L∞ algebras and field theory

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Abstract

We review and develop the general properties of L_∞ algebras focusing on the gauge structure of the associated field theories. Motivated by the L_∞ homotopy Lie algebra of closed string field theory and the work of Roytenberg and Weinstein describing the Courant bracket in this language we investigate the L_∞ structure of general gauge invariant perturbative field theories. We sketch such formulations for non-abelian gauge theories, Einstein gravity, and for double field theory. We find that there is an L_∞ algebra for the gauge structure and a larger one for the full interacting field theory. Theories where the gauge structure is a strict Lie algebra often require the full L_∞ algebra for the interacting theory. The analysis suggests that L_∞ algebras provide a classification of perturbative gauge invariant classical field theories.

Contents

1 Introduction and summary

In bosonic open string field theory [1] the interaction of strings is defined by a multiplication rule, a star product of string fields that happens to be associative. While this formulation is advantageous for finding classical solutions of the theory, associativity is not strictly necessary for the formulation of the string field theory; *homotopy* associativity is. This more intricate structure has been investigated in [2] and has appeared when one incorporates closed strings explicitly in the open string theory [3–5]. The homotopy associative A_∞ algebra of Stasheff [6] is the mathematical structure underlying the general versions of the classical open string field theory sector.

The product of closed string fields is analogous to Lie brackets in that they are graded commutative. But a strict Lie algebra does not appear to allow for the formulation of closed string field theory. Instead one requires a collection of higher products satisfying generalized versions of Jacobi identities. The classical string field theory is thus organized by a homotopy Lie algebra, an L_∞ algebra whose axioms and Jacobi-like identities were given explicitly in [7]. The axioms and identities were later given in different but equivalent conventions in [8]. These two formulations are related by a "suspension", an operation where the degree of all vector spaces is shifted by one unit. An earlier mathematical motivation for homotopy Lie algebras is found in [9]. The L_∞ algebra describes the structure of classical string field theory: the collection of string field products is all one needs to write gauge transformations and field equations. Equipped with a suitable inner product, one can also write an action. The interplay of A_∞ and L_∞ algebras feature in the study of open-closed string field theory [3, 5, 10]. L_∞ algebras have recently featured in a study of massive two-dimensional field theory [11].

The relevance of L_∞ to closed string field theory, which is a field theory for an infinite number of component fields, suggests that it should also be relevant to arbitrary field theories, and this provided motivation for the present study. In particular, in a recent work Sen has shown how to define consistent truncations for a set of closed string modes [12]. For these degrees of freedom one has an effective field theory organized by an L_∞ algebra that can be derived from the full algebra of the closed string field theory. This again suggests the general relevance of L_∞ to field theories.

A first look into the problem of identifying the L_∞ gauge structure of some field theories was given by Barnich *et.al.* [13]. The early investigation of non-linear higher-spin symmetries in Berends *et.al.* [14] eventually led to an analysis by Fulp *et.al.* [15] of *gauge* structures that under some assumptions define L_∞ algebras. More recently, Yang-Mills-type gauge theories were fully formulated as L_∞ algebras by Zeitlin using the BRST complex of open string field theory [16–18].

In an interesting paper, Roytenberg and Weinstein [19] systematically analyzed the Courant algebroid in the language of L_∞ algebras. The authors explicitly identified the relevant vector spaces, the products, and proceeded to show that the Jacobi-like homotopy identities are all satisfied. As it turns out, in this algebra there is a bracket $[\,\cdot\,,\cdot\,]$ that applied to two gauge parameters coincides with the Courant bracket, and a triple product $[\,\cdot\,,\cdot\,,\,\cdot\,]$ that applied to three gauge parameters gives a function whose gradient is the Jacobiator. No higher products exist in this particular L_∞ algebra. The L_∞ formulation of the C-bracket, the duality-covariant extension of the Courant bracket, was considered by Deser and Saemann [20]. The authors used 'derived brackets' in their construction, making use of the results of [21–23]. Since the Courant bracket underlies the gauge structure of double field theory [24–28], we were motivated by the above results to try to understand what would be the L_∞ algebra of double field theory.

For the study of L_∞ in field theory a fact about L_∞ in closed string field theory was puzzling. In this theory there is a triple product $[\,\cdot\,,\cdot\,,\,\cdot\,]$ and it controls several aspects of the theory. It enters in the quartic interactions of fields, as the inner product of Ψ with $[\,\Psi\,,\Psi\,,\Psi\,]$. It appears in the gauge transformations as a nonlinear contribution $\delta_\Lambda\Psi \sim \cdots + [\,\Lambda\,,\Psi\,,\,\Psi\,]$. It makes the commutator of two gauge transformations δ_{Λ_1} and δ_{Λ_2} a gauge transformation with a gauge parameter that includes a field-dependent term $[\Lambda_1, \Lambda_2, \Psi]$. Finally, the triple product implies that the gauge transformations only close on shell, the extra term being $[\Lambda_1, \Lambda_2, \mathcal{F}]$, where $\mathcal{F}$ is the string field equation. In closed string field theory all these peculiarities happen simultaneously because the triple product is non-vanishing; these facts are correlated. Moreover, the definition of the product is universal, valid for arbitrary input string fields.

These facts, however, are not correlated in ordinary field theories. Yang-Mills, for example, has a quartic interaction in the Lagrangian but the commutator of gauge transformations is a gauge transformation with a field-independent gauge parameter. The same is true for Einstein gravity in a perturbative expansion around a background. The gauge algebra of double field theory is field independent but there is a triple bracket associated to the failure of the Jacobi identity for the Courant bracket. The lack of correlation is quickly demystified by exploring such examples. Gauge parameters, fields, and field equations appear in different vector spaces, according to some relevant grading. In defining a triple product one must state its value for all possible gradings of the various inputs. While in the string field theory one has a universal definition controlled by some data about four-punctured Riemann spheres, the definition of products *in a given field theory* has to be done in a case by case approach as we vary the inputs. The product is non-vanishing for inputs with certain gradings and can vanish for other sets of inputs. The correlation observed in string field theory is not discernible in the various separate field theories.

Another important feature is revealed by the explicit analysis: there are at least two L_∞ algebras associated to a field theory, one a subalgebra of the other. There is an L_∞^{gauge} for the gauge structure which is a subalgebra of the larger L_∞^{full} that includes the interactions associated with the action and the field equations:

$$L_\infty^{\text{gauge}} \subset \; L_\infty^{\text{full}}\,. \tag{1.1}$$

Any algebra for the gauge structure must include a vector space for gauge parameters. If the algebra is field dependent, it must include a vector space for fields. This algebra does not include interactions. When including interactions, the gauge algebra is supplemented by a new vector space for field equations and a set of new products, including some defined on fields and field equations. One can easily have a theory where L_∞^{gauge} is a Lie algebra but L_∞^{full} has higher products, because the theories have quartic or higher-order interactions. This is the case for Yang-Mills theory and for Einstein gravity. For a discussion of L_∞ algebras associated with some class of gauge algebras with field-dependent structure constants see also [15].

If L_∞^{gauge} is field independent, there is a third, intermediate algebra

$$L_\infty^{\text{gauge+fields}} , \tag{1.2}$$

that includes the off-shell realization of the gauge transformations on fields but that does not include dynamics. Roytenberg and Weinstein [19], for example, computed the field-independent algebra L_∞^{gauge} for the Courant bracket. We will review that work, translated in $O(d,d)$ covariant language, and extend it to consider the algebra $L_\infty^{\text{gauge+fields}}$ and the algebra L_∞^{full} for the interacting double field theory.

Our work here points to an intriguing possibility: any gauge invariant perturbative field theory may be represented by an L_∞ algebra that encodes all the information about the theory, namely, the gauge algebra and its interactions. It is then possible that *gauge invariant perturbative field theories are classified by* L_∞ *algebras.* A few comments are needed here. Associated to any gauge structure, field theories can differ by the interactions. Since the interactions define some of the products in L_∞^{full}, that aspect of the theory is properly incorporated. Of course, field redefinitions establish equivalences between theories, and such equivalences must correspond to suitable isomorphisms of L_∞ algebras, presumably in the way discussed for A_∞ in [2]. We seem to be constrained to theories formulated in perturbative form, that is theories in which one can identify unambiguously terms with definite powers of the fields in the Lagrangian, although there may be exceptions. For Einstein gravity, which in the standard formulation contains both the metric and its inverse, one must expand around a background to obtain a perturbative expansion in terms of the fluctuating field. This formulation of gravity as L_∞ is completely straightforward, as will be clear to the reader of this paper, but real insight would come only if some elegant explicit definition of the products could be given. Constrained fields, such as the generalized metric of double field theory, are also problematic as the power of the field in any expression may be altered by use of the constraint. All in all, we do not attempt to show that *any* gauge invariant field theory has a description as an L_∞ algebra, although we suspect that the result is true for unconstrained fields. Perhaps a proof could be built using a different approach. Consider a perturbative theory that can be formulated in the Batalin-Vilkovisky formalism with a master action S satisfying the classical Batalin-Vilkovisky master equation $\{S,S\}=0$; see [29] for a review of these techniques. General arguments indicate that an L_∞ structure can be systematically extracted from the master action [30].

Since A_∞ is the algebraic setup for open string field theory, one can ask why is L_∞, the setup for closed string field theory, chosen for perturbative field theories. We have no general answer but it appears that the L_∞ setup is rather flexible. We show that for Chern-Simons theory both an A_∞ and a L_∞ formulation exists. The first formulation requires describing the Lie algebra in terms of an associative algebra of matrix multiplication. The second formulation requires the use of a background metric in the definition of the products. We have not investigated if other field theories have both formulations. See, however, the general discussion in [31] giving an A_∞ setup to Yang-Mills theory. The formulation of gauge theories as A_∞ algebras will also be investigated in [32].

A fraction of the work here deals with the structure of L_∞ algebras. Following [7] we discuss the axioms and main identities, but develop a bit further the analysis. We show explicitly that given an L_∞ algebra with multilinear products with $n \geq 1$ inputs, one can construct consistent *modified* products with $n \geq 0$ inputs. A product $[\cdot]'$ without an input is just a special vector $\mathcal{F}$ in the algebra and that vector is in fact the field equation for a field Ψ in the theory defined by the original products. The modified product with one input, $[B]' = Q'B$, defines a linear operator Q', built from Ψ and a Q operator that squares to zero and defines the one-input original product. We establish the L_∞ identity

$$Q'\mathcal{F} \; = \; 0\,, \tag{1.3}$$

which can be viewed as the *Bianchi identity* of the original theory, and Q' may be thought of as a *covariant derivative.* Indeed we also have $Q'^2 \sim \mathcal{F}$. The modified products simplify the analysis of the gauge structure of the theory. The gauge transformations take the form

$$\delta_\Lambda \Psi \; = \; Q'\Lambda\,, \tag{1.4}$$

and the computation of the gauge algebra $[\delta_{\Lambda_2}, \delta_{\Lambda_1}]$ can be simplified considerably. We also compute the 'gauge Jacobiator'

$$\mathcal{J}(\Lambda_1, \Lambda_2, \Lambda_3) \; \equiv \; \sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, [\delta_{\Lambda_2}\,,\, \delta_{\Lambda_1}]\big]\,. \tag{1.5}$$

The right hand side is trivially zero for any theory with well-defined gauge transformations; this is clear by expansion of the commutators. On the other hand, this vanishing is a nontrivial constraint on the form of the gauge algebra. This constraint is satisfied by virtue of the identities satisfied by the higher products in the L_∞ algebra.

In the above approach, called the b-picture of the L_∞ algebra, the signs in the field equation, gauge transformations, action, and gauge algebra are known. There is another picture, the ℓ-picture of the algebra [8] in which the signs of the Jacobi-like identities are more familiar. The two pictures are related by suspension, a shift in the degree of the various spaces involved. We use this suspension to derive the form of field equations, gauge transformation, action, and gauge algebra in the ℓ picture.

Here is a brief summary of this paper. We begin in section 2 with a description of the L_∞ algebra in the conventions of the original closed string field theory and discuss the gauge structure, particularly the closure of the algebra and the triviality of the Jacobiator. In sec. 3 we begin by defining the axioms of L_∞ algebras in two conventions, one (the ℓ-picture) that is conventional in the mathematics literature and one (the b-picture) that is conventional in string field theory and hence directly related to sec. 2. In sec. 3.3 we make some general remarks how to identify for a given field theory the corresponding structures of an L_∞ algebra. Moreover, we explain how gauge covariance of the field equations and closure of the gauge transformations imply that large classes of L_∞ identities hold. (Readers mainly interested in the applications to field theory can skip sec. 2 and sec. 3.2.) These results will be applied in sec. 4 in order to describe Yang-Mills-like gauge theories, both for Chern-Simons actions in 3D and for general Yang-Mills actions. In sec. 5 we discuss the L_∞ description of double field theory, which in turn is an extension of the construction by Roytenberg and Weinstein for the Courant algebroid. We finally compare these results with A_∞ algebras by giving the A_∞ description of Chern-Simons theory in sec. 6. We close with a summary and an outlook in sec. 7.

2 L_∞ algebra and gauge Jacobiator

In this section we review the definition of an L_∞ algebra, state the main identities,[1] and introduce the field equation and the action. We then turn to a family of identities for modified products, giving the details of a result anticipated in [7]. They correspond to the products that would arise after the expansion of the string field theory action around a background that does not solve the string field theory equations of motion.[2] With these products, the Bianchi identities of string field theory become clear and the modified BRST operator functions as a covariant derivative. We elaborate on the types of gauge transformations, and the modified products simplify the calculation of the gauge algebra. We are also able to verify that the gauge Jacobiator for a general field theory described with an L_∞ algebra vanishes, as required by consistency.

2.1 The multilinear products and main identity

In an L_∞ algebra we have a vector space V graded by a degree, which is an integer. We will typically work with elements $B_i \in V$ of fixed degree.[3] The degree enters in sign factors where, for convenience, we omit the 'deg' label. Thus, for example:

$$(-1)^{B_1 B_2} \;\equiv\; (-1)^{\deg(B_1)\cdot\deg(B_2)}\,. \tag{2.1}$$

In exponents, the degrees are relevant only mod 2. In an L_∞ algebra we have multilinear products. In the notation used for string field theory the multilinear products are denoted by brackets $[B_1, \ldots, B_n]$ and are graded commutative

$$[\cdots B_i, B_j, \cdots] = (-1)^{B_i B_j}[\cdots B_j, B_i, \cdots]\,. \tag{2.2}$$

All products are defined to be of intrinsic degree -1, meaning that the degree of a product of a given number of inputs is given by

$$\deg\left([B_1,\, \ldots\, , B_n]\right) \;=\; -1 + \sum_{i=1}^{n} \deg(B_i)\,. \tag{2.3}$$

The product with one input is sometimes called the Q operator (for BRST)

$$[B] \;\equiv\; QB\,. \tag{2.4}$$

We also have a product $[\cdot]$ with no input whose value is just some special vector in the vector space.

The L_∞ relations can be written in the form [7]:

$$\sum_{\substack{l,k\geq 0\\ l+k=n}} \sum_{\sigma_s} \sigma(i_l, j_k)\big[B_{i_1},\, \ldots\, , B_{i_l}\, [B_{j_1},\, \ldots\, , B_{j_k}]\big] \;=\; 0\,, \quad n\geq 0\,. \tag{2.5}$$

[1]The identities for gauge invariance of the classical theory first appeared in [35] and were re-cast as L_∞ identities in [7]. Note that the structure of quantum closed string field theory goes beyond L_∞ algebras.

[2]After expansion of the string field theory around a background that satisfies the equations of motion, the type of algebraic structure is not changed [34].

[3]In closed string field theory degree 'deg' is related to ghost number 'gh' as deg $= 2-$gh.

Here n is the number of inputs (if $n = 0$ we still get a nontrivial identity). The inputs $B_1, \ldots, B_n$ are split into two sets: a first set $\{B_{i_1} \ldots B_{i_l}\}$ with l elements and a second set $\{B_{j_1} \ldots B_{j_k}\}$ with k elements, where $l + k = n$. The first set is empty if $l = 0$ and the second set is empty if $k = 0$. The two sets do not enter the identity symmetrically: the second set has the inputs for a product nested inside a product that involves the first set of elements. The set of numbers $\{i_1, \ldots, i_l, j_1, \ldots, j_k\}$ is a permutation of the list $\{1, \ldots, n\}$.

The sums are over inequivalent splittings. Sets with different values of l and k are inequivalent, so we must sum over all possible values of k and l. Two splittings with the same values of l and k are equivalent if the first set $\{B_{i_1} \ldots B_{i_l}\}$ contains the same elements, regardless of order. The factor $\sigma(i_l, j_k)$ is the sign needed to rearrange the list $\{B_*, B_1, \ldots, B_n\}$ into $\{B_{i_1}, \ldots B_{i_l}, B_*, \, B_{j_1}, \ldots B_{j_k}\}$:

$$\{B_*, B_1, \ldots, B_n\} \quad \rightarrow \quad \{B_{i_1}, \ldots B_{i_l}, B_*, \, B_{j_1}, \ldots B_{j_k}\}\,, \tag{2.6}$$

using the degrees to commute the B's according to (2.2) and thinking of B_* as an element of odd degree. The element B_* is needed to take into account that the products are odd.

For classical string field theory, or for any field theory expanded around a classical solution, the value of the zeroth product $[\cdot]$ will be set equal to the zero vector:

$$[\,\cdot\,] \;\equiv\; 0\,. \tag{2.7}$$

Using the above rules for sign factors, we can write out the L_∞ identities (2.5). Note that in the absence of a zeroth product $k > 0$ and thus $n > 0$ to get a nontrivial identity. For $n = 1, 2, 3$ one gets:

$$\begin{aligned}
0 \;&=\; Q(QB)\,, \\
0 \;&=\; Q[B_1, B_2] + [QB_1, B_2] + (-1)^{B_1}\,[B_1, QB_2]\,, \\
0 \;&=\; Q[B_1, B_2, B_3] \\
&\quad + [QB_1, B_2, B_3] + (-1)^{B_1}[B_1, QB_2, B_3] + (-1)^{B_1+B_2}[B_1, B_2, QB_3] \\
&\quad + (-1)^{B_1}[B_1, [B_2, B_3]\,] \;+ (-1)^{B_2(1+B_1)}[B_2, [B_1, B_3]\,] \\
&\quad + (-1)^{B_3(1+B_1+B_2)}\,[B_3, [\,B_1, B_2]]\,.
\end{aligned} \tag{2.8}$$

We will now discuss how to define in this language equations of motion and actions for a field theory. To this end and for brevity, we write products with repeated inputs as powers. When there is no possible confusion we also omit the commas between the inputs:

$$[\Psi^3] \;\equiv\; [\Psi, \Psi, \Psi]\,, \qquad [B\Psi^3] \;\equiv\; [B, \Psi, \Psi, \Psi]\,. \tag{2.9}$$

Here Ψ, called the field, is an element of degree zero:

$$\deg \Psi \;=\; 0\,. \tag{2.10}$$

If Ψ had been of odd degree, the above products would vanish by the graded commutativity property.

Given a set of products satisfying the L_∞ conditions and a Grassmann even field Ψ we introduce a field equation $\mathcal{F}$ of degree minus one:

$$\mathcal{F} \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}\,[\Psi^n] \;=\; Q\Psi + \tfrac{1}{2}[\Psi^2] + \tfrac{1}{3!}[\Psi^3] + \ldots \;=\; Q\Psi + \tfrac{1}{2}[\Psi,\Psi] + \tfrac{1}{3!}[\Psi,\Psi,\Psi] + \ldots\,. \tag{2.11}$$

Again, we used that the term with $n=0$ vanishes, as it involves a product with no input. The field equation $\mathcal{F}$ is of degree minus one because Ψ is of degree zero and all products are of degree minus one. Certain infinite sums appear often when dealing with gauge transformations and make it convenient to define modified, primed products:

$$[A_1 \ldots A_n]' \;\equiv\; \sum_{p=0}^{\infty} \frac{1}{p!}\,[A_1 \ldots A_n\, \Psi^p]\,, \qquad n \geq 1\,. \tag{2.12}$$

Thus, for example,

$$\begin{aligned} [A]' \;\equiv\; Q'A \;&=\; QA + [A\Psi] + \tfrac{1}{2}[A\Psi^2] + \ldots\,, \\ [A_1 \ldots A_n]' \;&=\; [A_1 \ldots A_n] + [A_1 \ldots A_n\Psi] + \tfrac{1}{2}[A_1 \ldots A_n\Psi^2] \;+\; \ldots\,. \end{aligned} \tag{2.13}$$

The variation of those products is rather simple:

$$\delta[A_1 \ldots A_n]' \;=\; [\delta A_1 \ldots A_n]' + \ldots + [A_1 \ldots \delta A_n]' + [A_1 \ldots A_n \delta\Psi]'\,. \tag{2.14}$$

The identification of $[A]'$ with $Q'A$ is natural given (2.4). The variation of the field equation takes the form of a modified product. We have

$$\delta\mathcal{F} \;=\; Q'(\delta\Psi)\,, \tag{2.15}$$

which is readily established:

$$\delta\mathcal{F} \;=\; \delta\sum_{k=0}^{\infty}\frac{1}{k!}[\Psi^k] \;=\; \sum_{k=1}^{\infty}\frac{k}{k!}[\Psi^{k-1}\delta\Psi] \;=\; \sum_{k=0}^{\infty}\frac{1}{k!}[\Psi^k\delta\Psi] \;=\; [\,\delta\Psi\,]'\,. \tag{2.16}$$

Inner product and action: The action exists if there is a suitable inner product $\langle\cdot\,,\cdot\rangle$. One requires that

$$\langle A, B\rangle \;=\; (-1)^{(A+1)(B+1)}\langle B, A\rangle\,, \tag{2.17}$$

and that the expression

$$\langle\, B_1, [B_2, \ldots, B_n]\rangle\,, \tag{2.18}$$

for $n \geq 1$ is a multilinear *graded-commutative* function of all the arguments. From the above one can show, for example, that

$$\langle QA, B\rangle \;=\; (-1)^A\langle A, QB\rangle\,. \tag{2.19}$$

The action is given by

$$S \;=\; \sum_{n=1}^{\infty} \frac{1}{(n+1)!}\langle\Psi\,,\,[\Psi^n]\,\rangle\,. \tag{2.20}$$

A short calculation shows that that under a variation $\delta\Psi$ one has

$$\delta S \;=\; \langle\delta\Psi, \mathcal{F}\rangle\,, \tag{2.21}$$

confirming that $\mathcal{F} = 0$ is the field equation corresponding to the action.

2.2 A family of identities

In this subsection we will establish a number of identities that will be useful below when computing the Jacobiator. The products $[\ldots]'$ can in fact be viewed as a set of products satisfying a simple extension of the L_∞ identities. To see this and to get the general picture we consider a few examples.

Consider (2.5) when all B's are of even degree. The sign factor is then always equal to $+1$ and we have

$$\sum_{\substack{l,k\geq 0\\ l+k=n}} \sum_{\sigma_s} \big[B_{i_1}\ldots B_{i_l}\,[B_{j_1}\ldots B_{j_k}]\big] \;=\; 0\,, \quad n\geq 0 \quad B_k \text{ even } \forall k\,. \tag{2.22}$$

For l and k fixed there are $n!/(l!k!)$ inequivalent splittings of the inputs; this is the number of terms in the sum $\sum_{\sigma_s}$. Assume now that all the B's are the same: $B_1 = B_2 = \ldots = B$, so that all those terms are equal. We then have

$$\sum_{\substack{l,k\geq 0\\ l+k=n}} \frac{1}{l!\,k!}\big[B^l\,[B^k]\big] \;=\; 0\,, \quad n\geq 0\,, \quad B \text{ even}\,, \tag{2.23}$$

where we have taken out an overall factor of $n!$ from the numerator. If we now take $B = \Psi$ and sum over n this identity becomes

$$\sum_{n\geq 0}\sum_{\substack{l,k\geq 0\\ l+k=n}} \frac{1}{l!\,k!}\big[\Psi^l\,[\Psi^k]\big] \;=\; 0\,. \tag{2.24}$$

Reordering the double sum we have

$$\sum_{l,k\geq 0} \frac{1}{l!\,k!}\big[\Psi^l\,[\Psi^k]\big] \;=\; \sum_{l\geq 0}\frac{1}{l!}\big[\Psi^l\,\mathcal{F}\big] \;=\; 0\,, \tag{2.25}$$

where we summed over k in the second step and used (2.11). Recalling (2.13), the sum over l finally gives

$$Q'\mathcal{F} \;=\; 0\,. \tag{2.26}$$

If we view Q' as the analogue of the covariant derivative D and $\mathcal{F}$ as the analogue of the non-abelian field strength F in Yang-Mills theory, then this identity is the analogue of the Bianchi identity $DF = 0$.

Let us consider a second L_∞ identity again based on (2.5) but with $n+1$ inputs

$$B_1 = A\,,\; B_2 = \ldots = B_{n+1} \;=\; B\,, \quad B \text{ even}\,. \tag{2.27}$$

There are two possible classes of splittings, both of which involve separating the n copies of B into a set with l elements and a set with k elements, with $l+k=n$. These are

$$\{AB^l\}\,,\,\{B^k\} \quad \text{and} \quad \{B^l\}\,,\,\{AB^k\}\,. \tag{2.28}$$

The sign factors arise from reordering $B_*AB^lB^k$ into $AB^lB_*B^k$ for the first sequence, giving a sign $(-1)^A$, and into $B^lB_*AB^k$ for the second sequence, giving no sign. We thus have

$$\sum_{\substack{l,k\geq 0\\ l+k=n}} \frac{n!}{l!\,k!}\Big((-1)^A[AB^l[B^k]] + [B^l[AB^k]]\Big) \;=\; 0\,. \tag{2.29}$$

For $n=0$ all terms vanish. This is a fine identity but an alternative form is also useful. We cancel the $n!$ in the numerator and sum over n:

$$\sum_{n\geq 0}\sum_{\substack{l,k\geq 0\\ l+k=n}}\frac{1}{l!\,k!}\big((-1)^A[AB^l[B^k]]+[B^l[AB^k]]\big) \;=\; 0\,, \tag{2.30}$$

which becomes, reorganizing the sums and letting $B=\Psi$

$$\sum_{l,k\geq 0}\frac{1}{l!\,k!}\big((-1)^A[A\Psi^l[\Psi^k]]+[\Psi^l[A\Psi^k]]\big) \;=\; 0\,. \tag{2.31}$$

Performing the sum over k gives

$$\sum_{l,\geq 0}\frac{1}{l!}\big((-1)^A[A\Psi^l\mathcal{F}]+[\Psi^l Q'A]\big) \;=\; 0\,. \tag{2.32}$$

Doing the sum over l now gives

$$(-1)^A[A\mathcal{F}]'+Q'(Q'A) \;=\; 0\,. \tag{2.33}$$

Since $\mathcal{F}$ is odd, this result is equivalent to

$$Q'(Q'A)\;+\;[\,\mathcal{F}\,A\,]' \;=\; 0\,. \tag{2.34}$$

Again, if we view Q' and $\mathcal{F}$ as the analogues of covariant derivative D and field strength F in Yang-Mills theory, then this relation is the analogue of $D^2=F$.

Consider another L_∞ identity, again based on (2.5), but with $n+2$ inputs and two string fields A_1, A_2:

$$B_1=A_1\,,\;\; B_2=A_2\,, B_3=\ldots=B_{n+2} \;=\; \Psi\,. \tag{2.35}$$

This time the above procedure yields:

$$0 \;=\; Q'[A_1,A_2]'+[Q'A_1,A_2]'+(-1)^{A_1}\,[A_1,Q'A_2]'+[\,\mathcal{F}A_1A_2]'\,. \tag{2.36}$$

Comparing the above and (2.34) with the first two equations in (2.8) the pattern becomes clear. We are obtaining for the primed products the same L_∞ identities with one extra term. In fact, the extra term corresponds to having a zeroth product, as in (2.7), that this time is nonzero:

$$[\,\cdot\,]' \;\equiv\; \mathcal{F}\,. \tag{2.37}$$

As noted in [7] the identities (2.5) indeed give, for $n=0,1,2,3$

$$\begin{aligned}
0 \;&=\; Q'\mathcal{F}\,,\\
0 \;&=\; Q'(Q'A)+[\mathcal{F}A]'\,,\\
0 \;&=\; Q'[A_1A_2]'+[Q'A_1A_2]'+(-1)^{A_1}\,[A_1Q'A_2]'+[\mathcal{F}A_1A_2]'\,,\\
0 \;&=\; Q'[A_1A_2,A_3]'\\
&\quad+[Q'A_1A_2A_3]'+(-1)^{A_1}[A_1Q'A_2A_3]'+(-1)^{A_1+A_2}[A_1A_2Q'A_3]'\\
&\quad+(-1)^{A_1}[A_1[A_2A_3]']'\;+(-1)^{A_2(1+A_1)}[A_2[A_1A_3]']'\\
&\quad+(-1)^{A_3(1+A_1+A_2)}\,[A_3[\,A_1A_2]']'\;+\;[\mathcal{F}A_1A_2A_3]'\,.
\end{aligned} \tag{2.38}$$

We have thus demonstrated that the modified products satisfy an extended form of the L_∞ identities, one that includes a nontrivial zeroth product. These identities simplify some of the work that was done in [7] and allow us to examine the Jacobiator. The above identities would be needed to construct a classical field theory around a background that is not a solution of the field equations.

The inner product interacts nicely with the modified products. One can quickly use (2.19) and multi-linearity to show that, for example,

$$\langle Q'A, B\rangle \;=\; (-1)^{A}\langle A, Q'B\rangle\,. \tag{2.39}$$

2.3 Gauge transformations and algebra

We now describe the gauge transformations and their gauge algebra. Here our primed identities are very helpful. We also discuss trivial or equation-of-motion symmetries of two types. We introduce the notion of trivial gauge parameters, and that of extended gauge transformations.

Standard Gauge transformations: These take a very simple form in terms of the new product: they are simply the result of Q' acting on the gauge parameter Λ, an element of degree $+1$. Indeed

$$\delta_\Lambda\Psi \;=\; [\Lambda]' \;=\; Q'\Lambda \;=\; Q\Lambda + [\Lambda\Psi] + \tfrac{1}{2}[\Lambda\,\Psi^2] + \dots\ . \tag{2.40}$$

The key constraint is that the field equations must be gauge covariant. This requires that the gauge transformation of $\mathcal{F}$ vanishes when $\mathcal{F}=0$. With the help of the new products this is now a trivial computation. Using (2.15) and the second of (2.38)

$$\delta_\Lambda\mathcal{F} \;=\; Q'(\delta_\Lambda\Psi) \;=\; Q'(Q'\Lambda) \;=\; [\Lambda\,\mathcal{F}]'\,. \tag{2.41}$$

We see that covariance holds. Writing out the result more explicitly,

$$\delta_\Lambda\mathcal{F} \;=\; [\Lambda\,\mathcal{F}] + [\Lambda\mathcal{F}\Psi\,] + \tfrac{1}{2}[\Lambda\mathcal{F}\Psi^2\,] + \dots\,, \tag{2.42}$$

makes it clear that the bare field appears on the right-hand side. The action is, of course, gauge invariant:

$$\delta S \;=\; \langle\delta_\Lambda\Psi,\mathcal{F}\rangle \;=\; \langle Q'\Lambda,\mathcal{F}\rangle \;=\; -\langle\Lambda, Q'\mathcal{F}\rangle \;=\; 0\,, \tag{2.43}$$

making use of (2.39) and the first identity in (2.38).

Equations-of-motion symmetries: These are transformations that vanish when using the equations of motion and are invariances of the action. For example, $\delta\Psi=[\chi,\mathcal{F}]$, for even χ is a trivial gauge transformation. It vanishes on-shell and leaves the action invariant because

$$\delta S \;=\; \langle\delta\Psi,\mathcal{F}\rangle \;=\; \langle\,\mathcal{F}\,,\,[\chi,\mathcal{F}]\,\rangle \;=\; \langle\,\chi\,,\,[\mathcal{F},\mathcal{F}]\,\rangle \;=\; 0\,, \tag{2.44}$$

because $\mathcal{F}$ is Grassmann odd. Two types of equations-of-motion symmetries will play a special role, one parameterized by a Grassmann even single string field χ of ghost number zero and another parameterized by two gauge parameters Λ_1,Λ_2. They are:

$$\begin{aligned}\delta^{T}_{\chi}\Psi \;&\equiv\; [\chi\,\mathcal{F}]' \;=\; -Q'(Q'\chi)\,,\\ \delta^{T}_{\Lambda_1,\Lambda_2}\Psi \;&\equiv\; [\Lambda_1\Lambda_2\mathcal{F}]'\,,\end{aligned} \tag{2.45}$$

using (2.34) in the first line. The second type of equations-of-motion symmetries shows up in the commutator of two standard gauge transformations, as we will discuss now.

Gauge algebra: We claim that the standard gauge transformations form an algebra that includes the equations-of-motion symmetries of the second type. Indeed, assuming the gauge parameters Λ_1 and Λ_2 are field independent we have[4]

$$\big[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}\big] \;=\; \delta_{[\Lambda_1\,\Lambda_2]'} \;+\; \overset{T}{\delta}_{\Lambda_1,\Lambda_2}\,. \tag{2.46}$$

With the help of our identities, the proof of this claim is much simplified. Using the variation formula (2.14) we find

$$\delta_{\Lambda_2}\,\delta_{\Lambda_1}\Psi \;=\; \delta_{\Lambda_2}[\Lambda_1]' \;=\; [\Lambda_1\delta_{\Lambda_2}\Psi]' \;=\; [\Lambda_1 Q'\Lambda_2]'\,. \tag{2.47}$$

Thus, it follows that

$$\big[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}\big]\Psi \;=\; [\Lambda_1 Q'\Lambda_2]' - [\Lambda_2 Q'\Lambda_1]'\,. \tag{2.48}$$

The third identity in (2.38) gives

$$0 \;=\; Q'[\Lambda_1\Lambda_2]' + [Q'\Lambda_1\Lambda_2]' - \;[\Lambda_1 Q'\Lambda_2]' + [\mathcal{F}\Lambda_1\Lambda_2]'\,. \tag{2.49}$$

As a result

$$\big[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}\big]\Psi \;=\; Q'[\Lambda_1\Lambda_2]' + [\Lambda_1\Lambda_2\mathcal{F}]' \;=\; \delta_{[\Lambda_1\Lambda_2]'}\Psi + \overset{T}{\delta}_{\Lambda_1,\Lambda_2}\Psi\,, \tag{2.50}$$

which is what we wanted to prove.

Trivial gauge parameters: A field-dependent parameter Λ is said to be trivial if $\Lambda = Q'\chi$ for some Grassmann even χ. A standard transformation with a trivial gauge parameter is a equations-of-motion symmetry of the first kind:

$$\delta_{Q'\chi}\Psi \;=\; Q'(Q'\chi) \;=\; -[\mathcal{F}\,\chi]' \;=\; -\overset{T}{\delta}_{\chi}\Psi\,. \tag{2.51}$$

Extended gauge transformations: They are the sum of a standard gauge transformation with parameter Λ and a equations-of-motion symmetry of the first kind with parameter χ of ghost-number zero:

$$\overset{E}{\delta}_{\Lambda,\chi}\,\Psi \;\equiv\; \delta_\Lambda\,\Psi \,+\, \overset{T}{\delta}_{\chi}\Psi \;=\; Q'\Lambda \,+\, [\chi\,\mathcal{F}\,]'\,. \tag{2.52}$$

Null transformations: These are extended gauge transformations that give no variation of the field. Indeed, if $\Lambda = Q'\chi$ the transformation $\overset{E}{\delta}_{\Lambda,\chi}$ of the string field vanishes:

$$\overset{E}{\delta}_{Q'\chi,\,\chi}\,\Psi \;\equiv\; \delta_{Q'\chi}\Psi + \overset{T}{\delta}_{\chi}\Psi \;=\; Q'(Q'\chi) \,+\, [\chi\,\mathcal{F}\,]' \;=\; 0\,, \tag{2.53}$$

because χ is Grassmann even. We say that χ generates the null transformation $\overset{E}{\delta}_{Q'\chi,\,\chi}$. Since null transformations give no variation of the fields we declare they are identically zero: $\overset{E}{\delta}_{Q'\chi,\,\chi} = 0$.

[4]In [7] the equations-of-motion symmetry on the right hand side has a wrong sign.

2.4 Gauge Jacobiator

Given a set of gauge transformations one can consider the gauge algebra, as we did above. In addition, one can consider the 'gauge Jacobiator' $\mathcal{J}$,

$$\mathcal{J}(\Lambda_1, \Lambda_2, \Lambda_3) \equiv \sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, [\delta_{\Lambda_2}\,,\, \delta_{\Lambda_1}]\big]\,, \tag{2.54}$$

a definition inspired by that of the Jacobiator of a bracket. Here the cyclic sum involves the sum of three terms in which we cycle the three indices $1,2,3$. The Jacobiator $\mathcal{J}$, if non-zero, would be a gauge transformation because gauge transformations close. But in the above, the brackets are simply commutators, and if the gauge transformations are well defined, upon expansion one can see that all terms vanish and this gauge Jacobiator should vanish.

This vanishing, however, is not a trivial constraint from the viewpoint of the L_∞ algebra. One can compute $\mathcal{J}$ using the gauge algebra (2.46) and one finds that the vanishing requires the L_∞ identities for three and four inputs. We will do this calculation below. Indeed, we will find that the gauge Jacobiator is a null transformation and thus vanishes identically. Namely,

$$\sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, [\delta_{\Lambda_2}\,,\, \delta_{\Lambda_1}]\big] \;=\; 0\,. \tag{2.55}$$

Proof: Using the gauge algebra (2.46) we have

$$\sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, [\delta_{\Lambda_2}\,,\, \delta_{\Lambda_1}]\big] \;=\; \sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, \delta_{[\Lambda_1\Lambda_2]'}\big] + \sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, \overset{T}{\delta}_{\Lambda_1,\Lambda_2}\big]\,. \tag{2.56}$$

For the first term on the right-hand side, we can use the gauge algebra noticing, however, the presence of an extra term because the gauge parameter $[\Lambda_1\Lambda_2]'$ is now field dependent and must be varied using (2.14). Following the same steps as in the derivation of the gauge algebra one finds

$$\sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, \delta_{[\Lambda_1\Lambda_2]'}\big] \;=\; \delta_{\sum_{\rm cyc}\left(\big[\,[\Lambda_1\Lambda_2]'\Lambda_3\big]' + \big[\,\Lambda_1\Lambda_2 Q'\Lambda_3\big]'\right)} + \sum_{\rm cyc} \overset{T}{\delta}_{[\Lambda_1\Lambda_2]',\Lambda_3}\,, \tag{2.57}$$

where the extra term is the one involving $\big[\,\Lambda_1\Lambda_2 Q'\Lambda_3\big]'$. We now use the last identity in (2.38), with $A_i = \Lambda_i$ to simplify the first term on the above right-hand side:

$$\sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, \delta_{[\Lambda_1\Lambda_2]'}\big] \;=\; \delta_{\left(-Q'\,[\Lambda_1\Lambda_2\Lambda_3]' + \big[\,\Lambda_1\Lambda_2\Lambda_3\mathcal{F}\big]'\right)} + \sum_{\rm cyc} \overset{T}{\delta}_{[\Lambda_1\Lambda_2]',\Lambda_3}\,. \tag{2.58}$$

This completes our simplification of the first term on the right-hand side of (2.56). For the second term on that same right-hand side, acting on Ψ, we get

$$\sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, \overset{T}{\delta}_{\Lambda_1,\Lambda_2}\big]\,\Psi \;=\; \sum_{\rm cyc}\Big(-\big[\Lambda_1\Lambda_2\,[\mathcal{F}\Lambda_3]'\big]' + \big[\Lambda_1\Lambda_2\mathcal{F}\,Q'\Lambda_3\big]' - \big[\Lambda_3[\Lambda_1\Lambda_2\mathcal{F}]'\big]'\Big)\,, \tag{2.59}$$

where we had to use (2.41). We now need to use the L_∞ identity for four string-fields $\Lambda_1, \Lambda_2, \Lambda_3, \mathcal{F}$ and primed products. Although we did not include it in (2.38) it is readily obtained. Recalling also that $Q'\mathcal{F} = 0$ and that any product with more than one $\mathcal{F}$ vanishes we get

$$\sum_{\rm cyc} \big[\delta_{\Lambda_3}\,,\, \overset{T}{\delta}_{\Lambda_1,\Lambda_2}\big]\,\Psi \;=\; -\sum_{\rm cyc} \big[\,[\Lambda_1\Lambda_2]'\Lambda_3\mathcal{F}\big]' \;-\; Q'[\,\Lambda_1\Lambda_2\Lambda_3\mathcal{F}]' - \big[[\Lambda_1\Lambda_2\Lambda_3]'\mathcal{F}\big]'\,. \tag{2.60}$$

We can now write the right-hand side in terms of familiar transformations and there is then no need to keep the string field explicitly:

$$\sum_{\rm cyc}\big[\delta_{\Lambda_3}\,,\,\delta^T_{\Lambda_1,\Lambda_2}\big] \;=\; -\sum_{\rm cyc}\delta^T_{[\Lambda_1\Lambda_2]',\Lambda_3} \;-\; \delta_{[\,\Lambda_1\Lambda_2\Lambda_3\mathcal{F}]'} \;-\; \delta^T_{[\Lambda_1\Lambda_2\Lambda_3]'}\,. \tag{2.61}$$

This interesting equation can be thought of as part of the gauge algebra. It gives the commutator of a standard gauge transformation and an equations-of-motion transformation of the second type. The answer is an equations-of-motion transformation of the first type (last term), an equations-of-motion transformation of the second type (first term), and a middle term that could be thought of as a new, additional, equations-of-motion transformation.

Combining (2.61) and (2.58) we see the cancellation of two equations-of-motion symmetries and two ordinary transformations, leaving:

$$\sum_{\rm cyc}\big[\delta_{\Lambda_3}\,,\,[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}]\big] \;=\; -\delta_{\,Q'[\Lambda_1\Lambda_2\Lambda_3]'} - \delta^T_{[\Lambda_1\Lambda_2\Lambda_3]'} \;=\; -\delta^E_{Q'\chi,\,\chi}\,, \quad \text{with } \chi \;=\; [\Lambda_1\Lambda_2\Lambda_3]'\,. \tag{2.62}$$

The gauge Jacobiator is a null transformation and thus, as claimed, vanishes identically. □

If a gauge algebra is field independent and closes off-shell, as in the case of Courant brackets, we have

$$\big[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}\big] \;=\; \delta_{[\Lambda_1\,\Lambda_2]}\,. \tag{2.63}$$

This requires that

$$[\Lambda_1\Lambda_2\Psi^n\,] = 0\,,\; n\geq 1\,, \quad \text{and} \quad [\Lambda_1\Lambda_2\mathcal{F}\Psi^n\,] = 0\,,\; n\geq 0\,, \tag{2.64}$$

so that the extra terms in the gauge algebra (2.46) vanish. In this case the gauge Jacobiator is equal to a gauge transformation with parameter equal to the standard Jacobiator:

$$\mathcal{J} \;=\; \sum_{\rm cyc}\big[\delta_{\Lambda_3}\,,\,[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}]\big] \;=\; \sum_{\rm cyc}\big[\delta_{\Lambda_3}\,,\,\delta_{[\Lambda_1\,\Lambda_2]}\big] \;=\; \delta_{\sum_{\rm cyc}[\,[\Lambda_1\,\Lambda_2],\Lambda_3]}\,. \tag{2.65}$$

The third identity in (2.8) then gives

$$\mathcal{J} \;=\; -\delta_{Q[\Lambda_1\Lambda_2\Lambda_3]+[Q\Lambda_1\Lambda_2\Lambda_3]-[\Lambda_1 Q\Lambda_2\Lambda_3]+[\Lambda_1\Lambda_2 Q\Lambda_3]}\,. \tag{2.66}$$

The transformation on the right must vanish on fields. The way this works (as will be seen later in section 6.2) is that there are no 3-brackets between the field $Q\Lambda$ and two gauge parameters. Moreover, we also have

$$\delta_{Q\chi} \;=\; 0\,, \tag{2.67}$$

meaning that such gauge parameters simply generate no transformations. These two facts imply with (2.66) that $\mathcal{J}=0$, as required by consistency.

3 L_∞ algebra in ℓ picture and field theory

In the previous section we reviewed the axioms of L_∞ algebras, in the formulation where all products have degree minus one. We will return to this briefly in a slightly different notation, with elements $\tilde{x}_i$ and products written as

$$[\tilde{x}_1,\ldots,\tilde{x}_n] \quad\rightarrow\quad b_n(\tilde{x}_1,\ldots,\tilde{x}_n)\,. \tag{3.1}$$

We will call this the 'b-picture' of the L_∞ algebra. The sign conventions we have described in this picture result in a simple action, field equations, and gauge transformations. The signs for the Jacobi-like identities, however, are a bit unfamiliar. Shortly after the work in [7], the axioms of L_∞ algebras were presented in a different convention [8] and later reviewed nicely in [33]. In this 'ℓ-picture', the products ℓ_n satisfy Jacobi-like identities with more familiar signs. The action, field equations, and gauge transformations, however, have more intricate signs. These two pictures of the L_∞ algebra are related by suspension.

In this section we begin by stating the general identitites for products in the ℓ-picture. Recalling the analogous definitions for the b-picture we explain how suspension relates the two pictures. We are then able to use the familiar b-picture results for the field equations, action, and gauge transformations to obtain the corresponding formulae in the ℓ-picture. In the following sections all of our discussions and examples will be stated in the ℓ-picture. To set the stage for these examples, in the final subsection we discuss general features of field theories in this language. We show how to read large classes of products from the perturbative setup and identify large classes of L_∞ identities that hold when the field equations are gauge covariant and gauge transformations close.

3.1 L_∞ algebra identities; ℓ-picture

In an L_∞ algebra we have a vector space X graded by a degree:

$$X = \bigoplus_n X_n\,, \quad n \in \mathbb{Z}\,. \tag{3.2}$$

The elements of the vector space X_n are said to be of degree n. We use the notation $x_1, x_2, \ldots$ to denote arbitrary vectors in X, but each one having definite degree; each x_k belongs to some space X_p. The degree enters in sign factors where, for convenience, we omit the 'deg' label. Thus, for example:

$$(-1)^{x_1 x_2} \;\equiv\; (-1)^{\deg(x_1)\cdot\deg(x_2)}\,. \tag{3.3}$$

In exponents, the degrees are relevant only mod 2.

In an L_∞ algebra we have multilinear products $\ell_1, \ell_2, \ell_3, \cdots$. The multilinear product ℓ_k is said to have degree $k-2$:

$$\deg \ell_k \;=\; k-2\,, \tag{3.4}$$

meaning that when acting on a collection of inputs we find

$$\deg\left(\ell_k(x_1,\ldots,x_k)\right) \;=\; k-2+\sum_{i=1}^{k}\deg(x_i)\,. \tag{3.5}$$

Thus

$$\deg(\ell_1) \;=\; -1\,, \quad \deg(\ell_2) \;=\; 0\,, \quad \deg(\ell_3) \;=\; 1\,, \quad \text{etc.} \tag{3.6}$$

The products are defined to be *graded commutative.* For ℓ_2, for example,

$$\ell_2(x_1,x_2) \;=\; (-1)^{1+x_1x_2}\,\ell_2(x_2,x_1)\,. \tag{3.7}$$

Note the extra sign added in the exponent, when compared to the b-picture formula in section 2. More generally for any permutation σ of k labels we have

$$\ell_k(x_{\sigma(1)},\ldots,x_{\sigma(k)}) \;=\; (-1)^\sigma \epsilon(\sigma;x)\,\ell_k(x_1,\ldots,x_k)\,. \tag{3.8}$$

Here $(-1)^\sigma$ gives a plus sign if the permutation is even and a minus sign if the permutation is odd. The Koszul sign $\epsilon(\sigma;x)$ is defined by considering a graded commutative algebra $\Lambda(x_1,x_2,\cdots)$ with

$$x_i\wedge x_j \;=\; (-1)^{x_ix_j}\,x_j\wedge x_i\,,\quad \forall i,j, \tag{3.9}$$

and reading its value from the relation

$$x_1\wedge\ldots\wedge x_k = \epsilon(\sigma;x)\;x_{\sigma(1)}\wedge\ldots\wedge\,x_{\sigma(k)}\,. \tag{3.10}$$

The L_∞ identities given in b-language can be stated in ℓ-language and are enumerated by a positive integer n [33]:

$$\sum_{i+j=n+1}(-1)^{i(j-1)}\sum_\sigma(-1)^\sigma\epsilon(\sigma;x)\,\ell_j\left(\,\ell_i(x_{\sigma(1)}\,,\,\ldots\,,x_{\sigma(i)})\,,\,x_{\sigma(i+1)},\,\ldots\,x_{\sigma(n)}\right)\;=\;0\,. \tag{3.11}$$

Here $n\geq 1$ is the number of inputs. The sum over σ is a sum over "unshuffles" meaning that we restrict to permutations in which the arguments are partially ordered as follows

$$\sigma(1)<\,\cdots\,<\sigma(i)\,,\qquad \sigma(i+1)<\,\cdots\,<\sigma(n)\,. \tag{3.12}$$

Schematically, the identities are of the form

$$\sum_{i+j=n+1}(-1)^{i(j-1)}\ell_j\,\ell_i \;=\; 0\,. \tag{3.13}$$

For $n=1$ we have

$$\ell_1(\ell_1(x)) \;=\; 0\,. \tag{3.14}$$

This means that the iterated action of ℓ_1 gives zero. In string field theory ℓ_1 is identified with the BRST operator. For $n=2$, the constraint is, schematically,

$$\ell_1\ell_2 \;=\; \ell_2\ell_1\,, \tag{3.15}$$

and in detail it gives

$$\ell_1(\ell_2(x_1,x_2)) \;=\; \ell_2(\ell_1(x_1),x_2)+(-1)^1(-1)^{x_1x_2}\ell_2(\ell_1(x_2),x_1)\,, \tag{3.16}$$

where $(-1)^1$ is the sign of σ and $(-1)^{x_1x_2}$ the Koszul sign. The arguments in the last term can be exchanged to find

$$\ell_1(\ell_2(x_1,x_2)) \;=\; \ell_2(\ell_1(x_1),x_2)+(-1)^{x_1}\ell_2(x_1,\ell_1(x_2))\,. \tag{3.17}$$

We recognize this as the statement that ℓ_1 is a derivation of the product ℓ_2. The next identity arises for $n=3$,

$$0 \;=\; \ell_1\ell_3+\ell_3\ell_1+\ell_2\ell_2\,, \tag{3.18}$$

and explicitly reads:

$$
\begin{aligned}
0 = \;& \ell_1(\ell_3(x_1,x_2,x_3)) \\
&+ \ell_3(\ell_1(x_1),x_2,x_3) + (-1)^{x_1}\ell_3(x_1,\ell_1(x_2),x_3) + (-1)^{x_1+x_2}\ell_3(x_1,x_2,\ell_1(x_3)) \\
&+ \ell_2(\ell_2(x_1,x_2),x_3) + (-1)^{(x_1+x_2)x_3}\ell_2(\ell_2(x_3,x_1),x_2) + (-1)^{(x_2+x_3)x_1}\ell_2(\ell_2(x_2,x_3),x_1)\,.
\end{aligned}
\tag{3.19}
$$

The first four terms on the above right-hand side quantify the failure of ℓ_1 to be a derivation of the product ℓ_3. The last three terms are the Jacobiator for a bracket defined by ℓ_2. The failure of ℓ_2 to be a Lie bracket is thus related to the existence of the higher product ℓ_3.

Let us consider one more identity. For $n = 4$ we get, schematically,

$$
0 \;=\; \ell_1\ell_4 - \ell_2\ell_3 + \ell_3\ell_2 - \ell_4\ell_1\,. \tag{3.20}
$$

Explicitly we have

$$
\begin{aligned}
0 \;=\;\; & \ell_1(\,\ell_4(x_1,x_2,x_3,x_4)) \\
& - \ell_2(\,\ell_3(x_1,x_2,x_3),x_4) + \;(-1)^{x_3x_4}\,\ell_2(\,\ell_3(x_1,x_2,x_4),x_3) \\
& + (-1)^{(1+x_1)x_2}\ell_2(x_2,\ell_3(x_1,x_3,x_4)) \;-(-1)^{x_1}\ell_2(x_1,\ell_3(x_2,x_3,x_4)) \\
& + \ell_3(\ell_2(x_1,x_2),x_3,x_4) \;\;+(-1)^{1+x_2x_3}\;\ell_3(\ell_2(x_1,x_3),x_2,x_4) \\
& + (-1)^{x_4(x_2+x_3)}\ell_3(\ell_2(x_1,x_4),x_2,x_3) \;\;-\ell_3(x_1,\ell_2(x_2,x_3),x_4) \\
& + (-1)^{x_3x_4}\ell_3(x_1,\ell_2(x_2,x_4),x_3) \;\;+\ell_3(x_1,x_2,\ell_2(x_3,x_4)) \\
& - \ell_4(\ell_1(x_1),x_2,x_3,x_4) \;\;-(-1)^{x_1}\ell_4(x_1,\ell_1(x_2),x_3,x_4) \\
& - (-1)^{x_1+x_2}\ell_4(x_1,x_2,\ell_1(x_3),x_4) \;\;-(-1)^{x_1+x_2+x_4}\ell_4(x_1,x_2,x_3,\ell_1(x_4))\,.
\end{aligned}
\tag{3.21}
$$

We now turn to the b-picture and the relation between the two pictures.

3.2 From b-picture to ℓ-picture

In the b-picture of the L_∞ algebra we have a vector space $\tilde{X}$ graded by a degree:

$$
\tilde{X} = \bigoplus_n \tilde{X}_n\,, \quad n \in \mathbb{Z}\,. \tag{3.22}
$$

The elements of the vector space $\tilde{X}_n$ are said to be of degree n. We use the notation $\tilde{x}_1, \tilde{x}_2, \ldots$ to denote arbitrary fixed-degree vectors in $\tilde{X}$. In the b picture all products have degree minus one:

$$
\deg b_n \;=\; -1\,. \tag{3.23}
$$

As we have already explained all products are *graded commutative*, with no additional factors:

$$
b_n(\ldots\,,\tilde{x}_i,\tilde{x}_j,\,\ldots) \;=\; (-1)^{\tilde{x}_i\tilde{x}_j}\,b_n(\ldots\,,\,\tilde{x}_j,\tilde{x}_i,\ldots)\,, \tag{3.24}
$$

with exponents representing degrees. The inner product is completely graded commutative and has a simple exchange symmetry:

$$\begin{aligned}\langle \tilde{x}_1\,,\, b_n(\tilde{x}_2\,,\ldots\,,\tilde{x}_n)\rangle &= (-1)^{\tilde{x}_1\tilde{x}_2}\,\langle \tilde{x}_2\,,\, b_n(\tilde{x}_1\,,\ldots\,,\tilde{x}_n)\rangle \\ \langle \tilde{x}_1, \tilde{x}_2\rangle &= (-1)^{(\tilde{x}_1+1)(\tilde{x}_2+1)}\langle \tilde{x}_2, \tilde{x}_1\rangle\,.\end{aligned} \tag{3.25}$$

It follows from these that

$$\langle b_1(\tilde{x}_1), \tilde{x}_2\rangle = (-1)^{\tilde{x}_1}\langle \tilde{x}_1, b_1(\tilde{x}_2)\rangle\,. \tag{3.26}$$

In this new notation, the L_∞ identities are just a simple translation of those given in (2.8) and need not be repeated here. The gauge transformation and field equations, with $\Lambda \to \tilde{\Lambda}$ and the field denoted by $\tilde{\Psi}$, take the form (see (2.40) and (2.11))

$$\begin{aligned}\delta_{\tilde{\Lambda}}\tilde{\Psi} &= b_1(\tilde{\Lambda}) + b_2(\tilde{\Lambda},\tilde{\Psi}) + \tfrac{1}{2}b_3(\tilde{\Lambda},\tilde{\Psi},\tilde{\Psi}) + \tfrac{1}{3!}b_3(\tilde{\Lambda},\tilde{\Psi},\tilde{\Psi},\tilde{\Psi}) + \ldots\,, \\ \tilde{\mathcal{F}} &= b_1(\tilde{\Psi}) + \tfrac{1}{2}b_2(\tilde{\Psi},\tilde{\Psi}) + \tfrac{1}{3!}b_3(\tilde{\Psi},\tilde{\Psi},\tilde{\Psi}) + \ldots\end{aligned} \tag{3.27}$$

The degrees of the various vectors here are

$$\deg\tilde{\Lambda} \;=\; 1\,,\quad \deg\tilde{\Psi} \;=\; 0,\quad \deg\tilde{\mathcal{F}} \;=\; -1\,. \tag{3.28}$$

Suspension: Suspension is a map that starting with a graded vector space X gives us a graded vector space $\tilde{X}$. Acting on X_n suspension gives us the space $\tilde{X}_{n+1}$. The map simply copies the vectors in X_n into $\tilde{X}_{n+1}$. The degree of the elements is then 'suspended', or increased by one unit. To track properly the various vectors we will write the suspension map as s or sometimes as $\uparrow$ and say that

$$\tilde{x}_i \;=\; s\,x_i \;=\uparrow x_i\,, \tag{3.29}$$

leading to

$$\deg\tilde{x}_i \;=\; \deg x_i \;+\; 1\,. \tag{3.30}$$

The inverse map is well defined and we will write

$$x_i \;=\downarrow \tilde{x}_i\,. \tag{3.31}$$

For gauge parameters, fields and field equations we write,

$$\tilde{\Lambda} = s\,\Lambda \;=\uparrow\Lambda\,,\qquad \tilde{\Psi} = s\,\Psi \;=\uparrow\Psi\,,\qquad \tilde{\mathcal{F}} \;=\; s\,\mathcal{F} \;=\uparrow\mathcal{F}\,. \tag{3.32}$$

We note that given (3.28) we now have

$$\deg\Lambda \;=\; 0\,,\quad \deg\Psi \;=\; -1,\quad \deg\mathcal{F} \;=\; -2\,. \tag{3.33}$$

The products in the two pictures are related as follows. Up to a sign, $b_n(\tilde{x}_1,\ldots,\tilde{x}_n)$ is the same as $\ell_n(x_1,\ldots,x_n)$. As discussed in [2] and [33], we have

$$b_{n+1}(\tilde{x}_1,\ldots\tilde{x}_{n+1}) \;=\; (-1)^{x_1 n + x_2(n-1)+\ldots+x_n}\; s\,\ell_{n+1}(x_1,\ldots,x_{n+1})\,. \tag{3.34}$$

In the above, the values $x_1,\ldots,x_n$ in exponents denote the degrees as elements of X. Note that the degree of the right-hand side of (3.34) is

$$1 + ((n+1)-2) + \sum_{k=1}^{n+1}\deg x_k \;=\; -1 + \sum_{k=1}^{n+1}(\deg x_k + 1) \;=\; -1 + \sum_{k=1}^{n+1}\deg\tilde{x}_k\,, \tag{3.35}$$

showing that (3.34) is consistent with the stated degrees of ℓ and b products. The first few cases of (3.34) give

$$\begin{aligned} b_1(\tilde{x}) &= s\,\ell_1(x)\,, \\ b_2(\tilde{x}_1,\tilde{x}_2) &= (-1)^{x_1}\,s\,\ell_2(x_1,x_2)\,, \\ b_3(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) &= (-1)^{x_2}\,s\,\ell_3(x_1,x_2,x_3)\,, \\ b_4(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{x}_4) &= (-1)^{x_1+x_3}\,s\,\ell_4(x_1,x_2,x_3,x_4)\,. \end{aligned} \tag{3.36}$$

One can verify with some explicit computation that the Jacobi-like ℓ_n identities, upon suspension become the corresponding b_n identities.

It follows from (3.34), applied to a gauge parameter and n fields, that

$$b_{n+1}(\tilde{\Lambda},\tilde{\Psi}^n) \;=\; (-1)^{0n+(-1)(n-1)+(-1)(n-2)+\ldots+(-1)}\;s\,\ell_{n+1}(\Lambda,\Psi^n)\,. \tag{3.37}$$

Performing the sum in the exponent and applying $\downarrow$ we get

$$\downarrow b_{n+1}(\tilde{\Lambda},\tilde{\Psi}^n) \;=\; (-1)^{\frac{n(n-1)}{2}}\;\ell_{n+1}(\Lambda,\Psi^n)\,. \tag{3.38}$$

This formula allows us to translate the gauge transformations from the b picture to the ℓ picture. Consider

$$\delta_{\tilde{\Lambda}}\tilde{\Psi} \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}b_{n+1}(\tilde{\Lambda},\tilde{\Psi}^n)\,. \tag{3.39}$$

Applying $\downarrow$ to the gauge transformation above,

$$\downarrow\delta_{\tilde{\Lambda}}\tilde{\Psi} \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\;\downarrow b_{n+1}(\tilde{\Lambda},\tilde{\Psi}^n)\,, \tag{3.40}$$

and therefore

$$\delta_{\Lambda}\Psi \;\equiv\downarrow\delta_{\tilde{\Lambda}}\tilde{\Psi} \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\;(-1)^{\frac{n(n-1)}{2}}\;\ell_{n+1}(\Lambda,\;\Psi^n)\,. \tag{3.41}$$

Expanding, this gives a series whose signs alternate every two elements

$$\delta_{\Lambda}\Psi \;=\; \ell_1(\Lambda)+\ell_2(\Lambda,\Psi)-\tfrac{1}{2}\ell_3(\Lambda,\Psi,\Psi)-\tfrac{1}{3!}\ell_4(\Lambda,\Psi,\Psi,\Psi)+\ldots\;. \tag{3.42}$$

Let us now consider the action. For this we must consider the inner product. The identities for the inner product in the ℓ picture arise from the definition of this inner product in terms of the b-picture inner product:

$$\langle x_1,x_2\rangle \;\equiv\; \langle\tilde{x}_1,\tilde{x}_2\rangle\,. \tag{3.43}$$

Here, $x_1,x_2\in X$ and, with a slight abuse of notation, the inner product on the right-hand side is in the b-picture and the inner product on the left-hand side is in the ℓ picture. From the properties of the b-picture inner product (3.25) and the above definition we quickly derive the properties of the ℓ-picture inner product:

$$\begin{aligned} \langle x,\,\ell_n(x_1,\ldots x_n)\rangle &= (-1)^{xx_1+1}\langle x_1,\,\ell_n(x,x_2\ldots x_n)\rangle\,, \\ \langle x_1,x_2\rangle &= (-1)^{x_1x_2}\langle x_2,x_2\rangle\,. \end{aligned} \tag{3.44}$$

As we can see the inner product is totally graded symmetric, just as the products are. A short computation shows that we also have:

$$\begin{aligned}\langle \ell_1(x_1), x_2\rangle \; &= (-1)^{x_1+1}\langle x_1, \ell_1(x_2)\rangle\,,\\ \langle \ell_2(x_1,x_2)\,, x_3)\rangle \;&=\; \langle x_1,\; \ell_2(x_2,x_3)\rangle\,.\end{aligned} \tag{3.45}$$

The translation of the action

$$S \;=\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\,\langle \tilde\Psi, b_n(\tilde\Psi^n)\,\rangle\,, \tag{3.46}$$

is done using (3.34), which gives

$$\downarrow b_n(\tilde\Psi^n) \;=\; (-1)^{\frac{n(n-1)}{2}}\;\;\ell_n(\Psi^n)\,. \tag{3.47}$$

Indeed, together with (3.43) we have the closed form expression for the action in the ℓ picture:

$$S \;=\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\,\langle \Psi, \downarrow b_n(\tilde\Psi^n)\,\rangle \;=\; \sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n-1)}{2}}}{(n+1)!}\,\langle \Psi, \ell_n(\tilde\Psi^n)\,\rangle\,. \tag{3.48}$$

Again, if we expand we get alternating signs:

$$S \;=\; \tfrac{1}{2}\langle\Psi,\ell_1(\Psi) - \tfrac{1}{3!}\langle\Psi,\ell_2(\Psi^2)\rangle - \tfrac{1}{4!}\langle\Psi,\ell_3(\Psi^3)\rangle + \tfrac{1}{5!}\langle\Psi,\ell_4(\Psi^4)\rangle + \ldots \tag{3.49}$$

The field equation takes the form

$$\mathcal{F}(\Psi) \;=\; \sum_{n=1}^{\infty}\frac{(-1)^{\frac{n(n-1)}{2}}}{n!}\ell_n(\Psi^n) \;=\; \ell_1(\Psi) - \tfrac{1}{2}\ell_2(\Psi^2) - \tfrac{1}{3!}\ell_3(\Psi^3) \,+ \tfrac{1}{4!}\ell_4(\Psi^4) + \cdots\,. \tag{3.50}$$

The gauge transformation of the field equation can be translated starting from (2.41)

$$\delta\tilde{\mathcal{F}} \;=\; [\,\tilde\Lambda\,\tilde{\mathcal{F}}\,]' \tag{3.51}$$

together with

$$\downarrow b_{n+2}(\tilde\Lambda,\,\tilde{\mathcal{F}}\,,\tilde\Psi^n) \;=\; (-1)^{\frac{n(n-1)}{2}}\,\ell_{n+2}(\Lambda,\mathcal{F},\Psi^n) \tag{3.52}$$

This leads to

$$\delta_\Lambda\mathcal{F}(\Psi) \;=\; \ell_2(\Lambda,\mathcal{F}) \,+\, \ell_3(\Lambda,\mathcal{F}(\Psi),\Psi) \,-\, \tfrac{1}{2}\ell_4(\Lambda,\mathcal{F}(\Psi),\Psi^2) + \cdots \tag{3.53}$$

This expresses the gauge covariance of the field equation in the ℓ picture.

For the gauge algebra we had (2.46) stating that $\big[\delta_{\tilde\Lambda_2}\,,\,\delta_{\tilde\Lambda_1}\big]$ is a gauge transformation with parameter

$$\tilde\Lambda_{12} \;\equiv\; [\tilde\Lambda_1\,\tilde\Lambda_2]'\,, \tag{3.54}$$

in addition to a trivial gauge transformation. In the ℓ picture the commutator $\big[\delta_{\Lambda_2}\,,\,\delta_{\Lambda_1}\big]$ is a gauge transformation with parameter

$$\Lambda_{12} \;=\; \ell_2(\Lambda_1,\Lambda_2) + \ell_3(\Lambda_1,\Lambda_2,\Psi) - \tfrac{1}{2}\ell_4(\Lambda_1,\Lambda_2,\Psi,\Psi) - \ldots\,, \tag{3.55}$$

with the by-now-familiar alternating signs. This translation follows from the identity

$$\downarrow\, b_{n+2}(\tilde\Lambda_1,\tilde\Lambda_2,\tilde\Psi^n) \;=\; (-1)^{\frac{n(n-1)}{2}}\,\ell_{n+2}(\Lambda_1,\Lambda_2,\Psi^n)\,. \tag{3.56}$$

3.3 General remarks on the L_∞ algebra of field theories

Let us make a few general remarks about the extraction of products from a gauge invariant perturbative field theory.

We will focus on the part of the theory dealing with gauge parameters, fields, and field equations. We thus consider the graded vector space

$$\begin{array}{ccccccc} \ldots & \longrightarrow & X_0 & \longrightarrow & X_{-1} & \longrightarrow & X_{-2}\,. \\ & & \Lambda & & \Psi & & E \end{array} \tag{3.57}$$

The arrows are defined as the map ℓ_1. We will assume that there are no spaces X_{-d} with $d \geq 3$. Recall that the field equations (3.50) take the form

$$\ell_1(\Psi) - \tfrac{1}{2!}\ell_2(\Psi,\Psi) - \tfrac{1}{3!}\ell_2(\Psi,\Psi,\Psi) + \tfrac{1}{4!}\ell_4(\Psi,\Psi,\Psi,\Psi) + \ldots \;=\; 0\;. \tag{3.58}$$

It follows that knowledge of the field equations determines explicitly all products

$$\ell_n(\Psi,\, \ldots\, ,\Psi) \,\in\, X_{-2}\,, \quad n \geq 1\;, \tag{3.59}$$

that involve fields. Here all arguments are identical, but a general result (a polarization identity) implies that a multilinear symmetric form is completely determined by the values on the diagonal. For example, defining $L_2(\Psi) = \ell_2(\Psi,\Psi)$ and $L_3(\Psi) = \ell_3(\Psi,\Psi,\Psi)$ we have

$$\begin{aligned} 2\,\ell_2(\Psi_1,\Psi_2) &= L_2(\Psi_1+\Psi_2) - L_2(\Psi_1) - L_2(\Psi_2)\;, \\ 3!\,\ell_3(\Psi_1,\Psi_2,\Psi_3) &= L_3(\Psi_1+\Psi_2+\Psi_3) - L_3(\Psi_1+\Psi_2) - L_3(\Psi_1+\Psi_3) - L_3(\Psi_2+\Psi_3) \\ &\quad + L_3(\Psi_1) + L_3(\Psi_2) + L_3(\Psi_3)\;. \end{aligned} \tag{3.60}$$

More generally, defining $L_n(\Psi) = \ell_n(\Psi,\, \ldots\, ,\Psi)$ we have

$$\begin{aligned} n!\ell_n(\Psi_1,\, \ldots, \Psi_n) \;=\;& L_n(\Psi_1 + \ldots + \Psi_n) \\ & - \big[L_n(\Psi_1 + \ldots + \Psi_{n-1}) + \ldots\big] + \big[\ldots\big] - \ldots \\ & + (-1)^{n-k}[L_n(\Psi_1 + \ldots \Psi_k) + \ldots] + \ldots \\ & + (-1)^{n-1}[L_n(\Psi_1) + \ldots + L_n(\Psi_n)]\,. \end{aligned} \tag{3.61}$$

The pattern is clear. On the second line we subtract all terms with L_n evaluated on the sum of fields leaving one out. As we proceed we alternate signs and leave out two, three, four, until we leave out all fields except one. This shows we have determined completely the multilinear products acting on arbitrary fields.

Consider now the L_∞ identities acting on just fields. The first is

$$\ell_1(\ell_1(\Psi)) = 0\;. \tag{3.62}$$

Since $\ell_1(\Psi)$ is an element E of X_{-2} we can satisfy this constraint by setting

$$\ell_1(E) = 0\,. \tag{3.63}$$

For the second identity we have

$$\ell_1(\ell_2(\Psi,\Psi)) \;=\; 2\,\ell_2(\ell_1(\Psi),\Psi)\,. \tag{3.64}$$

The left-hand side is of the form $\ell_1(E)$ and thus vanishes. Thus the identity holds if we set

$$\ell_2(E,\Psi) = 0\,. \tag{3.65}$$

An inductive argument shows that all L_∞ identities acting on fields are satisfied if we take

$$\ell_{n+1}(E,\Psi_1,\,\dots\,,\Psi_n) \;=\; 0\,, \quad n = 0,1,\dots \tag{3.66}$$

This is not surprising, since all of the above are of degree $n+1-2+(-2)-n = -3$ and we have not introduced a space X_{-3}. If we did, we could contemplate setting some of these products to be nonzero: for example setting $\ell_1(E)$ to some value such that $\ell_1(\ell_1(\Psi)) = 0$.

Let us now consider the gauge transformations. From (3.42)

$$\delta_\xi\Psi \;=\; \ell_1(\Lambda) + \ell_2(\Lambda,\Psi) - \tfrac{1}{2}\ell_3(\Lambda,\Psi,\Psi) - \tfrac{1}{3!}\ell_4(\Lambda,\Psi,\Psi,\Psi) + \dots\;, \tag{3.67}$$

we are now able to read off the products

$$\ell_{n+1}(\Lambda,\Psi_1,\dots\Psi_n) \in X_{-1}\,, \qquad n \geq 0\,, \tag{3.68}$$

where we can use the polarization identities above to deduce the value of the product for non-diagonal field entries.

We can now examine the L_∞ identities when we input a list $(\Lambda,\Psi,\dots,\Psi)$ of arguments. The identity $\ell_1(\ell_1(\Lambda)) = 0$ is nontrivial but must hold due to gauge invariance of the linearized field equation. The next identity is

$$\ell_1(\ell_2(\Lambda,\Psi)) \;=\; \ell_2(\ell_1(\Lambda),\Psi) + \ell_2(\Lambda,\ell_1(\Psi))\;. \tag{3.69}$$

The left-hand side is already determined and so is the first term on the right-hand side. Thus this identity determines

$$\ell_2(\Lambda,E) \;\in\; X_{-2}\;. \tag{3.70}$$

The next identity can be seen to determine $\ell_3(\Lambda,E,\Psi)$. All in all, the set of L_∞ identities acting on $(\Lambda,\Psi,\dots,\Psi)$ determine the products

$$\ell_{n+2}(\Lambda,E,\Psi_1,\dots\Psi_n) \in X_{-2}\,, \qquad n \geq 0\,. \tag{3.71}$$

The identities that lead to this determination are in fact the ones relevant to the gauge covariance (2.42) of the field equation. We can now iterate this process and consider the L_∞ identities on a list $(\Lambda,E,\Psi,\dots,\Psi)$. This time this would lead us to consider products $\ell_{n+3}(\Lambda,E_1,E_2,\Psi,\dots,\Psi)$. But these products are all of degree minus three, and thus they vanish with the assumption that X_{-3} does not exist.

The gauge algebra commutator leads to the determination of the following products. From the field-dependent gauge parameter we read

$$\ell_{n+2}(\Lambda_1,\Lambda_2,\Psi_1,\dots\Psi_n) \in X_0\,, \qquad n \geq 0\,. \tag{3.72}$$

If we have only on-shell closure we then read

$$\ell_{n+3}(\Lambda_1, \Lambda_2, E,\, \Psi_1, \ldots \Psi_n) \in X_{-1}\,, \qquad n \geq 0\,. \tag{3.73}$$

Using the L_∞ identities for inputs of the form $(\Lambda_1, \Lambda_2, \Psi, \ldots, \Psi)$ we get constraints on the products $\ell_{n+3}(\Lambda_1, \Lambda_2, E, \Psi_1, \ldots \Psi_n) \in X_{-1}$ determined by on-shell closure. By use of the identities for inputs of the form $(\Lambda_1, \Lambda_2, E, \Psi, \ldots, \Psi)$ we can get information about products of the form $\ell_{n+4}(\Lambda_1, \Lambda_2, E_1, E_2, \Psi_1, \ldots \Psi_n) \in X_{-2}$. Note that the products vanish on Λ diagonals and on E diagonals.

We want to emphasize an important point. We have seen in detail how a consistent set of L_∞ products leads to gauge transformations under which the field equation transforms covariantly and to a gauge algebra that closes. We now want to explain that the reverse is true. More precisely:

1. If we have gauge transformations and gauge covariance properties of the field equations of a certain standard type, c.f. (3.74) below, L_∞ identities acting on inputs

   $$(\Lambda, \Psi, \ldots),$$

   with arbitrary numbers of Ψ's, are all satisfied.

2. If we have gauge transformations of the standard type and a standard-form gauge algebra, then the L_∞ identities acting on inputs

   $$(\Lambda_1, \Lambda_2, \Psi \ldots)\,,$$

   with arbitrary numbers of Ψ's, are all satisfied.

Consider the first item above, and work for simplicity in the b picture where all signs are simple. We recall the following equalities

$$\begin{aligned} \delta_\Lambda \Psi \;&=\; Q\Lambda + [\Lambda, \Psi] + \tfrac{1}{2}[\Lambda, \Psi, \Psi] + \tfrac{1}{3!}[\Lambda, \Psi, \Psi, \Psi] + \ldots\ , \\ \mathcal{F}(\Psi) \;&=\; Q\Psi + \tfrac{1}{2}[\Psi, \Psi] + \tfrac{1}{3!}[\Psi, \Psi, \Psi] \,+ \tfrac{1}{4!}[\Psi, \Psi, \Psi, \Psi] + \cdots\ , \\ \delta_\Lambda \mathcal{F}(\Psi) \;&=\; [\Lambda, \mathcal{F}] \;+\; [\Lambda, \mathcal{F}, \Psi] \;+\; \tfrac{1}{2}[\Lambda, \mathcal{F}, \Psi, \Psi] + \cdots \end{aligned} \tag{3.74}$$

The first equation is what we mean by standard gauge transformations and the last one what we mean by a standard-type field-equation covariance. Think of the first two equations as definitions. Then, we used some subset of the L_∞ identities to show that the last one holds. But in fact the last one holds *if and only if* that subset of the L_∞ identities hold. The equation is checked in powers of Ψ, and for each power Ψ^n *one* L_∞ identity is involved. It is also clear, because $\mathcal{F}$ is a sum of products of fields, that the relevant L_∞ identities are those with one Λ and any number of Ψ's.

For the second item now consider the gauge algebra (2.46) acting on a field,

$$\big[\delta_{\Lambda_2}\,,\, \delta_{\Lambda_1}\big]\Psi \;=\; \delta_{[\Lambda_1\, \Lambda_2]'}\Psi \;+\; [\Lambda_1 \Lambda_2 \mathcal{F}]'\,. \tag{3.75}$$

We call this a standard-form gauge algebra. We have checked before that using the above gauge transformations the gauge algebra above follows if the collection of L_∞ identities that involve inputs $(\Lambda_1, \Lambda_2, \Psi, \ldots)$ hold. In fact the gauge algebra holds *if and only if* those L_∞ identities hold. Again, equation (3.75) is checked in powers of Ψ and for each power Ψ^n one L_∞ identity with inputs $(\Lambda_1, \Lambda_2, \Psi^n)$ is involved.

The utility of the above remarks is that if we identify a perturbative field theory in which we have standard gauge transformations, field-equation covariance, and gauge algebra, we are guaranteed that the products that can be easily read off from those expressions will satisfy large subsets of the L_∞ identities.

4 Non-abelian gauge theories and L_∞ algebras

In this section we formulate Yang-Mills-type gauge theories as L_∞ algebras. In the first subsection we discuss the Yang-Mills gauge structure in this framework. As examples we then consider in the second subsection the dynamical theory based on the Chern-Simons action in three dimensions and, in the third subsection, the usual Yang-Mills theory that exists in arbitrary dimensions. Yang-Mills theories were first formulated as L_∞ algebras in [16–18] using the BRST complex of open string field theory, which is larger than the complex we use here.

4.1 Generalities on Yang-Mills theory

Consider a Lie algebra $\mathcal{G}$ with generators T_α:

$$[T_\alpha, T_\beta] \;=\; f_{\alpha\beta}{}^{\gamma}\, T_\gamma\,, \tag{4.1}$$

where $f_{\alpha\beta}{}^{\gamma}$ are the structure constants. We also consider Lie algebra valued gauge fields $A_\mu(x) = A_\mu^\alpha(x) T_\alpha$ and gauge parameters $\lambda(x) = \lambda^\alpha(x) T_\alpha$. The gauge field transformations are

$$\delta_\lambda A_\mu{}^\alpha \;=\; \partial_\mu \lambda^\alpha + \big[A_\mu, \lambda\big]^\alpha\;, \tag{4.2}$$

and they close according to the Lie algebra structure:

$$\big[\delta_{\lambda_1}, \delta_{\lambda_2}\big] \;=\; \delta_{[\lambda_1,\lambda_2]}\,. \tag{4.3}$$

We also have the field strength

$$F_{\mu\nu}{}^\alpha \;=\; \partial_\mu A_\nu{}^\alpha - \partial_\nu A_\mu{}^\alpha + \big[A_\mu, A_\nu\big]^\alpha\,, \tag{4.4}$$

that transforms covariantly under gauge transformations:

$$\delta_\lambda F_{\mu\nu} \;=\; \big[F_{\mu\nu}, \lambda\big]\,. \tag{4.5}$$

Our goal is now to determine the appropriate L_∞ algebra for Chern-Simons theory in 3D and for Yang-Mills theory in arbitrary dimensions. For both of these cases the total graded vector space X will be taken to contain three spaces of fixed degrees:

$$\begin{array}{ccc} X_0 & X_{-1} & X_{-2} \\ \lambda^\alpha & A_\mu{}^\alpha & E_\mu{}^\alpha \end{array} \tag{4.6}$$

The gauge parameters λ are of degree zero, gauge fields A have degree minus one, and equations of motion E have degree minus two. We write this as

$$\deg(\lambda) \;=\; 0\,, \qquad \deg(A) \;=\; -1\,, \qquad \deg(E) \;=\; -2\,. \tag{4.7}$$

Recalling that $\ell_2(x_1, x_2) \;=\; (-1)^{1+x_1x_2}\,\ell_2(x_2, x_1)$ we have that ℓ_2 is antisymmetric for gauge parameters, as it befits a Lie algebra, and symmetric for fields, as it befits the interactions of a bosonic field.

We define the inner product that is non-vanishing only when the total degree is minus three:

$$\langle A, E\rangle \;\equiv\; \int dx\;\kappa_{\alpha\beta}\,\eta^{\mu\nu}\,A_\mu{}^\alpha(x)\,E_\nu{}^\beta(x)\,, \tag{4.8}$$

where $\kappa_{\alpha\beta}$ is the Cartan-Killing form and $\eta_{\mu\nu}$ a fixed spacetime metric (say the Minkowski metric) and we include the integration over spacetime, as the inner product is supposed to give a number.

The homotopy Lie algebra implies an infinite number of identities. Of course, for polynomial gauge theories we only need to check a finite number of them. Here is a table of the identities, ordered by total degree of the identity, and showing the degrees of total inputs that must be checked given the relevant complex exists at degree zero, minus one and minus two.

$$\begin{aligned}
&\deg=-2\,,\quad \ell_1\ell_1=0, && \Big\{\deg=0:\ \lambda\\
&\deg=-1\,,\quad \ell_1\ell_2-\ell_2\ell_1=0, && \begin{cases}\deg=\ \ 0:\ \lambda\lambda\\ \deg=-1:\ \lambda A\end{cases}\\
&\deg=0\,,\quad \ell_3\ell_1+\ell_2\ell_2+\ell_1\ell_3=0, && \begin{cases}\deg=\ \ 0:\ \lambda\lambda\lambda\\ \deg=-1:\ \lambda\lambda A\\ \deg=-2:\ \lambda AA, \lambda\lambda E\end{cases}\\
&\deg=1\,,\quad \ell_1\ell_4-\ell_2\ell_3+\ell_3\ell_2-\ell_4\ell_1=0, && \begin{cases}\deg=\ \ -1:\ \lambda\lambda\lambda A\\ \deg=-2:\ \lambda\lambda AA,\ \lambda\lambda\lambda E\\ \deg=-3:\ \lambda AAA,\ \lambda\lambda AE\end{cases}\\
&\deg=2\,,\quad \ell_1\ell_5\pm\ell_2\ell_4\pm\ell_3\ell_3\pm\ldots=0, && \Big\{\ldots
\end{aligned} \tag{4.9}$$

For Chern-Simons theory there are only ℓ_1 and ℓ_2 products and thus just the first three identities must be checked. Yang-Mills theory has also an ℓ_3 and thus all identities above must be checked. As we will see, the last one ends up holding trivially, so we did not include the various subcases above.

Since the gauge structure is the same for Chern-Simons and Yang-Mills theories, we can read off some of the basic products. Comparing the gauge transformation (4.2) with the expression

$$\delta_\lambda A \;=\; \ell_1(\lambda) + \ell_2(\lambda, A) + \ldots \tag{4.10}$$

we infer:

$$\begin{aligned} \ell_1(\lambda) &= \partial_\mu \lambda \quad \in X_{-1}\,, \\ \ell_2(\lambda, A) &= \big[A, \lambda\big] \quad \in X_{-1}\,. \end{aligned} \tag{4.11}$$

All products involving a gauge parameter and two or more fields vanish. We can write the indices in these equations explicitly

$$\begin{aligned} [\ell_1(\lambda)]_\mu{}^\alpha &= \partial_\mu \lambda^\alpha \quad \in X_{-1}\,, \\ [\ell_2(\lambda, A)]_\mu{}^\alpha &= \big[A_\mu, \lambda\big]^\alpha \in X_{-1}\,. \end{aligned} \tag{4.12}$$

Note that to comply with the graded commutativity we must also define

$$\ell_2(A, \lambda) \equiv -\ell_2(\lambda, A) = -[A, \lambda]\,. \tag{4.13}$$

We can now use the gauge algebra to identify the product ℓ_2 acting on two gauge parameters. From (4.10) we quickly find that

$$\begin{aligned} [\delta_{\lambda_1}, \delta_{\lambda_2}]\, A &= \delta_{\lambda_1}(\ell_1(\lambda_2) + \ell_2(\lambda_2, A)) - (1 \leftrightarrow 2) \\ &= \ell_2(\lambda_2, \delta_{\lambda_1} A) - \ell_2(\lambda_1, \delta_{\lambda_2} A) \\ &= \ell_2(\lambda_2, \ell_1(\lambda_1)) - \ell_2(\lambda_1, \ell_1(\lambda_2)) + \mathcal{O}(A) \\ &= -\,\ell_2(\ell_1(\lambda_1), \lambda_2) - \ell_2(\lambda_1, \ell_1(\lambda_2)) + \mathcal{O}(A) \end{aligned} \tag{4.14}$$

We now use the $\ell_1\ell_2 = \ell_2\ell_1$ identity to identify the gauge transformation on the right-hand side:

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\, A = \ell_1(-\ell_2(\lambda_1, \lambda_2)) + \mathcal{O}(A) = \delta_{-\ell_2(\lambda_1,\lambda_2)} A\,. \tag{4.15}$$

The A dependent terms on the right-hand side are not needed for the identification. Comparing with (4.3) we infer

$$\ell_2(\lambda_1, \lambda_2) = -\big[\lambda_1, \lambda_2\big] \in X_0\,. \tag{4.16}$$

4.2 Chern-Simons Theory

We now turn to the Chern-Simons theory. In order to define an action we have to assume that for the Lie algebra there exists an invariant inner product. We write for this inner product of Lie algebra valued objects

$$\langle\langle A_\mu, B_\nu \rangle\rangle = \kappa_{\alpha\beta}\, A_\mu{}^\alpha B_\nu{}^\beta\,, \tag{4.17}$$

where $\kappa_{\alpha\beta}$ is the Cartan-Killing metric. With this definition, the full inner product (4.8) becomes

$$\langle A, E \rangle = \int d^3x\, \eta^{\mu\nu} \langle\langle A_\mu\,, E_\nu \rangle\rangle\,. \tag{4.18}$$

Consider now the gauge invariant 3D Chern-Simons action

$$S = \tfrac{1}{2} \int d^3x\, \varepsilon^{\mu\nu\rho} \langle\langle A_\mu, \partial_\nu A_\rho + \tfrac{1}{3}[A_\nu, A_\rho] \rangle\rangle\,. \tag{4.19}$$

The Chern-Simons action is topological and hence does not depend on the spacetime metric. The general variation of the action is given by

$$\begin{aligned}\delta S \;&=\; \tfrac{1}{2}\int \mathrm{d}^3x\, \langle\!\langle\, \delta A_\mu\,,\, \varepsilon^{\mu\nu\rho}(\partial_\nu A_\rho - \partial_\rho A_\nu + [A_\nu, A_\rho])\,\rangle\!\rangle \\ &=\; \int \mathrm{d}^3x\; \eta^{\mu\sigma}\langle\!\langle\, \delta A_\mu\,,\, \varepsilon_\sigma{}^{\nu\rho}(\partial_\nu A_\rho + \tfrac{1}{2}[A_\nu\,,\, A_\rho])\,\rangle\!\rangle \\ &=\; \langle\, \delta A\,,\, \varepsilon_*(\partial A + \tfrac{1}{2}[A, A])\,\rangle\,,\end{aligned} \tag{4.20}$$

where the star denotes the position of the free index on the epsilon symbol and we used the definition of the inner product. Comparing with the expected form of the field equation,

$$\ell_1(A) - \tfrac{1}{2}\,\ell_2(A, A) \;=\; 0\,, \tag{4.21}$$

we get

$$\begin{aligned}[\ell_1(A)]_\mu{}^\alpha \;&=\; \varepsilon_\mu{}^{\nu\rho}\,\partial_\nu A_\rho{}^\alpha \;\in\; X_{-2}\,, \\ [\ell_2(A_1, A_2)]_\mu{}^\alpha \;&=\; -\varepsilon_\mu{}^{\nu\rho}\,\big[A_{1\nu}, A_{2\rho}\big]^\alpha \;\in\; X_{-2}\,.\end{aligned} \tag{4.22}$$

In index free notation we would write

$$\begin{aligned}\ell_1(A) \;&=\; \varepsilon_*\partial A \;\in\; X_{-2}\,, \\ \ell_2(A_1, A_2) \;&=\; -\varepsilon_*\,\big[A_1, A_2\big] \;\in\; X_{-2}\,.\end{aligned} \tag{4.23}$$

As expected ℓ_2 is *symmetric* under the exchange of gauge fields.[5] Note that the inner product in (4.18) now contains the spacetime metric, which is also used in $\ell_1(A)$ to lower the index on the epsilon tensor. Thus, the L_n formulation obscures the topological nature of the Chern-Simons action, but that is unavoidable if we have spaces X_1 and X_2 with the same index structure.

We now confirm that the action has the expected form

$$S \;=\; \tfrac{1}{2}\langle A, \ell_1(A)\rangle - \tfrac{1}{3!}\langle A, \ell_2(A, A)\rangle \;=\; \langle A\,,\; \tfrac{1}{2}\ell_1(A) \;-\; \tfrac{1}{3!}\ell_2(A, A)\;\rangle\,. \tag{4.24}$$

The Chern-Simons action given above can be written as

$$\begin{aligned}S \;&=\; \int \mathrm{d}^3x\, \langle\!\langle\, A_\mu,\; \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho + \tfrac{1}{3!}\,\varepsilon^{\mu\nu\rho}[A_\nu, A_\rho]\;\rangle\!\rangle\,, \\ &=\; \langle\, A,\; \tfrac{1}{2}\varepsilon_*\partial A \,+\, \tfrac{1}{3!}\,\varepsilon_*[A, A]\;\rangle\,.\end{aligned} \tag{4.25}$$

Comparing with (4.23) we see that the action is indeed correctly reproduced.

Let us verify the L_n axioms.

Checking $\ell_1\ell_1 = 0$. This is only nontrivial at degree zero. Indeed, we have

$$[\ell_1(\ell_1(\lambda))]_\mu{}^\alpha \;=\; \varepsilon_\mu{}^{\nu\rho}\,\partial_\nu[\ell_1(\lambda)]_\rho{}^\alpha \;=\; \varepsilon_\mu{}^{\nu\rho}\,\partial_\nu\partial_\rho\lambda^\alpha \;=\; 0\,. \tag{4.26}$$

This is just linearized gauge invariance.

Checking $\ell_1\ell_2 = \ell_2\ell_1$. This means checking (3.17) at degree zero and minus one.

[5]This is the first instance where we derive the general product starting with the product evaluated on diagonals.

Degree zero. At this degree we must act on two gauge parameters:

$$\ell_1(\ell_2(\lambda_1,\lambda_2)) \;=\; \ell_2(\ell_1(\lambda_1),\lambda_2) \;+\; \ell_2(\lambda_1,\ell_1(\lambda_2))\,. \tag{4.27}$$

This gives

$$\begin{aligned} -\partial([\lambda_1,\lambda_2]) \;&=\; \ell_2(\partial\lambda_1,\lambda_2) \;+\; \ell_2(\lambda_1,\partial\lambda_2) \\ &=\; -\,[\partial\lambda_1,\,\lambda_2] \;-\; [\lambda_1,\partial\lambda_2]\,, \end{aligned} \tag{4.28}$$

which works out correctly.

Degree minus one. We must verify

$$\ell_1(\ell_2(A,\lambda)) \;=\; \ell_2(\ell_1(A),\lambda) \;-\; \ell_2(A,\ell_1(\lambda))\,. \tag{4.29}$$

We then have that the left-hand side is

$$\begin{aligned} [\ell_1(\ell_2(A,\lambda))]_\mu{}^\alpha \;&=\; \varepsilon_\mu{}^{\nu\rho}\,\partial_\nu[\ell_2(A,\lambda)]_\rho{}^\alpha \;=\; -\varepsilon_\mu{}^{\nu\rho}\,\partial_\nu\big[A_\rho,\lambda\big]^\alpha \\ &= -\varepsilon_\mu{}^{\nu\rho}\,\big[\partial_\nu A_\rho,\lambda\big]^\alpha - \varepsilon_\mu{}^{\nu\rho}\,\big[A_\rho,\partial_\nu\lambda\big]^\alpha \\ &=\; \big(-\big[\varepsilon_*\partial A,\lambda\big] + \varepsilon_*\,\big[A,\partial\lambda\big]\big)_\mu{}^\alpha\,. \end{aligned} \tag{4.30}$$

The right-hand side is

$$\ell_2(\varepsilon_*\partial A,\lambda) \;-\; \ell_2(A,\partial\lambda) \;=\; \ell_2(\varepsilon_*\partial A,\lambda) \;+\; \varepsilon_*[A,\partial\lambda]\,. \tag{4.31}$$

In order for this to agree with the left-hand side we have to define for $E\in X_{-2}$, $\lambda\in X_0$

$$\ell_2(E,\lambda) \;=\; -\big[E,\lambda\big] \;\in\; X_{-2}\,. \tag{4.32}$$

Checking $\ell_3\ell_1+\ell_1\ell_3+\ell_2\ell_2=0$. Since ℓ_3 is assumed to be zero this means checking that $\ell_2\ell_2=0$. From (3.19) this requires that

$$\ell_2(\ell_2(x_1,x_2),x_3)+(-1)^{(x_1+x_2)x_3}\ell_2(\ell_2(x_3,x_1),x_2)+(-1)^{(x_2+x_3)x_1}\ell_2(\ell_2(x_2,x_3),x_1) \;=\; 0\,. \tag{4.33}$$

As indicated in our table this identity can only be nontrivial acting on elements whose total degree equals zero, minus one, or minus two.

Degree zero. We must act on three gauge parameters. Since they have degree zero, we have

$$\ell_2(\ell_2(\lambda_1,\lambda_2),\lambda_3)+\ell_2(\ell_2(\lambda_3,\lambda_1),\lambda_2)+\ell_2(\ell_2(\lambda_2,\lambda_3),\lambda_1) \;=\; 0\,. \tag{4.34}$$

Using $\ell_2(\lambda_1,\lambda_2)=-[\lambda_1\,,\lambda_2]$ we obtain

$$-\,\ell_2([\lambda_1,\lambda_2],\lambda_3)-\ell_2([\lambda_3,\lambda_1],\lambda_2)-\ell_2([\lambda_2,\lambda_3],\lambda_1) \;=\; 0\,. \tag{4.35}$$

Since the bracket is another gauge parameter, we use the same expression for ℓ_2 to see that we must have

$$[[\lambda_1,\lambda_2],\lambda_3]+[[\lambda_3,\lambda_1],\lambda_2]+[[\lambda_2,\lambda_3],\lambda_1] \;=\; 0\,. \tag{4.36}$$

This holds because $\mathcal{G}$ is a Lie-algebra.

Degree minus one. Here we have two gauge parameters and one gauge field

$$\ell_2(\ell_2(\lambda_1,\lambda_2),A)+\ell_2(\ell_2(A,\lambda_1),\lambda_2)+\ \ell_2(\ell_2(\lambda_2,A),\lambda_1)\ =\ 0\,. \tag{4.37}$$

Again, first replacing the nested in products

$$-\,\ell_2([\lambda_1,\lambda_2],A)-\ell_2([A,\lambda_1],\lambda_2)-\ \ell_2([\lambda_2,A],\lambda_1)\ =\ 0\,. \tag{4.38}$$

Since $[A,\lambda]\in X_{-1}$ we can now take

$$[[\lambda_1,\lambda_2],A]+[[A,\lambda_1],\lambda_2]+\ [[\lambda_2,A],\lambda_1]\ =\ 0\,, \tag{4.39}$$

which holds by virtue of the Jacobi identity of the Lie algebra $\mathcal{G}$.

Degree minus two . We now act on two gauge fields and one gauge parameter $(AA\lambda)$ or two gauge parameters and one field equation $(\lambda\lambda E)$. First, for the former we have

$$\ell_2(\ell_2(A_1,A_2),\lambda)+\ell_2(\ell_2(\lambda,A_1),A_2)-\ell_2(\ell_2(A_2,\lambda),A_1)\ =\ 0\ , \tag{4.40}$$

and we compute

$$-\,\ell_2(\varepsilon_*[A_1,A_2],\lambda)-\ell_2([\lambda,A_1],A_2)+\ell_2([A_2,\lambda],A_1)\ =\ 0\ . \tag{4.41}$$

For the first term we use (4.32)

$$\varepsilon_*[[A_1,A_2],\lambda]+\varepsilon_*[[\lambda,A_1],A_2]-\varepsilon_*[[A_2,\lambda],A_1]\ =\ 0\ . \tag{4.42}$$

In the first two terms the second and third indices in ε are contracted with A_1 and A_2 respectively. Not so in the third, so we can factor out ε by changing the sign of the last term:

$$\varepsilon_*\big([[A_1,A_2],\lambda]+[[\lambda,A_1],A_2]+[[A_2,\lambda],A_1]\big)\ =\ 0\ . \tag{4.43}$$

This holds on account of the Jacobi identity of $\mathcal{G}$.

Now for the second case $(\lambda\lambda E)$ we have

$$\ell_2(\ell_2(\lambda_1,\lambda_2),E)+\ell_2(\ell_2(E,\lambda_1)\lambda_2)+\ell_2(\ell_2(\lambda_2,E)\lambda_1)\ =\ 0\,. \tag{4.44}$$

This simply gives

$$[[\lambda_1,\lambda_2],E]+[[E,\lambda_1],\lambda_2]+[[\lambda_2,E],\lambda_1]\ =\ 0\,, \tag{4.45}$$

which again holds by the Jacobi identity.

With all checks done, we list the complete set of *nonvanishing* L_2 products:

$$\boxed{\begin{aligned} \text{Chern-Simons:}\qquad \ell_1(\lambda)\ &=\ \partial\lambda\ \in\ X_{-1}\\ \ell_1(A)\ &=\ \varepsilon_*\,\partial A\ \in\ X_{-2}\\ \ell_2(\lambda_1,\lambda_2)\ &=\ -\big[\lambda_1,\lambda_2\big]\ \in\ X_0\\ \ell_2(A,\lambda)\ &=\ -\big[A,\lambda\big]\ \in\ X_{-1}\\ \ell_2(A_1,A_2)\ &=\ -\varepsilon_*\,\big[A_1,A_2\big]\ \in\ X_{-2}\\ \ell_2(E,\lambda)\ &=\ -\big[E,\lambda\big]\ \in\ X_{-2}\,. \end{aligned}} \tag{4.46}$$

The versions with explicit indices were given above.

With the identities one can verify that the field equations transform covariantly,

$$\delta_\lambda(\ell_1(A) - \tfrac{1}{2}\ell_2(A,A)) \;=\; \ell_2(\ell_1(A) - \tfrac{1}{2}\ell_2(A,A),\lambda)\,, \tag{4.47}$$

which is the correct covariant transformation.

In order to compute the field equations and check gauge invariance we need the invariance properties of the inner product: Assuming that we can integrate by parts under the integral implicit in the inner product (4.18), we have for $A, B \in X_{-1}$

$$\langle A, \ell_1(B)\rangle \;=\; \langle \ell_1(A), B\rangle\,. \tag{4.48}$$

Moreover, for $A, B, C \in X_{-1}$ we have explicitly

$$\langle A, \ell_2(B,C)\rangle \;=\; \int \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\,\kappa\big(A_\mu, [B_\nu, C_\rho]\big)\,. \tag{4.49}$$

The invariance of the Cartan-Killing form then implies cyclicity, i.e.,

$$\langle A, \ell_2(B,C)\rangle \;=\; \langle C, \ell_2(B,A)\rangle\,, \quad \text{etc.} \tag{4.50}$$

The general variation of the Chern-Simons action is then

$$\begin{aligned}\delta S \;&=\; \tfrac{1}{2}\langle \delta A, \ell_1(A)\rangle + \tfrac{1}{2}\langle A, \ell_1(\delta A)\rangle + \tfrac{1}{3!}\langle \delta A, \ell_2(A,A)\rangle + \tfrac{2}{3!}\langle A, \ell_2(A,\delta A)\rangle \\ &=\; \langle \delta A, \ell_1(A)\rangle + \tfrac{1}{3!}\langle \delta A, \ell_2(A,A)\rangle + \tfrac{2}{3!}\langle \delta A, \ell_2(A,A)\rangle \\ &=\; \langle \delta A, \ell_1(A) + \tfrac{1}{2}\ell_2(A,A)\rangle\,,\end{aligned} \tag{4.51}$$

implying the correct field equation.

4.3 Yang-Mills theory

We now turn to the dynamical Yang-Mills theory, for which we keep the general conventions for Yang-Mills gauge transformations as above. Consider the Yang-Mills Lagrangian and its expansion in powers of the gauge field:

$$\begin{aligned}\mathcal{L} \;&=\; -\tfrac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle \\ &=\; \tfrac{1}{2}\langle A^\mu, \partial^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu)\rangle \;-\; \langle \partial^\mu A^\nu, [A_\mu, A_\nu]\rangle \;-\; \tfrac{1}{4}\langle [A^\mu, A^\nu], [A_\mu, A_\nu]\rangle\,.\end{aligned} \tag{4.52}$$

To derive a few of the products we consider the field equations:

$$\begin{aligned}0 \;&=\; D^\mu F_{\mu\nu} \;=\; \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] \\ &=\; \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) + [A^\mu, \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]] \\ &=\; \square A_\nu - \partial_\nu\,\partial\cdot A \quad + \partial^\mu[A_\mu, A_\nu] + [A^\mu, \partial_\mu A_\nu - \partial_\nu A_\mu] \quad + [A^\mu\,, [A_\mu, A_\nu]]\,.\end{aligned} \tag{4.53}$$

We now compare with the expectation for the gauge transformations and the equations of motion

$$\begin{aligned}\delta_\lambda A \;&=\; \ell_1(\lambda) + \ell_2(\lambda, A)\,, \\ \mathcal{F}(A) \;&\equiv\; \ell_1(A) - \tfrac{1}{2}\ell_2(A,A) - \tfrac{1}{3!}\ell_3(A,A,A) \;=\; 0\,,\end{aligned} \tag{4.54}$$

and we read off

$$\begin{aligned}\ell_1(A) \;&=\; \Box A - \partial(\partial\cdot A)\,,\\ [\ell_2(A_1,A_2)]_\mu \;&=\; -\,\partial^\nu[A_{1\nu},A_{2\mu}] - [\partial_\mu A_{1\nu}-\partial_\nu A_{1\mu},A_2^\nu] + (1\leftrightarrow 2)\,,\\ \ell_3(A_1,A_2,A_3)_\mu \;&=\; -\,[A_1^\nu,[A_{2\nu},A_{3\mu}]] - [A_2^\nu,[A_{3\nu},A_{1\mu}]] - [A_3^\nu,[A_{1\nu},A_{2\mu}]]\\ &\quad -\,[A_2^\nu,[A_{1\nu},A_{3\mu}]] - [A_1^\nu,[A_{3\nu},A_{2\mu}]] - [A_3^\nu,[A_{2\nu},A_{1\mu}]]\,.\end{aligned} \tag{4.55}$$

Since the gauge field has degree minus one, the above products are symmetric under the exchange of any two gauge fields. We can confirm that ℓ_2, so defined, gives the correct cubic terms in the action:

$$\begin{aligned}-\tfrac{1}{3!}\langle A,\ell_2(A,A)\rangle \;&=\; \tfrac{2}{3!}\langle A^\mu,\partial^\nu[A_\nu,A_\mu]+[\partial_\mu A_\nu-\partial_\nu A_\mu,A^\nu]\rangle\\ &=\; -\tfrac{2}{3!}\langle\,\partial^\mu A^\nu,[A_\mu,A_\nu]\,\rangle - \tfrac{2}{3!}\langle\,A^\mu,[A^\nu\,,\partial_\mu A_\nu-\partial_\nu A_\mu]\rangle\\ &=\; -\tfrac{1}{3}\langle\,\partial^\mu A^\nu,[A_\mu,A_\nu]\,\rangle - \tfrac{2}{3}\langle\,[A^\mu,A^\nu]\,,\partial_\mu A_\nu]\rangle\\ &=\; -\,\langle\,\partial^\mu A^\nu\,,[A_\mu,A_\nu]\rangle\,,\end{aligned} \tag{4.56}$$

where from the first to second line we integrated by parts and used the invariance of the Cartan-Killing metric.

In the following we verify the L_∞ relations:

Checking $\ell_1\ell_1=0$. This must only be checked at degree zero, and it works out immediately:

$$\ell_1(\ell_1(\lambda)) \;=\; \ell_1(\partial\lambda) \;=\; \Box\partial_*\lambda - \partial_*(\Box\lambda) \;=\; 0\,. \tag{4.57}$$

Checking $\ell_1\ell_2=\ell_2\ell_1$. At degree zero the computation is identical to that for Chern-Simons. At degree minus one we must verify

$$\ell_1(\ell_2(A,\lambda)) \;=\; \ell_2(\ell_1(A),\lambda) \;-\; \ell_2(A,\ell_1(\lambda))\,. \tag{4.58}$$

All terms are calculable except for the first one on the right-hand side. This identity works out correctly if, again, we choose

$$\ell_2(E,\lambda) \;=\; -\big[E,\lambda\big] \;\in\; X_{-2}\,. \tag{4.59}$$

There are no more cases to check here.

Checking $\ell_3\ell_1+\ell_2\ell_2+\ell_1\ell_3=0$.

Since the products on the identity do not change degree, this identity is nontrivial only in degrees zero, minus one and minus two.

We set the following combinations to zero:

$$\ell_3(\lambda_1,\lambda_2,\lambda_3) \;=\; 0\,,\quad \ell_3(\lambda_1,\lambda_2,A) \;=\; 0\,,\quad \ell_3(A,A,\lambda) \;=\; 0\,,\quad \ell_3(\lambda_1,\lambda_2,E) \;=\; 0\,, \tag{4.60}$$

because there is no Lie algebra Jacobiator, no $\ell_3(A,A,\lambda)$ term in $\delta_\lambda A$, no field dependent structure constants $\ell_3(\lambda_1,\lambda_2,A)$ in the gauge algebra, and no $\ell_3(\lambda_1,\lambda_2,E)$ because the algebra closes on shell.

At degree zero we act on $(\lambda\lambda\lambda)$ and the ℓ_3 terms in the identity will vanish because the ℓ_3 acts on $(\lambda\lambda\lambda)$ or $(\lambda\lambda A)$. At degree minus one we act on $(\lambda\lambda A)$ and the ℓ_3 terms in the identity will vanish because ℓ_3 acts on $(\lambda\lambda A)$, (λAA) or $(\lambda\lambda E)$. Thus both at degree zero and minus one the computation reduces to $\ell_2\ell_2 = 0$ and is the same as in CS.

At degree minus two we have $AA\lambda$ and $\lambda\lambda E$. For the first one, (3.19) requires

$$\begin{aligned}&\ell_2(\ell_2(A_1,A_2),\lambda)+\ell_2(\ell_2(\lambda,A_1),A_2)-\ell_2(\ell_2(A_2,\lambda),A_1)\\ &\quad = -\ell_1(\ell_3(A_1,A_2,\lambda))-\ell_3(\ell_1(A_1),A_2,\lambda)+\ell_3(A_1,\ell_1(A_2),\lambda)-\ell_3(A_1,A_2,\ell_1(\lambda))\,.\end{aligned} \tag{4.61}$$

A computation of the l.h.s. gives

$$\text{l.h.s.} \;=\; [[\partial_\mu\lambda,A_{1\nu}],A_2^\nu]-[[\partial_\nu\lambda,A_{1\mu}],A_2^\nu]-[[A_{1\nu},A_{2\mu}],\partial^\nu\lambda]+(1\leftrightarrow 2)\,. \tag{4.62}$$

On the r.h.s. the first term is zero because of (4.60). The final term on the r.h.s. is

$$-\,\ell_3(A_1,A_2,\ell_1(\lambda))_\mu \;=\; [[A_1^\nu,[A_{2\nu},\partial_\mu\lambda]]+[A_1^\nu,[\partial_\nu\lambda,A_{2\mu}]]+[\partial^\nu\lambda,[A_{1\nu},A_{2\mu}]]+(1\leftrightarrow 2)\,. \tag{4.63}$$

This agrees precisely with the l.h.s., and so the identity holds in the form

$$\ell_2(\ell_2(A_1,A_2),\lambda)+\ell_2(\ell_2(\lambda,A_1),A_2)-\ell_2(\ell_2(A_2,\lambda),A_1) \;=\; -\ell_3(A_1,A_2,\ell_1(\lambda))\,, \tag{4.64}$$

implying that we can satisfy the equation by setting

$$\ell_3(E,A,\lambda) \;=\; 0\,. \tag{4.65}$$

The second one is $\lambda\lambda E$. The ℓ_3 terms in the identity will find ℓ_3 acting on $\lambda\lambda E$ and λAE, both of which vanish. Thus we again have to check only $\ell_2\ell_2 = 0$ and this is the same check as in Chern-Simons. At degree minus three there is nothing to check.

Checking $\ell_1\ell_4 - \ell_2\ell_3 + \ell_3\ell_2 - \ell_4\ell_1 = 0$. Since we will take $\ell_4 = 0$ we just need to check

$$\ell_2\ell_3 = \ell_3\ell_2\,. \tag{4.66}$$

Since $\ell_2\ell_3$ has degree plus one, this identity must only be checked on four arguments adding to degree $-1, -2$, or -3. Since ℓ_3 is only non-vanishing on three A's, we claim we need three A's and the last one must be a λ, giving total degree -3. Indeed, if there are two or less A's no term survives: when ℓ_3 acts first there are no three A's to give something nonzero. When it acts after ℓ_2 there are no three A's either, since ℓ_2 does not change degree. Thus the only nontrivial check is on $AAA\lambda$: The explicit form given in (3.21) then requires

$$\ell_2(\ell_3(A_1,A_2,A_3),\lambda) \;=\; \ell_3(\ell_2(A_1,\lambda),A_2,A_3)+\ell_3(A_1,\ell_2(A_2,\lambda),A_3)+\ell_3(A_1,A_2,\ell_2(A_3,\lambda))\,. \tag{4.67}$$

This relation can be proved by multiple use of the Jacobi identity.

Checking $\ell_1\ell_5 \pm \ell_2\ell_4 \pm \ell_3\ell_3 \pm \ell_4\ell_2 \pm \ell_5\ell_1 = 0$.

Here we only need to check the

$$\ell_3\ell_3 = 0\,. \tag{4.68}$$

Conceivably, having degree $+2$, this equation should be tested on degrees $-2, -3$, and -4. But acting on AAA, the product ℓ_3 produces an E and then the second ℓ_3 will always give zero. So this equation is trivially satisfied.

Checking $\sum_{i,j} \ell_i \ell_j = 0$ with $i + j \geq 7$. In here every term has an ℓ_k with $k \geq 4$ therefore these are trivially satisfied.

$$
\boxed{
\begin{array}{lrcl}
\text{Yang-Mills:} & \ell_1(\lambda) & = & \partial\lambda \ \in \ X_{-1} \\
& \ell_1(A) & = & \Box A - \partial(\partial\cdot A) \ \in \ X_{-2} \\
& \ell_2(\lambda_1,\lambda_2) & = & -\big[\lambda_1,\lambda_2\big] \ \in \ X_0 \\
& \ell_2(A,\lambda) & = & -\big[A,\lambda\big] \ \in \ X_{-1} \\
& \ell_2(A_1,A_2)_* & = & -\partial[A_1,A_{2*}] - [\partial_* A_1 - \partial A_{1*}\,,\, A_2] + (1\leftrightarrow 2) \ \in \ X_{-2} \\
& \ell_2(E,\lambda) & = & -\big[E,\lambda\big] \ \in \ X_{-2} \\
& \ell_3(A_1,A_2,A_3)_* & = & -[A_1,[A_2,A_{3*}]] \ + \ \text{sym.} \ \in \ X_{-2}
\end{array}
}
\tag{4.69}
$$

5 Double field theory and L_∞ algebras

In this section we discuss double field theory (DFT) in the framework of L_∞ algebras. In the first subsection, following the notation and setup of Roytenberg and Weinstein, we discuss the subalgebra corresponding to the pure gauge structure, given by the C-bracket algebra, which in turn is the $O(D,D)$ covariantization of the Courant algebroid. The results in this subsection were obtained by Deser and Saemann [20] in a geometrical setup that involves symplectic NQ-manifolds and a derived bracket construction [21]. In the second subsection we extend this to the L_∞ algebra that also encodes fields and their off-shell gauge transformations. Finally, in the third subsection, we discuss the full L_∞ algebra describing the complete DFT symmetries and dynamics, using perturbation theory around flat space.

5.1 DFT C-bracket algebra as an L_3 algebra

We begin by recalling a few generalities of DFT, which is manifestly $O(D,D)$ covariant. We denote $O(D,D)$ indices by $M,N = 1,\ldots,2D$, and the group-invariant inner product is defined on vectors by

$$
\langle V_1, V_2\rangle \ = \ \eta_{MN}\, V_1^M V_2^N \ , \qquad \eta_{MN} \ \equiv \ \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} . \tag{5.1}
$$

The role of infinitesimal gauge transformations will be played by generalized Lie derivatives w.r.t. to a gauge parameter ξ^M,

$$
\mathcal{L}_\xi V^M \ = \ \xi^N\partial_N V^M + \big(\partial^M\xi_N - \partial_N\xi^M\big)V^N \ . \tag{5.2}
$$

The generalized Lie derivatives form an algebra, $[\mathcal{L}_{\xi_1},\mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1,\xi_2]_c}$, which is governed by the antisymmetric C-bracket

$$
[\xi_1,\xi_2]_c^M \ \equiv \ \xi_1^K\partial_K\xi_2^M - \tfrac{1}{2}\xi_1^K\partial^M\xi_{2K} - (1\leftrightarrow 2)\,, \tag{5.3}
$$

where in the following we will sometimes leave out the sub-index c in order not to clutter the equations. Both the inner product and the C-bracket are covariant under the action of the generalized Lie derivative. Moreover, the generalized Lie derivative and the C-bracket are equal up to a total derivative of the inner product,

$$\mathcal{L}_V W \;=\; [V, W] + \tfrac{1}{2}\partial\langle V, W\rangle \;. \tag{5.4}$$

The generalized Lie derivative w.r.t. a gauge parameter that is a total derivative, $\xi^M = \partial^M \chi$, acts trivially on fields as a consequence of the strong constraint ($\partial^M\partial_M A = 0$ and $\partial^M A\,\partial_M B = 0$ for all A, B). Moreover, when one of the two gauge parameters (vectors) inside the C-bracket is trivial, i.e., $\xi_2 = \partial\chi$ for some function χ, one finds

$$[\xi, \partial\chi] \;=\; \tfrac{1}{2}\partial(\xi^K\partial_K\chi) \;=\; \partial\,\tfrac{1}{2}\langle\xi, \partial\chi\rangle \,. \tag{5.5}$$

The C-bracket satisfies a Jacobiator identity

$$J(\xi_1, \xi_2, \xi_3) \;\equiv\; 3\big[\,[\xi_{[1}, \xi_2]\,, \xi_{3]}\big] \;=\; \big[\,[\xi_1, \xi_2]\,, \xi_3\big] + \text{c.p.} \;=\; \partial\, T(\xi_1, \xi_2, \xi_3)\,, \tag{5.6}$$

where the antisymmetrization is over all three indices, c.p. denotes 'cyclic permutation', and T is defined by

$$T(\xi_1, \xi_2, \xi_3) \;\equiv\; \tfrac{1}{2}\big\langle\,[\xi_{[1}, \xi_2]\,, \xi_{3]}\big\rangle \;=\; \tfrac{1}{6}\big(\big\langle\,[\xi_1, \xi_2]\,, \xi_3\big\rangle + \text{c.p.}\big)\,. \tag{5.7}$$

Since the above identities take the same form as for the Courant algebroid, the setup of Roytenberg and Weinstein applies here, and we can next reformulate this as an L_3 homotopy Lie algebra. The total graded vector space X will be taken to contain three spaces of fixed degrees:

$$\begin{array}{ccccccc} 0 \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 \\ & c & & \chi & & \xi^M \end{array} \tag{5.8}$$

The space of degree zero contains the gauge parameters, the space of degree one contains functions, and the space of degree zero contains the constants. The above arrows define the ℓ_1 map. From X_2 to X_1 it is the inclusion map $\iota : c \to \iota c$, which is the same constant, now viewed as a trivial function in X_1. From X_1 to X_2 is the partial derivative ∂. Acting on X_0 the map ℓ_1 is defined to give zero

$$\ell_1(\xi) \;=\; 0\,, \tag{5.9}$$

in agreement with the fact that we are not taking fields into account.

The non-vanishing multilinear maps are

$$\boxed{\begin{aligned} \ell_1(\chi) \;&=\; \partial\chi \;\in X_0\;, \\ \ell_1(c) \;&=\; \iota\, c \;\in X_1\;, \\ \ell_2(\xi_1, \xi_2) \;&=\; [\xi_1, \xi_2] \;\in X_0\;, \\ \ell_2\,(\xi,\,\chi) \;&=\; \tfrac{1}{2}\,\langle\xi\,, \partial\chi\rangle \;=\; \tfrac{1}{2}\xi^K\partial_K\chi \;\in X_1\;, \\ \ell_3(\xi_1, \xi_2, \xi_3) \;&=\; -\,T(\xi_1, \xi_2, \xi_3) \;\in X_1\,. \end{aligned}} \tag{5.10}$$

Note that no product, except for ℓ_1, involves an input from the space X_2 nor does any product give an element of X_2. Additionally, we have the following interesting relations:

$$\begin{aligned} \partial \ell_3(\xi_1, \xi_2, \xi_3) &= - J(\xi_1, \xi_2, \xi_3)\,, \\ \ell_2(\xi, \partial\chi) &= \partial \ell_2(\xi, \chi)\,. \end{aligned} \tag{5.11}$$

The first relates the Jacobiator (5.6) to the derivative of ℓ_3. The second encodes the behavior (5.5) of the C-bracket when one of the inputs is a trivial vector.

Step by step construction. We now sketch how the construction of the above algebra is done, step by step, starting with the C-bracket.

1. We begin with the space X_0 of gauge parameters. From the C-bracket one sets the product $\ell_2(\xi_1, \xi_2)$ equal to the bracket itself. At this stage one does not know if any other products are needed or not.

2. With just $\ell_2 \neq 0$ the only nontrivial identity would be $\ell_2\ell_2 = 0$, acting on three ξ's. But $\ell_2\ell_2$ gives the Jacobiator $J(\xi_1, \xi_2, \xi_3)$, which does not vanish. This implies we have to introduce both ℓ_3 and ℓ_1 to fix this identity.

3. Since the Jacobiator can be written as $\partial T(\xi_1, \xi_2, \xi_3)$ this suggests setting ℓ_3 on three gauge parameters equal to the function $T(\xi_1, \xi_2, \xi_3)$. Since ℓ_3 has degree plus one, we now need a space X_1 of functions. On this space of functions ℓ_1 acts as a derivative and this fixes the homotopy identity $\ell_1\ell_3 + \ell_2\ell_2 + \ell_3\ell_1 = 0$, assuming the last term is zero.

4. The presence of ℓ_1 forces one to reconsider the lower identities. In order to guarantee that $\ell_1\ell_1 = 0$ acting on X_1 we now set $\ell_1 : X_0 \to 0$. This now confirms the last term in the previous item vanishes.

5. We then consider $\ell_1\ell_2 = \ell_2\ell_1$ which is only nontrivial acting on a $\chi \in X_1$ and a gauge parameter $\xi \in X_0$. That identity determines $\ell_2(\xi, \chi)$.

6. At this point all nontrivial products have been determined and one must verify that all homotopy identities hold without the need of additional products.

7. If one wishes to have an exact sequence of spaces then one can introduce the space X_2 of constants, and ℓ_1 acting on it simply gives the same constant, now as an element of the space of functions X_1. This completes the construction.

As indicated above, the only nontrivial computation is checking that no ℓ_4 is needed because the identity $\ell_3\ell_2 - \ell_2\ell_3 = 0$ holds when acting on four gauge parameters. Indeed, using (3.21) we see that

$$\ell_3\ell_2 - \ell_2\ell_3 \;=\; 6\,\ell_3([\xi_{[1}, \xi_2], \xi_3, \xi_{4]}) \;-\; 4\,\ell_2(\ell_3(\xi_{[1}, \xi_2, \xi_3), \xi_{4]})\,. \tag{5.12}$$

We will show that $\ell_3\ell_2 - \ell_2\ell_3$ must be a constant by proving that its derivative vanishes. But this means that $\ell_3\ell_2 - \ell_2\ell_3$ actually vanishes, because it is a local function of arbitrary space-dependent gauge parameters; if it did not vanish it would have to have space dependence and could not be a constant. Taking the derivative of the above equation and using both lines in (5.11), we compute

$$\partial(\ell_3\ell_2 - \ell_2\ell_3) \;=\; -6\,J([\xi_{[1},\xi_2],\xi_3,\xi_{4]}) \;+\; 4\,\ell_2(J(\xi_{[1},\xi_2,\xi_3),\xi_{4]})\,. \tag{5.13}$$

Rearranging the inputs on both terms and recalling the definition of ℓ_2 on two vectors we get

$$\partial(\ell_3\ell_2 - \ell_2\ell_3) \;=\; -6\,J(\,\xi_{[1},\xi_2\,,[\xi_3,\xi_{4]}]) \;+\; 4\,[\,\xi_{[1}\,,\,J(\xi_2,\xi_3,\xi_{4]})\,]\,. \tag{5.14}$$

It is straightforward to see that the right-hand side vanishes. It does so trivially, just upon using the definition $J(\xi_1,\xi_2,\xi_3) \equiv 3\big[\,[\xi_{[1},\xi_2]\,,\xi_{3]}\big] = \big[\,[\xi_1,\xi_2]\,,\xi_3\big] + \text{c.p.}$. Thus, as claimed, the derivative of $\ell_3\ell_2 - \ell_2\ell_3$ is guaranteed to vanish by our definitions. As argued above, this means that $\ell_3\ell_2 - \ell_2\ell_3 = 0$.

We can contemplate the possibility that in some other scenario $\ell_3\ell_2 - \ell_2\ell_3 \neq 0$ is a non-vanishing constant and we would require an ℓ_4 product that would contribute, for example, a term $\ell_1\ell_4$ to the identity. Note that acting on four gauge parameters $\ell_4 \in X_2$, which is correctly identified as the space of constants. That constant in X_2 would be mapped by ℓ_1 to the same constant in X_1, allowing the possibility of cancellation of the constant $\ell_3\ell_2 - \ell_2\ell_3 \neq 0$. The space X_2 would then play an important role. This, however, does not happen for the C-bracket.

A more conventional proof of the identity $\ell_3\ell_2 - \ell_2\ell_3 = 0$ is just by direct computation: Indeed, starting from (5.12) we can show that

$$\begin{aligned}
\ell_3\ell_2 - \ell_2\ell_3 \;&=\; 6\,\ell_3([\xi_{[1},\xi_2],\xi_3,\xi_{4]}) \;-\; 4\,\ell_2(\ell_3(\xi_{[1},\xi_2,\xi_3),\xi_{4]}) \\
&=\; -\,6\,T([\xi_{[1},\xi_2],\xi_3,\xi_{4]}) \;+\; 4\,\ell_2(T(\xi_{[1},\xi_2,\xi_3),\xi_{4]}) \\
&=\; -\,\langle\,[\,[\xi_{[1},\xi_2],\xi_3],\,\xi_{4]}\rangle - \langle\,[\xi_{[3},\xi_4]\;,\;[\xi_1\,\xi_{2]}]\rangle - \langle\,[\xi_{[4}\,,[\xi_{[1},\xi_2]\,],\,\xi_{3]}\rangle \\
&\quad\; -\,2\langle\,\xi_{[4}\,,\,\partial T(\xi_1,\xi_2,\xi_{3]})\,\rangle \\
&=\; -\,2\langle\;[\,[\xi_{[1},\xi_2],\xi_3],\,\xi_{4]}\rangle - \langle\;[\xi_{[1},\xi_2]\,,\,[\xi_3\,\xi_{4]}]\rangle \;-2\langle\,\xi_{[4}\,,\,J(\xi_1,\xi_2,\xi_{3]})\,\rangle \\
&=\; -\,\tfrac{2}{3}\langle\;J(\xi_{[1},\xi_2,\xi_3),\,\xi_{4]}\rangle - \langle\;[\xi_{[1},\xi_2]\,,\,[\xi_3\,\xi_{4]}]\rangle \;+2\langle\,J(\xi_{[1},\xi_2,\xi_3)\,,\,\xi_{4]}\,\rangle \\
&=\; -\,\tfrac{8}{3}\langle\;J(\xi_{[1},\xi_2,\xi_3),\,\xi_{4]}\rangle - \langle\;[\xi_{[1},\xi_2]\,,\,[\xi_3\,\xi_{4]}]\rangle \\
&=\; -\,\tfrac{2}{3}\mathbf{J} - \tfrac{1}{3}\mathbf{K} \;=\; -\tfrac{1}{3}(2\mathbf{J}+\mathbf{K})\,.
\end{aligned} \tag{5.15}$$

Here, following Roytenberg-Weinstein, we have defined the scalars

$$\begin{aligned}
\mathbf{J}(\xi_1,\xi_2,\xi_3,\xi_4) \;&\equiv\; 4\langle J(\xi_{[1},\xi_2,\xi_3),\xi_{4]}\rangle \;=\; -2\xi_{[1}\langle[\xi_2,\xi_3],\xi_{4]}\rangle\,. \\
\mathbf{K}(\xi_1,\xi_2,\xi_3,\xi_4) \;&\equiv\; 3\langle[\xi_{[1},\xi_2],[\xi_3,\xi_{4]}]\rangle\,.
\end{aligned} \tag{5.16}$$

Writing the antisymmetrizations out one has

$$\begin{aligned}
\mathbf{J} \;&\equiv \langle J(\xi_1,\xi_2,\xi_3)\,,\xi_4\rangle - \langle J(\xi_1,\xi_2,\xi_4)\,,\xi_3\rangle + \langle J(\xi_1,\xi_3,\xi_4)\,,\xi_2\rangle - \langle J(\xi_2,\xi_3,\xi_4)\,,\xi_1\rangle\,, \\
\mathbf{K} \;&\equiv \langle[\xi_1,\xi_2],[\xi_3,\xi_4]\rangle - \langle[\xi_1,\xi_3],[\xi_4,\xi_4]\rangle + \langle[\xi_1,\xi_4],[\xi_2,\xi_3]\rangle\,.
\end{aligned} \tag{5.17}$$

The requisite identity is satisfied because

$$\mathbf{K} + 2\mathbf{J} = 0\,, \tag{5.18}$$

as we show now. Using the covariance of the inner product and bracket, leaving the total antisymmetrization in the four arguments implicit from now on, we compute

$$\begin{aligned}
\xi_1\langle[\xi_2,\xi_3],\xi_4\rangle \;=\; \mathcal{L}_{\xi_1}\langle[\xi_2,\xi_3],\xi_4\rangle \;&=\; 2\,\langle[\mathcal{L}_{\xi_1}\xi_2,\xi_3],\xi_4\rangle + \langle[\xi_2,\xi_3],\mathcal{L}_{\xi_1}\xi_4\rangle \\
&=\; 2\,\langle[[\xi_1,\xi_2],\xi_3],\xi_4\rangle + \langle[\xi_2,\xi_3],[\xi_1,\xi_4]\rangle \\
&=\; \tfrac{2}{3}\langle J(\xi_1,\xi_2,\xi_3),\xi_4\rangle + \langle[\xi_1,\xi_2],[\xi_3,\xi_4]\rangle \\
&=\; -\tfrac{1}{3}\xi_1\langle[\xi_2,\xi_3],\xi_4\rangle + \langle[\xi_1,\xi_2],[\xi_3,\xi_4]\rangle\;,
\end{aligned} \tag{5.19}$$

where we used $\mathcal{L}_X Y = [X,Y] + \frac{1}{2}\partial\langle X,Y\rangle$, noting that under the total antisymmetrization the symmetric terms drop out. Thus, bringing the first term on the r.h.s. to the l.h.s.,

$$\tfrac{4}{3}\,\xi_1\langle[\xi_2,\xi_3],\xi_4\rangle \;=\; \langle[\xi_1,\xi_2],[\xi_3,\xi_4]\rangle\;, \tag{5.20}$$

and thus

$$2\,\mathbf{J}(\xi_1,\xi_2,\xi_3,\xi_4) \;=\; -4\,\xi_1\langle[\xi_2,\xi_3],\xi_4\rangle \;=\; -3\,\langle[\xi_1,\xi_2],[\xi_3,\xi_4]\rangle \;=\; -\mathbf{K}(\xi_1,\xi_2,\xi_3,\xi_4)\;, \tag{5.21}$$

proving (5.18). Thus, we have proved that all L_∞ identities are satisfied.

5.2 Off-shell DFT as extended L_3 algebra

Here we extend the L_3 algebra describing the C-bracket algebra to include the fields and gauge transformations of DFT, but still without taking dynamics into account. In other words, we consider the off-shell gauge structure of the DFT fields and build the algebra

$$L_\infty^{\text{gauge+fields}} \tag{5.22}$$

described in the introduction. We discuss two alternative formulations. In the first we include only the generalized metric $\mathcal{H}_{MN}$ and its gauge transformations. In the second we include the non-symmetric metric $\mathcal{E}_{ij}$ and its gauge transformations. Both formulations are background independent and non-perturbative: there is no need to consider expansions around some specific backgrounds.

Gauge structure in terms of $\mathcal{H}_{MN}$

We start by extending the total graded vector space as follows:

$$\begin{array}{ccccccc}
0 \longrightarrow X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} \\
c & & \chi & & \xi^M & & \mathcal{H}_{MN}
\end{array} \tag{5.23}$$

Since there is no vector space of degree minus two, we have

$$\ell_n(\mathcal{H}^n) \;=\; 0\,, \quad \text{for} \;\; n\geq 1\,. \tag{5.24}$$

This also implies that there are no field equations and no dynamics. The gauge transformation of $\mathcal{H}_{MN}$ is given by the generalized Lie derivative

$$\delta_\xi\mathcal{H}_{MN} \;\equiv\; \mathcal{L}_\xi\mathcal{H}_{MN} \;=\; \xi^K\partial_K\mathcal{H}_{MN} + K_M{}^K\mathcal{H}_{KN} + K_N{}^K\mathcal{H}_{MK}\;, \tag{5.25}$$

where we defined $K_{MN} \equiv \partial_M \xi_N - \partial_N \xi_M$. The generalized metric satisfies $\mathcal{H}\eta\mathcal{H} = \eta$, and this constraint is preserved by the generalized Lie derivative. In this background independent formulation the gauge transformations are homogenous in the fields. We now compare the above with the general form (3.42) of the gauge transformations. This comparison shows that we require one new non-trivial product:

$$\ell_2(\xi, \mathcal{H}) \equiv \mathcal{L}_\xi \mathcal{H} . \tag{5.26}$$

The gauge transformation then reads, by definition, $\delta_\xi \mathcal{H} = \ell_2(\xi, \mathcal{H})$. The lack of an inhomogeneous term and of terms nonlinear in the field imply that

$$\ell_1(\xi) = 0 , \quad \ell_{n+1}(\xi, \Psi^n) = 0 , \quad \text{for } n \geq 1 . \tag{5.27}$$

We claim that with the addition of the product (5.26) to the list of products (5.10) we have a consistent L_3 algebra structure on the vector space (5.23).

In order to prove this claim we have to verify the L_∞ relations. The relation $\ell_1^2 = 0$ does not need to be re-checked because the ℓ_1 product is not modified. The relation $\ell_1 \ell_2 = \ell_2 \ell_1$, c.f. (3.17), tells us that $\ell_2(\chi, \mathcal{H})$ and $\ell_2(c, \mathcal{H})$ can be taken to be zero. The first one is a bit nontrivial: take $x_1 = \chi$, $x_2 = \mathcal{H}$ for a scalar function χ:

$$\ell_1 \ell_2(\chi, \mathcal{H}) = \ell_2(\partial\chi, \mathcal{H}) , \tag{5.28}$$

using $\ell_1(\chi) = \partial\chi$. The left-hand side is zero because ℓ_1 is acting on a vector in X_0. But the right-hand side is also zero,

$$\ell_2(\partial\chi, \mathcal{H}) = \mathcal{L}_{\partial\chi} \mathcal{H} = 0 , \tag{5.29}$$

because $\partial^M \chi$ is a trivial parameter and the associated generalized Lie derivative (5.25) vanishes by the strong constraint. The consistency of setting $\ell_2(c, \mathcal{H}) = 0$ now follows quickly.

We now turn to the relation $0 = \ell_1 \ell_3 + \ell_3 \ell_1 + \ell_2 \ell_2$, c.f. (3.19), which requires three inputs. We must check it on inputs that include at least one $\mathcal{H}$. If we introduce no new ℓ_3 products, the $\ell_1 \ell_3$ and $\ell_3 \ell_1$ terms must vanish and the identity reduces to $\ell_2 \ell_2 = 0$. To get something non-vanishing when having an $\mathcal{H}$ we must then have two ξ's. Thus we find that for arguments $\xi_1, \xi_2, \mathcal{H}$, the identity reads

$$\ell_2(\ell_2(\xi_2, \xi_1), \mathcal{H}) = \ell_2(\xi_2, \ell_2(\xi_1, \mathcal{H})) - (1 \leftrightarrow 2) . \tag{5.30}$$

It is easy to see that this is precisely the closure condition of δ_ξ on $\mathcal{H}$ and hence satisfied by the general DFT results.

Finally, the L_∞ relation (3.21) and all higher ones are trivially satisfied, because they involve products like ℓ_3 or higher. Since all higher products vanish under our assumption, we need only concern ourselves with the appearance of ℓ_3. The ℓ_3 product is only non-zero evaluated for three gauge parameters and takes values in the space X_1 of scalar functions. But there is no non-zero product for $\mathcal{H}$ and an argument in X_1. This proves that the L_∞ or L_3 relations remain valid after the extension in (5.23) and the addition of the new product (5.26).

Gauge structure in terms of $\mathcal{E}_{ij}$

Let us now turn to a similar but somewhat more intriguing extension of the L_3 algebra. We still work with non-perturbative off-shell fields, in this case the 'non-symmetric metric' $\mathcal{E} = g + b$, so that the graded vector space is still given by (5.23), but with the elements of X_{-1} now being fields $\mathcal{E}$. The products (5.10) encoding the pure C-bracket algebra are unchanged, but we decompose the gauge parameter as $\xi^M = (\tilde{\xi}_i, \xi^i)$, with $i = 1, \ldots, D$. Despite being non-perturbative, the gauge transformations of $\mathcal{E}$ have inhomogeneous and higher order terms. Specifically, the gauge transformation can be written as

$$\delta_\xi \mathcal{E} \;=\; \ell_1(\xi) + \ell_2(\xi, \mathcal{E}) - \tfrac{1}{2}\ell_3(\xi, \mathcal{E}, \mathcal{E}) \;, \tag{5.31}$$

where according to eqn. (2.32) in [27]

$$\begin{aligned} [\ell_1(\xi)]_{ij} \;&=\; \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i \;, \\ [\ell_2(\xi, \mathcal{E})]_{ij} \;&=\; L_\xi \mathcal{E}_{ij} + \tilde{L}_{\tilde{\xi}} \mathcal{E}_{ij} \;, \\ [\ell_3(\xi, \mathcal{E}_1, \mathcal{E}_2)]_{ij} \;&=\; \mathcal{E}_{1ik}(\tilde{\partial}^k \xi^l - \tilde{\partial}^l \xi^k)\mathcal{E}_{2lj} + (1 \leftrightarrow 2) \;, \end{aligned} \tag{5.32}$$

using the notation of [27] for Lie derivatives L_ξ and dual Lie derivatives $\tilde{L}_{\tilde{\xi}}$, defined by

$$\begin{aligned} L_\xi \mathcal{E}_{ij} \;&=\; \xi^k \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \, \mathcal{E}_{kj} + \partial_j \xi^k \, \mathcal{E}_{ik} \;, \\ \tilde{L}_{\tilde{\xi}} \mathcal{E}_{ij} \;&=\; \tilde{\xi}_k \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \tilde{\xi}_i \, \mathcal{E}_{kj} - \tilde{\partial}^k \tilde{\xi}_j \, \mathcal{E}_{ik} \;. \end{aligned} \tag{5.33}$$

Since the gauge algebra is field independent and we have no dynamics,

$$\ell_{n+2}(\xi_1, \xi_2, \mathcal{E}^n) \;=\; 0 \;, \;\; n \geq 1 \;, \qquad \ell_n(\mathcal{E}, \ldots, \mathcal{E}) \;=\; 0 \quad \forall\, n \geq 1 \;. \tag{5.34}$$

We claim that with the addition of the products (5.32) to the list of products (5.10) we have a consistent L_3 algebra structure on the vector space (5.23) (with $\mathcal{E}$ replacing $\mathcal{H}$).

Let us now verify the L_∞ relations. First, the relations involving only gauge parameters ξ and functions χ still hold as in the first subsection, since we merely changed the notation by splitting ξ^M into $\tilde{\xi}_i$ and ξ^i. For instance, $\ell_1^2 = 0$ on X_1 holds for $\ell_1(\chi)_i = \partial_i \chi$:

$$\ell_1(\ell_1(\chi))_{ij} \;=\; \partial_i \ell_1(\chi)_j - \partial_j \ell_1(\chi)_i \;=\; 0 \;. \tag{5.35}$$

Second, the relation $\ell_1 \ell_2 = \ell_2 \ell_1$ of the same form as in (5.28), with $\mathcal{H}$ replaced by $\mathcal{E}$, follows again because $\tilde{\xi}_i = \partial_i \chi$, $\xi^i = \tilde{\partial}^i \chi$ are trivial parameters of (5.31), which can be easily verified by use of (5.33) and the strong constraint. So again we have that $\ell_2(\chi, \mathcal{E})$ and $\ell_2(c, \mathcal{E})$ can be set to zero.

Under our assumption that there are no additional products, the L_∞ identities of the form $\ell_i \ell_j + \ldots = 0$ with $i + j \geq 7$ are trivially satisfied, since we have no products ℓ_n with $n \geq 4$. We must consider the identities for $i + j = 4, 5, 6$, having 3,4, and 5 inputs, respectively. In each case we must have at least one field $\mathcal{E}$ among the inputs, otherwise the identity was checked before (note that the products in (5.10) can never generate an object in X_{-1}).

We sketch the procedure now. Take $i + j = 4$, which corresponds to the identity $\ell_1 \ell_3 + \ell_3 \ell_1 + \ell_2 \ell_2 = 0$, of intrinsic degree zero. This requires three inputs. Three $\mathcal{E}$ works trivially because of degree. The cases $(\star, \mathcal{E}\mathcal{E})$ for $\star = \xi, \chi, c$ also work trivially. From the six cases $(\star, \star, \mathcal{E})$ the only nontrivial one happens to occur for $(\xi\xi\mathcal{E})$. For $i + j = 5$ the identity is $\ell_3\ell_2 = \ell_2\ell_3$ and it requires four inputs. An enumeration shows that the only nontrivial case occurs for inputs $(\xi\xi\mathcal{E}\mathcal{E})$. Finally, for $i + j = 6$ the identity is $\ell_3\ell_3 = 0$ and requires 5 inputs, the only nontrivial one being $(\xi\xi\mathcal{E}\mathcal{E}\mathcal{E})$. It follows that the remaining non-trivial L_∞ relations are

$$\begin{aligned} \ell_2(\ell_2(\xi_2,\xi_1),\mathcal{E}) &= \ell_2(\xi_2,\ell_2(\xi_1,\mathcal{E})) - \ell_3(\xi_2,\ell_1(\xi_1),\mathcal{E}) - (1 \leftrightarrow 2) \;, \\ \ell_3(\ell_2(\xi_2,\xi_1),\mathcal{E},\mathcal{E}) &= \ell_2(\xi_2,\ell_3(\xi_1,\mathcal{E},\mathcal{E})) + 2\,\ell_3(\xi_2,\ell_2(\xi_1,\mathcal{E}),\mathcal{E}) - (1 \leftrightarrow 2) \;, \\ 0 &= \ell_3(\xi_2,\ell_3(\xi_1,\mathcal{E},\mathcal{E}),\mathcal{E}) - (1 \leftrightarrow 2) \;, \end{aligned} \tag{5.36}$$

with diagonal arguments for fields $\mathcal{E}$, which by the general discussion is sufficient for the validity of the L_∞ relations.[6] These relations, together with those in (5.10), are sufficient for closure of the gauge transformations,

$$\big[\delta_{\xi_1},\delta_{\xi_2}\big]\mathcal{E} \;=\; \ell_1(\xi_{12}) + \ell_2(\xi_{12},\mathcal{E}) - \tfrac{1}{2}\ell_3(\xi_{12},\mathcal{E},\mathcal{E}) \;, \tag{5.37}$$

where $\xi_{12} \equiv [\xi_2,\xi_1]_c \equiv \ell_2(\xi_2,\xi_1)$, c.f. (3.55). As explained at the end of section 3.3, the closure of the gauge algebra implies that the identities (5.36) hold. We have also verified (5.36) by direct computations.

5.3 Perturbative DFT as L_∞ algebra

We finally take dynamics into account, employing a perturbative formulation obtained by expanding DFT around a background. The fundamental fields are the dilaton ϕ and the generalized metric $\mathcal{H}$, which is expanded around a constant background as follows:

$$\mathcal{H}_{MN} \;=\; \bar{\mathcal{H}}_{MN} + h_{\underline{M}\bar{N}} + h_{\underline{N}\bar{M}} - \tfrac{1}{2}h^{\underline{K}}{}_{\bar{M}}\,h_{\underline{K}\bar{N}} + \tfrac{1}{2}h_{\underline{M}}{}^{\bar{K}}\,h_{\underline{N}\bar{K}} \;+\; \mathcal{O}(h^3) \;. \tag{5.38}$$

This expansion is compatible with the constraint $\mathcal{H}_M{}^K\mathcal{H}_K{}^N = \delta_M{}^N$. We use projected $O(D,D)$ indices defined for a vector by $V_{\underline{M}} = P_M{}^N V_N$, $V_{\bar{M}} = \bar{P}_M{}^N V_N$, with the projectors

$$P_M{}^N \;=\; \tfrac{1}{2}\big(\delta_M{}^N - \bar{\mathcal{H}}_M{}^N\big) \;, \qquad \bar{P}_M{}^N \;=\; \tfrac{1}{2}\big(\delta_M{}^N + \bar{\mathcal{H}}_M{}^N\big) \;. \tag{5.39}$$

Indeed, $P^2 = P$, $\bar{P}^2 = \bar{P}$, and $P\bar{P} = 0$ as a consequence of the constraint on the background generalized metric, $\bar{\mathcal{H}}_M{}^K\bar{\mathcal{H}}_K{}^N = \delta_M{}^N$.

Let us now discuss the L_∞ algebra encoding the symmetries and dynamics of the above perturbative field variables. To this end we extend the above sequence once more to

$$\begin{array}{ccccccccc} 0 \longrightarrow X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} & \longrightarrow & X_{-2} \\ c & & \chi & & \xi^M & & (h_{\underline{M}\bar{N}},\phi) & & (\mathcal{R}_{\underline{M}\bar{N}},\mathcal{R}) \end{array} \tag{5.40}$$

where X_{-1} encodes the fields and X_{-2} the field equations. More precisely, since we have the fundamental fields $h_{\underline{M}\bar{N}}$ and ϕ we have a further decomposition into direct sums:

$$X_{-1} \;=\; X_{-1,t} \,\oplus\, X_{-1,s} \;, \qquad X_{-2} \;=\; X_{-2,t} \,\oplus\, X_{-2,s} \;, \tag{5.41}$$

[6] These relations were applied recently in [36].

with subscript 's' denoting the dilaton (scalar) component and subscript 't' the tensor component. Collectively, we denote fields by $\Psi = (h_{\underline{M}\bar{N}}, \phi)$ and field equations by $E = (\mathcal{R}_{\underline{M}\bar{N}}, \mathcal{R})$, so that

$$[\Psi_t]_{\underline{M}\bar{N}} \; = \; h_{\underline{M}\bar{N}} \, , \quad \Psi_s \; = \; \phi \, , \qquad [E_t]_{\underline{M}\bar{N}} \; = \; \mathcal{R}_{\underline{M}\bar{N}} \, , \quad E_s \; = \; \mathcal{R} \, . \tag{5.42}$$

We will leave out the subindex of the grading if the tensor character is evident from the index structure.

Before starting the construction we must digress. The gauge parameters are ξ^M as above, and the gravitational field is $h_{\underline{M}\bar{N}}$, but the conventional perturbative DFT expressions are defined in terms of gauge parameters λ_i and $\bar{\lambda}_i$ and closed string field theory (CSFT) variables e_{ij} and $d = -\frac{1}{2}\phi$ [25]. By means of background frame fields it is straightforward to translate the indices i, j of the original perturbative DFT expressions to the projected $2D$ valued $O(D, D)$ indices. Indeed, we assume a constant background frame field $E_A{}^M$ (an anchor map), and use [37, 38]

$$E_a{}^i \, E_{\bar{b}}{}^j \, e_{ij} \; = \; \tfrac{1}{2} \, E_a{}^M \, E_{\bar{b}}{}^N \, h_{\underline{M}\bar{N}} \, , \tag{5.43}$$

and

$$\begin{aligned} \lambda_i \; &\equiv \; -E_i{}^a E_a{}^M \xi_M \, , \qquad & \bar{\lambda}_i \; &\equiv \; E_i{}^{\bar{a}} E_{\bar{a}}{}^M \xi_M \, , \\ D_i \; &\equiv \; E_i{}^a E_a{}^M \partial_M \, , \qquad & \bar{D}_i \; &\equiv \; E_i{}^{\bar{a}} E_{\bar{a}}{}^M \partial_M \, , \end{aligned} \tag{5.44}$$

where the sign in the first line is introduced for convenience, in order to comply with the CSFT conventions for $\lambda, \bar{\lambda}$. Note that we could also replace the $O(D, D)$ indices appearing above immediately as projected indices, so that e.g. $\bar{\lambda}_i = E_i{}^{\bar{a}} E_{\bar{a}}{}^M \xi_{\bar{M}}$. Moreover, one must recall that the flattened $O(D, D)$ metric $\mathcal{G}_{AB} = E_A{}^M E_{BM}$ is related to the background metric G via

$$G_{ij} \; = \; -\tfrac{1}{2} \, E_i{}^a \, E_j{}^b \, \mathcal{G}_{ab} \; = \; \tfrac{1}{2} \, E_i{}^{\bar{a}} \, E_j{}^{\bar{b}} \, \mathcal{G}_{\bar{a}\bar{b}} \, , \tag{5.45}$$

where $E_i{}^a$ is the inverse of $E_a{}^i$ and similarly for the other fields.

The L_∞ products governing the C-bracket algebra are given by (5.10) and still apply in this construction. If desired, they could be rewritten in terms of projected gauge parameters using (5.44). We must now determine what are the extra products that make up the complete dynamical L_∞^{full} of the interacting DFT.

We begin by inspecting the perturbative gauge transformations for the CSFT variable e_{ij}, given by [27]

$$\begin{aligned} \delta e_{ij} \; = \; & D_i \bar{\lambda}_j + \bar{D}_j \lambda_i + \tfrac{1}{2}(\lambda \cdot D + \bar{\lambda} \cdot \bar{D}) e_{ij} \\ & + \tfrac{1}{2}(D_i \lambda^k - D^k \lambda_i) e_{kj} + \tfrac{1}{2}(\bar{D}_j \bar{\lambda}^k - \bar{D}^k \bar{\lambda}_j) e_{ik} + \tfrac{1}{4} e_{ik} (D^l \bar{\lambda}^k - \bar{D}^k \lambda^l) e_{lj} \, . \end{aligned} \tag{5.46}$$

Converting to $O(D, D)$ indices by means of (5.43), this implies

$$\begin{aligned} \delta_\xi h_{\underline{M}\bar{N}} \; = \; & 2(\partial_{\underline{M}} \xi_{\bar{N}} - \partial_{\bar{N}} \xi_{\underline{M}}) + \xi^P \partial_P h_{\underline{M}\bar{N}} + K_{\underline{M}}{}^{\underline{K}} h_{\underline{K}\bar{N}} + K_{\bar{N}}{}^{\bar{K}} h_{\underline{M}\bar{K}} \\ & + \tfrac{1}{8} \, h_{\underline{M}\bar{K}} \, K^{\underline{L}\bar{K}} \, h_{\underline{L}\bar{N}} \, . \end{aligned} \tag{5.47}$$

This form of the gauge transformation can be taken to be exact. More precisely, there is a choice for the higher order terms in the expansion (5.38) of $\mathcal{H}$ so that $\delta_\xi \mathcal{H} = \mathcal{L}_\xi \mathcal{H}$ yields (5.47) exactly. Finally, the gauge transformation of the dilaton reads

$$\delta_\xi \phi \; = \; \xi^N \partial_N \phi + \partial_N \xi^N \, . \tag{5.48}$$

The products read off from this and (5.47) are

$$
\begin{aligned}
[\ell_1(\xi)]_{\underline{M}\bar{N}} &= 2(\partial_{\underline{M}}\xi_{\bar{N}} - \partial_{\bar{N}}\xi_{\underline{M}}) \,, \\
[\ell_2(\xi,h)]_{\underline{M}\bar{N}} &= \xi^P\partial_P h_{\underline{M}\bar{N}} + K_{\underline{M}}{}^{\underline{K}} h_{\underline{K}\bar{N}} + K_{\bar{N}}{}^{\bar{K}} h_{\underline{M}\bar{K}} \,, \\
[\ell_3(\xi,h_1,h_2)]_{\underline{M}\bar{N}} &= -\tfrac{1}{2}\, h_{1\underline{M}\bar{K}}\, K^{\underline{L}\bar{K}}\, h_{2\underline{L}\bar{N}} + (1\leftrightarrow 2) \,, \\
\ell_1(\xi)_s &= \partial_N\xi^N \,, \\
\ell_2(\xi,\phi)_s &= \xi^N\partial_N\phi \,.
\end{aligned} \tag{5.49}
$$

All other products involving only gauge parameters ξ and fields are zero.

Let us now turn to the field equations. The full field equations can be written in terms of the generalized metric and the dilaton,

$$
\mathcal{R}_{MN}(\mathcal{H},\phi) \;=\; 0 \,, \qquad \mathcal{R}(\mathcal{H},\phi) \;=\; 0 \,. \tag{5.50}
$$

Expanding around a constant background and taking ‘off-diagonal’ projections, one obtains

$$
0 \;=\; \mathcal{R}^{(1)}_{\underline{M}\bar{N}}(h,\phi) + \mathcal{R}^{(2)}_{\underline{M}\bar{N}}(h,\phi) + \cdots \,, \qquad 0 \;=\; \mathcal{R}^{(1)}(h,\phi) + \mathcal{R}^{(2)}(h,\phi) + \cdots \,, \tag{5.51}
$$

where the superscript denotes the number of fields. The linearized equations can be read off from eq. (6.7) in [38] (defining $\square = \partial^{\underline{M}}\partial_{\underline{M}}$)

$$
\begin{aligned}
\mathcal{R}^{(1)}_{\underline{M}\bar{N}} &= \square h_{\underline{M}\bar{N}} - \partial^{\underline{K}}\partial_{\underline{M}} h_{\underline{K}\bar{N}} + \partial^{\bar{K}}\partial_{\bar{N}} h_{\underline{M}\bar{K}} - 2\partial_{\underline{M}}\partial_{\bar{N}}\phi \,, \\
\mathcal{R}^{(1)} &= \partial^{\underline{M}}\partial^{\bar{N}} h_{\underline{M}\bar{N}} - 2\square\phi \,.
\end{aligned} \tag{5.52}
$$

We can now read off the ℓ products taking values in the space of field equations X_{-2}. Specifically, to lowest order we find the two projections

$$
\begin{aligned}
[\ell_1(\Psi)_t]_{\underline{M}\bar{N}} &= \mathcal{R}^{(1)}_{\underline{M}\bar{N}} \,, \\
\ell_1(\Psi)_s &= \mathcal{R}^{(1)} \,.
\end{aligned} \tag{5.53}
$$

The higher products taking values in X_{-2} can be determined algorithmically by expanding (5.50) to the desired order and using the polarization identities of sec. 3.3 to determine the product for arbitrary different arguments in the space of fields X_{-1}. For instance, the correction to second order in fields for $\mathcal{R}$ has been given explicitly in [27] in terms of the original CSFT variables. Writing

$$
\mathcal{R}^{(2)} \;=\; -\tfrac{1}{2}\ell_2(\Psi,\Psi)_s \,, \tag{5.54}
$$

we read off from eq. (4.28) in [27] for the (diagonal) product

$$
\begin{aligned}
\ell_2(\Psi,\Psi)_s \;=\;& 2\,D^i\phi\, D_i\phi - 4e^{ij}D_i\bar{D}_j\phi - 2D^ie_{ij}\,\bar{D}^j\phi - 2\bar{D}^je_{ij}\,D^i\phi + \tfrac{1}{2}D^pe^{ij}\,D_pe_{ij} \\
&+ e^{ij}\big(D_iD^ke_{kj} + \bar{D}_j\bar{D}^ke_{ik}\big) + \tfrac{1}{2}\big(D_le^{li}\,D^ke_{ki} + \bar{D}_le^{il}\,\bar{D}^ke_{ik}\big) \,,
\end{aligned} \tag{5.55}
$$

or, translating into $O(D,D)$ indices by means of the anchor map,

$$
\begin{aligned}
\ell_2(\Psi_1,\Psi_2)_s \;=\;& -2\,\partial^{\underline{M}}\phi_1\,\partial_{\underline{M}}\phi_2 + 4\,h_1^{\underline{M}\bar{N}}\partial_{\underline{M}}\partial_{\bar{N}}\phi_2 + 2\,\partial^{\underline{M}}h_{1\underline{M}\bar{N}}\,\partial^{\bar{N}}\phi_2 \\
&+ 2\,\partial^{\bar{N}}h_{1\underline{M}\bar{N}}\,\partial^{\underline{M}}\phi_2 + \tfrac{1}{2}\,\partial^{\underline{K}}h_1^{\underline{M}\bar{N}}\,\partial_{\underline{K}}h_{2\underline{M}\bar{N}} \\
&+ h_1^{\underline{M}\bar{N}}\big(\partial_{\underline{M}}\partial^{\underline{K}}h_{2\underline{K}\bar{N}} - \partial_{\bar{N}}\partial^{\bar{K}}h_{2\underline{M}\bar{K}}\big) \\
&+ \tfrac{1}{2}\big(\partial_{\underline{L}}h_1^{\underline{L}\bar{N}}\,\partial^{\underline{K}}h_{2\underline{K}\bar{N}} - \partial_{\bar{L}}h_1^{\underline{M}\bar{L}}\,\partial^{\bar{K}}h_{2\underline{M}\bar{K}}\big) + (1\leftrightarrow 2) \,,
\end{aligned} \tag{5.56}
$$

where we restored the two arbitrary field arguments. In general, the field equations contain arbitrary powers of the fields and hence are non-polynomial. Thus, all higher products for fields are expected to be non-vanishing,

$$\ell_n(\Psi, \ldots, \Psi) \;\neq\; 0 \quad \text{for } n \geq 1\,. \tag{5.57}$$

Finally, we employ the gauge covariance of the field equations in order to determine the products involving arguments in the space of field equations X_{-2}. The full field equations (5.50) transform covariantly according to generalized Lie derivatives:

$$\begin{aligned} \delta_\xi \mathcal{R}_{MN} \;&=\; \mathcal{L}_\xi \mathcal{R}_{MN} \;=\; \xi^K \partial_K \mathcal{R}_{MN} + K_M{}^K \mathcal{R}_{KN} + K_N{}^K \mathcal{R}_{MK}\;, \\ \delta_\xi \mathcal{R} \;&=\; \mathcal{L}_\xi \mathcal{R} \;\;=\;\; \xi^K \partial_K \mathcal{R}\;. \end{aligned} \tag{5.58}$$

Note that the right-hand sides are linear in the field equations and do not contain bare fields. On the other hand, as we showed in (3.53) one has a general formula for the gauge variation of the field equations

$$\delta_\xi \mathcal{F} \;=\; \ell_2(\xi, \mathcal{F}) \;+\; \ell_3(\xi, \mathcal{F}(\Psi), \Psi) + \ldots \tag{5.59}$$

Comparing the two equations above we see that only the first term on the right-hand side of the second equation is present. We learn therefore that

$$\ell_2(\xi, E) \;=\; \mathcal{L}_\xi E \;\in\; X_{-2}\;, \tag{5.60}$$

and higher products vanish

$$\ell_{n+2}(\xi, E, \Psi^n) \;=\; 0\;, \quad n \geq 1\;. \tag{5.61}$$

As a consistency check, we can show that (5.60) is required by the L_∞ relation $\ell_1\ell_2 = \ell_2\ell_1$. We focus on the dilaton and thus expand the second equation in (5.58) to first order in fields,

$$\xi^K \partial_K \mathcal{R}^{(1)} \;=\; \delta^{(1)}\mathcal{R}^{(1)} + \delta^{(0)}\mathcal{R}^{(2)}\;. \tag{5.62}$$

Here $\delta^{(n)}$ refers to the terms in the gauge variation with n powers of the fields, i.e., the terms encoded in the product $\ell_{n+1}(\xi, \Psi^n)$. For the dilaton component $\ell_1\ell_2 = \ell_2\ell_1$, gives

$$\ell_2(\xi, \ell_1(\Psi))_s \;=\; \ell_1(\ell_2(\xi, \Psi))_s - \ell_2(\ell_1(\xi), \Psi)_s\;. \tag{5.63}$$

To this end, we rewrite the two terms on the right-hand side of (5.62) as follows. First, using $\delta^{(1)}\Psi = \ell_2(\xi, \Psi)$, we compute with (5.53)

$$\delta^{(1)}\mathcal{R}^{(1)} \;=\; \delta^{(1)}\big(\ell_1(\Psi)_s\big) \;=\; \ell_1\big(\delta^{(1)}\Psi\big)_s \;=\; \ell_1(\ell_2(\xi, \Psi))_s\;. \tag{5.64}$$

Second, using $\delta^{(0)}\Psi = \ell_1(\xi)$, we compute with (5.54)

$$\delta^{(0)}\mathcal{R}^{(2)} \;=\; -\tfrac{1}{2}\delta^{(0)}\big(\ell_2(\Psi, \Psi)\big)_s \;=\; -\ell_2(\delta^{(0)}\Psi, \Psi)_s \;=\; -\ell_2(\ell_1(\xi), \Psi)_s\;, \tag{5.65}$$

where we used the symmetry of ℓ_2 for two arguments in the space of fields. We have thus shown that the right-hand side of (5.63) equals the right-hand side of (5.62). Therefore, (5.63) is satisfied if the left-hand sides are also equal, i.e.,

$$\ell_2(\xi, \ell_1(\Psi))_s \;=\; \xi^N \partial_N \mathcal{R}^{(1)}\;. \tag{5.66}$$

This relation is satisfied provided we define the product for a general $E \in X_{-2}$ to be

$$\ell_2(\xi, E)_s \;=\; \xi^N \partial_N E_s \;. \tag{5.67}$$

This is what we wanted to show. The above derivation goes through for the tensor component in the same way.

So far we have determined the non-trivial ℓ_2 product between gauge parameters and field equations, and it is easy to see that there are no higher products involving one gauge parameter and several field equations. For instance, an ℓ_3 product like

$$\ell_3(\xi, E, E) \;\in\; X_{-3} \;, \tag{5.68}$$

has to vanish because there is no space X_{-3}, and similarly for higher products. Moreover, there is no product $\ell_3(\xi_1, \xi_2, E)$ of two gauge parameters and a field equation, because this would imply that closure of the gauge algebra holds only on-shell, while in DFT we have off-shell closure. Similarly, there is no need for higher products with two gauge parameters and an arbitrary number of field equations, so those are also zero.

We now claim that the products we have identified so far are all the products that are non-zero. In summary, the non-vanishing products for the L_∞ algebra describing the full (perturbative) DFT are the following:

i) the products governing the pure gauge structure, i.e., gauge parameters ξ, 'trivial' functions χ and constants c, which are non-vanishing for

$$\ell_1(\chi) \;, \quad \ell_1(c) \;, \quad \ell_2(\xi_1, \xi_2) \;, \quad \ell_2(\xi, \chi) \;, \quad \ell_3(\xi_1, \xi_2, \xi_3) \;, \tag{5.69}$$

and given explicitly in (5.10);

ii) the products involving gauge parameters and fields describing the full gauge transformations of fields, which are non-vanishing for

$$\ell_1(\xi) \;, \quad \ell_2(\xi, \Psi) \;, \quad \ell_3(\xi, \Psi_1, \Psi_2) \;, \tag{5.70}$$

and given explicitly in (5.49);

iii) products ℓ_n for arbitrary n involving only fields Ψ,

$$\ell_n(\Psi_1, \ldots, \Psi_n) \qquad \text{for} \qquad \Psi_1, \ldots, \Psi_n \;\in\; X_{-1} \;, \tag{5.71}$$

such as given to lowest order in (5.53), (5.56); we did not attempt to write these products in a closed form, but we explained how they can be determined systematically from the field equations to any desired order n;

iv) the product between gauge parameter and field equation,

$$\ell_2(\xi, E) \;=\; \mathcal{L}_\xi E \;. \tag{5.72}$$

Proof. We now explain why all L_∞ identities are satisfied. For this we will consider the possible lists of inputs for the identities. The inputs, of course, can be various numbers of c's, χ's, ξ's, Ψ's, and E's. Note that given a list of n inputs there is a single L_∞ identity that must be cheched, the identity with sums of products of the form $\ell_i \ell_j$, with $i + j = n + 1$. For any term $\ell_i \ell_j$ we will call ℓ_j the *first* product and ℓ_i the *second* product, as they act first and second, respectively, on any list of inputs.

As discussed in section 3.3 we do not need to consider any identity on only pure Ψ inputs as they hold if all products involving one E and any number of Ψ's vanish, as they do here. Moreover it is explained there as well that we need not consider identities acting on input sets (ξ, Ψ^n) with one gauge parameter and multiple Ψ's as those identities hold when the field equations transform covariantly, as we have checked before. Finally, we need not consider identities acting on input sets $(\xi_1 \xi_2 \Psi^n)$ with two gauge parameters and multiple Ψ's as those identities hold when the gauge algebra takes a standard form.

Claim 1: Any list of inputs that includes one or more E's leads to correctly satisfied identities. Suppose there is one E. If the first product involves the E then it must be an ℓ_2 that couples it to a ξ and gives another E. The second product must then couple the new E to another ξ. The inputs in this case are $\xi_1 \xi_2 E$ and are relevant to the $\ell_1 \ell_3 + \ell_2 \ell_2 + \ell_3 \ell_1 = 0$ identity. The identity holds: terms with ℓ_3 vanish and $\ell_2 \ell_2 = 0$ because Lie derivatives form a Lie algebra. If the first product does not involve E then the second product must, and has to be of the form $\ell_2(E, \xi)$. This means that the first product must have taken some inputs and given a gauge parameter. The options for this are $\ell_1(\chi)$ and $\ell_2(\xi_1, \xi_2)$. Thus the possible lists to check are $E\chi$ and $E\xi_1\xi_2$. The second list was dealt with a few lines above. The first list is relevant to the identity $\ell_1 \ell_2 = \ell_2 \ell_1$ and holds because $\ell_2(E, \ell_1(\chi))$ vanishes as it is a generalized Lie derivative along a trivial parameter. Now consider the case when there are two E's in the original list. Since there is no product that includes two E's the first product must involve an E. But the product gives another E, and therefore the second product is faced with two E's and it must vanish. The case of more than two E's works for analogous reasons. $\square$

Claim 2: Any list of inputs that includes one or more c's leads to correctly satisfied identities. If there are two or more c's any sequence of products will give zero because the only product involving c is ℓ_1. If there is one c and no other inputs, this is trivially satisfied. If there is one c and some other inputs, the first product cannot act on the other inputs because then there would be no suitable second product. The only possibility is that the first product is ℓ_1 and acts on c to give a constant χ. That χ can be acted by another ℓ_1, in which case the identity is trivial, or appear in $\ell_2(\chi, \xi)$, which vanishes because χ is constant. $\square$

Claim 3: Any list of inputs that includes one or more χ's leads to correctly satisfied identities. Consider first the case when we have two χ's. Assume the first product is not acting on any of the χ's. Then there is no available second product that acts on a list with at least two χ's. If the first product acts on one of the χ's it could be in the form $\ell_1(\chi)$ or $\ell_2(\chi, \xi)$, because of (5.69). Since the latter gives another χ, it must be the former. The second product is then faced with at least a χ and a ξ. But there can be no more inputs, so that we can use ℓ_2. In summary, the only possibility for the original list is $\chi\chi$. This is certainly a trivially satisfied identity. The same argument holds for more than two χ's.

Consider now the case of a single χ in the list of inputs. If the first product does not act on χ then it must act on all of the other inputs to produce a ξ. Looking again at the list of products (5.69), the only option is $\ell_2(\xi_1, \xi_2)$ showing that the original list must have been $(\chi\xi\xi)$. The associated identity was already checked in the Courant algebroid. If the first product acts on the χ it may act with ℓ_2 or ℓ_1. If it acts with ℓ_2, it must be $\ell_2(\xi, \chi)$ giving another χ-type input. The second product can be ℓ_1 if the list is $(\chi\xi)$ or another ℓ_2 in which case the list is $(\chi\xi\xi)$. Both lists were checked in the Courant algebroid. If the first product acts on the χ with ℓ_1 it turns it into a ξ and for nontrivial products one must then have at most two Ψ's. So the only possibilities are inputs $(\chi\Psi)$ or $(\chi\Psi\Psi)$. In both cases these vanish because the ξ obtained as $\ell_1(\chi)$ is a trivial gauge parameter. □

Because of our earlier observations and the three claims above we need only check identities with inputs that have three or more ξ's and any number of Ψ's. The case when all inputs are ξ need not be considered because this was part of the analysis in the Courant algebroid.

Consider the lists $(\xi\xi\xi\Psi^n)$ with three ξ's and a number $n \geq 1$ of Ψ's. With three ξ's we cannot have both the first and the second product arise from the list $\ell_1(\xi), \ell_2(\xi, \Psi), \ell_3(\xi, \Psi, \Psi)$ that use one ξ. One of them must be a $\ell_3(\xi, \xi, \xi)$. Suppose the first product is $\ell_3(\xi, \xi, \xi)$. Then the second product is acting on $(\chi\Psi^n)$ and vanishes, as χ never couples to a field. Suppose the first product is not $\ell_3(\xi, \xi, \xi)$, then the chosen product uses one ξ and returns a Ψ, leaving for the second product inputs $(\xi\xi\Psi^k)$ with $k \geq 1$. But no product is non-zero for this list.

The lists $(\xi\xi\xi\xi\Psi^n)$ with four ξ's and a number of $n \geq 1$ of Ψ's also leads to no constraints. If the first product is one of $\ell_1(\xi), \ell_2(\xi, \Psi), \ell_3(\xi, \Psi, \Psi)$, then the second product is facing the list $(\xi\xi\xi\Psi^k)$ with $k \geq 1$ gives zero. If the first product is $\ell_3(\xi, \xi, \xi)$ the second product faces the list $(\chi\xi\Psi^n)$ and gives zero. It is clear that more than four ξ's and a number $n \geq 1$ of Ψ's will also give trivially satisfied identities. This concludes our proof that the products listed in (5.69 –5.72) define a consistent L_∞ algebra for DFT. □

5.4 Comments on Einstein gravity

We close this section be briefly commenting on the description of Einstein gravity as L_∞ algebra. Einstein gravity is contained in DFT, so this is a special case of our results above, but it is instructive to see how the L_∞ algebra simplifies for pure gravity. As before, we consider perturbative gravity, in which the Einstein-Hilbert theory is expanded around flat space, writing $g_{mn} = \eta_{mn} + h_{mn}$. The diffeomorphism symmetry acts on the massless spin-2 fluctuation h_{mn} as

$$\delta_\xi h_{mn} \;=\; \partial_m \xi_n + \partial_n \xi_m + L_\xi h_{mn} \,, \tag{5.73}$$

where L_ξ is the conventional Lie derivative, defined like in the first line of (5.33). These transformations close according to the Lie bracket $[\,,]$ of vectors fields, which in turn satisfies the Jacobi identity. Thus, there is no need for a space X_1 of 'trivial' functions, and it is sufficient to consider the graded vector space

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_{-1} & \longrightarrow & X_{-2} \\ \xi^m & & h_{mn} & & R_{mn} \end{array} \tag{5.74}$$

The non-zero products are the following:

i) the product governing the pure diffeomorphism Lie algebra for ξ,

$$\ell_2(\xi_1, \xi_2) \; = \; [\xi_1, \xi_2] \tag{5.75}$$

ii) the products involving gauge parameters and fields describing the gauge transformations,

$$\ell_1(\xi)_{mn} \; = \; \partial_m \xi_n + \partial_n \xi_m \; , \qquad \ell_2(\xi, h) \; = \; L_\xi h \tag{5.76}$$

iii) products ℓ_n for arbitrary n involving only the field h,

$$\ell_n(h, \ldots, h) \qquad \text{for} \qquad h \; \in \; X_{-1} \; , \tag{5.77}$$

as can be determined from the Einstein equations to any desired order;

iv) the product between gauge parameter and field equation,

$$\ell_2(\xi, E) \; = \; L_\xi E \; . \tag{5.78}$$

6 A_∞ algebras and revisiting Chern-Simons

In this final section we briefly contrast the L_∞ constructions of this paper with the A_∞ formulation of Chern-Simons theory. The A_∞ axioms relevant to the construction of a theory include a set of products m_n, with $n = 1, 2, 3, \ldots$. The product m_n, with n inputs, is of degree $n - 2$. The first couple of identities are [2]:

$$\begin{aligned} m_1(m_1(x)) \; &= \; 0 \, , \\ m_1(m_2(x_1, x_2) \; &= \; m_2(m_1(x_1), x_2) + (-1)^{x_1} m_2(x_1, m_1(x_2)) \, . \end{aligned} \tag{6.1}$$

For the example we want to discuss, and for Witten's open string field theory, the product m_3 and all higher ones vanish. In this case, the remaining identity in the algebra is the associativity condition for m_2:

$$m_2(\, m_2(x_1, x_2), x_3) \; = \; m_2(\, x_1, \, m_2(x_2, x_3)\,) \, . \tag{6.2}$$

If we supply an inner product one can also write an action. The inner product must satisfy

$$\begin{aligned} \langle x_1, x_2 \rangle \; &= \; (-1)^{x_1 x_2} \langle x_2, x_1 \rangle \, , \\ \langle m_1(x_1), x_2 \rangle \; &= \; - \, (-1)^{x_1} \, \langle x_1, \, m_1(x_2) \, \rangle \, , \\ \langle x_1, m_2(x_2, x_3) \rangle \; &= \; \langle m_2(x_1, x_2), \, x_3 \rangle \, . \end{aligned} \tag{6.3}$$

In writing a field theory with a field $A \in X_{-1}$ we have an action

$$S \; = \; \tfrac{1}{2} \langle A \, , m_1(A) \rangle + \tfrac{1}{3} \langle A, \, m_2(A, A) \, \rangle \, . \tag{6.4}$$

The field equation takes the form $\mathcal{F} = 0$ with

$$\mathcal{F}(A) \; \equiv \; m_1(A) + m_2(A, A) \, . \tag{6.5}$$

With a gauge parameter $\lambda \in X_0$ the gauge transformations leaving the action invariant take the form

$$\delta_\lambda A \;=\; m_1(\lambda) \,+\, m_2(A,\lambda) - m_2(\lambda, A)\,. \tag{6.6}$$

The gauge algebra takes the form

$$\big[\,\delta_{\lambda_1}\,,\,\delta_{\lambda_2}\big] \;=\; \delta_{m_2(\lambda_1,\lambda_2)-m_2(\lambda_2,\lambda_1)}\,. \tag{6.7}$$

The field equation is gauge covariant: we have

$$\delta_\lambda \mathcal{F} \;=\; m_2\,(\mathcal{F},\lambda) \;-\; m_2(\lambda,\mathcal{F})\,. \tag{6.8}$$

In order to formulate the Chern-Simons theory we consider the graded vector space

$$\begin{array}{ccc} X_0 & X_{-1} & X_{-2} \\ \lambda & A_\mu & E_{\mu\nu} \end{array} \tag{6.9}$$

Here we think of these objects as *matrix valued* fields:

$$\lambda \,\equiv\, \lambda^\alpha\, t_\alpha\,, \qquad A_\mu \,\equiv\, A_\mu^\alpha\, t_\alpha\,, \qquad E_{\mu\nu} \,\equiv\, E^\alpha_{\mu\nu}\, t_\alpha\,, \tag{6.10}$$

where the t_α can be chosen as the adjoint representation of the generators T_α of the Lie algebra. We have the commutator $[t_\alpha, t_\beta] = f_{\alpha\beta}{}^\gamma t_\gamma$ and the relation $\kappa_{\alpha\beta} = -\text{tr}(t_\alpha t_\beta)$.

The inner product is given by

$$\langle A, E\rangle \;\equiv\; \int d^3x\,\varepsilon^{\mu\nu\rho}\,\text{tr}(A_\mu E_{\nu\rho}) \;=\; \int d^3x\,\varepsilon^{\mu\nu\rho}\kappa_{\alpha\beta}\,A^\alpha_\mu\,E^\beta_{\nu\rho}\,. \tag{6.11}$$

We list the complete set of *nonvanishing* A_∞ products:

$$\boxed{\begin{aligned} A_\infty\ \text{Chern-Simons:}\qquad m_1(\lambda)_\mu \;&=\; \partial_\mu\lambda \;\in\; X_{-1}\\ m_1(A)_{\mu\nu} \;&=\; \partial_\mu A_\nu - \partial_\nu A_\mu \;\in\; X_{-2}\\ m_2(\lambda_1,\lambda_2) \;&=\; \lambda_1\lambda_2 \;\in\; X_0\\ m_2(A,\lambda)_\mu \;&=\; A_\mu\,\lambda \;\in\; X_{-1}\\ m_2(\lambda,A)_\mu \;&=\; \lambda\,A_\mu \;\in\; X_{-1}\\ m_2(A_1,A_2)_{\mu\nu} \;&=\; A_{1\mu}\,A_{2\nu} - A_{1\nu}\,A_{2\mu} \;\in\; X_{-2}\\ m_2(E,\lambda) \;&=\; E\,\lambda \;\in\; X_{-2}\,,\\ m_2(\lambda,E) \;&=\; \lambda\,E \;\in\; X_{-2}\,. \end{aligned}} \tag{6.12}$$

Note that the products m_2 have no exchange property: they are neither symmetric nor anti-symmetric; they are intrinsically non-commutative products, which are associative, however.

Let us briefly go over the derivation of such products and the check that they satisfy the relevant identities. Comparing the gauge transformation $\delta_\lambda A = m_1(\lambda) + m_2(A,\lambda) - m_2(\lambda, A)$ with $\delta_\lambda A_\mu = \partial_\mu\lambda + A_\mu\lambda - \lambda A_\mu$ (which follows from (4.2)) we read off expressions for $m_1(\lambda)$,

$m_2(A,\lambda)$ and $m_2(\lambda,A)$. Next, we compare the field equation $m_1(A)+m_2(A,A)=0$ to the explicit field equation $F=0$, which using (4.4) reads $\partial_\mu A_\nu-\partial_\nu A_\mu+A_\mu A_\nu-A_\nu A_\mu=0$. This allows us to read off $m_1(A)$ and $m_2(A,A)$. Comparing the gauge algebra (6.7) to the gauge algebra $[\delta_{\lambda_1},\delta_{\lambda_2}]=\delta_{\lambda_1\lambda_2-\lambda_2\lambda_1}$ we obtain the value of $m_2(\lambda_1,\lambda_2)$. The last two entries in the above table are obtained from the identities themselves.

The list (4.9) of gauge theory inputs to products applies here. The identity $m_1m_1=0$ need only be checked acting on X_0 and holds trivially. The identity $m_1m_2=-m_2m_1$ must be checked on (A,λ), (λ,A) and (λ_1,λ_2). The first two determine $m_2(E,\lambda)$ and $m_2(\lambda,E)$, respectively. The last holds trivially. The associativity condition (6.2) must be checked on $\lambda\lambda\lambda$, $\lambda\lambda A$, λAA and $\lambda\lambda E$. All those are readily verified.

The Chern-Simons action is also reproduced correctly:

$$\begin{aligned}\langle A\,,\tfrac{1}{2}m_1(A)+\tfrac{1}{3}m_2(A,A)\,\rangle &= \int d^3x\,\varepsilon^{\mu\nu\rho}\,\mathrm{tr}\big(\,A_\mu,\;\tfrac{1}{2}(\partial_\nu A_\rho-\partial_\rho A_\nu)+\tfrac{1}{3}(A_\nu A_\rho-A_\rho A_\nu)\big)\\ &=\int d^3x\,\varepsilon^{\mu\nu\rho}\,\mathrm{tr}\big(\,A_\mu,\;\partial_\nu A_\rho+\tfrac{1}{3}(A_\mu A_\nu-A_\nu A_\mu)\big)\,,\end{aligned}\tag{6.13}$$

which agrees with (4.19). As discussed before neither the inner product nor the products refer to a spacetime metric. In this sense the A_∞ formulation seems more natural than the L_∞ formulation for Chern-Simons theory.

7 Conclusions and outlook

Homotopy Lie algebras or L_∞ algebras are generalizations of Lie algebras that describe the underlying algebraic structure of classical closed string field theory. It might appear that L_∞ algebras are somewhat exotic, because the gauge symmetries of ordinary field theories, properly extended to include equations-of-motion symmetries, form a Lie algebra. We argued that, on the contrary, L_∞ algebras are the underlying algebraic structure for any consistent classical field theory. We illustrated this with examples and outlined a general algorithm to determine the L_∞ structures for a given field theory. It must be emphasized that this is not in conflict with the fact that in conventional classical field theories field products are naturally associative and symmetry variations always satisfy a Jacobi identity.

One possible application is to formulate the 'Wilsonian effective actions' recently described by Sen [12]. In principle, these can be obtained by integrating out all modes except for some specific sub-sectors that, along with massless fields, can also include arbitrarily massive fields. A particularly interesting case is that of double field theory as envisioned in [25], where one would include the Kaluza-Klein and winding modes associated with the massless fields of string theory in toroidal backgrounds. While in this paper we have made no attempt to construct such theory, the results here should be the proper starting point for any such endeavor.

Other possible applications are in M-theory, for which we have exceptional field theory [39], a formulation analogous to double field theory that makes the U-duality groups $\mathrm{E}_{d(d)}$, $d=2,\ldots,8$, manifest. Unlike double field theory, these theories require a 'split-formulation' in which the coordinates of $D=11$ supergravity are decomposed into external and internal coordinates in analogy to Kaluza-Klein. The internal coordinates are then enlarged to transform in the fundamental representation of $\mathrm{E}_{d(d)}$. The theory features p-forms of various ranks with respect to the *external* space, transforming as generalized tensors under the *internal* symmetries. The gauge structure of these p-forms is governed by so-called tensor hierarchies, which were originally introduced in gauged supergravity [40] and have various features in common with L_∞ algebras. Notably, the gauge algebra structure does not satisfy the Jacobi identity; rather, the failure of the Jacobi identity is 'absorbed' by higher-form gauge symmetries, with a 'generalized Cartan structure' emerging naturally [41, 42]. It thus appears likely that there is an L_∞ description of the tensor hierarchy, which in turn could shed light on a more fundamental formulation of exceptional field theory. Indeed, so far exceptional field theory has only been constructed on a case-by-case basis, for each duality group $\mathrm{E}_{d(d)}$ separately. One might hope that eventually there will be a formulation based on a larger algebraic structure, realizing the U-duality groups as sub-structures. This algebraic structure might well be an L_∞ algebra.

Finally, L_∞ algebras are important for higher-spin gravity. Indeed, the early investigation of the consistency of non-linear higher-spin symmetries in [14] naturally led to a structure that can be interpreted as a homotopy Lie algebra. It would therefore be interesting to reformulate or extend higher-spin theories such as constructed by Vasiliev (see [43] for a recent review) in terms of L_∞ algebras. Aspects of this relation have already been discussed in [44]. Specifically, in the formulation of higher-spin theories in [45] the gauge symmetries are governed by a Lie algebra (albeit infinite-dimensional), but this is achieved thanks to additional unphysical coordinates. Upon 'integrating them out' one should recover an L_∞ algebra. It would be interesting to see if other theories whose gauge symmetries need L_∞ structures can be reformulated with pure Lie algebras by using additional coordinates. Further illuminating the L_∞ description of higher-spin symmetries may also shed a new light on the open problem of constructing an action for higher-spin gravity, which would be important for holographic applications. Perhaps the L_∞ algebra can be naturally constructed by adding sets of free fields, in the way that the difficulties in constructing actions for superstring field theories were overcome in [46].

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