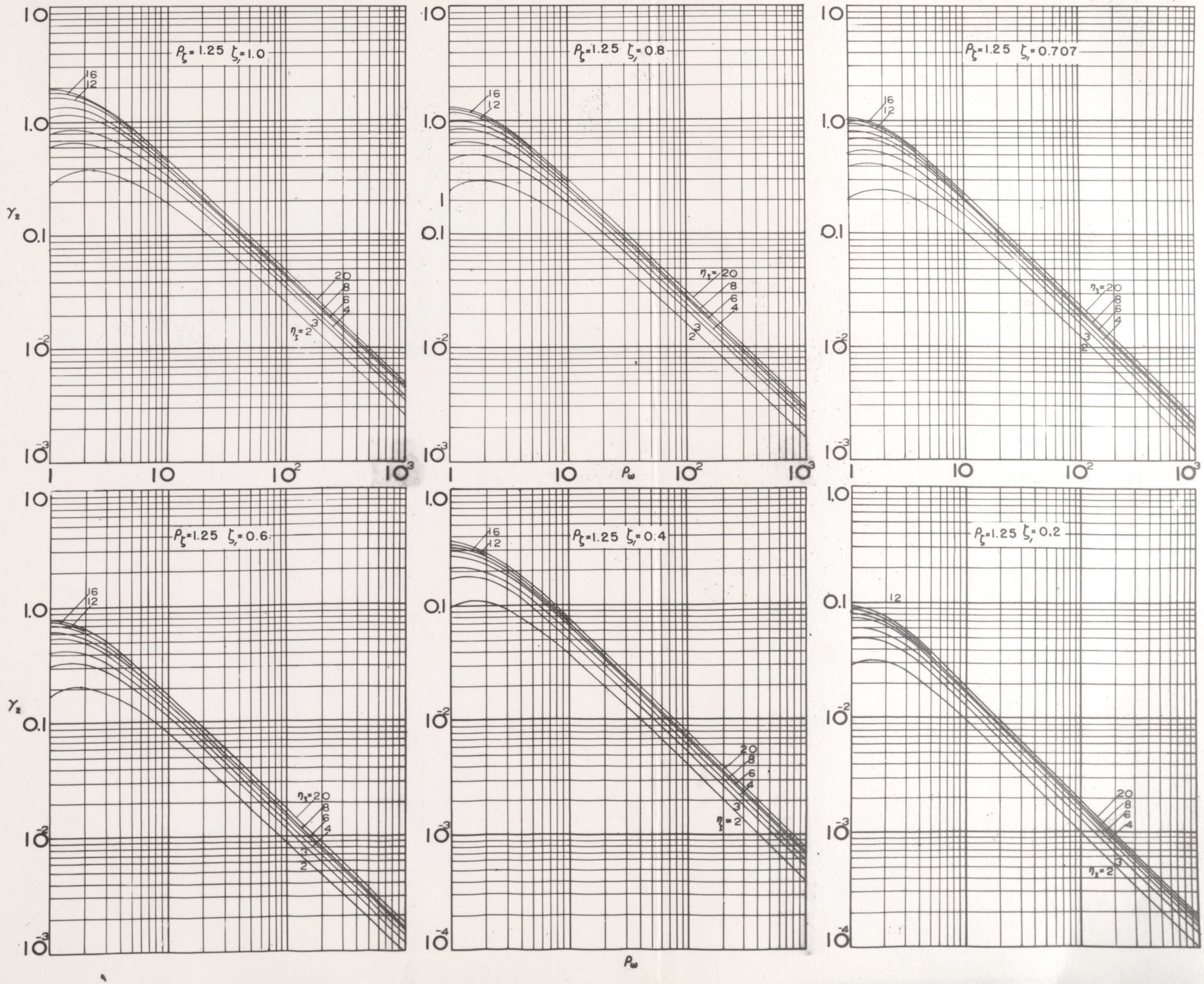


FIG. XII C $\gamma_1 - \gamma_2$ COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY

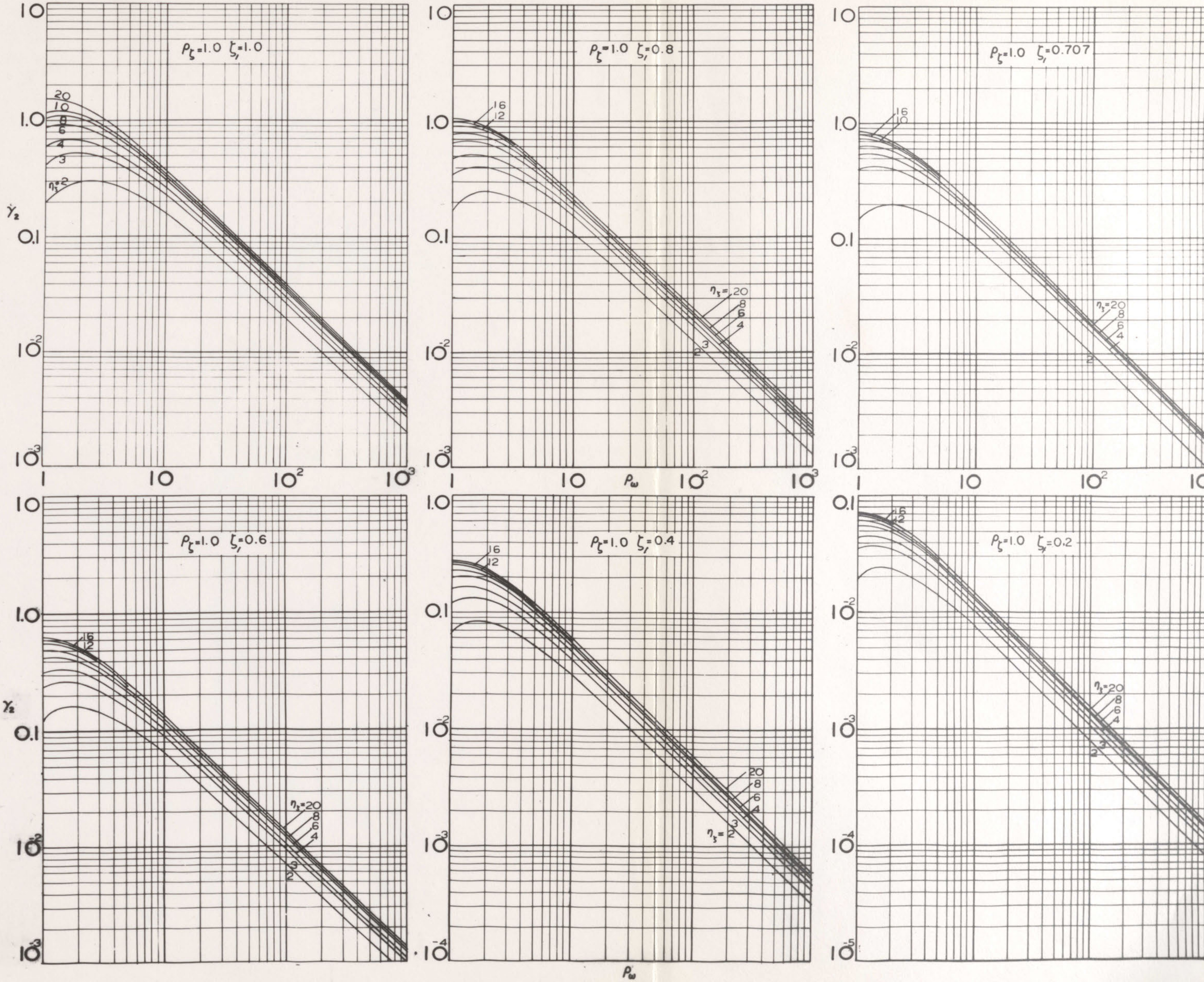


FIG. XII D $\gamma_1 - \gamma_2$ COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY

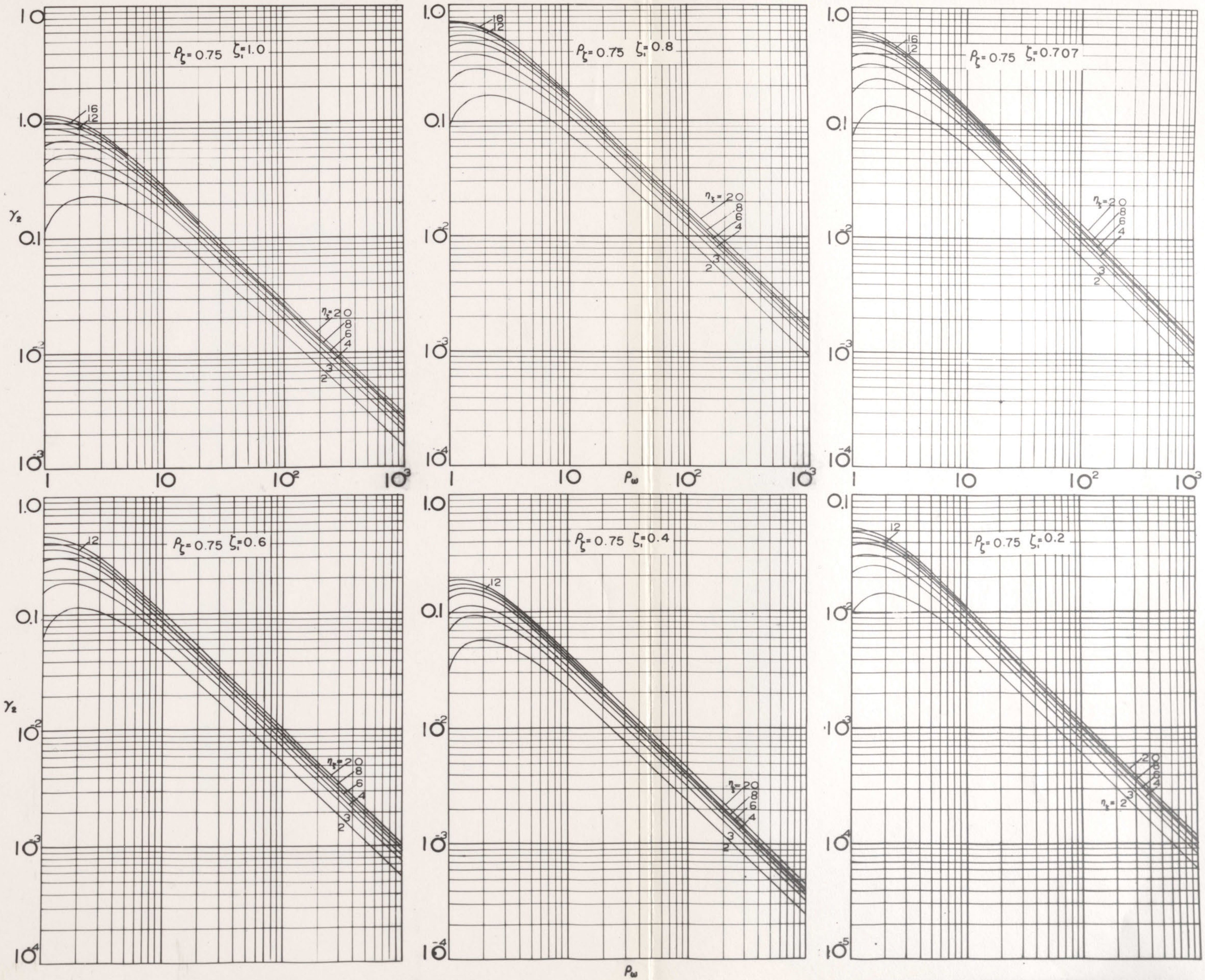
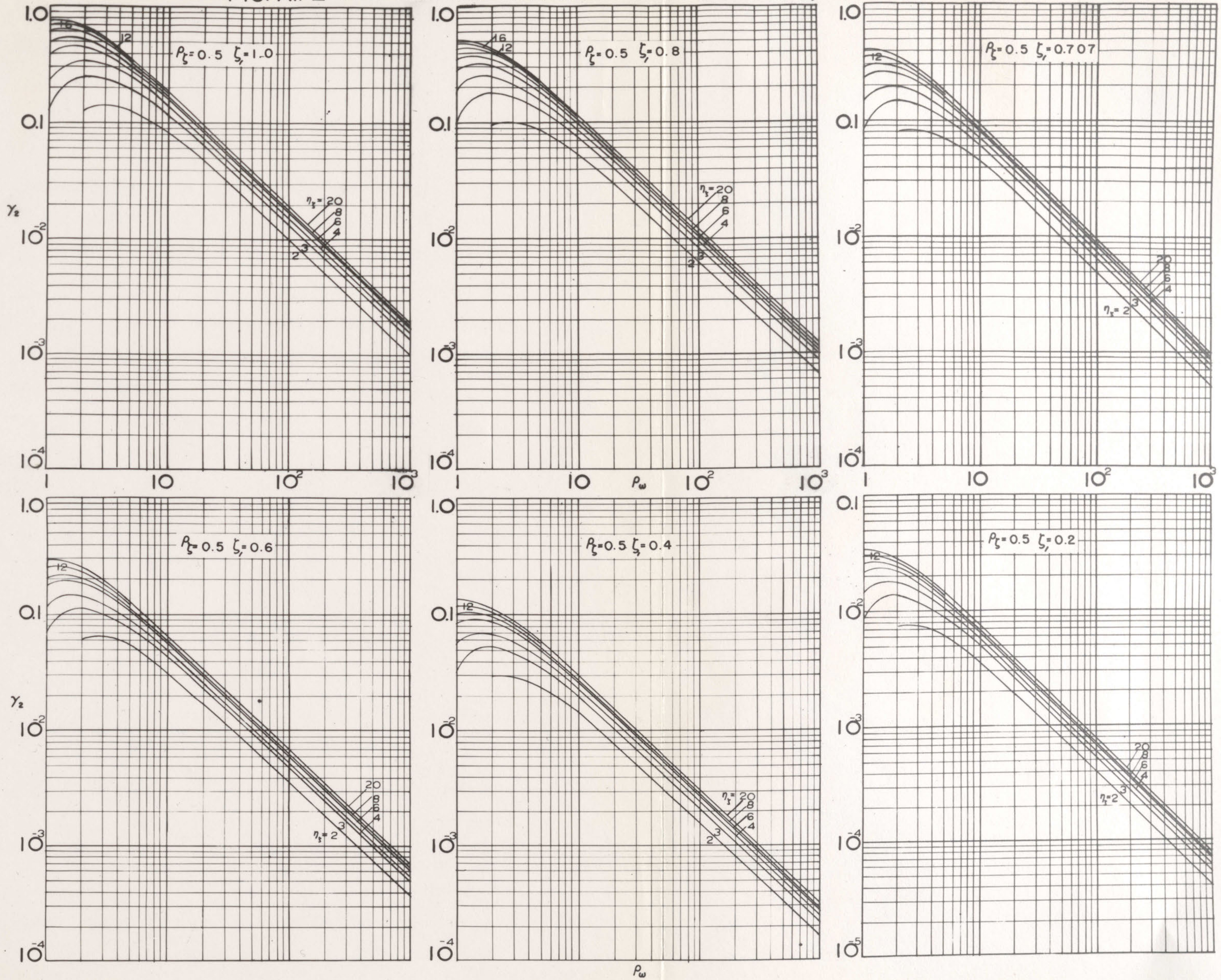


FIG. XII E $\gamma_1 - \gamma_2$ COMPOUND CONTROLS DAMPING & DAMPING RATIO IMPROVING ONLY



With known specification, ρ_ω, η_ζ, and ρ_ζ, $\rho_{\zeta o}$ can be found from Fig. 10, (Eq. 3.80), γ_1 from Fig. 11 (Eq. 3.81), and γ_2 from Figs. 12A to 12E.

From these figures, it is seen that the positive acceleration coupling is required together with a positive velocity coupling. With high frequency control, analysis has been made for singly coupled control and the results are plotted in Fig. 7A-D. Slight positive acceleration-coupling means slight reduction of undamped natural frequency of the controlled member and slight variation of the damping ratio. If the system is not strictly restricted to constant ω_1, neglect of γ_2, determined by Fig. 12, in most cases would act in favor of improving damping. Of course, if γ_2 is neglected, and γ_1 determined by Fig. 11 is applied along, the quadratic factors of the quartic equation are to be determined according to the method described in Part II, in order to get the true improvement of damping ratio, frequency ratio, etc.

36. Constant ω_1, First-and-Third Derivative Coupling Compound Controls

It is well known that $\frac{\alpha_3}{\alpha_1}$ affects the distribution of damping ratio between the high and low frequency components. If $\frac{\alpha_{30}}{\alpha_{10}}$, when the control is idle, deviates from unity, the first derivative and third derivative couplings may be applied compoundly in such a way that tends to make $\frac{\alpha_3}{\alpha_1}$ more or less equal to unity. The effect of third derivative coupling should be much more prominent than that due to acceleration coupling so far as improvement of the distribution of the damping ratio distribution is concerned. The coefficients of the standard quartic equation of such compound controls are

$$\begin{aligned} \alpha_3 &= \alpha_{30}\,(1+\gamma_3) \\ \alpha_2 &= \alpha_{20} \\ \alpha_1 &= \alpha_{10}\,(1+\gamma_1) \end{aligned} \tag{3.83}$$

in which

$$\rho_\omega + \frac{1}{\rho_\omega} + 4\zeta_1^2 \rho_\zeta = \alpha_2 = \alpha_{20} = \rho_\omega + \frac{1}{\rho_\omega} + 4\zeta_0^2 \rho_{\zeta 0} \qquad (\because \omega_1 = \text{Const.})$$

$$\therefore \quad \underline{\rho_{\zeta 0} = \rho_\zeta \eta_\zeta^2} \tag{3.84}$$

$$2\zeta_1\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega}\right) = \alpha_3 = \alpha_{30}(1+\gamma_3) = 2\zeta_0\left(\frac{1}{\sqrt{\rho_\omega}} + \rho_{\zeta 0}\sqrt{\rho_\omega}\right)(1+\gamma_3)$$

$$\text{or} \quad \underline{\gamma_3 = \left(\frac{\frac{1}{\rho_\omega} + \rho_\zeta}{\frac{1}{\rho_\omega} + \rho_\zeta \eta_\zeta^2}\right)\eta_\zeta - 1} \tag{3.85}$$

and

$$2\zeta_1\left(\sqrt{\rho_\omega} + \frac{\rho_\zeta}{\sqrt{\rho_\omega}}\right) = \alpha_1 = \alpha_{10}(1+\gamma_1) = 2\zeta_0\left(\sqrt{\rho_\omega} + \frac{\rho_{\zeta 0}}{\sqrt{\rho_\omega}}\right)(1+\gamma_1)$$

or

$$\underline{\gamma_1 = \eta_\zeta\left(\frac{\rho_\omega + \rho_\zeta}{\rho_\omega + \rho_\zeta \eta_\zeta^2}\right) - 1} \tag{3.86}$$

Eqs. (3.84), (3.85) and (3.86) are plotted as Fig. 13, Fig. 14, and Fig. 15, respectively. The leading parameter ρ_ζ is taken below unity, otherwise the damping ratio in the control would be too large. It is interesting to note that normally γ_3 needs negative sign, while γ_1 needs a positive one. But when ρ_ζ is too small, positive γ_3 is required for controls with moderate high frequency.

Even at normal ρ_ζ, negative γ_1 is required for controls with $\rho_\omega < 10$. It is hard to accept this result from the ordinary point of view that a negative velocity coupling could improve damping, but if the idle damping ratio of the control

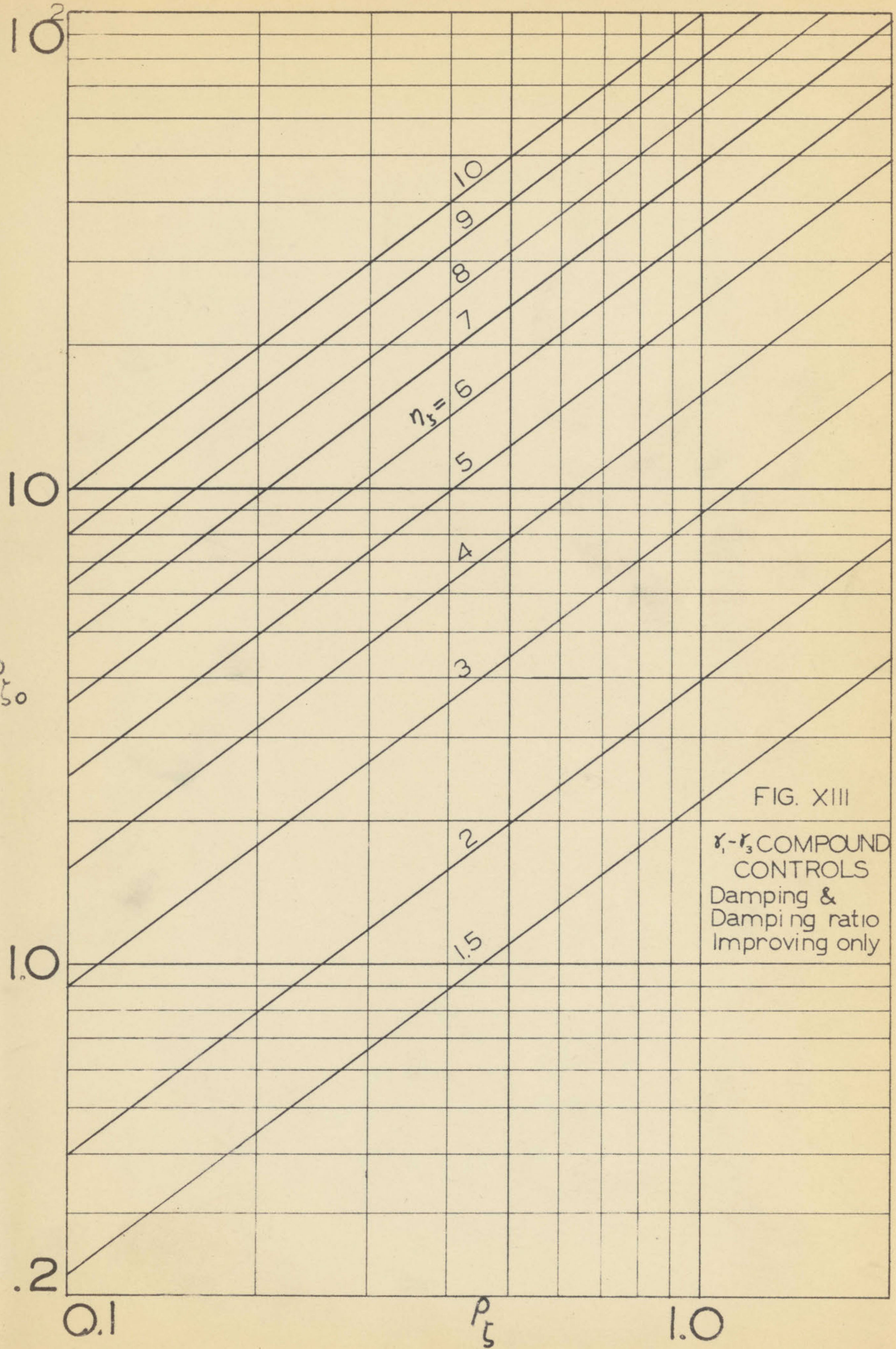


FIG. XIII

$\gamma_1 - \gamma_3$ COMPOUND CONTROLS
Damping & Damping ratio Improving only

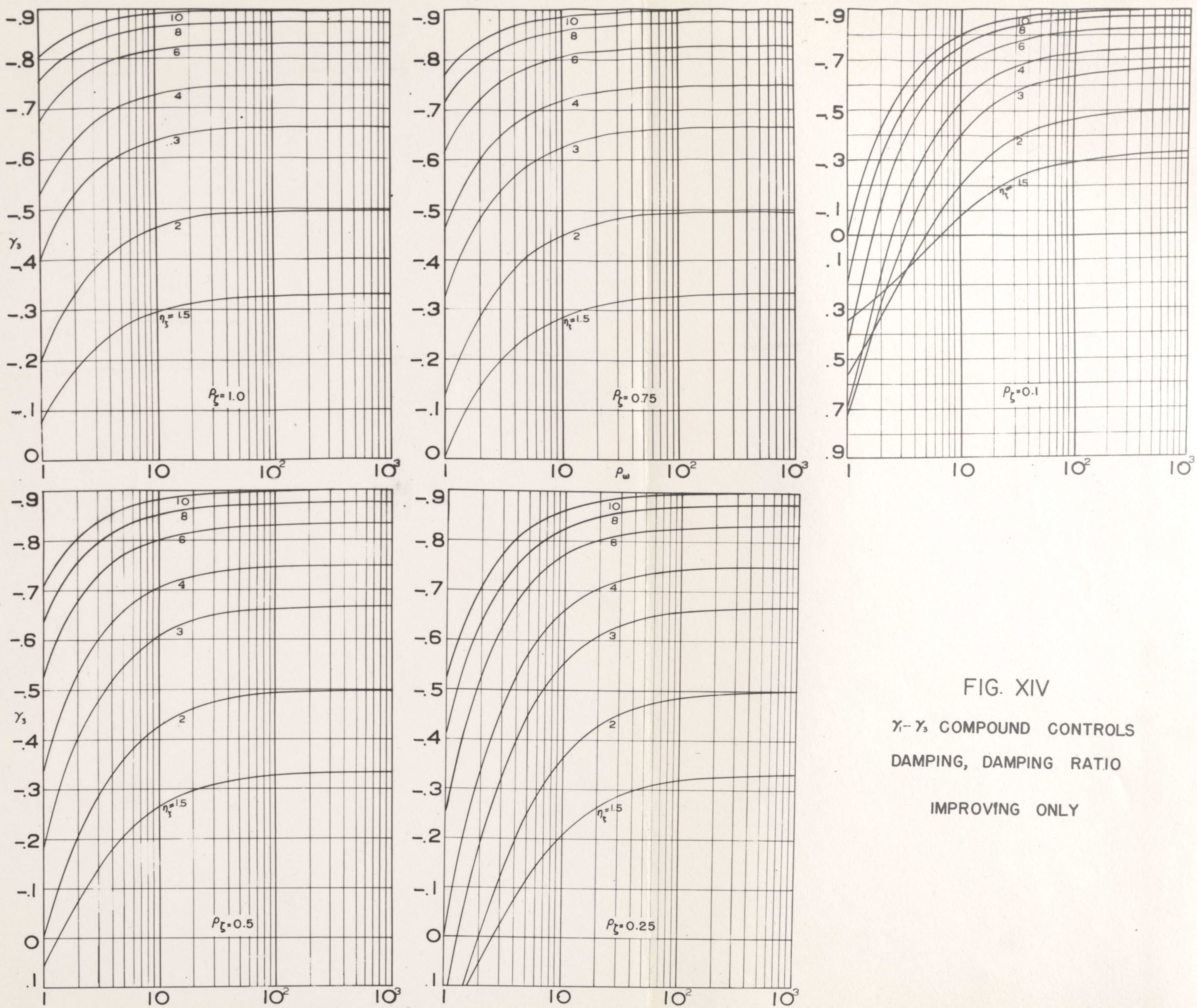


FIG. XIV

$\gamma_1 - \gamma_3$ COMPOUND CONTROLS

DAMPING, DAMPING RATIO

IMPROVING ONLY

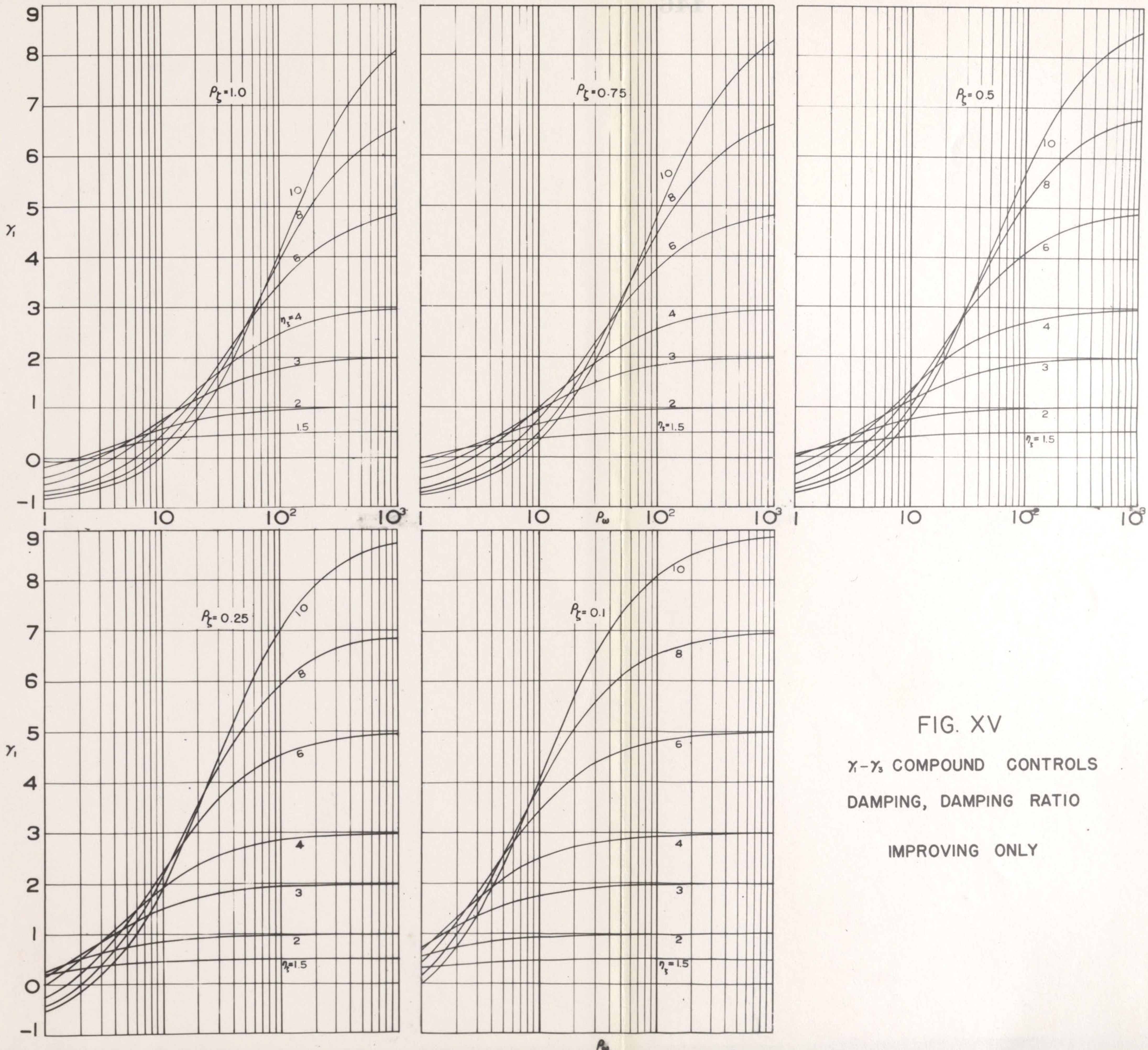


FIG. XV

$\gamma_1 - \gamma_3$ COMPOUND CONTROLS

DAMPING, DAMPING RATIO

IMPROVING ONLY

is considered, which should be high for such compound control, it can be understood that the improvement of damping of the controlled member is at the expense of decreasing the damping ratio of the control component.

CHAPTER TEN

TUNING CONTROLS

37. Tuned System and Tuning Control

A system, when controlled by a nonideal control, singly or compoundly coupled, gives two identical quadratic factors to the quartic equation of the stability function of the system which is then defined as a tuned system. Such control is defined as the tuning control of the system. The damping ratio of the quadratic factor shall be called tuning ratio, ζ_t. If ζ_t is equal to one, the system is said to be critically tuned. If $\zeta_t > 1$, it is overtuned, and if $\zeta_t < 1$, it is undertuned.

38. Tuning Control with Single Coupling and Theory of Tuning

It is more practical to design a singly coupled control than a compound one. If that control can be made a tuning one at desirable tuning ratio, it would receive more application than any other type of control. The following analysis verifies the possibility of such a tuning control.

It is believed that error-velocity coupling gives best results in improving the damping of the system. It is our purpose, therefore, to develop the theory for such a type of control in different degrees of tuning.

Let us assume:

ζ_o = damping ratio of the member to be controlled.

ζ_c = damping ratio of the control.

$\rho_{\zeta o}$ = ratio of damping ratios between the control and the member to be controlled.

$\rho_{\omega o} = \dfrac{\omega_{nc}}{\omega_{no}}$ = ratio of undamped natural frequencies between the control and the controlled member.

γ_1 = error-velocity coupling coefficient.

Γ_1 = dimensionless error-velocity coupling factor $\left(\Gamma_1 = \alpha_{10} \gamma_1 \rho_{\omega o}^{3/2} \right) = \dfrac{B_1'}{\omega_o^3}$

Also assume:

(1) The two quadratic factors are only of the same frequency so that $\rho_\omega = 1.0$.

(2) The damping ratios of the two components are ζ_1 and ζ_2 which may or may not be equal to each other.

(3) If $\zeta_1 = \zeta_2$, their designation is ζ_t, and sufficient conditions will be developed thereon.

It is understood that:

$$\alpha_3 = 2(\zeta_1 + \zeta_2); \qquad \alpha_{30} = 2\,\zeta_o \left(\frac{1}{\sqrt{\rho_{\omega o}}} + \rho_{\zeta o} \sqrt{\rho_{\omega o}} \right) \tag{3.87a}$$

$$\alpha_2 = 2 + 4\zeta_1\zeta_2; \qquad \alpha_{20} = \frac{1}{\rho_{\omega o}} + \rho_{\omega o} + 4\zeta_o^2 \rho_{\zeta o} \tag{3.87b}$$

$$\alpha_1 = \alpha_{10}(1 + \gamma_1); \qquad \alpha_{10} = 2\,\zeta_o \left(\frac{\rho_{\zeta o}}{\sqrt{\rho_{\omega o}}} + \sqrt{\rho_{\omega o}} \right) \tag{3.87c}$$

By equating $\alpha_3 = \alpha_{30}$, and $\alpha_2 = \alpha_{20}$, we have

$$\zeta_1 + \zeta_2 = \zeta_o \left(\frac{1}{\sqrt{\rho_{\omega o}}} + \rho_{\zeta o} \sqrt{\rho_{\omega o}} \right) \tag{3.88a}$$

$$2 + 4\zeta_1\zeta_2 = \frac{1}{\rho_{\omega o}} + \rho_{\omega o} + 4\zeta_o^2 \rho_{\zeta o} \tag{3.88b}$$

By solving Eqs. (3.88)a and (3.88)b for ζ_1 and ζ_2 , we have

$$\frac{\zeta_1}{\zeta_0},\ \frac{\zeta_2}{\zeta_0} = \frac{1}{2}\left[\frac{1}{\sqrt{\rho_{\omega 0}}} + \rho_{\zeta 0}\sqrt{\rho_{\omega 0}} \pm \left(\frac{1}{\sqrt{\rho_{\omega 0}}} - \rho_{\zeta 0}\sqrt{\rho_{\omega 0}}\right)\sqrt{1 - \frac{1}{\zeta_0^2}\left(\frac{\rho_{\omega 0} - 1}{\rho_{\zeta 0}\rho_{\omega 0} - 1}\right)^2} \qquad (3.89)$$

When $|\zeta_c \rho_{\omega 0} - \zeta_0| > |(\rho_{\omega 0} - 1)|$ the radical is real. With proper adjustment of γ_1 the controlled results would be two components of motion of the same frequency, but with different damping ratios. When $|\zeta_c \rho_{\omega 0} - \zeta_0| < |(\rho_{\omega 0} - 1)|$ the radical is imaginary, with proper adjustment of γ_1 the controlled results would be two components of the same frequency, but with complex damping ratio conjugate to one another. The statement is mathematically correct, but what is the physical meaning of such a pair of components of motion?

With some algebraic juggling the above statement can be changed to: the controlled results would be two components of different frequency, but with the same damping ratio. Let

$$S(\lambda) = \left[\lambda^2 + 2(a + ib)\lambda + 1\right]\left[\lambda^2 + 2(a - ib)\lambda + 1\right] \qquad (3.90)$$

$$\lambda_{1,2} = -(a + ib) \pm \sqrt{a^2 - b^2 + 2iab - 1} = -(a + ib) \pm (c + id) \qquad (3.91)a$$

$$\lambda_{3,4} = -(a - ib) \pm \sqrt{a^2 - b^2 - 2iab - 1} = -(a - ib) \pm (c - id) \qquad (3.91)b$$

where a, b, c, d, are real positive quantities and

$$(c + id)^2 = a^2 + b^2 - 1 + 2iab$$

or $$(c - id)^2 = a^2 + b^2 - 1 - 2iab$$

or $$c^2 - d^2 = a^2 + b^2 - 1, \text{ and } \underline{cd = ab} \qquad (3.92)$$

On developing,

$$\lambda_1 = -a + c - i(b - d)$$

$$\lambda_2 = -(a + c) - i(b + d)$$

$$\lambda_3 = -(a + c) + i(b - d)$$

$$\lambda_4 = -(a + c) + i(b + d)$$

Rearrange the sequence so that

$$S(\lambda) = \left[(\lambda-\lambda_1)(\lambda-\lambda_3)\right]\left[(\lambda-\lambda_2)(\lambda-\lambda_4)\right]$$
$$= \left[\lambda^2+2(c-a)\lambda + (c-a)^2 + (b-d)^2\right] \times$$
$$\left[\lambda^2+2(a+c)\lambda + (a+c)^2 + (b+d)^2\right] \qquad (3.93)$$

$$\therefore \qquad \rho_\omega = \left[\frac{(a+c)^2+(b+d)^2}{(c-a)^2+(b-d)^2}\right]^{\frac{1}{2}} \neq 1 \qquad (3.94)$$

It is necessary to prove

$$\zeta_1' = \frac{c-a}{\sqrt{(c-a)^2+(b-d)^2}} = \frac{a+c}{\sqrt{(a+c)^2+(b+d)^2}} = \zeta_2' \qquad (3.95)$$

or the same thing

$$\sqrt{\zeta_1'^2 - 1} = \frac{b-d}{c-a} = \frac{b+d}{c+a} = \sqrt{\zeta_2'^2 - 1} \qquad (3.95)a$$

Finally, the above equation can be reduced to cd = ab which is fundamentally true as indicated by Eq. (3.92). Therefore both Eqs. (3.94) and (3.95) are true, so physically the controlled results would consist of two components of different frequency, but with the same damping ratio.

When $\zeta_c \rho_{\omega o} - \zeta_o = \pm(\rho_{\omega o} - 1)$ (3.96)

the radical is zero, therefore, at that condition, $\zeta_1 = \zeta_2 = \zeta_t$ if the coupling coefficient γ_1 is properly adjusted, and

$$\zeta_t = \frac{1}{2}\zeta_o\left(\frac{1}{\sqrt{\rho_{\omega o}}} + \rho_{\zeta o}\sqrt{\rho_{\omega o}}\right) = \frac{1}{2}\left(\frac{\zeta_o}{\sqrt{\rho_{\omega o}}} + \zeta_c\sqrt{\rho_{\omega o}}\right) \qquad (3.97)$$

39. Tuning Control With Positive Error Velocity Coupling

Take the positive sign of Eq. (3.96) and solve for ζ_c

$$\zeta_c = 1 - \frac{1-\zeta_o}{\rho_{\omega o}} \qquad (3.98)$$

Squaring both sides of Eq. (3.97), we have

$$4\zeta_t^2 = \frac{\zeta_o^2}{\rho_{\omega o}} + \zeta_c^2 \rho_{\omega o} + 2\zeta_o \zeta_c \tag{3.99}$$

Substitute ζ_c of Eq. (3.98) into Eq. (3.99) and we have

$$\rho_{\omega o}^2 - 2(1 + 2\zeta_t - 2\zeta_o)\rho_{\omega o} + (1 - 2\zeta_o)^2 = 0$$

$$\therefore \quad \underline{\rho_{\omega o} = 1 + 2\zeta_t - 2\zeta_o + 2\zeta_t\sqrt{1 + \zeta_t^2 - 2\zeta_o}} \tag{3.100}$$

It is seen from Eq. (3.100) that the condition

$$1 + \zeta_t^2 - 2\zeta_o > 0 \tag{3.101}$$

must be fulfilled in order to get a real value of $\rho_{\omega o}$. The negative sign before the radical of Eq. (3.100) has been discarded, which not only gives negative value to $\rho_{\omega o}$ in most cases, but also yeilds negative value to ζ_c which is impossible to get from ordinary mechanical nonideal controls.

In Section 12, Chapter IV, it has been mentioned that when $\frac{\alpha_3}{\alpha_1} = 1$, it is possible that:

(C1) $\rho_\zeta = 1$ at any value of ρ_ω

(C2) $\rho_\omega = 1$ at any value of ρ_ζ

or (C3) $\rho_\zeta = \rho_\omega = 1$

Apparently the first case is only possible when

$$|\zeta_c \rho_{\omega o} - \zeta_o| < |\rho_{\omega o} - 1|$$

and the second case is only possible when

$$|\zeta_c \rho_{\omega o} - \zeta_o| > |\rho_{\omega o} - 1|$$

It is the third case that is held true by the condition

$\zeta_c \rho_{\omega o} - \zeta_o = \pm(\rho_{\omega o} - 1)$ (for the time being + sign is used)

Therefore,

$\frac{\alpha_3}{\alpha_1}$ must be unity, or

$$\frac{\alpha_{30}}{\alpha_{10}(1+\gamma_1)} = 1$$

or $$\alpha_{10}\gamma_1 = \alpha_{30} - \alpha_{10} \qquad (3.102)$$

Therefore $$\Gamma_1 = \frac{B_1'}{\omega_0^3} = \alpha_{10}\gamma_1 \rho_{\omega o}^{3/2} \qquad (3.102)$$

Substitute the value of α_{30} and α_{10} of Eqs. (3.87)a and (3.87)b into equation (3.102) and simplify the result by substituting $1 - \frac{1-\zeta_o}{\rho_{\omega o}}$ for ζ_c. The result will be

$$\Gamma_1 = 2(\rho_{\omega o} - 1)^2 (1 - \zeta_o)^2 \qquad (3.103)$$

where $\rho_{\omega o}$ is the value obtained from Eq. (3.100). This value of Γ_1, when multiplied by ω_o^3, gives the value of B_1' defined as error-velocity coupling factor in Eq. (2.13)c.

Eq. (3.100) is plotted as Fig. 16 with $\rho_{\omega o}$ as ordinate against ζ_o as abscissa with ζ_t as varying parameter (range plotted ζ_t = .4 -- 1.1). It is clearly seen that a constant ζ_t curve turns back when it reaches its furthermost point to the right. Beyond that furthermost ζ_o, it is impossible to get a ratio of tuning below that particular ζ_t. If the negative sign before the radical of Eq. (3.100) is retained, the curve will have its turned back position as shown by the two dotted branches. The locus of these horizontal vertices can be represented by a straight line with Eq. $\rho_{\omega o} = 2\zeta_o - 1$ as shown by the dotted one.

From the relation

$A_{oo}' = \rho_{\omega o}^2 \omega_o^4 = \rho_\omega^2 \omega_1^4 = A_o'$ for the constant term of the quartic equation

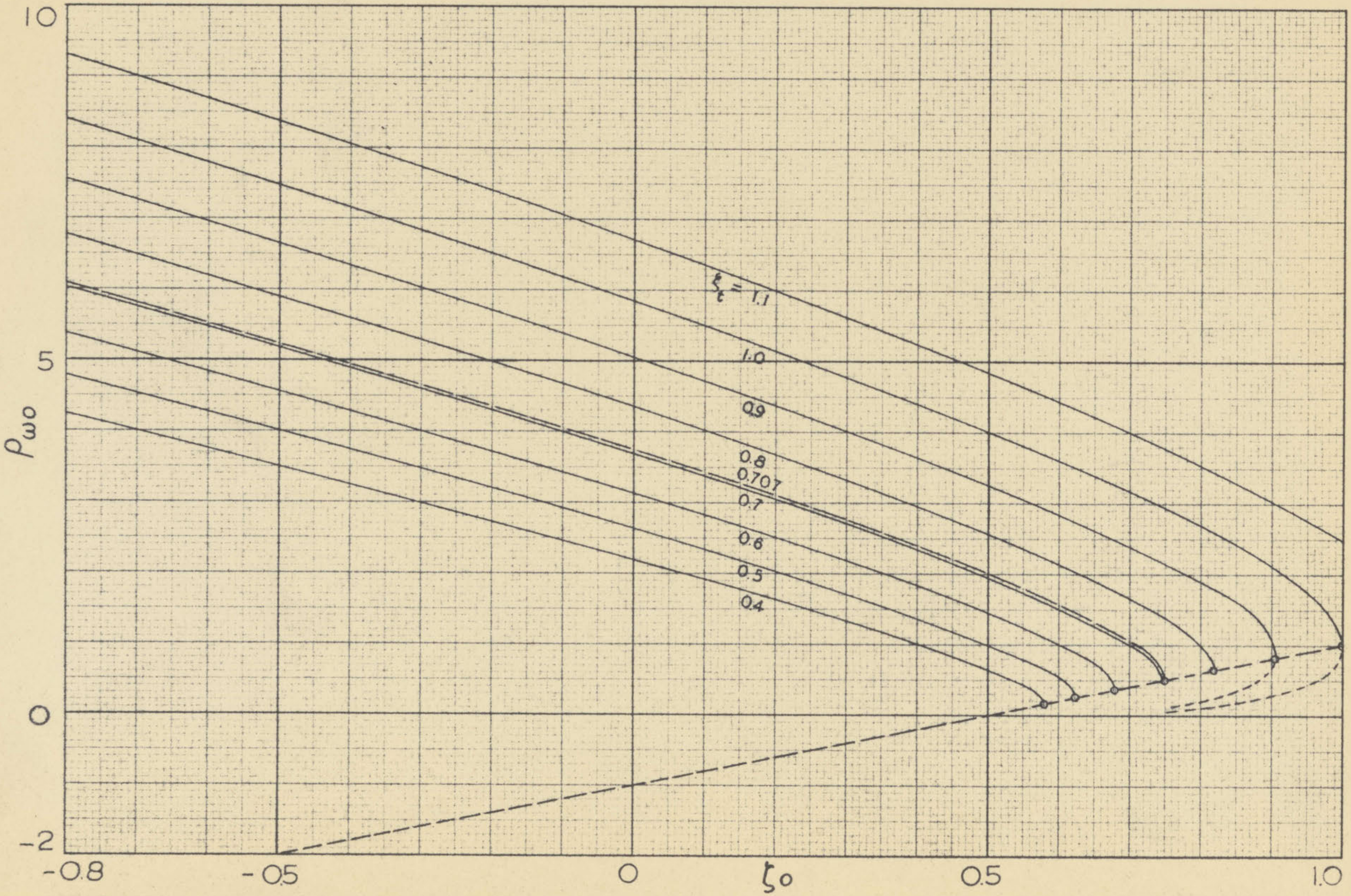


FIG. XVI DESIGN DATA OF TUNING CONTROLS
(+Γ_1 COUPLING) $\rho_{\omega o}$ VS. ζ_o AT VARIOUS ζ_t, ζ_t = TUNING RATIO

FIG. XVII DESIGN DATA OF TUNING CONTROLS ($+\Gamma$ COUPLING)

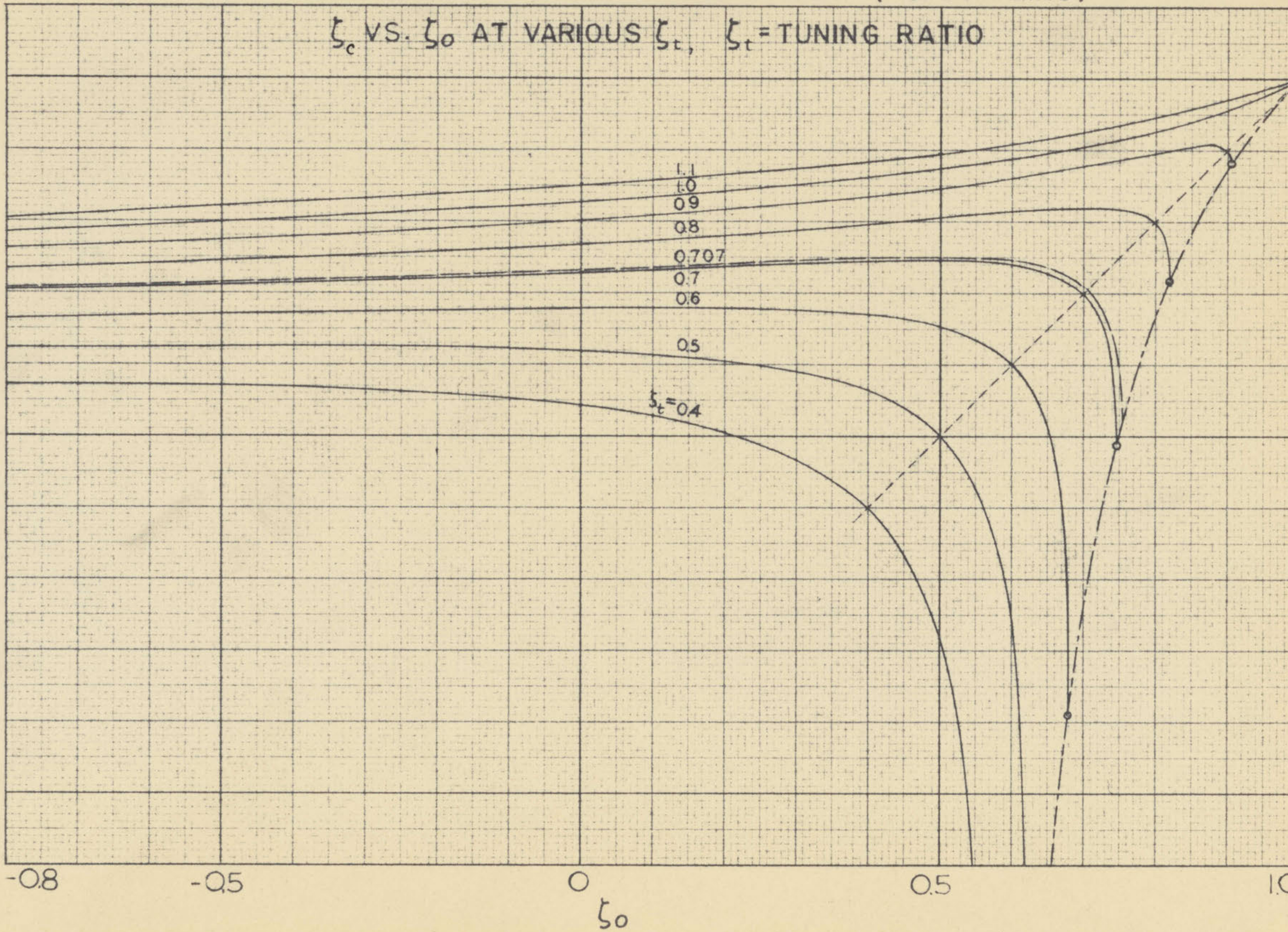
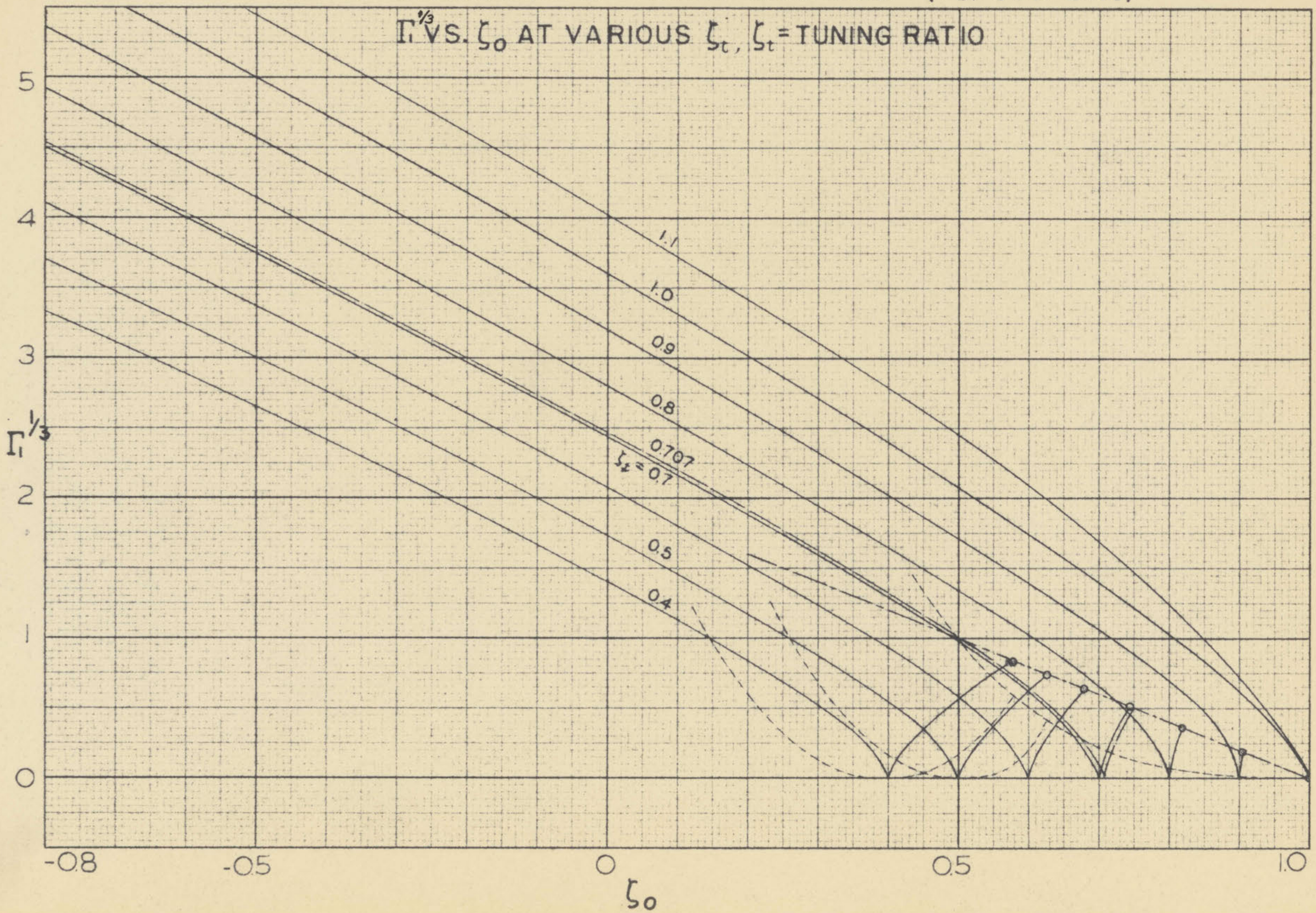


FIG. XVIII DESIGN DATA OF TUNING CONTROLS (+Γ_1 COUPLING)



$\eta_\omega = \frac{\omega_1}{\omega_0} = \sqrt{\rho_{\omega 0}}$ because $\rho_\omega = 1.0$

As all $\rho_{\omega 0}$'s are seen to be greater than 1.0, so $\eta_\omega > 1$, which means that such a tuned system possesses a favorable η_ω in improving damping. The conclusion may also be reached that if the member to be tuned is originally overdamped (that is, $\zeta_0 > 1.0$) it is impossible to tune this system back to $\zeta_t < 1.0$ by using a control having $\rho_{\omega 0} > 1.0$.

A mechanical member of 1 degree of freedom cannot possess negative damping. However, the exploration of Eq. (3.100) has been extended to negative ζ_0. The adequacy of such an extension will find its utility in a system of two-degree-of-freedom which is unstable without being controlled. Therefore, even an airplane which shows longitudinal instability on free flight can be tuned to have ζ_t if originally the frequency ratio between the two uncontrolled components is favorable for doing so.

Eq. (3.98) is plotted as Fig. 17 with ζ_c as ordinate against ζ_0 as abscissa with ζ_t as varying parameter. From Fig. 17 it is interesting to notice when $\rho_{\omega 0}$ reaches its horizontal vertex on Fig. 16 where ζ_c becomes tangential to the vertical line on Fig. 17. In approaching this region the damping ratio of the control is too sensitive to the damping ratio of the controlled member so that the calculated ζ_c may not be at tuned condition if the determination of ζ_0 is slightly in error. When $\rho_{\omega 0} = 1$, it corresponds to $\zeta_c = \zeta_0$ in Fig. 17 shown by the dotted 45° line. To the left of this line it corresponds to $\rho_{\omega 0} > 1$; to the right, $\rho_{\omega 0} < 1$. It is therefore seen that when $\rho_{\omega o} > 1$, ζ_c is very stable with respect to ζ_0, which means that with small error in determination of ζ_0, the calculated value of ζ_c for the tuned condition does not vary considerably.

In general, within the operating range the damping ratio of the control necessary for the tuned condition never exceeds 1.0; yet in some cases the tuned result is amazing in that the tuning ratio ζ_t is greater than either ζ_c or ζ_0. In common language, such an amazing case may be stated thus: that an oscillatory member can be tuned to give less or nonoscillatory motion by an oscillatory control.

Eq. (3.103) is plotted as Fig. 18 with $\Gamma_1^{1/3}$ as ordinate against ζ_0 as abscissa. When $\Gamma_1 = 0$ naturally $\rho_{\omega o}$ for tuning must be 1.0 and $\zeta_c = \zeta_0$; and the cusps of the solid curves form the locus $\Gamma_1 = 0$. If Γ_1 is plotted instead of $\Gamma_1^{1/3}$ there are no cusps, but continuous curves with minima at the cusps as shown by the dotted curves for part of $\zeta_t = 0.4$ and $\zeta_t = 0.5$. The dot-dash straight line on Fig. 18 is equivalent locus through the vertices of ζ_t = constant on Fig. 16. Such a dot-dash line can be represented by Eq. $\Gamma^{1/3} = 2 - 2\zeta_0$ (or $\Gamma_1 = 8(1 - \zeta_0)^3$ if Γ_1 is plotted instead of $\Gamma^{1/3}$).

It is evident that keeping the condition $\zeta_c \rho_{\omega o} - \zeta_0 = +\rho_{\omega o}^{-1}$ for the tuning, positive coupling (factor, dimensional or nondimensional) is required within the working range that $\rho_{\omega o}$ is greater than one and the control must be underdamped.

40. Tuning Controls with Negative Error-Velocity Coupling

By doing the same algebraic work as has been done in Section 39, only with

$$\zeta_c \rho_{\omega o} - \zeta_o = -(\rho_{\omega o} - 1) \qquad (3.104)$$

instead of

$$\zeta_c \rho_{\omega o} - \zeta_o = +(\rho_{\omega o} - 1) \qquad (3.98)$$

the following results are obtained

$$\zeta_c = -1 + \frac{1 + \zeta_o}{\rho_{\omega o}} \qquad (3.105)$$

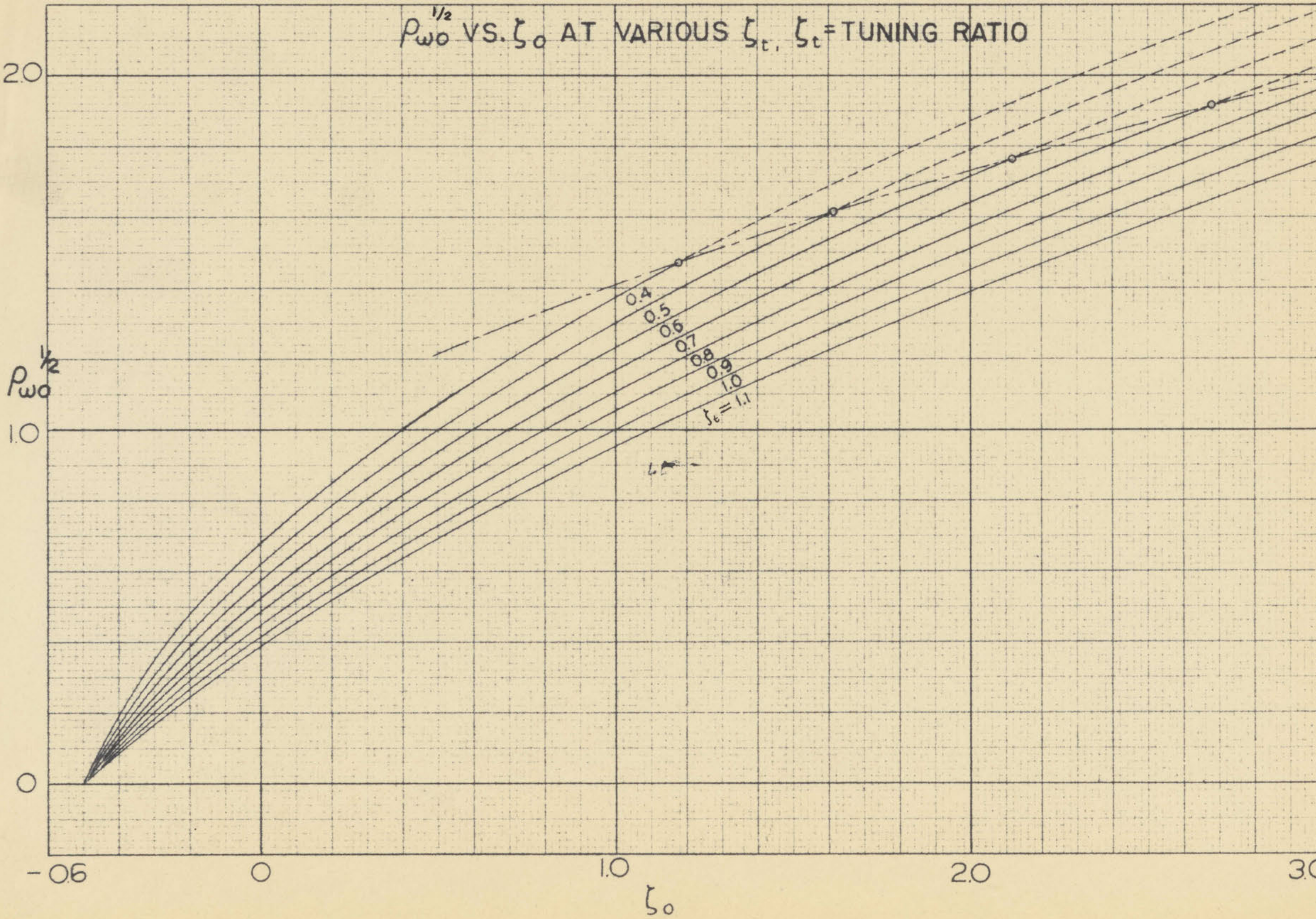
$$\rho_{\omega o} = 1 + 2\zeta_c^2 + 2\zeta_o - 2\zeta_c\sqrt{1 + \zeta_c^2 + 2\zeta_o} \qquad (3.106)$$

$$\Gamma_1 = -(1 + \zeta_o)(1 - \rho_{\omega o})^2 \qquad (3.107)$$

Apparently the + sign before the radical in Eq. (3.106) has been discarded. It is because otherwise the damping ratio of the control needs to be negative which is impossible from ordinary mechanical nonideal controls.

Eq. (3.106) is plotted as Fig. 19 with $\rho_{\omega o}^{1/2}$ as ordinate against ζ_o as abscissa. To tune a member with $\zeta_o = -0.5$ a control with no stiffness should be used; that is, $\rho_{\omega o} = 0$. Beyond $\zeta_o = -0.5$ ($\zeta_o < -0.5$) negative $\rho_{\omega o}$ is needed which has no physical significance. From Fig. 19 it is evident that no matter how much (+) damping is possessed by the controlled member, it is always possible to be tuned back to ζ_t of any + magnitude. The dot-dash locus represents the bound-

FIG. XIX DESIGN DATA OF TUNING CONTROLS (−Γ COUPLING)



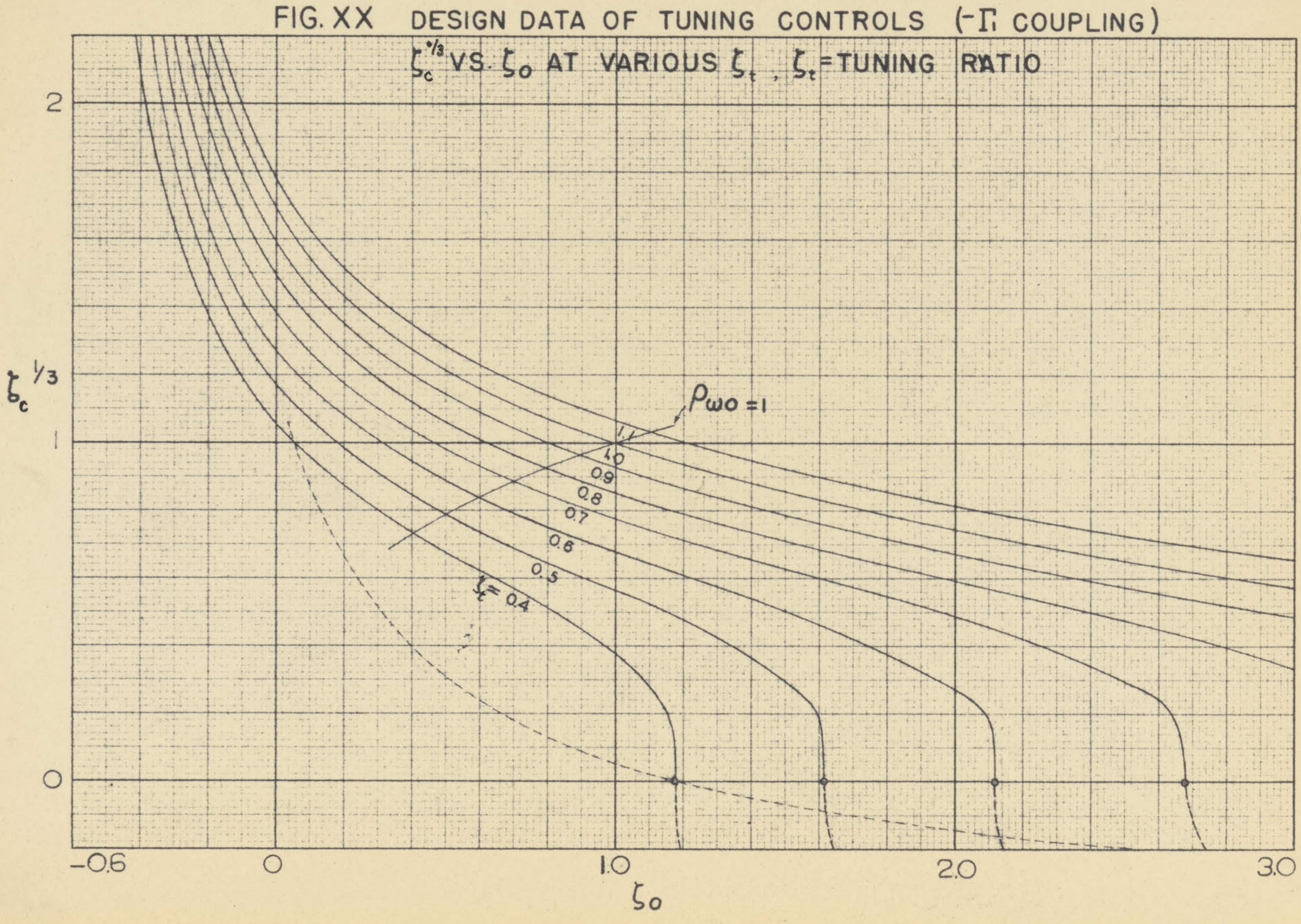
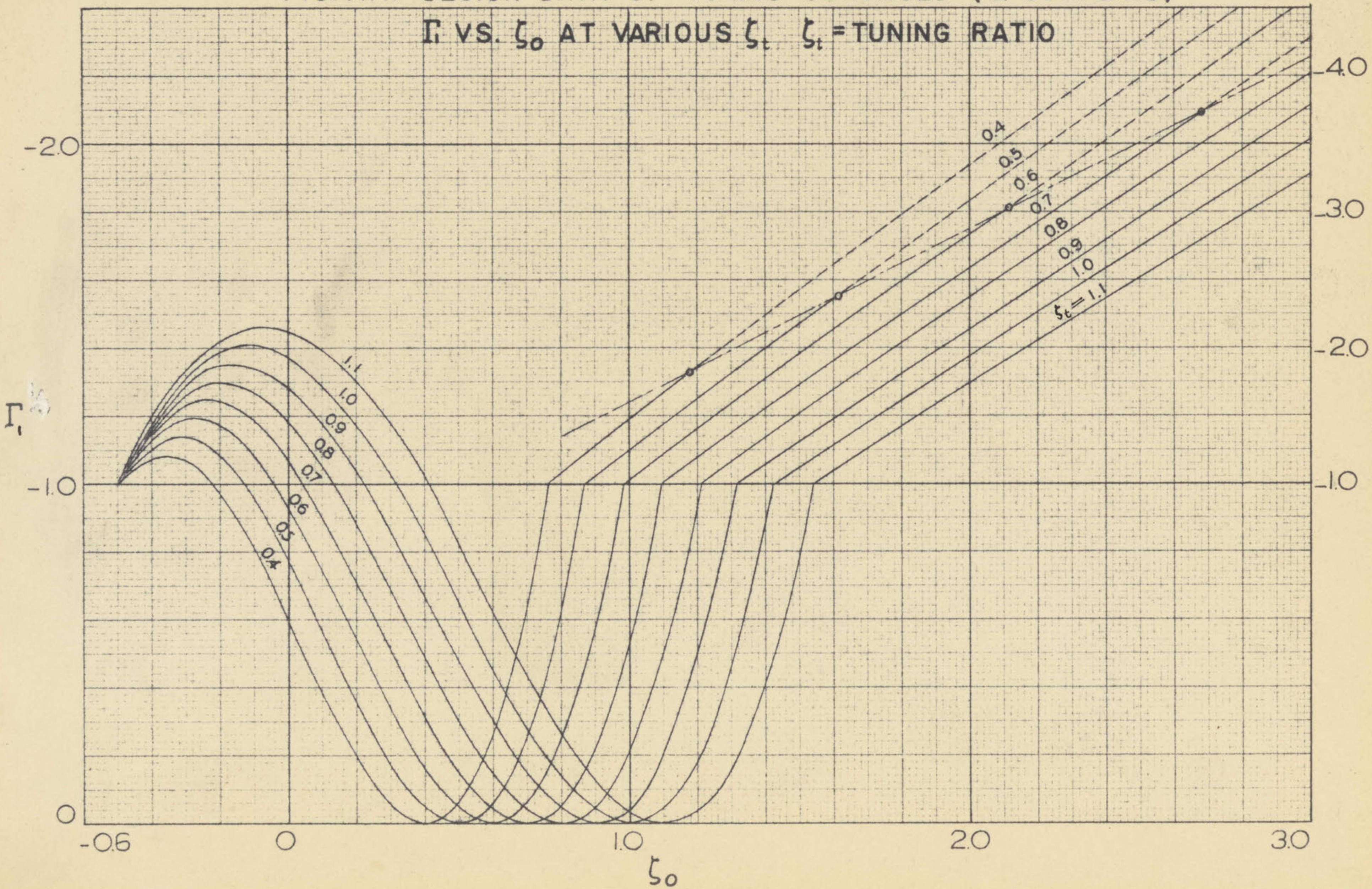


FIG. XX DESIGN DATA OF TUNING CONTROLS (-Γ COUPLING)
$\zeta_c^{1/3}$ VS. ζ_o AT VARIOUS ζ_t, ζ_t=TUNING RATIO
2
$\zeta_c^{1/3}$
1
0
$\rho_{\omega o}=1$
1.1
1.0
0.9
0.8
0.7
0.6
0.5
ζ_t= 0.4
-0.6
0
1.0
2.0
3.0
ζ_o

FIG. XXI DESIGN DATA OF TUNING CONTROLS ($-\Gamma_1$ COUPLING)



ary of $\zeta_c = 0$ beyond which negative ζ_c is required which is impossible from ordinary mechanical control, but possible with a vacuum tube circuit as the control element. The dotted curves indicate negative damping controls are required for tuning.

Eq. (3.105) is plotted as Fig. 20 with $\zeta_c^{1/3}$ as ordinate against ζ_o as abscissa with ζ_t as varying parameter. When $\zeta_o > 1.0$, tuning back to $\zeta_t < 1.0$ needs a control of $\rho_{\omega o} > 1.0$ and $\zeta_c < 1.0$. Such requirement is quite normal. To the left of $\rho_{\omega o} = 1$, $\rho_{\omega o}$ is less than 1.0; high damping ratio in control is therefore needed for tuned results. However, the damping coefficient is not excessive because of the small value of $\rho_{\omega o}$. When $\rho_{\omega o}$ is small or negative, ζ_c is too sensitive to be tuned with ζ_o or the tuned condition is not very stable in such a region. The abrupt inflection along constant ζ_t is caused by the scale of $\zeta_c^{1/3}$. If ζ_c is plotted as ordinate instead of $\zeta_c^{1/3}$, such inflection is missed and the curve is continuous at $\zeta_c = 0$ as shown by the dotted curve for $\zeta_t = 0.4$. Below $\zeta_c = 0$, only electric tuning control may possibly be practical.

Eq. (3.107) is plotted as Fig. 21 with Γ_1 as ordinate against ζ_0 as abscissa with ζ_t as varying parameter. $\Gamma_1^{1/3}$ is plotted instead of Γ_1 when $-\Gamma_1 > 1.0$ and $\rho_{\omega o} > 1.0$. It is seen that no matter how large the value of the $\rho_{\omega o}$ used, a negative coupling factor (nondimensional or dimensional) is needed for the error-velocity coupled control in tuned condition. The dot-dash locus separates the mechanical nonideal control below and electric nonideal control above (shown by dotted curves.)

41. Comparison Between $+\Gamma_1$ and $-\Gamma_1$ Tuning Controls

It is much simpler to make a table (as follows) for the comparison of the $+\Gamma_1$ and $-\Gamma_1$ tuning controls.

TABLE VI

Comparison Between $+\Gamma_1$ and $-\Gamma_1$ Tuning Controls

Characteristics	$+\Gamma_1$ Tuning Control	$-\Gamma_1$ Tuning Control
1. What makes the consistency of the sign	$\zeta_c \rho_{\omega o} - \zeta_o = +(\rho_{\omega o} - 1)$; $\alpha_{30} > \alpha_{10}$ $\zeta_c \gtreqless \zeta_o$ when $\rho_{\omega o} \gtreqless 1.0$ At idle condition, high freq. comp. has higher damping ratio	$\zeta_c \rho_{\omega o} - \zeta_c = -(\rho_{\omega o} - 1)$; $\alpha_{30} < \alpha_{10}$ $\zeta_c \lesseqgtr \zeta_o$ when $\rho_{\omega o} \gtreqless 1.0$ At idle condition, low freq. comp. has higher damping ratio
2. Best working range	$\rho_{\omega o} > 1.0$, $\zeta_o < 1.0$, $\zeta_c < 1.0$	$\rho_{\omega o} > 1.0$ $\zeta_o > 1.0$, $\zeta_c < 1.0$
3. Tuning result not very stable	$\rho_{\omega o} < 1.0$, $\zeta_o < 1.0$ $\zeta_c < 1.0$	$\rho_{\omega o} < 1.0$ $-0.5 < \zeta_o < 0$ $\zeta_c > 1.0$
4. Range of physical nonexistence	$\rho_{\omega o} < 1.0$ $\zeta_t^2 > 2\zeta_o - 1$	$\rho_{\omega o} < 1.0$, $\zeta_o < -0.5$

It is understood that control of higher frequency is preferable to control of lower frequency simply because at tuned condition the frequency of the controlled member is increased by $\sqrt{\rho_{\omega o}}$. However, if high frequency control is not available by some physical restriction, the tuning condition is obtainable by controls with $\rho_{\omega o} < 1.0$ only at the expense of (1) less assurance in tuned condition, and (2) the undamped natural frequency of the controlled member is lowered, which means that such control spoils the rapidity of response of the controlled member.

P A R T IV

ANALYSIS OF TRANSIENT

CHAPTER ELEVEN

Stability Function, Quality Function, Disturbance Function and Response Function

42. Stability and Transient Analysis of Controlled System

Sufficient knowledge of stability analysis of automatic controlled problem may assure (a) stable operation in general, and (b) better distribution of damping between the components by providing a well compromised control. However, such assurances do not inform us how the controlled system responds to a disturbance of any characteristic. Analysis of transient response is therefore important in order to obtain better performance when the system is subjected to sudden disturbance or to one which does not repeat periodically.

43. Stability Function, Quality Function, Disturbance Function and Response Function

From the analysis in Section 5, Chapter II, and in Section 8, Chapter III, Eqs. (1.102), (1.104) and (2.136) are noticed in the general form.

$$R_h(t) = \frac{Q_h(D)}{S(D)} I(t) 1 \quad * \qquad (4.01)a$$

or $$R_h(\tau) = \frac{Q_h(d)}{S(d)} I(\tau) 1 \qquad (4.01)$$

---

*Such a form of equation is originated by Heaviside in treating network responses. For the convenience of this thesis, the symbols are entirely different from the original form. The reader is referred to Chapter VII in Bush's "Operational Circuit Analysis".

where $D = \frac{D}{Dt}$ = time differential operator

$d = \frac{d}{d\tau}$ = dimensionless time differential operator

$S(D) = \sum_{k=0}^{k=n} A_k D^k$ and is defined as stability function of the controlled system (dimensional).

$S(d) = \sum_{k=0}^{k=n} a_k d^k$ also defined as stability function of the controlled system (non-dimensional).

$Q_h(D) = \sum_{k=0}^{k=m} B_{hk} D^k$ and is defined as quality function of h-wise motion with $m = n$.

$Q_h(d) = \sum_{k=0}^{k=m} b_{hk} d^k$ also defined as quality function of h-wise motion (non-dimensional) with $m = n$.

$I(t)$ = function of (t), defined as disturbance function or input function applied to the controlled system.

The symbol $\mathbf{1}$ is defined as unit function which specifies a discontinuous function of time which is zero until t equals zero and unity thereafter. Any function, such as $I(t)$, followed by the symbol $\mathbf{1}$ indicates its discontinuity at $t = 0$; and that the function is zero until t equals zero and equal to $I(t)$ thereafter.

$R_h(t)$ is function of (t) for h-wise response to the disturbance function $I(t)\mathbf{1}$. It is defined as response function of the h-wise motion.

$I(\tau)$ and $R_h(\tau)$ are defined in the same way but they are referred to non-dimensional quantities.

Eq. (4.01) has the same form as Eq. (4.01)a. Therefore full development of either equation will enable one to handle the other. Eq. (4.01) shall be fully developed into a form which is more familiar to engineers.

44. Expanded Form of Response Function when Unit Step Disturbance is Applied

When $I(\tau)1 = 1$, the response function is defined as unit response.

$$R_h(\tau) = \frac{Q_h(d)}{S(d)} 1 \tag{4.02)a$$

The subscript h is omitted hereafter temporarily for the simplicity in appearance.

$$R(\tau) = \frac{Q(d)}{S(d)} 1 \tag{4.02}$$

On expanding,

$$R(\tau) = \frac{Q(0)}{S(0)} + \sum_{k=1}^{k=n} \frac{Q(d_k)}{d_k S'(d_k)} \epsilon^{d_k \tau} \tag{4.03}$$

where $\frac{Q(0)}{S(0)}$ is the steady state response and $\sum_{k=1}^{k=n} C_k \epsilon^{d_k \tau}$ is the transient response. d_k's are the roots of $S(d) = 0$.

$S'(d_k)$ is the first derivative of the stability function, in purely algebraic form,

$$S'(d_k) = a_0(d_k-d_1)(d_k-d_2)\ldots(d_k-d_{k-1})(d_k-d_{k+1})\ldots(d_k-d_n) \tag{4.04}$$

which should be substituted in Eq.(4.03).

When all d_k's are real, solution (4.03) is really handy. But when some of them are complex quantities, Eq. (4.03) is too tedious to be rationalized. The following procedure will reduce the effort in doing so.

Physically, when one of the roots of S(d) is complex there must be another complex root which is conjugate to the first. Let the root be d_f and its conjugate be $\overline{d_f}$, where

$$d_f,\ \overline{d_f} = (-\zeta_f \pm j\sqrt{1-\zeta_f^2})\,\omega_{nnf} \qquad (4.05)$$

where ω_{nnf} = non-dimensional natural frequency of component f.

ζ_f = damping ratio of component f.

Now,

$$\sum_{k=1}^{k=n} \frac{Q(d_k)}{d_k S'(d_k)}\epsilon^{d_k\tau} = \sum_{f=1}^{f=\frac{n}{2}} \frac{Q(d_f)}{d_f S'(d_f)}\epsilon^{d_f\tau} + \sum_{f=1}^{f=\frac{n}{2}} \frac{Q(\overline{d_f})}{\overline{d_f} S'(\overline{d_f})}\epsilon^{\overline{d_f}\tau} \qquad (4.06)$$

$$= \sum_{f=1}^{f=\frac{n}{2}} \frac{1}{d_f\overline{d_f}S'(d_f)S'(\overline{d_f})}\left[Q(d_f)\overline{d_f}S'(\overline{d_f})\epsilon^{d_f\tau} + Q(\overline{d_f})d_f S'(d_f)\epsilon^{\overline{d_f}\tau}\right] \qquad (4.07)$$

Inside the bracket of (4.06), $Q(d_f)\overline{d_f}S'(\overline{d_f})$ and $Q(\overline{d_f})d_f S'(d_f)$ are conjugate functions, so let

$$\tfrac{1}{2}(\underline{R}_f \pm j\,\underline{I}_f) = Q(d_f)\overline{d_f}S'(\overline{d_f}),\ \ Q(\overline{d_f})d_f S'(d_f) \qquad (4.08)$$

where $\frac{1}{2}(\underline{R}_f)$ = real part of $Q(d_f)\overline{d_f}S'(\overline{d_f})$

$\frac{1}{2}(\underline{I}_f)$ = imaginary part of $Q(d_f)\overline{d_f}S'(\overline{d_f})$.

Subsitute (4.08) and (4.05) into (4.07). The result will be

$$\sum_{k=1}^{k=n} \frac{Q(d_k)}{d_k S'(d_k)} \epsilon^{d_k \tau} = \sum_{f=1}^{f=\frac{n}{2}} \left[\frac{\frac{1}{2}}{d_f \overline{d_f} S'(d_f) S'(\overline{d_f})} \left\{ (\underline{R}_f + j\underline{I}_f) \epsilon^{(-\zeta_f + i\sqrt{1-\zeta_f^2})\omega_{nnf}\tau} \right.\right.$$

$$\left.\left. + (\underline{R}_f - i\underline{I}_f) \epsilon^{(-\zeta_f - i\sqrt{1-\zeta_f^2})\omega_{nnf}\tau} \right\} \right] \qquad (4.09)$$

$$= \sum_{f=1}^{f=\frac{n}{2}} \frac{\epsilon^{-\zeta_f \omega_{nnf}\tau}}{d_f \overline{d_f} S'(d_f) S'(\overline{d_f})} \left[\underline{R}_f \cos\sqrt{1-\zeta_f^2}\,\omega_{nnf}\tau - I_f \sin\sqrt{1-\zeta_f^2}\,\omega_{nnf}\tau \right]$$

(4.10)

or

$$\sum_{k=1}^{k=n} \frac{Q(d_k)}{d_k S'(d_k)} \epsilon^{d_k \tau} = \sum_{f=1}^{f=\frac{n}{2}} \frac{\mu_f \epsilon^{-\zeta_f \omega_{nnf}\tau}}{d_f \overline{d_f} S'(d_f) S'(\overline{d_f})} \cos(\sqrt{1-\zeta_f^2}\,\omega_{nnf}\tau + \phi_f)$$

(4.11)

where $\mu_f = \sqrt{\underline{R}_f^2 + \underline{I}_f^2} = 2\sqrt{Q(d_f)Q(\overline{d_f})d_f\overline{d_f}S'(d_f)S'(\overline{d_f})}$ (4.11)a

$$\phi_f = \tan^{-1}\frac{\underline{I}_f}{\underline{R}_f} = \tan^{-1}\frac{\underline{I}_{Qf}}{\underline{R}_{Qf}} - \tan^{-1}\frac{\underline{I}_{S'f}}{\underline{R}_{S'f}} + \tan^{-1}\frac{\sqrt{1-\zeta_f^2}}{\zeta_f}$$

(4.11)b

with $\underline{R}_{Qf}$ = real part of $Q(d_f)$

$\underline{I}_{Qf}$ = imaginary part of $Q(d_f)$

$\underline{R}_{S'f}$ = real part of $S'(d_f)$

$\underline{I}_{S'f}$ = imaginary part of $S'(d_f)$.

Substitute (4.11)a in (4.11) and the transient response will appear as

$$2 \sum_{f=1}^{f=\frac{n}{2}} \frac{\sqrt{Q(d_f)Q(\overline{d_f})}}{d_f\overline{d_f}S'(d_f)S'(\overline{d_f})} \epsilon^{-\zeta_f \omega_{nnf}\tau} \cos(\sqrt{1-\zeta_f^2}\,\omega_{nnf}\tau + \phi_f) \qquad (4.12)$$

Eq. (4.12) indicates the physical conception of the transient response when all components are oscillatory so that each undergoes a damped sinusoidal motion with its own damping rate, $-\zeta_f\omega_{nnf}$, with a certain natural frequency and a certain phase shift. The phase shift apparently consists of three parts: (1) due to the quality function Q_f, (2) due to the stability derivative function S'_f, and (3) due to the damping ratio ζ_f.

Since $\sqrt{d_f\ \overline{d_f}} = \left[(-\zeta_f + j\sqrt{1-\zeta_f^2})(-\zeta_f - j\sqrt{1-\zeta_f^2})\right]^{\frac{1}{2}} \omega_{nnf}$

$$= \zeta_f^2 - j^2(1-\zeta_f^2)^{\frac{1}{2}} \omega_{nnf} = \sqrt{-(-1)}\ \omega_{nnf}$$

$$= -\omega_{nnf} \qquad (4.13)a$$

Therefore expression (4.12) can be simplified as

$$-2 \sum_{f=1}^{f=\frac{n}{2}} \left[\frac{Q(d_f)Q(\overline{d_f})}{S'(d_f)S'(\overline{d_f})}\right]^{\frac{1}{2}} \frac{\epsilon^{-\zeta_f\omega_{nnf}\tau}}{\omega_{nnf}} \cos(\sqrt{1-\zeta_f^2}\,\omega_{nnf}\tau + \phi_f) \qquad (4.13)$$

It should be noted that when the component f is non-oscillatory, that is, it is overdamped or $\zeta_f > 1$,

(1) both d_f and d'_f become real and their magnitude ratios are

$$\frac{Q(d_f)}{S'(d_f)} \text{ and } \frac{Q(d'_f)}{S'(d'_f)}.$$

(2) $\sqrt{1-\zeta_f^2} = j\sqrt{\zeta_f^2-1}$

and ϕ_f becomes $j\psi_f$ * where $\phi_f = \psi_f$.

Therefore expression (4.13) becomes

Transient response of component f =

$$-2\left[\frac{Q(d_f)Q(d'_f)}{S'(d_f)S'(d'_f)}\right]^{\frac{1}{2}} \frac{\epsilon^{-\zeta_f\omega_{nnf}\tau}}{\omega_{nnf}} \cosh(\sqrt{\zeta_f^2-1}\,\omega_{nnf}\tau + \psi_f) \qquad (4.14)$$

with $\zeta_f > 1$.

The above derivation is valid only for non-repeating roots. If some real root or complex root occurs more than once in S(d) = 0, the evaluation is much more complicated. It will be analyzed in the next chapter.

---

*because

$$j \tanh^{-1}\frac{\sqrt{\zeta^2-1}}{\zeta} = \tan^{-1}\frac{j\sqrt{\zeta^2-1}}{\zeta}$$

and all the imaginary quantities for $\zeta < 1$ become real when $\zeta > 1$ and vice versa; therefore (4.11)b may be used to find ψ_f.

CHAPTER TWELVE

Characteristic Decomposition

45. Characteristic Decomposition or Quadratically Partial Fraction

For engineers' better understanding of transient response of controlled system it is better to fractionalize the expression $\frac{Q(d)}{S(d)}$ before the unit function or any disturbance function is attached to it. The partial fractions will be done according to physical significance, so that each one of them will represent a mode of vibration when disturbance is applied.

For the purpose of this thesis, the highest power of d in S(d) will be limited to six. The highest power of d in Q(d) is limited either to six or less. In case the highest powers of d in Q(d) and in S(d) are the same, their coefficient is usually the same (and equal to unity) as shown in Eq. (1.104), and the fraction $\frac{Q(d)}{S(d)}$ may be changed to the form $1 + \frac{Q(d)}{S(d)}$ in which the highest power of d in Q(d) is at least one less than that in S(d). It is this $\frac{Q(d)}{S(d)}$ which will be partial fractionalized.

Assume $$\frac{Q(d)}{S(d)} = \frac{q_1(d)}{s_1(d)} + \frac{q_2(d)}{s_2(d)} + \frac{q_3(d)}{s_3(d)} \qquad (4.15)$$

where $$s_1(d) = d^2 + 2\,\zeta_1\,\omega_{nn1}\,d + \omega^2_{nn1}, \text{ etc.} \qquad (4.15)a$$

(ω_{nn1} represents the non-dimensional natural frequency of component 1.)

$$q_1(d) = \zeta_1 \omega_{nn1} \kappa_{11} d + \kappa_{01} \omega^2_{nn1}, \text{ etc.} \qquad (4.15)b$$

(First subscript of κ refers to component number; second subscript of κ indicates the associated exponential power of d.)

On summing up the right sides of (4.15), the following form is obtained.

$$\frac{Q(d)}{S(d)} = \frac{q_1(d)s_2(d)s_3(d) + q_2(d)s_1(d)s_3(d) + q_3(d)s_1(d)s_2(d)}{S(d)}$$

Let d_1, $\overline{d_1}$ = roots of $S(d) = 0$, where d_1 and $\overline{d_1}$ are conjugate pair, because $s_1(d) = 0$.

Substitute d_1 into (4.16). It appears that

$$Q(d_1) = q_1(d_1)s_2(d_1)s_3(d_1)$$

or

$$q_1(d_1) = \frac{Q(d_1)}{s_2(d_1)s_3(d_1)} = F_1 \qquad (4.17)$$

From Eq. (4.15)a, we obtain

$$d_1 = (-\zeta_1 + j\sqrt{1-\zeta_1^2})\,\omega_{nn1} \qquad (4.18)$$

Substitute (4.18) into (4.15)b, and then (4.17).

$$\omega^2_{nn1}(\kappa_{01} - \zeta_1^2 \kappa_{11} + i\,\kappa_{11}\,\zeta_1\sqrt{1-\zeta_1^2}) = F_1$$

or

$$\kappa_{01} - \zeta_1^2\,\kappa_{11} + i\,\kappa_{11}\,\zeta_1\sqrt{1-\zeta_1^2} = \frac{1}{\omega^2_{nn1}} F_1 \qquad (4.19)$$

Therefore

$$\underline{\kappa_{11} = \frac{1}{\omega^2_{nn1}\,\zeta_1\sqrt{1-\zeta_1^2}}\, I_{F_1}} \qquad (4.20)$$

$$\kappa_{01} = \frac{1}{\omega^2_{nn1}} \underline{R}_{F_1} + \zeta_1^2 \; \kappa_{11} \qquad (4.21)$$

or

$$\kappa_{01} = \frac{1}{\omega^2_{nn1}} \left[\underline{R}_{F_1} + \frac{\zeta_1}{\sqrt{1-\zeta_1^2}} \underline{I}_{F_1} \right] \qquad (4.21)a$$

where $\underline{I}_{F_1}$ and $\underline{R}_{F_1}$ represent imaginary and real parts of F_1 respectively.

Now we have to break F_1 into its real and imaginary parts.

$$F_1 = \frac{Q(d_1)}{s_2(d_1)s_3(d_1)} = \frac{Q(d_1)s_2(\overline{d_1})s_3(\overline{d_1})}{s_2(d_1)s_2(\overline{d_1})s_3(d_1)s_3(\overline{d_1})} \qquad (4.22)$$

where the denominator is rationalized and equal to

$$s_2(d_1)s_2(\overline{d_1})s_3(d_1)s_3(\overline{d_1}) = \left[(\zeta_2\omega_2^* - \zeta_1\omega_1^*)^2 + (\sqrt{1-\zeta_1^2}\,\omega_1 + \sqrt{1-\zeta_2^2}\,\omega_2)^2 \right]$$

$$\times \left[(\zeta_2\omega_2 - \zeta_1\omega_1)^2 + (\sqrt{1-\zeta_1^2}\,\omega_1 - \sqrt{1-\zeta_2^2}\,\omega_2)^2 \right]$$

$$\times \left[(\zeta_3\omega_3^* - \zeta_1\omega_1)^2 + (\sqrt{1-\zeta_1^2}\,\omega_1 + \sqrt{1-\zeta_3^2}\,\omega_3)^2 \right]$$

$$\times \left[(\zeta_3\omega_3 - \zeta_1\omega_1)^2 + (\sqrt{1-\zeta_1^2}\,\omega_1 - \sqrt{1-\zeta_3^2}\,\omega_3)^2 \right] \qquad (4.23)$$

and the denominator can be expanded by the binomial theorem:

---

*For convenience in typing, ω_1, ω_2 and ω_3 are used for ω_{nn1}, ω_{nn2} and ω_{nn3} respectively.

$$Q(d_1) = Q(-\zeta_1\omega_1 + j\omega_1\sqrt{1-\zeta_1^2})^{*}$$

$$= Q(-\zeta_1\omega_1) - \omega_1^2(1-\zeta_1^2)\frac{Q''(-\zeta_1\omega_1)}{2!} + \omega_1^4(1-\zeta_1^2)^2\frac{Q^{iv}(-\zeta_1\omega_1)}{4!} + \omega_1^6(1-\zeta_1^2)^3\frac{Q^{vi}(-\zeta_1\omega_1)}{6!}$$

$$+ j\left[\omega_1(1-\zeta_1^2)^{1/2}\frac{Q'(-\zeta_1\omega_1)}{1!} - \omega_1^3(1-\zeta_1^2)^{3/2}\frac{Q'''(-\zeta_1\omega_1)}{3!} + \omega_1^5(1-\zeta_1^2)^{5/2}\frac{Q^{v}(-\zeta_1\omega_1)}{5!}\right] \quad (4.24)$$

$$s_2(\overline{d_1}) = s_2(-\zeta_1\omega_1 - j\omega_1\sqrt{1-\zeta_1^2}) = 2(\zeta_2\omega_2 - \zeta_1\omega_1)(-\zeta_1\omega_1 - j\omega_1\sqrt{1-\zeta_1^2})$$

$$= 2(\zeta_1\omega_1 - \zeta_2\omega_2)(\zeta_1\omega_1 + j\omega_1\sqrt{1-\zeta_1^2}) \quad (4.25)$$

$$s_3(\overline{d_1}) = s_3(-\zeta_1\omega_1 - j\omega_1\sqrt{1-\zeta_1^2}) = 2(\zeta_3\omega_3 - \zeta_1\omega_1)(-\zeta_1\omega_1 - j\omega_1\sqrt{1-\zeta_1^2})$$

$$= 2(\zeta_1\omega_1 - \zeta_3\omega_3)(\zeta_1\omega_1 + j\omega_1\sqrt{1-\zeta_1^2}) \quad (4.26)$$

With the help of Eqs. (4.23) to (4.26), the real and imaginary parts of (4.22) can be evaluated. It looks very messy but when numerical values are substituted in, the work is much simplified.

Likewise, we can evaluate κ_{12}, κ_{02}, κ_{13} and κ_{03}.

$$\kappa_{12} = \frac{1}{2\,\omega_{nn2}^2\,\zeta_2\sqrt{1-\zeta_2^2}}\,\underline{I}_{F_2} \quad (4.27)$$

$$\kappa_{02} = \frac{1}{\omega_{nn2}^2}\left(\underline{R}_{F_2} + \frac{\zeta_2}{\sqrt{1-\zeta_2^2}}\,\underline{I}_{F_2}\right) \quad (4.28)$$

---

*Practical evaluation of polynomial function of complex variables is much simplified by applying DeMoivre's theorem graphically. See Appendix D.

$$\kappa_{13} = \frac{1}{2\,\omega^2_{nn3}\ \zeta_3\sqrt{1-\zeta_3^2}}\ \underline{I}_{F_3} \qquad (4.29)$$

$$\kappa_{03} = \frac{1}{\omega^2_{nn3}}\left(\underline{R}_{F_3} + \frac{\zeta_3}{\sqrt{1-\zeta_3^2}}\ \underline{I}_{F_3} \right) \qquad (4.30)$$

where $$F_2 = \frac{Q(d_2)}{s_1(d_2)s_3(d_2)} = \frac{Q(d_2)s_1(\overline{d_2})s_3(\overline{d_2})}{s_1(d_2)s_1(\overline{d_2})s_3(d_2)s_3(\overline{d_2})} \qquad (4.31)$$

$$F_3 = \frac{Q(d_3)}{s_1(d_3)s_2(d_3)} = \frac{Q(d_3)s_1(\overline{d_3})s_2(\overline{d_3})}{s_1(d_3)s_1(\overline{d_3})s_2(d_3)s_2(\overline{d_3})} \qquad (4.32)$$

The expansion of F_2 and F_3 can be made with the same procedure as has been used for F_1.

When the numerical evaluation of all the constants κ_{01}, κ_{11}, κ_{02}, κ_{12}, κ_{03} and κ_{13} is completed, Eq. (4.15) can be written in the following form.

$$\frac{Q(d)}{S(d)} = \frac{\zeta_1\omega_{nn1}\kappa_{11}d+\kappa_{01}\omega^2_{nn1}}{d^2+2\zeta_1\omega_{nn1}d+\omega^2_{nn1}} + \frac{\zeta_2\omega_{nn2}\kappa_{12}d+\kappa_{02}\omega^2_{nn2}}{d^2+2\zeta_2\omega_{nn2}d+\omega^2_{nn2}}$$

$$+ \frac{\zeta_3\omega_{nn3}\kappa_{13}d+\kappa_{03}\,{}^{2}_{nn3}}{d^2+2\zeta_3\omega_{nn3}d+\omega^2_{nn3}} \qquad (4.33)$$

Hence

$$\frac{Q(d)}{S(d)}_1 = \frac{\zeta_1\omega_{nn1}\kappa_{11}d+\kappa_{01}\omega^2_{nn1}}{d^2+2\zeta_1\omega_{nn1}d+\omega^2_{nn1}}_1 + \frac{\zeta_2\omega_{nn2}\kappa_{12}d+\kappa_{02}\omega^2_{nn2}}{d^2+2\zeta_2\omega_{nn2}d+\omega^2_{nn2}}_1$$

$$+ \frac{\zeta_3\omega_{nn3}\kappa_{13}d+\kappa_{03}\omega^2_{nn3}}{d^2+2\zeta_3\omega_{nn3}d+\omega^2_{nn3}}_1 \qquad (4.34)$$

Each of the three terms on the right side of Eq. (4.34) represents one of the three components of vibratory motion (steady and transient response). Each one has its own steady state response and the respective transient response transient response is referred to this component steady state response. In symbols,

$$\text{Steady state} = \frac{Q(0)}{S(0)} = \frac{q_1(0)}{s_1(0)} + \frac{q_2(0)}{s_2(0)} + \frac{q_3(0)}{s_3(0)}$$

$$\text{or} \qquad \frac{Q(0)}{S(0)} = o_1 + o_2 + o_3 \qquad (4.35)$$

Such fractionalization (as into several quadratic fractions) which is made before operating on the disturbance function is defined as characteristic decomposition of the controlled system.

After being characteristically decomposed, each component then operates on the unit function and the following formula is common to all:

$$\frac{\zeta\kappa_1\omega_{nn} d + \kappa_0\omega^2_{nn}}{d^2 + 2\zeta\omega_{nn} d + \omega^2_{nn}} 1 = \kappa_0\left[1 - \frac{\epsilon^{-\zeta\omega_{nn}\tau}}{\sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2}\,\omega_{nn}\tau + \phi)\right]$$

$$+ \zeta\kappa_1\left[\frac{\epsilon^{-\zeta\omega_{nn}\tau}}{\overline{1-\ ^2}} \sin(\sqrt{1-\zeta^2}\,\omega_{nn}\tau)\right]^* \qquad (4.36)$$

$$\text{or} \qquad \frac{\zeta\kappa_1\omega_{nn} d + \kappa_0\omega^2_{nn}}{d^2 + 2\zeta\omega_{nn} d + \omega^2_{nn}} 1$$

$$= \kappa_0\left[1 - \sqrt{\frac{\zeta^2}{/1-\zeta^2}\left(\frac{\kappa_1}{\kappa_0} - 1\right)^2 + 1}\ \epsilon^{-\zeta\omega_{nn}\tau} \cos(\sqrt{1-\zeta^2}\,\omega_{nn}\tau + \phi')\right. \ {}^{**}(4.37)$$

$$\text{where } \phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} \qquad (4.36)\text{a}$$

---

*Derivation based upon basic operational formula.

**Same result can be obtained by carefully manipulating Eqs. (4.13) and (4.11)b and counting the steady state response in them.

and $\phi' = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} (\frac{\kappa_1}{\kappa_0} - 1)$ (4.37)a

When the forcing function is of sinusoidal shape continuous with respect to time: that is,

$$I(\tau) = \sin(\Omega_{Fn}\tau) \tag{4.38}$$

where the magnitude of the forcing sinusoidal function is assumed to be unity, Ω_{Fn} means the angular frequency of the forcing function in non-dimensional units matched to non-dimensional time unit (τ).

The steady state response from each characteristic component to such sinusoidal forcing function will be

$$R(\tau)_{ss} = \mu_F \sin(\Omega_{Fn}\tau + \phi_F) \tag{4.39}$$

where $\mu_F = \frac{\kappa_0\sqrt{1 + (\frac{\kappa_1}{\kappa_0}\zeta\beta)^2}}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$ is defined as sinusoidal magnification factor. (4.40)

$\phi_F = \tan^{-1} \frac{\kappa_1}{\kappa_0}\zeta\beta - \tan^{-1} \frac{2\zeta\beta}{1-\beta^2}$ is defined as sinusoidal phase shift. (4.41)

$\beta = \frac{\Omega_{Fn}}{\omega_{nn}} = \frac{\Omega_F}{\omega_n}$ is defined as forcing frequency ratio. (4.42)

Both μ_F and ϕ_F are referred to steady state response of the particular characteristic component.

When all the (three) components are evaluated, they can be put together with proper attention to magnitude and phase shift and the response curve, whether steady state or transient, can be plotted.

46. Characteristic Decomposition with One or More Components Overdamped

Oftentimes some of the components are overdamped. The process of decomposing is somewhat simplified. Now assuming component 1 of Eq. (4.15) being an overdamped one,

$$\frac{q_1(d)}{S_1(d)} = \frac{q_{a1}(d)}{S_{a1}(d)} + \frac{q_{b1}(d)}{S_{b1}(d)} \tag{4.42}$$

where

$$S_{a1}(d) = d + (\zeta_1 + \sqrt{\zeta_1^2-1}\ )\omega_{nn1}, \quad d_{a1} = -(\zeta_1 + \sqrt{\zeta_1^2-1}\ ) \tag{4.43)a}$$

$$S_b\ (d) = d + (\zeta_1 - \sqrt{\zeta_1^2-1}\ )\omega_{nn1}, \quad d_{b1} = -(\zeta_1 - \sqrt{\zeta_1^2-1}\ ) \tag{4.43)b}$$

$$q_a\ (d) = \kappa_{a1}\ (\zeta_1 + \sqrt{\zeta_1^2-1}\ )\omega_{nn1}, \tag{4.43)c}$$

$$q_b\ (d) = \kappa_{b1}\ (\zeta_1 - \sqrt{\zeta_1^2-1}\ )\omega_{nn1} \tag{4.43)d}$$

Rewrite Eq. (4.16) in the following form

$$\frac{Q(d)}{S(d)} \equiv \frac{q_{a1}(d)S_{b1}(d)S_3(d)S_2(d) + q_{b1}(d)S_{a1}(d)S_2(d)S_3(d) + q_2(d)S_1(d)S_3(d) + q_3(d)S_1(d)S_2(d)}{S(d)} \quad \text{Eq}(4.45)$$

Substituting d_a into (4.45) we have

$$Q(d_{a1}) = q_{a1}(d_{a1})S_{b1}(d_{a1})S_2(d_{a1})S_3(d_{a1})$$

or

$$q_{a1}(d_{a1}) = \frac{Q(d_{a1})}{S_{b1}(d_{a1})S_2(d_{a1})S_3(d_{a1})} \tag{4.46}$$

$$\kappa_{a1}(\zeta_1 + \sqrt{\zeta_1^2-1}\ )\omega_{nn1} = \frac{Q(-\zeta_1 - \sqrt{\zeta_1^2-1}\ )}{-2\sqrt{\zeta_1^2-1}\quad S_2(-\zeta_1 - \sqrt{\zeta_1^2-1}\ )S_3(-\zeta_1 - \sqrt{\zeta_1^2-1}\ )\omega_{nn1}}$$

$$\therefore\ \kappa_{a1} = \frac{Q(d_{a1})}{2\sqrt{\zeta_1^2-1}\ \ d_{a1}\ S_2(d_{a1})\ S_3(d_{a1})\ \omega_{nn1}^2} \tag{4.47}$$

and likewise

$$\kappa_{b1} = \frac{Q(d_{b1})}{-2\sqrt{\zeta_1^2-1}\ \ d_{b1}S_2(d_{b1})S_3(d_{b1})\ \omega_{nn1}^2} \tag{4.48}$$

When this is done, the fraction $\frac{Q(d)}{S(d)}$ can be reduced to a simpler one with two degrees reduced in d in the denominator. Let us represent this reduced fraction by

$$\frac{\dot{Q}(d)^{*}}{\dot{S}(d)} = \frac{Q(d)}{S(d)} - \left[\frac{q_{a1}(d)}{S_{a1}(d)} + \frac{q_{b1}(d)}{S_{b1}(d)}\right] \tag{4.49}$$

The reduced fraction contains one less degree of freedom than the original one. It can be decomposed with much less effort through the same procedure, but in a simpler way, should be followed as described in Section 45.

Evaluation of $\frac{q_a(d)}{S_a(d)}1$ and $\frac{q_b(d)}{S_b(d)}1$ is very simple:

$$\frac{q_a(d)}{S_a(d)} = \frac{K_{a1}(\zeta_1 + \sqrt{\zeta_1^2 - 1})\omega_{nn1}}{d + (\zeta_1 + \sqrt{\zeta_1^2 - 1})\omega_{nn1}} 1 = K_{a1}(1 - \epsilon^{-(\zeta_1 + \sqrt{\zeta_1^2 - 1})\omega_{nn1}\tau}) \tag{4.50}$$

Likewise,

$$\frac{q_{b1}(d)}{S_{a1}(d)} 1 = K_{b1}\left[1 - \epsilon^{-(\zeta_1 - \sqrt{\zeta_1^2 - 1})\omega_{nn1}\tau}\right] \tag{4.50a}$$

47. Characteristic Decomposition when S(d) Has Repeating Quadratic Factors

Let us again limit our scope to $\frac{Q(d)}{S(d)}$ in which the highest power in S(d) is six and that in Q(d) five or less. And assume $S_2(d) = S_3(d)$ and $S_1(d)$ is oscillatory.

$$\therefore \quad \frac{Q(d)}{S(d)} \equiv \frac{q_1(d)}{S_1(d)} + \frac{q_2(d)}{S_2(d)} + \frac{q_{2r}(d)}{S_2^2(d)} \tag{4.51}$$

---

*The reduction of $\frac{Q(d)}{S(d)}$ into $\frac{\dot{Q}(d)}{\dot{S}(d)}$ is also applicable to the case where all components are oscillatory. In such a case,

$$\frac{\dot{Q}(d)}{\dot{S}(d)} = \frac{Q(d)}{S(d)} - \frac{q_1(d)}{S_1(d)}$$

where $S_2^2(d) = \left[S_2(d)\right]^2 = (d^2 + 2\zeta_2\omega_{nn2}\, d + \omega_{nn2}^2)^2$ (4.51)a

$$q_{2r}(d) = \zeta_2\,\omega_{nn2}^3 K_{12r}\, d + K_{02r}\,\omega_{nn2}^4 \quad (4.51)b$$

with subscript r indicating the belonging of repeating factor

$$\therefore \quad Q(d) \equiv q_1(d)S_2^2(d) + q_2(d)S_1(d)S_2(d) + q_{2r}(d)S_1(d) \quad (4.52)$$

Substitute d_2 in equation (4.52) so that $S_2(d_2) = 0$

$$\therefore \quad q_{2r}(d_2) = \frac{Q(d_2)}{S_1(d_2)} = F_{2r} \quad (4.53)$$

By the same procedure used in Section 45 we obtain

$$K_{12r} = \frac{1}{\omega_{nn2}^4\,\zeta_2\sqrt{1-\zeta_2^2}}\,\underline{I}\,F_{2r} \quad (4.54)$$

$$K_{02r} = \frac{1}{\omega_{nn2}^4}\left[\underline{R}\,F_{2r} + \frac{\zeta_2}{\sqrt{1-\zeta_2^2}}\,\underline{I}\,F_{2r}\right] \quad (4.55)$$

Now the reduced fraction can be obtained as

$$\frac{\dot{Q}(d)}{\dot{S}(d)} = \frac{Q(d)}{S(d)} - \frac{q_{2r}(d)}{S_2^2(d)} \quad (4.55)a$$

From this reduced fraction the constants K_{11}, K_{01}, K_{12}, and K_{02} can be evaluated with less effort.

48. Transient Response with Repeating Binomial Factors in S(d) = 0

Let the highest power of the binomial factor be r, then $\frac{K_r\,\omega_{nn}^r}{(d + d_0\omega_{nn})^r}\,1$ will be the response for the r^{th} component of the repeating factor.

$$\frac{K_r\,\omega_{nn}^r}{(d + d_0\omega_{nn})^r}\,1 = \frac{K_r}{d_0^r}\left\{1 - \epsilon^{-d_0\omega_{nn}\tau}\left[1 + d_0\omega_{nn}\tau + \frac{(d_0\omega_{nn}\tau)^2}{2!} + \cdots + \frac{(d_0\omega_{nn}\tau)^{r-1}}{(r-1)!}\right]\right\} \quad (4.56)$$

49. Transient Response with Repeating Quadratic Factors in S(d) = 0

Suppose $\frac{q\,(d)}{S\,(d)}$ takes the following form:

$$\frac{\omega_{nn}^4}{(d^2+2\zeta\omega_{nn}d+\omega_{nn}^2)^2} = \frac{\omega_{nn}^4}{(d+d_a\omega_{nn})^2(d+d_b\omega_{nn})^2} \qquad (4.57)$$

where $d_a = \zeta + \sqrt{\zeta^2-1}$, $d_b = \zeta - \sqrt{\zeta^2-1}$ when $\zeta > 1$ (4.57)a

or $d_a = \zeta + j\sqrt{1-\zeta^2}$, $d_b = \zeta - j\sqrt{1-\zeta^2}$ when $\zeta < 1$ (4.57)b

$$\frac{\omega_{nn}^4}{(d+d_a\omega_{nn})^2(d+d_b\omega_{nn})^2}1 = \frac{1}{(d_b-d_a)^2}\left\{\frac{\omega_{nn}^2}{(d+d_a\omega_{nn})^2}+\frac{\omega_{nn}^2}{(d+d_b\omega_{nn})^2}-\frac{2}{d_b-d_a}\left[\frac{\omega_{nn}}{d+d_a\omega_{nn}}-\frac{\omega_{nn}}{d+d_b\omega_{nn}}\right]\right\}1$$

$$= \frac{1}{d_a^2d_b^2}-\frac{1}{(d_a-d_b)^3}\left\{\frac{d_a-3d_b}{d_b^2}\epsilon^{-d_b\omega_{nn}\tau}-\frac{d_b-3d_a}{d_a^2}\epsilon^{-d_a\omega_{nn}\tau} + (d_a-d_b)\omega_{nn}\tau\left[\frac{\epsilon^{-d_b\omega_{nn}\tau}}{d_b}+\frac{\epsilon^{-d_a\omega_{nn}\tau}}{d_a}\right]\right\}$$

$$\frac{\omega_{nn}^4}{(d+d_a\omega_{nn})^2(d+d_b\omega_{nn})^2}1 = 1-\frac{1}{(d_a-d_b)^3}\left[\frac{d_a-3d_b}{d_b^2}\epsilon^{-d_b\omega_{nn}\tau}-\frac{d_b-3d_a}{d_a^2}\epsilon^{-d_a\omega_{nn}\tau}\right] - \frac{1}{(d_a-d_b)^2}\omega_{nn}\tau\left[\frac{\epsilon^{-d_b\omega_{nn}\tau}}{d_b}+\frac{\epsilon^{-d_a\omega_{nn}\tau}}{d_a}\right] \qquad (4.58)$$

and

$$\frac{\omega_{nn}^3 d}{(d+d_a\omega_{nn})^2(d+d_b\omega_{nn})^2}1 = \frac{2}{(d_a-d_b)^3}\left(\epsilon^{-d_b\omega_{nn}\tau}-\epsilon^{-d_a\omega_{nn}\tau}\right) + \frac{\omega_{nn}\tau}{(d_a-d_b)^2}\left(\epsilon^{-d_b\omega_{nn}\tau}+\epsilon^{-d_a\omega_{nn}\tau}\right) \qquad (4.59)$$

Hence

$$\frac{K_{1r}\omega_{nn}^3 d + K_{or}\omega_{nn}^4}{(d+d_a\omega_{nn})^2(d+d_b\omega_{nn})^2}1$$

$$= K_{or}\left\{1-\frac{1}{(d_a-d_b)^3}\left[\left(\frac{d_a-3d_b}{d_b^2}+\frac{2K_{1r}}{K_{or}}\right)\epsilon^{-d_b\omega_{nn}\tau}-\left(\frac{d_b-3d_a}{d_a^2}+\frac{2K_{1r}}{K_{or}}\right)\epsilon^{-d_a\omega_{nn}\tau}\right] - \frac{\omega_{nn}\tau}{(d_a-d_b)^2}\left[\left(\frac{1}{d_b}-\frac{K_{1r}}{K_{or}}\right)\epsilon^{-d_b\omega_{n}\tau}+\left(\frac{1}{d_a}-\frac{K_{1r}}{K_{or}}\right)\epsilon^{-d_a\omega_{nn}\tau}\right]\right\} \qquad (4.60)$$

Eqs. (4.58), (4.59) and (4.60) are convenient for application when $\zeta > 1$.

When $\zeta < 1$, substitute the complex values of d_a and d_b into Eq. (4.58). After long manipulation, we get

$$\frac{\omega_{nn}^4}{(d^2 + 2\zeta\omega_{nn} d + \omega_{nn}^2)^2} 1 = 1 - \frac{\epsilon^{-\zeta\omega_{nn}\tau}}{1-\zeta^2}\left[\frac{1-\frac{\zeta^2}{2}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\omega_{nn}\tau + \phi_{or}\right) + \frac{\omega_{nn}\tau}{2} \sin\left(\sqrt{1-\zeta^2}\omega_{nn}\tau + \phi_{or}'\right)\right] \quad (4.61)$$

where $\phi_{or} = \tan^{-1} \dfrac{2(1-\zeta^2)^{3/2}}{\zeta(3-2\zeta^2)}$; $\phi_{or}' = -\tan^{-1}\dfrac{\zeta}{\sqrt{1-\zeta^2}}$ (4.61)a

Eq. (4.59) can be transformed into the following form for $\zeta < 1$

$$\frac{\omega_{nn}^3 d}{(d^2 + 2\zeta\omega_{nn} d + \omega_{nn}^2)^2} 1 = \frac{\epsilon^{-\zeta\omega_{nn}\tau}}{2(1-\zeta^2)}\left[\omega_{nn}\tau \cos\left(\sqrt{1-\zeta^2}\omega_{nn}\tau\right) - \frac{1}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\omega_{nn}\tau\right)\right] \quad (4.62)$$

Careful manipulation of Eq. (4.60) gives the following result for $\zeta < 1$

$$\frac{K_{1r}\omega_{nn}^3 d + K_{or}\omega_{nn}^4}{(d^2 + 2\zeta\omega_{nn}^2 d + \omega_{nn}^2)^2} 1 = K_{or}\left\{1 - \frac{\epsilon^{-\zeta\omega_{nn}\tau}}{2(1-\zeta^2)}\left[-\sigma'\omega_{nn}\tau(\cos\omega_{nn}\tau + \phi_r') + \sigma \sin(\sqrt{1-\zeta^2}\omega_{nn}\tau + \phi_r)\right]\right\} \quad (4.63)$$

where $\sigma' = \left[\left(\dfrac{K_{1r}}{K_{or}} + \zeta\right)^2 + 1 - \zeta^2\right]^{1/2}$ (4.63)a

$$\sigma = \left\{\left[\frac{K_{1r}}{K_{or}} + \frac{(2-\zeta^2)\zeta(3-2\zeta^2)}{2\sqrt{1-\zeta^2}}\right]^2 + (2-\zeta^2)^2(1-\zeta^2)^2\right\}^{1/2} \frac{1}{\sqrt{1-\zeta^2}} \quad (4.63)b$$

$$\phi_r = \tan^{-1} \frac{(2-\zeta^2)(1-\zeta^2)^2}{\frac{K_{1r}}{K_{or}} + \frac{(2-\zeta^2)\zeta(3-2\zeta^2)}{2\sqrt{1-\zeta^2}}} \quad (4.63)c$$

$$\phi_r' = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\frac{K_{1r}}{K_{or}} + \zeta} \quad (4.63)d$$

Summary

From the above analysis, the response of an automatically controlled system on unit function can be calculated first by decomposing the function $\frac{Q(d)}{S(d)}$ into elementary characteristic component of one degree of freedom and then each of the components will respond to the step function simultaneously. For one-degree-of-freedom component, the unit response can be easily found by the nondimensional operational formulae developed in this chapter. Nondimensional operation formulae of unit response with repeating roots in $S(d) = 0$ are also developed for both $\zeta \gtrless 1$ and $\zeta = 1$

CHAPTER THIRTEEN

SURGING ERROR AND SURGING DISTURBANCE

50. Surging Error or Surge

When a system is disturbed, its motion goes off from equilibrium state and gradually comes back to the original state of equilibrium, or continues on until it reaches another equilibrium state. During this course of change, the deviation from the equilibrium state (original or the new one) may reach a largest magnitude which shall never be exceeded until another disturbance comes to affect it. Such maximum deviation is defined as surge error, or often it is named the first surging error. Mathematically it can be expressed as:

$$R(t_1) = \frac{Q(0)}{S(0)} + \sum_1^k \frac{Q(d_k)}{d_k S'(d_k)} \epsilon^{-d_k t_1}$$

where t_1 satisfies $R'(t_1) = 0$ for the first occurrence.

However, when there are a number of components, it is difficult to solve t_1 from the condition $R'(t) = 0$ analytically. When the response $R(t)$ is plotted, the surging error can be measured then. Actually, only the predominant component, usually the low frequency one, plays an important role on this surging error. Differentiation may therefore be applied to that component only and solve t_1' from $r_f'(t) = 0$ and find $r_f(t_1')$ instead of $R(t_1)$. Such approximate value usually gives a good check to that measured from the response curve.

51. Surging Disturbance and Unit Surging Disturbance Function

Step function disturbance is justifiable in many cases. Sudden application of D.C. voltage to a network falls in this type of disturbance. When an automatic direction finder is called to action by suddenly switching in, this also belongs to this type of function. However, "rough air" does not possess step-function characteristics unless the airplane is flown into a storm where air current rises steadily. A gust, whether horizontal, vertical or rotary, does not keep its magnitude. In fact, it rises to a certain magnitude and then dies away. Later on a second gust follows. Physically, the rise of a gust from zero to a certain magnitude takes time no matter how fast it is. That is, $\left(\frac{dW}{dt}\right)_{t=0} \neq \infty$ as what step function is. Such rise and fall of a gust is evident, however, due to lack of experimental data of such rise and fall; true representation of a gust train is impossible.

However, a single gust probably can be represented by Kte^{-bt}, for such function has its surging phenomenon; that is, it rises to maximum at $t = \frac{1}{b}$ and then dies away gradually to zero. When b is larger, it reaches its maximum sooner and then dies away faster and vice versa.

It is more convenient to study the effect of such surging disturbance by keeping its maximum magnitude at unity.

Let $\quad I_s(t)1 = ebte^{-bt}1$ be such function. $\qquad$ (4.64)

Then $\quad I_s'(t)1 = eb\left[e^{-bt}(1-bt)\right]$ for $t > 0$.

Put $\quad I_s'(t)1 = 0$; we have $t = \frac{1}{b}$.

Substitute $t = \frac{1}{b}$ into equation (4.64), and we have:

$$I_s(t)1_{max} = 1 \tag{4.65}$$

where subscript s refers to "surging", and the function $I_s(t)1 = \epsilon bt \epsilon^{-bt}1$ is defined as unit surging disturbance function or unit surging input function. The larger the constant b is, the faster the surging phenomenon is. When b approaches ∞, the surging phenomenon approaches a quick surging impulse with peak value equal to unity.

The unit surging disturbance can be put in non-dimensional form:

$$\text{Let } \frac{t}{T} = \tau \tag{4.66}$$

where $T = \frac{m}{\frac{\rho}{2} SU}$, equation (1.67), for the longitudinal motion of the airplane. Then $bt = bT\tau$ where bT is non-dimensional damping.

$$\text{Let } bT = \xi_1 \omega_{nn1} = \xi_2 \omega_{nn2} = \xi_f \omega_{nnf} \tag{4.67}$$

where ξ_1 = damping ratio of the surging disturbance by assuming its natural frequency to be the same as that of the characteristic component 1, etc.

$$I_s(\tau)1 = \epsilon\, \xi_f \omega_{nnf} \tau \epsilon^{-\xi_f \omega_{nnf} \tau}\, 1 \tag{4.68}$$

For simplicity, just drop the subscript f:

$$I_s(\tau)1 = \epsilon\, \xi \omega_{nn} \tau \epsilon^{-\xi \omega_{nn} \tau}\, 1 \tag{4.68a}$$

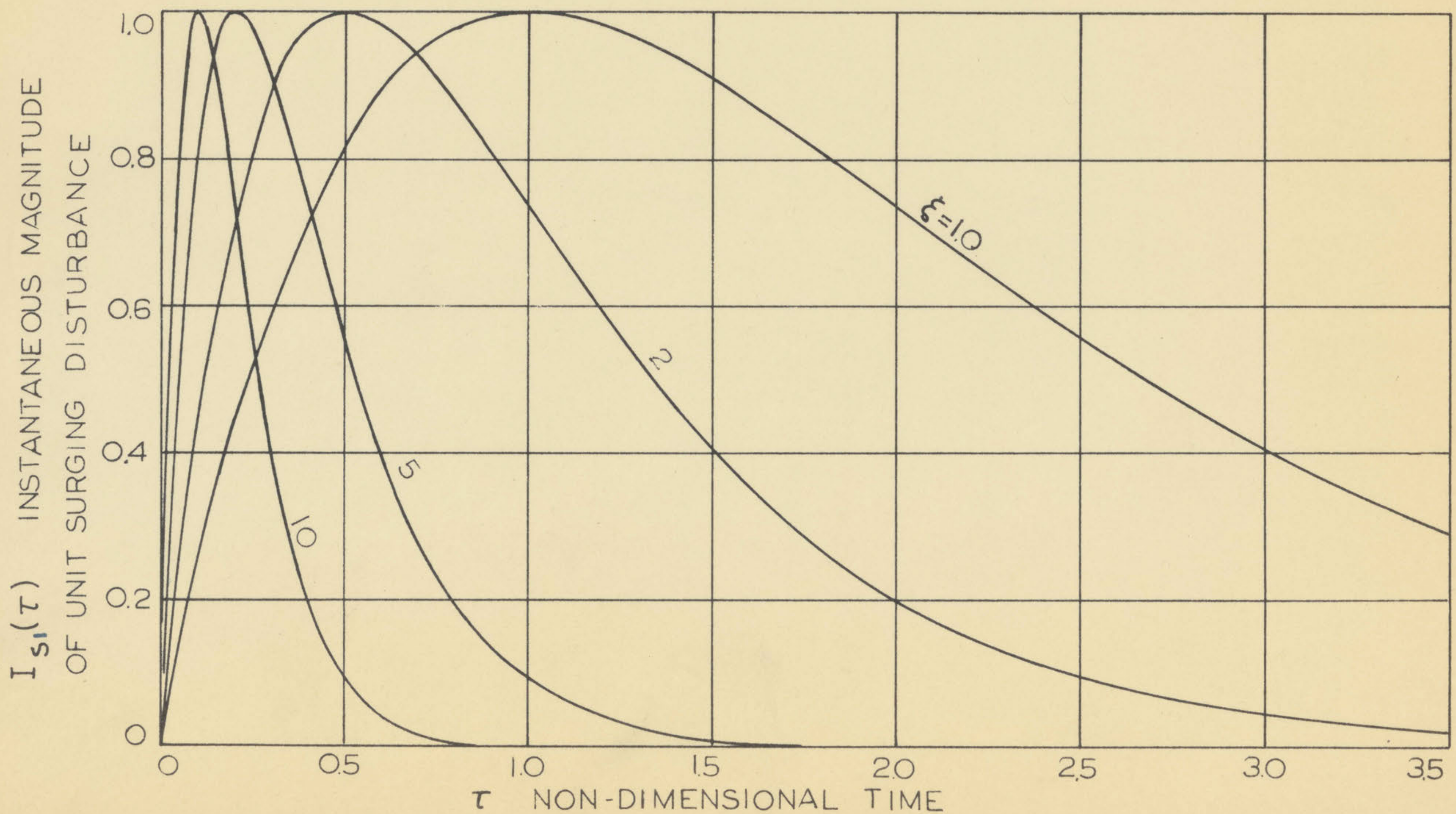
A plot showing $I_{s_1}(\tau)1 = \epsilon \xi \epsilon^{-\xi\tau} 1$ (4.68)b

is shown in Figure XXII with ξ as varying parameter to show its effect upon the rapidity of surging.

FIG. XXII

UNIT SURGING DISTURBANCE FUNCTION WITH

VARIOUS APPARENT SURGING FACTORS ξ



As the rapidity of surging is determined by ξ when the disturbance is applied to the characteristic component, and because $\xi \omega_{nn}$ = constant for a certain disturbance, therefore it will be a fast surging disturbance when the dealt component is a slow oscillatory one, and a slow surging disturbance when the dealt component is a fast one.

Differentiate equation (4.68)b:

$$I_{s1}'(\tau)1 = \epsilon\, \xi \left[\epsilon^{-\xi\tau}(1-\xi\tau)\right] \qquad \text{for } \tau > 0 \qquad (4.69)$$

When τ approaches zero from the positive side, we have:

$$I_{s1}'(\tau)1_{\tau\to 0} = \epsilon\,\xi \qquad (4.69)a$$

Equation (4.69)a indicates the fact that the apparent (apparent to the characteristic component we are dealing) initial surging speed is proportional to ξ . Therefore, ξ is defined as apparent surging factor and $\epsilon\xi$ is defined as apparent initial surging speed.

52. Response Due to Surging Disturbance

Repeat equation (4.01) with addition of subscripts; it will be the operational form of unit surging response:

$$R_{hs}(\tau) = \frac{Q_h(d)}{S(d)} I_s(\tau)1 \qquad (4.70)$$

To evaluate equation (4.70), several directions of approach can be equally applied. But for the interest of engineers, we shall first follow the method of characteristic decomposition to have:

$$\frac{Q(d)}{S(d)} = \sum_{f=1}^{f=f} \frac{q_f(d)}{S_f(d)} + \sum_{n=1}^{n=n}\sum_{r=f+1}^{r=r} \frac{q_{nr}(d)}{S_r^n(d)} \qquad (4.71)$$

When the above procedure has been done, the effect of surging disturbance can be studied component after component.

Now assuming one of the components may be represented by the expression:

$$\frac{q(d)}{S(d)} = \frac{K_o \omega_{nn}^2}{d^2 + 2\zeta\omega_{nn} d + \omega_{nn}^2} \tag{4.71a}$$

Then applying the surging disturbance function, we have:

$$r_s(\tau) = \frac{K_o \omega_{nn}^2}{d^2 + 2\zeta\omega_{nn} d + \omega_{nn}^2} \epsilon \xi \omega_{nn}\tau \epsilon^{-\xi\omega_{nn}\tau} \tag{4.72}$$

By application of transformation formulae of operational calculus we may derive the expression $r_s(\tau)$ according to the following steps:

$$r_s(\tau) = \left(\epsilon^{-\xi\omega_{nn}\tau}\right) \frac{\epsilon K_o \omega_{nn}^2}{\left[d + (d_1 - \xi)\omega_{nn}\right]\left[d + (\bar{d}_1 - \xi)\omega_{nn}\right]} \xi\omega_{nn}\tau\, 1 \quad * \tag{4.72a}$$

where $d_1 , \bar{d}_1 = \zeta \pm i\sqrt{1-\zeta^2}$

$$r_s(\tau) = \left(\epsilon^{-\xi\omega_{nn}\tau}\right) \frac{\epsilon K_o \omega_{nn}^2 \xi\omega_{nn}}{\left[d + (d_1 - \xi)\omega_{nn}\right]\left[d + (\bar{d}_1 - \xi)\omega_{nn}\right] d}\, {}^{**} 1 \tag{4.72c}$$

$$r_s(\tau) = \epsilon K_o \xi \epsilon^{-\xi\omega_{nn}\tau} \left\{ \frac{\omega_{nn}}{(d_1 - \xi)(\bar{d}_1 - \xi) d} {}^{***} - \frac{\omega_{nn}}{(\bar{d}_1 - d_1)(d_1 - \xi)\left[d + (d_1 - \xi)\omega_{nn}\right]} - \frac{\omega_{nn}}{(d_1 - \bar{d}_1)(\bar{d}_1 - \xi)\left[d + (\bar{d}_1 - \xi)\omega_{nn}\right]} \right\} \tag{4.72d}$$

$$\therefore\ r_s(\tau) = K_o \epsilon \xi \left[\mu_\zeta \frac{\epsilon^{-\zeta\omega_{nn}\tau}}{\sqrt{1-\zeta^2}} \cos\left(\sqrt{1-\zeta^2}\,\omega_{nn}\tau + \phi\right) + \epsilon^{-\xi\omega_{nn}\tau}\left(\mu_{0\xi} + \mu_{1\xi}\omega_{nn}\tau\right) \right] \tag{4.73}$$

* by formula 293, p. 134, Bush's "Operational Circuit Analysis"
** by formula 205, p. 117, Bush's "Operational Circuit Analysis"
*** obtained by characteristic decomposition

where $$\mu_\zeta = \frac{1}{(\zeta-\xi)^2 + 1 - \zeta^2} = \mu_{1\xi} \qquad (4.73)a$$

$$\mu_{o\xi} = \frac{2(\xi-\zeta)}{\left[(\zeta-\xi)^2 + 1-\zeta^2\right]^2} = 2\mu_{1\xi}^2(\xi-\zeta) \qquad (4.73)b$$

$$\mu_{1\xi} = \frac{1}{(\zeta-\xi)^2 + 1-\zeta^2} = \mu_\zeta \qquad (4.73)c$$

$$\phi = \tan^{-1}\frac{(\zeta-\xi)^2 - (1-\zeta^2)}{2(\zeta-\xi)\sqrt{1-\zeta^2}} \qquad (4.73)$$

It is interesting to notice that:

(1) The oscillatory component keeps its essential characteristic; that is, its damping, and angular velocity.

(2) The surging disturbance is magnified by a factor equal to

$$K_o\mu_{1\xi}$$

(3) An additional subsiding component takes place of the steady state (when unit step disturbance is applied).

(4) However, the guiding magnitude factor* of the oscillatory component is multiplied by a factor equal to $\epsilon\xi\mu_\zeta$, and the phase shift is modified too.

(5) When ξ is very large or it is an apparent fast surging disturbance, all these factors $\xi\mu_\zeta$, $\xi\mu_{o\xi}$ and $\xi\mu_{1\xi}$ become small and approach $\frac{1}{\xi}$, $\frac{2}{\xi}$ and $\frac{1}{\xi}$ as limits. Physically it means that

* Compare to equation (4.36)

this component is so inert to the surging disturbance which is very fast apparent to the system (or rather to this component we are dealing). On the other hand, when ξ is small, their magnitudes are small too because of this small factor ξ . Physically it means a very sluggish surging disturbance would be so gentle that it can only affect or disturb the system unnoticeably.

It is therefore believed that the system will be disturbed most violently by a surging disturbance of certain particular apparent surging factor.

At first glance, one would suggest determining such maximum disturbance by maximizing $\xi \mu_\xi$, $\xi \mu_{o\xi}$ and $\mu_{1\xi}$. Unfortunately, this cannot be done for simultaneous occurrence. Besides, when a system is comprised of several degrees of freedom, analytical maximization is impossible. The conclusion can be safely obtained after the response is plotted.

(6) If the numerator of the right side of equation (4.71) contains a d term such as $\kappa_1 \omega_{nn} d$, the complete solution of such component when disturbed by surging disturbance shall be:

$$r_s(\tau) + \frac{\kappa_1}{\kappa_o} r_s'(\tau)$$

where $r_s(\tau)$ is the equation of (4.73) and $r_s'(\tau) = \frac{d}{d\tau} r_s(\tau)$.

(7) The simultaneous effect of the same surging disturbance upon all the characteristic components will yield as many surging components and as many subsiding components as the number of characteristic components. Each of them has the

same factor $\epsilon^{1-\xi\omega_{nn}\tau}$, but the factors $K_0\xi\mu_{0\xi}$ and $K_0\xi\mu_{1\xi}$ are all different. The total surging and subsiding components can be written as:

$$\epsilon^{1-\xi\omega_{nn}\tau}\left\{\sum(K_0\xi\mu_{0\xi})_f+\left[\sum(K_0\mu_{1\xi})_f\right]\xi\omega_{nn}\tau\right\}$$

Additional terms that contain $(K_1)_f$ should be included.

53. Substitution of Operational Expression for the Unit Surging Disturbance

It is quite tedious to follow the analysis given in Paragraph 52. An alternative method is briefly formulated here.

The unit surging disturbance $\epsilon\xi\omega_{nn}\tau\epsilon^{-\xi\omega_{nn}\tau}1$ can be substituted by its operational form:

$$I_s(\tau)1=\epsilon\xi\omega_{nn}\tau\epsilon^{-\xi\omega_{nn}\tau}1=\frac{\epsilon\xi\omega_{nn}d}{(d+\xi\omega_{nn})^2}1 \qquad (4.74)$$

It should be noted that $\xi\omega_{nn}$ is a constant for a particular unit surging disturbance.

Equation (4.74) is then substituted into equation (4.70) and we have:

$$R_{hs}(\tau)=\frac{Q_h(d)}{S(d)}\cdot\frac{\epsilon\xi\omega_{nn}d}{(d+\xi\omega_{nn})^2}1 \qquad (4.75)$$

The procedure of characteristic decomposition can be applied with two additional terms, as:

$$\frac{Q_h(d)\epsilon\xi\omega_{nn}d}{S(d)(d+\xi\omega_{nn})^2}=\epsilon\xi\omega_{nn}\left[\frac{K_s\xi\omega_{nn}}{d+\xi\omega_{nn}}+\frac{K_{sr}\xi\omega_{nn}}{(d+\xi\omega_{nn})^2}+\sum_f\frac{g_{sf}(d)}{S_f(d)}+\sum_r\sum_n\frac{g_{snr}(d)}{S_r^n(d)}\right] \qquad (4.76)$$

Equation (4.76) gives the separation of the characteristic components, each of which can be solved upon the unit step function.

PART V

PERFORMANCE OF TYPICAL AIRPLANE WITH NONIDEAL CONTROL AND SOME REFINEMENT CONSIDERATIONS

INTRODUCTION

In the foregoing chapters, the general procedure in analyzing an automatically controlled problem has been well established. Stability improvement is usually analyzed before the transient response. In fact, with the aid of the foregoing chapters, we may specify the stability improvement for the problem in which we are interested, and determine the control constant as well as the coupling coefficients thereof. With such control and coupling coefficients, the transient response can be analyzed. If the transient is considered satisfactory, the problem is solved; otherwise the specification of stability improvement should be revised until a satisfactory transient is also obtained. Therefore, actual designing of control is a matter of compromise.

In order to gain the freedom to control the uncontrolled quartic stability function of the longitudinal motion of an airplane, the control specification is fixed by the parasite* minor of the stability determinant of the uncontrolled longitudinal motion. The compromise between the stability improvement and the transient improvement will be left entirely to the variation of coupling coefficients.

A numerical example becomes necessary to show the validity and facility of the theory and analysis which have been established in the foregoing chapters. We shall take the Fair-

---

*See Chapter Two, Part I.

child 22 as a studying subject of which the aerodynamic characteristics have been thoroughly investigated and are believed in the average region of a good many modern airplanes so far as nondimensional derivatives are concerned.

We shall assume the disturbance to be a surging veritcal gust of unit magnitude.

CHAPTER FOURTEEN

AIRPLANE CONTROLLED BY PITCHING VELOCITY-ELEVATOR CONTROL COUPLING

51. Dimensional Data, Nondimensional Aerodynamic Derivatives, and Flight Conditions of the Fairchild 22 [39]

(a) Dimensional Data:

Wing area (S) (high wing, monoplane)	171	sq. ft.
Span (b)	32.83	ft.
Stabilizer area	15.8	sq. ft.
Elevator area	10.4	sq. ft.
Tail length (L)	14.69	ft.
Weight (W)	1600	pounds
Radius of gyration in pitch	4.41	ft.
Wing setting	1	degree

(b) Horizontal Flight Condition [40]

Power-off, horizontal flight

Airspeed (U_o)	133	f.p.s.
Altitude (H) (above sea level)	3000	ft.
Air density at 3000 ft. above sea level	0.00218	$\frac{\text{slug}}{\text{ft}^3}$
C_L	-0.45	

(c) Fundamental Units in Nondimensional System

Unit of mass (m)	50	slugs
Unit of time (T)	2	seconds
Unit of length (L)	15	feet
Compact parameter μ	20	

(d) The Nondimensional Aerodynamic Derivatives*

x_u	-0.15**
x_w	0.40
z_u	-1.00
z_w	-4.50
m_w	-3.00
m_q	-6.00

θ_0, α_0, x_q, z_q and m_u are all assumed negligible compared with the other terms in the stability equations.

52. Uncontrolled Pitching Motion with Vertical Surging Gust

We shall examine the response in pitching and in vertical motion when a vertical surging gust of unit magnitude acts on the uncontrolled airplane.

$$\frac{\theta}{W_0} = \frac{1}{\left[\frac{L}{T}\right]} \cdot \frac{d\begin{vmatrix} d-x_u & -x_w \\ -m_u & -m_w \end{vmatrix}}{\Delta_0} \cdot \frac{\epsilon \xi \omega_{nn} d}{(d + \xi\omega_{nn})^2} 1$$

$$= \frac{1}{\left[\frac{L}{T}\right]} \cdot \frac{d(-dm_w + x_u m_w - m_u x_w)}{\Delta_0} \cdot \frac{\epsilon \xi_{nn} d}{(d + \xi\,\omega_{nn})^2} 1$$

$$= \frac{1}{\left[\frac{L}{T}\right]} \cdot \frac{3d(d + .15)}{d^4 + 10.65d^3 + 89.0d^2 + 15.5d + 27.0} \times \frac{\xi\,\omega_{nn} d}{(d + \xi\,\omega_{nn})^2} 1 \qquad (5.01)$$

Here, $Q(d) = 3d(d + .15) = 3d^2 + .45d$ (5.01)a

$S(d) = d^4 + 10.65d^3 + 89.0d^2 + 15.5d + 27.0$ (5.01)b

---

* See Table II, Chapter Two, Part I.

**Figures are transformed from the measured results (Klemin's T.N. 666) and rounded off to the nearest significant figures.

Convert S(d) into $S_\lambda(\lambda)$

$$S_\lambda(\lambda) = \lambda^4 + 4.67\lambda^3 + 17.09\lambda^2 + 1.31\lambda + 1 \qquad (5.02)$$

Here $\alpha_3 = 4.67$, $\alpha_2 = 17.09$, $\alpha_1 = 1.31$

Immediately we know that the high frequency component has a larger damping ratio, (because $\alpha_3 > \alpha_1$) and the system is stable (because $\alpha_2 > \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}$).

Then obtain $M = \frac{\alpha_3\alpha_1 - 4}{\alpha_2^2} = 0.0072$

$$N = \frac{\alpha_3^2 + \alpha_1^2 - 4\alpha_2}{\alpha_2^3} = -0.009$$

Because M is positive and N is negative the system is doubly oscillatory; that is, both high and low components are oscillatory.

Follow the direction of the quartic chart and we shall get

$\rho_\omega = 16.72$

$\omega_r = 00.244$, $\omega_{nn1} = \omega_r 27^{\frac{1}{4}} = 0.556$, $\omega_{nn2} = 16.72\,\omega_{nn1} = 9.31$

$\rho_\zeta = 4.52$ (5.03)

$\zeta_1 = 0.1245$

$\zeta_2 = \zeta_1 \rho_\zeta = 0.561$

Therefore, the characteristic roots in d are:

$$\begin{aligned} d_1 &= (-0.1245 + i.992) \\ \bar{d}_1 &= (-0.1245 - i.992) \\ d_2 &= (-0.561 + i.827) \\ \bar{d}_2 &= (-0.561 - i.827) \end{aligned} \qquad (5.03)a$$

Since there is no repeating root in the uncontrolled quartic stability equation, it can be shown* that

---

*See Appendix C

$$\frac{Q(d)}{S(d)}\,\frac{\epsilon\xi\omega_{nn}d}{(d+\xi\omega_{nn})^2} = 1$$

$$= \sum_1^k \epsilon\xi\omega_{nn}A_k\,\epsilon^{d_k\omega_{nnk}\tau} + \sum_1^f \epsilon\xi\omega_{nn}2\sqrt{A_f\bar{A}_f}\,\epsilon^{-\zeta_f\omega_{nnf}\tau}\cos(\omega_{nnf}\sqrt{1-\zeta_f^2}\,\tau+\phi_f)$$

$$+(B_k+B_f)\,\epsilon\xi\omega_{nn}\epsilon^{-\xi\omega_{nn}\tau}$$

$$+\left[C_k+C_f+\frac{Q(0)}{S(0)}\right]\epsilon\xi\omega_{nn}\tau\epsilon^{-\xi\omega_{nn}\tau} \qquad (5.04)$$

where $A_k = \dfrac{Q(d_k\,\omega_{nnk})}{S'(d_k\,\omega_{nnk})(\xi_k+d_k)\,\omega_{nnk}}$

$$A_f = \frac{Q(d_f\,\omega_{nnf})}{S'(d_f\,\omega_{nnf})(\xi_f+d_f)^2\,\omega_{nnf}^2}\ *$$

$$\bar{A}_f = \frac{Q(\bar{d}_f\,\omega_{nnf})}{S'(\bar{d}_f\,\omega_{nnf})(\xi_f+\bar{d}_f)^2\,\omega_{nnf}^2}$$

$$B_k = -\sum_1^k A_k$$

$$B_f = -\sum_1^f (A_f+\bar{A}_f)$$

$$C_k = \sum_1^k \frac{Q(d_k\,\omega_{nnk})}{S'(d_k\omega_{nnk})(d_k\omega_{nnk})}\cdot\frac{\xi_k}{(\xi_k+d_k)}$$

$$C_f = \sum_1^f\left[\frac{Q(d_f\,\omega_{nnf})}{S'(d_f\,\omega_{nnf})(d_f\,\omega_{nnf})}\cdot\frac{\xi_f}{\xi_f+d_f}+\frac{Q(\bar{d}_f\,\omega_{nnf})}{S'(\bar{d}_f\omega_{nnf})(\bar{d}_f\omega_{nnf})}\cdot\frac{\xi_f}{(\bar{d}_f+\xi_f)}\right.$$

$$\phi_f = \tan^{-1}\frac{\underline{I}_{Qf}}{\underline{R}_{Qf}} - \tan^{-1}\frac{\underline{I}_{S'f}}{\underline{R}_{S'f}} - \tan^{-1}\frac{2(\xi_f-\zeta_f)\sqrt{1-\zeta_f^2}}{(\xi_f-\zeta_f)^2-(1-\zeta_f^2)} \qquad (5.04)a$$

where k goes with nonoscillatory components and f oscillatory one.

---

* Practical evaluation of polynomial function of complex variable is greatly simplified by applying the De Moivre theorem graphically. See Appendix D.

In case the stability function only shows oscillatory characteristic, $k = 0$, and $A_k = B_k = C_k = 0$.

Assuming the surging disturbance has an apparent surging factor equal to unity with respect to the predominant component or component 1 (low frequency component), it follows that

$$\xi_1 \omega_{nn1} = \omega_{nn1}, \text{ or } \quad \xi_1 = 1; \text{ therefore}$$

$$\xi_2 \omega_{nn2} = \omega_{nn1}, \text{ or } \quad \xi_2 = \frac{\omega_{nn1}}{\omega_{nn2}} = \frac{1}{\rho_\omega} = .0592 \qquad (5.05)$$

that is, with the same surging disturbance, it appears to be surging slowly with respect to the high frequency component.

Substitute the values of d_1, $\bar{d}_1$, d_2, $\bar{d}_2$, ξ_1 and ξ_2 of Eq. (5.05) into Eq. (5.04)a, and we have

$$2\sqrt{A_1\bar{A}_1} = 0.0366, \quad \phi_1 = -3.0^\circ \qquad A_k = B_k = C_k = 0$$

$$2\sqrt{A_2\bar{A}_2} = 0.00532, \quad \phi_2 = 28.5^\circ \qquad \frac{Q(0)}{S(0)} = 0 \qquad (5.06)$$

$$B_f = -0.0405, \quad c_f = 0.0145$$

When these values are substituted into Eq. (5.04) and multiplied by $\frac{1}{\left[\frac{L}{T}\right]} \times 57.3$, we have

$$\begin{aligned}\frac{\theta}{W_0} = {} & 0.421\, \epsilon^{-0.0693\tau} \cos(0.533\tau - 3.0^\circ) \\ & + 0.0612\, \epsilon^{-5.22\tau} \cos(7.7\tau + 28.5^\circ) \\ & - 0.466\, \epsilon^{-0.556\tau} \\ & + 0.167\tau\, \epsilon^{-0.556\tau}\end{aligned} \qquad (5.07)$$

In degrees per unit surging disturbance (max. 1 ft./sec. with apparent surging factor equal to one with respect to low frequency component.)

The second component, (quick oscillation) owing to its small initial magnitude and fast damping characteristic, dies away in less than one (or two seconds for our particular airplane at the assumed flight condition.) However it is an important component to adjust the initial condition of the response.

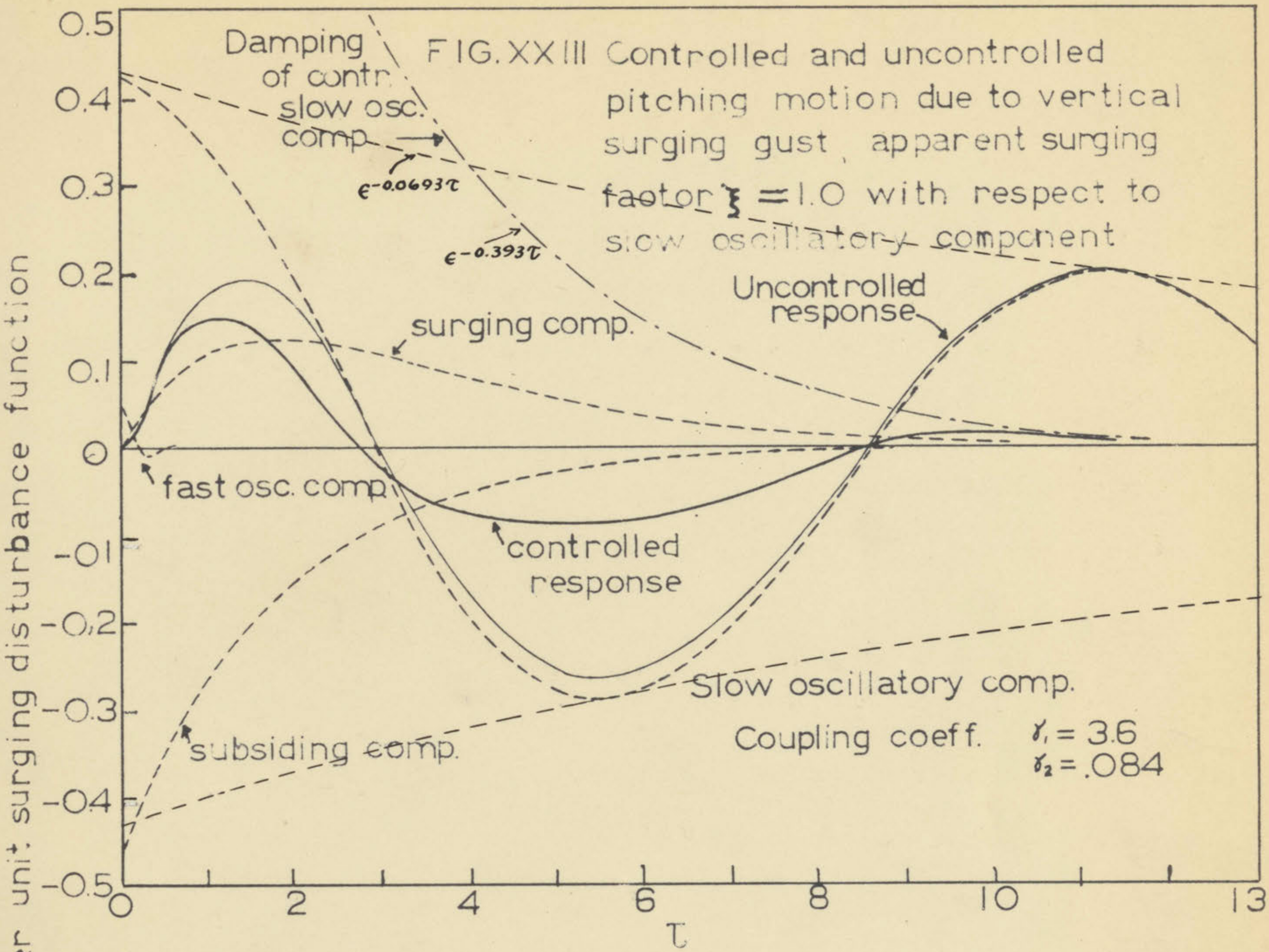


FIG. XXIII Controlled and uncontrolled pitching motion due to vertical surging gust, apparent surging ~~factor~~ $\xi = 1.0$ with respect to slow oscillatory component

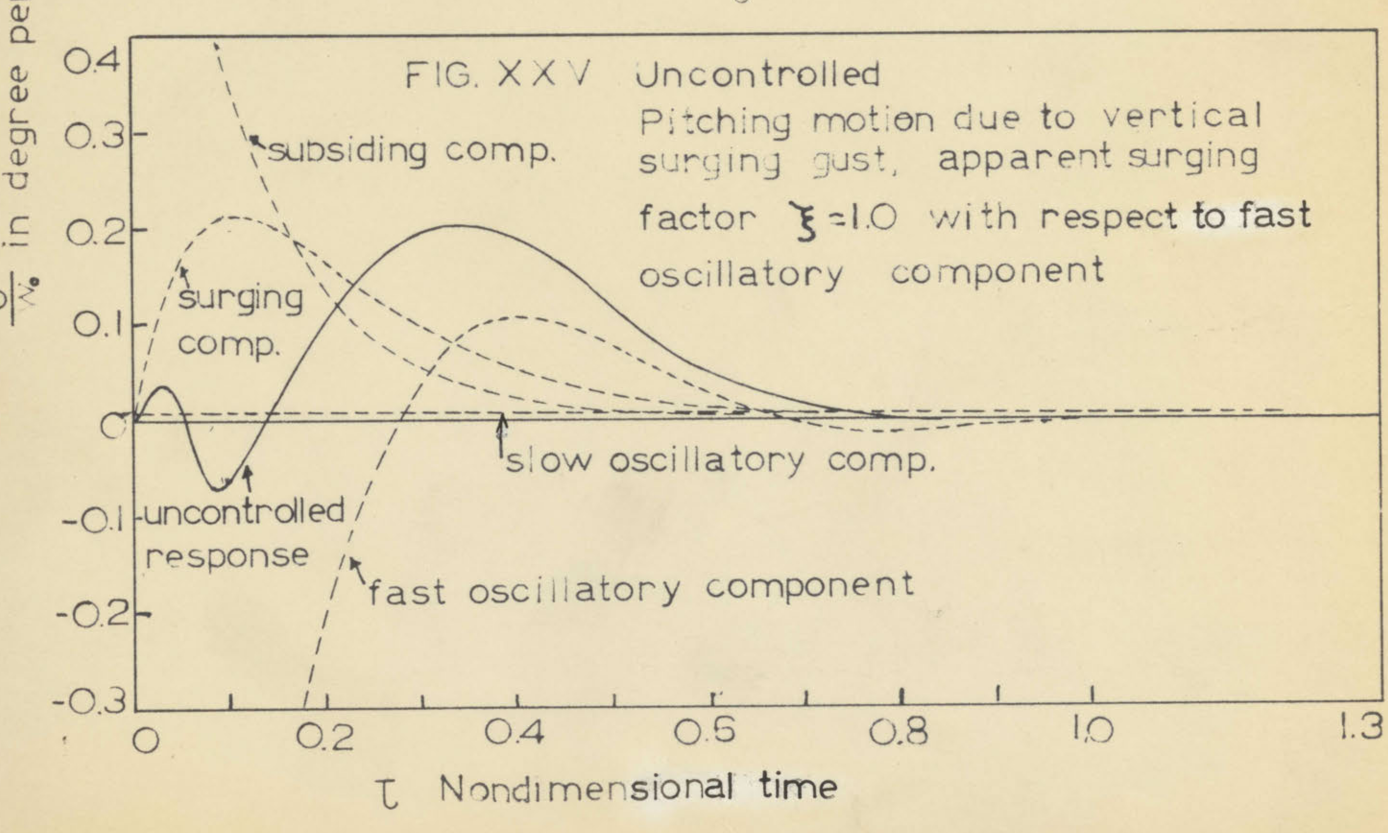
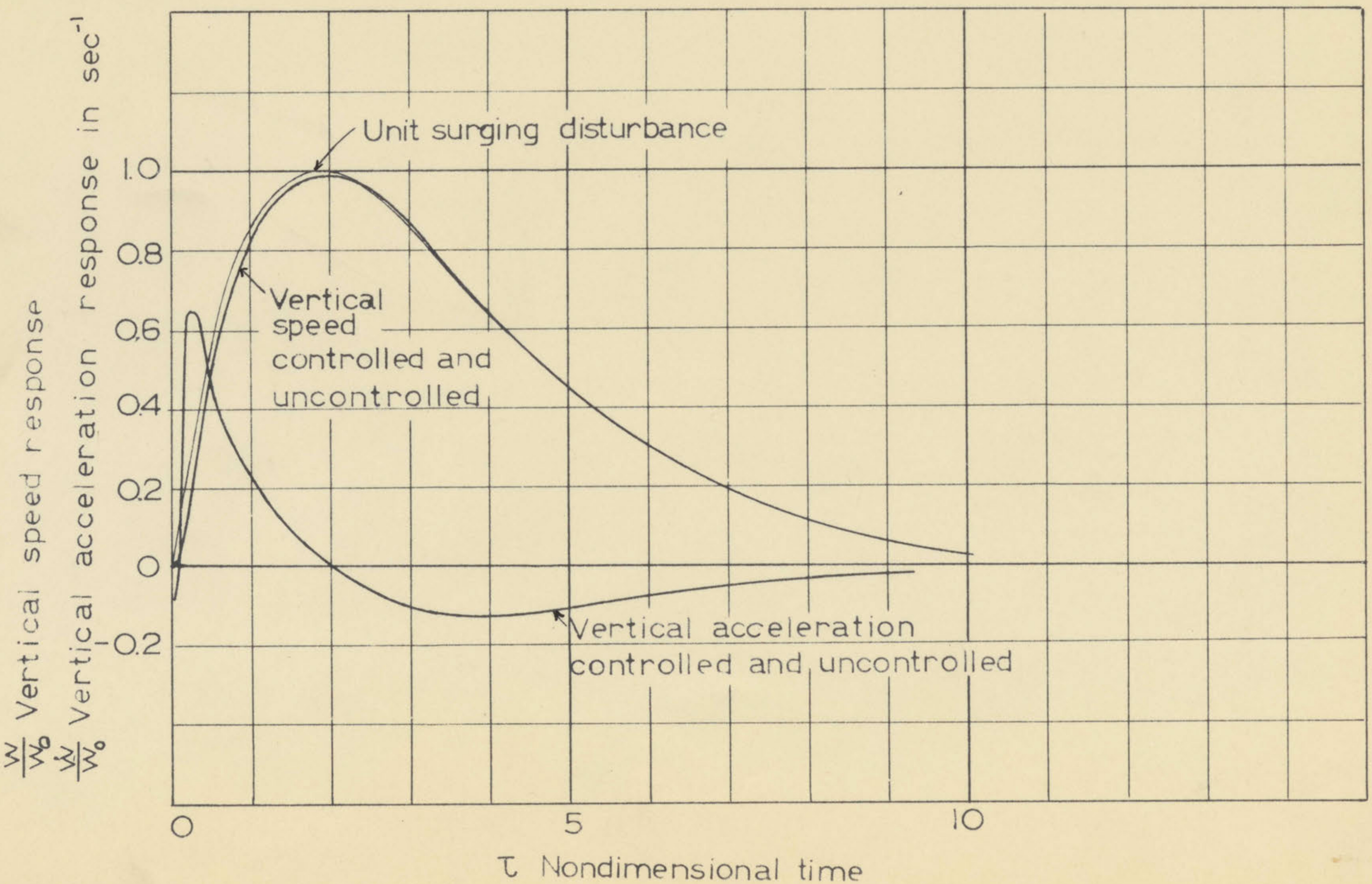


FIG. XXV Uncontrolled Pitching motion due to vertical surging gust, apparent surging factor $\xi = 1.0$ with respect to fast oscillatory component

FIG. XXIV

CONTROLLED AND UNCONTROLLED VERTICAL SPEED AND ACCELERATION RESPONS DUE TO VERTICAL SURGING DISTURBANCE, ξ =1.0 WITH RESPECT TO SLOW OSCILLATORY COMPONENT



All four components are plotted in Fig. 23 with the same scale separately in dotted curves to show their relative magnitudes and phases and their resultant is plotted in a light solid curve. It is interesting to notice, due to the surging characteristic of disturbance, the resultant curve shows a slightly apparent divergence at the first two peak values. But after the second peak the resultant is essentially the same as the component of slow oscillation with slow convergence indicated by guiding curve of damping $\epsilon^{-0.0693\tau}$.

53. Uncontrolled Vertical Motion with Vertical Surging Disturbance

The uncontrolled vertical motion when an airplane is encountered with a vertical surging gust can be expressed by

$$\frac{W}{W_O} = \frac{\Delta_o - d\begin{vmatrix} d-x_u & -C_L \\ -m_u & d^2 - dm_q \end{vmatrix}}{\Delta_o} \cdot \frac{\xi\omega_{nn}d}{(d + \xi\omega_{nn})^2}\, 1 \qquad (5.08)$$

With substitution of the particular constants of the airplane and the assumed flight condition and surging disturbance, Eq. (5.08) can be written in the following form

$$\frac{W}{W_O} = \frac{4.5d^3 + 88.1d^2 + 15.5d + 27}{d^4 + 10.65d^3 + 89d^2 + 15.5d + 27} \times \frac{.556d}{(d + .556)}1 \qquad (5.08)a$$

which gives the following solution:

$$\begin{aligned} \frac{W}{W_O} &= 0.01283\,\epsilon^{-0.0693\tau}\cos(0.533\tau - 9.08^\circ) \\ &+ 0.192\,\epsilon^{-5.22\tau}\cos(7.7\tau + 56.4^\circ) \\ &- 0.1055\,\epsilon^{-0.556\tau} \\ &+ 1.546\,\tau\epsilon^{-0.556\tau} \end{aligned} \qquad (5.09)$$

It is very interesting to notice that the slowly oscillatory component is of negligible importance in the vertical motion because of its minute magnitude. The response is highly predominated by the last surging term of the same shape with the disturbance. The second and third terms of (5.09) play as important a role in adjusting the initial condition of the response with a slight lagging in the rising part of the gust. This solution verifies the good following-up characteristic of the airplane in longitudinal disturbance which is contributed by the component of quick oscillation. As the apparent surging of the disturbance with respect to quick oscillation is very slow, such slow disturbance should be easily followed.

It is more important to know the vertical acceleration when the airplane is subjected to vertical surging gust. By differentiating Eq. (5.09) and dividing the result by T we have:

$$\frac{\dot{W}}{W_0} = \frac{1}{T}\Big[-0.01283\,\epsilon^{-0.0693\tau}\,\omega_{nn1}\sin(0.533\tau - 90.8^\circ + \phi_1') \\ -0.192\,\epsilon^{-5.22\tau}\,\omega_{nn2}\sin(7.7\tau + 56.4^\circ + \phi_2') \\ +(1.546 + .1055 \times .556)\,\epsilon^{-.556\tau} \\ - 1.546 \times .556\,\tau\,\epsilon^{-.556\tau}\Big]\,\frac{\text{ft./sec.}^2}{\text{unit surging disturbance}}$$

where $\phi_1' = \tan^{-1}\frac{\zeta_1}{\sqrt{1-\zeta_1^2}}$ and $\phi_2' = \tan^{-1}\frac{\zeta_2}{\sqrt{1-\zeta_2^2}}$

$$\frac{\dot{W}}{W_0} = -.00358\,\epsilon^{-0.0693\tau}\sin(0.533\tau - 83.6^\circ) \\ - .902\,\epsilon^{-5.22\tau}\sin(7.7\tau + 90.7^\circ) \\ + .815\,\epsilon^{-.556\tau} \\ -0.43\,\tau\,\epsilon^{-.556\tau} \qquad (5.10)$$

Again, the slow oscillation is negligible, while the quick oscillatory term is very important in adjusting the zero time condition, although after $\tau = 3$ its effect is negligible.

Both the vertical speed and vertical acceleration are plotted as Fig. 24. The oscillation of the vertical motion is unnoticeable. The acceleration shows a peak value of .65 ft. per second per second with unit surging gust of 1 ft. per second of unit apparent surging factor with respect to the slow component. During the decaying part of the gust, the vertical acceleration of the aircraft is negative and then gradually dies away with the gust.

With a vertical surging gust of apparent surging factor equal to unity with respect to the quick oscillation, the slow oscillation in pitch becomes unnoticeable while the quick oscillation gives maximum peak of .2 degree per unit surging disturbance, but it dies away before $\tau = 1$ (2 seconds) as shown in Fig. 25.

54. Disturbed Pitching Motion of the Airplane with Nonideal Control of Deparasitized Type When Encountered By Vertical Surging Gust

By the theory developed in Chapter Two we shall adopt the deparasitized control to allow better controlability. Such control should have the nondimensional undamped natural frequency ω_{nnp} and damping ratio ζ_p according to the following equation.

$$d^2 + 2\,\zeta_p\,\omega_{nnp}\,d + \omega_{nnp}^2 = \begin{vmatrix} d - x_u & -x_w \\ -z_u & d - z_w \end{vmatrix}$$

With the substitution of particular values of x_u, x_w, z_u and z_w, it is found:

$$\omega_{nnp} = 1.035$$

$$\zeta_p = 2.25$$

With such control, we can apply different θ derivative exciting forces and couple the control movement to the elevator. Now let the two components of uncontrolled motion be designated by the subscripts 0 and c instead of 1 and 2. The subscript zero is for the low frequency component, while c is for the high frequency component which is acting as a control component especially to the pitching motion, because the component introduced by the parasitized control is of zero magnitude. Through the action of control, the component 0 becomes compon- 1 and the component c becomes component 2 as usual. For the vertical motion there shall be an additional component which will be designated by subscript p.

The purpose of applying control is to regulate the distribution of damping ratio between the two components. For the uncontrolled motion $\rho_{\zeta_0} = \frac{\zeta_c}{\zeta_0} = 4.52$. It is reasonable to specify the controlled ρ_ζ less than unity because in such a case the decreased damping ratio in high frequency component is not serious for there is a time factor to affect the rate of decay. Now let us assume, the coupling shall not affect the frequency (natural undamped) of either component (so that ρ_ω is kept at 16.72) while ρ_ζ is expected to be 0.75. From Fig. 10 it is found that the damping advantage η_ζ arising from such specification is 5.7, so that $\zeta_1 = 0.707$ and $\zeta_2 = 0.531$.

Because the frequencies are not to be changed, it is necessary not to use the error in θ itself as a quantity to excite the control; error derivative coupling is therefore needed. From Fig. 11 with $\rho_\zeta = .75$, $\eta_\zeta = 5.7$, $\rho_\omega = 16.72$ it is found that γ_1, the first derivative coupling coefficient, must be 3.6 and from Fig. 12D γ_2, the second derivative coupling coefficient, must be 0.084.

Therefore the stability equation of the controlled longitudinal stability becomes

$$\Delta_c = (d^2+2\zeta_p\omega_{nnp}d+\omega_{nnp}^2)\left[d^4+10.65d^3+89(1+.084)d^2+15.5(1+3.6)d+27.0\right] \qquad \text{Eq.}(5.11)$$

$$= (d^2+2\zeta_p\omega_{nnp}d+\omega_{nnp}^2)(d^4+10.65d^3+96.5d^2+71.3d+27.0)$$

$$= (d^2+2\zeta_p\omega_{nnp}d+\omega_{nnp}^2)S_c(d)$$

with $f_{\dot\theta}m_\sigma = 15.5 \times 3.6 = 55.8$ (5.12)

and $f_{\ddot\theta}m_\sigma = 89 \times .084 = 7.5$ (5.13)

From Eqs. (5.12) and (5.13) the coefficient of exciting forces $f_{\dot\theta}$ and $f_{\ddot\theta}$ (nondimensional) can be evaluated because m_σ is fixed by the design of the elevator.

For the pitching motion, the response to a vertical surging disturbance can be expressed as

$$\frac{\theta}{W_0} = \frac{1}{\left[\frac{L}{T}\right]}\frac{(d^2+2\zeta_p\omega_{nnp}d+\omega_{nnp}^2)d\begin{vmatrix} d-x_u & -x_w \\ -m_u & -m_w \end{vmatrix}}{(d^2+2\zeta_p\omega_{nnp}d+\omega_{nnp}^2)S_c(d)} \times \frac{\xi\omega_{nn}d}{(d+\xi\omega_{nn})^2}1$$

$$= \frac{1}{\left[\frac{L}{T}\right]}\frac{d\begin{vmatrix} d-x_u & -x_w \\ -m_u & -m_w \end{vmatrix}}{S_c(d)} \times \frac{\xi\omega_{nn}d}{(d+\xi\omega_{nn})^2}1 \qquad (5.14)$$

It is noticed that the numerator of the operational expression does not change; the only difference is the denominator, which changes from $S(d)$ to $S_c(d)$.

Here the characteristic of $S_c(d)$ has been known from the specification of the problem, so it is not necessary to use the quartic chart to find those physical nondimensional quantities.

The solution to (5.14) can be written as

$$\begin{aligned} \frac{\theta}{W_0} = & \; 1.242\, \epsilon^{-0.393\tau} \cos(0.393\tau + 28.3^{\circ}) \\ & + 0.057\, \epsilon^{-4.93\tau} \cos(7.38\tau + 34.1^{\circ}) \\ & - 1.385\, \epsilon^{-0.556\tau} \\ & + 0.52\, \tau\, \epsilon^{-0.556\tau} \text{ in } \frac{\text{degrees}}{\text{unit surging disturbance with } \xi_1 = 1} \end{aligned} \qquad (5.15)$$

Compare Eq. (5.15) with Eq. (5.07). It may be concluded that

(a) the quick oscillation is essentially unchanged,

(b) the surging component is increased because the approaching of ζ_1 with ξ_1 (refer to Eq. (4.73)c).

(c) the slow oscillation is also magnified due to the same reason and with approximately the same ratio (should be the same as (4.73)a and (4.73)c indicate). However, the magnification does not do any harm because it converges rapidly for having a large damping ratio,

(d) the simple exponential term is also magnified to suit the zero condition.

Eq. (5.15) is plotted superimposed on Fig. 23 as a heavy solid curve. It is interesting to notice that the beginning part of the curve coincides with the uncontrolled disturbed pitching motion. This is no doubt due to the unavoidable control lag that the control cannot produce noticeable effect when the disturbance just begins. But after two seconds or one τ the control turns down the motion very rapidly; at the end of one cycle the disturbed pitching motion almost disappears along with the disappearance of the disturbance itself.

55. Disturbed Vertical Motion of an Airplane with Nonideal Control of Deparasitized Type when Encountered with Vertical Surging Gust

The qualitative function of the disturbed motion can be written as

$$\Delta_w = \Delta_{p\theta}\,(\Delta_{w0} + f_{\ddot{\theta}}m_\sigma d^2 + f_{\dot{\theta}}m_\sigma d)$$
$$- d(d - x_u)(f_{\ddot{\theta}}m_\sigma d^2 + f_{\dot{\theta}}m_\sigma d) \qquad (5.16)$$

Substitute the value of $f_{\ddot{\theta}}m_\sigma$ and $f_{\dot{\theta}}m_\sigma$ into Eq. (5.16) and expand the equation numerically so that we have

$$\Delta_w = 4.5d^4 + 116.5d^3 + 467.0d^2 + 223.3d + 29 - 7.5d^3 - 57.0d^2 - 8.4d \qquad (5.16)a$$

The negative terms are derived from $-d(d-x_u)(f_{\ddot{\theta}}m_\sigma d^2 + f_{\dot{\theta}}m_\sigma d)$ the presence of which causes the addition of overdamped component. However, by the examination on relative magnitude of the numerical coefficients, it can be roughly stated that the additional control component is of small magnitude. With the neglect of these negative terms, the response of vertical motion can be simplified as

$$\frac{W}{W_0} = \frac{\Delta_{p\theta}\,(\Delta_{w0} + f_{\ddot{\theta}}m_\sigma d^2 + f_{\dot{\theta}}m_\sigma d)}{\Delta_{p\theta} \quad \Delta_c} \times \frac{\xi\omega_{nn}}{(d + \xi\omega_{nn})^2}\,1$$

$$= \frac{\Delta_{w0} + f_{\ddot{\theta}}m_\sigma d^2 + f_{\dot{\theta}}m_\sigma d}{\Delta_0 + f_{\ddot{\theta}}m_\sigma d^2 + f_{\dot{\theta}}m_\sigma d} \times \frac{\xi\omega_{nn}}{(d + \xi\omega_{nn})^2}\,1 \qquad (5.17)$$

The solution to Eq. (5.17) should be a few per cent off the true value (greater than the true value). When it is plotted superimposedly on Fig. 23 with the allowance of being overestimated, the difference between the controlled and uncontrolled motion is almost undistinguishable. And the variation in the vertical acceleration is approximately the same in both cases.

It is therefore believed that with such (θ) deparasitized control the pitching motion is greatly eased on surging disturbance while the vertical motion cannot be appreciably improved.

CHAPTER FIFTEEN

SOME REFINEMENT CONSIDERATIONS IN THE θ DEPARASITIZED NONIDEAL LONGITUDINAL CONTROL

56. Possibility of Reducing Vertical Acceleration by Introducing Flap Control

The θ deparasitized control is effective in reducing the pitching oscillation discussed in the last chapter. Due to the high value of z_w, the vertical acceleration cannot be appreciably reduced. Weiss[4] has pointed out that even with a fully restrained pitching motion by a powerful fast θ control, the vertical motion cannot be eased unless z_w has been reduced. But due to the requirement of airplane efficiency, z_w has to be high by using large aspect ratio. When the airplane is disturbed, or is operating under disturbing air conditions, the only way to reduce z_w is to use flap control with the flap up when the airplane is struck by an upward gust. The moment variation of wing due to the flap movement is somewhat neutralized by the variation of the downward angle which affects the tail moment. There is some variation in x_w due to the same flap movement. Such flap control should be excited by the relative vertical velocity between the airplane and the vertical gust; in other words, the detecting instrument must be an airspeed meter with pitot tube heading along the vertical axis. The detailed analysis can be done by the aid of mathematics developed in this thesis, but due to lack of time the conclusion is not yet reached.

57. Effect of Time Lag on the Detecting Instrument

In the θ deparasitized longitudinal control, the control lag is entirely offset by the parasite minor so far as the pitching motion is concerned; however, the detecting instrument itself usually possesses certain lagging effects. If such lagging is counted in the operational form of the response, it would become more complicated. However, if such time lag is very short compared to the natural frequencies of the system to be controlled, approximation can be made from the expanded form of Taylor's theorem.

$$\theta(\tau - \tau_1) = \theta(\tau) - \tau_1 \theta'(\tau) + \frac{\tau_1^2}{2!}\theta''(\tau)$$

$$\theta'(\tau - \tau_2) = \theta'(\tau) - \tau_2 \theta''(\tau) + \frac{\tau_2^2}{2!}\theta'''(\tau), \text{ etc.}$$

where τ_1 and τ_2 are respectively time lag of error and error derivative of the detecting instrument. In operational form, they appear as

$$\theta(\tau - \tau_1) = (1 - \tau_1 d + \frac{\tau_1^2}{2!}d^2)\theta(\tau)$$

$$\theta'(\tau - \tau_2) = (d - \tau_2 d^2 + \frac{\tau_2^2}{2!} d^3)\theta(\tau)$$

Therefore, with an error-sensitive control and coupling factor $f_\theta m_\sigma$, a slight negative damping coupling factor $-\tau_1 f_{\dot\theta} m_\sigma$ and a slight positive accelerating coupling of coupling factor $\frac{\tau_1^2}{2!} f_{\ddot\theta} m_\sigma$ are naturally involved due to the time lag of the detecting instrument. It therefore acts like a compound control and the advantages of the lag-compounding can be evaluated by the theory of compounding developed in Chapter Nine. With a second derivative coupling control, the lagging effect may reach the fourth derivative of the stability function. It is for this reason that Table V is made up to the fourth derivative coupling coefficient γ_4 .

CONCLUSION
AND SUGGESTION TO FURTHER DEVELOPMENT

In the present thesis, attention has been centered on the stability of a controlled system involving the fourth order linear differential equation. Ordinary nonideal control for the longitudinal motion of aircraft involves a sixth order linear differential equation, but when the control is properly designed, the stability function in pitching remains as a fourth order differential equation.

The damping ratios and the undamped natural frequencies of the two components (between which there may be wide difference when in their original uncontrolled state) are free to be adjusted. The stability function alone does not reveal the whole story of the response. Transient response must also be analyzed. In reviewing the present thesis, the writer feels the following points are worth while developing or investigating further.

(1) Simple pitching velocity elevator control coupling of the θ deparasitized type should be investigated thoroughly to compare with ordinary θ-elevator control.

(2) Experimental method of determining the resultant lagging from the variable, which is needed to work on the control, up to the valve movement which produces the exciting force on the control. By knowing this lagging time the compounding theory can be applied to determine the effect of lagging.

(3) To investigate the stability and transient response of vertical velocity flap control. If possible, actual tests should be conducted in order to know the effectiveness of reducing vertical acceleration when the airplane encounters a vertical surging gust.

(4) To develop an instrument for recording gusts in bumpy air from which the spectrum of the apparent surging factor can be determined thereby enabling the designer to attain a compromise in selecting the most suitable coupling coefficient.

(5) To investigate a simple course for following-up an equation for disturbed lateral motion of aircraft parallel to what has been done by Minorsky for steamship course stabilization by assuming full restraint in roll and pitch.

(6) For constant azimuth control investigate the relative merits in using a control of high natural frequency and a positive first derivative tuning control including the transient analysis of the response.

(7) Theoretically an overdamped slow control of first derivative coupling is advantageous to distributing more evenly the damping ratio between the two components. It is not easily seen. For this reason, actual tests should be conducted in seeking convincible evidence.

(8) To investigate, following the method of attack on the quartic equation, the property of the sextic equation. If possible, summarize the stability criteria in the form of a plot and develop a sextic chart for evaluating the nondimensional physical constants of the system involving the sextic equation.

B I B L I O G R A P H Y

1. "The Sperry Pilot for Automatic Flying", a publication of the Sperry Gyroscope Company, Brooklyn, New York

   E. A. Sperry, Jr., "Description of the Sperry Automatic Pilot", Aviation Engineering, pp 16-18, January 1932

   M. Huggins, "Gyropilot Goes Cross Country", Aero Digest Vol. XVII, No. 1, pp 51-52, July 1930

2. H. K. Weiss, "Automatic Control Theory", M.I.T. Thesis 1938

3. S. N. Lin, "A Mathematical Study of Controlled Motion of Airplanes", M.I.T. Thesis, 1938 1939

4. Fr. Haus, "Automatic Stability of Airplanes", N.A.C.A. T.M. 695, December 1932

5. G.R.M.Garratt, "The British Automatic Pilot for Aircraft" The Engineer, Vol. CLXI, No. 4188, 4189, April 17 and 24 1936

   E. W. Meredith and P. A. Cooke, "Aeroplane Stability and Automatic Pilot", Journal of the Royal Aeronautic Society Vol. XLI, No. 318, p 415, June 1937

6. N. Minorsky, "Directional Stability of Automatically Steered Bodies", Journal of Am. Soc. of Naval Engineers Vol. XXXIV, No. 2, pp 280-309, May 1922

   N. Minorsky, "Automatic Steering Tests", Journal of Am. Soc. of Naval Engineers, Vol. XLII, No. 2, pp 285-310, May 1930

7. W. J. Milne, "Advanced Algebra for Colleges and Schools" pp 479-495, American Book Company, New York, 1902.

8. V. Bush, "Operational Circuit Analysis", Chapter IV, John Wiley & Sons, Inc., 1929, New York.

9. A. G. B. Metcalf, "Airplane Longitudinal Stability",(a resume), Journal of the Aero. Sci., Vol. IV, No. 2, December, 1936

10. O. C. Koppen, "Lateral Control at High Angles of Attack" Journal of the Aero. Sci., January, 1935

11. H. Glauert, "A Nondimensional Form of the Stability Equation of an Airplane", R & M, 1093, March 1927

12. A. R. Maxwell, "Some Design Considerations for Nonideal Continuous Controls", M.I.T. Thesis, 1940

13. W. S. Diehl, "Engineering Aerodynamics", Chapters 7 and 8 Ronald Press, New York, 1928

14. W. S. Diehl, "Engineering Aerodynamics", pp 9-11, Ronald Press, New York, 1928

15. R. T. Jones, "Letter to Editor," Journal of the Aero. Sci. Vol. IV, No. 4, pp. 153, February 1937

16. H. K. Weiss, "Theory of Automatic Control", M.I.T. Thesis, pp 114-115, 1938

17. S. H. Lin, "A Mathematical Study of Controlled Motion of Airplanes", M.I.T. Thesis, p 82, 1939

18. C. S. Draper and G. V. Schliestett, "General Principles of Instrument Analysis", Instruments, p 137, May 1939

19. J. P. Den Hartog, "Mechanical Vibrations", Chapter II, McGraw-Hill Book Company, New York, 1st Ed., 1934

20. H. K. Weiss, "Theory of Automatic Control", M.I.T. Thesis, p 119, 1938

21. S. N. Lin, "A Mathematical Study of Controlled Motion of Airplanes", M.I.T. Thesis, p 7, 1939

21. H. L. Hazen "Theory of Servo-Mechanism", Journ. of the Franklin Institute, Vol. CCXVIII, No. 3, September 1934

22. J. P. Den Hartog, "Mechanical Vibrations", pp 306-323, McGraw-Hill Book Company, New York, 1st Ed., 1934

23. E. J. Routh, "Advanced Rigid Mechanics", Chapter Six, Macmillan, London, 1930

24. S. N. Lin, "A Mathematical Study of Controlled Motion", M. I. T. Thesis, 1939, pp 68-70

25. Don Fink, "Bendix Automatic Direction Finder", McGraw-Hill Publishing Company, October 1940

26. C. S. Draper and G. V. Schliestett, "General Principles of Instrument Analysis", Instruments, p 137, May 1939

27. W. J. Milne, " Advanced Algebra for Colleges and Schools" Section 672, American Book Company, 1902

28. R. E. Doherty and E. G. Keller, "Mathematics of Modern Engineering", Vol. I, Section 46, pp 105-128, Graeffe's General Theory, John Wiley & Sons, New York, 1936

29. W. V. Lyon, "Note on a Method of Evaluating the Complex Roots of a Quartic Equation", Journal of Mathematics and Physics, Vol. III, No. 3, April 1924

30. L. F. Woodruff, "Notes on a Method of Evaluating the Complex Roots of Six and Higher Order Equations", Journal of Mathematics and Physics, Vol. VI, No. 3, May 1925

31. Y. H. Ku, "Note on a Method of Evaluating the Complex Roots of a Quartic Equation," Journal of Mathematics and Physics, Vol. V, No. 2, February 1926

32. L. E. Dickson, "Elementary Theory of Equations", Chapter V, p 38, Ferrari's Method, Wiley & Sons, New York, 1914

33. S. N. Lin, "A Mathematical Study of Controlled Motions of Airplanes", M.I.T. Thesis, 1939, Appendix I, Successive Approximations of Factoring of Higher Degree Equations.

34. L. E. Dickson, "Elementary Theory of Equations", Chapter IV, Sections 2 and 3, Wiley & Sons, New York, 1914

35. H. K. Weiss, "Constant Speed Control Theory", Journal of the Aero. Sci., Vol. VI, No. 4, February, 1939

36. A. R. Maxwell, "Some Design Considerations for Nonideal Continuous Control", M.I.T. Thesis, Fig. 11, 1940

37. H. L. Hazen, "Theory of Servo-Mechanism", Journal of the Franklin Institute, Vol. CCXVIII, No. 3, p 322

38. T. V.Karman and M. A. Biot, "Mathematical Methods in Engineering", p 274, Eq. 7.4, McGraw-Hill Book Company, New York, 1940

39. A. Klemin, N.A.C.A. Tech. Note No. 666

40. S. N. Lin, "A Mathematical Study of Controlled Motion of Airplanes", M.I.T. Thesis, p 15, 1939

41. H. K. Weiss, "Theory of Automatic Control", M.I.T. Thesis pp 144-145, 1938

B I O G R A P H Y

The author, Thomas Yee-Ying Liu, of the present thesis, was born in Jukao, Kiangsu, China on January 28, 1909. He received his elementary education in his home town until 1924 when he went to Nanking to have his senior middle school education. There his English was promoted one year ahead of the rest of the courses. He felt proud of such a promotion then, but later on he found it was a terrible ~~missing~~ loss that his English could never be advanced beyond the basic language.

On 1926 he entered Chiao Tung University at Shanghai to study Electrical Engineering and was graduated in July, 1930. From then he spent five years in teaching Physics and Chemistry at Kiangsu Provincial Taitsang and Hawaiian middle schools. He was honored twice in being appointed by the Provincial Education Bureau to take the charge of joint examination in Physics and Chemistry for the middle schools. With the five year patience in the provincial education circle he was qualified to take the competitive examination for abroad study. In winning such scholarship on 1935 he was sent to United States of America on January 1936.

In attempting to improve the safety and comfort of flying, he was permitted to study Aeronautical Engineering at Massachusetts Institute of Technology with special attention to Aeronautical Instrumentation. He received his Master of Science degree in 1938 at the Institute.

Due to the invasion of Japanese, Kiangsu has been occupied by the invader since December 1937. The author could not get support from the Provincial Government. In seeking financial aid for his further study, he applied the scholarship of the Institute and the scholarship of the China Foundation for the Promotion of Education and Culture. Fortunately he was granted by both for two years up to 1940, when the present thesis has been partially worked out. For the present academic year Tsieng Hua University grants him a partial scholarship for his completion of writing up of the present thesis titled "Stability and Transient Analysis of Longitudinal Controlled Motion of Aircraft with Nonideal Automatic Controls".

During the course in working the thesis, frequent stalls were encountered. But being inspired by L. Kronecker's famous statement, "God made the integers -- the rest is the work of man", he was much encouraged by the Catholicism. The fruitfulness of the present thesis in fact is made after he was baptized on December 8, 1940, when he received his Christian name, Thomas.

The year after he entered Chiao Tung University he suffered the loss of his father. By the struggle of his mother in earning money he was able to finish the college education. Recently, when hoping to be back soon to have a reunion with his dear mother, he is again struck by the news of her passing away. The only dream he is seeking now is hoping to see his wife and their two children soon after his completion of the thesis.

APPENDIX A

Nondimensional Cubic Equation and the Cubic Chart

by

Y. J. Liu

In the analysis of the performance of nonideal constant speed control, a cubic equation* representing the stability function is often met. In terms of natural frequency ω_n, the damping ratio ζ_c of the control and the control constant S_v, the stability function can be written as

$$\frac{d^3U}{dt} + 2\zeta_c\omega_n\frac{d^2U}{dt} + \omega_n^2\frac{dU}{dt} + S_v\omega_n^3 U = 0 \qquad \text{A1}$$

Let $\frac{d}{dt} = D$, then

$$D^3 + 2\zeta_c\omega_n D^2 + \omega_n^2 D + S_v\omega_n^3 = 0 \qquad \text{A2}$$

Eq. A2 can be nondimensionalized by introducing nondimensional operator d for $\frac{D}{\omega_n}$ giving the form

$$d^3 + 2\zeta_c d^2 + d + S_v = 0 \qquad \text{A3}$$

The roots of Eq. A3 are then evaluated.

Before the process of evaluating the roots of Eq. A3, stability can be verified from the form

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0 \qquad \text{A4}$$

where $\lambda = \frac{d}{S_v^{1/3}}$, $\alpha_2 = \frac{2\zeta_c}{S_v^{1/3}}$, & $\alpha_1 = \frac{1}{S_v^{2/3}}$ A5

The condition necessary for stable performance is

$$\alpha_2 > \alpha_1$$

For nonoscillatory performance, the reader is referred to Fig. A1. Assume that Eq. A3 can be factored into the following form.

$$d^3 + 2\zeta_c d^2 + d + S_v = (d^2 + 2\zeta\omega_r d + \omega_r^2)(d + \xi\omega_r) \qquad \text{A6}$$

where ω_r is the reference angular frequency of the oscillatory component of the controlled motion, ζ the damping ratio of this

---

*Constant Speed Control Theory, by H. K. Weiss, Journal of the Aeronautical Sciences, Vol. VI, No. 4, February 1939.

component, and ξ the apparent subsiding coefficient of the subsiding component. On developing,

$$2\zeta_c = (\xi + 2\zeta)\,\omega_r \qquad \text{A7}$$

$$1 = (2\zeta\xi + 1)\,\omega^2 \qquad \text{A8}$$

$$S_v = \xi\,\omega_r^3 \qquad \text{A9}$$

From Eq. A8, $\omega_r = \dfrac{1}{\sqrt{2\zeta\xi+1}}$ A8'

Substitute Eq. A8' into A7 and A9. We then have

$$2\zeta_c = \frac{\xi + 2\zeta}{\sqrt{2\zeta\xi+1}} \qquad \text{A7'}$$

and $$S_v = \frac{\xi}{(2\zeta\xi+1)^{3/2}} \qquad \text{A9'}$$

From Eq. A9' we may solve for

$$\zeta = \frac{1}{2\xi}\left[\left(\frac{\xi}{S_v}\right)^{2/3} - 1\right] \qquad \text{A10}$$

From Eq. A7' we may also solve for

$$\zeta = \xi\left[\zeta_c^2 - \frac{1}{2} \pm \zeta_c\sqrt{\zeta_c^2 - 1 + \frac{1}{\xi^2}}\right. \qquad \text{A11}$$

Eq. A10 is plotted with ζ as ordinate against ξ as abscissa and $(S_v)^{1/3}$ as varying parameter. Eq. A11 is plotted also with ζ as ordinate against ξ as abscissa, but with ζ_c as varying parameter. These two sets of curves mutually intersect one another and form the cubic chart. Therefore, when a controlled system is known by its damping ratio in control (ζ_c) and control constant (S_v), the nondimensional characteristic of the controlled system can be found from the cubic chart by locating the intersection of ζ_c and $S_v^{1/3}$ which determines the damping ratio ζ of the oscillatory component of the controlled result and the apparent subsiding coefficient of the subsiding component, ξ.

When ξ is known,

$$\omega_r = \left(\frac{S_v}{\xi}\right)^{1/3} \qquad \text{A12}$$

The dimensional natural angular frequency is then

$$\omega_n \omega_r = \omega_n \left(\frac{S_v}{\xi} \right)^{1/3} \qquad \text{A13}$$

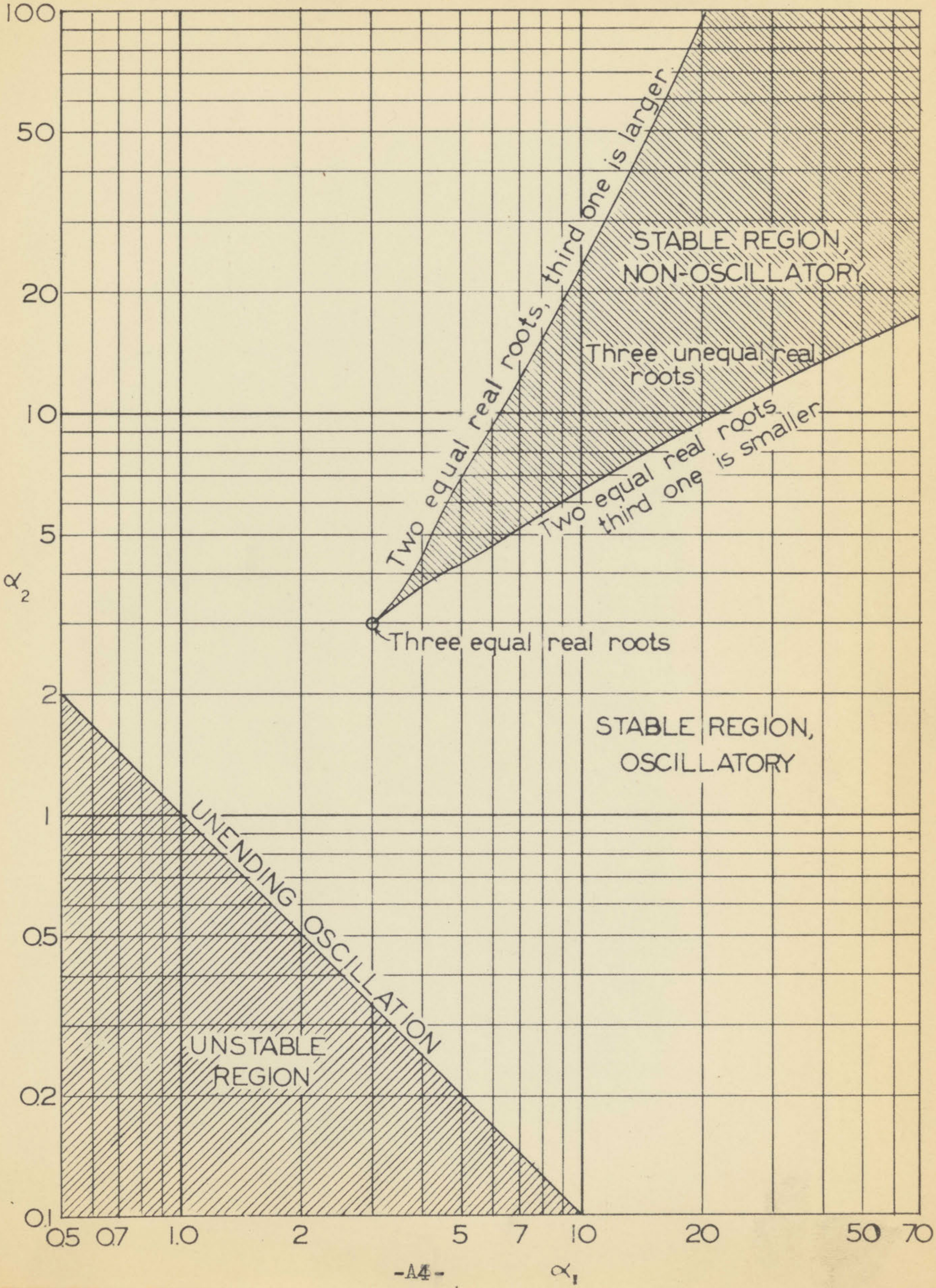
With ζ, ξ, and ω_r known, Eq. A6 can be written immediately and Eq. A2 can be factored as

$$(D^2 + 2\zeta\, \omega_n \omega_r\, D + \omega_n^2 \omega_r^2)(D + \xi\, \omega_n \omega_r) = 0 \qquad \text{A14}$$

It can be seen from the cubic chart that in the region $\zeta > 1$, three intersections can be found from a pair of ζ_c and $S_v^{1/3}$. This is naturally true because the three binomial factors can be arranged in three different ways to form the factorized Eq. A14.

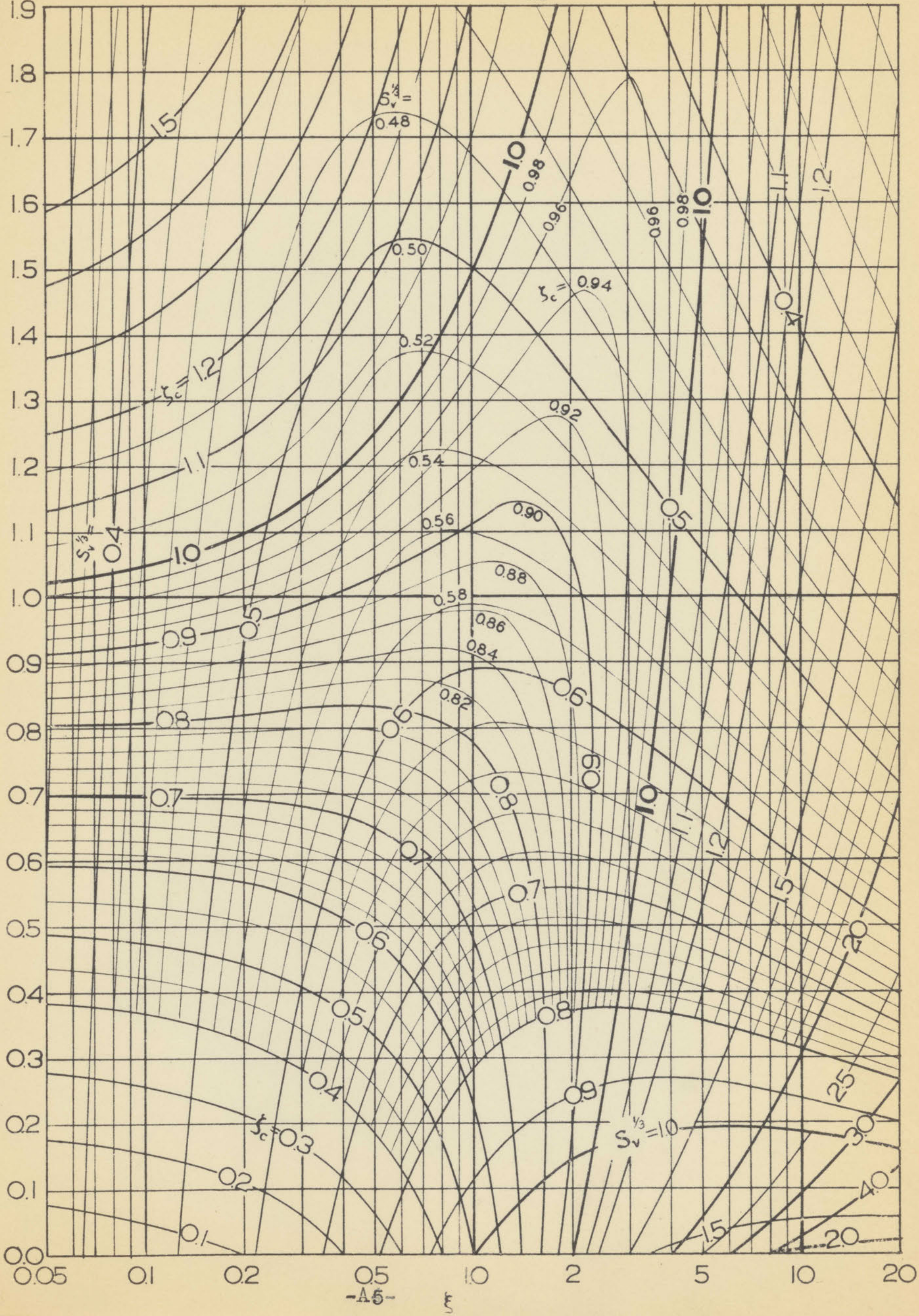
STABILITY CRITERIA α_2 VS α_1 OF THE CUBIC EQUATION

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$$



THE CUBIC CHART

For equation $d^3 + 2\zeta_c d^2 + d + S_v = 0$



APPENDIX B

DIRECTIONS FOR THE QUARTIC CHART

By Y. J. LIU

SUMMARY: In the analysis of servo-mechanism or automatic follow-up systems linear differential equations of fourth order are often met when a non-ideal controller is used (that is, a controller in which the effects of inertia damping and spring constants are not negligible). Physically a linear differential equation of fourth order as

$$a_4 \frac{d^4x}{dt^4} + a_3 \frac{d^3x}{dt^3} + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \quad \text{----------(1)}$$

represents two vibratory (divergent, critically or over damped) components of motion or other quantities provided at least the first and the last coefficients are of the same sign. Equation (1) can be transformed into an algebraic one by introducing p as the time operator symbol for $\frac{d}{dt}$, giving the form:

$$p^4 + a'_3p^3 + a'_2p^2 + a'_1p + a'_0 = 0 \quad \text{----------------------------(2)}$$

Where, $a'_3 = \frac{a_3}{a_4}$, $a'_2 = \frac{a_2}{a_4}$, $a'_1 = \frac{a_1}{a_4}$, and $a'_0 = \frac{a_0}{a_4}$.

Again, equation (2) can be non-dimensionalized by introducing a non-dimensional operation λ for $\frac{p}{a_0'^{\frac{1}{4}}}$ giving the form:

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0 \quad \text{-----------------------------(3)}$$

Where, $\alpha_3 = a'_3/a_0'^{\frac{1}{4}}$; $\alpha_2 = a'_2/a_0'^{\frac{1}{2}}$ $\alpha_1 = a'_1/a_0'^{\frac{3}{4}}$.

The roots of equation (3) are then to be evaluated.

However, instead of finding the roots of equation (3) directly, we may according to the physical state of the original differential equation factorize equation (3) into two quadratics indicating the dynamic characteristics of the two components respectively. The factorized equation can be written as:

$$(\lambda^2 + 2\zeta_1\omega_{r1}\lambda + \omega_{r1}^2)(\lambda^2 + 2\zeta_2\omega_{r2}\lambda + \omega_{r2}^2) = 0 \quad \text{--------------(4)}$$

Where, ω_{r1} = dimensionless natural frequency[1] of component 1 (angular)

ω_{r2} = dimensionless natural frequency of component 2 (angular)

ζ_1 = damping ratio[1] of component 1

ζ_2 = damping ratio of component 2

and, $\omega_{r1}a_0'^{\frac{1}{4}} = \omega_{n1}$ = undamped natural frequency of component 1 of equation (1)

$\omega_{r2}a_0'^{\frac{1}{4}} = \omega_{n2}$ = undamped natural frequency of component 2 of equation (1)

It is advisable to have one component as reference. Arbitrarily component I is considered as the reference. Then the following symbols are defined:

$\omega_r = \omega_{r1}$ = dimensionless natural frequency of reference component

----

Superscripts are referred to notes in sub-appendices

$\zeta_r = \zeta_1 =$ damping ratio of reference component

$\rho_\omega = \frac{\omega_{r2}}{\omega_{r1}} = \frac{\omega_{n2}}{\omega_{n1}} =$ ratio of undamped natural frequency between components

$\rho_\zeta = \frac{\zeta_2}{\zeta_1} =$ ratio of damping ratios between components

Hence equation (4) can be written as

$$(\lambda^2 + 2\zeta_r\omega_r\lambda + \omega_r^2) \quad [\lambda^2 + 2\,(\zeta_r\rho_\zeta)\,(\omega_r\rho_\omega)\,\lambda + (\omega_r\rho_\omega)^2] = 0 \quad \text{---(5)}$$

Evidently it is of primary interest to know these four important physical or rather characteristic constants, namely ω_r, ζ_r, ρ_ω, and ρ_ζ; because the dynamic characteristics of the system are well defined if these four quantities are known.

To facilitate the evaluation of these four quantities a quartic chart has been designed by the writer based on the non-dimensionalized equation (3). Three auxiliary figures are presented together with the quartic chart for the verification of stability of the original system.

The quartic chart itself is composed of two parts; the left part (or Chart I) is designed for the ratio of undamped natural frequencies and the dimensionless natural frequency of the reference component, and the right part (or Chart II) for the ratio of damping ratios and the damping ratio of the reference component. To minimize the effort in applying the chart, steps are listed below and should be followed orderly. Symbols and definitions of other interesting quantities that have not been introduced in the summary will be given as they appear along with the steps. A list of symbols and definitions is given as Table I.

1. Non-dimensionalization Of Equation

(1a) Given equation in the general form

$$a_4\frac{d^4x}{dt^4} + a_3\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = 0 \quad \text{------------------(1)}$$

(1b) Introduce time operator p for $\frac{d}{dt}$, then

$$p^4 + a'_3p^3 + a'_2p^2 + a'_1p + a'_0 = 0 \quad \text{-------------------------(2)}$$

Where, $a'_3 = \frac{a_3}{a_4}$, $a'_2 = \frac{a_2}{a_4}$, $a'_1 = \frac{a_1}{a_4}$, and $a'_0 = \frac{a_0}{a_4}$.

(1c) Introduce non-dimensional operator λ,

with $\lambda = \frac{p}{a_0'^{\frac{1}{4}}}$, then

$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0 \quad \text{-------------------------(3)}$$

Where, $\alpha_3 = \frac{a'_3}{a_0'^{\frac{1}{4}}}$, $\alpha_2 = \frac{a'_2}{a_0'^{\frac{1}{2}}}$, $\alpha_1 = \frac{a'_1}{a_0'^{\frac{3}{4}}}$.

2. Verification of The Stability of The Original Equation [Equation (1)] with the non-dimensional coefficients α_3, α_2, and α_1.

(2a) Obtain stability criteria M (Greek capital Mu) and N (Greek capital Nu), and damping parameters

$\frac{\alpha_3}{\alpha_1}$ and $\frac{1}{2}(\alpha_3 + \alpha_1)$;

where, $M = \frac{\alpha_3\alpha_1 - 4}{\alpha_2^2}$, $N = \frac{\alpha_3^2+\alpha_1^2-4\alpha_2}{\alpha_2^3}$.

(2b) Verify the stability with the stability criteria curves M vs. N (Fig. IIA, IIB, IIC)[2]

3. Application Of Chart I To Find The Ratio Of Undamped Natural Frequencies Between The Two Components, ρ_ω, And The Dimensionless Frequency of The Reference Component, ω_r

(3a) Locate intersention (3a)[3] of the particular pair of M and N on Chart I (see sketch on next page). Draw a line through (3a) and parallel to the 135° inclined lines until it intersects at the 45° inclined scale. The intersection on this scale gives the value of ρ_α; where

$$\rho_\alpha = \frac{1}{\alpha_2}\left(\rho_\omega + \frac{1}{\rho_\omega}\right).$$

The value of ρ_α, however, is not required for further application of the chart, but it serves as a principal datum from which other data can be evaluated[4] thereon by calculation in a more elaborate way.

(3b) Pick up the particular value of α_2 from the left hand scale[5] and draw a horizontal line until it meets the particular 135° inclined line. Call this intersection (3b)

(3c) From the intersection (3b) draw a vertical line which will intersect the curve P[5] at (3P) and curve Q at (3Q).

(3d) A horizontal line then drawn through the intersection (3P) intersects on the immediate right scale of ordinates at (3d) showing the value of ρ_ω and on the next right scale (that is, the left hand scale of ordinates of Chart II) at (3d') showing the value of ω_r. Record the values of ρ_ω and ω_r.

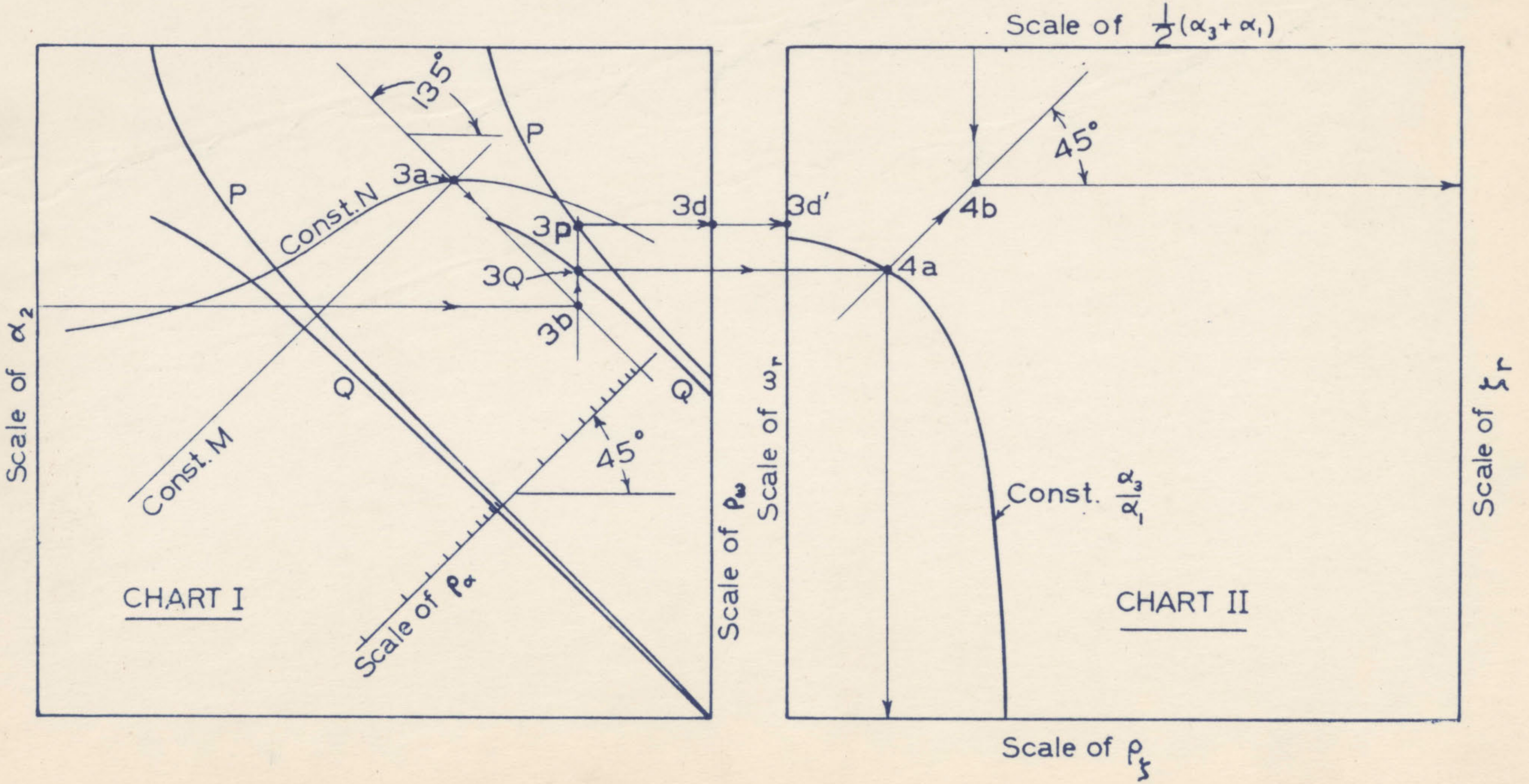
4. Application Of Chart II To Find The Ratio Of Damping Ratios Between Components, ρ_ζ, And The Damping Ratio of The Reference Component, ζ_r

(4a). Starting from the intersection (3Q) on Chart I run a horizontal line until it meets a curve of the particular value of $\frac{\alpha_3}{\alpha_1}$[6] [which has been found in (2a)] on Chart II. This intersection (4a) projected onto the abscissa scale gives the value of ρ_ζ. Record this value of ρ_ζ.

(4b) Through the intersection (4a) run a line parallel to those 45° inclined lines until it meets a vertical line of the corresponding particular value of $1/2(\alpha_3+\alpha_1)$ [which has been found in (2a)]. This intersection (4b)[7] projected onto the extreme right hand ordinates of Chart II gives the value of ζ_r. Record this value of ζ_r.

5. Factorization of Equation (3) By Utilization Of The Values Of ρ_ω, ω_r, ρ_ζ, And ζ_r.

SKETCH OF THE QUARTIC CHART



As pointed in the summary equation (3) can be written in terms of the four characteristic constants in the form of equation (5)

$$(\lambda^2+2\zeta_r\omega_r\lambda+\omega_r^2)\ [\lambda^2+2(\zeta_r\rho_\omega)\lambda+(\omega_r\rho_\omega)^2] = 0 \quad \text{------------(5)}$$

6. Factorization Of Equation (2) By Utilization Of The Values of ρ_ω, ω_r, ρ_ζ, ζ_r, And $a_0'^{\frac{1}{4}}$.

By writing:

$$\omega_{n1} = \omega_r a_0'^{\frac{1}{4}}, \qquad \omega_{n2} = \omega_{n1}\,\rho_\omega,$$

$$\zeta_1 = \zeta_r, \quad \text{and} \quad \zeta_2 = \zeta_1\,\rho_\zeta,$$

equation (2) can be written as:

$$(p^2+2\zeta_1\omega_{n1}p+\omega_{n1}^2)\ (p^2+2\zeta_2\omega_{n2}p+\omega_{n2}^2) = 0 \quad \text{--------------(6)}$$

or one step further as:

$$[p+(\zeta_1+\sqrt{\zeta_1^2-1})\omega_{n1}]\ [p+(\zeta_1-\sqrt{\zeta_1^2-1})\omega_{n1}]\ [p+(\zeta_2+\sqrt{\zeta_2^2-1})\omega_{n2}]\ [p+(\zeta_2-\sqrt{\zeta_2^2-1})\omega_{n2}] = 0 \quad \text{---(7)}$$

The values of $\sqrt{\zeta^2-1}$ can be obtained easily by means of Fig. III. It is evident that in case ζ is less than unity, $\sqrt{\zeta^2-1}$ comes out naturally with imaginary value which indicates the presence of oscillatory component.

(1) Natural frequency ω_n and damping ratio ζ are two very significant quantities of the dynamic characteristic of vibratory system with one degree of freedom. In symbol, $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{c_c}$; where k is the spring coefficient of the simple system, m the vibratory mass of the system, c the actual damping present in the system and c_c is the amount of damping which would cause critical damping to the system. As our problem is one which may be resolved into two components of motion, so each has its own natural frequency and damping ratio. Further reference may be found in various papers published by C. S. Draper and his co-authors such as "General Principles of Instrument Analysis" by C. S. Draper and G. V. Schliestett and "An Instrument for Measuring Low Frequency Accelerations in Flight" by C. S. Draper and W. Wrigley.

(2) Fig. IIA covers wide range of M and N. Stability can be verified practically with every possible combination of M and N. Fig. IIB is an enlargement of the non-oscillatory region where four unequal real roots are present. Fig. IIC is plotted in logarithmic scales to render better the visualization of very small quantities of M and N forming part of the boundary between the oscillatory and the non-oscillatory regions. Fig. I can be referred to as an indication of relative degree of damping between the two components; such indication is not available in Fig. II. Table II is a summary of Figs. I, IIA, IIB, & IIC. With the table and these figures one can find out whether the system is oscillatory, stable or not immediately after he gets the values of M and N.

(3) It is preferable to take the right-and-uppermost intersection of a given pair of M and N; because in case oscillatory component or components are present the left intersection will lead to a complex frequency ratio which, though reasonable from the mathematical point of view, is not usable. In case two non-oscillatory components are present, three intersections may be observed with one pair of M and N. They are equally usable, but the rightmost one gives the largest value of frequency ratio which makes the further application of the Chart easier. The dotted curve appearing on Chart I is nothing but equivalent to the boundary lines AB and ACD on Fig. IIA. To the left of the vertex, the dotted curve is equivalent to the boundary line AB and to the right, equivalent to the part ACD. Discard any intersection of M and N which appears below and to the left of the dotted curve. The significance of the boundary lines (and so of the dotted curve) is summarized in Table I.

(4) When ρ_α is obtained, the following formulae can be used for the evaluation of ρ_ω, ρ_ζ, ω_r, and ζ_r:

$$\rho_\omega = \tfrac{1}{2}[\alpha_2\rho_\alpha + \sqrt{(\alpha_2\rho_\alpha)^2 - 4}] \qquad (8)$$

$$\omega_r = \frac{1}{\sqrt{\rho_\omega}} \qquad (9)$$

$$\rho_\zeta = \frac{\rho_\omega(\frac{\alpha_3}{\alpha_1}) - 1}{\rho_\omega - (\frac{\alpha_3}{\alpha_1})} \quad \text{---------------------(10)}$$

$$\zeta_r = \frac{\frac{1}{2}(\alpha_3 + \alpha_1)}{(1+\rho_\zeta)(\sqrt{\rho_\omega} + \frac{1}{\sqrt{\rho_\omega}})} \quad \text{---------------------(11a)}$$

$$= \frac{1}{2}\sqrt{\frac{\alpha_2(1-\rho_\alpha)}{\rho_\zeta}} \quad \text{---------------------(11b)}$$

$$= \frac{\alpha_3}{2(\frac{1}{\sqrt{\rho_\omega}} + \rho_\zeta\sqrt{\rho_\omega})} \quad \text{---------------------(11c)}$$

$$= \frac{\alpha_1}{2(\frac{\rho_\zeta}{\sqrt{\rho_\omega}} + \sqrt{\rho_\omega})} \quad \text{---------------------(11d)}$$

(5) Curves P and Q appearing to the right upper corner (in the crowded zone) of Chart I are matched with the right scale of α_2 (range: 1-20). The centered P and Q are matched with left scale of α_2 (range: 10-1000). In case no intersection on the particular 135° inclined line and the horizontal line of particular value of α_2 can be found within Chart I, one is at liberty to shift the proper 135° inclined line one logarithmic cycle left or right, but the scale of α_2 should never be changed. By this process the matched P and Q curves are automatically shifted one logarithmic cycle left or right with the shifted 135° inclined line. So the local P and Q curves are available.

(6) As curve Q on Chart I has its top value corresponding to $\omega_r = 0.5$, any linear differential equation of 4th order with positive real coefficients does not go beyond the limit. Therefore any horizontal line run from Q can always intersect with constant $\frac{\alpha_3}{\alpha_1}$ curves. In case $\frac{\alpha_3}{\alpha_1}$ is greater than 4, $\frac{\alpha_1}{\alpha_3}$ may be used instead of $\frac{\alpha_3}{\alpha_1}$. Then the ratio of damping ratios ρ_ζ read from the abscissa scale of Chart II will be referred to the damping ratio of component 2 and the value of ζ_r on the rightmost ordinate scale is still the damping ratio of reference component, but it is the damping ratio of component 2. As ρ_ω is always given greater than unity, component 1 is always referred as the low frequency one, and component 2 as the high frequency one.

(7) Shifting of the 45° lines on Chart II one logarithmic cycle up or down on Chart II is permissible. However the decimal points of the ordinates scale for ζ_r must be shifted one figure left when the 45° line is shifted up one logarithmic cycle or one figure right when the latter shifted one cycle down. Moreover, the scales of $1/2\,(\alpha_3 + \alpha_1)$ and of ζ_r can be multiplied by a common factor, for instance, 10, simultaneously.

TABLE I

List of Symbols And Their Definitions

t	time, independent variable
x	dependent variable
a_k	physical coefficient associated with $\frac{d^k x}{dt^k}$, kth time derivative of the dependent variable
p	time operator symbol, $p = \frac{d}{dt}$
p^k	$p^k = \frac{d^k}{dt^k}$
a'_k	$a'_k = \frac{a_k}{a_4}$, time coefficient associated with p^k ($a'_4 = 1$)
λ	dimensionless operator, $\lambda = \frac{p}{a_0'^{\frac{1}{4}}}$
λ^k	$\lambda^k = (\frac{p}{a_0'^{\frac{1}{4}}})^k$
α_k	$\alpha_k = \frac{a'_k}{a_0'^{\frac{4-k}{4}}}$ dimensionless coefficient associated with λ^k ($\alpha_4 = 1$, and $\alpha_0 = 1$)
$\frac{\alpha_3}{\alpha_1}$	damping parometer 1
$\frac{1}{2}(\alpha_3+\alpha_1)$	damping parometer 2
M (Greek Capital Mu)	$M = \frac{\alpha_3\alpha_1 - 4}{\alpha_2^2} = \frac{a_3a_1 - 4a_4a_0}{a_2^2}$ Stability criterion 1
N (Greek Capital Nu)	$N = \frac{\alpha_3^2 + \alpha_1^2 - 4\alpha_2}{\alpha_2^2} = \frac{a_3^2a_0 + a_4a_1^2 - 4a_4a_2a_0}{a_2^3}$ Stability criterion 2
ω_{r1}	dimensionless angular frequency of component 1 (Low freq.)
ω_{r2}	dimensionless angular frequency of component 2 (High freq.)
ω_r	dimensionless angular frequency of reference component (component 1, arbitrarily)
ω_n	undamped natural angular frequency in general
ω_{n1}	undamped natural angular frequency of component 1
ω_{n2}	undamped natural angular frequency of component 2
ζ	damping ratio in general
ζ_1	damping ratio of component 1
ζ_2	damping ratio of component 2
ζ_r	damping ratio of reference component (component 1, arbitrarily)
ρ_ω	$\rho_\omega = \omega_{r2}/\omega_{r1} = \omega_{n2}/\omega_{n1}$, ratio of undamped natural frequency
ρ_ζ	$\rho_\zeta = \zeta_2/\zeta_1$, ratio of damping ratios
ρ_α	$\rho_\alpha = (1/\alpha_2)(\rho_\omega + 1/\rho_\omega)$

TABLE II

STABILITY BEHAVIOR OF THE QUARTIC EQUATION IN NONDIMENSONAL FORM

$$\lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0$$

$$\text{with } M = \frac{\alpha_3 \alpha_1 - 4}{\alpha_2^2}, \quad N = \frac{\alpha_3^2 + \alpha_1^2 - 4\alpha_2}{\alpha_2^3}$$

N > M	Unstable				
N = M	Unending oscillation		N > 0		One component only
			N < 0		One or both components
N=M= -1	Unending oscillation of both components				
N < M	Outside region BABC	At least one component is oscillatory	N > 0		One component oscillatory
			N = 0		Nonphysical existence
			N < 0		Both components oscillatory
Stable	Along boundary BABC $N = \frac{2\zeta^2 - 1}{(1+2\zeta^2)^3}$ $M = \frac{4\zeta^2 - 1}{(1+2\zeta)^2}$ ζ may be any real number	Two equal quadratic factors or one critically damped quadratic factor accompanying another of any damping ratio	Along AB $\zeta > 1$	$\alpha_3 = \alpha_1$	Two pair of equal roots Two equally overdamped quadratics
				$\alpha_3 \neq \alpha_1$	Two equal roots. Another component may be overdamped.
			Along AC	$\alpha_3 = \alpha_1$	Two equal conjugate-paired roots
			Also CD	$\alpha_3 \neq \alpha_1$	Two equal roots. Another component may be overdamped.
			At Vertex A $\zeta = 1$ $M = \frac{1}{3}$ $N = \frac{1}{27}$	$\alpha_3 = \alpha_1$	Four equal roots
				$\alpha_3 \neq \alpha_1$	Three equal roots
			Along CB	$\zeta_1 = \zeta_2 = .707$ at $\rho_\omega > 1$ or $\rho_\omega = 1$ $\quad \zeta_2 = \sqrt{1-\zeta_1^2}$	
	Inside region ABC	Four distinct real roots or both components oscillatory limited by $\zeta_{2,1} < \sqrt{1-\zeta_{1,2}^2}$			
$\alpha_3 > \alpha_1$		High frequency component has greater damping ratio			
$\alpha_3 < \alpha_1$		High frequency component has lesser damping ratio			
$\alpha_3 = \alpha_1$		Two components of same frequency, but with different damping Two components of different frequency, but with same damping or two components of same frequency and same damping			

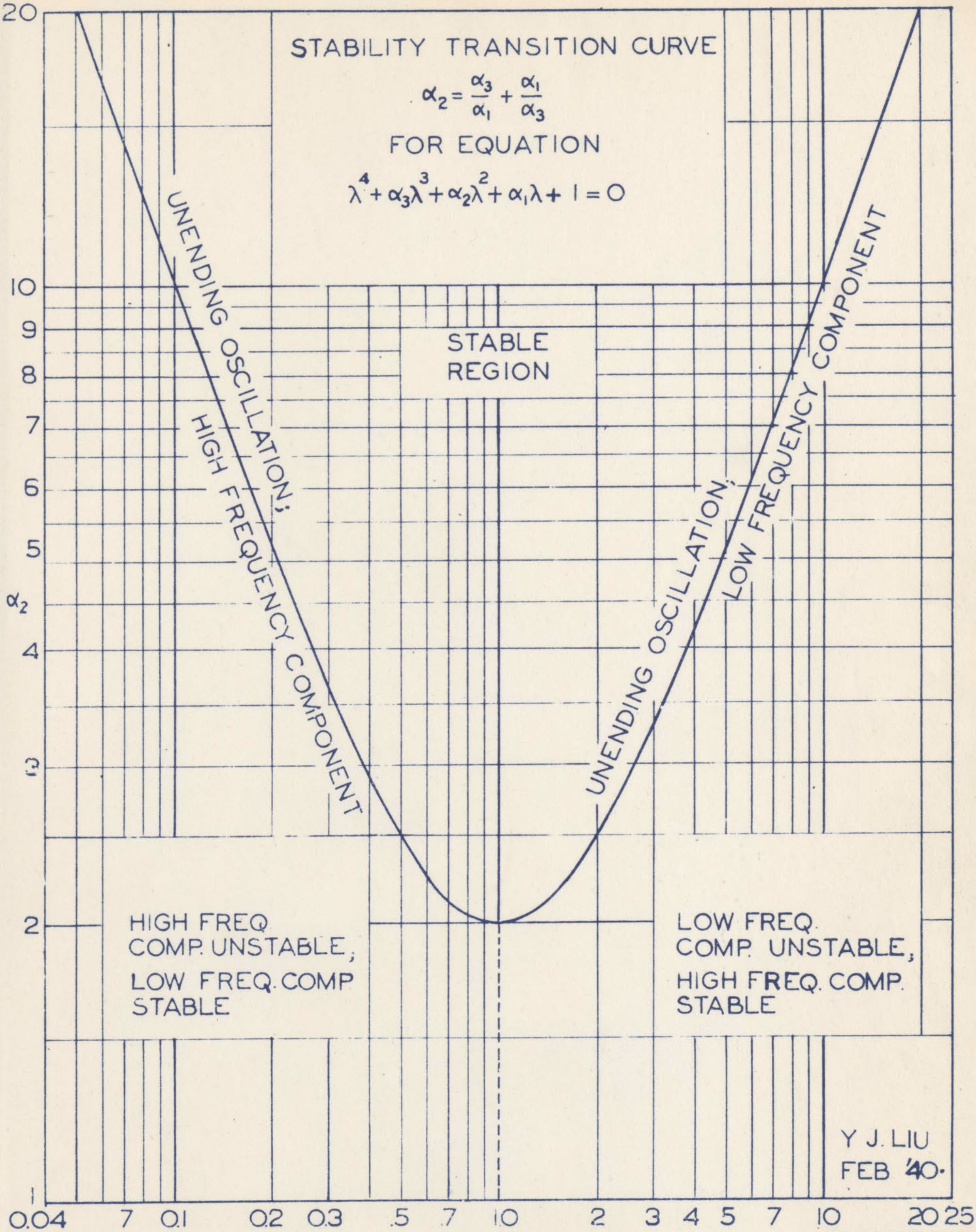
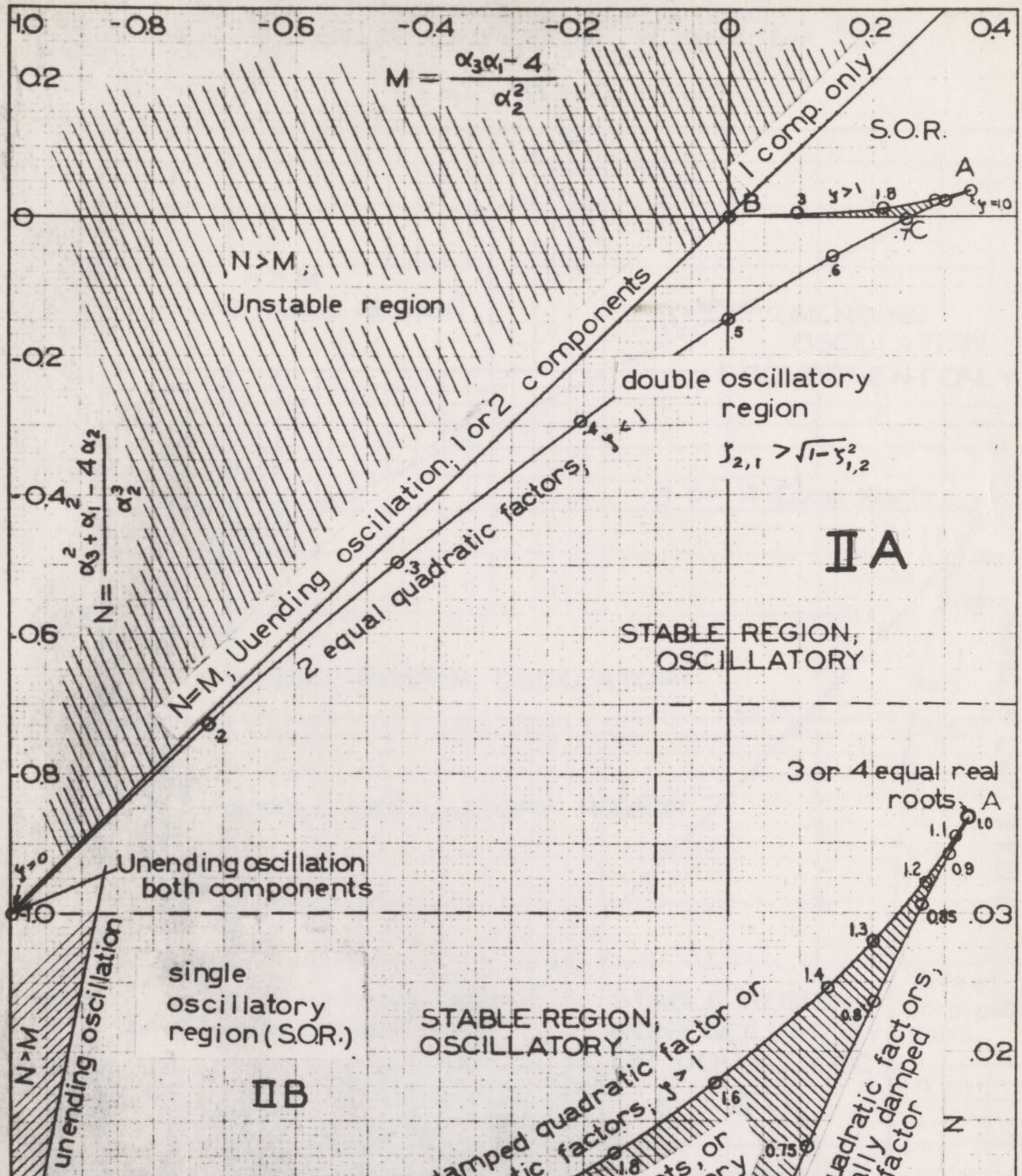

STABILITY TRANSITION CURVE
$\alpha_2 = \frac{\alpha_3}{\alpha_1} + \frac{\alpha_1}{\alpha_3}$
FOR EQUATION
$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$
STABLE REGION
UNENDING OSCILLATION; HIGH FREQUENCY COMPONENT
UNENDING OSCILLATION; LOW FREQUENCY COMPONENT
HIGH FREQ COMP. UNSTABLE; LOW FREQ. COMP STABLE
LOW FREQ. COMP. UNSTABLE; HIGH FREQ. COMP. STABLE
α_2
20
10
9
8
7
6
5
4
3
2
1
0.04
7
0.1
0.2
0.3
.5
.7
1.0
2
3
4
5
7
10
20
25
$\frac{\alpha_3}{\alpha_1}$
Y J. LIU
FEB '40.

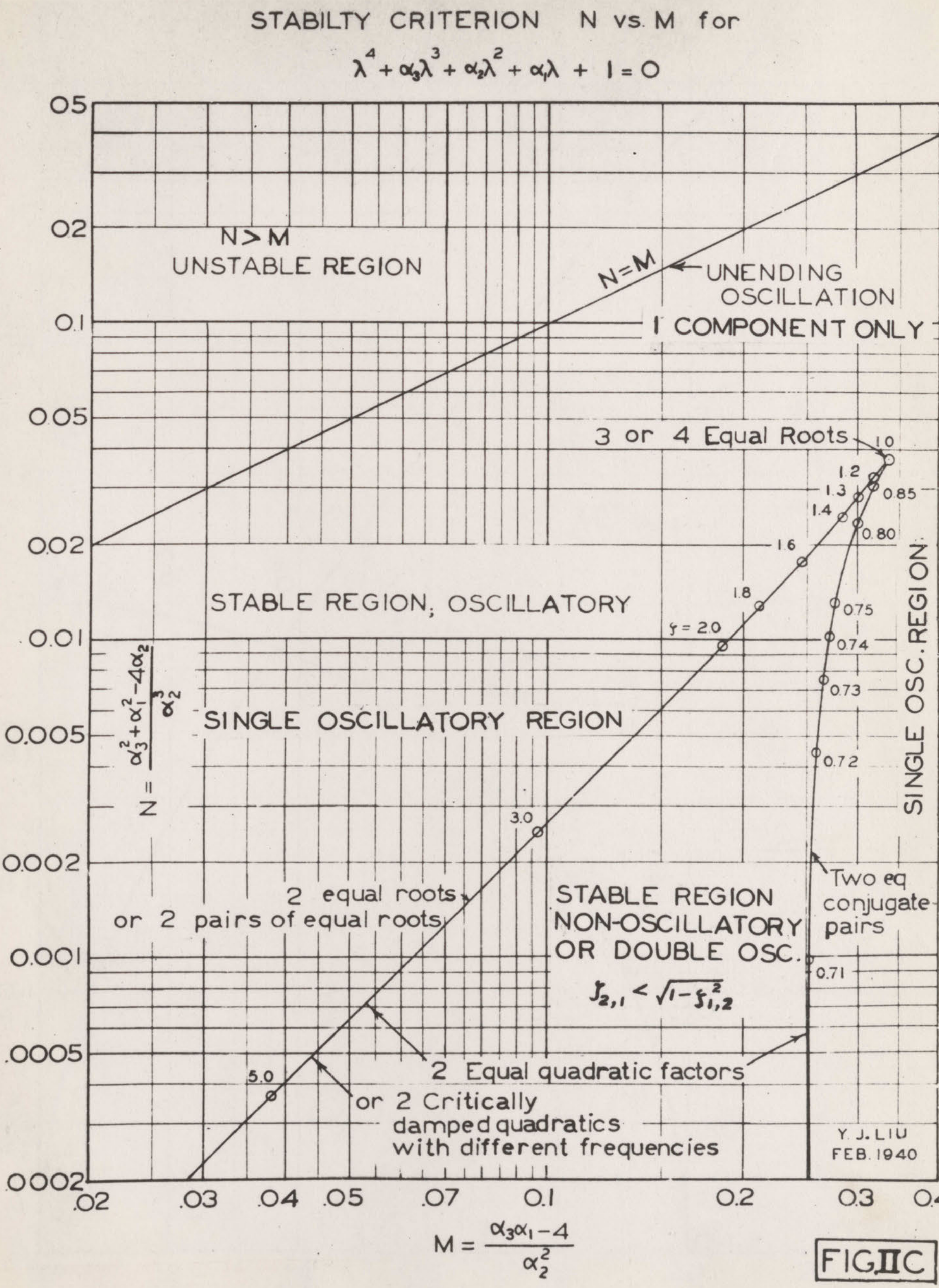
FIG. I



STABILITY CRITERION N vs. M for

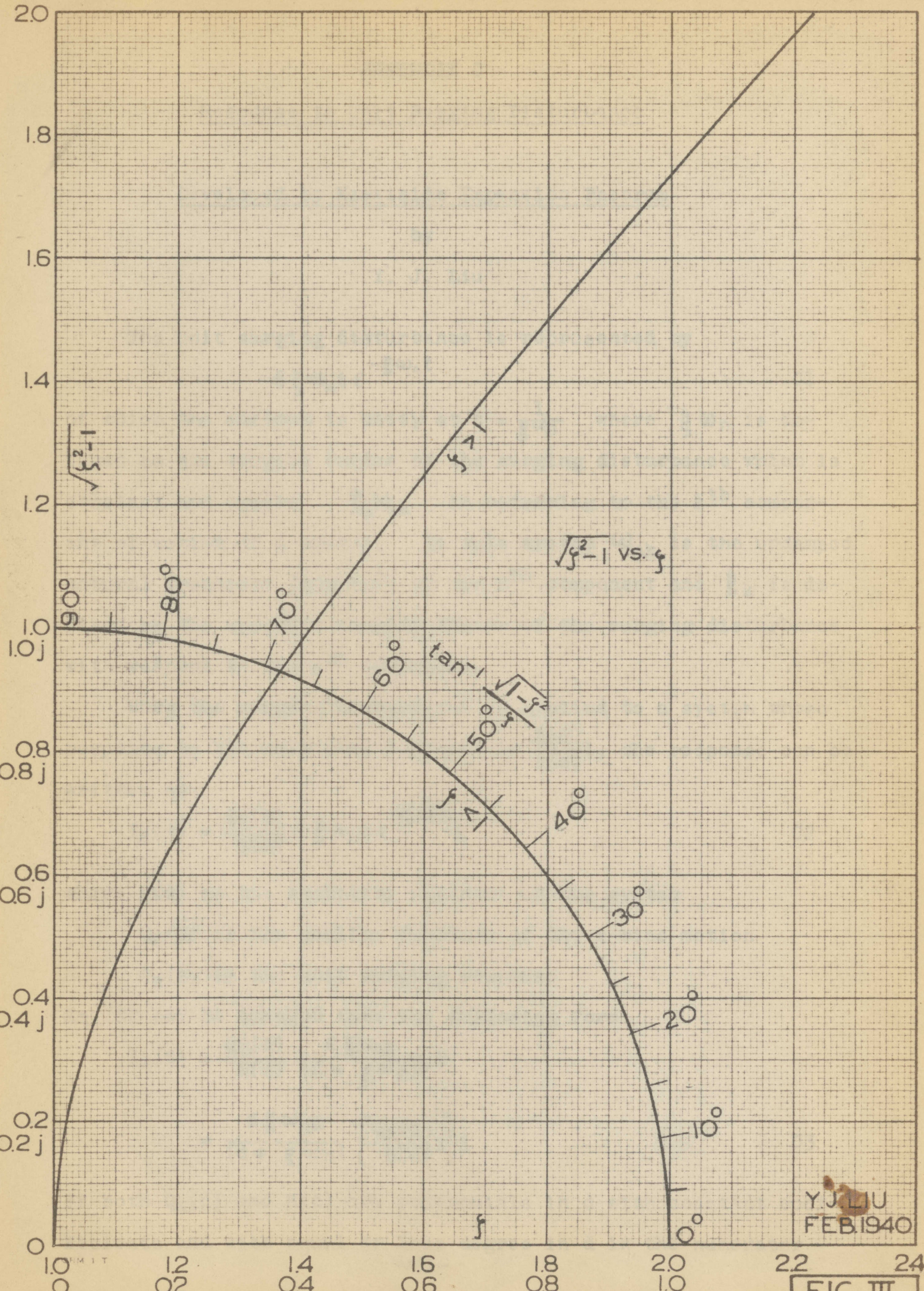
$$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$$

FIG. IIA & IIB



STABILTY CRITERION N vs. M for
$\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + 1 = 0$
N>M
UNSTABLE REGION
N=M
UNENDING OSCILLATION
1 COMPONENT ONLY
3 or 4 Equal Roots
STABLE REGION, OSCILLATORY
SINGLE OSCILLATORY REGION
SINGLE OSC. REGION
$N = \frac{\alpha_3^2+\alpha_1^2-4\alpha_2}{\alpha_2^3}$
2 equal roots or 2 pairs of equal roots
STABLE REGION NON-OSCILLATORY OR DOUBLE OSC.
$\zeta_{2,1} < \sqrt{1-\zeta_{1,2}^2}$
Two eq conjugate pairs
2 Equal quadratic factors
or 2 Critically damped quadratics with different frequencies
Y. J. LIU
FEB. 1940
$M = \frac{\alpha_3\alpha_1-4}{\alpha_2^2}$

FIG.IIC



2.0
1.8
1.6
1.4
1.2
1.0
1.0 j
0.8
0.8 j
0.6
0.6 j
0.4
0.4 j
0.2
0.2 j
0
$\sqrt{\xi^2-1}$
$\xi > 1$
$\sqrt{\xi^2-1}$ VS. ξ
$\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$
$\xi < 1$
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
ξ
Y. J. LIU
FEB. 1940
1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4
0 0.2 0.4 0.6 0.8 1.0

FIG. III

APPENDIX C

Response to Unit Surging Disturbance

Developed by Heaviside Expansion Theorem

by

Y. J. Liu

The unit surging disturbance is represented by

$$\epsilon \xi \omega_n t \epsilon^{-\xi \omega_n t} \qquad C1$$

of which the maximum is unity at $t = \frac{1}{\xi \omega_n}$ where $\xi \omega_n$ is defined as the surging factor of the surging disturbance which is constant and equals $\xi_k \omega_{nk}$ in referring to the k^{th} component of motion of a system. In this system ω_{nk} is the undamped natural frequency (angular) of the k^{th} component and ξ_k is defined as the apparent surging factor of the surging disturbance with respect to the k^{th} component.

When the surging disturbance is applied to a system whose response to the unit step function is $\frac{Q(d)}{S(d)}1$, the response can be written as

$$I_s(t) = \frac{Q_x(d)}{S(d)} \epsilon \xi \omega_n t \epsilon^{-\xi \omega_n t} 1 \qquad C2$$

where S(d) is the stability function of the system

$Q_x(d)$ is the quality function of the x-wise motion

$I_s(t)$ is the unit surging response

Eq. C2 can be changed into the following form

$$I_s(t) = \frac{Q_x(d)}{S(d)} \cdot \frac{\epsilon \xi \omega_n d}{(d + \xi \omega_n)^2} 1$$

$$= \frac{\epsilon \xi \omega_n d}{(d + \xi \omega_n)} \left[\frac{Q_x(d)}{S(d)} 1\right] 1 \qquad C3$$

If both $Q_x(d)$ and S(d) are polynomials in d with constant coefficients and with equal or less degree in d in the numerator than that in the denominator, C3 can be expanded by the Heaviside Expansion Theorem as follows provided the S(d) = 0 has no repeating roots

$$I_s(t) = \frac{\epsilon \xi \omega_n d}{(d+\xi\omega_n)^2}\left[\frac{Q_x(0)}{S(0)} + \sum_1^k \frac{d}{(d+\xi_k\omega_{nk})^2}\frac{Q_x(d_k\omega_{nk})}{d_k\omega_{nk}S'(d_k\omega_{nk})}\right]^* 1$$

$$= \epsilon\xi\omega_n\left[\frac{d}{(d+\xi\omega_n)^2}\frac{Q_x(0)}{S(0)}1 + \sum_1^k \frac{d}{(d+\xi_k\omega_{nk})^2}\cdot\frac{Q_x(d_k\omega_{nk})}{d_k\omega_{nk}S'(d_k\omega_{nk})}1\right] \quad \text{C4}$$

where $d_k\omega_{nk}$= the k^{th} root of **S(d) = 0** real or complex S' is the first derivative of the stability function. The first term in the bracket of C4 can be written as:

$$\frac{Q_x(0)}{S(0)}t\epsilon^{-\xi\omega_n t} \quad \text{C5}$$

and the second term, by applying the shifting formula** can be written as:

$$\sum_1^k \epsilon^{d_k\omega_{nk}t}\frac{Q_x(d_k\omega_{nk})}{d_k\omega_{nk}S'(d_k\omega_{nk})}\cdot\frac{d+d_k\omega_{nk}}{(d+\xi_k\omega_{nk}+d_k\omega_{nk})^2}1 \quad \text{C6}$$

On developing, C6 gives

$$\sum_1^k \frac{Q_x(d_k\omega_{nk})}{d_k\omega_{nk}S'(d_k\omega_{nk})}\left[\frac{\xi}{\xi_k+d_k}t\epsilon^{-\xi_k\omega_{nk}t} - \frac{d_k}{(\xi_k+d_k)^2\omega_{nk}}\epsilon^{-\xi_k\omega_{nk}t} + \frac{d_k}{(\xi_k+d_k)^2\omega_{nk}}\epsilon^{d_k\omega_{nk}t}\right] \quad \text{C7}$$

$$I_s(t) = \epsilon\xi\omega_n\left\{\begin{aligned}&\sum_1^k \frac{Q_x(d_k\omega_{nk})}{S'(d_k\omega_{nk})(\xi_k+d_k)^2\omega_{nk}^2}\epsilon^{d_k\omega_{nk}t}\\ &-\left[\sum_1^k \frac{Q_x(d_k\omega_{nk})}{S'(d_k\omega_{nk})(\xi_k+d_k)^2\omega_{nk}^2}\right]\epsilon^{-\xi\omega_n t}\\ &+\left[\frac{Q_x(0)}{S(0)} + \sum_1^k \frac{Q_x(d_k\omega_{nk})}{d_k\omega_{nk}S'(d_k\omega_{nk})}\cdot\frac{\xi_k}{\xi_k+d_k}\right]t\epsilon^{-\xi\omega_n t}\end{aligned}\right\} \quad \text{C8}$$

---

* V. Bush, "Operational Circuit Analysis", Chapter VII
**V. Bush, "Operational Circuit Analysis", Chapter VIII, p. 130.

when the f^{th} pair of roots is a conjugate pair, d_f and $\bar{d}_f$ can be used to represent the conjugate pair;

$$-\zeta_f \pm i\sqrt{1-\zeta_f^2}$$

Then
$$I_s(t) = \sum_1^k \epsilon\xi\omega_n A_k \epsilon^{d_k\omega_{nk}t} + \sum_1^f \epsilon\xi\omega_n 2\sqrt{A_f\bar{A}_f}\,\epsilon^{-\zeta_f\omega_n t}\cos(\sqrt{1-\zeta_f^2}\,\omega_{nf}t+\phi_f)$$
$$+(B_k + B_f)\,\epsilon\xi\omega_n\epsilon^{-\xi\omega_n t}$$
$$+\left[C_k + C_f + \frac{Q_x(0)}{S(0)}\right]\epsilon\xi\omega_n t\,\epsilon^{-\xi\omega_n t} \qquad \text{C9}$$

where

$$A_k = \frac{Q_x(d_k\omega_{nk})}{S'(d_k\omega_{nk})(\xi_k + d_k)^2\omega_{nk}^2}$$

$$A_f = \frac{Q_x(d_f\omega_{nf})}{S'(d_f\omega_{nf})(\xi_f + d_f)^2\omega_{nf}^2}, \quad \bar{A}_f = \frac{Q_x(\bar{d}_f\omega_{nf})}{S'(\bar{d}_f\omega_{nf})(\xi_f + \bar{d}_f)^2\omega_{nf}^2}$$

$$B_k = -\sum_1^k A_k$$

$$B_f = -\sum_1^f (A_f + \bar{A}_f)$$

$$C_k = \sum_1^k \frac{Q_x(d_k\omega_{nk})}{S'(d_k\omega_{nk})(d_k\omega_{nk})}\cdot\frac{\xi_k}{(\xi_k + d_k)}$$

$$C_f = \sum_1^f\left[\frac{Q_x(d_f\omega_{nf})}{S'(d_f\omega_{nf})(d_f\omega_{nf})}\cdot\frac{\xi_f}{\xi_f + d_f} + \frac{Q_x(\bar{d}_f\omega_{nf})}{S'(\bar{d}_f\omega_{nf})(\bar{d}_f\omega_{nf})}\cdot\frac{\xi_f}{\xi_f + \bar{d}_f}\right]$$

$$\phi_f = \tan^{-1}\frac{\underline{I}_{Qf}}{\underline{R}_{Qf}} - \tan^{-1}\frac{\underline{I}_{S'f}}{\underline{R}_{S'f}} - \tan^{-1}\frac{2(\xi_f - \zeta_f)\sqrt{1-\zeta_f^2}}{(\xi_f - \zeta_f)^2 - (1-\zeta_f^2)} \qquad \text{C10}$$

and $\underline{I}_{Qf}$ means the imaginary part of the quality function $Q_x(d_f\omega_{nf})$ and $\underline{R}$ means the real part, etc.

From the above solution, it is seen that the surging disturbance does not affect the stability of the system. However, it produces a surging component with a magnitude factor $\left[C_k + C_f + \frac{Q_x(0)}{S(0)}\right]$ comparable to the disturbance function and another subsiding component which adjusts the zero condition of the response.

APPENDIX D

Semigraphical Application of De Moivre's Theorem in Evaluating Polynomial Functions with a Complex Number

by
Y. J. Liu

In evaluating the polynomial Q(d) or S'(d) with the substitution of a complex number $(-\zeta + i\sqrt{1-\zeta^2})\,\omega_n$, the work is tedious. However, much time may be saved by using De Moivre's Theorem graphically. Assume:

$$S'(d) = a_m d^m + a_{m-1} d^{m-1} + \dots + a_4 d^4 + a_3 d^3 + a_2 d^2 + a_1 d + a_0 \quad \text{D1}$$

and $$d_k \omega_{nk} = (-\zeta_k + i\sqrt{1-\zeta_k^2})\,\omega_{nk} \quad \text{D2}$$

Then $$S'(d_k\omega_{nk}) = a_m \omega_{nk}^m d_k^m + \dots + a_3 \omega_{nk}^3 d_k^3 + a_2 \omega_{nk}^2 d_k^2 + a_1 \omega_{nk} d_k + a_0 \quad \text{D3}$$

$\because\ d_k = -\zeta_k + i\sqrt{1-\zeta_k^2}$

$$\therefore\ d_k = \cos\theta_k + i\sin\theta_k \quad \text{D4}$$

where $$\cos\theta_k = -\zeta_k \text{ and } \sin\theta_k = \sqrt{1-\zeta_k^2} \quad \text{D5}$$

θ_k is in the second quadrant or

$$\theta_k = \frac{\pi}{2} + \tan^{-1}\frac{\zeta_k}{\sqrt{1-\zeta_k^2}} \quad \text{D6}$$

where θ_{kp} is defined as proper angle

By De Moivre's Theorem

$$\begin{aligned}
d_k^m &= \cos m\theta_k + i \sin m\theta_k \\
&= \cos m\left(\frac{\pi}{2} + \theta_{kp}\right) + i \sin m\left(\frac{\pi}{2} + \theta_{kp}\right) \\
&= \cos\left(\frac{m\pi}{2} + m\theta_{kp}\right) + i \sin\left(\frac{m\pi}{2} + m\theta_{kp}\right) \\
&= \cos\frac{m\pi}{2}\cos m\theta_{kp} - \sin\frac{m\pi}{2}\sin m\theta_{kp} \\
&\quad + i \sin\frac{m\pi}{2}\cos m\theta_{kp} + i\cos\frac{m\pi}{2}\sin m\theta_{kp} \\
&= \left(\cos\frac{m\pi}{2} + i\sin\frac{m\pi}{2}\right)(\cos m\theta_{kp} + i \sin m\theta_{kp}) \\
&= e^{i\frac{m\pi}{2}}(\cos m\theta_{kp} + i\sin m\theta_{kp}) \\
&= i^m(\cos m\theta_{kp} + i \sin m\theta_{kp})
\end{aligned} \quad \text{D7}$$

From Eq. D7 a simple graphical method can be developed for finding d_k^m by a circle superimposed on rectangular coordinate paper.

At the upper left hand part of Fig. D1 is the scale of ζ. Pick up ζ_k and drop it down to the circle. The arc between this point and zenith of the circle represents the value of θ_{kp}. The coordinates of this point are $-\zeta_k + i\sqrt{1-\zeta_k^2}$. Use a pair of dividers to pick up the chord length of the arc and divide the circle with this chord length counterclockwisely until the m^{th} point or m times the θ_{kp} is obtained, where

$$\begin{aligned} d_k^m &= i^m(\cos m\theta_{kp} + i \sin m\theta_{kp}) \\ &= i^m(y_m + ix_m) \\ &= i^{m-1}(-x_m + iy_m) \end{aligned} \qquad \text{D8}$$

where $-x_m$ and y_m are coordinates of $m\theta_{kp}$

For instance, let $\zeta_k = .372$ (See Fig. D1)

$$d_k = -0.372 + i0.922$$

$$d_k^2 = i(-0.69 + i0.72) = -.72 + i0.69$$

$$d_k^3 = i^2(-0.908 + i0.42) = +.908 - i0.42$$

$$d_k^4 = i^3(-1.0 + i0.03) = +0.03 + i1.0$$

$$d_k^5 = i^4(-0.93 - i0.36) = -0.93 - i0.36$$

Such a graphical application to evaluate d_k^m does not need a known angle, but a dividers and circle diagram should be provided. The result is accurate enough for engineering purposes, yet the method is so simple that not even a trigonometric table is needed.

With all d_k^m known, $S'(d_k\omega_{nk})$ and $Q(d_k\omega_{nk})$ can be easily evaluated in the form of Eq. D3.

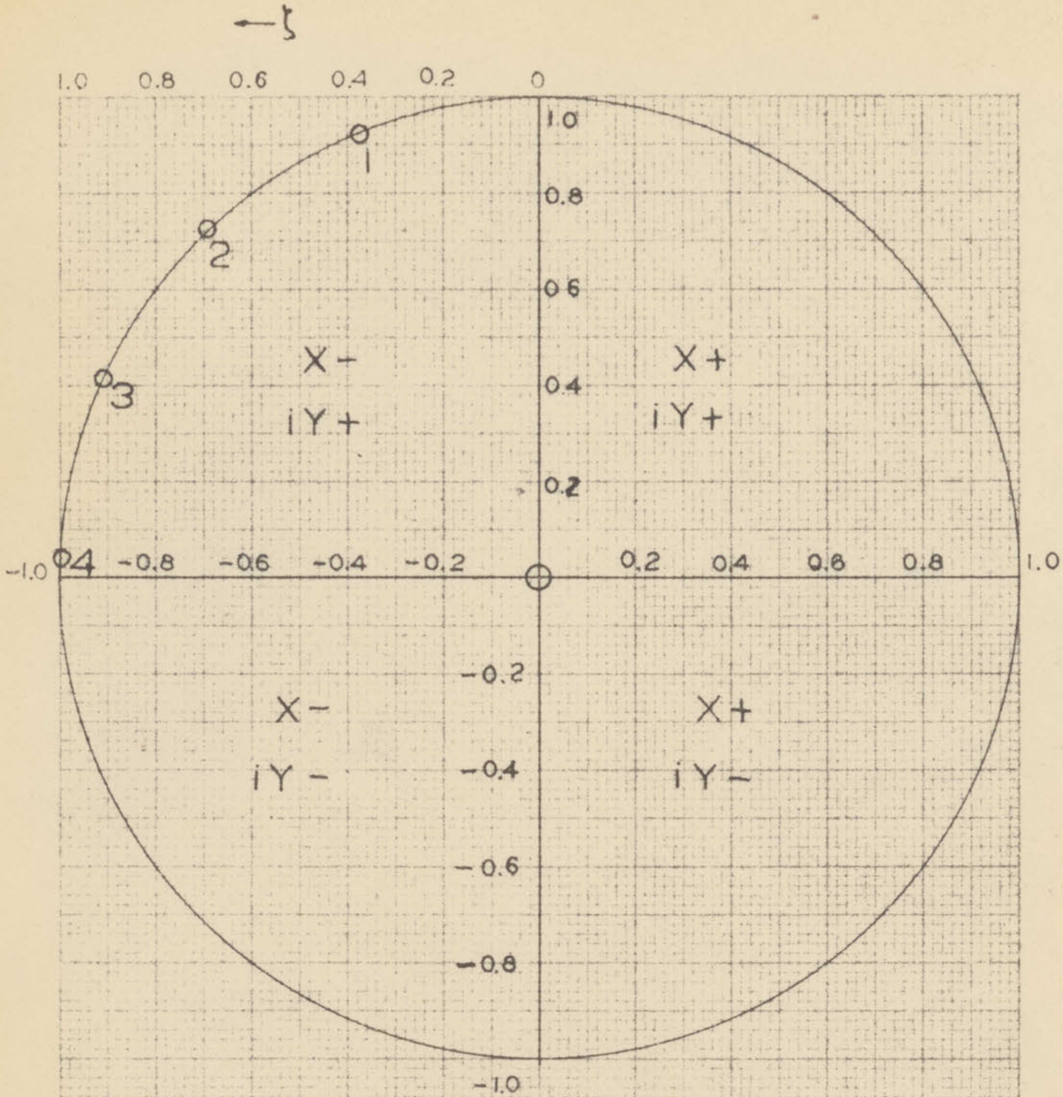


FIG. D1

Semigraphical Application of DeMoivre's Theorem in Evaluating Polynomial Functions with complex number