Aspects of double geometry and supersymmetric gauge theories

by

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Abstract

This thesis explores aspects of gravity and quantum field theory (QFT). The first part is devoted to the interplay between manifest T-duality symmetry and higher-derivative corrections to the low-energy effective action of string theory. The second part focuses on exact results in supersymmetric gauge theories.

We first discuss T-duality symmetry of string theory from various perspectives. We next review the manifest-duality-symmetric formulation of low-energy effective actions: double field theory (DFT) and its higher-derivative generalization, HSZ (Hohm-Siegel-Zwiebach) theory. We then compute on-shell three-point amplitudes in the HSZ theory. We show that these amplitudes factorize, as in bosonic and heterotic string theories, but they differ from both. Further, we analyze degrees of freedom in the HSZ theory. The spectrum of the theory contains massive spin-2 ghosts and massive scalars, in addition to massless fields. The massive modes can be integrated out exactly at the quadratic level, leading to an infinite series of higher-derivative corrections. Finally, we give a ghost-free massive extension of linearized DFT, which employs novel mass terms for the dilaton and the graviton.

In the second part, we start by reviewing the exact results for partition functions of supersymmetric gauge theories on spheres. Exact results, however, are not available for minimally supersymmetric theories on S^4 and S^6. Minahan conjectured the form of perturbative partition functions for theories on S^d with eight and sixteen supersymmetries. We show that this form gives the correct one-loop divergences of the flat-space super Yang-Mills (SYM) upon taking the radius of the sphere to infinity. We also prove the conjecture explicitly for theories with eight supersymmetries. Further, we extend our results to theories with four supersymmetries for $d < 4$. We then propose an analytic continuation to $d = 4$ to obtain the partition function for a certain $\mathcal{N} = 1$-preserving mass deformation of $\mathcal{N} = 4$ SYM. This analytic continuation gives the correct β-function and agrees with the result for free vector and chiral multiplets.

Thesis Supervisor: Barton Zwiebach
Title: Professor of Physics and MacVicar Faculty Fellow

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امی جی کے نام اور ابو جی کی یاد کے نام
جن کی بے لوث محبت سے میں اس لائق بنا

and to Lily.

Contents

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Introduction

What is the geometry of space and time? This question can be traced back to ancient Greek philosophers. Plato viewed time as inseparable from periodic motion. This was a reasonable reflection on what he saw in nature: the repetition of seasons, alternation of day and night, and motion of visible planets. Aristotle rejected the existence of empty space. He argued: *just as every body is in its place, so, too, every place has a body in it.* While our understanding of the nature of space and time has been improved since then, it is far from complete. On the macroscopic scale, Einstein's general theory of relativity, which describes gravity, is the answer. It has withstood externsive experimental testing, from the precession of Mercury's orbit to the detection of gravitational waves [1]. The structure of spacetime on microscopic scales, however, remains elusive. Our understanding of physics at small scales is based on quantum field theory (QFT). Its predictions have seen striking agreement with experiments, e.g., the prediction for the anomalous magnetic moment of the electron agrees with experimentally measured value to more than ten significant figures. Despite its remarkable success, QFT does not encompass gravity. The failure to reconcile general relativity and QFT points to our incomplete understanding of both.

Strings see spacetime differently from point particles. Our usual understanding of spacetime geometry is based on manifolds of fixed topology with particle motion described by geodesics. The realm of "stringy geometry," however, allows the topology of the spacetime to change. It also allows distinct-looking spacetimes to describe the same physics. String theory suggests that our notions of spacetime geometry have to be modified. The simplest manifestation of this arises in low-energy effective actions of string theory, where general relativity is modified by including other massless fields and an infinite number of higher-derivative corrections.

Families of spacetimes which describe the same physics are often connected by target-space-duality (T-duality) transformations. In the first part of the thesis we use T-duality to explore the geometry of spacetime in string theory. We investigate, in particular, the interplay between T-duality and higher-derivative corrections.

When a string propagates along a periodic direction of radius R, its momentum is quantized like that of a particle. Due to its extended nature, a string can also wind around the periodic direction, a feature absent in a particle theory. This leads to another quantum number: the winding number. The energy spectrum of the string does not change if one swaps the momentum and winding quantum numbers while changing the radius from R to $\frac{1}{R}$. This is T-duality in its most primitive form. For motion on a toroidal background T^d, T-duality transformations form the group $O\left(d, d : \mathbb{Z}\right)$, which shuffles $d + d$ quantum numbers associated with momenta and windings. For the low-energy effective action of string theory, T-duality is realized, surprisingly, as a global $O\left(d, d : \mathbb{R}\right)$ symmetry. Double field theory (DFT) makes the duality symmetry manifest albeit only for two-derivative effective actions. There is only one known example that includes higher-derivative interactions and has exact duality invariance, the so-called HSZ (Hohm-Siegel-Zwiebach) theory [2]. A key aspect of duality-manifest formulations is that they employ field variables that transform linearly under the duality group and encode the physical fields, e.g., graviton, in a non-trivial way. These developments are reviewed in chapter 1.

We explore on-shell physical properties of the HSZ theory in chapters 2 and 3. Chapter 2 is based on [3], where we compute three-point amplitudes in the theory. Our methods are instructive; they illuminate the relation between the duality-covariant field variables and the physical ones. These amplitudes factorize as in bosonic and heterotic string theories, but are different from both.

This motivates the analysis of the spectrum of the HSZ theory presented in [4], which is what chapter 3 is based on. The HSZ theory has exact duality symmetry and higher-derivative interactions – the hallmark of string theories – but is not a string theory. We discover that in addition to the usual massless fields, the spectrum includes two massive spin-2 ghosts and scalars. Such inconsistencies are expected in generic higher-derivative theories. We also show that the massive modes can be integrated out exactly at the quadratic level, leading to an infinite number of higher derivative corrections. One expects this to be required for exact duality invariance. We then give a ghost-free massive extension of linearized DFT, which employs novel mass terms for the dilaton and the metric. Our analysis illuminates the interplay among α'-corrections, massive modes of higher spin, and T-duality.

The second part of the thesis focuses on exact results for supersymmetric gauge theories. Exact results in QFT are as hard to find as they are desired. Much of our understanding of QFT is based on perturbative and approximate methods. While these methods work well when the effective strength of the interactions is weak, they are of no use when the theory is strongly interacting.

It is difficult to obtain exact results for generic strongly coupled QFTs, but in the presence of supersymmetry on compact spaces such as a spheres, progress can be made. A compact space allows a systematic way to regulate IR divergences and supersymmetry allows the use of supersymmetric localization. Observables in QFT are computed by doing an infinite dimensional path integral, $\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi \, \mathcal{O} e^{-S[\Phi]}$. Here $S[\Phi]$ is the action functional, which can contain cubic or higher order interaction terms in fields Φ. In general it is impossible to compute this integral exactly. Supersymmetric localization reduces it to a finite dimensional integral.

Tremendous progress has been made in obtaining exact results for supersymmetric gauge theories based on the localization principle, as pioneered in [5]. These results, however, remain elusive in theories such as those on S^4 with four supercharges and on S^6 with eight supercharges. In the former case, the standard localization procedure fails, and in the latter, no explicit Lagrangian for the theory is known. We examine these missing cases in chapters 4 to 6.

In chapter 4, we review all known results for partition functions of supersymmetric gauge theories on spheres S^d. These results admit an analytic continuation where one can treat d as an analytic parameter. Based on this analytic continuation, Minahan conjectured the form of partition functions of theories with eight and sixteen supersymmetries [6]. In chapter 5, based on [7], we perform non-trivial consistency checks on the analytic continuation. We show that the dimensional regularization of the analytically continued partition functions have the same logarithmic divergences as known partition functions with a hard cutoff. We also provide a consistency check on this analytic continuation when explicit partition functions are not known: in the limit when the radius of the sphere goes to infinity, the analytically continued partition function correctly gives the one-loop divergences of the corresponding flat space SYM.

Consistency checks of the conjecture motivates [8]. Chapter 6 is based on this work, where we compute the perturbative partition functions for gauge theories with eight supersymmetries on spheres of dimension $d \leq 5$. This proves the conjecture in [6]. We apply similar techniques to compute partition functions for theories with four supersymmetries for $d \leq 3$. This provides a unified approach to the localization of supersymmetric gauge theories on spheres. We propose an analytic continuation to $d = 4$ that gives the partition function for an $\mathcal{N} = 1$ gauge theory. We show that it is consistent with the free multiplets and the one-loop β-functions for general $\mathcal{N} = 1$ gauge theories. We also show that the general structure of the real part of the free energy obtained from the analytic continuation is consistent with the holographic predictions for $\mathcal{N} = 1^*$ theory.

Other works which are not included in this thesis

In DFT, unlike in general relativity, the exponential map of infinitesimal gauge transformations had no known formulation in terms of the Jacobian matrix for coordinate transformations. Hohm and Zwiebach conjectured an expression in reference [9]. In [10] we showed that their proposal coincides with the exponentiation of the Lie derivative along a particular parameter and gave explicit expression for it. This established the geometric form of finite gauge transformations in DFT.

In [11], we devised a canonical formulation of DFT. We showed that the dynamics is subject to primary and secondary constraints, as expected in a theory with gauge invariance. We also showed that the Poisson bracket algebra of secondary constraints closes on-shell. Further, we appled the formalism in a variety of solutions of DFT to obtain conserved charges. This gave a duality-covariant description of conserved charges associated with diffeomorphisms and the gauge invariance of Kalb-Ramond field.

Part I

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An overview of T-duality and double geometry

In this chapter we develop basic elements of the double geometry. We start by discussing the T-duality symmetry of string theory from various perspectives. This discussion naturally leads us to introduce ingredients of DFT.

A remarkable result from the early days of string theory is that the conformal invariance on the world-sheet implies equations of motions for the target space fields [12]. Consider the following world sheet action:

$$S_{\text{ws}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\left[\left(\sqrt{\gamma}\gamma^{ab}g_{mn}(X)+\epsilon^{ab}b_{mn}(X)\right)\partial_a X^m\partial_b X^n+\alpha'\sqrt{\gamma}R^{(2)}\phi(X)\right], \tag{1.1}$$

where γ_{ab} is the world sheet metric, $R^{(2)}$ the scalar curvature of the world sheet and $\sigma^a, a=1,2$ the world sheet coordinates. $g_{mn}(X)$ is the target space metric, $b_{mn}(X)$ the target space Kalb-Ramond field (b-field henceforth), $\phi(X)$ the target space dilaton and X^m the target space coordinates. Treating the target space fields as world sheet couplings, one can derive the corresponding one-loop beta functions $\beta^g_{mn}, \beta^b_{mn}$ and β^Φ:

$$\begin{aligned}
\beta^g_{mn}+8\pi^2 g_{mn}\frac{\beta^\phi}{\alpha'} &= \left(R_{mn}-\frac{1}{2}g_{mn}R\right)-T^{\text{matter}}_{mn}=0,\\
8\pi^2\frac{\beta^\phi}{\alpha'}+\frac{1}{2}g^{mn}\beta^g_{mn} &= 2\nabla_m\phi\nabla^m\phi-\nabla^2\phi-\frac{1}{12}H^2=0,\\
&\beta^b_{mn}=\nabla_\lambda H^\lambda_{mn}-2\nabla_\lambda\phi H^\lambda_{mn}=0.
\end{aligned} \tag{1.2}$$

Imposing the vanishing of the one-loop beta functions is the requirement that the world-sheet theory is conformally invariant. $H=\mathrm{d}b$ is the three-form field strength for the b-field and T^{matter}_{mn} is given by:

$$T^{\text{matter}}_{mn} = \frac{1}{4}\left[H^2_{mn}-\frac{1}{6}g_{mn}H^2\right]-2\nabla_m\nabla_n\phi+2g_{mn}\nabla^2\phi-2g_{mn}\nabla_p\phi\nabla^p\phi. \tag{1.3}$$

Equations (1.2) can be derived from the following *target space* action:

$$S = \int d^D x \sqrt{g} e^{-2\phi} \left[R + 4\nabla_m \phi \nabla^m \phi - \frac{1}{12} H^2 \right]. \tag{1.4}$$

This is the low-energy effective action for the bosonic string theory. This is also the universal bosonic part of the NS-NS sector of low-energy effective actions for superstring theories. An observation due to Buscher [13, 14] lets us rewrite these equations in the following form:

$$\begin{aligned} 2\left(\hat{\nabla}\phi\right)^2 - 2\hat{\nabla}^2\phi - \tfrac{1}{2}\hat{R} - \tfrac{1}{12}H^2 = 0, \\ \hat{R}_{mn} + 2\hat{\nabla}_m \hat{\nabla}_n \phi = 0, \end{aligned} \tag{1.5}$$

where $\hat{R}_{mn}$ is non-symmetric the Ricci tensor associated with the torsional connection $\hat{\nabla} = \nabla + \frac{1}{2} g^{-1} H$. This rewriting implies that a torsionless target space with a b-field and a torsionful target space are completely equivalent. Therefore it is clear that the metric and the b-field are in general not separate entities. This is obvious from the closed string point of view where both fields arise at the same excitation-level. From the target space perspective, this means that a more *stringy* way of dealing with these fields is to treat them in a unified framework.

1.1 T-duality in string theory

The mixing of the b-field and the metric is due to one of the most important duality symmetries of the string theory, the T-duality. In this section we explore various aspects of T-duality. We start by describing the T-duality from the perspective of both the classical and the first quantized bosonic string theory. We also discuss the T-duality from the point of view of the low-energy effective action for massless fields.

1.1.1 Classical T-duality à la Buscher rules

The duality symmetry can be made apparent by using a first-order form of the action. This is achieved by including a Lagrange multiplier field $\hat{X}^0$ and choosing $\partial_a X^0 = V_a$. The world sheet action becomes:

$$\begin{aligned} S_{\text{ws}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sum_{\alpha,\beta=1}^{D-1} & \sqrt{\gamma}\gamma^{ab} \left[g_{00} V_a V_b + 2 g_{0\alpha} V_a \partial_b X^\alpha + g_{\alpha\beta}(X) \partial_a X^\alpha \partial_b X^\beta \right] \\ & + \epsilon^{ab} \left(2 b_{0\alpha} V_a \partial_b X^\alpha + b_{\alpha\beta}(X) \partial_a X^\alpha \partial_b X^\beta \right) \\ & + \epsilon^{ab} \hat{X}^0 \partial_a V_b + \alpha' \sqrt{\gamma} R^{(2)} \phi(X). \end{aligned} \tag{1.6}$$

The equation of motion for the Lagrange multiplier field ensures that V_a can be written as $\partial_a f(\sigma)$ for some function $f(\sigma)$ of the world sheet coordinates. Upon plugging this solution back into the action, we recover the standard world sheet action with $f(\sigma)$ identified with X^0. However one can also treat V_a as a Lagrange multiplier field. Solving the algebraic equation of motion for V_a and plugging the result back in the action leads to a *dual* action:

$$S_{\text{dual}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{\gamma}\gamma^{ab}\hat{g}_{mn}\left(\hat{X}\right) + \epsilon^{ab}\hat{b}_{mn}\left(\hat{X}\right)\right) \partial_a \hat{X}^m \partial_b \hat{X}^n + \alpha' \sqrt{\gamma} R^{(2)} \hat{\phi}\left(\hat{X},\right) \tag{1.7}$$

where the dual background fields are related to the fields in the original theory as follows;

$$\begin{aligned} \hat{g}_{00} &= \frac{1}{g_{00}}, \qquad \hat{g}_{0\alpha} = \frac{b_{0\alpha}}{g_{0\alpha}}, \qquad \hat{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta}}{g_{00}}, \\ \hat{b}_{0\alpha} &= \frac{g_{0\alpha}}{g_{00}}, \qquad \hat{b}_{\alpha\beta} = b_{\alpha\beta} + \frac{g_{0\alpha}b_{0\beta} - b_{0\beta}b_{0\alpha}}{g_{00}}, \qquad \hat{\phi} = \phi - \frac{1}{2}\log g_{00}. \end{aligned} \tag{1.8}$$

The index m is split into $m = (0, \alpha)$ where α takes values $1, \cdots, D-1$. The relations between the dual fields and the original fields are known as *Buscher rules*. Buscher rules tell us how the duality symmetry acts on the world sheet action. For example consider consider reducing the theory on a circle of radius R with vanishing b-field and $g_{00} = R$. Then according to the Buscher rules $\hat{g}_{00} = \frac{1}{R}$, i.e., the theory on a circle of radius R and $\frac{1}{R}$ are dual.

This argument establishes the T-duality at the level of the classical world sheet action.

1.1.2 T-duality from first quantization

We now describe the T-duality from the perspective of first quantized bosonic string theory. This can be done via either the path integral approach - by computing the one-loop partition function in a toroidal background - or via canonical quantization by explicitly analyzing the spectrum of the theory. We take the latter route here as it naturally leads to the notion of *generalized metric*, an important ingredient in double geometry.

Consider the world sheet action describing strings propagating in a target space $\mathcal{M}_{\text{target}} = T^D$[1], where T^D is a D-dimensional torus. For simplicity we set $\alpha' = 1$ and the world sheet metric to be the two-dimensional Minkowski metric. The action in eq. (1.1) takes the form:

$$S_{\text{ws}} = -\frac{1}{4\pi} \int_0^{2\pi} d\sigma \int_{-\infty}^{\infty} d\tau \left(\eta^{ab} \partial_\alpha X^m \partial_\beta X^n g_{mn} + \epsilon^{ab} \partial_\alpha X^m \partial_\beta X^n b_{mn} \right), \tag{1.9}$$

[1]One can perform the analysis for $\mathcal{M}_{\text{target}} = \mathcal{M} \times T^d$ in a completely analogous fashion. This choice is merely to keep the notation simple

where $X^m \sim X^m + 2\pi$ are periodic coordinates. The closed string background fields g and b are $D \times D$ matrices. It is straightforward to obtain the canonical momenta P_m and the Hamiltonian density $\underline{H}$ from action (1.9).

$$
\begin{aligned}
P_m &= g_{mn}\dot{X}^n + b_{mn}X'^m \,, \\
4\pi\underline{H} &= \begin{pmatrix} X' \,, & 2\pi P \end{pmatrix} \mathcal{H}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix} \,,
\end{aligned}
\tag{1.10}
$$

where a 'dot' denotes derivative with respect to τ and a 'prime' denotes derivative with respect to σ. We have defined a $D \times D$ matrix $E \equiv g + b$. $\mathcal{H}(E)$ is $2D \times 2D$ symmetric matrix constructed out of the metric and the b-field. It is called the *'generalized metric.'*

$$
\mathcal{H}(E) \equiv \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} \,. \tag{1.11}
$$

To quantize this theory canonically we expand the string coordinate X^m in terms of the momenta, winding modes and oscillators, which satisfy a set of commutation relations consistent with $[X^m, P_n] = i\delta^m{}_n$. For our purposes it is sufficient to focus only on the zero modes in that expansion and ignore the oscillators. We have

$$
X^m = x^m + w^m\sigma + \tau g^{mn}p_n + \cdots \,, \tag{1.12}
$$

where $\cdots$ denote the oscillator expansion. This expansion can be split into left moving and a right moving pieces as usual. The zero-modes α_0 and $\bar{\alpha}_0$ are given by

$$
\begin{aligned}
\alpha_0^m &= \frac{1}{\sqrt{2}} g^{mn}(p_n - E_{np}w^p) \,, \\
\bar{\alpha}_0^m &= \frac{1}{\sqrt{2}} g^{mn}(p_n + E_{pn}w^p) \,.
\end{aligned}
\tag{1.13}
$$

The zero-mode Virasoro operators are given by

$$
\begin{aligned}
L_0 &= \frac{1}{2}\alpha_0^m g_{mn}\alpha_0^n + N - 1 \,, \\
\bar{L}_0 &= \frac{1}{2}\bar{\alpha}_0^m g_{mn}\bar{\alpha}_0^n + \bar{N} - 1 \,,
\end{aligned}
\tag{1.14}
$$

where N and $\bar{N}$ are number operators counting the excitations. The invariance of the closed string under reparametrization $\sigma \to \sigma +$const., leads to the so-called *level-matching constraint* that requires $L_0 - \bar{L}_0 = 0$, which can be expressed as:

$$L_0 - \bar{L}_0 = N - \bar{N} + \partial_m \tilde{\partial}^m \,, \tag{1.15}$$

where [2]

$$p_m \equiv -i\frac{\partial}{\partial x^m}, \qquad w^m \equiv -i\frac{\partial}{\partial \tilde{x}_m}. \tag{1.16}$$

The level-matching condition can now be expressed as a constraint on the number operator

$$N - \bar{N} \equiv -\partial \cdot \tilde{\partial} \,. \tag{1.17}$$

For massless fields in bosonic string theory the level-matching constraint can be implemented as follows. A general massless state can be written as

$$\begin{aligned} &\sum_{p,w} e_{mn}(p,w)\alpha_{-1}^m \bar{\alpha}_{-1}^n c_1 \bar{c}_1 |p,w\rangle, \\ &\sum_{p,w} \Phi(p,w)(c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|p,w\rangle \,, \end{aligned} \tag{1.18}$$

with momentum space wavefunctions $e_{mn}(p,w)$ and $\Phi(p,w)$. Here the matter and ghost oscillators act on a vacuum $|p,w\rangle$ with momentum p and winding number w. Level matching condition then requires that the Fourier transformed fields $e_{mn}(x,\tilde{x})$ and $\Phi(x,\tilde{x})$ satisfy the constraint

$$\partial \cdot \tilde{\partial}\, e_{mn}(x,\tilde{x}) = \partial \cdot \tilde{\partial}\, \Phi(x,\tilde{x}) = 0 \,. \tag{1.19}$$

By integrating the Hamiltonian density $\underline{H}$ in (1.10) we get the following Hamiltonian.

$$H = \int_0^{2\pi} d\sigma \underline{H} = \frac{1}{2} Z^t \mathcal{H}(E) Z + N + \bar{N} + \ldots \tag{1.20}$$

where the dots contain irrelevant terms and

$$Z = \begin{pmatrix} w^m \\ p_m \end{pmatrix}$$

[2] $\tilde{x}_m$ is the coordinate canonically conjugate to the winding numbers w^m.

is a $2D$ column vector consisting of integer winding and momentum quantum numbers. The level matching condition can now be expressed as

$$N - \bar{N} = \frac{1}{2} Z^t \eta Z \,, \tag{1.21}$$

where η is the matrix defined as follows:

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,. \tag{1.22}$$

Consider now a reshuffling of the quantum numbers

$$Z = h^t Z' \,,$$

with some $2D \times 2D$ invertible matrix h with integer entries. Under such a transformation the physics should not change. In particular the constraint (1.21) should be unchanged. For this it is then necessary that

$$Z'^t \eta Z' = Z^t \eta Z = Z'^t h \eta h^t Z' \,, \tag{1.23}$$

which requires

$$h \eta h^t = \eta \,. \tag{1.24}$$

The h matrices belong to the $O(D, D; \mathbb{Z})$ group. We write

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D, D) \,, \tag{1.25}$$

where a, b, c and d are $D \times D$-matrices. The conditions on a, b, c, and d following from (1.24) are

$$ab^t + ba^t = cd^t + dc^t = 0 \,, \;\; ad^t + bc^t = 1 \,. \tag{1.26}$$

Under the shuffling of quantum numbers, not only the level matching condition but the physical spectrum is also not changed. This requires a change of the background field E. The shuffled quantum numbers are associated to a background field E' [15].

$$E' = h(E) = (aE+b)(cE+d)^{-1} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} E\,. \tag{1.27}$$

The generalized metric corresponding to the background E' is

$$\mathcal{H}(E') = h\mathcal{H}(E)h^t\,. \tag{1.28}$$

This proves that

$$Z^t\mathcal{H}(E)Z \;=\; Z'^t\mathcal{H}(E')Z'\,, \tag{1.29}$$

hence the Hamiltonian in eq. (1.20) is not changed. We have thus shown that there is a $O\,(D, D; \mathbb{Z})$ duality symmetry, which leaves the spectrum of the string theory invariant. [3]

This establishes the duality at the level of first quantization. This is indeed a symmetry of the full quantum string theory. One can show the partition function of the world sheet theory on all genus g Riemann surfaces has an $O\,(d, d; \mathbb{Z})$ symmetry if target space has the form $\mathcal{M}_{\text{target}} = \mathcal{M} \times T^d$ [16].

1.1.3 Duality in the low-energy effective action

What are the consequences of this duality symmetry for the low-energy effective actions of string theory? This question was answered in the seminal work [17]. The low-energy effective action, upon compactification over T^d, has an enhanced global $O(d, d; \mathbb{R})$ symmetry.

Consider the effective action in eq. (1.4) compactified on a T^d, i.e., $\mathcal{M}_{\text{target}} = \mathcal{M} \times T^d$. We denote the D-dimensional fields with an *overhat* and the D-dimensional indices with latin letters $i, j, \cdots$. The D-dimensional indices split as $i = (m, \alpha)$, where $m, n, \cdots$ are indices along T^d and $\alpha, \beta, \cdots$ are indices along $\mathcal{M}$. The D-dimensional metric splits as follows,

$$\hat{g}_{ij} = \begin{pmatrix} g_{\alpha\beta} + A^{(1)m}_{\alpha} A^{(1)}_{\beta m} & A^{(1)}_{\alpha n} \\ A^{(1)}_{\beta m} & G_{mn} \end{pmatrix}, \tag{1.30}$$

where G_{mn} is the metric along the compactified directions. The determinant of the metric and the dilaton

[3] It is $O\,(d, d; \mathbb{Z})$ when d is the number of toroidal dimensions in the target space.

are related by

$$\sqrt{-\operatorname{Det}\hat{g}} = \sqrt{-\operatorname{Det}g}\sqrt{\operatorname{Det}G}, \qquad \phi = \hat{\phi} - \frac{1}{2}\log\operatorname{Det}G. \tag{1.31}$$

Different components of the B-field can be arranged as follows:

$$\begin{aligned} \hat{b}_{mn} &= b_{mn}, \qquad \hat{b}_{\alpha m} = A^{(1)}_{\alpha m} - b_{mn}A^{(1)n}_{\alpha}, \\ \hat{b}_{\alpha\beta} &= b_{\alpha\beta} - A^{(1)m}_{[\alpha}A^{(1)}_{\beta]m} + A^{(1)m}_{\alpha}b_{mn}A^{(1)n}_{\beta}. \end{aligned} \tag{1.32}$$

The dimensionally reduced action takes the form:

$$S = \int d^{D-d}x\sqrt{-g}\left(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\right), \tag{1.33}$$

where

$$\begin{aligned} \mathcal{L}_1 &= R + g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \\ \mathcal{L}_2 &= \frac{1}{4}g^{\alpha\beta}\left(\partial_\alpha G_{mn}\partial_\beta G^{mn} - G^{mn}G^{pq}\partial_\alpha b_{mp}\partial_\beta b_{nq}\right), \\ \mathcal{L}_3 &= -\frac{1}{4}g^{\alpha\gamma}g^{\beta\delta}\left(G_{mn}F^{(1)m}_{\alpha\beta}F^{(1)n}_{\gamma\delta} + G^{mn}H_{\alpha\beta m}H_{\gamma\delta n}\right), \\ \mathcal{L}_4 &= -\frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}. \end{aligned} \tag{1.34}$$

The three-form field strength in the last term is given by

$$H_{\alpha\beta\gamma} = \partial_\alpha b_{\beta\gamma} - \frac{1}{2}\left(A^{(1)m}_{\alpha}F^{(2)}_{\beta\gamma m} + A^{(2)m}_{\alpha}F^{(1)}_{\beta\gamma m}\right) + \text{cyc. permutations}. \tag{1.35}$$

Each of the four terms above are invariant under global $O(d,d;\mathbb{R})$ transformations. The first term is trivially invariant because the metric $g_{\alpha\beta}$ and the dilaton field ϕ are invariant. The second term can be written in the following way, which makes its invariance manifest.

$$\mathcal{L}_2 = \frac{1}{8}\partial_\alpha\mathcal{H}_{MN}\partial^\alpha\mathcal{H}^{MN}, \tag{1.36}$$

where $\mathcal{H}^{MN}$ is a $2d \times 2d$ symmetric matrix, the generalized metric introduced earlier in eq. (1.11) [4]. Indices M, N are contracted using the $O(d,d)$ invariant metric η. Under an $O(d,d;\mathbb{R})$ transformation h, the

[4] This generalized metric only contains metric and B-field components along T^d directions.

generalized metric $\mathcal{H}$ changes as

$$\mathcal{H} \to h\mathcal{H}h^t. \tag{1.37}$$

Since $h^t \eta h = \eta$, we see that $\mathcal{L}_2$ is invariant under this transformation. Note that the action of $O(d,d)$ transformations on the *physical* metric and the b-field is rather complicated and non-linear, but the generalized metric transforms linearly under these transformations.

The third term in the Lagrangian can be written as follows:

$$\mathcal{L}_3 = -\frac{1}{4}\mathcal{F}^M_{\alpha\beta}\mathcal{H}_{MN}\mathcal{F}^{\alpha\beta N}, \tag{1.38}$$

where $\mathcal{F}^M_{mn}$ is a $2d$-component vector of field strength.

$$\mathcal{F}^M_{\alpha\beta} \equiv \begin{pmatrix} F^{(1)m}_{\alpha\beta} \\ F^{(2)}_{\alpha\beta m} \end{pmatrix} \tag{1.39}$$

This is an $O(d,d)$ vector and transforms linearly under the action of $h \in O(d,d)$

$$\mathcal{F}^M_{\alpha\beta} \to h^M{}_N\mathcal{F}^N_{\alpha\beta}. \tag{1.40}$$

This, along with the transformation property of the generalized metric, guarantees the invariance of $\mathcal{L}_3$ under $O(d,d;\mathbb{R})$ transformations.

The invariance of $\mathcal{L}_4$ can also be made manifest by writing $H_{\alpha\beta\gamma}$ in the following form

$$H_{\alpha\beta\gamma} = \partial_\alpha b_{\beta\gamma} - \frac{1}{2}\mathcal{A}^M_\alpha\eta_{MN}\mathcal{F}^N_{\beta\gamma} + \text{(cyc. permutations)}. \tag{1.41}$$

Since $b_{\alpha\beta}$ does not transform under $O(d,d;R)$ and the second term is manifestly invariant under the transformation, we conclude that $H_{\alpha\beta\gamma}$ and hence $\mathcal{L}_4$ are invariant under $O(d,d;R)$ transformations.

Hence for the low-energy effective action of string theory the duality symmetry is enhanced from $O(d,d;\mathbb{Z})$ to $O(d,d;\mathbb{R})$. It is useful to decompose the action of the duality group in terms of its various subgroups to understand it better.

$O(d,d;\mathbb{R})$ group

Recall that the group $O(d,d;\mathbb{R})$ is defined as the group of matrices which leave the matrix η, defined in 1.22, invariant [5]. The dimension of the group is $d(2d-1)$.

We first examine the part of the $O(d,d;\mathbb{R})$ given by the matrices of the form

$$h = \begin{pmatrix} A & 0 \\ 0 & \left(A^t\right)^{-1} \end{pmatrix}, \qquad A \in GL(d). \tag{1.42}$$

This forms a d^2-dimensional subgroup. From eq. (1.27) we can see that it acts on the fields g and b as follows:

$$g \to g' = AgA^t, \qquad b \to b' = AbA^t. \tag{1.43}$$

Note that this group arises under the following coordinate transformation along the T^d.

$$x^i \to x'^i = (A.x)^i. \tag{1.44}$$

Hence this subgroup only generates gauge transformations.

Next we look at the matrices of the form

$$h = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \qquad B = -B^t. \tag{1.45}$$

This generates an abelian $\frac{d(d-1)}{2}$-dimensional subgroup denoted by $\mathbb{R}^{\frac{d(d-1)}{2}}$. Under its action the metric remains unchanged and only the b-field changes: $b \to b' = b + B$. This is the same as the gauge transformation of the b-field with respect to the gauge parameter $B_{ab}x^b$.

Finally we discuss the subgroup $\frac{O(d)\times O(d)}{O(d)_{\text{diag}}}$ [6]. This is given by the set of matrices

$$h = \frac{1}{2}\begin{pmatrix} R+S & R-S \\ R-S & R+S \end{pmatrix}, \qquad R^tR = 1 = S^tS. \tag{1.46}$$

This acts on the background E in a non-trivial and non-linear manner ,which involve mixing the b-field and the metric. To see this, we compute the action on the background with a b-field, $b_{12} = -b_{21} = b$ and a torus T^2. We take the two cycles of the torus to have the same length 1 unit. The metric on the torus is δ_{mn}. We

[5] It is understood that η has the same form as in eq. (1.22) but is a $2d \times 2d$ matrix for the current discussion.

[6] The diagonal subgroup of $O(d) \times O(d)$ is contained in $GL(d)$ as in eq. (1.42) with $A^tA = 1$.

now do a duality transformation on this background. We take R to be the identity and S to be a rotations by angle $\theta_{R,S}$. Then under the action of the transformation h:

$$\begin{aligned}\delta_{mn} \to g'_{mn} &= \frac{2}{2+b^2\left(1-\cos\left(\theta_S-\theta_R\right)\right)+2b\sin\left(\theta_S-\theta_R\right)}\delta_{mn},\\ b_{mn} \to b'_{mn} &= \frac{b\sin\left(\theta_S-\theta_R\right)+2\cos\left(\theta_S-\theta_R\right)}{2+b^2\left(1-\cos\left(\theta_S-\theta_R\right)\right)+2b\sin\left(\theta_S-\theta_R\right)}b_{mn}.\end{aligned} \tag{1.47}$$

Notice however that if $b = 0$, the background remains invariant. This illustrates the complexity and non-linearity of the action of the duality group, even in the simple case of compactification on a torus. Moreover the transformed metric and the b-field are not related to the original ones by usual notions of diffeomorphism or gauge symmetryl.

To summarize, the "geometric" subgroup $GL\left(d;\mathbb{R}\right) \ltimes \mathbb{R}^{\frac{d(d-1)}{2}}$ of the full duality group $O\left(d,d;\mathbb{R}\right)$ is attributed to usual gauge deformations of fields. The subgroup $O\left(d\right) \times O\left(d\right)/O\left(d\right)_{\text{diag}}$ relates dual configurations of the metric and the b-field. Physics in backgrounds related by a duality transformation is completely equivalent. This is the essence of the duality.

Our analysis dealt with only the leading low-energy dynamics of the string theory. However the general picture is true even when subleading corrections are considered. It was shown by A. Sen in [18] that the space of classical solutions of the string field theory has the same duality symmetry as discussed above. Recall that the classical string field theory involves an infinite number of higher derivative terms and they enter in such a way that the T-duality is maintained.

1.2 Elements of DFT

The T-duality symmetry is present in the *full* low-energy effective action of string theory. By *full* effective action we mean the effective action obtained after integrating out all the massive fields of the string theory at tree level. This effective action involves only massless fields and have an infinite number of higher derivative (α'-) corrections. However, even for the case of α'^0, the duality only becomes apparent after a convoluted procedure of dimensional reduction. This raises two natural questions:

1. *Is there a formulation of low-energy effective actions of string theory that makes manifest this $O\left(d,d;\mathbb{R}\right)$ symmetry before compactification?*
2. *Can we impose the requirement of manifest T-duality to constrain α'-corrections to the low-energy effective actions?*

The answer to both these questions is *yes*, as we review below.

1.2.1 Courant bracket

It is clear from the preceding discussion that any attempt to find a manifestly duality-invariant formulation must start by treating the metric and the b-field in a uniform manner. We start by unifying the gauge symmetries associated with the two fields.

A theory with a metric and two-form field has gauge symmetries: diffeomorphism generated by $v \in T\mathcal{M}$ and the gauge transformations of the b-field generated by one forms $\xi \in T^*\mathcal{M}$. By formally adding the vector and the one-form as $v + \xi \in T\mathcal{M} \oplus T^*\mathcal{M}$, we can write the gauge transformations as

$$\delta_{v+\xi} g = \mathcal{L}_v g, \qquad \delta_{v+\xi} b = \mathcal{L}_v g + d\xi, \tag{1.48}$$

where $\mathcal{L}_v$ is the Lie derivative with respect to the vector v. By computing two successive gauge transformations one finds that the algebra of the gauge transformations is

$$[\delta_{v_2+\xi_2}, \delta_{v_1+\xi_1}] \;=\; \delta_{[v_1,v_2]+\mathcal{L}_{v_1}\xi^2-\mathcal{L}_{v_2}\xi_1}. \tag{1.49}$$

The last expression defines a *bracket* on $T\mathcal{M} \oplus T^*\mathcal{M}$ given by

$$[v_1 + \xi_1, v_2 + \xi_2] = [v_1, v_2] + \mathcal{L}_{v_1}\xi_2 - \mathcal{L}_{v_2}\xi_1. \tag{1.50}$$

This is in fact a Lie bracket as it is anti-symmetric and the Jacobi identity is satisfied. However eq. (1.50) is not the only choice consistent with the gauge algebra eq. (1.49). To see this note that

$$\mathcal{L}_{v_1}\xi_1 - \mathcal{L}_{v_2}\xi_1 = di_{v_1}\xi_1 + i_{v_1}d\xi_1 - (1 \leftrightarrow 2)\,, \tag{1.51}$$

where $i_v\xi$ is the standard interior product of vector v and the one-form ξ. Recall that an exact one-form does not generate a gauge transformation of the b-field. The gauge algebra is independent of the term $di_{v_1}\xi_2 - (1 \leftrightarrow 2)$ appearing in eq. (1.50). We can consistently modify the bracket by replacing the coefficient of this term from 1 to $\left(1 - \frac{\beta}{2}\right)$.

$$[v_1 + \xi_1, v_2 + \xi_2] = [v_1, v_2] + \mathcal{L}_{v_1}\xi_2 - \mathcal{L}_{v_2}\xi_1 - \frac{1}{2}\beta d\left(i_{v_1}\xi_2 - i_{v_2}\xi_1\right). \tag{1.52}$$

We will choose a specific value of β by appealing to an important part of the duality symmetries: the $\mathbb{B}$-shifts. Consider a closed two-form B. Shifting the b-field by such a two-form B is a symmetry of the theory. How does B act on the gauge parameters? The two-form B defines a natural map[7]

$$B : T\mathcal{M} \to T^*\mathcal{M}, \qquad v \in T\mathcal{M} \to i_v B \in T^*\mathcal{M}. \tag{1.53}$$

This map has a straightforward extension to $v + \xi \in T\mathcal{M} \oplus T^*\mathcal{M}$, where it leaves the vector as it is and shifts the one-form by $i_v B$. We fix the parameter β in eq. (1.52) by requiring that the action of the $\mathbb{B}$-shifts on the gauge parameters is an automorphism of the bracket. This fixes β to 1. The resulting bracket is known as the *Courant bracket* after T. Courant.

1.2.2 Gauge algebra, fields and action

For uniform treatment of diffeomorphism and the b-field gauge transformations, we introduce the following notation

$$\xi^M \equiv \begin{pmatrix} \tilde{\xi}_m \\ \xi^m \end{pmatrix} \tag{1.54}$$

to denote the gauge transformation parameters $\xi + \tilde{\xi} \in T\mathcal{M} \oplus T^*\mathcal{M}$. This suggest a *doubling* of coordinates and corresponding derivatives.

$$X^M \equiv \begin{pmatrix} \tilde{x}_m \\ x^m \end{pmatrix} \qquad \partial_M \equiv \begin{pmatrix} \tilde{\partial}^m \\ \partial_m \end{pmatrix}. \tag{1.55}$$

Here $M = 1, 2 \cdots 2D$ is an $O(D, D)$ index, which is raised and lowered by the $O(D, D)$-invariant metric η as defined in eq. (1.22). It is useful to repeat the definition of η with indices here:

$$\eta^{MN} = \begin{pmatrix} 0 & \delta_m{}^n \\ \delta^m{}_n & 0 \end{pmatrix}, \qquad \eta_{MN} = \begin{pmatrix} 0 & \delta^m{}_n \\ \delta_m{}^n & 0 \end{pmatrix}, \qquad \eta_M{}^N = \eta_{MP}\eta^{PN} = \eta^{NP}\eta_{MP} = \delta_M{}^N. \tag{1.56}$$

Next we need to identify the right choice of field variables to make the duality manifest. The generalized metric, which has already appeared in 1.49, is a natural candidate. With indices it takes the following form:

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{mn} & -g^{mp}b_{pn} \\ b_{mp}g^{kn} & g_{mn} - b_{mp}g^{pq}b_{qn} \end{pmatrix}, \qquad \mathcal{H}^{MN} = \begin{pmatrix} g_{mn} - b_{mp}g^{pq}b_{qn} & b_{mp}g^{kn} \\ -g^{mp}b_{pn} & g^{mn} \end{pmatrix}. \tag{1.57}$$

[7] Another way to obtain this map is to note that $\delta_{v+\xi}(b + B) = \delta_{v+(\xi+i_v B)} b$.

The generalized metric satisfies the constraint

$$\mathcal{H}_{MN}\mathcal{H}^{NP} = \delta_M{}^P. \tag{1.58}$$

In matrix notation this constraint is $\mathcal{H}\eta\mathcal{H}^t = \eta$, i.e., $\mathcal{H} \in O(D, D)$. We want to treat the generalized metric as the fundamental field variable without referring to its parameterization in terms of the metric and the b-field. Since $\mathcal{H}$ is a $2D \times 2D$ matrix we need to impose further constraints for it to have same number of degrees of freedom as $g_{mn} + b_{mn}$. Demanding $\mathcal{H}$ to be a symmetric element of $O(D, D; \mathbb{R})$ reduces the number of degrees of freedom from $2D \times 2D$ to D^2.

The determinant of the metric is combined with the dilaton to form a duality covariant dilaton defined as

$$e^{-2\Phi} = e^{-2\phi}\sqrt{-g}. \tag{1.59}$$

Infinitesimal gauge transformations for the fundamental fields $\mathcal{H}_{MN}$ and Φ are given by

$$\begin{aligned} \delta_\xi \mathcal{H}_{MN} &= \widehat{\mathcal{L}}_\xi \mathcal{H}_{MN} \equiv \xi^P \partial_P \mathcal{H}_{MN} + \left(\partial_M \xi^K - \partial^K \xi_M\right) \mathcal{H}_{KN} + \left(\partial_N \xi^K - \partial^K \xi_N\right) \mathcal{H}_{KM}, \\ \delta_\xi \Phi &= \xi^P \partial_P \Phi - \frac{1}{2}\partial_P \xi^P. \end{aligned} \tag{1.60}$$

To describe closed string theory consistently, string fields are subject to the level matching condition.

$$\left(L_0 - \bar{L}_0\right) f(x, \tilde{x}) = 0, \tag{1.61}$$

In the massless subsector, this implies the following constraint on double fields

$$\partial_M \partial^M f(x, \tilde{x}) = 0. \tag{1.62}$$

However, in general given two fields $f_1(x, \tilde{x})$ and $f_2(x, \tilde{x})$ satisfying the constraint eq. (1.61), their product does not satisfy this constraint. Hence one has to impose the so-called *strong constraint*. It is the statement that all fields, gauge parameters and their product are annihilated by $\partial_M \partial^M$.

$$\partial_M \partial^M (\cdots) = 0. \tag{1.63}$$

To write down an action we take into account another symmetry: the $\mathbb{Z}_2$ symmetry which acts on the b-field as $b \to -b$. This symmetry acts on the coordinates, derivatives and the generalized metric as follows [19]:

$$X^{\bullet} \to ZX^{\bullet}, \qquad \partial_{\bullet} \to Z\partial_{\bullet}, \qquad \mathcal{H}^{\bullet\bullet} \to Z\mathcal{H}^{\bullet\bullet}Z, \qquad \mathcal{H}_{\bullet\bullet} \to Z\mathcal{H}_{\bullet\bullet}Z,$$
$$\eta^{\bullet\bullet} \to Z\eta^{\bullet\bullet}, \qquad \eta^{\bullet\bullet} \to Z\eta^{\bullet\bullet}, \tag{1.64}$$

where $\bullet$'s indicate $O(D,D)$ indices and Z is given by the matrix

$$Z = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.65}$$

The two-derivative action is then completely fixed by the gauge symmetry and the $\mathbb{Z}_2$ invariance.

$$\begin{aligned} S_{\text{DFT}} = & \int d^D x d^D \tilde{x} e^{-2\Phi} \big(\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{PQ}\partial_N\mathcal{H}_{PQ} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{PQ}\partial_Q\mathcal{H}_{MP} \\ & - 2\partial_M\Phi\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M\Phi\partial_N\Phi\big). \end{aligned} \tag{1.66}$$

We now comment on important features of this action:

- Upon taking $\tilde{\partial} = 0$ to be the solution of the strong constraint, it reduces to the effective action in 1.4 (up to an over all factor), hence it is a manifestly $O(D,D;\mathbb{R})$-invariant formulation of the low-energy effective action.

- Notice that the group $O(D,D;\mathbb{R})$ is much bigger than actual global symmetry group arising upon compactifying on T^d! Strong constraint is required to break it to a smaller subgroup.

- If the D-dimensional target space has no isometries then a solution of the strong constraint breaks $O(D,D;\mathbb{R})$ to the geometric subgroup $GL(D) \ltimes \mathbb{R}^{\frac{D(D-1)}{2}}$. To see this, note that under $GL(D)$ transformation

$$X^M \to X'^M = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \begin{pmatrix} \tilde{x}_m \\ x^m \end{pmatrix}. \tag{1.67}$$

  Then $\tilde{\partial}^m = 0$ implies that $\tilde{\partial}'^m = 0$. A similar conclusion is drawn for transformations in $\mathbb{R}^{\frac{D(D-1)}{2}}$.

- In the case of compactification on T^d, we divide the double coordinates as follows:

$$X^M = \left(\tilde{x}_\mu, \tilde{y}_i, x^\mu, y^i\right), \tag{1.68}$$

where y^i are the coordinates along T^d. Since fields are independent of y^i, the solution $\tilde{\partial} = 0$ of strong constraint is invariant under $O(d, d; R)$ transformations that mix y^i and $\tilde{y}_i$, hence the global symmetry group of the compactified theory is

$$O(d, d; \mathbb{R}) \times GL(n) \ltimes \mathbb{R}^{\frac{n(n-1)}{2}}, \tag{1.69}$$

where $n = D - d$ is the dimension of the compactified space.

1.3 Higher derivative corrections

Having written down an action for *DFT* we have answered the first question asked at the beginning of section 1.2. Now we turn to the second question.

A detailed review of how to carry out this procedure in practice is out of the scope of this thesis. Here we illustrate the principle using an elementary example. Suppose that we are only given the metric and the dilaton part of the effective action in eq. (1.4). We want to *duality complete* this action. We will show that this duality completion lets us determine the correct contribution of the B-field to the effective action. We write

$$\begin{aligned} S_* &= \int d^d x \sqrt{-g} e^{-2\phi} \left(R + 4\nabla_m \phi \nabla^m \phi\right) \\ &= \int d^D x e^{-2\Phi} \left[2\partial^m \Phi \partial^n g_{mn} + 4\partial_m \Phi \partial^m \Phi + \frac{1}{4}\partial_m g^{np} \partial^m g_{np} - \frac{1}{2}\partial^m g_{mn} \partial_p g^{pn}\right], \end{aligned} \tag{1.70}$$

where in the second line we have used the explicit expression for R and integration by parts to write the action in a suggestive form. The requirement of manifest T-duality then implies that the action S_* is the restriction to $b = 0$ sector of a manifestly $O(D, D)$-invariant functional S. Next we notice that the different terms can be written in terms of the generalized metric as follows.

$$\begin{aligned} \partial_m g^{np} \partial^m g_{np} &= g^{mq} \partial_m g^{np} \partial_q g_{np} = \frac{1}{2} \mathcal{H}^{MQ} \partial_M \mathcal{H}^{NP} \partial_Q \mathcal{H}_{NP} \Big|_{b=0,\tilde{\partial}=0}, \\ \partial^m g_{mn} \partial_p g^{pn} &= g^{mq} \partial_q g_{mn} \partial_p g^{pn} = \mathcal{H}^{MQ} \partial_Q \mathcal{H}_{MN} \partial_P \mathcal{H}^{PN} \Big|_{b=0,\tilde{\partial}=0}, \\ \partial^m \Phi \partial^n g_{mn} &= g^{mp} g^{nq} \partial_p \Phi \partial_q g_{mn} = -\partial_n \Phi \partial_q g^{nq} = -\partial_N \Phi \partial_Q \mathcal{H}^{NQ} \Big|_{b=0,\tilde{\partial}=0}, \\ g^{mn} \partial_m \Phi \partial_n \Phi &= \mathcal{H}^{MN} \partial_M \Phi \partial_N \Phi \Big|_{b=0,\tilde{\partial}=0}. \end{aligned} \tag{1.71}$$

Hence the appropriate duality completion of S_* is

$$S_* + \text{duality completion} = \int d^dx e^{-2\Phi} \Big(\frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{PQ} \partial_N \mathcal{H}_{PQ} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{PQ} \partial_Q \mathcal{H}_{MP} \\ -2 \partial_M \Phi \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M \Phi \partial_N \Phi \Big) \Big|_{\tilde{\partial}=0}, \tag{1.72}$$

which is the DFT action after choosing $\tilde{\partial} = 0$ as a solution of the strong constraint. As discussed in section 1.2, this action is completely equivalent to the one in eq. (1.4) and hence gives the correct contribution of the B-field.

In principle, this procedure can be carried out at higher orders in α', e.g., see [20] for an analysis at $\mathcal{O}(\alpha')$. A more ambitious goal is to use the manifest duality invariance and gauge invariance as guiding principles and construct a higher-derivative theory from first principles. This was achieved in [2] by Hohm, Siegel and Zwiebach, (the *HSZ* theory). The first part of this thesis is devoted to exploring this theory. We will introduce relevant details along the way. Its key features are the following.

- The theory is formulated in terms the dilaton Φ and a *double metric* $\mathcal{M}_{MN}$, which is a rank-2 $O(D, D)$ tensor. Unlike the generalized metric $\mathcal{M} \notin O(D, D)$.
- The gauge transformation of the dilaton remains the same, but the gauge transformation of $\mathcal{M}_{MN}$ has higher derivative contributions. Consequently the gauge algebra also has higher derivation contributions.
- The action is cubic in $\mathcal{M}$.
- The action involves terms with up to six derivatives.
- The theory has higher derivatives, gauge invariance and an *exact* global duality invariance.

1.4 Summary

In this chapter we reviewed how the discrete duality symmetries of toroidally compactified string theories imply continuous duality symmetries of the classical effective field theory for the massless string degrees of freedom [21, 18]. DFT formulates the higher-dimensional two-derivative massless effective field theory in a way that the duality symmetry can be anticipated before dimensional reduction [22, 23, 24, 25, 19]. When higher-derivative corrections (or α'-corrections) are included it becomes much harder to provide a duality covariant formulation. It is generally expected that as soon as higher-derivatives are included, an infitite number of them are required for exact duality invariance.

At present, there is only one known example of an effective gravitational theory with higher-derivatives and exact duality invariance: the HSZ theory. It is formulated in terms of a double metric $\mathcal{M}$, an unconstrained version of the generalized metric $\mathcal{H}$.

2

Three point functions in HSZ theory

The purpose of this chapter is to calculate on-shell three-point amplitudes for the metric and b field in the HSZ theory. While this is a relatively simple matter in any gravitational theory described in terms of a metric and a b field, it is a rather nontrivial computation in a theory formulated in terms of a double metric $\mathcal{M}$. This is so because metric and b-field fluctuations are encoded nontrivially in $\mathcal{M}$ fluctuations and because $\mathcal{M}$ also contains other non-familiar degrees of freedom. These amplitudes will be obtained using the $\mathcal{M}$ field Lagrangian. The procedure is instructive: it requires us to obtain the explicit α' expansion of the Lagrangian and to discuss the extraneous fields contained in $\mathcal{M}$.

In both bosonic string theory and heterotic theory, on-shell three-point amplitudes factorize into factors that involve left-handed indices and right-handed indices (see eqn.(2.6)). We show that in HSZ these amplitudes also factorize (see eqn.(2.7)). The explicit form of the result has implications for the low-energy effective field theory. In the bosonic string the terms in the low-energy effective action needed to reproduce its three-point amplitudes include Riemann-squared (or Gauss-Bonnet) [26, 27] and HHR terms to first order in α', and Riemann-cubed to second order in α' [28, 29]. To reproduce the (gravitational) heterotic three-point amplitudes the theory has only order α' terms: Gauss-Bonnet, HHR and a b-odd term $b\,\Gamma\partial\Gamma$, with Γ the Christoffel connection. At order α' HSZ theory contains only the b-odd term with twice the coefficient in heterotic string, and to second order in α' the bosonic string Riemann-cubed term with *opposite* sign. To order $(\alpha')^2$, the following is the gauge invariant action that reproduces the on-shell cubic amplitudes of HSZ theory:

$$S = \int d^Dx\sqrt{-g}\,e^{-2\phi}\Big(R + 4(\partial\phi)^2 - \tfrac{1}{12}\widehat{H}_{\mu\nu\rho}\widehat{H}^{\mu\nu\rho} - \tfrac{1}{48}\,\alpha'^2\,R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\rho\sigma}R_{\rho\sigma}{}^{\mu\nu}\Big)\,. \tag{2.1}$$

The $\mathcal{O}(\alpha')$ terms above arise from the kinetic term for the three-form curvature [30]. We have $\widehat{H}_{\mu\nu\rho} =$

$H_{\mu\nu\rho} + 3\,\alpha'\Omega_{\mu\nu\rho}(\Gamma)$, where $H_{\mu\nu\rho} = 3\,\partial_{[\mu}b_{\nu\rho]}$ with the Chern Simons term Ω given by:

$$\Omega_{\mu\nu\rho}(\Gamma) \;=\; \Gamma^{\sigma}_{[\mu|\delta|}\partial_{\nu}\Gamma^{\delta}_{\rho]\sigma} + \tfrac{2}{3}\Gamma^{\sigma}_{[\mu|\delta|}\Gamma^{\delta}_{[\nu|\gamma|}\Gamma^{\gamma}_{[\rho]\sigma|}\,. \tag{2.2}$$

2.1 Bosonic, heterotic, and HSZ three-point amplitudes

In this section we motivate and state our main claim: In HSZ theory, on-shell three-point amplitude for gravity and b fields exhibits a factorization structure analogous to that of the bosonic and heterotic string. For this purpose we consider these amplitudes. Let k_1, k_2, and k_3 denote the momenta of the particles. Since we are dealing with massless states, the on-shell condition and momentum conservation imply that for all values of $a, b = 1, 2, 3$:

$$k_a \cdot k_b = 0\,. \tag{2.3}$$

We also have three polarization tensors $e_{a\,mn}$ with $a = 1, 2, 3$. Symmetric traceless polarizations represent gravitons, and antisymmetric polarizations represent b fields. Dilaton states are encoded by polarizations proportional to the Minkowski metric [31]. The polarizations satisfy transversality

$$k_a^m e_{a\,mn} = 0\,, \quad k_a^n e_{a\,mn} = 0\,, \qquad a \text{ not summed.} \tag{2.4}$$

To construct the three-point amplitudes one defines the auxiliary three-index tensors T and W:

$$\begin{aligned} T^{mnp}(k_1,k_2,k_3) \;&\equiv\; \eta^{mn}\,k_{12}^p \;+\; \eta^{np}\,k_{23}^m \;+\; \eta^{pm}\,k_{31}^n\;, \\ W^{mnp}(k_1,k_2,k_3) \;&\equiv\; \tfrac{1}{8}\,\alpha'\,k_{23}^m\,k_{31}^n k_{12}^p\;, \end{aligned} \tag{2.5}$$

with $k_{ab} = k_a - k_b$. Note the invariance of T and W under simultaneous cyclic shifts of the spacetime indices and the $1, 2, 3$ labels. For bosonic and heterotic strings the on-shell amplitudes for three massless closed string states with polarizations $e_{a\,mn}$ are given by (see, for example, eqn.(6.6.19) in [32] and eqn. (12.4.14) in [33]):

$$\begin{aligned} S_{bos} \;&=\; \tfrac{i}{2}\kappa\,(2\pi)^D\delta^D(\textstyle\sum p)\,e_{1mm'}e_{2nn'}e_{3pp'}\;(T+W)^{mnp}\,(T+W)^{m'n'p'}\,, \\ S_{het} \;&=\; \tfrac{i}{2}\kappa\,(2\pi)^D\delta^D(\textstyle\sum p)\,e_{1mm'}e_{2nn'}e_{3pp'}\;(T+W)^{mnp}\,T^{m'n'p'}\,. \end{aligned} \tag{2.6}$$

Note the factorization of the amplitude into a factor that involves the first indices on the polarization tensors and a factor that involves the second indices on the polarization tensors.[1] We claim that in HSZ theory the on-shell amplitudes also factorize:

$$S_{hsz} \;=\; \tfrac{i}{2}\kappa\,(2\pi)^D\delta^D(\textstyle\sum p)\, e_{1mm'}e_{2nn'}e_{3pp'}\;(T+\,W)^{mnp}\,(T-W)^{m'n'p'}\,. \tag{2.7}$$

For the bosonic string $(T+W)^{mnp}\,(T+W)^{m'n'p'}$ is symmetric under the simultaneous exchange of primed and unprimed indices. As a result, the amplitude for any odd number of b fields vanishes. Expanding out

$$S_{bos} \;=\; \tfrac{i}{2}\kappa\,(2\pi)^D\delta^D(\textstyle\sum p)\, e_{1mm'}e_{2nn'}e_{3pp'}\Big(T^{mnp}T^{m'n'p'}+[W^{mnp}T^{m'n'p'}+W^{m'n'p'}T^{mnp}]+W^{mnp}W^{m'n'p'}\Big)\,, \tag{2.8}$$

making clear the separation into two-, four-, and six-derivative structures, all of which are separately invariant under the simultaneous exchange of primed and unprimed indices. The four-derivative structure indicates the presence of Riemann-squared or Gauss-Bonnet terms [26, 27]. The six-derivative structure implies the presence of Riemann-cubed terms [29]. For the heterotic string we write the amplitude as

$$\begin{aligned} S_{het} \;=\; \tfrac{i}{2}\kappa\,(2\pi)^D\delta^D(\textstyle\sum p)\, e_{1mm'}e_{2nn'}e_{3pp'}\,\Big(\,&T^{mnp}T^{m'n'p'}\;+\tfrac{1}{2}\,[W^{mnp}T^{m'n'p'}\,+\,W^{m'n'p'}T^{mnp}] \\ &+\tfrac{1}{2}[W^{mnp}T^{m'n'p'}\,-\,W^{m'n'p'}T^{mnp}]\Big)\,. \end{aligned} \tag{2.9}$$

We have split the four-derivative terms into a first group, symmetric under the simultaneous exchange of primed and unprimed indices, and a second group, antisymmetric under the simultaneous exchange of primed and unprimed indices. The first group is one-half of the four-derivative terms in bosonic string theory, a well-known result. The second group represents four-derivative terms that can only have an odd number of b fields. In fact, only one b-fieldis allowed. The term with three b fields would have to be of the form $HH\partial H$, with $H=db$ and it can be shown to vanish by Bianchi identities. The term that one gets is of the form $H\Gamma\partial\Gamma$, and arises from the kinetic term of the Chern-Simons corrected b-field field strength. This kind of term also appears in HSZ theory, as discussed in [34].

[1] The on-shell conditions satisfied by the momenta imply that there are no candidates for three-point amplitudes with more than six derivatives.

Expanding the HSZ amplitude above one finds

$$S_{hsz} = \tfrac{i}{2}\kappa\,(2\pi)^D\delta^D(\sum p)\, e_{1mm'}e_{2nn'}e_{3pp'}\Big(T^{mnp}T^{m'n'p'}+[W^{mnp}T^{m'n'p'}-T^{mnp}W^{m'n'p'}]-W^{mnp}W^{m'n'p'}\Big)\,, \tag{2.10}$$

implying that there is no Gauss-Bonnet term, that the term with four derivatives has a single b-field and is the same as in heterotic string but with twice the magnitude. The six-derivative term is the same as in bosonic string, but with opposite sign. This implies that the Riemann-cubed term in the HSZ action and in bosonic strings have opposite signs. Most of the work in the rest of the chapter deals with the computation of the g and b three-point amplitudes that confirms (2.10) holds.

It is useful to have simplified expressions for the amplitudes. For later use we record the following results, with 'cyc.' indicating that two copies of the terms to the left must be added with cyclic permutations of the 1,2, and 3 labels:

$$\begin{aligned} e_{1mm'}e_{2nn'}e_{3pp'}T^{mnp}T^{m'n'p'} &= \text{tr}(e_1^Te_2)(k_{12}e_3k_{12}) + k_{12}(e_3e_2^Te_1 + e_3^Te_2e_1^T)k_{23} + \text{cyc.} \\ e_{1mm'}e_{2nn'}e_{3pp'}(W^{mnp}T^{m'n'p'} \pm T^{mnp}W^{m'n'p'}) &= \tfrac{1}{8}\alpha'\left[k_{12}(e_3e_1^T \pm e_3^Te_1)k_{23}(k_{31}e_2k_{31}) + \text{cyc.}\right], \\ e_{1mm'}e_{2nn'}e_{3pp'}W^{mnp}W^{m'n'p'} &= \tfrac{1}{64}\alpha'^2(k_{12}e_3k_{12})(k_{23}e_1k_{23})(k_{31}e_2k_{31})\,. \end{aligned} \tag{2.11}$$

The formulae (2.6) for massless on-shell three-point amplitudes also hold for amplitudes that involve the dilaton. For the dilaton one must use a polarization tensor proportional to the Minkowski metric. Although we will not use the HSZ action to compute dilaton amplitudes, the predictions from the factorized amplitude (2.7) are exactly what we expect for the the dilaton. We explain this now.

Let $\hat{\phi}$ denote the physical dilaton field. For cubic dilaton interactions $\hat{\phi}^3$ there is no on-shell candidate at two, four, or six derivatives. For $\hat{\phi}^2 e$ interactions there is no on-shell candidate at four or six derivatives, but there is one at two derivatives: $\partial^m\hat{\phi}\partial^n\hat{\phi}\, e_{mn} \sim \partial^m\hat{\phi}\partial^n\hat{\phi}\, h_{mn}$. This term does arise from the first line in (2.11) when we take $e_{1mm'} \sim \eta_{mm'}\hat{\phi}$, $e_{2nn'} \sim \eta_{nn'}\hat{\phi}$, and $e_{3pp'} = h_{pp'}$. It is present in all three theories as it is the universal coupling of a scalar to gravity.

For $\hat{\phi}ee$ there are no on-shell candidates with six derivatives, but there are candidates with two and with four derivatives. Let's consider first the on-shell candidates with two derivatives. Again, an examination of the first line in (2.11) shows that $\hat{\phi}hh$ vanishes. This is expected: the physical dilaton does not couple to the scalar curvature. There is also no $\hat{\phi}hb$ coupling. On the other hand one can check that $\hat{\phi}bb$ does not vanish. This is also expected, as an exponential of $\hat{\phi}$ multiplies the b-field kinetic term. Again, all this is valid for the three theories.

We now consider $\hat{\phi}ee$ on-shell couplings with four derivatives. There is one on-shell candidate: $\hat{\phi}\partial^{mn}e_{pq}\partial^{pq}e_{mn}$. Due to the commutativity of derivatives this term requires both e's to be gravitons. This coupling arises both in bosonic and heterotic string theory because an exponential of $\hat{\phi}$ multiplies Riemann-squared terms. As expected, it can be seen from the second line in (2.11), using the top sign. It does not arise in HSZ theory because in this theory the four-derivative terms are odd under the $\mathbb{Z}_2$ transformation $b \to -b$ [30], and thus must involve a b field. In conclusion, HSZ theory only has on-shell couplings of dilatons at two derivatives, and shares them with heterotic and bosonic strings. The latter two have a single on-shell coupling of the dilaton at four derivatives. These are indeed the predictions of the three factorized formulae.

2.2 Derivative expansion of HSZ theory

Our first goal is to give the action for $\mathcal{M}$ and ϕ in explicit form and organized by the number of derivatives, a number that can be zero, two, four, and six. While the parts with zero and two derivatives are known and take relatively simple forms [2, 30], the parts with four and six derivatives are considerably longer. We give their partially simplified forms and then their fully simplified forms when the dilaton field is set to zero. This will suffice for our later computation of on-shell three-point amplitudes for gravity and b-field fluctuations.

We will define actions S as integrals over the double coordinates of the density e^{ϕ} times the Lagrangian L. For the theory in question [2] we have

$$S = \int e^{-2\Phi} L\,, \qquad L \;=\; \tfrac{1}{2}\text{tr}(\mathcal{T}) \;-\; \tfrac{1}{6}\,\langle \mathcal{T}|\mathcal{T}\star\mathcal{T}\rangle\,. \tag{2.12}$$

The field $\mathcal{T}$ is a tensor operator and encodes the double metric. For arbitrary tensor operators T we have the expansion

$$T \;=\; \tfrac{1}{2}T^{MN}Z_M Z_N - \tfrac{1}{2}(\hat{T}^M Z_M)'\,, \tag{2.13}$$

here T^{MN} and $\hat{T}^M$ are, respectively, the tensor part and the pseudo-vector part of the tensor operator. The trace of the tensor operator T is a scalar operator $\text{tr}\,T$ defined by (eqn.(5.17), [2])

$$\text{tr}\,T \;\equiv\; \eta^{MN}T_{MN} + 6(T^{MN}\partial_M\partial_N\Phi - \tfrac{1}{2}\partial\cdot\hat{T} + \hat{T}\cdot\partial\Phi)\,. \tag{2.14}$$

If a tensor operator T is divergenceless, the pseudo-vector part is determined as a dilaton dependent function

G linear in the tensor component:

$$\hat{T}^M = G^M(T_{PQ}) = G_1^M(T) + G_3^M(T), \tag{2.15}$$

where G_1 and G_3 have one and three derivatives, respectively (eqn.(5.37), [2]):

$$\begin{aligned} G_1^M(T) &= \partial_N T^{MN} - 2T^{MN}\partial_N\Phi \\ G_3^M(T) &= T^{NP}\partial_N\partial_P\partial^M\Phi - \tfrac{1}{2}\partial^M\big(\partial_N\partial_P T^{NP} - 2T^{NP}(\partial_N\partial_P\Phi - 2\partial_N\Phi\partial_P\Phi) - 4\partial_N\, T^{NP}\,\partial_P\Phi\big)\,. \end{aligned} \tag{2.16}$$

We make the following remarks:

1. The free index on G_3 is carried by a derivative.
2. $G_1(T)$ and $G_3(T)$ both vanish if the two indices in T_{MN} are carried by derivatives,
3. $G_3(T)$ vanishes if one index on T_{MN} is carried by a derivative.

The tensor operator $\mathcal{T}$ featuring in the action is parametrized by a double metric $\mathcal{M}^{MN}$, and the pseudo-vector part $\hat{\mathcal{M}}^M$ is determined by the condition that $\mathcal{T}$ is divergenceless:

$$\mathcal{T} = \tfrac{1}{2}\mathcal{M}^{MN}Z_MZ_N - \tfrac{1}{2}(\hat{\mathcal{M}}^N Z_M)', \quad \hat{\mathcal{M}}^M = G^M(\mathcal{M})\,. \tag{2.17}$$

A short calculation gives

$$\mathrm{tr}\mathcal{T} = \eta^{MN}\mathcal{M}_{MN} - 3\partial_M\partial_N\mathcal{M}^{MN} + 12\mathcal{M}^{MN}\partial_M\partial_N\Phi + 12\partial_M\mathcal{M}^{MN}\partial_N\Phi - 12\mathcal{M}^{MN}\partial_M\Phi\partial_N\Phi\,, \tag{2.18}$$

which contains terms linear, quadratic and cubic in fields, and no more than two derivatives. We now use the star product $\star$ of two tensors, which gives a divergenceless tensor, to define

$$W \equiv \mathcal{T}\star\mathcal{T} = \tfrac{1}{2}W^{MN}Z_MZ_N - \tfrac{1}{2}\big(\hat{W}^M Z_M\big)', \tag{2.19}$$

where the last equality defines the components of W. The definition of the star product ([2], sect.6.2) implies that

$$W^{MN} \equiv (\mathcal{T}\circ_2\mathcal{T})^{MN}, \quad W^M \equiv G^M(W^{PQ}), \tag{2.20}$$

the second following because W is divergenceless. The formula for product $\circ_2$ is given in (6.67) of [2].[2] The field W^{MN} has an expansion on derivatives,

$$W^{MN} \;=\; W_0^{MN} + W_2^{MN} + W_4^{MN} + W_6^{MN}\,, \tag{2.21}$$

which, using the notation $\partial_{M_1\cdots M_k} \equiv \partial_{M_1}\cdots\partial_{M_k}$, takes the form

$$\begin{aligned}
W_{0MN} \;&=\; 2\mathcal{M}_{MK}\mathcal{M}^K{}_N\,,\\
W_{2MN} \;&=\; -\tfrac{1}{2}\partial_M\mathcal{M}^{PQ}\partial_N\mathcal{M}_{PQ} + \mathcal{M}^{PQ}\partial_{PQ}\mathcal{M}_{MN} + 4\partial_{(M}\mathcal{M}^{KL}\partial_L\mathcal{M}_{N)K}\\
&\quad - 2\partial_Q\mathcal{M}_M{}^P\partial_P\mathcal{M}_N{}^Q + G_1^K(\mathcal{M})\partial_K\mathcal{M}_{MN} + 2\big(\partial_{(N}G_1^K(\mathcal{M}) - \partial^K G_{1(N}(\mathcal{M})\big)\mathcal{M}_{M)K}\,,\\
W_{4MN} \;&=\; \partial_{MP}\mathcal{M}^{LK}\partial_{NL}\mathcal{M}_K{}^P - 2\partial_{K(M}\mathcal{M}^{PQ}\partial_{PQ}\mathcal{M}_{N)}{}^K\\
&\quad + 2\big(\partial_{(M}G_3^K(\mathcal{M}) - \partial^K G_{3(M}(\mathcal{M})\big)\mathcal{M}_{N)K} - 2\partial_{P(M}G_1^K(\mathcal{M})\partial_K\mathcal{M}^P{}_{N)}\\
&\quad + \partial_P\big(\partial_{(M}G_{1Q}(\mathcal{M}) - \partial_Q G_{1(M}(\mathcal{M})\big)\partial_{N)}\mathcal{M}^{PQ}\,,\\
W_{6MN} \;&=\; -\tfrac{1}{4}\partial_{MPQ}\mathcal{M}^{KL}\partial_{NKL}\mathcal{M}^{PQ} + \partial_P\big(\partial_{(M}G_{3Q}(\mathcal{M}) - \partial_Q G_{3(M}(\mathcal{M})\big)\partial_{N)}\mathcal{M}^{PQ}\\
&\quad - \tfrac{1}{2}\partial_{PQ(M}G_1^K(\mathcal{M})\partial_{N)K}\mathcal{M}^{PQ}\,.
\end{aligned} \tag{2.22}$$

We note that

1. On W_{4MN} at least one index is carried by a derivative.

2. On W_{6MN} both indices are carried by derivatives.

We now turn to the pseudo-vector components $\hat{W}^K$ which, by definition are given by

$$\hat{W}^K \;=\; G^K(W_{MN}) \;=\; G_1^K(W_0 + W_2 + W_4 + W_6) + G_3^K(W_0 + W_2 + W_4 + W_6)\,. \tag{2.23}$$

It then follows by the remarks that the only terms in W^K are:

$$\begin{aligned}
\hat{W}_1^K \;&=\; G_1^K(W_0)\,,\\
\hat{W}_3^K \;&=\; G_1^K(W_2) \;+ G_3^K(W_0)\,,\\
\hat{W}_5^K \;&=\; G_1^K(W_4) \;+ G_3^K(W_2)\,.
\end{aligned} \tag{2.24}$$

[2] In [2] symmetrizations or antisymmetrizations carry no weight, in this chapter they do.

These are terms with one, three, and five derivatives. Note that on $G_1(W_4)$ the free index is on a derivative because it is an index on W_4 and the other index on W_4 must be the non-derivative one to have a non vanishing contribution. Thus the free index in $\hat{W}_5$ is on a derivative.

It is now possible to evaluate the full Lagrangian in (2.12). For the cubic term we need the inner product formula that follows from eqn.(6.67) of [2]

$$\begin{aligned}\langle T_1|T_2\rangle \;=\;& \tfrac{1}{2}T_1^{PQ}T_{2PQ} - \partial_P T_1^{KL}\partial_L T_{2K}{}^P + \tfrac{1}{4}\partial_{PQ}T_1^{KL}\partial_{KL}T_2^{PQ} \\ & - \tfrac{3}{2}(\hat{T}_1^M\hat{T}_2^N\eta_{MN} - \partial_N\hat{T}_1^M\partial_M\hat{T}_1^N) - \tfrac{3}{2}(\partial_P\hat{T}_1^K T_{2K}{}^P + \partial_P\hat{T}_2^K T_{1K}{}^P) \\ & + \tfrac{3}{4}\big(\partial_{PQ}\hat{T}_1^K\partial_K T_2^{PQ} + \partial_{PQ}\hat{T}_2^K\partial_K T_1^{PQ}\big)\,.\end{aligned} \tag{2.25}$$

This formula must be used for $T_1 = \mathcal{T}$ and $T_2 = W$. A useful identity, easily derived by integration by parts, reads

$$\int e^{-2\Phi}\, f^K G_{(1)K}(T) \;=\; \int e^{-2\Phi}(-\partial^P f^K\, T_{KP})\,. \tag{2.26}$$

Using this identity and the earlier results we find the following terms in the Lagrangian

$$\begin{aligned} L_0 \;=\;& -\tfrac{1}{6}\mathcal{M}_{MN}\mathcal{M}^{NP}\mathcal{M}_P{}^M \; + \tfrac{1}{2}\mathcal{M}_M{}^M\,, \\ L_2 \;=\;& -\big(\mathcal{M}^2-1\big)^{MP}\mathcal{M}_P{}^N\partial_M\partial_N\Phi \; + \tfrac{1}{8}\mathcal{M}^{MN}\partial_M\mathcal{M}^{PQ}\partial_N\mathcal{M}_{PQ} \\ & -\tfrac{1}{2}\mathcal{M}^{MN}\partial_N\mathcal{M}^{KL}\partial_L\mathcal{M}_{KM} \; + \mathcal{M}^{MN}\partial_M\partial_N\Phi\,, \\ L_4 \;=\;& -\tfrac{1}{12}\,\mathcal{M}^{MN}W_{4MN} \; + \tfrac{1}{6}\,\partial^P\mathcal{M}^{KL}\partial_L W_{2KP} + \tfrac{1}{4}\partial_P G_1^K(W_2)\mathcal{M}_K{}^P \\ & -\tfrac{1}{24}\partial_P\partial_Q\mathcal{M}^{KL}\partial_K\partial_L W_0^{PQ} \\ & -\tfrac{1}{4}\partial_N G_1^M(\mathcal{M})\partial_M G_1^N(W_0) - \tfrac{1}{8}\partial_P\partial_Q G_1^K(\mathcal{M})\partial_K W_0^{PQ} - \tfrac{1}{8}\partial_P\partial_Q G_1^K(W_0)\partial_K\mathcal{M}^{PQ}, \\ L_6 \;=\;& -\tfrac{1}{12}\,\mathcal{M}^{MN}W_{6MN} \; + \tfrac{1}{6}\,\partial^P\mathcal{M}^{KL}\partial_L W_{4KP} + \tfrac{1}{4}\partial_P G_1^K(W_4)\mathcal{M}_K{}^P \\ & -\tfrac{1}{24}\partial_P\partial_Q\mathcal{M}^{KL}\partial_K\partial_L W_{(2)}^{PQ} \\ & -\tfrac{1}{4}\partial_N G_1^M(\mathcal{M})\partial_M G_1^N(W_2) - \tfrac{1}{8}\partial_P\partial_Q G_1^K(\mathcal{M})\partial_K W_2^{PQ} - \tfrac{1}{8}\partial_P\partial_Q G_1^K(W_2)\partial_K\mathcal{M}^{PQ}\,. \end{aligned} \tag{2.27}$$

The results for the zero and two-derivative part of the Lagrangian were given in [2, 30] and cannot be simplified further. One can quickly show that the last line of L_4 and L_6 vanish if we have zero dilaton derivatives. Also the last two terms in the first lines of L_4 and L_6 admit simplification. Still keeping all terms, we can simplify L_4 and L_6 to read

$$
\begin{aligned}
L_4 \;=\;& -\tfrac{1}{12}\mathcal{M}^{MN}W_{4MN} + \tfrac{1}{12}\partial^M G_1^N(\mathcal{M})W_{2MN} - \tfrac{1}{3}\mathcal{M}^{MK}W_{2K}{}^N\partial_{MN}\Phi \\
& -\tfrac{1}{24}\partial_P\partial_Q\mathcal{M}^{KL}\partial_K\partial_L W_0^{PQ} \\
& -\tfrac{1}{2}G_1^M(\mathcal{M})G_1^N(W_0)\partial_{MN}\Phi + \tfrac{1}{8}\big(G_1^K(\mathcal{M})W_0^{PQ} - 2G_1^K(W_0)\mathcal{M}^{PQ}\big)\partial_{KPQ}\Phi, \\
L_6 \;=\;& -\tfrac{1}{12}\mathcal{M}^{MN}W_{6MN} + \tfrac{1}{12}\partial^M G_1^N(\mathcal{M})W_{4MN} - \tfrac{1}{3}\mathcal{M}^{MK}W_{4K}{}^N\partial_{MN}\Phi \\
& -\tfrac{1}{24}\partial_P\partial_Q\mathcal{M}^{KL}\partial_K\partial_L W_2^{PQ} \\
& -\tfrac{1}{2}G_1^M(\mathcal{M})G_1^N(W_2)\partial_{MN}\phi + \tfrac{1}{8}\big(G_1^K(\mathcal{M})W_2^{PQ} - 2G_1^K(W_2)\mathcal{M}^{PQ}\big)\partial_{KPQ}\Phi\,.
\end{aligned}
\tag{2.28}
$$

The fourth and sixth derivative part of the Lagrangian, written explicitly in terms of $\mathcal{M}$ and Φ are rather long. Since we will focus in this chapter on gravity and b-field three-point amplitudes, we will ignore the dilaton. With dilaton fields set to zero a computation gives:

$$
\begin{aligned}
L_4|_{\Phi=0} \;=\;& \mathcal{M}^{MN}\Big(\tfrac{1}{6}\partial_{ML}\mathcal{M}^{PQ}\partial_{PQ}\mathcal{M}_N{}^L - \tfrac{1}{12}\partial_{NP}\mathcal{M}^{LQ}\partial_{ML}\mathcal{M}^P{}_Q \\
& \quad + \tfrac{1}{12}\partial_{MN}\mathcal{M}_{KQ}\partial_P^K\mathcal{M}^{PQ} - \tfrac{1}{12}\partial_{MP}\mathcal{M}^{PQ}\partial_{NK}\mathcal{M}_Q{}^K \\
& \quad + \tfrac{1}{3}\partial_P\mathcal{M}_M{}^K\partial_{NKQ}\mathcal{M}^{PQ} - \tfrac{1}{6}\partial_M\mathcal{M}^{PQ}\partial_{PK[N}\mathcal{M}_{Q]}{}^K\Big) \\
& + \partial^{MP}\mathcal{M}_P{}^N\big(\tfrac{1}{6}\partial_N\mathcal{M}^{KL}\partial_L\mathcal{M}_{MK} - \tfrac{1}{6}\partial_Q\mathcal{M}_M{}^K\partial_K\mathcal{M}_N{}^Q + \tfrac{1}{12}\partial_L\mathcal{M}^{KL}\partial_K\mathcal{M}_{MN}\big), \\
L_6|_{\Phi=0} \;=\;& \mathcal{M}^{MN}\big(\tfrac{1}{48}\partial_{MPQ}\mathcal{M}^{KL}\partial_{NKL}\mathcal{M}^{PQ} + \tfrac{1}{24}\partial_{MPQL}\mathcal{M}^{KL}\partial_{NK}\mathcal{M}^{PQ} \\
& \quad - \tfrac{1}{24}\partial_{PQKL}\mathcal{M}^{KL}\partial_{MN}\mathcal{M}^{PQ} + \tfrac{1}{12}\partial_{MPKL}\mathcal{M}^{KL}\partial_{NQ}\mathcal{M}^{PQ}\big) \\
& - \tfrac{1}{24}\partial_{MNKL}\mathcal{M}^{KL}\big(\partial_P\mathcal{M}^{PQ}\partial_Q\mathcal{M}^{MN} - 2\partial_P\mathcal{M}^{MQ}\partial_Q\mathcal{M}^{NP}\big) \\
& - \tfrac{1}{24}\partial_{NL}\mathcal{M}^{ML}\big(2\partial_{MK}\mathcal{M}^{PQ}\partial_{PQ}\mathcal{M}_N{}^K + 2\partial_{MPQ}\mathcal{M}^{KQ}\partial_K\mathcal{M}^P{}_N + \partial_{PQR}\mathcal{M}^{NR}\partial_M\mathcal{M}^{PQ}\big).
\end{aligned}
\tag{2.29}
$$

2.3 Perturbative expansion of HSZ theory

In this section we discuss the perturbative expansion of the Lagrangian obtained in the previous section around a *constant* background $\langle\mathcal{M}\rangle$ that can be identified with a constant generalized metric, as discussed in

[30]. We define projected $O(D,D)$ indices as follows:

$$V_{\underline{M}} = P_M{}^N V_N, \; V_{\bar{M}} = \bar{P}_M{}^N V_N, \tag{2.30}$$

where the projectors are defined as:

$$P_M{}^N = \tfrac{1}{2}(\eta - \bar{\mathcal{H}})_M{}^N, \quad \bar{P}_M{}^N = \tfrac{1}{2}(\eta + \bar{\mathcal{H}})_M{}^N. \tag{2.31}$$

Here $\bar{\mathcal{H}}$ is the background, constant, generalized metric. We expand the double metric $\mathcal{M}$ as follows:

$$\mathcal{M}_{MN} = \bar{\mathcal{H}}_{MN} + m_{MN} = \bar{\mathcal{H}}_{MN} + m_{\underline{M}\underline{N}} + m_{\underline{M}\bar{N}} + m_{\bar{M}\underline{N}} + m_{\bar{M}\bar{N}}, \tag{2.32}$$

where we have decomposed the fluctuations m_{MN} into projected indices. The physical part of the metric and the b-field fluctuations are encoded in $m_{\underline{M}\bar{N}} = m_{\bar{N}\underline{M}}$. The projections $m_{\bar{M}\bar{N}}$ and $m_{\underline{M}\underline{N}}$ can be treated as auxiliary fields as far as the three-point amplitudes graviton and b-field are concerned. This is essentially due to that tree level three-point amplitudes come from contact vertices and there are no propagators involved. Note that this will not be true for higher point or loop amplitudes. To obtain the Lagrangian in terms of physical fields, we need to expand it in fluctuations and then eliminate the auxiliary fields using their equations of motion. To illustrate this procedure more clearly, and for ease of readability we will write

$$a_{\underline{M}\underline{N}} \equiv m_{\underline{M}\underline{N}}, \qquad a_{\bar{M}\bar{N}} \equiv m_{\bar{M}\bar{N}}, \tag{2.33}$$

where the label a for the field reminds us that it is auxiliary. With this notation the $\mathcal{M}$ field expansion reads

$$\mathcal{M}_{MN} = \bar{\mathcal{H}}_{MN} + a_{\underline{M}\underline{N}} + m_{\underline{M}\bar{N}} + m_{\bar{M}\underline{N}} + a_{\bar{M}\bar{N}}. \tag{2.34}$$

We now argue that for the amplitudes that we are interested in, we can simply set the a fields to zero.

2.3.1 Treatment of auxiliary fields

Here we argue that for the purposes of three-point on-shell amplitudes and, with the dilaton set to zero, the auxiliary field does not affect the Lagrangian and can safely be ignored. To prove the claim we must use on-shell conditions (2.51): we will argue that any contribution from auxiliary fields vanishes upon use of these conditions. It is straightforward to translate these on-shell conditions in terms of the double metric fluctuations. They can be written as:

$$\partial_{\underline{M}} m^{\underline{M}\bar{N}} \;=\; \partial_{\bar{N}} m^{\underline{M}\bar{N}} \;=\; \partial_{\bar{M}} m^{\cdot\cdot} \partial^{\bar{M}} m^{\cdot\cdot} \cdots \;=\; 0\,. \tag{2.35}$$

Setting all dilatons to zero, the only physical field is $m_{\underline{M}\bar{N}}$, which we symbolically represent by m. The most general form of the Lagrangian involving at least one auxiliary field is as follows:

$$L[a,m] \;=\; am \;+a^2 \;+a^3 \;+a^2 m \;+am^2\,. \tag{2.36}$$

Since the theory is cubic in $\mathcal{M}$ and the dilaton is set to zero, this is all there is. In here we are leaving derivatives implicit; all the above terms can carry up-to six derivatives. We now show that there is no 'am' term which does not vanish using the on-shell conditions. The general term of this kind would be

$$a_{\bar{M}\bar{N}}\big(\cdots m_{\underline{P}\bar{Q}}\big)\,, \tag{2.37}$$

where the dots represent derivatives or metrics η that contract same type indices, barred or un-barred. These are required to contract all indices and yield an $O(D,D)$ invariant. Since integration by parts is allowed we have assumed, without loss of generality that all derivatives are acting on the physical field. Since the un-barred index P is the only un-barred index, it must be contracted with a derivative. Thus the term must be of the form

$$a_{\bar{M}\bar{N}}\big(\cdots \partial^{\underline{P}} m_{\underline{P}\bar{Q}}\big)\,. \tag{2.38}$$

Regardless of what we do to deal with the other barred indices, we already see that this coupling vanishes using the on-shell conditions, proving the claim.

The Lagrangian (2.36) then reduces to the following:

$$L[a,m] \;=\; a^2 \;+a^3 \;+a^2 m \;+am^2\,. \tag{2.39}$$

The equation of motion for the auxiliary field is, schematically, $a \sim m^2 + am + a^2$, which implies that a perturbative solution in powers of physical fields begins with terms quadratic on the physical fields. Thus we write

$$a(m) \;=\; a_2(m) \;+a_3(m) + \cdots\,, \tag{2.40}$$

where dots indicate terms with quartic or higher powers of m. But now it is clear that substitution back into (2.39) can only lead to terms with quartic or higher powers of m. This concludes our argument that the elimination of auxiliary fields is not required for the computation of on-shell three-point amplitudes for metric and b fields. Note however that this argument is only valid for tree level on-shell three-point amplitudes. For higher point amplitudes and/or loop amplitudes auxiliary fields need to be dealt with appropriately.

2.3.2 Perturbative expansion of the two-derivative Lagrangian

We use $L^{(i,j)}$ to denote the part of the Lagrangian with i fields and j derivatives. In what follows, we are only interested in the Lagrangian up to cubic order in fields, so we will ignore all terms with more than three fields. Also note that the Langrangian appears in the action multiplied with a factor of $e^{-2\Phi}$. Using the expansion (2.34) we see that the zero derivative Lagrangian L_0 has terms quadratic and cubic in field fluctuations:

$$e^{-2\Phi} L_0 \;=\; L^{(2,0)} + L^{(3,0)} + \cdots\,, \tag{2.41}$$

where the dots denote terms quartic in fields and

$$\begin{aligned} L^{(2,0)} &= \tfrac{1}{2}\, a^{\underline{M}\underline{N}} a_{\underline{M}\underline{N}} - \tfrac{1}{2}\, a^{\bar{M}\bar{N}} a_{\bar{M}\bar{N}}, \\ L^{(3,0)} &= -\tfrac{1}{2}\, a^{\underline{M}\underline{N}} m_{\underline{M}}{}^{\bar{P}} m_{\underline{N}\bar{P}} - \tfrac{1}{6}\, a^{\underline{M}\underline{N}} a_{\underline{M}}{}^{\underline{P}} a_{\underline{N}\underline{P}} - \tfrac{1}{2}\, a^{\bar{M}\bar{N}} m^{\underline{P}}{}_{\bar{M}} m_{\underline{P}\bar{N}} - \tfrac{1}{6}\, a^{\bar{M}\bar{N}} a_{\bar{M}}{}^{\bar{P}} a_{\bar{N}\bar{P}} \\ &\quad - \Phi \left(a^{\underline{M}\underline{N}} a_{\underline{M}\underline{N}} - a^{\bar{M}\bar{N}} a_{\bar{M}\bar{N}} \right). \end{aligned} \tag{2.42}$$

Here we can explicitly see that solving for a in terms of m and plugging back into the action gives $\mathcal{O}\left(m^4\right)$ or higher order terms.

The perturbative expansion for the two-derivative Lagrangian L_2 in (2.27) is more involved but the same conclusion follows. It decomposes into a quadratic and a cubic part in fluctuations:

$$e^{-2\Phi} L_2 \;=\; L^{(2,2)} + L^{(3,2)} + \cdots\,. \tag{2.43}$$

and we find

$$
\begin{aligned}
L^{(2,2)} &= \tfrac{1}{2}\partial^{\bar{M}} m^{\underline{P}\bar{Q}} \partial_{\bar{M}} m_{\underline{P}\bar{Q}} + \tfrac{1}{2}\partial^{\underline{M}} m^{\underline{P}\bar{Q}} \partial_{\underline{P}} m_{\underline{M}\bar{Q}} - \tfrac{1}{2}\partial^{\bar{M}} m^{\underline{P}\bar{Q}} \partial_{\bar{Q}} m_{\underline{P}\bar{M}} \\
&\quad + 4 m^{\underline{M}\bar{N}} \partial_{\underline{M}} \partial_{\bar{N}} \Phi - 8\Phi\, \partial^{\bar{M}} \partial_{\bar{M}} \Phi \\
&\quad + \tfrac{1}{4}\partial^{\bar{M}} a^{\bar{P}\bar{Q}}\, \partial_{\bar{M}} a_{\bar{P}\bar{Q}} \; + \; \tfrac{1}{4}\partial^{\bar{M}} a^{\underline{P}\underline{Q}}\, \partial_{\bar{M}} a_{\underline{P}\underline{Q}} \\
&\quad + \tfrac{1}{2}\partial^{\underline{M}} a^{\underline{P}\underline{Q}}\, \partial_{\underline{Q}} a_{\underline{P}\underline{M}} \; - \tfrac{1}{2}\partial^{\bar{M}} a^{\bar{P}\bar{Q}}\, \partial_{\bar{Q}} a_{\bar{P}\bar{M}} \,, \\
L^{(3,2)} &= \tfrac{1}{2} m^{\underline{M}\bar{N}} \big(\partial_{\underline{M}} m^{\underline{P}\bar{Q}} \partial_{\bar{N}} m_{\underline{P}\bar{Q}} - \partial_{\underline{M}} m^{\underline{P}\bar{Q}} \partial_{\bar{Q}} m_{\underline{P}\bar{N}} - \partial_{\bar{N}} m^{\underline{P}\bar{Q}} \partial_{\underline{P}} m_{\underline{M}\bar{Q}}\big) \\
&\quad - \Phi\big(\partial_{\bar{M}} m^{\underline{P}\bar{Q}} \partial^{\bar{M}} m_{\underline{P}\bar{Q}} \; - \partial_{\bar{M}} m^{\underline{P}\bar{Q}} \partial_{\bar{Q}} m_{\underline{P}}{}^{\bar{M}} \; + \partial^{\underline{M}} m^{\underline{P}\bar{Q}} \partial_{\underline{P}} m_{\underline{M}\bar{Q}}\big) \\
&\quad + \big(m_{\underline{M}}{}^{\bar{P}} m_{\underline{N}\bar{P}} \partial^{\underline{M}} \partial^{\underline{N}} \Phi - m^{\underline{P}}{}_{\bar{M}} m_{\underline{P}\bar{N}} \partial^{\bar{M}} \partial^{\bar{N}} \Phi\big) \\
&\quad - 8\Phi^2 \partial_{\bar{M}} \partial^{\bar{M}} \Phi \; - 8\Phi m^{\underline{M}\bar{N}} \partial_{\underline{M}} \partial_{\bar{N}} \Phi \; + \; L^{(3,2)}_{\text{aux}} \,,
\end{aligned}
\tag{2.44}
$$

where $L^{(3,2)}_{\text{aux}}$ denotes the terms that contain at least one a field. Next, we eliminate the auxiliary fields to obtain the two-derivative Lagrangian which is almost cubic in fields.

$$
L^{(\leq 3,2)} \; = \; L^{(2,0)} + L^{(3,0)} + L^{(2,2)} + L^{(3,2)}\Big|_{a=0} . \tag{2.45}
$$

Next, we write the action in terms of DFT (or string field theory) variables e_{mn}. The way to translate from $m_{\underline{M}\bar{N}}$ variables to e_{mn} variables is explained in Sec. 5.3 of [30]. Here is the rule that follows:

- Replace $m_{\underline{M}\bar{N}}$ by e_{mn}, $a_{\underline{M}\underline{N}}$ by a_{mn} and $a_{\bar{M}\bar{N}}$ by $\bar{a}_{mn}$.

- Replace under-barred derivatives by D and barred derivatives by $\bar{D}$ defined as in [30],

$$
D_m \; = \; \partial_m - E_{mn} \tilde{\partial}^n \,, \qquad \bar{D}_m \; = \; \partial_m \, + E_{nm} \tilde{\partial}^n \,, \tag{2.46}
$$

  where $E_{mn} = G_{mn} \, + B_{mn}$ is given in terms of the constant background metric and the b-field. The strong constraint takes the form $D^m D_m = \bar{D}^m \bar{D}_m$, acting on arbitrary fields and all their products.

- Multiply by a coefficient, which is the product of a factor of 2 for each m, a, or $\bar{a}$ field, a factor of $+\frac{1}{2}$ for each barred contraction and a factor of $-\frac{1}{2}$ for each under-barred contraction.

As an example, consider the second term on the first line of $L^{(2,2)}$, after integration by parts, it becomes:

$$\tfrac{1}{2}\partial^{\underline{M}} m^{\underline{P}\bar{Q}} \partial_{\underline{P}} m_{\underline{M}\bar{Q}} = \tfrac{1}{2}\partial_{\underline{P}} m^{\underline{P}\bar{Q}} \partial^{\underline{M}} m_{\underline{M}\bar{Q}} \quad \rightarrow \tfrac{1}{2}\cdot 2^2 \cdot \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^2 D_p e^{pq} D^m e_{mq} = \tfrac{1}{4} D_p e^{pq} D^m e_{mq}. \tag{2.47}$$

Using this technique for all the terms appearing in the Lagrangian (2.45) we obtain:

$$\begin{aligned} L^{(\leq 3,2)} = \; & \tfrac{1}{4}\left(e^{mn}\bar{D}^2 e_{mn} + (D^m e_{mn})^2 + \left(\bar{D}^m e_{mn}\right)^2\right) - 2e^{mn} D_m \bar{D}_n \Phi - 4\Phi \bar{D}^2 \Phi. \\ & + \tfrac{1}{4} e_{mn}\Big(D^m e_{pq} \bar{D}^n e^{pq} - D^m e_{pq} \bar{D}^q e^{pn} - D^p e^{mq} \bar{D}^n e_{pq}\Big) \\ & + \tfrac{1}{2}\Phi\left(\left(D^m e_{mn}\right)^2 + \left(\bar{D}^n e_{mn}\right)^2 + \tfrac{1}{2}\left(D_p e_{mn}\right)^2 + \tfrac{1}{2}\left(\bar{D}_p e_{mn}\right)^2 + 2e^{mn}\left(D_m D^p e_{pn} + \bar{D}_n \bar{D}^p e_{mp}\right)\right) \\ & + 4\Phi e_{mn} D^m \bar{D}^n \Phi + 4\Phi^2 \bar{D}^2 \Phi. \end{aligned} \tag{2.48}$$

This is precisely the DFT Lagrangian in equation (3.25) of [24]. From the quadratic part of the above action, we see that the kinetic term of Φ has wrong sign. This is, because the action (2.48) is in the string frame and Φ is not the physical dilaton. To obtain the action in terms of physical fields $\hat{e}_{mn}$ and $\hat{\phi}$ that decouple at the quadratic level, we need a field re-definition. Physical fields $\hat{e}_{mn}$ and $\hat{\phi}$ are obtained in the Einstein frame as a linear combination of e_{mn} and Φ. We write schematically:

$$e_{mn} \sim \hat{e}_{mn} + \hat{\phi}\,\eta_{mn}, \qquad \Phi \sim \hat{\phi} + \hat{e}_m^{\ m}. \tag{2.49}$$

If we are looking for pure gravitational three-point amplitudes the first redefinition need not be performed in the action, as it would give rise to terms that involve the dilaton. The second one is not needed either, since on-shell gravitons have traceless polarizations.

After solving the strong constraint by setting $\tilde{\partial}^m = 0$ and setting the dilaton to zero, the above Lagrangian becomes:

$$L^{(\leq 3,2)}\Big|_{\Phi=0} = \tfrac{1}{4}\left(e^{mn}\partial^2 e_{mn} + 2\left(\partial^m e_{mn}\right)^2\right) + \tfrac{1}{4} e_{mn}\left(\partial^m e_{pq} \partial^n e^{pq} - \partial^m e_{pq} \partial^q e^{pn} - \partial^p e^{mq} \partial^n e_{pq}\right). \tag{2.50}$$

For an off-shell three-point vertex all terms in the cubic Lagrangian must be kept. But for the computation of *on-shell* three-point amplitudes we may use the on-shell conditions to simplify the cubic Lagrangian. These conditions can be stated as follows in terms of e_{mn}.

$$\partial^m e_{mn} = \partial^n e_{mn} = 0, \qquad \partial_m e^{\cdot\cdot} \partial^m e^{\cdot\cdot} \cdots = 0. \tag{2.51}$$

The first condition is transversality and the second condition follows from the momentum conservation and masslessness. For the cubic terms in (2.50) the on-shell conditions do not lead to any further simplification and we record:

$$L^{(3,2)}\Big|_{\Phi=0,\text{ on-shell}} = \tfrac{1}{4}e_{mn}\left(\partial^m e_{pq}\partial^n e^{pq} - \partial^m e_{pq}\partial^q e^{pn} - \partial^p e^{mq}\partial^n e_{pq}\right). \tag{2.52}$$

Three-point on-shell amplitudes can now be computed from this expression.

2.3.3 Higher-derivative Lagrangian and on-shell amplitudes

In this subsection we perform the perturbative expansion of the four and six derivative Lagrangian and compute the on-shell three-point amplitudes. We use the on-shell conditions (2.35) and ignore the auxiliary field in light of our earlier discussion. We note that

$$\begin{aligned} \partial^M \mathcal{M}_{M\bar{N}} &= \partial^{\bar{M}} a_{\bar{M}\bar{N}} + \partial^{\underline{M}} m_{\underline{M}\bar{N}}\,, \\ \partial^M \mathcal{M}_{M\underline{N}} &= \partial^{\bar{M}} m_{\bar{M}\underline{N}} + \partial^{\underline{M}} a_{\underline{M}\underline{N}}\,. \end{aligned} \tag{2.53}$$

Since we are allowed to set auxiliary fields to zero and to use the on-shell conditions (2.35), both $\partial^M \mathcal{M}_{M\bar{N}}$ and $\partial^M \mathcal{M}_{M\underline{N}}$ can be set to zero, and as a result, we are allowed to set

$$\partial^M \mathcal{M}_{MN} \to 0\,, \tag{2.54}$$

in simplifying the higher-derivative cubic interactions! This is a great simplification.

Now we use (2.54) in the four derivative Lagrangian L_4 given in (2.29). Only the terms on the first line survive and we get:

$$L_4\Big|_{\Phi=0} = \tfrac{1}{6}\mathcal{M}^{MN}\partial_{NL}\mathcal{M}^{PQ}\partial_{PQ}\mathcal{M}_M{}^L - \tfrac{1}{12}\mathcal{M}^{MN}\partial_{NP}\mathcal{M}^{LQ}\partial_{ML}\mathcal{M}^P{}_Q\,. \tag{2.55}$$

Now we plug in the expansion (2.34) and keep only the cubic terms which do not vanish on-shell. After a short computation we obtain the four derivative cubic Lagrangian in terms of the physical fields

$$L^{(3,4)}\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = \tfrac{1}{3}\, m^{\underline{M}\bar{N}}\Big(\partial_{\bar{N}\bar{L}} m^{\underline{P}\bar{Q}}\left[\partial_{\underline{P}\bar{Q}} m_{\underline{M}}{}^{\bar{L}} - \tfrac{1}{2}\partial_{\underline{M}\bar{Q}} m_{\underline{P}}{}^{\bar{L}}\right] + \partial_{\underline{M}\underline{L}} m^{\underline{P}\bar{Q}}\left[\partial_{\underline{P}\bar{Q}} m^{\underline{L}}{}_{\bar{N}} - \tfrac{1}{2}\partial_{\bar{N}\underline{P}} m^{\underline{L}}{}_{\bar{Q}}\right]\Big). \tag{2.56}$$

Translating this to e fluctuations (three m's and 5 contractions):

$$L^{(3,4)}\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = \tfrac{1}{12}\, e^{mn}\Big(\partial_{nl}e^{pq}\, \big[\partial_{pq}e_m{}^{l} - \tfrac{1}{2}\partial_{mq}e_p{}^{l}\big] - \partial_{ml}e^{pq}\big[\partial_{pq}e^{l}{}_{n} - \tfrac{1}{2}\partial_{pn}e^{l}{}_{q}\big]\Big)\,. \tag{2.57}$$

Using integration by parts, gauge conditions and $e_{mn} = h_{mn} + b_{mn}$ this simplifies to

$$L^{(3,4)}\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = \tfrac{1}{8}\, e_{mn}\big(\partial^{nq}e_{kl}\partial^{kl}e^{m}{}_{q} - \partial^{mp}e_{kl}\partial^{kl}e_{p}{}^{n}\big) = \tfrac{1}{2}\, b^{mn}\, \partial_{np}h^{pl}\, \partial_{pl}h^{p}{}_{m}\,. \tag{2.58}$$

A short computation confirms that this is precisely the on-shell value of the term

$$-\tfrac{1}{2}H^{mnp}\Gamma^{q}_{ml}\partial_{n}\Gamma^{l}_{pq}\,, \tag{2.59}$$

arising from the expansion of the kinetic term for the Chern-Simons improved field strength $\widehat{H}$. There is no Riemann-squared term appearing, as has been argued before.

In the six-derivative Lagrangian L_6 given in (2.29) only the first term survives after we impose the on-shell condition. Integrating by parts the ∂_N derivative we have

$$L_6\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = -\tfrac{1}{48}\, \mathcal{M}^{MN}\partial_{MNPQ}\mathcal{M}^{KL}\partial_{KL}\mathcal{M}^{PQ}. \tag{2.60}$$

Using the $\mathcal{M}$ field expansion and keeping only cubic terms which are non-vanishing on-shell, we get:

$$L^{(3,6)}\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = -\tfrac{1}{6}\, m^{\underline{M}\bar{N}}\partial_{\underline{M}\bar{N}\underline{P}\bar{Q}}m^{\underline{K}\bar{L}}\partial_{\underline{K}\bar{L}}m^{\underline{P}\bar{Q}}. \tag{2.61}$$

In term of e_{mn} this takes the form:

$$L^{(3,6)}\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = \tfrac{1}{48}\, e_{mn}\, \partial^{mnpq}e_{kl}\, \partial^{kl}e_{pq}\,. \tag{2.62}$$

The structure of the six-derivative term is such that only the symmetric part of e_{mn} contributes. In terms of the metric fluctuations we get:

$$L^{(3,6)}\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = \tfrac{1}{48}\, h_{mn}\, \partial^{mnpq}h_{kl}\, \partial^{kl}h_{pq}\,. \tag{2.63}$$

This term is produced by the perturbative on-shell evaluation of the following Riemann-cubed term:

$$-\tfrac{1}{48}R_{mn}{}^{pq}R_{pq}{}^{kl}R_{kl}{}^{mn}, \tag{2.64}$$

where the linearized Riemann tensor is: $R_{mnpq} = \frac{1}{2}\left(\partial_{np}h_{mq} + \partial_{mq}h_{np} - \partial_{pq}h_{mn} - \partial_{nq}h_{mp}\right)$. .

Collecting results (2.52), (2.58) and (2.62) for the cubic interactions with two, four, and six derivatives, we have:

$$
\begin{aligned}
L_3\Big|_{\substack{\Phi=0\\ \text{on-shell}}} = \; & \tfrac{1}{4}e_{mn}\Big[\; \partial^m e_{pq}\partial^n e^{pq} - \partial^m e_{pq}\partial^q e^{pn} - \partial^p e^{mq}\partial^n e_{pq} \\
& + \tfrac{1}{2}\alpha'\left(\partial^{nq}e_{kl}\partial^{kl}e^{i}{}_{q} - \partial^{mp}e_{kl}\partial^{kl}e_{p}{}^{n}\right) + \tfrac{1}{12}\,\alpha'^2\,\partial^{mnpq}e_{kl}\,\partial^{kl}e_{pq}\Big],
\end{aligned}
\tag{2.65}
$$

where we have made explicit the α' factors in the various contributions. To compute the on-shell amplitude we pass to momentum space. We need not concern ourselves with overall normalization; all that matters here is the relative numerical factors between the two, four, and six-derivative terms. We thus have an on-shell amplitude $\mathcal{A}$ proportional to

$$
\begin{aligned}
\mathcal{A} = e_{1mm'}e_{2nn'}e_{3pp'}\Big[& -k_2^m k_3^{m'}\,\eta^{np}\eta^{n'p'} + k_2^m k_3^{n'}\,\eta^{np}\eta^{n'p'} + k_2^p k_3^{m'}\,\eta^{mn}\eta^{m'p'} + \text{permutations} \\
& + \tfrac{1}{2}\alpha'\left(k_2^{m'}k_2^{p'}k_3^{n}k_3^{n'}\,\eta^{mp} - k_2^m k_2^p k_3^n k_3^{n'}\,\eta^{n'p'}\right) + \text{permutations} \\
& - \tfrac{1}{12}\,\alpha'^2\,k_2^m k_2^{m'}k_2^p k_2^{p'}\,k_3^n k_3^{n'} + \text{permutations}\Big],
\end{aligned}
\tag{2.66}
$$

where we have used three different lines to list the terms with two, four, and six derivatives. By 'permutations' here we mean adding, in each line, the five copies with index permutations required to achieve full Bose symmetry. In order to show that the above has the conjectured factorized form we must rewrite the momentum factors in terms of momentum differences k_{12}, k_{23}, and k_{31}. This is possible because momentum factors must contract with polarization tensors, and using momentum conservation and transversality ensure they can be converted into momentum differences. For example,

$$
e_{2nn'}k_1^{n'} = \tfrac{1}{2}e_{2nn'}(k_1^{jn} + k_1^{n'}) = \tfrac{1}{2}e_{2nn'}(k_1^{n'} - k_2^{n'} - k_3^{n'}) = -\tfrac{1}{2}e_{2nn'}k_{31}^{n'}. \tag{2.67}
$$

After rewriting all momenta as momentum differences the sum over permutations simplify and with modest work one can show that the two, four, and six derivative terms can be written as sum of products of the T and W tensors introduced in (2.5). Indeed, making use of (2.11) one finds,

$$
\begin{aligned}
\mathcal{A} &= \tfrac{1}{2}e_{1mm'}(k_1)e_{2nn'}(k_2)e_{3pp'}(k_3)\Big[\,T^{mnp}T^{m'n'p'} + \left(W^{mnp}\,T^{m'n'p'} - T^{mnp}W^{m'n'p'}\right) - W^{mnp}\,W^{m'n'p'}\Big] \\
&= \tfrac{1}{2}e_{1mm'}(k_1)e_{2nn'}(k_2)e_{3pp'}(k_3)\,(T^{mnp} + W^{mnp})(T^{m'n'p'} - W^{m'n'p'})\,,
\end{aligned}
\tag{2.68}
$$

in agreement with (2.7) and thus proving the claimed factorization.

2.4 Conclusions and remarks

Our work has determined the form (2.1) of the gauge invariant HSZ action that reproduces the on-shell cubic amplitudes of the theory. The $\mathcal{O}(\alpha')$ terms arise from the kinetic term for the three-form curvature $\widehat{H}$, which contains the Chern-Simons correction. Our work in section 2.3.3 reconfirmed that the cubic on-shell four-derivative couplings arise correctly – see (2.59). The kinetic term $\widehat{H}^2$ also contains $\mathcal{O}(\alpha'^2)$ contributions, but those would only affect six and higher-point amplitudes. The full HSZ action may contain other $\mathcal{O}(\alpha')$ terms that do not contribute to three-point amplitudes. The action includes the Riemann-cubed term derived in (2.64). Its coefficient is minus the coefficient of the same term in bosonic string theory. In bosonic string theory there is also a non-zero 'Gauss-Bonnet' Riemann-cubed term, but its presence can only be seen from four-point amplitudes [29]. Neither the Riemann-cubed nor its related Gauss-Bonnet term are present in heterotic string theory. It would be interesting to see if the cubic-curvature Gauss-Bonnet interaction is present in HSZ theory. The physical effects of Riemann-cubed interactions were considered in [35] and, regardless of the sign of the term, they lead to causality violations that require the existence of new particles.

The action (2.1), while exactly gauge invariant, is unlikely to be *exactly* duality-invariant. It is not, after all, the full action for HSZ theory. Reference [36] showed that the action (2.1), without the Riemann-cubed term, is not duality-invariant to order α' squared. It may be possible to use the methods in [36] to find out what other terms (that do not contribute to cubic amplitudes) are needed for duality invariance to order α' squared. We continue to expect that, in terms of a metric and a b-field, an action with infinitely many terms is required for exact duality invariance.

It is natural to ask to what degree global duality determines the classical effective action for the massless fields of string theory. Additionally, given an effective field theory of the metric, the b-field and the dilaton, it is also natural to ask if the theory has a duality symmetry. HSZ theory is useful as it is the simplest gravitational theory with higher derivative corrections and exact global duality. By investigating HSZ theory we will better understand the constraints of duality and its role in the effective field theory of strings.

3

Spectrum of HSZ theory

The goal of the present chapter is to analyze the dynamical content (the particle spectrum) of HSZ theory, including higher derivatives but restricting to the quadratic approximation around flat space. Motivation for this is two fold. On one hand we have asserted that the HSZ theory is the only known higher derivative gravitational theory with manifest duality and gauge invariance. These features are expected to be present in *full* classical string theory. On the other hand, we have established that the HSZ theory is different from both bosonic and heterotic string theories. Finding the physical spectrum will help elucidate this difference. A second motivation for this analysis is that generically higher derivative theories (which are not string theories) lead to inconsistencies. An example of such an inconsistency at the level of physical spectrum is the appearance of ghost degrees of freedom[1]. This is indeed what we find in this case.

Our analysis is simplified by introducing further fields that allow us to reduce the number of derivatives to two. In order to elucidate the structure of these theories, we find it convenient to compare them with a massive deformation of the original (massless) linearized DFT. This theory, which seems interesting in its own right, is given by the Lagrangian

$$L_{\rm mDFT} \;=\; \tfrac{1}{2}e^{mn}\mathcal{R}_{mn}(e,\Phi) - \Phi\mathcal{R}(e,\Phi) - \tfrac{1}{4}M^2(e^{mn}e_{mn} - 16\Phi^2)\;, \tag{3.1}$$

where $\mathcal{R}_{mn}$ and $\mathcal{R}$ are the linearized Ricci tensor and scalar curvature of DFT, whose explicit forms are given in (3.7). We will show that this model propagates precisely a massive spin-2 mode, a massive two-form field, and a massive scalar, without any undesired or ghost-like modes. This result hinges on the structure of both the mass terms and the kinetic terms, which are such that in the massless limit $M^2 \to 0$ the theory is invariant

[1]Tachyonic instabilities are not that severe generally as they can be cured by supersymmetry

under the DFT gauge symmetry,

$$\delta e_{mn} = D_i\bar{\lambda}_j + \bar{D}_j\lambda_i , \qquad \delta\Phi = -\tfrac{1}{4}(D_i\lambda^i + \bar{D}_i\bar{\lambda}^i) . \tag{3.2}$$

Intriguingly, this model seems new as it is *not* field-redefinition equivalent to the Fierz-Pauli-theory of (linearized) massive gravity augmented by a massive two-form and a massive scalar. Indeed, while the kinetic terms in (3.1) can be diagonalized (returning to Einstein frame) in order to write the model as a sum of linearized gravity, massless two-form and massless scalar, one cannot simultaneously diagonalize the above mass terms. Nevertheless, the above model is ghost-free, and this may shed a new light on the old problem of finding a consistent non-linear theory of massive gravity (see [37] for a recent review).

Remarkably, the six-derivative HSZ quadratic Lagrangian can be rewritten as a two-derivative Lagrangian by introducing two auxiliary scalars φ and $\bar{\varphi}$, which pair up with a_{mn} and $\bar{a}_{mn}$, to play a role largely analogous to that which the dilaton ϕ plays for e_{mn}. In particular, thanks to these new fields, the kinetic terms are 'improved' relative to the original two-derivative terms and the number of degrees of freedom does not increase. The massive spin-2 modes are ghost-like, as can be seen from the overall sign of the kinetic terms. The presence of ghost-like massive spin-2 modes is in qualitative agreement with the chiral string theory [38] but, again, the detailed spectrum differs.

The improved structure of the kinetic terms is reflected by an enhanced gauge invariance in the massless limit, as for the massive DFT theory above. This symmetry reads

$$\begin{aligned} \delta_\zeta a_{mn} &= D_i\zeta_j + D_j\zeta_i , \qquad & \delta_\zeta\varphi &= -D_i\zeta^i , \\ \delta_{\bar{\zeta}}\bar{a}_{mn} &= \bar{D}_i\bar{\zeta}_j + \bar{D}_j\bar{\zeta}_i , \qquad & \delta_{\bar{\zeta}}\bar{\varphi} &= \bar{D}_i\bar{\zeta}^i , \end{aligned} \tag{3.3}$$

and thus takes the form of two additional diffeomorphism-like symmetries with parameter ζ_i and $\bar{\zeta}_i$. Note that the massless limit corresponds to the tensionless limit $\alpha' \to \infty$ and hence this model confirms the general expectation that string theory exhibits an enlarged gauge symmetry in this limit [39].

We close this introduction with some general remarks. Given the presence of ghost-like modes in the spectrum, it follows that this theory is problematic — at least around flat space and to the extent that the quadratic theory provides a reliable approximation. It should be recalled, however, that the inclusion of more than two derivatives generically leads to additional propagating degrees of freedom, which are typically ghost-like and massive. For instance, the addition of curvature-squared terms to the Einstein-Hilbert action generally leads to a massive spin-2 ghost and a massive scalar, thereby violating unitarity. Can the spin-2 ghosts in HSZ theory be interpreted similarly? We will show in sec. 6 that in the quadratic theory the massive fields can be integrated out exactly. Due to the presence of *two* massive spin-2 fields this leads to an *infinite* number of higher-derivative corrections.

In the usual string field theories one can always choose a field basis for which the propagator is not modified, making manifest that there is no conflict with unitarity. To first order in α', one employs the Gauss-Bonnet combination [27], which is a total derivative at the quadratic level.[2] In contrast, there is evidence that any theory that is not a complete string theory (like generic higher-derivative gravity) is problematic at some level, see e.g. [35]. Our findings here seem to confirm this.

3.1 Full quadratic theory and non-derivative terms

In this section we compute the full quadratic Lagrangian and the potential of HSZ theory [2]. From the quadratic Lagrangian we will see that the theory has both 'ghost-like' and 'healthy' degrees of freedom. By analyzing the potential, we show that the theory admits two vacua with constant backgrounds. Both of these vacua have the same number of degrees of freedom; 'ghost-like' fields of one vacuum, however, correspond to 'healthy' fields of the other vacuum and vice versa.

The zero- and two-derivative parts of the HSZ quadratic action have been computed previously in eqs. (2.42) and (2.44). In terms of conventional fields, they are

$$\begin{aligned} L^{(2,0)} &= \tfrac{1}{2}\, a^{mn} a_{mn} \,-\, \tfrac{1}{2}\, \bar{a}^{mn} \bar{a}_{mn}\,, \\ L^{(2,2)} &= \tfrac{1}{4} e^{mn} \Box e_{mn} + \tfrac{1}{4}\left(D_p e^{mn}\right)^2 + \tfrac{1}{4}\left(\bar{D}_j e^{mn}\right)^2 - 2 e^{mn} D_i \bar{D}_j \Phi - 4\Phi\, \Box \Phi \\ &\quad - \tfrac{1}{8} a^{mn}\, \Box a_{mn} - \tfrac{1}{4}\left(D_i a^{mn}\right)^2 \,-\, \tfrac{1}{8} \bar{a}^{mn}\, \Box \bar{a}_{mn} \,-\, \tfrac{1}{4}\left(\bar{D}_j \bar{a}^{mn}\right)^2\,, \end{aligned} \tag{3.4}$$

where $\Box \equiv D_i D^i = \bar{D}_i \bar{D}^i$. From the two-derivative Lagrangian, we note that the kinetic terms for a_{mn} and $\bar{a}_{mn}$ appear with the 'wrong' sign and hence describe ghost-like degrees of freedom.

The four- and six-derivative parts of the quadratic Lagrangian can be computed explicitly starting from eq. (2.28). The computation can be simplified by noting that any term which involves derivatives acting on more than two fields will not contribute to the quadratic Lagrangian. Further, terms of the form $\left(\mathcal{M}^2\right)_{MN} \partial^M (\cdots)\, \partial^N (\cdots)$ can also be ignored, because upon expanding around the background generalized metric, such a term would vanish at quadratic level due to the strong constraint. After excluding such terms, one gets the following expressions for the four- and six-derivative terms that can contribute to the

[2]Other higher-derivative theories that do not propagate ghosts are Einstein-Hilbert plus the square of the pure Ricci scalar, which is equivalent to a massive scalar coupled to gravity and currently a favored model for inflation (Starobinsky model) [40], and new massive gravity in $2+1$ dimensions, which augments a 'wrong-sign' Einstein-Hilbert term with a particular curvature-squared term [41].

quadratic Lagrangian:

$$
\begin{aligned}
L^{(\cdot,4)} &= \tfrac{1}{12}\,\mathcal{M}^{MN}\Big[\partial_M\mathcal{M}^{PQ}\,\partial_{PQ}{}^{K}\mathcal{M}_{NK} + 2\,\partial^P\mathcal{M}_M{}^{Q}\,\partial_{NQ}{}^{K}\mathcal{M}_{PK} - \partial_M\mathcal{M}^{PQ}\,\partial_{NP}{}^{K}\mathcal{M}_{QK} \\
&\qquad + \partial_{MN}\mathcal{M}^{PQ}\,\partial_P{}^{K}\mathcal{M}_{QK} - \partial_M{}^{P}\mathcal{M}^{QK}\,\partial_{NQ}\mathcal{M}_{PK} - \partial_M{}^{P}\mathcal{M}_P{}^{Q}\,\partial_N{}^{K}\mathcal{M}_{QK} \\
&\qquad + \mathcal{M}^{PQ}\Big(\partial_M\mathcal{M}_P{}^{K}\,\partial_{NQK}\phi - 2\,\partial_M{}^{K}\mathcal{M}_{PK}\,\partial_{NQ}\phi + 3\,\partial_{MN}\mathcal{M}_P{}^{K}\,\partial_{QK}\phi \\
&\qquad\qquad + 3\,\partial_N\mathcal{M}_M{}^{K}\,\partial_{PQK}\phi + 3\,\partial^K\mathcal{M}_{MK}\,\partial_{NPQ}\phi\Big)\Big] + \cdots\,, \\
L^{(\cdot,6)} &= \tfrac{1}{48}\,\mathcal{M}^{MN}\big(\partial_M{}^{PQ}\mathcal{M}^{KL}\,\partial_{NKL}\mathcal{M}_{PQ} + 6\,\partial_M{}^{P}\mathcal{M}^{QK}\,\partial_{NQK}{}^{L}\mathcal{M}_{PL} - 2\partial^{PQ}\mathcal{M}^{KL}\,\partial_{MNKL}\mathcal{M}_{PQ}\big) \\
&\quad + \tfrac{1}{8}\,\mathcal{M}^{MN}\mathcal{M}^{PQ}\,\big(\partial_{MP}\mathcal{M}^{KL}\,\partial_{NQKL}\phi - \partial_{MKL}\mathcal{M}^{KL}\,\partial_{NPQ}\phi - 2\,\mathcal{M}^{KL}\partial_{MNP}\phi\,\partial_{QKL}\phi\big) + \cdots\,,
\end{aligned}
\tag{3.5}
$$

where '$\cdots$' denotes terms which do not contribute to the quadratic Lagrangian and $\partial_{M_1M_2\cdots M_k} \equiv \partial_{M_1}\partial_{M_2}\cdots\partial_{M_k}$. In computing this Lagrangian from eq. (2.28) no integrations by part have been performed. After expanding around the background generalized metric and keeping only terms quadratic in fields, we get:

$$
\begin{aligned}
L^{(2,4)} &= -\tfrac{1}{4}\,a^{\bar{M}\bar{N}}\partial_{\bar{M}\bar{N}\bar{P}\bar{Q}}a^{\bar{P}\bar{Q}} + \tfrac{1}{4}\,a^{\underline{MN}}\partial_{\underline{MNPQ}}a^{\underline{PQ}} + \tfrac{1}{4}\mathcal{R}\,\partial_{\bar{P}\bar{Q}}a^{\bar{P}\bar{Q}} - \tfrac{1}{4}\mathcal{R}\,\partial_{\underline{PQ}}a^{\underline{PQ}}\,, \\
L^{(2,6)} &= \tfrac{1}{16}\,\big(\partial_{\bar{M}\bar{N}}a^{\bar{M}\bar{N}} + \partial_{\underline{MN}}a^{\underline{MN}} - \mathcal{R}\big)\Box\big(\partial_{\bar{M}\bar{N}}a^{\bar{M}\bar{N}} + \partial_{\underline{MN}}a^{\underline{MN}} - \mathcal{R}\big)\,.
\end{aligned}
\tag{3.6}
$$

Here $\mathcal{R}$ is the linearized scalar curvature, which can be written in terms of the double metric fluctuation $m_{\underline{M}\bar{N}}$ or e_{mn} as follows:

$$
\begin{aligned}
\mathcal{R} &\equiv -2\partial_{\underline{M}\bar{N}}m^{\underline{M}\bar{N}} - +4\Box\Phi = D_m\bar{D}_n e^{mn} + 4\Box\Phi\,, \\
\mathcal{R}_{mn} &\equiv \tfrac{1}{2}\Box e_{mn} - \tfrac{1}{2}\,D_m D^p e_{pn} - \tfrac{1}{2}\,\bar{D}_n\bar{D}^p e_{mp} - 2D_m\bar{D}_n\Phi\,,
\end{aligned}
\tag{3.7}
$$

where we included the definition of the linearized Ricci tensor for future use. These tensors are invariant under (3.2). The above four- and six-derivative Lagrangians can be written in terms of the conventional fields and spacetime indices following the rules stated in previous chapter..

$$
\begin{aligned}
L^{(2,4)} &= -\tfrac{1}{16}\,\big(\bar{D}_m\bar{D}_n\bar{a}^{mn}\big)^2 + \tfrac{1}{16}\,\big(D_m D_n a^{mn}\big)^2 + \tfrac{1}{8}\mathcal{R}\,\bar{D}_m\bar{D}_n\bar{a}^{mn} - \tfrac{1}{8}\mathcal{R}\,D_m D_n a^{mn}\,, \\
L^{(2,6)} &= \tfrac{1}{64}\,\big(\bar{D}_m\bar{D}_n\bar{a}^{mn} + D_m D_n a^{mn} - 2\mathcal{R}\big)\,\Box\,\big(\bar{D}_m\bar{D}_n\bar{a}^{mn} + D_m D_n a^{mn} - 2\mathcal{R}\big)\,.
\end{aligned}
\tag{3.8}
$$

Full non-derivative Lagrangian and vacua

The full non-derivative part $L^{(0)}$ of the HSZ Lagrangian is given by

$$L^{(0)} = e^{-2\Phi}\left(\tfrac{1}{2}\mathcal{M}_M{}^M - \tfrac{1}{6}\mathcal{M}_{MN}\mathcal{M}^{NP}\mathcal{M}_P{}^M\right). \tag{3.9}$$

After expanding around the generalized metric, it can be written as:

$$\begin{aligned} L^{(0)} = \tfrac{1}{2}\, e^{-2\Phi}\Big(& a^{\underline{M}\underline{N}} a_{\underline{M}\underline{N}} - a^{\underline{M}\underline{N}} m_{\underline{M}}{}^{\bar{P}} m_{\underline{N}\bar{P}} - \tfrac{1}{3}\, a^{\underline{M}\underline{N}} a_{\underline{M}}{}^{\underline{P}} a_{\underline{N}\underline{P}} \\ & - a^{\bar{M}\bar{N}} a_{\bar{M}\bar{N}} - a^{\bar{M}\bar{N}} m^{\underline{P}}{}_{\bar{M}} m_{\underline{P}\bar{N}} - \tfrac{1}{3}\, a^{\bar{M}\bar{N}} a_{\bar{N}}{}^{\bar{P}} a_{\bar{N}\bar{P}}\Big)\,. \end{aligned} \tag{3.10}$$

Translating to conventional variables we get:

$$L^{(0)} = \tfrac{1}{2}e^{-2\Phi}\left(a_{mn}a^{mn} - a^{mn}e_{mp}e_n{}^p + \tfrac{1}{3}a^{mn}a_m{}^p a_{np} - \bar{a}_{mn}\bar{a}^{mn} + \bar{a}^{mn}e_{pm}e^p{}_n - \tfrac{1}{3}\bar{a}^{mn}\bar{a}_i{}^p\bar{a}_{jp}\right). \tag{3.11}$$

We now analyze the critical points of this potential. Specifically, we look at the critical points with $\langle e_{mn}\rangle = 0$, where $\langle A\rangle$ denote the value of A at the critical point. The dilaton independent part of the potential has four critical points:

$$\begin{aligned} \langle a_{mn}\rangle &= 0\,, & \langle \bar{a}_{mn}\rangle &= 0\,, \\ \langle a_{mn}\rangle &= -2\eta_{mn}\,, & \langle \bar{a}_{mn}\rangle &= -2\eta_{mn}\,, \\ \langle a_{mn}\rangle &= 0\,, & \langle \bar{a}_{mn}\rangle &= -2\eta_{mn}\,, \\ \langle a_{mn}\rangle &= -2\eta_{mn}\,, & \langle \bar{a}_{mn}\rangle &= 0\,. \end{aligned} \tag{3.12}$$

It is easy to see that the potential vanishes at the first two of these critical points and is non-vanishing at the other two. Moreover, extremizing the potential with respect to the dilaton requires the potential to be zero at the critical point. Hence, only the first two critical points correspond to true vacua. The first of these critical points leads to the quadratic Lagrangian discussed in the previous subsection.

The second critical point corresponds to expanding the double metric around a background generalized metric with the overall sign reversed, $\langle\mathcal{M}\rangle = -\bar{\mathcal{H}}$. The physical consequence of expanding around this critical point is to swap the ghost-like and healthy degrees of freedom. This is analogous to the phenomenon of 'ghost-condensation' [42], where kinetic terms for fields have different signs in different vacua.

3.2 Spectrum of the quadratic theory

In this section we give a complete analysis of the degrees of freedom in HSZ theory as determined by the full quadratic Lagrangian around flat space. We begin with the two-derivative quadratic theory and determine its spectrum. Then we turn to the full six-derivative quadratic theory and reconsider the spectrum. The calculations are significantly simplified by the observation that the six derivative theory can be rewritten as a two-derivative theory with additional scalar fields. The analysis of the spectrum reveals that, in this case, higher derivatives do not alter the number of degrees of freedom. The masses of some fields, however, are changed.

3.2.1 Spectrum of the two-derivative quadratic theory

The two-derivative quadratic theory is defined by the Lagrangian in (3.4), where we combine all quadratic terms with two or less derivatives:

$$\begin{aligned} L^{(2,\leq 2)} &= \tfrac{1}{4}e^{mn}\Box e_{mn} + \tfrac{1}{4}\left(D_m e^{mn}\right)^2 + \tfrac{1}{4}\left(\bar{D}_n e^{mn}\right)^2 - 2e^{mn}D_m\bar{D}_n\Phi \; - 4\Phi\,\Box\Phi \\ &- \tfrac{1}{8}a^{mn}\,\Box a_{mn} - \tfrac{1}{4}\left(D_m a^{mn}\right)^2 + \tfrac{1}{2\alpha'}\,a^{mn}a_{mn} \\ &- \tfrac{1}{8}\bar{a}^{mn}\,\Box\bar{a}_{mn} - \tfrac{1}{4}\left(\bar{D}_n\bar{a}^{mn}\right)^2 - \tfrac{1}{2\alpha'}\,\bar{a}^{mn}\bar{a}_{mn}\;. \end{aligned} \qquad (3.13)$$

The first line in this Lagrangian contains the familiar massless degrees of freedom. There is a massless graviton, a massless two-form field and a massless scalar dilaton.

On the second and third lines we have two symmetric tensors a_{mn} and $\bar{a}_{mn}$ with mass terms. This quadratic two-derivative action does not match the Fierz-Pauli Lagrangian by a long shot. In that theory the non-derivative terms are those of a massless spin two field, and we do not have those terms. Moreover, the two-derivative terms have the wrong sign, as can be seen comparing with those for e_{mn}. The Fierz-Pauli mass terms are not present either. In such an unfamiliar setting a straightforward method to ascertain the degrees of freedom involves coupling to sources [43]. As shown in in appendix A.1 the field a_{mn} in the two-derivative approximation propagates:

1. Ghost spin-two with $m^2 = 4/\alpha'$.

2. Ghost scalar with $m^2 = 4/\alpha'$.

3. Scalar tachyon with $m^2 = -4/\alpha'$.

The field $\bar{a}_{mn}$ in the two-derivative approximation propagates exactly the same degrees of freedom but with opposite value of mass-squared.

3.2.2 Spectrum of the full six-derivative quadratic theory

We now extend the above analysis to the full quadratic action including the higher derivative terms. Consider the four-derivative terms calculated before in (3.8). The signs in this expression are such that we can rewrite it as a difference of squares:

$$L^{(2,4)} = \tfrac{1}{16}\,(D_m D_n a^{mn} - \mathcal{R})^2 - \tfrac{1}{16}\,(\bar{D}_m \bar{D}_n \bar{a}^{mn} - \mathcal{R})^2 \,. \tag{3.14}$$

Note now that the six-derivative terms in (3.8) are also of a similar form

$$L^{(2,6)} = \tfrac{1}{64}\left(\bar{D}_m \bar{D}_n \bar{a}^{mn} + D_m D_n a^{mn} - 2\mathcal{R}\right)\square\left(\bar{D}_m \bar{D}_n \bar{a}^{mn} + D_m D_n a^{mn} - 2\mathcal{R}\right). \tag{3.15}$$

With the help of two auxiliary scalar fields φ and $\bar{\varphi}$ the Lagrangian can be written as

$$\begin{aligned} L^{(2,4)} + L^{(2,6)} = & - \varphi^2 - \tfrac{1}{2}\, D_m a^{mn} D_n \varphi - \tfrac{1}{2}\,\varphi\mathcal{R} \\ & + \bar{\varphi}^2 + \tfrac{1}{2}\,\bar{D}_m \bar{a}^{mn} \bar{D}_n \bar{\varphi} + \tfrac{1}{2}\,\bar{\varphi}\mathcal{R} \\ & + \tfrac{1}{4}(\varphi + \bar{\varphi})\square(\varphi + \bar{\varphi}) \,. \end{aligned} \tag{3.16}$$

Including the original two-derivative terms, we have the full quadratic action

$$\begin{aligned} L = & \tfrac{1}{4} e^{mn}\square e_{mn} + \tfrac{1}{4}\,(D_m e^{mn})^2 + \tfrac{1}{4}\left(\bar{D}_n e^{mn}\right)^2 - 2e^{mn} D_m \bar{D}_n \Phi - 4\Phi\,\square\Phi \\ & - \tfrac{1}{8}\, a^{mn}\,\square\, a_{mn} - \tfrac{1}{4}\,(D_m a^{mn})^2 - \tfrac{1}{2}\, D_m a^{mn} D_n \varphi + \tfrac{1}{4}\varphi\square\varphi \; + \tfrac{1}{2}\, a^{mn} a_{mn} - \varphi^2 \\ & - \tfrac{1}{8}\,\bar{a}^{mn}\,\square\,\bar{a}_{mn} - \tfrac{1}{4}\left(\bar{D}_m \bar{a}^{mn}\right)^2 + \tfrac{1}{2}\,\bar{D}_m \bar{a}^{mn} \bar{D}_n \bar{\varphi} + \tfrac{1}{4}\bar{\varphi}\square\bar{\varphi} \; - \tfrac{1}{2}\,\bar{a}^{mn}\bar{a}_{mn} + \bar{\varphi}^2 \\ & + \tfrac{1}{2}\bar{\varphi}\square\varphi \; - \tfrac{1}{2}\,\varphi\mathcal{R} + \tfrac{1}{2}\,\bar{\varphi}\mathcal{R} \,. \end{aligned} \tag{3.17}$$

Now the terms in the second and third lines are improved compared to the two-derivative Lagrangian (3.13). They have the derivative terms needed for a proper kinetic term and also mass terms for the new 'dilatons' φ and $\bar{\varphi}$.

The above action is not diagonal: it has a $\bar{\varphi}\square\varphi$ term and $\varphi - \bar{\varphi}$ is coupled to the original DFT fields via $\mathcal{R}$. It turns out, however, that the action can be completely diagonalized by an exact field redefinition of the dilaton. We let

$$\Phi \to \Phi' \equiv \Phi + \tfrac{1}{4}(\varphi - \bar{\varphi}) \,, \tag{3.18}$$

leaving all other fields unchanged. Note that this redefinition is local and exactly invertible; hence there is no danger of inducing infinitely many terms. Using this field redefinition in the Lagrangian (dropping the prime) and after some algebra one finds that theory is fully equivalent to

$$\begin{aligned} L \;=\; & \tfrac{1}{4}e^{mn}\Box e_{mn} + \tfrac{1}{4}\left(D_m e^{mn}\right)^2 + \tfrac{1}{4}\left(\bar{D}_n e^{mn}\right)^2 - 2e^{mn}D_m\bar{D}_n\Phi - 4\Phi\,\Box\Phi \\ & - \tfrac{1}{8}\,a^{mn}\,\Box\,a_{mn} - \tfrac{1}{4}\left(D_i a^{mn}\right)^2 - \tfrac{1}{2}\,D_i a^{mn}D_j\varphi + \tfrac{1}{2}\varphi\Box\varphi + \tfrac{1}{2}\,a^{mn}a_{mn} - \varphi^2 \\ & - \tfrac{1}{8}\,\bar{a}^{mn}\,\Box\,\bar{a}_{mn} - \tfrac{1}{4}\left(\bar{D}_i\bar{a}^{mn}\right)^2 + \tfrac{1}{2}\,\bar{D}_i\bar{a}^{mn}\bar{D}_j\bar{\varphi} + \tfrac{1}{2}\bar{\varphi}\Box\bar{\varphi} - \tfrac{1}{2}\,\bar{a}^{mn}\bar{a}_{mn} + \bar{\varphi}^2 \,, \end{aligned} \tag{3.19}$$

which is now diagonal, so that we can readily study the physical content. The analysis in appendix A.2 shows that the fields (a_{mn}, φ) propagate:

1. Ghost spin-two with $m^2 = 4/\alpha'$.

2. Ghost scalar with $m^2 = 4/\alpha'$.

3. Scalar with $m^2 = 4/\alpha'$.

These are the same degrees of freedom as in the two-derivative approximation, except that the scalar tachyon turned into a healthy massive scalar. The fields $(\bar{a}_{mn}, \bar{\varphi})$ propagate exactly the same degrees of freedom as the un-barred pair but with opposite value of mass-squared.

We conclude by noting that a further redefinition of φ and the trace a of a_{mn} allows us to fully diagonalize into massive spin-2 and a massive scalar in the Lagrangian (3.19). We let

$$\varphi \;=\; \varphi' - \tfrac{1}{2}a' \,, \qquad a_{mn} \;=\; a'_{mn} - \varphi'\eta_{mn} \,. \tag{3.20}$$

Inserting this into the second line of the action above and dropping primes at the end, one obtains

$$\begin{aligned} L \;=\; & -\tfrac{1}{8}a^{mn}\,\Box\,a_{mn} - \tfrac{1}{4}(D_i a^{mn})^2 - \tfrac{1}{4}a^{mn}\,D_iD_j a + \tfrac{1}{8}a\Box a + \tfrac{1}{2}\left(a^{mn}a_{mn} - \tfrac{1}{2}a^2\right) \\ & - \tfrac{1}{4}(D-2)\left(\tfrac{1}{2}\varphi\Box\varphi - 2\varphi^2\right) \,. \end{aligned} \tag{3.21}$$

The second line implies that φ is a ghost with mass $M^2 = 4$. The first line has the right kinetic terms as in the

Fierz-Pauli theory, but the mass term has the wrong relative coefficient.[3] Thus, in addition to the (ghostly) massive spin-2 it propagates a scalar mode, given by the trace a.

3.3 Massive linearized DFT

The linearized DFT action describes massless gravity, a massless two-form field, and a massless dilaton. We find here a duality-invariant mass term that gives the same mass to all these three fields, without introducing ghosts or spurious degrees of freedom. For linearized Einstein gravity a consistent massive deformation requires a judicious choice of mass terms: the Fierz-Pauli mass term, that involves both the trace $h^{mn}h_{mn}$ of the square of the metric fluctuation and the square of the trace h. The latter is required to guarantee that h is non-propagating, for otherwise it would be a scalar ghost. In DFT, the trace of the field e_{mn} is not available because there is no $O(D, D)$ covariant notion of taking this trace, but one can give a novel mass term involving the dilaton, which also avoids all scalar ghosts.

Consider the linearized two-derivative DFT action, given on the first line in (3.13):

$$\begin{aligned} L_{\text{DFT}} &= \tfrac{1}{4}e^{mn}\Box e_{mn} + \tfrac{1}{4}\left(D_m e^{mn}\right)^2 + \tfrac{1}{4}\left(\bar{D}_n e^{mn}\right)^2 - 2e^{mn}D_m\bar{D}_n\Phi \; - 4\Phi\,\Box\Phi \\ &= \tfrac{1}{2}e^{mn}\mathcal{R}_{mn}(e,\Phi) - \Phi\,\mathcal{R}(e,\Phi)\,, \end{aligned} \tag{3.22}$$

where we rewrote the kinetic terms geometrically, discarding total derivatives, in terms of the linearized Ricci tensor $\mathcal{R}_{mn}$ and the scalar curvature $\mathcal{R}$ defined in (3.7).[4] We add to this linearized two-derivative DFT action mass terms in the following way:

$$L_{\text{mDFT}} = \tfrac{1}{2}e^{mn}\mathcal{R}_{mn}(e,\Phi) - \Phi\,\mathcal{R}(e,\Phi) - \tfrac{1}{4}M^2(e^{mn}e_{mn} - 16\Phi^2)\,. \tag{3.23}$$

Note that $O(D, D)$ covariance does not restrict the relative coefficient between the mass terms of e_{mn} and the dilaton Φ, but we will show in the following that the specific choice made here leads to a ghost-free model. One way to see this is to inspect the field equations for e_{mn} and ϕ,

$$\mathcal{R}_{mn} = \tfrac{1}{2}M^2 e_{mn}\,, \qquad \mathcal{R} = 4M^2\Phi\,. \tag{3.24}$$

[3]Curiously, the mass term obtained here coincides with the 'mass term' obtained by expanding a cosmological constant term proportional to $\sqrt{|g|}$ around flat space, c.f. [44]

[4]Note that the total variation takes the form $\delta L_{\text{DFT}} = \delta e^{mn}\,\mathcal{R}_{mn} - 2\delta\Phi\,\mathcal{R}$, discarding total derivatives as usual.

The generalized Ricci tensor and scalar curvature satisfy the Bianchi identities

$$D^m \mathcal{R}_{mn} \;=\; -\tfrac{1}{2}\bar{D}_n \mathcal{R}\,, \qquad \bar{D}^n \mathcal{R}_{mn} \;=\; -\tfrac{1}{2}D_i \mathcal{R}\,, \tag{3.25}$$

so that taking the divergence and derivative of the field equations we obtain

$$\begin{aligned} 0 \;&=\; D^m \mathcal{R}_{mn} + \tfrac{1}{2}\bar{D}_n \mathcal{R} \;=\; \tfrac{1}{2}M^2(D^m e_{mn} + 4\bar{D}_n \Phi)\,, \\ 0 \;&=\; \bar{D}^n \mathcal{R}_{mn} + \tfrac{1}{2}\bar{D}_n \mathcal{R} \;=\; \tfrac{1}{2}M^2(\bar{D}^n e_{mn} + 4D_m \Phi)\,. \end{aligned} \tag{3.26}$$

Taking another divergence, this implies

$$D^m \bar{D}^n e_{mn} + 4\Box\Phi \;=\; 0 \quad\Rightarrow\quad \mathcal{R} \;=\; 0\,, \tag{3.27}$$

where we used the explicit expression for the scalar curvature. Thus, thanks to the specific choice of mass terms, the scalar curvature vanishes on-shell, which in turn removes propagating degrees of freedom that would otherwise be present. Indeed, from this we conclude with (3.24) that $\Phi = 0$ and hence with (3.26) that both barred and unbarred divergences of e_{mn} vanish on-shell:

$$D^m e_{mn} \;=\; \bar{D}^n e_{mn} \;=\; \Phi \;=\; 0\,. \tag{3.28}$$

This should be compared to on-shell constraints of the Fierz-Pauli theory for massive (linearized) gravity, which are $\partial^\mu h_{\mu\nu} = 0$ and $h^\mu{}_\mu = 0$, and the on-shell constraint of the massive two-form field, which is $\partial^\mu b_{\mu\nu} = 0$. We note that (3.28) gives as many constraints as needed in order to describe a massive graviton, a massive two-form field, and a massive scalar. Indeed, with the on-shell constraints the field equation becomes $(\Box - M^2)e_{mn} = 0$ and, in a frame where $p_\mu = (M, \vec{0})$, we see that $e_{0i} = e_{i0} = 0$, resulting in $(D-1)^2$ degrees of freedom describing a graviton, a two-form field and a scalar, all of mass M. Interestingly, in DFT variables the massive scalar is *not* encoded in the dilaton density Φ, which vanishes on-shell, but rather in the trace of e_{mn}, which can only be accessed after breaking manifest $O(D,D)$ covariance. It should also be noted that although the kinetic terms of massive DFT can be diagonalized (after abandoning manifest $O(D,D)$ invariance), this field redefinition does not diagonalize the mass terms. Therefore, this model is not simply the Fierz-Pauli theory of massive gravity supplemented by a massive 2-form and a massive scalar.

It is instructive to make this point a little more explicit. Since the b-field plays no role in this discussion, we will set it to zero and, having thus abandoned $O(D,D)$ invariance, denote the spacetime indices by $\mu, \nu, \ldots$, take the derivatives D and $\bar{D}$ to be partial derivatives and $\Box = \partial^2$. The Lagrangian (3.23) then gives:

$$L = \tfrac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} + \tfrac{1}{2}\left(\partial_\mu h^{\mu\nu}\right)^2 - 2h^{\mu\nu}\partial_\mu\partial_\nu\Phi - 4\Phi\,\Box\Phi - \tfrac{1}{4}M^2\left(h^{\mu\nu}h_{\mu\nu} - 16\Phi^2\right). \qquad (3.29)$$

We want to see if this is field redefinition equivalent to the Fierz-Pauli action supplemented by a massive scalar[5]:

$$\begin{aligned} L_{FP} + L_s \;=\; & \tfrac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} + \tfrac{1}{2}\left(\partial_\mu h^{\mu\nu}\right)^2 + \tfrac{1}{2}\,h^{\mu\nu}\partial_\mu\partial_\nu h - \tfrac{1}{4}h\,\Box h - \tfrac{1}{4}M^2\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) \\ & + 4\Phi\,\Box\Phi - 4M^2\Phi^2\,. \end{aligned} \qquad (3.30)$$

The most general field redefinition can be parameterized as follows:

$$h_{\mu\nu} = A_1 h'_{\mu\nu} + \eta_{\mu\nu}\left(A_2 h' - 2A_3\Phi'\right), \qquad \Phi = -\frac{1}{2}A_4 h' + A_5\Phi'. \qquad (3.31)$$

For this field redefinition to be invertible A_1 has to be non-zero. We will now show that there is no choice of coefficients $A_1, \cdots, A_5$ that define an invertible redefinition and *simultaneously* diagonalize the kinetic and mass terms of massive DFT. Using (3.31) in the Lagrangian (3.29) we get:

$$\begin{aligned} L = & -2A_1\left(-A_3 + A_5\right)h'^{\mu\nu}\partial_\mu\partial_\nu\Phi' - 2\left(\tfrac{1}{2}A_3\left(A_1 + (D-2)A_2 + 2A_4\right) + A_2A_5 - 2A_4A_5\right)h'\Box\Phi' \\ & + M^2\left(A_1A_3 + A_2A_3D - 4A_4A_5\right)h'\Phi' + \cdots, \end{aligned} \qquad (3.32)$$

where '$\cdots$' indicates diagonal terms. Requiring the off-diagonal terms to vanish, we find two solutions

$$A_3 = 0 = A_5\,, \qquad \text{or} \qquad A_2 = -\frac{A_1}{D}\,, \qquad A_3 = A_5, \qquad A_4 = 0\,. \qquad (3.33)$$

In the first solution the field redefinition (3.31) does not involve Φ' and hence is not invertible. With the second solution,

$$h_{\mu\nu} = A_1\left(h'_{\mu\nu} - \tfrac{1}{D}\eta_{\mu\nu}h'\right) - 2\,A_3\eta_{\mu\nu}\Phi'\,, \qquad \Phi = A_3\Phi'. \qquad (3.34)$$

Only the traceless part of $h'_{\mu\nu}$ appears and hence the redefinition is non-invertible. We conclude that there is no field redefinition which diagonalizes *both* the kinetic and the mass term of massive DFT.

[5]Notice that we are using a *funny* normalization for the scalar field and hence its kinetic and mass terms have a coefficient of 4. This normalization is consistent with the normalization of the dilaton field though.

We now perform a field redefinition which diagonalizes the kinetic term of massive DFT:

$$h_{\mu\nu} \;=\; h'_{\mu\nu} + \phi'\eta_{\mu\nu}\,, \qquad \phi \;=\; \phi' + \tfrac{1}{2}h'\,, \tag{3.35}$$

after which the Lagrangian (3.29), upon dropping primes, reads

$$\begin{aligned} L \;=\; & -\tfrac{1}{2}h^{\mu\nu}G_{\mu\nu}(h) \;-\; \tfrac{1}{4}M^2(h^{\mu\nu}h_{\mu\nu} - h^2) \\ & + \tfrac{1}{4}(D-2)\phi\Box\phi \;-\; \tfrac{1}{4}(D-4)M^2\phi^2 \;-\; \tfrac{1}{2}M^2\phi h\,. \end{aligned} \tag{3.36}$$

Here $G_{\mu\nu}(h)$ is the linearized Einstein tensor,

$$G_{\mu\nu}(h) \;=\; R_{\mu\nu}(h) - \tfrac{1}{2}R(h)\eta_{\mu\nu}\,, \tag{3.37}$$

where the linearized Ricci tensor and scalar curvatures are

$$R_{\mu\nu}(h) \;=\; -\tfrac{1}{2}(\Box h_{\mu\nu} - 2\partial_{(\mu}\partial^\rho h_{\nu)\rho} + \partial_\mu\partial_\nu h)\,, \quad R(h) \;=\; -\Box h + \partial^\mu\partial^\nu h_{\mu\nu}\,. \tag{3.38}$$

Under the integral one quickly checks that

$$-\tfrac{1}{2}h^{\mu\nu}G_{\mu\nu}(h) \;=\; \tfrac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} + \tfrac{1}{2}\,(\partial_\mu h^{\mu\nu})^2 + \tfrac{1}{2}\,h^{\mu\nu}\partial_\mu\partial_\nu h \;-\; \tfrac{1}{4}h\,\Box h\,. \tag{3.39}$$

The linearized Einstein tensor is self-adjoint: under an integral, $a^{\mu\nu}G_{\mu\nu}(b) \;=\; b^{\mu\nu}G_{\mu\nu}(a)$, for arbitrary symmetric tensors a and b.

As claimed, the kinetic terms in (3.36) are now diagonal, but the mass terms contain the non-removable term ϕh. Nevertheless, it is As a further check appendix A.3 gives a source analysis of the model. The results confirm that massive DFT describes a massive graviton, massive dilaton, and massive 2-form field and does not propagate any undesired (ghost-like) modes. We also show that the particular combination $e^{mn}e_{mn} - 16\Phi^2$ of mass terms is strictly necessary: for any other combination one finds an additional ghost scalar.

Part II

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4 An overview of supersymmetric localization

Non-abelian gauge theories play an important role in our understanding of particle physics. At high energies their dynamics are understood well. The effective strength of interactions can be weak at high energies, a phenomenon known as asymptotic freedom. In this regime perturbation theory can be used to compute quantities of physical interest. At low energies, however, the effective strength of interactions can increase. It can become strong enough to prevent the separation of the quarks in a hadron, a phenomenon known as confinement. In this regime, perturbative methods are of no use and one needs other techniques. For example, effective field theory methods involving the low-energy degrees of freedom as in chiral perturbation theory or numerical methods as in lattice gauge theory. In general, it is a formidable task to analytically understand the dynamics of strongly coupled gauge theories. From the perspective of perturbation theory we require a complete resummmation of all perturbative and non-perturbative effects.

The strong-coupling dynamics of generic gauge theories is an important open question in QFT, but progress can be made by adding supersymmetry to the problem. Supersymmetry enables the use of supersymmetric localization. In this chapter we review the basic argument underlying supersymmetric localization. We then give a survey of all known partition functions obtained by using localization on spheres. Finally we discuss two important missing cases.

4.1 Localization of supersymmetric gauge theories

Consider the deformed expectation value of a supersymmetric observable $\mathcal{O}$

$$\langle \mathcal{O} \rangle (t) \equiv \int \mathcal{D}\Phi \exp\left(-S\left[\Phi\right] - tQV\left[\Phi\right]\right) \mathcal{O}, \tag{4.1}$$

where Φ represents all fields in the theory. Q is a supersymmetry generator, which squares to a bosonic symmetry of the theory, i.e., $Q^2 = Q_{\text{Bos}}$. $S[\Phi]$ is the action for the theory which is invariant under supersymmetry, i.e., $QS = 0$. Finally, $V[\Phi]$ is an *auxiliary* Fermionic functional which respects all bosonic symmetries of the theory. If the integration measure is also invariant under supersymmetry, i.e., $Q[\mathcal{D}\Phi] = 0$, then Equation (4.1) is independent of the parameter t. In particular

$$\langle \mathcal{O} \rangle = \lim_{t\to\infty} \int \mathcal{D}\Phi e^{-S-tQV} \mathcal{O}. \tag{4.2}$$

If QV has a positive semi-definite bosonic part, the only non-vanishing contribution to the path integral comes from the field configurations Φ_0 satisfying $QV[\Phi_0] = 0$, i.e., the localization locus. In practice one makes the following choice for the functional V

$$V = \int \sum_{\psi} \psi (Q\psi)^\dagger + \text{h.c.} \tag{4.3}$$

Next, we expand fields around their value at the locus as $\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\Phi'$. This implies

$$S + tQV = S[\Phi_0] + QV[\Phi']\,|_{\text{quadratic}} + \mathcal{O}\left(t^{-\frac{1}{2}}\right), \tag{4.4}$$

where the second term is restricted to only quadratic order in fluctuations Φ'. The integral over fluctuations can be performed exactly and we obtain[1]

$$\langle \mathcal{O} \rangle = \int d\Phi_0 e^{-S[\Phi_0]} \mathcal{O}[\Phi_0]\, Z_{\text{1-loop}}(\Phi_0), \tag{4.5}$$

where $Z_{\text{1-loop}}(\Phi_0)$ is the one-loop determinant obtained by integrating over the fluctuations Φ'

$$Z_{\text{1-loop}}(\Phi_0) \equiv \int \mathcal{D}\Phi' e^{-QV[\Phi']|_{\text{quadratic}}}. \tag{4.6}$$

On a compact space, fields Φ' can be expanded in a discrete set of basis functions f_n, i.e., $\Phi' = \sum_{n=1}^{\infty} \Phi'_n f_n$. The integration measure $\mathcal{D}\Phi'$ can now be defined unambiguously as

$$\mathcal{D}\Phi' \equiv \prod_{n=1}^{\infty} d\Phi'_n. \tag{4.7}$$

[1]In general, there can be an infinite number of loci and one has to sum over the contribution from all loci. In this thesis, we will not consider this possibility.

Notice that the integral over Φ_0 is a finite dimensional integral. Hence supersymmetric localization on compact manifolds replaces an infinite dimensional integral of an arbitrary integrand by a finite dimensional integral.

4.2 A survey of results

Following the seminal work of Pestun [5], tremendous progress has been made in obtaining exact results for supersymmetric gauge theories on spheres and other curved backgrounds. Compact manifolds provide a natural way of regulating IR divergences of the theory. We will mainly focus on exact partition functions for theories on spheres. This quantity contains information about the conformal manifold and the rennormalization group flow [?]. A recent review [45] contains much of the results for sphere partition functions for supersymmetric theories. Table 4.1 summarizes the number of dimensions corresponding to the minimum number of supersymmetries for which the partition functions are known.

Manifold	Multiplet	Number of supersymmetries
S^2	$\mathcal{N}=2$	4
S^3	$\mathcal{N}=2$	4
S^4	$\mathcal{N}=2$	8
S^5	$\mathcal{N}=1$	8
S^6	$\mathcal{N}=2$	16
S^7	$\mathcal{N}=1$	16

Table 4.1: Spheres and the minimum number of supercharges with known partition functions.

The partition function of a theory on a d-dimensional sphere S^d of radius r takes the following form:

$$Z = \int [d\sigma]_{\text{Cartan}} \prod_{\alpha} i\langle\alpha,\sigma\rangle \exp\left(-\frac{8\pi^{\frac{d+1}{2}} r^{d-4} \text{Tr}\left(\sigma^2\right)}{g^2\Gamma\left(\frac{d-3}{2}\right)}\right) Z^{\text{vec}}_{1-\text{loop}}(\sigma)\, Z^{\text{mat}}_{1-\text{loop}}(\sigma) . \tag{4.8}$$

Here $Z^{\text{vec}}_{1-\text{loop}}$ and $Z^{\text{mat}}_{1-\text{loop}}$ are one-loop determinants associated with vector- and matter-multiplet (hypermultiplet for eight supersymmetries and chiral multiplet for four supersymmetries) respectively. If the theory has only a vector multiplet then $Z^{\text{mat}}_{1-\text{loop}} = 1$. σ is an element of the Cartan subalgebra, e.g., $\sigma \equiv \sum_{i\in\text{Cartan}} \sigma_i H_i$ with H_i begin Cartan generators. σ is the value of a dimensionless adjoint scalar at the localization locus. α are roots of the Lie algebra. Given a weight ξ of any representation, we define$\langle\xi,\sigma\rangle \equiv \xi_i\sigma_i$.

One-loop determinants in different dimensions were obtained in different papers using different techniques. We present the results for one-loop determinants here in a *unified* form based on our current understanding. One-loop determinants for vector multiplets with four supersymmetries on S^2 and S^3 (first computed in [46] and [47]) have the form[2]:

$$Z_{1-\text{loop}}^{\text{vec}} \prod_{\alpha} i\langle\alpha,\sigma\rangle = \prod_{\alpha} \prod_{k=0}^{k=\infty} \left[\frac{(k+i\langle\alpha,\sigma\rangle)}{(k+d-1-i\langle\alpha,\sigma\rangle)}\right]^{\frac{\Gamma(k+d-1)}{\Gamma(k+1)\Gamma(d-1)}} . \tag{4.9}$$

One-loop determinants for chiral multiplets on S^2 and S^3 take the form:

$$Z_{1-\text{loop}}^{\text{chi}} = \prod_{\xi} \prod_{k=0}^{k=\infty} \left[\frac{k+\frac{d}{2}-i\mu-i\langle\xi,\sigma\rangle}{k+\frac{d-2}{2}+i\mu+i\langle\xi,\sigma\rangle}\right]^{\frac{\Gamma(k+d-1)}{\Gamma(k+1)\Gamma(d-1)}} , \tag{4.10}$$

where ξ are weights in the representation of the chiral multiplet and $\mu = mr$ is a dimensionless mass parameter, with m the mass of the chiral multiplet.

For S^2 and S^3, the one-loop determinant for the vector multiplets with eight supersymmetries can be obtained. We do so by multiplying a vector multiplet determinant with four supersymmetries with a massless chiral multiplet determinant in the adjoint representation of the gauge group. Similarly the hypermultiplet determinant can be obtained by multiplying two chiral multiplet determinants with the complex conjugate masses and in the same representation of the gauge group.

Partition functions for theories on S^4 and S^5 with eight supersymmetries were computed in [5, 48, 49]. One-loop determinants with eight supersymmetries on S^d, where $3 \leq d \leq 5$, can be written in the following form:

$$Z_{1-\text{loop}}^{\text{vec}}(\sigma) \prod_{\alpha} i\langle\alpha,\sigma\rangle = \prod_{\alpha} \prod_{k=0}^{\infty} \left[(k+i\langle\alpha,\sigma\rangle)(k+d-2+i\langle\alpha,\sigma\rangle)\right]^{\frac{\Gamma(k+d-2)}{\Gamma[k+1]\Gamma[d-2]}} . \tag{4.11}$$

$$Z_{1-\text{loop}}^{\text{hyp}}(\sigma) = \prod_{\xi} \prod_{k=0}^{\infty} \left[\left(k+\frac{d-2}{2}\right)^2 + (\langle\xi,\sigma\rangle+\mu)^2\right]^{-\frac{\Gamma(k+d-2)}{\Gamma[k+1]\Gamma[d-2]}} . \tag{4.12}$$

On S^4 and S^5, theories with sixteen supercharges are obtained by taking the vector multiplet with eight supercharges and adding a massless hypermultiplet in the adjoint representation of the gauge group. Hence

[2]The result of [46] also includes a contribution from a non-zero flux on S^2 which is allowed at the localization locus. We consider it as a *non-perturbative* contribution, much like non-zero gauge field configurations in higher dimensions. Here we present only the perturbative part.

the one-loop determinant can be obtained by multiplying the results for eight supersymmetries.

$$Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}} \prod_{\alpha} i\langle\alpha,\sigma\rangle = \prod_{\alpha} \prod_{k=0}^{\infty} \left[\frac{k + i\langle\alpha,\sigma\rangle}{k+d-3+i\langle\alpha,\sigma\rangle} \right]^{\frac{\Gamma(k+d-3)}{\Gamma(k+1)\Gamma(d-3)}} . \tag{4.13}$$

This not only gives the correct expression for theories with sixteen supercharges in $3 \leq d \leq 5$, but also agrees with one-loop determinants on S^6 and S^7 for sixteen supercharges, which were explicitly computed in [50].

This completes the list of known results for partition functions on S^d.

4.3 $\mathcal{N} = 1$ on S^4 and S^6

There are two important cases missing from the list of results: $\mathcal{N} = 1$ on S^4 (four supersymmetries) and $\mathcal{N} = 1$ on S^6 (eight supersymmetries). We comment on these cases.

In the first case, one can construct explicit Lagrangians for theories realizing four supersymmetries on S^4. For example, a $U(1)$ multiplet in flat space in four dimensions is free and conformal. It can be conformally coupled to S^4, preserving four supersymmetries. A general non-abelian theory can be constructed by the dimensional reduction of five-dimensional $\mathcal{N} = 1$ SYM [51]. One can also start from the flat-space $\mathcal{N} = 1$ theory and then explicitly modify the action and supersymmetry transformations to put it on S^4 [52]. The difficulty arises in constructing a positive definite localization term. It is possible to construct a positive semi-definite Q-exact term from a supersymmetry generator, but the known generators that give such terms do not close to a symmetry of the Lagrangian.

For the case of $\mathcal{N} = 1$ on S^6, the situation is worse. In the four-dimensional minimal supersymmetric case, one expects to put the theory on S^4 because an appropriate superalgebra exists. It is $OSp(1|4)$, which has four supercharges and a bosonic $SO(5)$ subalgebra corresponding to the isometry of S^4. For six dimensions we want a superalgebra with a bosonic $SO(7)$ subalgebra and eight supercharges transforming in a spinor representation of $SO(7)$, but no such superalgebra exists. The $F(4)$ supergroup has an $SO(7) \times SU(1,1)$ bosonic subalgebra and 16 supercharges, so it is appropriate for $\mathcal{N} = 2$ supersymmetry.[3]

The remainder of this thesis aims at advancing our knowledge of exact results for these *missing* cases.

[3]In a recent work [52] we have been able to construct theories on S^6 with eight supersymmetries by employing a non-constant profile for the gauge coupling.

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5
1-loop tests of supersymmetric gauge theories

It was pointed out in [6] by Minahan that the perturbative partition functions for super Yang-Mills with 8 supersymmetries on S^3, S^4 and S^5 have a natural analytic continuation (see eqs. (4.11) and (4.12)), such that one can continue up to six dimensions. Likewise theories with 16 supersymmetries on S^d with $d = 3, 4, 5, 6, 7$ also have a natural analytic continuation (see eq. (4.13)) which can then be continued up to $d = 8, 9$. Although, we do not have an explicit construction of Lagrangians for these theories, it is reasonable to assume that in the decompactification limit, they reduce to usual gauge theories in flat space. The main objective of this chapter is to demonstrate that the partition functions are consistent with this picture. These partition functions include a dependence on one-loop determinants. We show that in the decompactification limit these one-loop determinants produce the well known physics of the flat space theories.

5.1 One-loop divergences from partition functions

In this section we will use the analytically continued expressions for one-loop determinants to compute effective couplings for theories with eight and sixteen supersymmetries in diverse dimensions. The ultraviolet divergences of the gauge coupling at one-loop can then be compared with the counter terms for supersymmetric theories at one-loop. In four dimensions, upon taking the decompactification limit one can compute the beta function of the theory. We show that results obtained from the analytically continued one-loop determinants are in agreement with explicit one-loop computations in these theories.

5.1.1 Eight supersymmetries in $4d$

Recall that for a gauge theory in four dimensions with N_f Dirac fermions in representation $\mathbf{R_f}$ and N_s complex scalars in representation $\mathbf{R_s}$ of a semi-simple gauge group, the one loop beta function is given by

$$\beta(g) = -\frac{1}{16\pi^2}g^3\left(\frac{11}{3}C_2(\mathbf{Adj}) - \frac{4}{3}N_f C_2(\mathbf{R_f}) - \frac{1}{3}N_s C_2(\mathbf{R_s})\right), \tag{5.1}$$

$C_2(\mathbf{R})$ is the quadratic Casimir in the representation $\mathbf{R}$ of the gauge group. For $\mathcal{N} = 2$ theory with N_h hypermultiplets in the representation $\mathbf{R}$ of the gauge group the beta function becomes

$$\beta(g) = \frac{g^3}{8\pi^2}\left(N_h\, C_2(\mathbf{R}) - C_2(\mathbf{Adj})\right). \tag{5.2}$$

The contribution from the vector multiplet was previously found in [5] by taking the hypermultiplet mass to infinity in the $\mathcal{N} = 2^*$ theory. We want to reproduce (5.2) by using the analytically continued one-loop determinant for the vector and hyper multiplets given in equations (4.11) and (4.12).

To do so we need to determine $\mathcal{O}\left(\sigma^2\right)$ terms appearing in the one-loop determinants. To proceed we replace σ by $t\sigma$ in the expressions for the one-loop determinants. The parameter t keeps track of the order of σ. Focusing only on the vector multiplet, one can easily find that

$$\begin{aligned}\frac{d\log Z^{\text{vec}}_{1-\text{loop}}}{dt^2} + \sum_{\beta>0}\frac{1}{t^2} = \\ \sum_{\beta>0}\langle\beta,\sigma\rangle^2\left(\mathcal{F}\left(d-2,0,t\langle\beta,\sigma\rangle\right) + \mathcal{F}\left(d-2,d-2,t\langle\beta,\sigma\rangle\right)\right),\end{aligned} \tag{5.3}$$

where

$$\mathcal{F}(x,y,z) \equiv \sum_{n=0}^{\infty}\frac{\Gamma(n+x)}{\Gamma(n+1)\Gamma(x)}\frac{1}{(n+y)^2+z^2} = \frac{i}{2z}\left(\frac{1}{y+iz}{}_2F_1(x, y+iz; y+iz+1; 1) - c.c\right).$$

For $d = 4-\epsilon$, we expand the R.H.S in powers of t and ϵ. Keeping only the leading terms, we find

$$\frac{d\log Z^{\text{vec}}_{1-\text{loop}}}{dt^2} = \frac{2}{\epsilon}C_2(\mathbf{Adj})\,\sigma^2 + \cdots. \tag{5.4}$$

From this we can easily obtain

$$\log Z^{\text{vec}}_{1-\text{loop}} = \frac{2}{\epsilon}C_2(\mathbf{Adj})\,\sigma^2 + \cdots. \tag{5.5}$$

A completely analogous calculation for a hypermultiplet in representation $\mathbf{R}$ of the gauge group gives

$$\log Z^{\text{hyp}}_{1-\text{loop}} = -\frac{2}{\epsilon}\sigma^2 C_2\left(\mathbf{R}\right) + \cdots . \tag{5.6}$$

For a gauge multiplet and N_h hypermultiplets, the contribution to the $\mathcal{O}\left(\sigma^2\right)$ term from the one-loop determinants can be combined with the $\mathcal{O}\left(\sigma^2\right)$ term in the fixed point action as given in equation (4.8) to get

$$\frac{8\pi^2}{g^2\left(\Lambda\right)} = \left(\frac{8\pi^2}{g_0^2} + \frac{2}{\epsilon}C_2\left(\mathbf{Adj}\right) - \frac{2}{\epsilon}N_h C_2\left(\mathbf{R}\right)\right)\Lambda^{-\epsilon}, \tag{5.7}$$

where $g\left(\Lambda\right)$ is the running coupling constant at the renormalization scale $\Lambda \sim r^{-1}$ [5], g_0 is the bare coupling. From the above equation one can easily obtain the beta function,

$$\beta\left(g\right) \;=\; \frac{g^3}{8\pi^2}\left(N_h\, C_2\left(\mathbf{R}\right) - C_2\left(\mathbf{Adj}\right)\right). \tag{5.8}$$

This matches precisely with equation (5.2). For one hypermultiplet in the adjoint representation the beta function vanishes. This is to be expected since it corresponds to $\mathcal{N} = 4$ SYM.

5.1.2 Eight supersymmetries in 6d

Since the explicit expression for one loop determinants for eight supersymmetries in $4d$ are known in terms of infinite products, the above results can be reproduced by regularizing those expressions by introducing a finite cut off parameter Λr and then taking the decompactification limit $r \to \infty$. As explained earlier, it is not known how to localize a six dimensional theory with eight supersymmetries. In this case the expression (4.11) is a genuine ansatz. In this subsection we will perform a non trivial check on that ansatz by computing the effective coupling. It is well known that the six dimensional theory with eight supersymmetries has a quadratic divergence at one-loop [53, 54]. We will compute the effective coupling using the one loop determinant (4.11) and show that it has a quadratic divergence in the decompactification limit.

Since dimensional regularization is only sensitive to logarithmic divergences we will use a hard cutoff to isolate the quadratic divergence. At leading order in the divergence this is expected to be consistent with supersymmetry. However, there could be issues with sub-leading divergences, if for example imposing the cutoff leaves off the super-partners of modes at or near the cutoff. However, assuming that the proposed dimensional regularization respects the supersymmetry we can show that the logarithmic divergences coming from the hard cutoff are consistent with the result coming from dimensional regularization, even if the log divergence is sub-leading.

We use $d = 6$ in (4.11) and truncate the infinite product at $n_{\mathbf{max}} = \Lambda r$ to find quadratic dependence on the energy cutoff Λ. It is straightforward to find that the divergent contribution to the σ^2 term from the vector one-loop determinant is

$$\log Z_{1-\text{loop}}^{\text{vec}} = \left(\frac{\Lambda^2 r^2 + 5\Lambda r}{6} + \frac{11}{3}\log(\Lambda r)\right) C_2\left(\mathbf{Adj}\right)\sigma^2 + \cdots . \tag{5.9}$$

By combining this with the fixed point action, we find the effective coupling given by

$$\frac{16\pi^3 r^2}{g^2} = \frac{16\pi^3 r^2}{g_0^2} - \left(\frac{\Lambda^2 r^2 + 5\Lambda r}{6} + \frac{11}{3}\log(\Lambda r)\right) C_2\left(\mathbf{Adj}\right). \tag{5.10}$$

In the $r \to \infty$ limit only the leading terms in r survive and one obtains

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{\Lambda^2}{96\pi^3} C_2\left(\mathbf{Adj}\right). \tag{5.11}$$

We see that the effective coupling diverges quadratically with the scale Λ.

It is also known that the six dimensional theory can be made finite at one loop by adding a suitable hypermultiplet. This would be the case if the hypermultiplet and the vector multiplet contribute to the quadratic divergence with opposite sign. This is also consistent with the one-loop determinant (4.12). For a hypermultiplet in representation $\mathbf{R}$, the contribution to $\mathcal{O}\left(\sigma^2\right)$ term is given by

$$\log Z_{1-\text{loop}}^{\text{hyp}} = \left(-\frac{\Lambda^2 r^2 + 5\Lambda r}{6} + \frac{1}{3}\log(\Lambda r)\right) C_2\left(\mathbf{R}\right)\sigma^2 + \cdots . \tag{5.12}$$

So that the effective coupling with N_h hypermultiplets in the representation $\mathbf{R}$ is given by:

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{\Lambda^2}{96\pi^3}\left(C_2\left(\mathbf{Adj}\right) - N_h C_2\left(\mathbf{R}\right)\right). \tag{5.13}$$

In particular, for a single hypermultiplet in the adjoint representation the quadratic divergence vanishes as expected [53].

5.1.3 Sixteen supersymmetries in 4d and 6d

In four and six dimensions, explicit expressions for the one-loop determinants for sixteen supersymmetries are known. We will compute the effective coupling at one-loop using both of these expressions and show that it is consistent.

For four dimensions, we compute effective coupling from (4.13) using $d = 4$. We will truncate the infinite product at $n_{\mathbf{max}} = \Lambda r$. By doing so one finds that the contribution to $\mathcal{O}\left(\sigma^2\right)$ term vanishes.

$$\log\left(Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}}\right) = 0 + \cdots . \tag{5.14}$$

Hence the coupling is not affected by one-loop effects and is independent of the cutoff scale Λ. Now we can easily check that this result is consistent with analytically continued expression, i.e., expanding (4.13) in powers of ϵ for $d = 4 - \epsilon$. Replacing σ by $t\sigma$ one can easily obtain

$$\frac{d}{dt^2}\log\left(Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}}\right) = \mathcal{O}\left(d-4\right) \implies \log\left(Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}}\right) = \mathcal{O}\left(d-4\right). \tag{5.15}$$

Which is consistent with the result obtained from the explicit expression.

Similarly, in six dimensions the contribution to the $\mathcal{O}\left(\sigma^2\right)$ term from (4.13) takes the form

$$\log\left(Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}}\right) = 3\left(\log\left(\Lambda r\right) - \frac{1}{2} + \gamma\right) C_2\left(\mathbf{Adj}\right)\sigma^2 + \cdots . \tag{5.16}$$

Combining this with the fixed point action, we obtain the effective coupling which is given by:

$$\frac{1}{g^2\left(\Lambda\right)} = \frac{1}{g_0^2} - \frac{3}{16\pi^3 r^2}\left(\log\left(\Lambda r\right) - \frac{1}{2} + \gamma\right) C_2\left(\mathbf{Adj}\right), \tag{5.17}$$

We see that the coupling has a logarithmic dependence on the scale Λ. It is easy to see that Logarithmic dependence is produced by using dimensional regularization for $d = 6 - \epsilon$ in the analytically continued expression. Doing so we find that:

$$\log\left(Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}}\right) = \frac{3}{\epsilon} C_2\left(\mathbf{Adj}\right)\sigma^2 + \cdots . \tag{5.18}$$

Combining this with the contribution from the fixed point action and noting that in $6-\epsilon$-dimensions $\frac{r^2}{g^2}$ has mass dimension $-\epsilon$ we get:

$$\frac{1}{g^2(\Lambda)} = \Lambda^{-\epsilon}\left(\frac{1}{g_0^2} + \frac{3}{16\pi^3 r^2 \epsilon}\right) = \frac{1}{g_0^2} - \frac{3}{16\pi^3 r^2}\log(\Lambda), \tag{5.19}$$

where a Λ-independent infinite piece is absorbed in $\frac{1}{g_0^2}$. This gives the same logarithmic dependence on the energy scale Λ. Note that in the decompactification limit this logarithmic divergence vanishes, consistent with that the six dimensional theory with 16 supersymmetries is finite at one-loop.

5.1.4 Sixteen supersymmetries in 8d and 9d

For $d = 8, 9$, it is not known how to localize. Here we show that the analytically continued expression for the one-loop determinant is consistent with known results. It is known that for $d = 8, 9$, none of the terms present in the tree level Lagrangian need a counter term at one-loop [54, 55]. Hence, the effective coupling determined from the analytically continued expressions for one-loop determinants should not have any divergences. This can be easily demonstrated by using the methods of this section. A short calculation shows that the contribution to $\mathcal{O}(\sigma^2)$ term from the one-loop determinant (4.13) for $d = 8$ is

$$\log\left(Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}}\right) = \frac{5}{6}\left(\frac{(\Lambda r)^2}{4} + \frac{3\Lambda r}{2} + 5\log(\Lambda r)\right) C_2(\mathbf{Adj})\,\sigma^2. \tag{5.20}$$

This leads to the effective ccoupling

$$\frac{1}{g^2(\Lambda)} = \frac{1}{g_0^2} - \frac{5}{64\pi^4 r^4}\left(\frac{(\Lambda r)^2}{4} + \frac{3\Lambda r}{2} + 5\log(\Lambda r)\right) C_2(\mathbf{Adj}). \tag{5.21}$$

Here we see that in the decompactification limit the dependence on the energy scale vanishes. A similar computation for $d = 9$ yields

$$\log\left(Z^{\text{vec}}_{1-\text{loop}} Z^{\text{hyp}}_{1-\text{loop}}\right) = \frac{\Lambda r}{10}\left(\frac{(\Lambda r)^2}{3} + \frac{7\Lambda r}{2} + \frac{151}{6}\right) C_2(\mathbf{Adj})\,\sigma^2, \tag{5.22}$$

which leads to following expression for effective coupling:

$$\frac{1}{g^2(\Lambda)} = \frac{1}{g_0^2} - \frac{\Lambda}{40\pi^5 r^4}\left(\frac{(\Lambda r)^2}{3} + \frac{7\Lambda r}{2} + \frac{151}{6}\right) C_2(\mathbf{Adj}). \tag{5.23}$$

This is independent of the UV scale Λ in the decompactification limit. The same calculation can be repeated for $d = 10$ and it can be shown that the one-loop determinants do not contribute any divergences to the gauge coupling in the decompactification limit.

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6

Analytic continuation of dimensions in supersymmetric localization

In this chapter we verify the conjecture in [6] for perturbative partition function on S^d with eight supersymmetries. Our methods are generalizations of the procedures used in[47] and [49]. When localizing with eight supersymmetries on S^d, we will choose a spinor whose vector bilinear leaves an S^{4-d} sphere fixed. So for example, on S^5 it acts freely, on S^4 there is a fixed S^0, namely the north and south poles, while on S^3 there is a fixed S^1. In the last case this is a different choice than the one used in [47], where the vector bilinear acts freely on S^3. Of course, the two procedures must give the same result.

We consider theories with four supersymmetries. Actions for gauge theories on S^4 preserving four supersymmetries have been constructed [56], but a direct localization procedure has not yet been found. Hence, our starting point is on S^3. Here we follow the prescription in [47] to generate a vector field that acts freely. We show how to generalize the construction to $d \leq 3$ and write down an explicit expression for the determinant factors. In the generalization the fixed point set for the vector field is S^{2-d}, hence S^2 will have fixed points at the poles.

We then make a proposal for analytically continuing gauge theories with four supersymmetries up to $d = 4$. The pitfalls of dimensionally regularizing supersymmetric gauge theories have been known for a long time [57, 58]. However, except perhaps for anomalies, it appears to work in one- and two-loop calculations [59]. Analytical continuation of the dimension has also been successfully applied to conformal field theories [60, 61, 62, 63, 64]. With this proposal for minimal supersymmetry on S^4 we test it against various cases. We first show that the continuation is consistent with the partition functions for a $U(1)$ vector multiplet or a free massless chiral multiplet. Both of these situations are conformal and so can be mapped from flat space onto S^4. Since they are free, their partition functions on the sphere are calculable. We next consider a general gauge theory with $\mathcal{N} = 1$ supersymmetry. We show that in the limit of large radius we can extract the correct one-loop β-function.

Lastly, we investigate a mass deformation of $\mathcal{N}=4$ super Yang-Mills. Here we concentrate on $\mathcal{N}=1^*$ theories with three chiral multiplets in the adjoint representation and masses m_i, with $i=1,2,3$. The superpotential also has a term cubic in the chiral fields that stays fixed as the mass parameters are varied. A straightforward dimensional reduction of $\mathcal{N}=1^*$ gives a three dimensional gauge theory with complex masses for chiral multiplets. In our analytic continuation we start with a vector multiplet and three chiral multiplets. However, the three dimensional mass deformed gauge theory that we can analytically continue requires real masses. Such terms appear explicitly as central charges in the superalgebra. The presence of the cubic term in the superpotential forces the sum of the three real masses to be zero in order to maintain supersymmetry.

Despite these subtleties, one can compare the general structure of the analytically continued partition function with the $\mathcal{N}=1^*$ partition function. We make a straightforward identification of the real masses of the analytically continued theory with the masses, up to a sign, that appear in the $\mathcal{N}=1^*$ superpotential. $\mathcal{N}=1$ superconformal theories on S^4 are scheme dependent [65]. However, in [66] it was argued that the the fourth derivatives of the free energy with respect to the mass parameters are scheme independent. This is in line with our observations here. We compute the corrections to the free energy to sixth order in the chiral masses at strong coupling. At least for the real part of the free energy we find no inconsistencies with the holographic results in [66]. In fact, having the sum of the real masses be zero turns out to play a crucial role.

6.1 Supersymmetric gauge theories on S^d by dimensional reduction

In this section we review and extend the procedure in [50] to construct supersymmetric gauge theories on S^d. This is a generalization of Pestun's study in four dimensions [45], and includes further details to reduce the number of supersymmetries to eight and four respectively.

As in [45] our starting point is the 10 dimensional $\mathcal{N}=1$ SYMLagrangian[1]

$$\mathcal{L}=-\frac{1}{g_{10}^2}\mathrm{Tr}\left(\tfrac{1}{2}F_{MN}F^{MN}-\Psi \not{D}\Psi\right)\,, \tag{6.1}$$

The space-time indices M,N run from 0 to 9 and Ψ^a is a Majorana-Weyl spinor in the adjoint representation. Properties of Γ^M_{ab} and $\tilde{\Gamma}^{M\,ab}$ are given in appendix B. The 16 independent supersymmetry transformations

[1]As in [45] we consider the real form of the gauge group so that the group generators are anti-Hermitian and independent generators satisfy $\mathrm{Tr}(T^aT^b)=-\delta^{ab}$.

that leave eq. (6.1) invariant are

$$\begin{aligned}\delta_\epsilon A_M &= \epsilon\,\Gamma_M\Psi\,,\\ \delta_\epsilon\Psi &= \tfrac{1}{2}\Gamma^{MN}F_{MN}\,\epsilon\,,\end{aligned} \tag{6.2}$$

where ϵ is a constant bosonic real spinor, but is otherwise arbitrary.

We next dimensionally reduce this theory to d dimensions by choosing Euclidean spatial indices $\mu = 1,\dots d$ with gauge fields A_μ and scalars ϕ_I with $I = 0, d+1,\dots 9$. The field strengths with scalar indices become $F_{\mu I} = D_\mu\phi_I$ and $F_{IJ} = [\phi_I,\phi_J]$. As in [45] we are choosing one scalar component to come from dimensionally reducing the time direction, leading to a wrong-sign kinetic term for this field.

We take the d-dimensional Euclidean space to be the round sphere S^d with radius r with the metric

$$ds^2 = \frac{1}{(1+\beta^2x^2)^2}\,dx_\mu dx^\mu\,. \tag{6.3}$$

The supersymmetry parameters are modified to be conformal Killing spinors on the sphere, satisfying

$$\nabla_\mu\epsilon = \tilde{\Gamma}_\mu\tilde{\epsilon}\,, \qquad\qquad \nabla_\mu\tilde{\epsilon} = -\beta^2\Gamma_\mu\epsilon\,. \tag{6.4}$$

where $\beta = \frac{1}{2r}$. We impose the further condition

$$\nabla_\mu\epsilon = \beta\,\tilde{\Gamma}_\mu\Lambda\,\epsilon\,, \tag{6.5}$$

leaving 16 independent supersymmetry transformations. To be consistent with eq. (6.4) Λ must satisfy $\tilde{\Gamma}^\mu\Lambda = -\tilde{\Lambda}\Gamma^\mu$, $\tilde{\Lambda}\Lambda = 1$, $\Lambda^T = -\Lambda$. The simplest choice has $\Lambda = \Gamma^0\tilde{\Gamma}^8\Gamma^9$. The solution to eq. (6.4) and eq. (6.5) is

$$\epsilon = \frac{1}{(1+\beta^2x^2)^{1/2}}\left(1+\beta\,x\cdot\tilde{\Gamma}\,\Lambda\right)\epsilon_s\,, \tag{6.6}$$

where ϵ_s is constant. On the sphere the supersymmetry transformations for the bosons are unchanged, but those for the fermions are modified to

$$\delta_\epsilon\Psi = \tfrac{1}{2}\Gamma^{MN}F_{MN}\epsilon + \frac{\alpha_I}{2}\Gamma^{\mu I}\phi_I\nabla_\mu\,\epsilon\,, \tag{6.7}$$

where the constants α_I are given by

$$
\begin{aligned}
\alpha_I =& \frac{4(d-3)}{d}, \qquad I = 8,9,0, \\
\alpha_I =& \frac{4}{d}, \qquad I = d+1, \ldots 7.
\end{aligned}
\tag{6.8}
$$

The index I in eq. (6.7) is summed over. This particular choice preserves all 16 supersymmetries. One needs to add following extra terms to get a supersymmetric Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\Psi\Psi} &= -\frac{1}{g_{YM}^2} \mathrm{Tr} \frac{(d-4)}{2r} \Psi\Lambda\Psi, \\
\mathcal{L}_{\phi\phi} &= -\frac{1}{g_{YM}^2} \left(\frac{d\,\Delta_I}{2\,r^2} \mathrm{Tr}\phi_I \phi^I \right), \\
\mathcal{L}_{\phi\phi\phi} &= \frac{1}{g_{YM}^2} \frac{2}{3r} (d-4) \varepsilon_{ABC} \mathrm{Tr} \left([\phi^A, \phi^B] \phi^C \right).
\end{aligned}
\tag{6.9}
$$

Here Δ_I is defined as

$$
\Delta_I = \alpha_I, \qquad \text{for} \qquad I = 8,9,0, \qquad \Delta_I = 2\frac{d-2}{d} \qquad \text{for} \qquad I = d+1, \cdots 7. \tag{6.10}
$$

The scalars split into two groups, ϕ^A, $A = 0,8,9$ and ϕ^i, $i = d+1, \cdots 7$ and the R-symmetry is manifestly broken from $SO(1,9-d)$ to $SO(1,7-d)$. The full supersymmetric Lagrangian is the dimensionally reduced version of eq. (6.1) supplemented with $\mathcal{L}_{\Psi\Psi}$, $\mathcal{L}_{\phi\phi}$ and $\mathcal{L}_{\phi\phi\phi}$.

6.1.1 Eight supersymmetries

In this chapter we are interested in theories with less supersymmetry. To construct theories with eight supersymmetries when $d \leq 5$ we put a further condition on ϵ.

$$
\Gamma\epsilon = +\epsilon, \qquad \Gamma \equiv \Gamma^{6789}. \tag{6.11}
$$

This reduces the number of independent supersymmetry transformations to eight. We divide the spinor Ψ as

$$
\Psi = \psi + \chi, \qquad \Gamma\psi = +\psi, \qquad \Gamma\chi = -\chi. \tag{6.12}
$$

ψ and χ fields will be the fermionic components of the vector multiplet and the hypermultiplet respectively. The scalars ϕ^I, $I = 6,7,8,9$ are in the hypermultiplet, while the remaining scalars belong to the vector multiplet. Given a hypermultiplet mass m, the constants in eq. (6.8) paired with the hypermultiplet scalars are modified to

$$\alpha_I = \frac{2(d-2)}{d} + \frac{4i\sigma_I\, m\, r}{d}, \qquad I = 6\ldots 9,$$
$$\sigma_6 = \sigma_7 = -\sigma_8 = -\sigma_9 = 1. \tag{6.13}$$

To preserve supersymmetry we must modify the cubic scalar terms in the Lagrangian to

$$\mathcal{L}_{\phi\phi\phi} = -\frac{4}{g_{YM}^2}\left((\beta(d-4)+im)\,\mathrm{Tr}(\phi^0[\phi^6,\phi^7]) - (\beta(d-4)-im)\,\mathrm{Tr}(\phi^0[\phi^8,\phi^9])\right). \tag{6.14}$$

We also need to change the quadratic term for the hypermultiplet fermion to

$$\mathcal{L}_{\chi\chi} = -\frac{1}{g_{YM}^2}\left(-im\mathrm{Tr}\chi\Lambda\chi\right). \tag{6.15}$$

The quadratic term for the hypermultiplet scalars is modified by changing the value of the constant Δ_I

$$\Delta_I = \frac{2}{d}\left(mr(mr+i\sigma_I) + \frac{d(d-2)}{4}\right), \qquad \text{for} \qquad I = 6,7,8,9. \tag{6.16}$$

The quadratic term for the vector multiplet fermion is the same as in the case of 16 supersymmetries with Ψ replaced by ψ. The full supersymmetric Lagrangian is then the dimensional reduction of eq. (6.1) supplemented with $\mathcal{L}_{\phi\phi} + \mathcal{L}_{\psi\psi} + {}_{\chi\chi} + \mathcal{L}_{\phi\phi\phi}$.

6.1.2 Four supersymmetries

If $d \leq 3$ then we can further reduce the number of supersymmetries by imposing the extra condition

$$\Gamma'\epsilon = +\epsilon, \qquad \Gamma' \equiv \Gamma^{4589}. \tag{6.17}$$

Now we decompose the spinor Ψ into four parts

$$\Psi = \psi + \sum_{\ell=1}^{3}\chi_\ell, \tag{6.18}$$

where ψ belongs to the vector multiplet and the χ_ℓ belong to three different types of chiral multiplets. If we write ℓ in binary form as $\ell = 2\beta_2(\ell) + \beta_1(\ell)$, where $\beta_s(\ell)$ are the binary digits for ℓ, then we can write the chirality conditions as

$$\Gamma\chi^{(\ell)} = (-1)^{\beta_1(\ell)}\chi^{(\ell)}, \qquad \Gamma'\chi^{(\ell)} = (-1)^{\beta_2(\ell)}\chi^{(\ell)}, \qquad \Gamma'\psi \;=\; \Gamma\psi \;=\; +\psi. \tag{6.19}$$

We also split the scalar fields into 4 groups. The fields ϕ^0 and ϕ^i, $i = d{+}1, \dots 3$ belong to the vector multiplet. Each chiral multiplet contains two scalar fields ϕ_{I_ℓ}, where the index I_ℓ takes two values $I_\ell \;=\; 2\ell+2, 2\ell+3$. Given the chiral multiplet masses m_ℓ, the constants in eq. (6.8) are further split into

$$\alpha_{I_\ell} = \frac{2(d-2)}{d} + \frac{4i\sigma_{I_\ell} m_\ell\, r}{d} \equiv \alpha_{(\ell)}\,, \qquad \sigma_{I_\ell} = (-1)^{\beta_2(\ell)\beta_1(\ell)} \equiv \sigma_{(\ell)}. \tag{6.20}$$

It is instructive to look at the individual supersymmetry transformations of the fermions in the vector and chiral multiplets. For the fermion ψ in the vector multiplet the transformations in eq. (6.7) reduces to

$$\delta_\epsilon\psi = \tfrac{1}{2}F_{M'N'}\Gamma^{M'N'}\epsilon + \tfrac{1}{2}\sum_{\ell=1}^{3}[\phi_{I_\ell},\phi_{J_\ell}]\Gamma^{I_\ell J_\ell}\epsilon + \frac{\alpha_a}{2}\Gamma^{\mu I}\phi_a\nabla_\mu\,\epsilon\,, \tag{6.21}$$

where $M', N' = 0, \dots, 3$ and $a = 0, d+1\dots 3$. Likewise, for the chiral multiplet fermions we have

$$\delta_\epsilon\chi_\ell = D_\mu\phi_{I_\ell}\Gamma^{\mu I_\ell}\epsilon + [\phi_a,\phi_{I_\ell}]\Gamma^{aI_\ell}\epsilon + \tfrac{1}{2}\varepsilon^{\ell mn}[\phi_{I_m},\phi_{J_n}]\Gamma^{I_m J_n}\epsilon + \frac{\alpha_{(\ell)}}{2}\Gamma^{\mu I_\ell}\phi_{I_\ell}\nabla_\mu\,\epsilon\,. \tag{6.22}$$

Notice that eq. (6.21) and eq. (6.22) have terms that contain fields outside of their respective multiplets. In the usual construction for four supersymmetries, the transformations of the fermions would contain the auxiliary fields D and F_ℓ. The terms outside the multiplets arise from evaluating the auxiliary fields on-shell[2]. In our construction we will still use auxiliary fields, but in this case they equal zero on-shell.

With the modification in eq. (6.20) the Lagrangian is *almost* supersymmetric under four supersymmetries if the mass terms have the form

$$\begin{aligned}\mathcal{L}_{\chi\chi} &= -\frac{1}{g_{YM}^2}\sum_{\ell=1}^{3}\left(-im_\ell \mathrm{Tr}\chi_\ell\Lambda\chi_\ell\right),\\ \mathcal{L}_{\phi\phi} &= -\frac{1}{g_{YM}^2}\sum_{\ell=1}^{3}\left(\frac{d\,\Delta_{(\ell)}}{2\,r^2}\mathrm{Tr}\phi_{I_\ell}\phi^{I_\ell}\right),\end{aligned} \tag{6.23}$$

where

$$\Delta_{(\ell)} \equiv \Delta_{I_\ell} = \frac{2}{d}\left(m_\ell r\big(m_\ell r + i\sigma_{(\ell)}\big) + \frac{d(d-2)}{4}\right), \tag{6.24}$$

[2]We thank Guido Festuccia for a helpful discussion on this point.

and we include the cubic terms

$$\mathcal{L}_{\phi\phi\phi} = -\frac{4}{g_{YM}^2}\sum_{\ell=1}^{3}\left(\left(im_\ell + \beta\sigma_{(\ell)}\,(d-4)\right)\mathrm{Tr}(\phi^0[\phi_{2\ell+2},\phi_{2\ell+3}])\right)\,. \tag{6.25}$$

However, under a supersymmetry transformation the Lagrangian changes by

$$\delta_\epsilon\mathcal{L} = \frac{1}{2g_{YM}^2}\left(\beta(d-4) + i\sum_{\ell=1}^{3}\sigma_{(\ell)}m_\ell\right)\mathrm{Tr}\left(\epsilon\Lambda\Gamma^{I_m I_n}\chi_\ell[\phi_{I_m},\phi_{J_n}]\right)\varepsilon^{\ell mn}\,. \tag{6.26}$$

The only way to get rid of this term is to set

$$\beta(d-4) + i\sum_{\ell=1}^{3}\sigma_{(\ell)}m_{(\ell)} = 0\,. \tag{6.27}$$

One might have expected that the leftover term in $\delta_\epsilon\mathcal{L}$ could have been cancelled by modifying the Lagrangian with a cubic term of the form $\sim \phi_{I_m}\phi_{I_n}\phi_{I_l}$. However, one can quickly check that this will not work because of the reality conditions imposed on the original spinors Ψ.

Another way to understand the origin of (6.27) is to consider the reduction of $\mathcal{N}=4$ in four dimensions down to three dimensions. To avoid unnecessary complications we assume the space is flat. In three dimensions, $\mathcal{N}=2$ SYMcan have two types of mass terms, real and complex [67, 68]. Complex masses descend directly from an $\mathcal{N}=1$ superpotential in four dimensions. However, a real mass arises from a Wilson line of a background $U(1)$ gauge field [68][3]. Writing the 4-dimensional Lagrangian in terms of $\mathcal{N}=1$ superfields, one has the term

$$\int d^2\theta d^2\bar{\theta}\exp(q_iU)\mathrm{Tr}(Q_i^\dagger e^V Q_i e^{-V})\,, \tag{6.28}$$

where V is the vector superfield for the $SU(N)$ gauge theory and U is the superfield for the background $U(1)$. The q_i's are the charges of the chiral multiplets under this $U(1)$. If we then compactify down to three dimensions, turn on the background Wilson line and integrate around the compactified dimension, (6.28) becomes

$$R\int d^2\theta d^2\bar{\theta}\mathrm{Tr}(Q_i^\dagger e^V Q_i e^{-V}) + \int d^2\theta'(q_i\Delta\Phi)\mathrm{Tr}(Q_i^\dagger e^V Q_i e^{-V}) \tag{6.29}$$

where R is the size of the compactified circle, which can be absorbed into the gauge coupling. The three-dimensional Grassmann variables are of the form θ_α and $\bar{\theta}_\alpha$, while $d^2\theta' \equiv (d\theta + d\bar{\theta})^2$. For the Wilson line we assume that $U_\mu = \nabla_\mu\Phi$ along the compactified direction. The second term in (6.29) is the contribution for a

[3]In Euclidean space the real masses do not have to be real, but we will continue to use this term.

real mass, $m_i^R = q_i \Delta\Phi/R$. In the large r limit, (6.23) and (6.25) arise from such a term, with $m_\ell^R = \sigma_{(\ell)} m_{(\ell)}$.

However, the four-dimensional $\mathcal{N} = 4$ Lagrangian has a term in the superpotential proportional to $\mathrm{Tr}(Q_i Q_j Q_k)\varepsilon^{mnk}$ which descends directly to the three-dimensional superpotential. In order to couple the background $U(1)$ field to the theory, this term in the superpotential needs to be gauge invariant. This requires setting $q_1 + q_2 + q_3 = 0$, which immediately means that the sum of the real masses is zero. Putting the theory on the sphere modifies this condition to (6.27).

We can also understand (6.27) using the three-dimensional $\mathcal{N} = 2$ superalgebra [67, 68],

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\} = i\,\sigma^\mu_{\alpha\beta} P_\mu + i\,m^R \varepsilon_{\alpha\beta}\,, \tag{6.30}$$

where the real mass appears explicitly in the algebra as a central charge. The contribution of the superpotential to the action is

$$\int d^3x\, d^2\theta\, W + \text{c.c.}\,. \tag{6.31}$$

If the superpotential has the term $\mathrm{Tr}(Q_i Q_j Q_k)\varepsilon^{mnk}$ then acting with $\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\}$ on (6.31) gives a term proportional to $m_1^R + m_2^R + m_3^R$. Hence, supersymmetry requires the sum to be zero.

6.1.3 Off-shell supersymmetry

We need an off-shell formulation of supersymmetry in order to localize. One must also ensure that the supersymmetry transformations close in the algebra. To this end we select a particular Killing spinor ϵ and introduce seven auxiliary fields K_m and bosonic pure spinors ν_m with $m = 1 \ldots 7$. These pure spinors satisfy the orthonormality conditions (B.8). The off-shell Lagrangian has the additional term

$$\mathcal{L}_{aux} = \frac{1}{g_{YM}^2}\,\mathrm{Tr} K^m K_m\,. \tag{6.32}$$

When reducing the number of supersymmetries we split the pure spinors accordingly. With 16 supersymmetries the full set of transformations are [50]

$$\begin{aligned}
\delta_\epsilon A_M =& \epsilon\,\Gamma_M \Psi\,,\\
\delta_\epsilon \Psi =& \tfrac{1}{2}\Gamma^{MN} F_{MN}\epsilon + \frac{\alpha_I}{2}\Gamma^{\mu I}\phi_I \nabla_\mu\,\epsilon + K^m \nu_m\,,\\
\delta_\epsilon K^m =& -\nu^m \not{D}\Psi + \beta(d-4)\nu^m \Lambda\Psi\,.
\end{aligned} \tag{6.33}$$

Acting twice with the supersymmetry transformation on the gauge fields one finds

$$\delta_\epsilon^2 A_\mu = -v^\nu F_{\nu\mu} + [D_\mu, v^I \phi_I]\,, \tag{6.34}$$

which is the Lie derivative of A_μ along the $-v^\nu$ direction, plus a gauge transformation. Likewise, the action on the scalar fields is

$$\delta_\epsilon^2 \phi_I = -v^\nu D_\nu \phi_I - [v^J \phi_J, \phi_I] - \frac{1}{2}\alpha_I \beta d\, \epsilon \tilde{\Gamma}_{IJ} \Lambda \epsilon\, \phi^J\,, \tag{6.35}$$

where again we have a Lie derivative plus a gauge transformation. The last term in eq. (6.35) is an R-symmetry transformation. The transformation on the fermions is

$$\begin{aligned} \delta_\epsilon^2 \Psi = &- v^N D_N \Psi - \frac{1}{4}(\nabla_{[\mu} v_{\nu]})\Gamma^{\mu\nu}\Psi \\ &- \frac{1}{2}\beta(\epsilon\tilde{\Gamma}^{mn}\Lambda\epsilon)\Gamma_{mn}\Psi - \frac{1}{2}(d-3)\beta(\epsilon\tilde{\Gamma}^{AB}\Lambda\epsilon)\Gamma_{AB}\Psi\,, \end{aligned} \tag{6.36}$$

where the terms in the last line are R-symmetry transformations. Finally, the transformation on the auxiliary fields is

$$\delta_\epsilon^2 K^m = -v^M D_M K^m - (\nu^{[m}\Gamma^\mu \nabla_\mu \nu^{n]})K_n + (d-4)\beta(\nu^{[m}\Lambda\nu^{n]})K_n\,, \tag{6.37}$$

where the last two terms are generators of an internal $SO(7)$ symmetry.

With fewer supersymmetries the fields divide up into vector, hyper or chiral multiplets along with the accompanying modifications to the α_I. For the case of eight supersymmetries, we split the pure spinors such that $\Gamma\nu_m = +\nu_m$ for $m = 1,2,3$, while $\Gamma\nu_m = -\nu_m$ for $m = 4,5,6,7$. The associated auxiliary fields K^m belong to the vector and hypermultiplet respectively. Their transformations are

$$\begin{aligned} \delta_\epsilon K^m &= -\nu^m \not{D}\psi + \beta\,(d-4)\,\nu^m \Lambda\psi, \qquad \text{for} \qquad m = 1,2,3, \\ \delta_\epsilon K^m &= -\nu^m \not{D}\chi - 2i\mu\beta\nu^m\Lambda\chi, \qquad \text{for} \qquad m = 4,5,\cdots,7. \end{aligned} \tag{6.38}$$

Here $\mu \equiv mr$ is a dimensionless parameter.

With reduced supersymmetry, the transformations in eq. (6.34) are unchanged while those in eq. (6.35) are modified by the change in the α_I. For fermions in the vector multiplet eq. (6.36) holds with Ψ replaced by ψ. For fermions in the hypermultiplet eq. (6.36) becomes

$$\begin{aligned} \delta_\epsilon^2 \chi = &- v^N D_N \chi - \frac{1}{4}(\nabla_{[\mu} v_{\nu]})\Gamma^{\mu\nu}\chi \\ &- \frac{1}{2}\beta(\epsilon\tilde{\Gamma}^{IJ}\Lambda\epsilon)\Gamma_{IJ}\chi - 2i\mu\beta(\epsilon\tilde{\Gamma}^A\Lambda\epsilon)\tilde{\Gamma}_A\Lambda\chi\,. \end{aligned} \tag{6.39}$$

For the auxiliary fields, equation (6.37) splits into two:

$$\begin{aligned}\delta_\epsilon^2 K^m &= -v^M D_M K^m - (\nu^{[m}\Gamma^\mu \nabla_\mu \nu^{n]})K_n + (d-4)\beta(\nu^{[m}\Lambda\nu^{n]})K_n \\ \delta_\epsilon^2 K^m &= -v^M D_M K^m - (\nu^{[m}\Gamma^\mu \nabla_\mu \nu^{n]})K_n - 2i\mu\beta(\nu^{[m}\Lambda\nu^{n]})K_n\,,\end{aligned} \tag{6.40}$$

where the first equation is for $m = 1,2,3$ and the second is for $m = 4,5,6,7$. Invariance under off-shell supersymmetry for the Lagrangian supplemented with $\mathcal{L}_{aux}$ can be shown by a computation that is almost identical to the one in [50] for 16 supersymmetries.

Reducing the number of supersymmetries to four, we split the pure spinors further as follows.

$$\Gamma'\nu_m = +\nu_m \qquad \text{for} \qquad m = 1,4,5, \qquad \Gamma'\nu_m = -\nu_m \qquad \text{for} \qquad m = 2,3,6,7. \tag{6.41}$$

The transformations of the auxiliary fields are

$$\begin{aligned}\delta_\epsilon K^m &= -\nu^m \not{D}\psi + \beta\,(d-4)\,\nu^m\Lambda\psi, && \text{for} \quad m=1,\\ \delta_\epsilon K^m &= -\nu^m \not{D}\chi_1 - 2i\mu_1\beta\nu^m\Lambda\chi_1, && \text{for} \quad m=2,3,\\ \delta_\epsilon K^m &= -\nu^m \not{D}\chi_2 - 2i\mu_2\beta\nu^m\Lambda\chi_2, && \text{for} \quad m=4,5,\\ \delta_\epsilon K^m &= -\nu^m \not{D}\chi_3 - 2i\mu_3\beta\nu^m\Lambda\chi_3, && \text{for} \quad m=6,7.\end{aligned} \tag{6.42}$$

With $\mu_\ell \equiv m_\ell r$ being dimensionless parameters. As before, equation eq. (6.34) is unchanged and eq. (6.35) is modified by the change in α_I. For two supersymmetry variations of the auxiliary field we have a straightforward generalization of eq. (6.40), where we split the auxiliary fields into four different types. Two supersymmetry variations of the chiral multiplet fermions take the following form

$$\begin{aligned}\delta_\epsilon^2\chi_\ell = -v^N D_N\chi_\ell &- \frac{1}{4}(\nabla_{[\mu}v_{\nu]})\Gamma^{\mu\nu}\chi_\ell \\ &- \frac{1}{2}\beta(\epsilon\tilde{\Gamma}^{IJ}\Lambda\epsilon)\Gamma_{IJ}\chi_\ell - 2i\mu_\ell\beta(\epsilon\tilde{\Gamma}^A\Lambda\epsilon)\tilde{\Gamma}_A\Lambda\chi_\ell\,.\end{aligned} \tag{6.43}$$

Invariance of the Lagrangian under off-shell supersymmetry follows as in the case of eight and 16 supersymmetries.

6.2 The localization Lagrangian

In this section we present the localization argument and compute the quadratic fluctuations about the fixed point locus. We also add a gauge fixing term in the Lagrangian and give the precise form of the partition function in terms of the determinants of the quadratic fluctuations around the fixed point locus. We only consider contributions in the zero instanton sector where the fixed point locus has a vanishing gauge field.

6.2.1 Fixed point locus

We modify the partition function path integral as follows:

$$\mathcal{Z}\left[t\right] \equiv \int \mathcal{D}\Phi\, e^{-S-tQV}, \tag{6.44}$$

where $\mathcal{D}\Phi$ denotes the integration measure for all the fields, Q is a fermionic symmetry of both the integration measure and the action and QV is positive semi-definite. The partition function is then independent of the parameter t. This allows us to evaluate the partition function at $t \to \infty$, where it only receives contributions from quadratic fluctuations of the fields about the locus of the zeros of QV.

For our purposes we choose Q to be the supersymmetry transformation generated by ϵ, and V to be

$$V = \int d^d x \sqrt{g}\ \mathrm{Tr}'\left(\Psi \overline{\delta_\epsilon \Psi}\right), \tag{6.45}$$

where Tr′ is a positive definite inner product on the Lie algebra, which can be different than the product used in the original action. We will drop the Tr′ sign henceforth for notational simplicity. $\overline{\delta_\epsilon \Psi}$ is given by

$$\overline{\delta_\epsilon \Psi} = \frac{1}{2}\tilde{\Gamma}^{MN} F_{MN}\Gamma^0\epsilon + \frac{\alpha_I}{2}\tilde{\Gamma}^{\mu I}\phi_I\Gamma^0\nabla_\mu\epsilon - K^m\Gamma^0\nu_m. \tag{6.46}$$

So, QV will be

$$QV = \int d^d x\, \sqrt{g}\, \delta_\varepsilon \Psi \overline{\delta_\varepsilon \Psi} - \int d^d x\, \sqrt{g}\, \Psi \delta_\varepsilon \left(\overline{\delta_\varepsilon \Psi}\right) \equiv \int d^d x \sqrt{g}\, \mathcal{L}^{\mathrm{b}} + \int d^d x \sqrt{g} \mathcal{L}^{\mathrm{f}}. \tag{6.47}$$

The first and second terms in the above equation contain the bosonic and fermionic part of the localization Lagrangian respectively. We now find the locus where the path integral localizes when $t \to \infty$. The bosonic part is [50]

$$\begin{aligned}\mathcal{L}^{\mathrm{b}} &= \frac{1}{2}F_{MN}F^{MN} - \frac{1}{4}F_{MN}F_{M'N'}\left(\epsilon\Gamma^{MNM'N'0}\epsilon\right) \\ &+ \frac{\beta d\alpha_I}{4}F_{MN}\phi_I\left(\epsilon\Lambda(\tilde{\Gamma}^I\tilde{\Gamma}^{MN}\Gamma^0 - \tilde{\Gamma}^0\Gamma^I\Gamma^{MN})\epsilon\right) \\ &- K^m K_m v^0 - \beta d\alpha_0\phi_0 K^m\left(\nu_m\Lambda\epsilon\right) + \frac{\beta^2 d^2}{4}\sum_I(\alpha_I)^2\phi_I\phi^I v^0.\end{aligned} \tag{6.48}$$

We choose the spinor ϵ such that $v^0 = 1$ and $v^8 = v^9 = 0$. Then the fixed point condition in the zero instanton sector can be written as

$$\nabla_\mu \phi^I \nabla^\mu \phi^I - \left(K^m + 2\beta(d-3)\phi_0\,(\nu_m \Lambda \epsilon)\right)^2 + \frac{\beta^2 d^2}{4} \sum_{I \neq 0} (\alpha_I)^2 \phi_I \phi^I = 0, \tag{6.49}$$

All terms on the lefthand side of the above equation are positive definite if fields K^m and ϕ_0 are imaginary. So the fixed point locus is given by

$$K^m = -2\beta(d-3)\phi_0\,(\nu_m \Lambda \epsilon)\,, \qquad \phi_0 = \text{const} = \phi_0^{\text{cl}} \equiv \frac{\sigma}{r}, \qquad \phi_J = 0 \quad (J \neq 0)\,. \tag{6.50}$$

The dimensionless variable σ is an element of the Lie algebra and parameterizes the fixed point locus. The action evaluated at the fixed point becomes

$$S_{\text{fp}} = \frac{V_d}{g_{\text{YM}}^2} \frac{(d-1)(d-3)}{r^2} \text{Tr}\left(\phi_0^{\text{cl}} \phi_0^{\text{cl}}\right) = \frac{8\pi^{\frac{d+1}{2}} r^{d-4}}{g_{\text{YM}}^2 \Gamma\left(\frac{d-3}{2}\right)} \text{Tr}\, \sigma^2, \tag{6.51}$$

where V_d is the volume of the d-dimensional sphere.

6.2.2 Quadratic fluctuations

The next step is to move away from the localization locus by perturbing the fields about their fixed point values. We write

$$\Phi' = \Phi^{\text{cl}} + \frac{1}{\sqrt{t}} \Phi, \tag{6.52}$$

for all fields Φ' in QV, with Φ^{cl} being their value at the fixed point. In the $t \to \infty$ limit, the only terms that survive in the localization Lagrangian are quadratic in the perturbations Φ. Details of the computation of quadratic fluctuations about the fixed point locus are given in Appendix C. Here we briefly summarize our results.

The bosonic fluctuations for the vector multiplet takes the following form

$$\begin{aligned} \mathcal{L}^{\text{b}}_{\text{v.m}} &= A^{\tilde{M}}\, \mathcal{O}_{\tilde{M}}{}^{\tilde{N}}\, A_{\tilde{N}} - [A_{\tilde{M}}, \phi_0^{\text{cl}}][A^{\tilde{M}}, \phi_0^{\text{cl}}] \\ &\quad - K^m K_m - 4\beta(d-3)\phi_0 K^m (\nu_m \Lambda \epsilon) - \phi_0 \left(-\nabla^2 + 4\beta^2 (d-3)^2\right) \phi_0. \end{aligned} \tag{6.53}$$

The indices with a tilde take the values as defined below

$$\tilde{M} = \{\mu, i\}, \qquad \mu = 1, 2, \cdots, d, \qquad i = d+1, \cdots D, \tag{6.54}$$

where $D = 5(3)$ for theories with eight(four) supersymmetries. A_μ is the usual vector field, while fields A_i denote scalars in the vector multiplet other than ϕ_0. The operator $\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}$ is defined as

$$\mathcal{O}_{\tilde{M}}{}^{\tilde{N}} = -\delta_{\tilde{M}}{}^{\tilde{N}}\nabla^2 + \alpha_{\tilde{M}}{}^{\tilde{N}} - 2\beta(d-3)\epsilon\Gamma_{\tilde{M}}{}^{\nu\tilde{N}89}\epsilon\nabla_\nu. \tag{6.55}$$

$\alpha_{\tilde{M}}{}^{\tilde{N}}$ is a diagonal matrix given by

$$\alpha_{\tilde{M}}{}^{\tilde{N}} = 4\beta^2 \begin{pmatrix} (d-1)\,\delta_\mu^\nu & 0 \\ 0 & \delta_i^j \end{pmatrix}. \tag{6.56}$$

The fermionic fluctuations for the vector multiplet can be written as

$$\begin{aligned} \mathcal{L}^{\mathrm{f}}_{\mathrm{v.m}} &= (\psi \not{\nabla} \psi) + \left(\psi\Gamma^0\left[\phi_0^{\mathrm{cl}}, \psi\right]\right) - \frac{1}{2}(d-3)\beta v^{\tilde{M}}\left(\psi\Gamma^0\tilde{\Gamma}_{\tilde{M}}\Lambda\psi\right) \\ &\quad - \frac{1}{4}(d-3)\beta\left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\left(\psi\Gamma^0\Gamma_{\tilde{M}\tilde{N}}\psi\right) + m_\psi\left(\psi\Lambda\psi\right). \end{aligned} \tag{6.57}$$

Here $m_\psi = \frac{d-1}{2}$ for eight supersymmetries and $m_\psi = (d-2)$ for four supersymmetries.

For theories with eight supersymmetries we have one hypermultiplet. The bosonic part contains four scalars. Their contribution to the quadratic fluctuations can be written as

$$\begin{aligned} \mathcal{L}^{\mathrm{b}}_{\mathrm{h.m}} &= \sum_{i=6}^{9}\left[\phi_i\left(-\nabla^2 + \beta^2(d-2+2i\sigma_i\mu)^2\right)\phi_i - [\phi_0^{\mathrm{cl}}, \phi_i][\phi_0^{\mathrm{cl}}, \phi_i]\right] \\ &\quad + 4\beta\,(2i\mu - 1)\,\phi_6 v^\mu\nabla_\mu\phi_7 + 4\beta\,(2i\mu+1)\,\phi_8 v^\mu\nabla_\mu\phi_9. \end{aligned} \tag{6.58}$$

For the hypermultiplet fermions we have

$$\mathcal{L}^{\mathrm{f}}_{\mathrm{h.m}} = (\chi\not{\nabla}\chi) + (\chi\Gamma^0[\phi_0^{\mathrm{cl}}, \chi]) - \frac{1}{2}\beta\left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\left(\chi\Gamma^0\Gamma_{\tilde{M}\tilde{N}}\chi\right) + 2i\mu\beta v^{\tilde{N}}\left(\chi\Gamma^0\tilde{\Gamma}_{\tilde{N}}\Lambda\chi\right). \tag{6.59}$$

For the case of four supersymmetries we have three chiral multiplets. The chiral multiplet part contains six scalars. Their contribution to the quadratic fluctuations is given by

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}_{\mathrm{c.m}} &= \sum_{\ell=1}^{3}\left[\phi_{I_\ell}\left(-\nabla^2+\beta^2(d-2+2i\sigma_{(\ell)}\mu_\ell)^2\right)\phi^{I_\ell}-[\phi_0^{\mathrm{cl}},\phi_{I_\ell}][\phi_0^{\mathrm{cl}}\phi^{I_\ell}]\right]\\ &\quad +4\beta\left(2i\mu_\ell-\sigma_{(\ell)}\right)\phi_{2\ell+2}v^\mu\nabla_\mu\phi_{2\ell+3}.\end{aligned} \tag{6.60}$$

Finally the contribution from the chiral multiplet fermions is

$$\begin{aligned}\mathcal{L}^{\mathrm{f}}_{\mathrm{c.m}} &= \sum_{\ell=1}^{3}\left(\chi_\ell\not\nabla\chi_\ell\right)+\left(\chi_\ell\Gamma^0[\phi_0^{\mathrm{cl}},\chi_\ell]\right)-\frac{1}{2}\beta\left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\left(\chi_\ell\Gamma^0\Gamma_{\tilde{M}\tilde{N}}\chi_\ell\right)\\ &\quad +\sigma_{(\ell)}\beta\left(2i\mu_\ell v^{\tilde{N}}\left(\chi_i\Gamma^0\tilde{\Gamma}_{\tilde{N}}\Lambda\chi_\ell\right)+\chi_i\Lambda\chi_\ell\right).\end{aligned} \tag{6.61}$$

6.2.3 Gauge fixing

With the expressions for quadratic fluctuations in hand, we give the precise form of the partition function in terms of quadratic fluctuations. To compute the partition function we need to add a gauge fixing term. In the computation of the quadratic fluctuations, we employed the Lorenz gauge, so we need to use the following gauge fixing term

$$S_{\mathrm{g.f}} = -\int d^dx\sqrt{g}\,\mathrm{Tr}\left(b\nabla_\mu A'^\mu-\bar{c}\nabla^2c\right). \tag{6.62}$$

Here b is the Lagrange multiplier which enforces the Lorenz gauge condition in the path integral. $c, \bar{c}$ are the usual Fadeev-Popov ghosts. A'_μ denotes the off-shell gauge field which can be decomposed as

$$A'_\mu = A_\mu+\nabla_\mu\phi, \tag{6.63}$$

where A_μ is divergenceless and ϕ encodes the pure divergence part.

To compute the partition function one now has to integrate over the following set of fields:

$$b, c, \bar{c}, \phi, K_m, \phi_0, A_\mu, \phi_{I\neq 0}, \Psi. \tag{6.64}$$

The first six give the following contributions:

- The b ghosts give a factor of $\delta\left(\nabla_\mu A'^\mu\right)=\delta\left(\nabla^2\phi\right)$.
- The c and $\bar{c}$ ghosts give a factor of $\det\left(\nabla^2\right)$.
- The gauge parameter ϕ has two contributions. There is a Jacobian factor $\sqrt{\det\nabla^2}$ coming from the change of integration measure $\mathcal{D}\nabla_\mu\phi\to\mathcal{D}\phi$, while the integration over ϕ gives a factor of $\left(\det\nabla^2\right)^{-1}$

coming from the delta function $\delta\left(\nabla^2\phi\right)$.

- The contribution of the auxiliary fields K_m is trivial. It gets rid of the mass term for the scalar field ϕ_0 in the quadratic fluctuations.

- The scalar ϕ_0 gives a factor of $(\sqrt{\det \nabla^2})^{-1}$.

These factors cancel and the partition function reduces to

$$\mathcal{Z} = \int d\sigma e^{-S_{\text{fp}}(\sigma)} \int \mathcal{D}A_\mu \mathcal{D}\phi_{I\neq 0}\mathcal{D}\Psi e^{-S_{\text{quad}}(\phi_0=2\beta\sigma)}. \tag{6.65}$$

Since the integrand is invariant under the adjoint action of the gauge group, we can replace the integral over the entire Lie algebra with an integral over a Cartan subalgebra. This introduces a Vandermonde determinant and we can write the partition function, with some convenient normalization as follows:

$$\mathcal{Z} = \int [d\sigma]_{\text{Cartan}}\, e^{-S_{\text{fp}}(\sigma)} \prod_\alpha i\langle\alpha,\sigma\rangle \int \mathcal{D}A_\mu \mathcal{D}\phi_{I\neq 0}\mathcal{D}\Psi e^{-S_{\text{quad}}(\phi_0=2\beta\sigma)}. \tag{6.66}$$

Now, what is left to be computed is the integral over the fields A_μ, $\Phi_{I\neq 0}$ and Ψ. Before doing that, we comment on the decomposition of the fields and quadratic fluctuations in terms of the root vectors of the Lie algebra. Schematically, bosonic quadratic fluctuations are given by

$$\mathcal{L}^{\text{b}} = \text{Tr}'\left(\Phi\cdot \text{O}^{\text{b}}\cdot\Phi - \left[\Phi,\phi_0^{\text{cl}}\right]\left[\Phi,\phi_0^{\text{cl}}\right]\right). \tag{6.67}$$

We expand the field Φ in the Cartan-Weyl basis. The component of Φ along the Cartan generators only contributes an uninteresting ϕ_0^{cl} independent overall constant to the partition function, and so we do not need to focus on that part. Next, we can write Φ as:

$$\Phi = \sum_\alpha \Phi^\alpha E_\alpha, \tag{6.68}$$

where E_α are the root vectors of the Lie algebra. They are normalized so that $\text{Tr}'\left(E_\alpha E_\beta\right) = \delta_{\alpha+\beta}$. Using $[\sigma, E_\alpha] = \langle\alpha,\sigma\rangle E_\alpha$, the quadratic fluctuations can be written as

$$\mathcal{L}^{\text{b}} = \sum_\alpha \Phi^{-\alpha}\cdot\left(\text{O}^{\text{b}} + 4\beta^2\langle\alpha,\sigma\rangle^2\right)\cdot\Phi^\alpha. \tag{6.69}$$

Similarly the fermionic quadratic fluctuations can be decomposed as

$$\mathcal{L}^{\rm f} = {\rm Tr}'\left(\Psi\Gamma^0 {\rm O}^{\rm f}\Psi + \Psi\Gamma^0\left[\phi_0^{\rm cl},\Psi\right]\right) = \sum_\alpha \Psi^{-\alpha}\Gamma^0\left({\rm O}^{\rm f}+2\beta\langle\alpha,\sigma\rangle\right)\Psi^\alpha. \tag{6.70}$$

After integrating over the quadratic fluctuations in $\mathcal{L}^{\rm b,f}$ one gets:

$$\int \mathcal{D}\Phi\mathcal{D}\Psi e^{-\int d^dx\sqrt{g}(\mathcal{L}^{\rm b}+\mathcal{L}^{\rm f})} = \prod_\alpha \frac{\det\left({\rm O}^{\rm f}+2\beta\langle\alpha,\sigma\rangle\right)_\Psi}{\sqrt{\det\left({\rm O}^{\rm b}+4\beta^2\langle\alpha,\sigma\rangle^2\right)_\Phi}}. \tag{6.71}$$

Hence to compute the one-loop determinants one needs to diagonalize the action of the "quadratic" operators $\rm O^{f,b}$ appearing in the quadratic fluctuations. We turn to this computation in the next section.

6.3 Determinants for eight supersymmetries

In this section we compute the determinants for theories with eight supersymmetries. We compute the determinants for bosons and fermions separately and then combine them to see that after a large cancellation we get

$$\begin{aligned} Z^{\rm vec}_{1-\rm loop}\prod_\alpha i\langle\alpha,\sigma\rangle &= \prod_\alpha\prod_{k=0}^{\infty}\left[\left(k+i\langle\alpha,\sigma\rangle\right)\left(k+d-2+i\langle\alpha,\sigma\rangle\right)\right]^{N_{k,d}}, \\ Z^{\rm hyp}_{1-\rm loop} &= \prod_\alpha\prod_{k=0}^{k=\infty}\left[\left(k+\frac{d-2}{2}+i\mu+i\langle\alpha,\sigma\rangle\right)\left(k+\frac{d-2}{2}-i\mu-i\langle\alpha,\sigma\rangle\right)\right]^{-N_{k,d}}. \end{aligned} \tag{6.72}$$

The factor $N_{k,d}$ in the exponent is given by

$$N_{k,d} = \binom{k+\frac{d-k'+1}{2}-1}{k} = \frac{\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-2)}. \tag{6.73}$$

This matches exactly with the conjectured form in [6]. We now provide the derivation of eq. (6.72).

6.3.1 Vector multiplet

We first compute the determinant for the vector multiplet. We start by introducing a complete set of basis elements that span spinor and vector harmonics on S^d. Then we diagonalize the action of the quadratic operator on these basis elements.

Complete set of basis elements

To compute the determinants we need to diagonalize the action of the quadratic operator. This can be done by using a suitable set of basis elements. To this end, we define spinors

$$\eta_\pm \equiv \left(1 \pm i\Gamma^{67}\right)\epsilon = \left(1 \mp i\Gamma^{89}\right)\epsilon, \qquad \tilde{\eta}_\pm \equiv \left(\Gamma^{68} \pm i\Gamma^{69}\right)\epsilon, \tag{6.74}$$

which satisfy

$$\Gamma^{89}\eta_\pm = \pm i\eta_\pm, \qquad \tilde{\Gamma}_0 v^{\tilde{M}}\Gamma_{\tilde{M}}\eta_\pm = \eta_\pm, \tag{6.75}$$

$$\Gamma^{89}\tilde{\eta}_\pm = \pm i\tilde{\eta}_\pm, \qquad \tilde{\Gamma}_0 v^{\tilde{M}}\Gamma_{\tilde{M}}\tilde{\eta}_\pm = \tilde{\eta}_\pm. \tag{6.76}$$

We can now build a basis for the vector multiplet fermions by using the spinors $\eta_\pm$, $\tilde{\eta}_\pm$ and the scalar spherical harmonics Y^k_m. Scalar spherical harmonics are labelled by the eigenvalues of the Laplacian and the Cartan generator along the vector v^μ:

$$\nabla^2 Y^k_m = -4\beta^2 k(k+d-1), \qquad v^\mu\nabla_\mu Y^k_m = 2i\beta m Y^k_m. \tag{6.77}$$

The definitions of our spinor harmonics and their eigenvalues under operators Γ^{89} and $\tilde{\Gamma}_0 v^{\tilde{M}}\Gamma_{\tilde{M}}$ are given in the following table.

Spinor harmonics	$\mathbf{\Gamma^{89}}$**-eigenvalue**	$\mathbf{\tilde{\Gamma}_0 v^{\tilde{M}}\Gamma_{\tilde{M}}}$**-eigenvalue**
$\mathcal{X}^1_\pm \equiv Y^k_m\eta_\pm$	$\pm i$	$+1$
$\tilde{\mathcal{X}}^1_\pm \equiv Y^k_m\tilde{\eta}_\pm$	$\pm i$	-1
$\mathcal{X}^2_\pm \equiv \tilde{\Gamma}_0\Gamma^{\tilde{M}}\hat{\nabla}_{\tilde{M}}Y^k_m\eta_\pm$, for $m \neq \pm k$	$\pm i$	$+1$
$\tilde{\mathcal{X}}^2_\pm \equiv \tilde{\Gamma}_0\Gamma^{\tilde{M}}\hat{\nabla}_{\tilde{M}}Y^k_m\tilde{\eta}_\pm$, for $m \neq \mp k$	$\pm i$	-1

Table 6.1: Spinor harmonics basis and corresponding eigenvalues.

Here $\hat{\nabla}_{\tilde{M}}$ is defined as

$$\hat{\nabla}_{\tilde{M}} \equiv \nabla_{\tilde{M}} - v_{\tilde{M}}v\cdot\nabla. \tag{6.78}$$

$\mathcal{X}^2_\pm$ and $\mathcal{X}^2_\mp$ vanish identically for $m = \pm k$ (see section D.1 for a proof). The set of spinors with a '+' subscript is related to the set with a '−' subscript via complex conjugation. We take the standard approach [5] that the Euclidean action is an analytical functional in the space of complexified fields and integrate over a certain half-dimensional subspace in the path integral. With this in mind, we will focus on the basis for spinors with Γ^{89} eigenvalue $+i$.

We show that set of spinors in Table 6.1 provide a complete set of basis elements for the vector multiplet fermions on S^d. To do so, we compute the action of the Dirac operator, $\tilde{\Gamma}_0 \not{\nabla}$, on these spinors using

$$\tilde{\Gamma}_0 \not{\nabla} \eta_+ = +id\eta_+, \qquad \tilde{\Gamma}_0 \not{\nabla} Y_m^k = \tilde{\Gamma}_0 \Gamma^{\tilde{M}} \hat{\nabla}_{\tilde{M}} Y_m^k + 2im\beta Y_m^k. \tag{6.79}$$

This gives

$$\tilde{\Gamma}_0 \Gamma^\mu \nabla_\mu \mathcal{X}_+^1 = 2i\beta \left(m + \frac{d}{2}\right) \mathcal{X}_+^1 + \mathcal{X}_+^2. \tag{6.80}$$

Next we note that $\mathcal{X}_+^2$ can be written as

$$\mathcal{X}_+^2 = \tilde{\Gamma}_0 \not{\nabla} Y_m^k \eta_+ - 2im\beta \mathcal{X}_+^1. \tag{6.81}$$

The action of $\tilde{\Gamma}_0 \not{\nabla}$ can now be worked out by using eq. (6.79), eq. (6.80) and that $\left(\tilde{\Gamma}_0 \not{\nabla}\right)^2 = \nabla^2$, which gives

$$\tilde{\Gamma}_0 \Gamma^\mu \nabla_\mu \mathcal{X}_+^2 = -4\beta^2 (k-m)(k+m+d-1)\mathcal{X}_+^1 - 2i\beta \left(m + \frac{d-2}{2}\right) \mathcal{X}_+^2. \tag{6.82}$$

Similarly, for $\tilde{\mathcal{X}}_+^{1,2}$ one finds

$$\tilde{\Gamma}_0 \Gamma^\mu \nabla_\mu \tilde{\mathcal{X}}_+^1 = 2i\beta \left(m - \frac{d}{2}\right) \tilde{\mathcal{X}}_+^1 + \tilde{\mathcal{X}}_+^2, \tag{6.83}$$

$$\tilde{\Gamma}_0 \Gamma^\mu \nabla_\mu \tilde{\mathcal{X}}_+^2 = -4\beta^2 (k+m)(k-m+d-1)\tilde{\mathcal{X}}_+^1 - 2i\beta \left(m - \frac{d-2}{2}\right) \tilde{\mathcal{X}}_+^2. \tag{6.84}$$

Now we diagonalize the action of $\tilde{\Gamma}_0 \not{\nabla}$ on the spinor basis to get the eigenvalues

$$\pm 2i\beta \left(k + \frac{d}{2}\right), \qquad \mp 2i\beta \left(k - 1 + \frac{d}{2}\right) \qquad \text{for} \qquad m \neq +k. \tag{6.85}$$

By shifting k in the second set of eigenvalues, we can arrange the spinor harmonics into two sets of eigenstates of the Dirac operator, with eigenvalues $\pm 2i\beta \left(k + \frac{d}{2}\right)$ whose degeneracy $\deg_f(k,d)$, is given by

$$\deg_f(k,d) = \mathcal{D}_k(d,0) + \mathcal{D}_{k+1}(d,0) - N_{k+1,d}, \tag{6.86}$$

where $\mathcal{D}_k(d,r)$ is the total degeneracy of symmetric traceless, divergence-less rank-r tensors defined on S^d[69]. $N_{m,d}$ is the number of scalar harmonics Y_m^k for the case of eight supersymmetries. The explicit expressions for these degeneracies are given in chapter D. Using these expressions we get

$$\deg_f(k,d) = 4\frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)}. \tag{6.87}$$

For $d = 4, 5$ this is equal to the degeneracy of spinor harmonics on S^d [70] and for $d = 2, 3$ this is twice the degeneracy of spinor harmonics, as expected. Hence, we conclude that the set of spinors defined in Table 6.1 provides a complete basis for the vector multiplet fermions in the case of eight supersymmetries.

Next we use the spinor basis to construct a basis for the fields $A_{\tilde{M}}$. We define

$$\begin{aligned}
\mathcal{A}^1_{\tilde{M}} &\equiv (\epsilon\Gamma_{\tilde{M}}\mathcal{X}^1_+) + c^1\nabla_{\tilde{M}}Y^k_m \;=\; (\epsilon\Gamma_{\tilde{M}}\mathcal{X}^1_-) + c^1\nabla_{\tilde{M}}Y^k_m \;=\; v_{\tilde{M}}Y^k_m + c^1\nabla_{\tilde{M}}Y^k_m, \\
\mathcal{A}^2_{\tilde{M}} &= \frac{i}{2}\left(\epsilon\Gamma_{\tilde{M}}\mathcal{X}^2_+ - \epsilon\Gamma_{\tilde{M}}\mathcal{X}^2_-\right) + c^2\nabla_{\tilde{M}}Y^k_m \;=\; \epsilon\Gamma_{\tilde{M}}{}^{\mu}\Lambda\epsilon\nabla_\mu Y^k_m + c^2\nabla_{\tilde{M}}Y^k_m, \\
\mathcal{A}^3_{\tilde{M}} &\equiv \epsilon\Gamma_{\tilde{M}}{}^{\mu}\Gamma^{069}\epsilon\nabla_\mu Y^k_m \;=\; \frac{-i}{2}\left(\epsilon\Gamma_{\tilde{M}}\tilde{\mathcal{X}}^2_+ - \epsilon\Gamma_{\tilde{M}}\tilde{\mathcal{X}}^2_-\right), \\
\mathcal{A}^4_{\tilde{M}} &\equiv \epsilon\Gamma_{\tilde{M}}{}^{\mu}\Gamma^{079}\epsilon\nabla_\mu Y^k_m \;=\; \frac{1}{2}\left(\epsilon\Gamma_{\tilde{M}}\tilde{\mathcal{X}}^2_+ + \epsilon\Gamma_{\tilde{M}}\tilde{\mathcal{X}}^2_-\right).
\end{aligned} \tag{6.88}$$

Here c^1, c^2 are constants which are determined by the condition that $\mathcal{A}^1_\mu$ and $\mathcal{A}^2_\mu$ should be divergenceless:

$$c^1 = \frac{im}{2\beta k(k+d-1)}, \qquad c^2 = \frac{(d-1)im}{k(k+d-1)}. \tag{6.89}$$

There is another bilinear involving spinors $\mathcal{X}^2_\pm$, which is equal to a linear combination of a pure divergence term and $\mathcal{A}^1_{\tilde{M}}$

$$\epsilon\Gamma_{\tilde{M}}\mathcal{X}^2_+ + \epsilon\Gamma_{\tilde{M}}\mathcal{X}^2_- = 2\nabla_{\tilde{M}}Y^k_m - 4im\beta v_{\tilde{M}}Y^k_m. \tag{6.90}$$

Since $\mathcal{X}^2_\pm$ vanishes identically for $m = \pm k$, we see that $\mathcal{A}^1$ and $\mathcal{A}^2$ are not linearly independent for $m = \pm k$:

$$\mathcal{A}^2_{\tilde{M}} = -2k\beta\mathcal{A}^1_{\tilde{M}}, \qquad \text{for} \qquad m = \pm k. \tag{6.91}$$

Similarly, $\mathcal{A}^3$ and $\mathcal{A}^4$ are proportional to each other for $m = \pm k$.

We now show that the bosonic fields defined in eq. (6.88) provide a complete basis for bosons in the vector multiplet[4]. We do so by diagonalizing the action of ∇^2 on $\mathcal{A}_{\tilde{M}}$. It acts on the vector field $v_{\tilde{M}}$ to give

$$\nabla^2 v_\mu = -4\beta^2(d-1)v_\mu, \qquad \nabla^2 v_i = -4\beta^2 d v_i. \tag{6.92}$$

[4]Excluding the scalar field ϕ_0.

Using this along with

$$\nabla^2\nabla_\mu Y^k_m = -4\beta^2\left(k(k+d-1)-(d-1)\right)\nabla_\mu Y^k_m, \qquad \nabla^\mu v_{\tilde{M}} = 2\beta\epsilon\Gamma_{\tilde{M}}{}^\mu\Lambda\epsilon, \tag{6.93}$$

gives us the action of ∇^2 on the $\mathcal{A}^1_{\tilde{M}}$

$$\begin{aligned}\nabla^2\mathcal{A}^1_\mu &= -4\beta^2\left[k(k+d-1)+d-1\right]\mathcal{A}^1_\mu + 4\beta\mathcal{A}^2_\mu,\\ \nabla^2\mathcal{A}^1_i &= -4\beta^2\left[k(k+d-1)+d\right]\mathcal{A}^1_i + 4\beta\mathcal{A}^2_i.\end{aligned} \tag{6.94}$$

To find the action of ∇^2 on $\mathcal{A}^2_{\tilde{M}}$ we need to know how the operators ∇^λ and ∇^2 act on more complicated bilinears. Using the Killing spinor equation and that $\tilde{\epsilon} = \beta\Lambda\epsilon$, one gets

$$\begin{aligned}\nabla^2\epsilon\Gamma_\mu{}^\nu\Lambda\epsilon = -8\beta^2\epsilon\Gamma_\mu{}^\nu\Lambda\epsilon, &\qquad \nabla^2\epsilon\Gamma_i{}^\nu\Lambda\epsilon = -4\beta^2\epsilon\Gamma_i{}^\nu\Lambda\epsilon,\\ \nabla^\lambda\left(\epsilon\Gamma_{\tilde{M}}{}^\mu\Lambda\epsilon\right)\nabla_\lambda\nabla_\mu Y^k_m \;=\; 8\beta^3 k(k+d-1)v_{\tilde{M}}Y^k_m &+ 4\beta^2\delta_{\tilde{M}}{}^\mu\left(im\nabla_\mu Y^k_m + \epsilon\Gamma_\mu{}^\nu\Lambda\epsilon\nabla_\nu Y^k_m\right).\end{aligned} \tag{6.95}$$

Using these results we find that

$$\begin{aligned}\nabla^2\mathcal{A}^2_\mu &= 16\beta^3 k(k+d-1)\mathcal{A}^1_\mu - 4\beta^2\left(k(k+d-1)-(d-1)\right)\mathcal{A}^2_\mu,\\ \nabla^2\mathcal{A}^2_i &= 16\beta^3 k(k+d-1)\mathcal{A}^1_i - 4\beta^2\left(k(k+d-1)-(d-2)\right)\mathcal{A}^2_i.\end{aligned} \tag{6.96}$$

The action of ∇^2 on $\mathcal{A}^{3,4}$ can computed in a similar way. The following results are necessary for this calculation:

$$\nabla^2\epsilon\Gamma_{\tilde{M}}{}^\nu\Gamma^{0I9}\epsilon = -4\beta^2(d-2)\epsilon\Gamma_{\tilde{M}}{}^{\nu 0I9}\epsilon, \qquad \nabla^\lambda\epsilon\Gamma_{\tilde{M}}{}^\nu\Gamma^{0I9}\epsilon\nabla_\lambda\nabla_\nu Y^k_m = 0, \qquad \text{for} \qquad I = 6,7. \tag{6.97}$$

This gives

$$\begin{aligned}\nabla^2\mathcal{A}^{3,4}_\mu &= -4\beta^2\left(k(k+d-1)-1\right)\mathcal{A}^{3,4}_\mu,\\ \nabla^2\mathcal{A}^{3,4}_i &= -4\beta^2 k(k+d-1)\mathcal{A}^{3,4}_i.\end{aligned} \tag{6.98}$$

The eigenvalues of ∇^2 acting on the vector and the scalar parts of $\mathcal{A}_{\tilde{M}}$ are given below. The first term in each row corresponds to $\mathcal{A}_\mu$ and the second to $\mathcal{A}_i$:

$$\begin{aligned}
&-4\beta^2\left(k(k-3)+d(k-1)+1\right), && -4\beta^2\left(k-1\right)\left(k+d-2\right), \\
&-4\beta^2\left(k\left(k+d+1\right)+d-1\right), && -4\beta^2\left(k+1\right)\left(k+d\right), \\
&-4\beta^2\left(k\left(k+d-1\right)-1\right), && -4\beta^2 k\left(k+d-1\right).
\end{aligned} \tag{6.99}$$

These eigenvalues correspond to the following linear combinations of the basis

$$\mathcal{A}^1_{\tilde{M}}+2\beta\left(k+d-1\right)\mathcal{A}^2_{\tilde{M}}, \qquad \mathcal{A}^1_{\tilde{M}}-2\beta k\mathcal{A}^2_{\tilde{M}}, \qquad \mathcal{A}^3_{\tilde{M}}\mp i\mathcal{A}^4_{\tilde{M}}. \tag{6.100}$$

For $m=\pm k$, we use that $\mathcal{A}^2_{\tilde{M}} = -2\beta k\mathcal{A}^1_{\tilde{M}}$ to see that the first eigenvalue in eq. (6.99) does not contribute. Similarly, $\mathcal{A}^3_{\tilde{M}}\mp i\mathcal{A}^4_{\tilde{M}}$ vanish identically for $m=\pm k$ so corresponding eigenvalues do not contribute. By shifting k, we can rearrange the basis into vector and scalar harmonics with eigenvalues

$$-4\beta^2\left(k\left(k+d-1\right)-1\right), \qquad -4\beta^2 k\left(k+d-1\right), \tag{6.101}$$

respectively. The total number of harmonics is given by

$$\deg_{\mathrm{b}}\left(k,d\right) = \mathcal{D}_{k+1}\left(d,0\right)+\mathcal{D}_{k-1}\left(d,0\right)+2\mathcal{D}_k\left(d,0\right)-2N_{k+1,d}-2N_{k,d}. \tag{6.102}$$

Using the explicit expressions for the degeneracies provided in chapter D we get

$$\deg_{\mathrm{b}}\left(k,d\right) = \left(5-d\right)\mathcal{D}_k\left(d,0\right) + \mathcal{D}_k\left(d,1\right). \tag{6.103}$$

So we deduce that the basis defined in eq. (6.88) provides a complete set of harmonics for the vector multiplet in the case of eight supersymmetries.

One-loop determinant of bosons

We now compute the one-loop determinant for vector multiplet bosons. We need to diagonalize the action of the operator $\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}$ defined in equation eq. (6.55)

$$\mathcal{O}_{\tilde{M}}{}^{\tilde{N}} = -\delta^{\tilde{N}}_{\tilde{M}}\nabla^2+\alpha_{\tilde{M}}{}^{\tilde{N}}-2\beta(d-3)\epsilon\Gamma_{\tilde{M}}{}^{\nu\tilde{N}89}\epsilon\nabla_\nu. \tag{6.104}$$

The matrix $\alpha_{\tilde{M}}{}^{\tilde{N}}$ is defined in eq. (6.56). The action of ∇^2 on the basis is given in equations (6.94), (6.96) and (6.98). The next non-trivial part of the operator $\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}$ involves $\epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda$. For $\mathcal{A}^1$ we have

$$\epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\mathcal{A}^1_{\tilde{N}} \;=\; 2\beta\epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\,\epsilon\Gamma_{\tilde{N}\lambda}\Lambda\epsilon Y^k_m + \epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\, v_{\tilde{N}}\nabla_\lambda Y^k_m. \tag{6.105}$$

The term multiplying Y^k_m and its derivative can be simplified using triality.

$$\epsilon\Gamma_{\tilde{\mu}}{}^{\lambda\tilde{N}89}\epsilon\,\epsilon\Gamma_{\tilde{N}\lambda}\Lambda\epsilon = -(d-1)v_\mu, \qquad \epsilon\Gamma_i{}^{\lambda\tilde{N}89}\epsilon\,\epsilon\Gamma_{\tilde{N}\lambda}\Lambda\epsilon = -dv_i, \qquad \epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\,\epsilon\Gamma_{\tilde{N}}\epsilon = \epsilon\Gamma_{\tilde{M}}{}^{\lambda}\Lambda\epsilon. \tag{6.106}$$

Using these relations, we get

$$\begin{aligned}\epsilon\Gamma_\mu{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\mathcal{A}^1_{\tilde{N}} &= -2\beta(d-1)\mathcal{A}^1_\mu + \mathcal{A}^2_\mu,\\ \epsilon\Gamma_i{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\mathcal{A}^1_{\tilde{N}} &= -2\beta d\mathcal{A}^1_i + \mathcal{A}^2_i.\end{aligned} \tag{6.107}$$

The action on $\mathcal{A}^2_{\tilde{N}}$ can be computed in a similar manner:

$$\begin{aligned}\epsilon\Gamma_\mu{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\mathcal{A}^2_{\tilde{N}} &= 4\beta^2k(k+d-1)\mathcal{A}^1_\mu,\\ \epsilon\Gamma_i{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\mathcal{A}^2_{\tilde{N}} &= 4\beta^2k(k+d-1)\mathcal{A}^1_i - 2\beta\mathcal{A}^2_i.\end{aligned} \tag{6.108}$$

However, the computation for $\mathcal{A}^{3,4}_{\tilde{M}}$ is slightly different. We have

$$\epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\,\mathcal{A}^{I-3}_{\tilde{N}} \;=\; \epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\left[\epsilon\Gamma_{\tilde{N}}{}^{\nu 0I9}\epsilon\nabla_\nu Y^k_m\right], \tag{6.109}$$

where $I=6,7$ corresponds to $\mathcal{A}^3, \mathcal{A}^4$ respectively. First we note that

$$\nabla_\lambda\left(\epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\right) = 2\beta d\epsilon\Gamma_{\tilde{M}}{}^{\tilde{N}0}\epsilon = 0, \tag{6.110}$$

So we can write the right-hand side of equation (6.109) as a total derivative. Next we use the following relation due to triality,

$$\epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\,\epsilon\Gamma_{\tilde{N}}{}^{\nu 0I9}\epsilon = -\epsilon\Gamma_{\tilde{M}}{}^{\nu\lambda I8}\epsilon - v^\nu\epsilon\Gamma_{\tilde{M}}{}^{\lambda 0I8}\epsilon\,, \tag{6.111}$$

which allows us to write

$$\epsilon\Gamma_{\tilde{M}}{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\,\mathcal{A}^{I-3}_{\tilde{N}} \;=\; -\nabla_\lambda\left(\epsilon\Gamma_{\tilde{M}}{}^{\nu\lambda I8}\epsilon\nabla_\nu Y^k_m + 2im\beta\epsilon\Gamma_{\tilde{M}}{}^{\lambda 0I8}Y^k_m\right). \tag{6.112}$$

This can be now computed using the Killing spinor equation and the triality identity, resulting in

$$\begin{aligned}
\epsilon\Gamma_\mu{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\begin{pmatrix}\mathcal{A}^3_{\tilde{N}}\\ \mathcal{A}^4_{\tilde{N}}\end{pmatrix} &= -2\beta\begin{pmatrix}d-2 & im\\ -im & d-2\end{pmatrix}\begin{pmatrix}\mathcal{A}^3_\mu\\ \mathcal{A}^4_\mu\end{pmatrix}.\\
\epsilon\Gamma_i{}^{\lambda\tilde{N}89}\epsilon\nabla_\lambda\begin{pmatrix}\mathcal{A}^3_{\tilde{N}}\\ \mathcal{A}^4_{\tilde{N}}\end{pmatrix} &= -2\beta\begin{pmatrix}d-1 & im\\ -im & d-1\end{pmatrix}\begin{pmatrix}\mathcal{A}^3_i\\ \mathcal{A}^4_i\end{pmatrix}.
\end{aligned} \tag{6.113}$$

The action of the complete operator on the set of basis vectors can be written in the following compact form:

$$\begin{aligned}
\left(\mathcal{O}\mathcal{A}^1\right)_{\tilde{M}} &= 4\beta^2\left[k(k+d-1)+(d-1)^2\right]\mathcal{A}^1_{\tilde{M}} - 2\beta\,(d-1)\,\mathcal{A}^2_{\tilde{M}},\\
\left(\mathcal{O}\mathcal{A}^2\right)_{\tilde{M}} &= -8\beta^3k(d-1)(k+d-1)\mathcal{A}^1_{\tilde{M}} + 4\beta^2k(k+d-1)\mathcal{A}^2_{\tilde{M}},\\
\left(\mathcal{O}\mathcal{A}^3\right)_{\tilde{M}} &= 4\beta^2\left[k(k+d-1+(d-2)^2\right]\mathcal{A}^3_{\tilde{M}} + 4i\beta^2m(d-3)\mathcal{A}^4_{\tilde{M}},\\
\left(\mathcal{O}\mathcal{A}^4\right)_{\tilde{M}} &= 4\beta^2\left[k(k+d-1)+(d-2)^2\right]\mathcal{A}^4_{\tilde{M}} - 4i\beta^2m(d-3)\mathcal{A}^3_{\tilde{M}}.
\end{aligned} \tag{6.114}$$

The corresponding eigenvalues are

$$4\beta^2k^2, \qquad 4\beta^2\,(k+d-1)^2, \qquad 4\beta^2\left[k(k+d-1)+(d-2)^2\pm m(d-3)\right]. \tag{6.115}$$

Including the contribution from different roots and taking into account the degeneracy of the basis, we get the one-loop determinant for the bosonic part of the vector multiplet:

$$\begin{aligned}
Z^{\text{vec}}_{1-\text{loop}}\Big|_{\text{b}} = \prod_\alpha\prod_{k=1}^{\infty}\left[4\beta^2\left(k^2+\langle\alpha,\sigma\rangle^2\right)\right]^{\frac{\mathcal{D}_k(d,0)}{2}-N_{k,d}}\prod_{k=0}^{\infty}\left[4\beta^2\left((k+d-1)^2+\langle\alpha,\sigma\rangle^2\right)\right]^{\frac{\mathcal{D}_k(d,0)}{2}}\\
\prod_{k=1}^{k=\infty}\prod_{m=-k}^{m=k-1}\left[4\beta^2\left(k\,(k+d-1)+(d-2)^2+(d-3)\,m+\langle\alpha,\sigma\rangle^2\right)\right]^{N_{m,d}}.
\end{aligned} \tag{6.116}$$

One-loop determinant of fermions

Next, we calculate the contribution to the one-loop determinant from the vector multiplet fermions. We will use the basis with the '+' subscript introduced in Table 6.1. We need to diagonalize the action of the following operator:

$$\mathcal{O}^{\rm f}_{\rm v.m} = \tilde{\Gamma}_0 \nabla\!\!\!\!/ - \frac{1}{2}(d-3)\beta v^{\tilde{M}} \tilde{\Gamma}_{\tilde{M}} \Lambda - \frac{1}{4}(d-3)\beta \left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right) \Gamma_{\tilde{M}\tilde{N}} + \frac{1}{2}(d-1)\Gamma^{89}. \tag{6.117}$$

The action of $\tilde{\Gamma}_0 \nabla\!\!\!\!/$ has been computed in eqs. (6.80), (6.82) and (6.83). The second operator can be written as

$$v^{\tilde{M}} \Gamma_{\tilde{M}} \Lambda = \Gamma^{89} \left(\tilde{\Gamma}_0 v^{\tilde{M}} \Gamma_{\tilde{M}}\right). \tag{6.118}$$

The spinor basis elements have definite eigenvalues under the action of Γ^{89} and $\tilde{\Gamma}_0 v^{\tilde{M}} \Gamma_{\tilde{M}}$ as given in Table 6.1. Hence the action of the second and the last operator on the righthand side of eq. (6.117) is trivial to evaluate.

The action of the third term appearing in $\mathcal{O}^{\rm f}_{\rm v.m}$ can be obtained using triality.

$$\begin{gathered} \epsilon \Gamma^{\tilde{M}\tilde{N}} \Lambda \epsilon \Gamma_{\tilde{M}\tilde{N}} \mathcal{X}^1_+ = -4i \mathcal{X}^1_+, \qquad \epsilon \Gamma^{\tilde{M}\tilde{N}} \Lambda \epsilon \Gamma_{\tilde{M}\tilde{N}} \tilde{\mathcal{X}}^1_+ = +4i \tilde{\mathcal{X}}^1_+, \\ \epsilon \Gamma^{\tilde{M}\tilde{N}} \Lambda \epsilon \Gamma_{\tilde{M}\tilde{N}} \mathcal{X}^2_+ = \epsilon \Gamma^{\tilde{M}\tilde{N}} \Lambda \epsilon \Gamma_{\tilde{M}\tilde{N}} \tilde{\mathcal{X}}^2_+ = 0. \end{gathered} \tag{6.119}$$

We get the action of the full operator on the spinor basis to be

$$\begin{aligned} \mathcal{O}^{\rm f}_{\rm v.m} \mathcal{X}^1_+ &= 2i\beta\left(m + (d-1)\right) \mathcal{X}^1_+ + \mathcal{X}^2_+, \\ \mathcal{O}^{\rm f}_{\rm v.m} \mathcal{X}^2_+ &= -4\beta^2 (k-m)(k+m+d-1) \mathcal{X}^1_+ - 2i\beta m \mathcal{X}^2_+, \\ \mathcal{O}^{\rm f}_{\rm v.m} \tilde{\mathcal{X}}^1_+ &= 2i\beta\left(m - (d-2)\right) \tilde{\mathcal{X}}^1_+ + \tilde{\mathcal{X}}^2_+, \\ \mathcal{O}^{\rm f}_{\rm v.m} \tilde{\mathcal{X}}^2_+ &= -4\beta^2 (k+m)(k-m+d-1) \tilde{\mathcal{X}}^1_+ - 2i\beta\left(m - (d-2)\right) \tilde{\mathcal{X}}^2_+. \end{aligned} \tag{6.120}$$

For $m \neq \pm k$, all of the above spinors contribute to the determinant. The contribution from $\mathcal{X}^{1,2}_+$ and $\tilde{\mathcal{X}}^{1,2}_+$ is

$$4\beta^2\, k\,(k+d-1)\,, \qquad 4\beta^2 \left[k(k+d-1) - m(d-3) + (d-2)^2\right], \tag{6.121}$$

respectively. However, as discussed earlier, $\mathcal{X}^2_+(\tilde{\mathcal{X}}^2_+)$ vanishes identically for $m = k(-k)$. So for $m = k(-k)$, the first(second) term in eq. (6.121) is replaced by the eigenvalue corresponding to $\mathcal{X}^1_+(\tilde{\mathcal{X}}^1_+)$:

$$\begin{aligned} \mathcal{O}^{\rm f}_{\rm v.m} \mathcal{X}^1_+ &= +2i\beta\left(k + (d-1)\right) \mathcal{X}^1_+, \\ \mathcal{O}^{\rm f}_{\rm v.m} \tilde{\mathcal{X}}^1_+ &= -2i\beta\left(k + (d-2)\right) \tilde{\mathcal{X}}^1_+. \end{aligned} \tag{6.122}$$

Including the contribution from different roots, the one-loop determinant for the fermions is given by

$$
\begin{aligned}
Z^{\text{vec}}_{1-\text{loop}}\Big|_{\text{f}} &= \prod_{\alpha}\prod_{k=0}^{\infty}\left[2i\beta\left(k+d-1-i\langle\alpha,\sigma\rangle\right)\right]^{\mathcal{D}_k(d,0)}\prod_{k=1}^{\infty}\left[-2i\beta\left(k+i\langle\alpha,\sigma\rangle\right)\right]^{\mathcal{D}_k(d,0)-N_{k,d}} \\
&\prod_{k=0}^{\infty}\left[-2i\beta\left(k+d-2+i\langle\alpha,\sigma\rangle\right)\right]^{N_{k,d}}\prod_{m=-k}^{k-1}\left[4\beta^2\left(k\left(k+d-1\right)+m\left(d-3\right)+\left(d-2\right)^2+\langle\alpha,\sigma\rangle^2\right)\right]^{N_{m,d}}.
\end{aligned}
\tag{6.123}
$$

Combining this with the bosonic determinant, we see that most terms cancel and in the end we are left with:

$$
Z^{\text{vec}}_{1-\text{loop}}\prod_{\alpha}i\langle\alpha,\sigma\rangle = \prod_{\alpha}\prod_{k=0}^{\infty}\left[\left(k+i\langle\alpha,\sigma\rangle\right)\left(k+d-2+i\langle\alpha,\sigma\rangle\right)\right]^{N_{k,d}}. \tag{6.124}
$$

With $N_{k,d}$ given in eq. (D.7) this matches exactly with the conjecture in [6].

6.3.2 Hypermultiplet

In this section we compute one-loop determinants for a hypermultiplet with eight supersymmetries. We proceed in the same manner as for the vector multiplet by introducing a complete set of states and then computing the eigenvalues and degeneracies of the quadratic operator.

One-loop determinant for bosons

The bosonic part of the quadratic fluctuations about the fixed point locus for the hypermultiplet is given in eq. (6.58).

$$
\begin{aligned}
\mathcal{L}^{\text{b}}_{\text{h.m}}(\mu) &= \sum_{i=6}^{9}\left[\phi_i\left(-\nabla^2+\beta^2(d-2+2i\sigma_i\mu)^2\right)\phi_i-[\phi_0^{\text{cl}},\phi_i][\phi_0^{\text{cl}},\phi_i]\right] \\
&\quad + 4\beta\left(2i\mu-1\right)\phi_6 v^{\mu}\nabla_{\mu}\phi_7+4\beta\left(2i\mu+1\right)\phi_8 v^{\mu}\nabla_{\mu}\phi_9.
\end{aligned}
\tag{6.125}
$$

We see that $\phi_{6,7}$ and $\phi_{8,9}$ mix under the action of the kinetic operator. We use Y^k_m to diagonalize the action of the operator appearing in eq. (6.125). The eigenvalues for $\phi_{6,7}$ are

$$
4\beta^2\left(k\left(k+d-1\right)+\left(\frac{d-2}{2}+i\mu\right)^2\pm m\left(2i\mu-1\right)\right). \tag{6.126}
$$

The eigenvalues for $\phi_{8,9}$ are the same as above with $\mu \to -\mu$. Including the contribution from different roots, the bosonic part of the one-loop determinant is given by

$$Z^{\text{hyp}}_{1-\text{loop}}\Big|_{\text{b}} = \prod_{\alpha}\prod_{k=0}^{k=\infty}\prod_{m=-k}^{k}\left[4\beta^2\left(k\left(k+d-1\right)+\left(\frac{d-2}{2}+i\mu\right)^2+\langle\alpha,\sigma\rangle^2+m\left(2i\mu-1\right)\right)\right]^{N_{m,d}}$$
$$\left[4\beta^2\left(k\left(k+d-1\right)+\left(\frac{d-2}{2}-i\mu\right)^2+\langle\alpha,\sigma\rangle^2-m\left(2i\mu+1\right)\right)\right]^{N_{m,d}}, \tag{6.127}$$

where we have used that in the product positive and negative values of m come in pairs, so the product is invariant under $m \leftrightarrow -m$.

One-loop determinant for fermions

The relevant part of quadratic fluctuations is given in eq. (6.59). We need to compute the determinant of the operator

$$\mathcal{O}^{\text{f}}_{\text{h.m}} = \tilde{\Gamma}_0\nabla\!\!\!/ - \frac{1}{2}\beta\left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\Gamma_{\tilde{M}\tilde{N}} + 2i\mu\beta v^{\tilde{N}}\tilde{\Gamma}_{\tilde{N}}\Lambda. \tag{6.128}$$

To diagonalize the action of this operator, we construct a complete basis for the hypermultiplet fermions. We define the spinors

$$\lambda_+ = \left(\Gamma_6 + i\Gamma_7\right)\epsilon, \qquad \tilde{\lambda}_+ = \left(\Gamma_8 + i\Gamma_9\right)\epsilon, \tag{6.129}$$

which satisfy

$$\Gamma^{89}\lambda_+ = +i\lambda_+, \qquad \tilde{\Gamma}_0\Gamma^{\tilde{M}}v_{\tilde{M}}\lambda_+ = -\lambda_+\,, \tag{6.130}$$
$$\Gamma^{89}\tilde{\lambda}_+ = +i\tilde{\lambda}_+, \qquad \tilde{\Gamma}_0\Gamma^{\tilde{M}}v_{\tilde{M}}\tilde{\lambda}_+ = -\tilde{\lambda}_+\,. \tag{6.131}$$

Now we define the spinor harmonics, using the spinors $\lambda_+^{1,2}$ and the scalar spherical harmonics Y^k_m:

$$\begin{aligned} \mathcal{X}^1_+ &= Y^k_m\lambda_+, \qquad \mathcal{X}^2_+ = \tilde{\Gamma}^0\Gamma^\mu\left(\hat{\nabla}_\mu Y^k_m\right)\lambda_+, \\ \tilde{\mathcal{X}}^1_+ &= Y^k_m\tilde{\lambda}_+, \qquad \tilde{\mathcal{X}}^2_+ = \tilde{\Gamma}^0\Gamma^\mu\left(\hat{\nabla}_\mu Y^k_m\right)\tilde{\lambda}_+. \end{aligned} \tag{6.132}$$

The spinors $\mathcal{X}^2_+(\tilde{\mathcal{X}}^2_-)$ vanish identically for $m = k(-k)$. An analysis similar to the one in section 6.3.1 shows that the basis defined above, provides a complete set of spinor harmonics on S^d for hypermultiplet fermions. The action of the operator on these basis elements can be computed in an analogous fashion to the vector multiplet fermions, resulting into

$$\begin{aligned}
\mathcal{O}^{\text{f}}_{\text{h.m}}\mathcal{X}^1_+ &= -2i\beta\left(m + \left(\frac{d-2}{2} + i\mu\right)\right)\mathcal{X}^1_+ + \mathcal{X}^2_+, \\
\mathcal{O}^{\text{f}}_{\text{h.m}}\mathcal{X}^2_+ &= -4\beta^2(k-m)(k+m+d-1)\mathcal{X}^1_+ + 2i\beta\left(m + \left(\frac{d-2}{2} + i\mu\right)\right)\mathcal{X}^2_+, \\
\mathcal{O}^{\text{f}}_{\text{h.m}}\tilde{\mathcal{X}}^1_+ &= -2i\beta\left(m - \left(\frac{d-2}{2} - i\mu\right)\right)\tilde{\mathcal{X}}^1_+ + \tilde{\mathcal{X}}^2_+, \\
\mathcal{O}^{\text{f}}_{\text{h.m}}\tilde{\mathcal{X}}^2_+ &= -4\beta^2(k+m)(k-m+d-1)\tilde{\mathcal{X}}^1_+ + 2i\beta\left(m - \left(\frac{d-2}{2} - i\mu\right)\right)\tilde{\mathcal{X}}^2_+.
\end{aligned} \tag{6.133}$$

For $m \neq \pm k$, the contribution to the determinant from these basis elements is given by

$$\begin{aligned}
\mathcal{X}^{1,2}_+ : \quad & 4\beta^2\left(k(k+d-1) + m\,(2i\mu - 1) + \left(\frac{d-2}{2} + i\mu\right)^2\right), \\
\tilde{\mathcal{X}}^{1,2}_+ : \quad & 4\beta^2\left(k(k+d-1) + m\,(2i\mu + 1) + \left(\frac{d-2}{2} - i\mu\right)^2\right).
\end{aligned} \tag{6.134}$$

For $m = k\,(-k)$, only $\mathcal{X}^1_+(\tilde{\mathcal{X}}^1_+)$ contributes to the first(second) term in eq. (6.134).

$$\begin{aligned}
\mathcal{O}^{\text{f}}_{\text{h.m}}\mathcal{X}^1_+ &= -2i\beta\left(k + \left(\frac{d}{2} + i\mu - 1\right)\right)\mathcal{X}^1_+, \\
\mathcal{O}^{\text{f}}_{\text{h.m}}\tilde{\mathcal{X}}^1_+ &= -2i\beta\left(k + \left(\frac{d}{2} - i\mu - 1\right)\right)\tilde{\mathcal{X}}^1_+.
\end{aligned} \tag{6.135}$$

After including the contribution from roots, the fermionic part of the one-loop determinant is given by

$$\begin{aligned}
Z^{\text{hyp}}_{1-\text{loop}}\Big|_{\text{f}} = \prod_{\alpha}\prod_{k=0}^{k=\infty}\prod_{m=-k}^{k-1} &\left[4\beta^2\left(k(k+d-1) + m\,(2i\mu - 1) + \left(\frac{d-2}{2} + i\mu\right)^2 + \langle\alpha,\sigma\rangle^2\right)\right]^{N_{m,d}} \\
&\left[4\beta^2\left(k(k+d-1) - m\,(2i\mu + 1) + \left(\frac{d-2}{2} - i\mu\right)^2 + \langle\alpha,\sigma\rangle^2\right)\right]^{N_{m,d}} \\
&\left[4\beta^2\left(\left(k + \frac{d-2}{2} + i\mu\right) - i\langle\alpha,\sigma\rangle\right)\left(\left(k + \frac{d-2}{2} - i\mu\right) + i\langle\alpha,\sigma\rangle\right)\right]^{N_{k,d}}.
\end{aligned} \tag{6.136}$$

Combining this with the bosonic determinant and after many cancellations, we are left with:

$$Z^{\text{hyp}}_{1-\text{loop}} = \prod_{\alpha}\prod_{k=0}^{k=\infty}\left[\left(k+\frac{d-2}{2}+i\mu+i\langle\alpha,\sigma\rangle\right)\left(k+\frac{d-2}{2}-i\mu-i\langle\alpha,\sigma\rangle\right)\right]^{-N_{k,d}}. \tag{6.137}$$

This matches with the conjectured form in [6].

6.4 Determinants for four supersymmetries

In this section we will compute one-loop determinants for theories with four supersymmetries. Most of the computation is similar to the case of eight supersymmetries. However there is an additional subtlety in the construction of complete sets of basis elements. Before discussing computation in detail, we summarize our results here. The one-loop determinants take the following form

$$\begin{aligned} Z^{\text{vec}}_{1-\text{loop}}\prod_{\alpha} i\langle\alpha,\sigma\rangle &= \prod_{\alpha}\prod_{k=0}^{k=\infty}\left[\frac{(k+i\langle\alpha,\sigma\rangle)}{(k+d-1-i\langle\alpha,\sigma\rangle)}\right]^{n_{k,d}}, \\ Z^{\text{chi}}_{1-\text{loop}} &= \prod_{\ell=1}^{3}\prod_{\alpha}\prod_{k=0}^{k=\infty}\left[\frac{k+\frac{d}{2}-i\sigma_{(\ell)}\mu_\ell-i\langle\alpha,\sigma\rangle}{k+\frac{d-2}{2}+i\sigma_{(\ell)}\mu_\ell+i\langle\alpha,\sigma\rangle}\right]^{n_{k,d}}. \end{aligned} \tag{6.138}$$

The factor n_{kid} in the exponent is given by

$$n_{k,d} = \frac{\Gamma(k+d-1)}{\Gamma(k+1)\Gamma(d-1)}. \tag{6.139}$$

We now give a derivation of eq. (6.138).

6.4.1 The complete set of basis elements

One can verify that only the first two of the spinors defined in Table 6.1 have $+1$ eigenvalue for Γ' and hence belong to the vector multiplet of theories with four supersymmetries. However, they do not provide a complete set of basis elements for spinor harmonics. To see this, recall that eigenvalues of the Dirac operator acting on $\mathcal{X}_+^{1,2}$ are given in equation eq. (6.85). By shifting the value of k, one can arrange them into spinor harmonics with eigenvalues $\pm 2i\beta\left(k+\frac{d}{2}\right)$. However the degeneracy of positive and negative eigenvalues is not the same.

$$\deg_+ = \mathcal{D}_k(d,0), \qquad \deg_- = \mathcal{D}_{k+1}(d,0) - n_{k+1,d}. \tag{6.140}$$

Here $n_{m,d}$ denotes the number of scalar harmonics Y^k_m for the case of four supersymmetries. This differs from $N_{m,k}$, as the vector field v_μ now vanishes only on an S^{2-d}. An explicit expression for $n_{k,d}$ is provided in chapter D. Using that, we get

$$\deg_- = 2\frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)}, \tag{6.141}$$

which is equal to the degeneracy of spinor harmonics on S^d for $d=2,3$. Clearly $\deg_+$ is different. Moreover one can show that:

$$\deg_-(k,d) - \deg_+(k,d) = n_{k,d}. \tag{6.142}$$

So $\mathcal{X}^1_+$ and $\mathcal{X}^2_+$ do not provide a complete basis for the spinor harmonics. This can be fixed by including another spinor $\mathcal{X}^{1'}_+$, which has the correct eigenvalue and degeneracy,

$$\mathcal{X}^{1'}_\pm = \Gamma^{0579}\eta_\mp Y^k_{\mp k}, \qquad \tilde{\Gamma}_0 \nabla\!\!\!\!/ \mathcal{X}^{1'}_\pm = \pm 2i\beta\left(k+\frac{d}{2}\right)\mathcal{X}^{1'}_\pm. \tag{6.143}$$

So a complete basis for spinor harmonics is provided by $\mathcal{X}^1_+, \mathcal{X}^2_+$ and $\mathcal{X}^{1'}_+$.

For the vector multiplet bosons we use the following basis

$$\begin{aligned}\mathcal{A}^1_{\bar{M}} &= v_{\bar{M}} Y^k_m + c^1 \nabla_{\bar{M}} Y^k_m \\ \mathcal{A}^2_{\bar{M}} &= \epsilon\Gamma_{\bar{M}}{}^\mu \Lambda \epsilon \nabla_\mu Y^k_m + c^2 \nabla_{\bar{M}} Y^k_m\end{aligned} \tag{6.144}$$

These are the first two basis elements that we used for theories with eight supersymmetries as defined in eq. (6.88)). As discussed in section 6.3.1, these basis elements can be arranged into vector and scalar harmonics on S^d with the total number given by

$$\deg_b(k,d) = \mathcal{D}_{k+1}(d,0) + \mathcal{D}_{k-1}(d,0) - 2n_{k+1,d}. \tag{6.145}$$

Using explicit values one can show that

$$\deg_b(k,d) - \mathcal{D}_k(d,1) - (3-d)\mathcal{D}_k(d,0) = -2n_{k-1,d} \neq 0. \tag{6.146}$$

Hence the above basis is not complete. We can complete it by including

$$\mathcal{A}^\pm_{\bar{M}} = \epsilon\Gamma_{\bar{M}}\mathcal{X}^{1'}_\pm. \tag{6.147}$$

The elements $\mathcal{A}^{\pm}_{\mu}$ defined above are divergenceless. Their eigenvalues under action of Laplacian are

$$\nabla^2 \mathcal{A}^{\pm}_{\mu} = -4\beta^2 \left(k\left(k+d+1\right)+d-1\right)\mathcal{A}^{\pm}_{\mu}, \qquad \nabla^2 \mathcal{A}^{\pm}_{i} = -4\beta^2 \left(k+1\right)\left(k+d\right)\mathcal{A}^{\pm}_{i}. \tag{6.148}$$

By shifting $k \to k-1$, we can put eigenvalues in the canonical form with the total number of harmonics given by $2n_{k-1,d}$, precisely what is needed to complete the basis.

6.4.2 Vector multiplet

One-loop determinant for bosons

To compute the one loop determinant we need the action of the operator $\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}$ on the basis elements $\mathcal{A}^{1,2}_{\tilde{M}}$ and $\mathcal{A}^{\pm}_{\tilde{M}}$. The computation for $\mathcal{A}^{1,2}_{\tilde{M}}$ was performed in detail in section 6.3.1. Their contribution to the one-loop determinant is given by

$$\prod_{\alpha}\prod_{k=1}^{\infty}\left[4\beta^2\left(k^2+\langle\alpha,\sigma\rangle^2\right)\right]^{\mathcal{D}_k(d,0)-2n_{k,d}}\prod_{k=0}^{\infty}\left[4\beta^2\left(\left(k+d-1\right)^2+\langle\alpha,\sigma\rangle^2\right)\right]^{\mathcal{D}_k(d,0)}.$$

The action of $\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}$ on $\mathcal{A}^{\pm}_{\tilde{M}}$ can be calculated using the same techniques as were employed in section 6.3.1.

$$\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}\mathcal{A}^{\pm}_{\tilde{N}} = 4\beta^2\left(k+d-1\right)^2\mathcal{A}^{\pm}_{\tilde{M}}. \tag{6.149}$$

Including the contributions from all basis elements, we get the bosonic part of the one-loop determinant:

$$Z^{\text{vec}}_{\text{1-loop}}\Big|_{\text{b}} = \prod_{\alpha}\prod_{k=1}^{\infty}\left[4\beta^2\left(k^2+\langle\alpha,\sigma\rangle^2\right)\right]^{\frac{\mathcal{D}_k(d,0)}{2}-n_{k,d}}\prod_{k=0}^{\infty}\left[4\beta^2\left(\left(k+d-1\right)^2+\langle\alpha,\sigma\rangle^2\right)\right]^{\frac{\mathcal{D}_k(d,0)}{2}+2n_{k,d}}. \tag{6.150}$$

One-loop determinant for fermions

The quadratic fluctuations for the vector multiplet fermions for the case of four supersymmetries are given in eq. (6.57). We need to diagonalize the operator

$$\mathcal{O}^{\text{f}}_{\text{v.m}} = \tilde{\Gamma}_0\Gamma^{\nu}\nabla_{\nu} - \frac{1}{2}(d-3)\beta v^{\tilde{M}}\Gamma_{\tilde{M}}\Lambda - \frac{1}{4}\beta(d-3)\epsilon\Gamma^{\tilde{M}\tilde{N}}\Lambda\epsilon\Gamma_{\tilde{M}\tilde{N}} + (d-2)\,\beta\Gamma^{89} \tag{6.151}$$

acting on $\mathcal{X}_+^{1,2}$ and $\mathcal{X}_+^{1'}$. The details of this computation are similar to the case of eight supersymmetries. One gets

$$\begin{aligned}
\mathcal{O}^{\text{f}}_{\text{v.m}}\mathcal{X}_+^1 &= 2i\beta\,(m+(d-1))\ \mathcal{X}_+^1 + \mathcal{X}_+^2, \\
\mathcal{O}^{\text{f}}_{\text{v.m}}\mathcal{X}_+^2 &= -4\beta^2(k-m)(k+m+d-1)\mathcal{X}_+^1 - 2i\beta m\mathcal{X}_+^2, \\
\mathcal{O}^{\text{f}}_{\text{v.m}}\mathcal{X}_+^{1'} &= +2i\beta\,(k+d-1)\,\mathcal{X}_+^{1'}.
\end{aligned} \tag{6.152}$$

From this we get the one-loop determinant

$$\begin{aligned}
Z^{\text{vec}}_{1-\text{loop}}\Big|_{\text{f}} = &\prod_\alpha \prod_{k=1}^{k=\infty} \left[-2i\beta\,(k+i\langle\alpha,\sigma\rangle)\right]^{\mathcal{D}_k(d,0)-n_{k,d}} \prod_{k=0}^{\infty} \left[2i\beta\,(k+d-1-i\langle\alpha,\sigma\rangle)\right]^{\mathcal{D}_k(d,0)} \\
&\prod_{k=0}^{k=\infty} \left[-2i\beta\,(k+d-1-i\langle\alpha,\sigma\rangle)\right]^{n_{k,d}}.
\end{aligned} \tag{6.153}$$

Combining this result with the bosonic determinant, we get the the full one-loop determinant for the vector multiplet:

$$Z^{\text{vec}}_{1-\text{loop}} \prod_\alpha i\langle\alpha,\sigma\rangle = \prod_\alpha \prod_{k=0}^{k=\infty} \left[\frac{(k+i\langle\alpha,\sigma\rangle)}{(k+d-1-i\langle\alpha,\sigma\rangle)}\right]^{n_{k,d}}. \tag{6.154}$$

One can check that for $d = 3$, this gives the correct one-loop determinant which matches with the results in [47].

6.4.3 Chiral multiplet

We now compute the one-loop determinants for the chiral multiplet. For the case of four supersymmetries, the mass-deformed Lagrangian contains three chiral multiplets.

One-loop determinant for bosons

We consider the chiral multiplet containing the scalar fields ϕ_4, ϕ_5. The relevant bosonic part of the quadratic fluctuations is given by

$$\sum_{i=4,5} \left[\phi_i \left(-\nabla^2 + \beta^2(d-2+2i\mu_1)^2\right)\phi_i\right] - 4\beta\,(1-2i\mu_1)\,\phi_4 v^\mu \nabla_\mu \phi_5. \tag{6.155}$$

Using the scalar spherical harmonics, the action of the kinetic operator can be diagonalized to obtain the one-loop determinant

$$Z^{\text{chi}}_{1-\text{loop}}(\mu_1)\Big|_{\text{b}} = \prod_{\alpha}\prod_{k=0}^{\infty}\prod_{m=-k}^{k}\left[4\beta^2\left(k(k+d-1)+\left(\frac{d-2}{2}+i\mu_1\right)^2+\langle\alpha,\sigma\rangle^2+m(1-2i\mu_1)\right)\right]^{n_{m,d}}. \tag{6.156}$$

The determinant for scalar fields $\phi_{6,7}$ ($\phi_{8,9}$) is the same as the above expression, but with $\mu_1 \to \mu_2(-\mu_3)$.

One-loop determinant for fermions

To compute the one loop determinant, we introduce a basis for the spinor harmonics as before. We introduce three sets of basis elements for three types of chiral multiplets:

$$\begin{aligned} \mathcal{X}^1_{+\ell} &\equiv Y^k_m\lambda_{+\ell}, \qquad \mathcal{X}^2_{+\ell} \equiv \Gamma_0\Gamma^{\tilde{M}}\hat{\nabla}_{\tilde{M}}\, Y^k_m\lambda_{+\ell}, \\ \mathcal{X}^{1'}_{+\ell} &\equiv \Gamma^{0579}Y^k_m\lambda_{-\ell}, \qquad \text{for} \qquad m = -(-1)^{\beta_1(\ell)\beta_2(\ell)}\,k, \end{aligned} \tag{6.157}$$

where $\lambda_{\pm\ell}$ is defined as

$$\lambda_{\pm\ell} = \Gamma_0\left(\Gamma_{2\ell+2}\pm\Gamma_{2\ell+3}\right)\epsilon. \tag{6.158}$$

The index $\ell = 1,2,3$ corresponds to the three chiral multiplets. Now, we need to diagonalize the action of the following operator

$$\begin{aligned} \mathcal{O}^{\text{f}}_{\text{c.m}} &= \sum_{\ell=1}^{3}\mathcal{O}^{\text{f}}_{\text{c.m},\ell}, \\ \mathcal{O}^{\text{f}}_{\text{c.m},\ell} &= \tilde{\Gamma}_0\nabla\!\!\!/ - \frac{1}{2}\beta\left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\Gamma_{\tilde{M}\tilde{N}} + \sigma_{(\ell)}\beta\left(2i\mu_\ell v^{\tilde{N}}\tilde{\Gamma}_{\tilde{N}}\Lambda+\Gamma^{89}\right). \end{aligned} \tag{6.159}$$

We give the result for the $\ell = 1$ explicitly:

$$\begin{aligned} \mathcal{O}^{\text{f}}_{\text{c.m},1}\mathcal{X}^1_{+1} &= -2i\beta\left(m+\frac{d-2}{2}+i\mu_1\right)\mathcal{X}^1_{+1} + \mathcal{X}^2_{+1}, \\ \mathcal{O}^{\text{f}}_{\text{c.m},1}\mathcal{X}^2_{+1} &= -4\beta^2(k-m)(k+m+d-1)\,\mathcal{X}^1_{+1} + 2i\beta\left(m+\frac{d-2}{2}+i\mu_1\right)\mathcal{X}^2_{+1}, \\ \mathcal{O}^{\text{f}}_{\text{c.m},1}\mathcal{X}^{1'}_{+1} &= +2i\beta\left(k+\frac{d}{2}-i\mu_1\right)\mathcal{X}^{1'}_{+1}. \end{aligned} \tag{6.160}$$

From this, one gets the one-loop determinant for fermions:

$$Z^{\text{chi}}_{1-\text{loop}}(\mu_1)\Big|_{\text{f}} =$$

$$\prod_{\alpha}\prod_{k=0}^{k=\infty}\prod_{m=-k}^{m=k-1}\left[4\beta^2\left(\left(\frac{d-2}{2}+i\mu_1\right)^2+m\,(2i\mu_1-1)+\langle\alpha,\sigma\rangle^2+k\,(k+d-1)\right)\right]^{n_{m,d}}$$
$$\prod_{k=0}^{k=\infty}\left[-2i\beta\left(k+\frac{d-2}{2}+i\mu_1+i\langle\alpha,\sigma\rangle\right)\right]^{n_{k,d}}\left[2i\beta\left(k+\frac{d}{2}-i\mu_1-i\langle\alpha,\sigma\rangle\right)\right]^{n_{k,d}}. \tag{6.161}$$

Combining this with the bosonic determinant, we get the full one-loop determinant for the chiral multiplet:

$$Z^{\text{chi}}_{1-\text{loop}}\,(\mu_1) = \prod_{\alpha}\prod_{k=0}^{k=\infty}\left[\frac{k+\frac{d}{2}-i\mu_1-i\langle\alpha,\sigma\rangle}{k+\frac{d-2}{2}+i\mu_1+i\langle\alpha,\sigma\rangle}\right]^{n_{k,d}}. \tag{6.162}$$

$n_{k,d}$ is given in eq. (D.8) The one-loop determinant for $\chi_{2(3)}$ can be obtained by simply replacing μ_1 with $\mu_2\,(-\mu_3)$. Hence, the full one-loop determinant for the chiral multiplet part is given by

$$Z^{\text{chi}}_{1-\text{loop}}\,(\mu_1,\mu_2,\mu_3) \;=\; Z^{\text{chi}}_{1-\text{loop}}\,(\mu_1)\,Z^{\text{chi}}_{1-\text{loop}}\,(\mu_2)\,Z^{\text{chi}}_{1-\text{loop}}\,(-\mu_3)\,. \tag{6.163}$$

6.5 Analytic continuation to $d = 4$ with four supersymmetries

Now that we have obtained expressions for partition functions with eight supersymmetries in $d \le 5$ dimensions and four supersymmetries in $d \le 3$ dimensions, it is tempting to continue the results to higher dimensions. In [7] this was done for eight supersymmetries where it was shown that the results were consistent with the one-loop running of coupling constants in flat space. In this section we consider continuing theories with four supersymmetries up to $d = 4$ using the expressions in section 6.4.

6.5.1 Consistency checks of analytic continuation

In this subsection we perform consistency checks on the analytic continuation with four supersymmetries. We will show that in the $g_{YM} \to 0$ limit, the analytic continuation gives the correct partition function for a free vector and free chiral multiplets on S^4. We also show that the analytic continuation gives the correct one-loop divergence for theories with four supersymmetries in the decompactification limit.

Partition function of $U(1)$ theory on S^4

A $U(1)$ gauge theory with four supersymmetries and massless adjoint matter in four dimensions is free and conformal. Hence it can be conformally coupled to S^4 and the partition function can be explicitly computed. This matches with the result of our analytical continuation as we demonstrate now.

Consider the chiral multiplet in the adjoint representation of the $U(1)$ gauge group. Our expressions for the one-loop determinants can then be simplified to take the form

$$Z^{\text{chi}}_{1-\text{loop}} = \prod_{k=0}^{\infty} \left(\frac{k+2}{k+1} \right)^{\frac{(k+1)(k+2)}{2}}, \qquad Z^{\text{vec}}_{1-\text{loop}} = \prod_{k=0}^{\infty} (k+1)^{3(k+1)}. \tag{6.164}$$

The full partition function in this case is equal to the product of the one-loop determinants up to an overall constant.

The chiral multiplet of $\mathcal{N} = 1$ supersymmetry in four dimensions contains a two component Weyl fermion and two real scalars. The conformally coupled action for a free chiral multiplet on the sphere takes the following form:

$$S^{\text{chi}}_{U(1)} = \int d^4x \sqrt{g} \left(\frac{1}{2} \left[\phi_1 \left(-\nabla^2 + 8\beta^2 \right) \phi_1 + \phi_2 \left(-\nabla^2 + 8\beta^2 \right) \phi_2 \right] - \psi \not{\nabla} \psi \right). \tag{6.165}$$

The partition function for the matter part is then given by

$$\mathcal{Z}^{\text{chi}}_{U(1)} = \frac{\det \not{\nabla}}{\det \left(-\nabla^2 + 8\beta^2 \right)}. \tag{6.166}$$

The eigenvalues and the degeneracies of these operators are given in appendix D. Using these we get

$$\begin{aligned} \det \not{\nabla} &= \prod_{k=0}^{\infty} \left[4\beta^2 (k+2)^2 \right]^{\frac{(k+1)(k+2)(k+3)}{3}}, \\ &= \prod_{k=0}^{\infty} \left[2\beta (k+2) \right]^{\frac{(k+1)(k+2)(k+3)}{3}} \left[2\beta (k+1) \right]^{\frac{k(k+1)(k+2)}{3}}, \end{aligned} \tag{6.167}$$

where the last equality follows by splitting the product into two parts and shifting $k \to k-1$ in one of the parts. Similarly, we have

$$\det \left(-\nabla^2 + 8\beta^2 \right) = \prod_{k=0}^{\infty} \left[4\beta^2 (k+1)(k+2) \right]^{\frac{(2k+3)(k+2)(k+1)}{6}}. \tag{6.168}$$

Combing the the two factors of determinants, one gets

$$\mathcal{Z}^{\text{chi}}_{U(1)} = Z^{\text{chi}}_{1-\text{loop}} = \prod_{k=0}^{\infty} \left(\frac{k+2}{k+1}\right)^{\frac{(k+1)(k+2)}{2}}, \tag{6.169}$$

which matches the analytic continuation.

Next we compute the partition function for the vector multiplet. The $\mathcal{N} = 1$ vector multiplet in four dimensions contains a gauge field and a two-component Weyl fermion. The relevant action on S^4, with the gauge fixing term included is given by

$$\begin{aligned} S^{\text{vec}}_{U(1)} = \int d^4x\sqrt{g}\Big(&A'^{\nu}\left[\delta_\nu{}^\mu\left(-\nabla^2+12\beta^2\right)+\nabla_\nu\nabla^\mu\right]A'_\mu - \psi\not{\nabla}\psi \\ &+ b\nabla_\mu A'^\mu - \bar{c}\nabla^\mu\partial_\mu c\Big). \end{aligned} \tag{6.170}$$

We split the vector field as follows

$$A'_\mu = A_\mu + \nabla_\mu\phi, \qquad \text{such that} \qquad \nabla_\mu A^\mu = 0. \tag{6.171}$$

By using that $\mathcal{D}\left(\nabla_\mu\phi\right) = \mathcal{D}'\phi\sqrt{\det\left(-\nabla^2\right)}$, we can write the partition function as follows

$$\mathcal{Z}^{\text{vec}}_{U(1)} = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}'\phi\mathcal{D}b\mathcal{D}c\mathcal{D}\bar{c}\sqrt{\det\left(-\nabla^2\right)}\exp\left(-S_{U(1),\text{v.m}}\right). \tag{6.172}$$

Integration over b gives a factor of $\delta\left(-\nabla^2\phi\right)$. This, upon integrating over ϕ gives a factor of $\left[\det\left(-\nabla^2\right)\right]^{-1}$which cancels against the contribution coming from integrating over ghosts. Hence the partition function becomes

$$\mathcal{Z}^{\text{vec}}_{U(1)} = \frac{\sqrt{\det'\left(-\nabla^2\right)}\det\left(\not{\nabla}\right)}{\sqrt{\det\left(-\nabla^2+12\beta^2\right)}}, \tag{6.173}$$

where the operator in the denominator acts on divergence less vector fields. Using the formulae for eigenvalues and degeneracies of the operators, the above expression reduces to the following infinite product:

$$\mathcal{Z}^{\text{vec}}_{U(1)} = \frac{1}{\sqrt{3}}\prod_{k=0}^{\infty}(k+1)^{3(k+1)}. \tag{6.174}$$

This is the same as the analytically continued $Z^{\text{vec}}_{1-\text{loop}}$ up to an overall finite constant.

Beta function from analytic continuation

For an $\mathcal{N}=1$ supersymmetric theory with a vector multiplet and N_c chiral multiplets in the representation R_c of the gauge group the above expression for beta function reduces to:

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(3C_2(\mathbf{Adj}) - N_c C_2(\mathbf{R_c})\right). \tag{6.175}$$

We will reproduce this result by dimensional regularization of the analytically continued expression. To do, so we need to determine the $\mathcal{O}(\sigma^2)$ terms appearing in the one-loop determinants. We proceed by replacing $\sigma \to t\sigma$ in the expressions for the one-loop determinants. Focusing only on the vector multiplet, one can easily find that

$$\frac{d\log Z^{\text{vec}}_{1-\text{loop}}}{dt^2} + \sum_{\alpha>0}\frac{1}{t^2} = \sum_{\alpha>0}\langle\beta,\sigma\rangle^2\left(\mathcal{F}\left(d-1,0,t\langle\alpha,\sigma\rangle\right) + \mathcal{F}\left(d-1,d-1,t\langle\alpha,\sigma\rangle\right)\right), \tag{6.176}$$

where

$$\mathcal{F}(x,y,z) \equiv \sum_{n=0}^{\infty}\frac{\Gamma(n+x)}{\Gamma(n+1)\Gamma(x)}\frac{1}{(n+y)^2+z^2} = \frac{i}{2z}\left(\frac{1}{y+iz}{}_2F_1(x,y+iz;y+iz+1;1) - c.c\right). \tag{6.177}$$

For $d = 4-\epsilon$, we expand the R.H.S in powers of t and ϵ. Keeping only the leading terms, we find

$$\frac{d\log Z^{\text{vec}}_{1-\text{loop}}}{dt^2} = \frac{3}{\epsilon}C_2(\mathbf{Adj})\,\sigma^2 + \cdots. \tag{6.178}$$

From this we can easily obtain

$$\log Z^{\text{vec}}_{1-\text{loop}} = \frac{3}{\epsilon}C_2(\mathbf{Adj})\,\sigma^2 + \cdots. \tag{6.179}$$

A completely analogous calculation for a chiral multiplet in the representation $\mathbf{R_c}$ of the gauge group gives

$$\log Z^{\text{chi}}_{1-\text{loop}} = -\frac{1}{\epsilon}\sigma^2 C_2(\mathbf{R_c}) + \cdots. \tag{6.180}$$

We combine the $\mathcal{O}(\sigma^2)$ contribution from one-loop determinants with the $\mathcal{O}(\sigma^2)$ term in the fixed point action as given in equation (6.51), to get

$$\frac{8\pi^2}{g^2(\Lambda)} = \left(\frac{8\pi^2}{g_0^2} - \frac{3}{\epsilon}C_2(\mathbf{Adj}) + \frac{1}{\epsilon}N_c C_2(\mathbf{R_c})\right)\Lambda^{-\epsilon}, \tag{6.181}$$

where Λ is the renormalization scale and g_0 is the bare coupling. Differentiating the above equation with respect to the $\log \Lambda$, one obtains the beta function

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(3C_2(\mathbf{Adj}) - N_c C_2(\mathbf{R_c})\right), \tag{6.182}$$

which is exactly what we wanted to show.

6.5.2 Free energy of mass-deformed $\mathcal{N} = 4$ SYM

In this subsection we compare results from analytic continuation to a recent holographic analysis for $\mathcal{N} = 1^*$ super Yang-Mills [66]. There are some caveats which we explain below, but to the extent that we can make a comparison our results are consistent with the holographic results.

The $\mathcal{N} = 4$ super Yang-Mills multiplet decomposes into an $\mathcal{N} = 1$ vector multiplet and three massless adjoint chiral multiplets. The superpotential also has a cubic term which is the product of all three chiral fields. We can give masses m^j, $j = 1 \ldots 3$, to the three chiral multiplets and still preserve $\mathcal{N} = 1$ supersymmetry. If we choose $m^{(1)} = 0$ and $m^{(2)} = m^{(3)}$ then we preserve $\mathcal{N} = 2$ supersymmetry, with the massless chiral multiplet joining with the $\mathcal{N} = 1$ vector multiplet to form an $\mathcal{N} = 2$ vector multiplet, while the two massive chiral multiplets combine into a hypermultiplet. The cubic term in the superpotential remains unchanged. The supersymmetry is broken to $\mathcal{N} = 1$ if the third chiral multiplet is given a mass or the first two multiplets have unequal masses. The theory is called $\mathcal{N} = 1^*$ if the cubic term in the superpotential is left unchanged.

It was shown explicitly in [56] how to put an $\mathcal{N} = 1$ theory on S^4, and the $\mathcal{N} = 1^*$ theory is no exception. However, there are some subtleties. First for a Lorentzian $\mathcal{N} = 1$ theory, every chiral superfield Φ has a complex conjugate superfield $\bar{\Phi}$. In Euclidean space, these fields should be considered independent. Likewise, for a flat Lorentzian $\mathcal{N} = 1$ theory, a mass term would appear in the superpotential, $W_m = \frac{1}{2}m\Phi^2$. The conjugate fields would have a complex conjugate mass $\bar{m}$. In Euclidean space these masses are independent. In the holographic analysis in [66] $m^{(j)}$ is set equal to $\bar{m}^{(j)}$.

There is no known localization procedure for $\mathcal{N} = 1^*$ on S^4. Instead we propose analytically continuing the mass deformed theory in $d \leq 3$ up to $d = 4$. There is an important warning in doing this. If we consider $\mathcal{N} = 1^*$ on flat space and compactify down to three dimensions, the resulting three-dimensional chiral multiplets have complex masses. As explained in section 6.1, the mass deformed theory we use in the analytic continuation has real masses. Hence, it is not obvious that the analytic continuation of the perturbative mass-deformed partition function actually equals the perturbative partition function for $\mathcal{N} = 1^*$ on S^4, where the continuation of each real mass is set equal to the mass, or its negative, of the corresponding $\mathcal{N} = 1^*$ chiral multiplet[5]. Perhaps there is a more involved relation between the two sets of the mass parameters for which the analytically continued partition function equals that of the $\mathcal{N} = 1^*$. We leave this question for future work. Here we simply explore the consequences of analytically continuing to $d = 4$ and find that the general form of the real part of the free energy at large N is consistent with the holographic results.

In three dimensions the mass parameters that appear in the partition function are written as $\mu_j^{(3)} = i\,\Delta_j + r\,m_j^R$ where m_j^R is the real three dimensional mass and Δ_j is a charge under a corresponding flavor symmetry. When continuing up to four dimensions we assume that this becomes $\mu_j^{(3)} \to \sigma_{(j)}\mu_j$ where $\sigma_{(j)}$ is defined in (6.20) and μ_j the four-dimensional complex mass multiplied by r. If we then set $d = 4$ in eq. (6.154) and eq. (6.162) for three massive adjoint chiral multiplets, we find the perturbative partition function

$$\begin{aligned} \mathcal{Z}_{\text{pert}} &= \int d\sigma_i e^{-\frac{8\pi^2}{g_{YM}^2}\text{Tr}\,\sigma^2} \prod_\alpha \prod_{k=0}^{\infty} \left[\frac{(k - i\langle\alpha,\sigma\rangle)}{(k + i\langle\alpha,\sigma\rangle + 3)} \prod_{j=1}^{3} \frac{(k - i\langle\alpha,\sigma\rangle - i\sigma_{(j)}\mu_j + 2)}{(k + i\langle\alpha,\sigma\rangle + i\sigma_{(j)}\mu_j + 1)} \right]^{\frac{(k+1)(k+2)}{2}} \\ &= \int d\sigma_i e^{-\frac{8\pi^2}{g_{YM}^2}\text{Tr}\sigma^2} \prod_\alpha i\langle\alpha,\sigma\rangle Z_{\text{mass}}, \end{aligned} \tag{6.183}$$

where Z_{mass} is the mass correction to the $\mathcal{N} = 4$ partition function,

$$Z_{\text{mass}} = \prod_\alpha \prod_{k=0}^{\infty} \prod_{j=1}^{3} \left[\frac{(k - i\langle\alpha,\sigma\rangle - i\sigma_{(j)}\mu_j + 2)(k + i\langle\alpha,\sigma\rangle + 1)}{(k + i\langle\alpha,\sigma\rangle + i\sigma_{(j)}\mu_j + 1)(k - i\langle\alpha,\sigma\rangle + 2)} \right]^{\frac{(k+1)(k+2)}{2}} . \tag{6.184}$$

This last expression collapses to $Z_{\text{mass}} = 1$ if all $\mu_j = 0$. In deriving the second line in eq. (6.183) we used the identity

$$\prod_{k=0}^{\infty} \left[\frac{(k + i\langle\alpha,\sigma\rangle)(k + i\langle\alpha,\sigma\rangle + 2)^3}{(k + i\langle\alpha,\sigma\rangle + 3)(k + i\langle\alpha,\sigma\rangle + 1)^3} \right]^{\frac{(k+1)(k+2)}{2}} = i\langle\alpha,\sigma\rangle \tag{6.185}$$

and that every root in the product comes with its negative. The σ are $N \times N$ matrices and the root vectors are all possible combinations $\sigma_i - \sigma_j$, $i \neq j$ where σ_i are the N eigenvalues of σ.

[5]Note that these concerns do not apply to $\mathcal{N} = 2^*$ theories, which correspond to $\mathcal{N} = 4$ in three dimensions. In decomposing the three dimensional $\mathcal{N} = 4$ vector multiplet into an $\mathcal{N} = 2$ vector and chiral multiplet, one can choose to have the scalar field ϕ_0 be part of the vector multiplet, which leads to real mass terms. However, we could have also chosen ϕ_4 to be part of the vector multiplet and ϕ_0 to pair up with ϕ_5 in the chiral multiplet. If at the same time one changes the pairings of the other four scalar fields, then the mass terms and the cubic term proportional to the mass in (6.14) would come from the superpotential.

This term is divergent if any $\mu_j \neq 0$ and needs to be regularized. To this end we define

$$Z_k(\sigma-\sigma',\mu) \equiv \left[\frac{(k-i(\sigma-\sigma')-i\mu+2)(k+i(\sigma-\sigma')+1)}{(k+i(\sigma-\sigma')+i\mu+1)(k-i(\sigma-\sigma')+2)}\right]^{\frac{(k+1)(k+2)}{2}}. \tag{6.186}$$

For $k \gg 1$ we expand $\log[Z_k(\sigma-\sigma',\mu)]$ in $1/k$, where we find

$$\log(Z_k(\sigma-\sigma',\mu)) = -i\left(k+\frac{1}{2}+\frac{(\sigma-\sigma')^2}{k}\right)\mu-\frac{1}{2k}\mu^2+\frac{i}{3k}\mu^3+\mathrm{O}\left(\frac{1}{k^2}\right). \tag{6.187}$$

Hence, if we expand $\log Z_{\text{mass}}$ in powers of μ_j, the terms up to cubic order in the masses will be divergent. The term linear in μ can be dropped as it eventually will cancel because of the mass condition (6.27), which in terms of the μ_j is

$$\mu_1+\mu_2-\mu_3 = 0. \tag{6.188}$$

The remaining divergent terms are independent of $\sigma - \sigma'$ and can be removed by adding constant local counterterms to the Lagrangian.

In the large N-limit the free energy can be found by saddle point. We are particularly interested in the behavior at strong coupling, where the 't Hooft coupling $\lambda \equiv g_{YM}^2 N \gg 1$. In this case, the saddle point will have the separation between two generic eigenvalues $|\sigma_i - \sigma_j|$ to be much greater than 1. One can then check that for $|\sigma - \sigma'| \gg 1$,

$$\begin{aligned}\sum_{j=1}^{3}\sum_{k=0}^{\infty}\log(Z_k(\sigma-\sigma',\sigma_{(j)}\mu_j)_{\text{reg}}) \sim &+\frac{1}{4}\log(\sigma-\sigma')^2(\mu_1^2+\mu_2^2+\mu_3^2)\\ &-\frac{i}{6}\log(\sigma-\sigma')^2(\mu_1^3+\mu_2^3-\mu_3^3).\end{aligned} \tag{6.189}$$

Using (6.188) we can reexpress the cubic term as

$$\mu_1^3+\mu_2^3-\mu_3^3 = -3\,\mu_1\mu_2\mu_3. \tag{6.190}$$

Then, when eq. (6.189) is combined with the $\mathcal{N} = 4$ part of the partition function, the saddle point equation reduces to

$$\frac{16\pi^2}{\lambda}\sigma \approx 2\,⨍ d\sigma'\rho(\sigma')\frac{1+\frac{1}{2}(\mu_1^2+\mu_2^2+\mu_3^2)+i\,\mu_1\mu_2\mu_3}{\sigma-\sigma'}, \tag{6.191}$$

where $\rho(\sigma')$ is the eigenvalue density. Notice that eq. (6.191) is similar to the $\mathcal{N} = 2^*$ saddle point equation [71, 72] which has the same form as the saddle point equation for a Gaussian matrix model. One then solves for $\rho(\sigma)$ in the standard way, where one finds the Wigner semi-circle distribution,

$$\rho(\sigma) = \frac{2}{\pi A^2}\sqrt{A^2-\sigma^2}\,, \tag{6.192}$$

with

$$A^2 = \frac{\lambda(1+\frac{1}{2}(\mu_1^2+\mu_2^2+\mu_3^2)+i\,\mu_1\mu_2\mu_3)}{8\pi^2}\,. \tag{6.193}$$

Because of the imaginary part in eq. (6.193) the eigenvalue distribution runs at an angle off of the real axis. One then substitutes $\rho(\sigma)$ back into the free energy, where the dominant part is given by

$$\begin{aligned} F \approx& -\frac{N^2}{2}\int d\sigma d\sigma' \log(\sigma-\sigma')^2 \\ \approx& -\frac{N^2}{2}\left(1+\frac{1}{2}(\mu_1^2+\mu_2^2+\mu_3^2)+i\,\mu_1\mu_2\mu_3\right) \\ &\times \log\left(\lambda\left(1+\frac{1}{2}(\mu_1^2+\mu_2^2+\mu_3^2)+i\,\mu_1\mu_2\mu_3\right)\right)\,, \end{aligned} \tag{6.194}$$

Expanding about small μ_i and dropping terms up to cubic order which are not universal [65, 66], eq. (6.194) becomes

$$\begin{aligned} F \approx -N^2\bigg(&\frac{1}{16}(\mu_1^2+\mu_2^2+\mu_3^2)^2+\frac{i}{4}(\mu_1^2+\mu_2^2+\mu_3^2)\mu_1\mu_2\mu_3 \\ &-\frac{1}{96}(\mu_1^2+\mu_2^2+\mu_3^2)^3-\frac{1}{4}(\mu_1\mu_2\mu_3)^2+\mathrm{O}(\mu^7)\bigg)\,, \end{aligned} \tag{6.195}$$

In [66] it was argued that the terms in the free energy could only come with factors of $m^{(1)}m^{(2)}m^{(3)}$, $\tilde{m}^{(1)}\tilde{m}^{(2)}\tilde{m}^{(3)}$, or $\sum_j(m^{(j)}\tilde{m}^{(j)})^n$ where n is a positive integer in order to be consistent with supersymmetry. If $m^{(j)} = \tilde{m}^{(j)}$ then this translates into terms of the form $\mu_1\mu_2\mu_3$ or $\mu_1^{2n}+\mu_2^{2n}+\mu_3^{2n}$. Equation (6.195) is consistent with this observation. One should also note that the regularization should preserve the supersymmetry. If equation (6.188) had not been in effect, we would have had to add counterterms linear in μ_j, which violates this supersymmetry prescription.

Assuming that a regularization can be performed, one expects the free energy for a general choice of μ_j to have the form [66]

$$\begin{aligned} F = -N^2\bigg(&A_1(\mu_1^4+\mu_2^4+\mu_3^4)+A_2(\mu_1^2+\mu_2^2+\mu_3^2)^2+i\,B_1(\mu_1^2+\mu_2^2+\mu_3^2)\mu_1\mu_2\mu_3 \\ &-C_1(\mu_1^6+\mu_2^6+\mu_3^6)-C_2(\mu_1^2+\mu_2^2+\mu_3^2)^3-C_3(\mu_1\mu_2\mu_3)^2+\mathrm{O}(\mu^7)\bigg) \end{aligned} \tag{6.196}$$

Comparing with eq. (6.195) and using eq. (6.188)), we find that

$$A_1 + 2A_2 = \frac{1}{8}, \quad B_1 = -\frac{1}{4}, \quad C_1 + C_2 = \frac{1}{24}, \quad -12C_2 + C_3 = \frac{1}{8}. \tag{6.197}$$

The first and third relations were derived in [66] using the $\mathcal{N} = 2^*$ results, where one has $\mu_1 = 0$. The second relation differs from [66] since their free energy is real. The fourth relation is a new prediction.

One feature that is different here compared to the holographic dual is that the free energy in (6.196) has an imaginary piece, while the holographic result has a real free energy [66]. Since the theory is Euclidean and nonconformal it is not reflection positive [56], so it is not obvious on general grounds why the supergravity dual gives a real free energy. This issue deserves further investigation.

One further issue is that a gaugino condensate appears in the holographic analysis if all three chiral multiplet masses are nonzero [66]. It is not clear how one sees the condensate in the analytic continuation.

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Degrees of freedom

The purpose of this appendix is to determine the physical degrees of freedom propagated by the HSZ theory and the massive deformation of DFT. We will abandon manifest $O\,(D,D)$ invariance by taking the derivatives D and $\bar{D}$ to be partial derivatives, using indices as $\mu, \nu, \ldots$, and $\Box = \partial^2$. We start by determining the spectrum of the two-derivative part of the HSZ theory. Then we compare it with the spectrum of the full HSZ theory and show that no extra degrees of freedom appear upon adding higher derivative terms. Finally, we consider the spectrum for the massive deformation of DFT given in section 4 and show that it does not propagate any ghost-like degree of freedom.

A.1 Degrees of freedom of two-derivative HSZ theory

Consider the two-derivative quadratic Lagrangian given in equation (3.4). The part of the Lagrangian involving e_{mn} and ϕ is trivial to analyze and it describes massless graviton, dilaton and two-form field. We thus focus on the part of the Lagrangian involving the a-field. After putting in explicit factors of α', we have:

$$L \;=\; -\tfrac{1}{8}\, a^{\mu\nu}\, \Box\, a_{\mu\nu} - \tfrac{1}{4}(\partial_\mu a^{\mu\nu})^2 \;+ \tfrac{1}{2\alpha'}\, a^{\mu\nu} a_{\mu\nu}\;. \tag{A.1}$$

We rescale the field as $a \;\to\; 2a$ to get the canonical normalization for the kinetic term and also define $m^2 = \frac{4}{\alpha'}$. After coupling to a source $J_{\mu\nu}$ the Lagrangian becomes:

$$L \;=\; -\tfrac{1}{2}\, a^{\mu\nu}\, (\Box - m^2)\, a_{\mu\nu} + a^{\mu\nu}\partial_\mu\partial^\rho a_{\rho\nu} + a^{\mu\nu} J_{\mu\nu}\;. \tag{A.2}$$

The equations of motion in momentum space take the following form

$$(p^2+m^2)\,a_{\mu\nu} - p_\mu(p\cdot a)_\nu - p_\nu(p\cdot a)_\mu \;=\; -J_{\mu\nu}\,. \tag{A.3}$$

Using the equations of motion in the Lagrangian it takes the following form in the momentum space:

$$L \;=\; \tfrac{1}{2}\,J^{\mu\nu}\,(-p)\,a_{\mu\nu}\,(p)\;. \tag{A.4}$$

We introduce the notation $(pap) \equiv p_\mu a^{\mu\nu} p_\nu$ and $(p\cdot a)_\mu = p^\nu a_{\nu\mu}$. Contracting the above equation with $p^\mu p^\nu$ and solving for (pap), we get:

$$(pap) \;=\; \frac{(pJp)}{p^2-m^2}\;. \tag{A.5}$$

Contracting equation (A.3) with p^ν and using the expression for (pap), we can solve for $(pa)_\mu$:

$$(p\cdot a)_\mu \;=\; \frac{1}{m^2}\Big(\frac{p_\mu}{p^2-m^2}(pJp) - (p\cdot J)_\mu\Big)\,. \tag{A.6}$$

Using these expressions for (pap) and $(pa)_\mu$ in (A.3) we can solve for $a_{\mu\nu}$ and obtain:

$$a_{\mu\nu} \;=\; -\frac{1}{p^2+m^2}\Big(J_{\mu\nu} + \frac{1}{m^2}\Big(p_\mu(p\cdot J)_\nu + p_\nu(p\cdot J)_\mu\Big)\Big) + \frac{2\,p_\mu p_\nu}{m^2}\frac{(pJp)}{(p^2-m^2)(p^2+m^2)}\,. \tag{A.7}$$

Decomposing the last term into partial fractions, we get

$$a_{\mu\nu} \;=\; -\frac{1}{p^2+m^2}\widetilde{J}_{\mu\nu} + \frac{1}{(p^2-m^2)}\frac{p_\mu p_\nu(pJp)}{m^4}\,, \tag{A.8}$$

where

$$\widetilde{J}_{\mu\nu} \;\equiv\; J_{\mu\nu} + \frac{1}{m^2}\left(p_\mu(p\cdot J)_\nu + p_\nu(p\cdot J)_\mu\right) + \frac{p_\mu p_\nu(pJp)}{m^4}\;. \tag{A.9}$$

Back in (A.4) the Lagrangian becomes

$$L \;=\; -\tfrac{1}{2}J^{\mu\nu}\,(-p)\,\frac{1}{p^2+m^2}\widetilde{J}_{\mu\nu}\,(p) + \tfrac{1}{2m^4}(pJp)(-p)\frac{1}{p^2-m^2}(pJp)\,(p)\,. \tag{A.10}$$

The nature of the degrees of freedom is determined by the residues at the poles. At the pole $p^2+m^2=0$, it is easy to see that $\tilde{J}^{\mu\nu}$ is transverse, i.e., $p_\mu\tilde{J}^{\mu\nu} = 0$. Using this, we can write the Lagrangian as follows:

$$L \;=\; -\tfrac{1}{2}\tilde{J}^{\mu\nu}\,(-p)\,\frac{1}{p^2+m^2}\widetilde{J}_{\mu\nu}\,(p) + \tfrac{1}{2m^4}(pJp)(-p)\frac{1}{p^2-m^2}(pJp)\,(p)\,. \tag{A.11}$$

The first term implies that we are propagating a ghostly (overall minus sign) massive spin two mode (the traceless part of $\tilde{J}_{\mu\nu}$) and a ghostly, massive scalar (the trace of $\tilde{J}_{\mu\nu}$), both with mass squared equal to m^2. The second term shows a proper tachyonic scalar with mass squared given by $-m^2$.

The analysis of the sector involving $\bar{a}_{\mu\nu}$ can be done similarly. Note that the kinetic terms for $\bar{a}_{\mu\nu}$ and $a_{\mu\nu}$ have the same sign but their mass terms have opposite signs. Hence, the $\bar{a}_{\mu\nu}$ sector describes a ghostly tachyonic spin-2, a ghostly tachyonic scalar and a healthy massive scalar.

A.2 Degrees of freedom of full quadratic HSZ theory

Here we consider the full quadratic theory as given in the Lagrangian (3.19). We see that the three sectors, $(e_{\mu\nu}, \phi)$, $(a_{\mu\nu}, \varphi)$ and $(\bar{a}_{\mu\nu}, \bar{\varphi})$ are completely decoupled. The sector $(e_{\mu\nu}, \phi)$ is well known and describes a massless graviton, dilaton, and b-field. We focus on the $(a_{\mu\nu}, \varphi)$ sector of the Lagrangian given by:

$$L = -\tfrac{1}{8}\, a^{\mu\nu}\,\Box\, a_{\mu\nu} - \tfrac{1}{4}(\partial_\mu a^{\mu\nu})^2 - \tfrac{1}{2}\,\partial_\mu a^{\mu\nu}\partial_\nu\varphi \;+ \tfrac{1}{2\alpha'}\, a^{\mu\nu} a_{\mu\nu} \;+\; \tfrac{1}{2\alpha'}\varphi\Box\varphi - \varphi^2\;, \tag{A.12}$$

where we have put explicit factors of α'. We rescale the field $a_{\mu\nu} \to 2a_{\mu\nu}$ to get a canonical kinetic term and define $m^2 \equiv \frac{4}{\alpha'}$. After coupling to sources $J_{\mu\nu}$ and K the Lagrangian takes the following form

$$L \;=\; -\tfrac{1}{2}\, a^{\mu\nu}\,(\Box - m^2)\, a_{\mu\nu} + a^{\mu\nu}\partial_\mu\partial^\rho a_{\rho\nu} + \, a^{\mu\nu}\partial_\mu\partial_\nu\varphi + \tfrac{1}{2}\varphi(\Box - \tfrac{1}{2}m^2)\varphi + a^{\mu\nu}J_{\mu\nu} + \varphi K\;. \tag{A.13}$$

The equations of motion in momentum space are given by:

$$\begin{aligned}(p^2 + m^2)\, a_{\mu\nu} - p_\mu(p\cdot a)_\nu - p_\nu(p\cdot a)_\mu - p_\mu p_\nu\varphi \;&=\; -\,J_{\mu\nu}\,,\\ (p^2 + \tfrac{1}{2}m^2)\varphi + (pap) \;&=\; K\,.\end{aligned} \tag{A.14}$$

Using the equations of motion in the Lagrangian it takes the following form in the momentum space:

$$L \;=\; \tfrac{1}{2}\, J^{\mu\nu}\,(-p)\, a_{\mu\nu}\,(p) \;+\; \tfrac{1}{2}\, K\,(-p)\,\varphi\,(p)\;. \tag{A.15}$$

Contracting the first equation in (A.14) with p^μ we get

$$m^2\,(p\cdot a)_\nu - p_\nu\big((pap) + p^2\varphi\big) \;=\; -(p\cdot J)_\nu\;. \tag{A.16}$$

Contracting this with p^μ and solving for (pap) we obtain:

$$(pap) \; = \; \frac{(pJp)}{p^2 - m^2} \; - \; \frac{p^4 \varphi}{p^2 - m^2} \, , \tag{A.17}$$

where we notice tachyonic poles (that will disappear later). Using this in the equation of motion for φ (second one in (A.14)), we can solve for φ and obtain:

$$\varphi \; = \; \frac{2}{m^2} \cdot \frac{(pJp)}{p^2 + m^2} \; - \; \frac{2}{m^2} \cdot \frac{p^2 - m^2}{p^2 + m^2} K \, . \tag{A.18}$$

We now reconsider the first contraction (A.16) to find

$$(p \cdot a)_\mu \; = \; \frac{1}{m^2} \Big(p_\mu \big((pap) + p^2 \varphi \big) - (p \cdot J)_\mu \Big) \, . \tag{A.19}$$

Using this and the expression for φ in the equation of motion for $a_{\mu\nu}$ we can solve for $a_{\mu\nu}$ in terms of sources and get:

$$a_{\mu\nu} \; = \; - \frac{1}{p^2 + m^2} \widetilde{J}_{\mu\nu} + \frac{p_\mu p_\nu}{m^2 \left(p^2 + m^2\right)} \left(\frac{(pJp)}{m^2} + 2K \right) \, , \tag{A.20}$$

where $\widetilde{J}_{\mu\nu}$ is the transverse part of $J_{\mu\nu}$ as defined in equation (A.9). Back in (A.15) the Lagrangian becomes

$$\begin{aligned} L \; = \; & - \tfrac{1}{2} J^{\mu\nu} \left(-p\right) \frac{1}{p^2 + m^2} \widetilde{J}_{\mu\nu} \left(p\right) - \; K \left(-p\right) \frac{p^2 - m^2}{m^2 \left(p^2 + m^2\right)} K \left(p\right) \\ & + \left(pJp\right) \left(-p\right) \frac{1}{m^2 \left(p^2 + m^2\right)} K \left(p\right) + K(-p) \frac{1}{m^2 \left(p^2 + m^2\right)} \left(pJp\right) \left(p\right) \, . \end{aligned} \tag{A.21}$$

We now have to look at the pole $p^2 + m^2 = 0$. Using that $\tilde{J}_{\mu\nu}$ is transverse at the pole, the Lagrangian can be written in the following form at the pole:

$$L \; = \; - \tfrac{1}{2} \widetilde{J}^{\mu\nu} \left(-p\right) \frac{1}{p^2 + m^2} \widetilde{J}_{\mu\nu} \left(p\right) + \tfrac{1}{2} \left(2K \; + \frac{(pJp)}{m^2} \right) \left(-p\right) \frac{1}{p^2 + m^2} \left(2K \; + \frac{(pJp)}{m^2} \right) \left(p\right) \, . \tag{A.22}$$

The first term tells us that we are propagating a ghostly (overall minus sign) massive spin-2 mode (the traceless part of $\widetilde{J}_{\mu\nu}$) and a ghostly, massive scalar (the trace of $\widetilde{J}_{\mu\nu}$). The second term shows a proper massive scalar.

The analysis of the $(\bar{a}_{\mu\nu}, \bar{\varphi})$ sector can be done similarly. Since the mass terms of the two sectors have opposite signs and the kinetic terms have the same sign, the $(\bar{a}_{\mu\nu}, \bar{\varphi})$ sector propagates a ghostly tachyonic spin-2, a ghostly tachyonic scalar and a proper tachyonic scalar. If we compare this spectrum with that of the two-derivative theory we see that the *full* spectrum remains unchanged.

A.3 Degrees of freedom of massive DFT

We start with the Lagrangian for the massive DFT as given in equation (3.23). We scale the fields as $e_{\mu\nu} \to \sqrt{2} e_{\mu\nu}$ and $\Phi \to -\frac{1}{2\sqrt{2}}\phi$ to get canonical normalization for the kinetic terms. By using $e_{\mu\nu} = h_{\mu\nu} + b_{\mu\nu}$, the Lagrangian for the massive DFT can be written as:

$$L_{\text{mDFT}} = L_{h,\phi} + L_b \,, \tag{A.23}$$

where

$$\begin{aligned} L_{h,\phi} &= \tfrac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + (\partial_\mu h^{\mu\nu})^2 + h^{\mu\nu}\partial_\mu\partial_\nu\phi - \tfrac{1}{2}\phi\Box\phi - \tfrac{1}{2}M^2(h^{\mu\nu}h_{\mu\nu} - \phi^2)\,, \\ L_b &= \tfrac{1}{2}b^{\mu\nu}\Box b_{\mu\nu} + (\partial_\mu b^{\mu\nu})^2 - \tfrac{1}{2}M^2 b^{\mu\nu}b_{\mu\nu}\,. \end{aligned} \tag{A.24}$$

The Lagrangian L_b is well known to describe a massive two-form field and will not be discussed further. In order to make it clear that the mass terms in $L_{h,\phi}$ are special, we modify one of the coefficients by introducing a parameter γ. We will indeed find that the value $\gamma = 1$ is selected by the condition that we have no ghosts in the spectrum. We thus take, henceforth,

$$L_{h,\phi} = \tfrac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + (\partial_\mu h^{\mu\nu})^2 + h^{\mu\nu}\partial_\mu\partial_\nu\phi - \tfrac{1}{2}\phi\Box\phi - \tfrac{1}{2}M^2(h^{\mu\nu}h_{\mu\nu} - \gamma\phi^2)\,. \tag{A.25}$$

After coupling to sources $J_{\mu\nu}$ and K for $h_{\mu\nu}$ and ϕ, we have:

$$L_{h,\phi} = \tfrac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} + (\partial_\mu h^{\mu\nu})^2 + h^{\mu\nu}\partial_\mu\partial_\nu\phi - \tfrac{1}{2}\phi\Box\phi - \tfrac{1}{2}M^2(h^{\mu\nu}h_{\mu\nu} - \gamma\phi^2) + J^{\mu\nu}h_{\mu\nu} + K\phi\,. \tag{A.26}$$

In momentum space, the equations of motion take the following form:

$$\begin{aligned} J_{\mu\nu} - \left(p^2 + M^2\right) h_{\mu\nu} + 2p_{(\mu}p_\rho h^\rho{}_{\nu)} - p_\mu p_\nu \phi &= 0\,, \\ K + \left(p^2 + \gamma M^2\right)\phi - p_\mu p_\nu h^{\mu\nu} &= 0\,. \end{aligned} \tag{A.27}$$

Using these the Lagrangian takes the following form in the momentum space:

$$L_{h,\phi} = \tfrac{1}{2}J^{\mu\nu}(-p)\, h_{\mu\nu}(p) + \tfrac{1}{2}K(-p)\,\phi(p)\,. \tag{A.28}$$

Contracting the top equation of motion with $p^\mu p^\nu$ we get:

$$(pJp) \; + \left(p^2 - M^2\right)(php) - p^4\phi \; = \; 0 \, . \tag{A.29}$$

The above equation and the equation of motion for ϕ can now be used to eliminate ϕ and (php) in favor of sources:

$$\begin{aligned} \phi \; &= \; -\frac{(pJp)}{M^2 A} - \frac{p^2 - M^2}{M^2 A} K \\ (php) \; &= \; -\frac{p^4}{M^2 A} K - \frac{p^2 + \gamma M^2}{M^2 A} (pJp) \; , \end{aligned} \tag{A.30}$$

where A is given by

$$A \; = \; p^2 \left(\gamma - 1\right) \; - \gamma M^2 \, . \tag{A.31}$$

Contracting the equation of motion for $h_{\mu\nu}$ with p^ν we get:

$$(pJ)_\mu \; + p_\mu \left(php - p^2\phi\right) \; = \; M^2 \, (ph)_\mu \, . \tag{A.32}$$

Using eqns. (A.30) yields

$$(ph)_\mu = \frac{1}{M^2} (pJ)_\mu \; - \frac{p_\mu}{M^2 A} \left(p^2 K + \gamma \, (pJp)\right) . \tag{A.33}$$

Finally, using eqns. (A.33) and (A.30) in the equation of motion for $h_{\mu\nu}$ we obtain

$$h_{\mu\nu} \; = \widetilde{J}_{\mu\nu} \, \frac{1}{p^2 + M^2} \; - \frac{p_\mu p_\nu}{M^4 A} \left(M^2 K \; + (\gamma - 1)\,(pJp)\right) \; , \tag{A.34}$$

where $\tilde{J}_{\mu\nu}$ is defined by

$$\widetilde{J}_{\mu\nu} \; = \; J_{\mu\nu} \; + \frac{2}{M^2} p_{(\mu} \, (ph)_{\nu)} \; + \frac{p_\mu p_\nu}{M^4} \, (pJp) \; . \tag{A.35}$$

It is easy to see that on the mass-shell $p^2 = -M^2$, the tensor $\tilde{J}_{\mu\nu}$ is transverse,

$$p^\mu \widetilde{J}_{\mu\nu} \; = \; 0 \quad (p^2 = -M^2) \, . \tag{A.36}$$

This will be useful below. Inserting these expressions back into $L_{h,\phi}$, we get:

$$L_{h,\phi} = \tfrac{1}{2} J_{\mu\nu}(-p) \frac{1}{p^2 + M^2} \widetilde{J}^{\mu\nu}(p) - \tfrac{1}{2} \frac{(pJp)^2 (\gamma - 1) + 2M^2 (pJp) K + M^2 K^2 (p^2 - M^2)}{M^4 (\gamma - 1) \left(p^2 + \frac{\gamma M^2}{1-\gamma}\right)} . \tag{A.37}$$

For the case of interest, $\gamma = 1$, the second term above is completely regular and we need only focus on the first term. Using the transversality condition (A.36) we can rewrite $L_{h,\phi}$ as

$$L_{h,\phi} = \widetilde{J}_{\mu\nu}(-p) \frac{1}{p^2 + M^2} \widetilde{J}^{\mu\nu}(p) + \dots , \tag{A.38}$$

near $p^2 = -M^2$ and where the dots indicate terms that are regular. At the mass-shell we can choose $p = (M, \vec{0})$ and thus the transversality condition implies that $\widetilde{J}_{0\mu} = \widetilde{J}_{\mu 0} = 0$. The only non vanishing components of $\widetilde{J}_{\mu\nu}$ are those where both indices represent spatial directions. We are thus propagating $(D-1)D/2$ positive-norm degrees of freedom, associated with a symmetric $(D-1) \times (D-1)$ matrix. The trace-less part corresponds to the massive spin-2 and the trace corresponds to the massive scalar.

For the case $\gamma \neq 1$ the above degrees of freedom are still present but we now have more, due to the pole in the second term of (A.37). This time the mass-shell is $p^2 = -\frac{\gamma M^2}{1-\gamma}$ and we go to a Lorentz frame where $p^0 = \sqrt{\frac{\gamma M^2}{1-\gamma}}$. Near the pole we now find

$$L_{h,\phi}|_{\text{second pole}} = -\tfrac{1}{2} \frac{(K - \gamma J_{00})(-p)\,(K - \gamma J_{00})(p)}{(\gamma - 1)^2 \left(p^2 + \frac{\gamma M^2}{1-\gamma}\right)} + \dots , \tag{A.39}$$

making it manifest that for $\gamma \neq 1$ we propagate an additional ghostly massive scalar. We conclude that the model constructed in section (4.1) describes massive graviton, dilaton and b-field and does not propagate any extra undesired degrees of freedom.

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B

Conventions and useful properties of Gamma matrices

We use 10-dimensional Majorana-Weyl spinors ϵ_α and Ψ_α, etc. The 10-dimensional Γ-matrices are chosen to be real and symmetric:

$$\Gamma^{M\alpha\beta} = \Gamma^{M\beta\alpha}, \qquad \tilde{\Gamma}^{M\alpha\beta} = \tilde{\Gamma}^{M\beta\alpha}. \tag{B.1}$$

Products of Γ-matrices are given by:

$$\begin{aligned} \Gamma^{MN} &\equiv \tilde{\Gamma}^{[M}\Gamma^{N]}, \qquad \tilde{\Gamma}^{MN} \equiv \Gamma^{[M}\tilde{\Gamma}^{N]} \\ \Gamma^{MNP} &\equiv \Gamma^{[M}\tilde{\Gamma}^{N}\Gamma^{P]}, \qquad \tilde{\Gamma}^{MNP} \equiv \tilde{\Gamma}^{[M}\Gamma^{N}\tilde{\Gamma}^{P]}, \qquad \text{etc.} \end{aligned} \tag{B.2}$$

we also have that $\Gamma^{MNP\alpha\beta} = -\Gamma^{MNP\beta\alpha}$, hence:

$$\epsilon\Gamma^{MNP}\epsilon = 0 \tag{B.3}$$

for any bosonic spinor ϵ. We also introduce:

$$\tilde{\epsilon} = \beta\Lambda\epsilon, \tag{B.4}$$

where $\beta = \frac{1}{2r}$ and $\Lambda = \Gamma^{089}$. A useful relation is the triality condition,

$$\Gamma^M_{\alpha\beta}\Gamma_{M\gamma\delta} + \Gamma^M_{\beta\delta}\Gamma_{M\gamma\alpha} + \Gamma^M_{\delta\alpha}\Gamma_{M\gamma\beta} = 0. \tag{B.5}$$

Using eq. (B.5) one can show

$$\epsilon\Gamma^M\epsilon\,\epsilon\Gamma_M\chi = 0, \tag{B.6}$$

where χ is any spinor. It immediately follows that $v^M v_M = 0$, where v^M is the vector field

$$v^M \equiv \epsilon\Gamma^M\epsilon. \tag{B.7}$$

We define another set of bosonic spinors, ν_m for $m = 1, 2, \cdots, 7$. They satisfy the following properties.

$$\begin{aligned}
\nu_m\Gamma^M\epsilon &= 0, \\
\nu_m\Gamma^M\nu_n &= \delta_{mn}v^M, \\
\nu_\alpha^m\nu_\beta^n + \epsilon_\alpha\epsilon_\beta &= \frac{1}{2}v^M\tilde{\Gamma}_{M\alpha\beta}.
\end{aligned} \tag{B.8}$$

They are invariant under an internal $SO(7)$ symmetry, which can be enlarged to $SO(8)$ by including ϵ.

To reduce to eight supersymmetries we impose the condition $\epsilon = +\Gamma^{6789}\epsilon$. Furthermore, for $d \leq 5$, Ψ can be split up into even and odd eigenstates of Γ^{6789}. The even eigenstates, $\psi = \frac{1}{2}\left(1 + \Gamma^{6789}\right)\Psi$, make up the fermions in the vector multiplet, while the odd eigenstates, $\chi = \frac{1}{2}\left(1 - \Gamma^{6789}\right)\Psi$, make up the fermions in the hypermultiplet. The scalars ϕ^I, $I = 6, \ldots 9$ constitute the bosonic fields of the hyper multiplet. The gauge fields A^μ and the rest of the scalars ϕ_I, $I = 0, d+1, \ldots 5$ make up the bosonic fields in the vector multiplet. Finally, the auxiliary fields split up, with K^m, $m = 1, 2, 3$, being in the vector multiplet, and K_m, $m = 4, 5, 6, 7$, being in the hypermultiplet. The same is true for the pure-spinors ν^m. Reduction to four supersymmetries can be done similarly by imposing $\epsilon = +\Gamma^{4589}\epsilon$.

C

Quadratic fluctuations about the fixed point locus

In this appendix we give details of the computation of quadratic fluctuations about the fixed point locus. We focus on bosonic and fermionic parts separately.

C.1 Bosonic part

The bosonic part of fluctuations about the fixed point locus is equal to [6]:

$$
\begin{aligned}
\mathcal{L}^{\mathrm{b}} &= \delta_\epsilon \Psi \overline{\delta_\epsilon \Psi} \\
&= \frac{1}{2} F_{MN} F^{MN} - \frac{1}{4} F_{MN} F_{M'N'} \left(\epsilon \Gamma^{MNM'N'0} \epsilon\right) + \frac{\beta d \alpha_I}{4} F_{MN} \phi_I \left(\epsilon \Lambda \left(\tilde{\Gamma}^I \tilde{\Gamma}^{MN} \Gamma^0 - \tilde{\Gamma}^0 \Gamma^I \Gamma^{MN}\right) \epsilon\right) \\
&- K^m K_m v^0 - \beta d \alpha_0 \phi_0 K^m (\nu_m \Lambda \epsilon) + \frac{\beta^2 d^2}{4} \sum_I (\alpha_I)^2 \phi_I \phi^I v^0.
\end{aligned}
\tag{C.1}
$$

Expanding the first term in eq. (C.1) we get

$$
\begin{aligned}
\frac{1}{2} F_{MN} F^{MN} &= \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + F_{\mu 0} F^{\mu 0} - [\phi_0^{\mathrm{cl}}, \phi_J][\phi_0^{\mathrm{cl}}, \phi^J] + \nabla_\mu \phi_J \nabla^\mu \phi^J \\
&= \nabla_\mu A_\nu \nabla^\mu A^\nu - \nabla_\mu A_\nu \nabla^\nu A^\mu + \nabla_\mu \phi_0 \nabla^\mu \phi^0 + 2 \nabla_\mu \phi_0 [A^\mu, \phi^0_{cl}] \\
&- [A_\mu, \phi_0^{\mathrm{cl}}][A^\mu, \phi_0^{\mathrm{cl}}] - [\phi_0^{\mathrm{cl}}, \phi_J][\phi_0^{\mathrm{cl}}, \phi^J] + \nabla_\mu \phi_J \nabla^\mu \phi^J,
\end{aligned}
\tag{C.2}
$$

where $J = d+1, \ldots, 9$. The second term in eq. (C.1) can be expanded to get:

$$
\begin{aligned}
-\frac{1}{4} F_{MN} F_{M'N'} \left(\epsilon \Gamma^{MNM'N'0} \epsilon\right) &= -\nabla_\mu A_\nu \nabla_{\mu'} A_{\nu'} \left(\epsilon \Gamma^{\mu\nu\mu'\nu'0} \epsilon\right) - 2 \nabla_\mu A_\nu \nabla_{\mu'} \phi_J \left(\epsilon \Gamma^{\mu\nu\mu' J 0} \epsilon\right) \\
&- \nabla_\mu \phi_J \nabla_{\mu'} \phi_{J'} \left(\epsilon \Gamma^{\mu J \mu' J' 0} \epsilon\right).
\end{aligned}
\tag{C.3}
$$

The third term in eq. (C.1) is

$$\frac{\beta d\alpha_I}{4} F_{MN}\phi_I \left(\epsilon\Lambda\left(\tilde{\Gamma}^I\tilde{\Gamma}^{MN}\Gamma^0 - \tilde{\Gamma}^0\Gamma^I\Gamma^{MN}\right)\epsilon\right) = \frac{\beta d\alpha_J}{2}\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right)\phi_J\left(\epsilon\Lambda\tilde{\Gamma}^J\Gamma^0\tilde{\Gamma}^\mu\Gamma^\nu\epsilon\right) + \beta d\alpha_{J'}\nabla_\mu\phi_J\phi_{J'}\left(\epsilon\Lambda\tilde{\Gamma}^{J'}\Gamma^0\tilde{\Gamma}^\mu\Gamma^J\epsilon\right). \tag{C.4}$$

Collecting our results, we find that the bosonic part is:

$$\begin{aligned}\mathcal{L}^{\rm b} &= \nabla_\mu A_\nu\nabla^\mu A^\nu - \nabla_\mu A_\nu\nabla^\nu A^\mu - \nabla_\mu\phi_0\nabla^\mu\phi_0 - 2\nabla_\mu\phi_0[A^\mu,\phi_0^{\rm cl}] - [A_\mu,\phi_0^{\rm cl}][A^\mu,\phi_0^{\rm cl}]\\ &- [\phi_0^{\rm cl},\phi_J][\phi_0^{\rm cl},\phi^J] + \nabla_\mu\phi_J\nabla^\mu\phi^J\\ &- \nabla_\mu A_\nu\nabla_{\mu'}A_{\nu'}\left(\epsilon\Gamma^{\mu\nu\mu'\nu'0}\epsilon\right) - 2\nabla_\mu A_\nu\nabla_{\mu'}\phi_J\left(\epsilon\Gamma^{\mu\nu\mu'J0}\epsilon\right) - \nabla_\mu\phi_J\nabla_{\mu'}\phi_{J'}\left(\epsilon\Gamma^{\mu J\mu'J'0}\epsilon\right)\\ &+ \frac{\beta d\alpha_J}{2}\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right)\phi_J\left(\epsilon\Lambda\tilde{\Gamma}^J\Gamma^0\tilde{\Gamma}^\mu\Gamma^\nu\epsilon\right) + \beta d\alpha_{J'}\nabla_\mu\phi_J\phi_{J'}\left(\epsilon\Lambda\tilde{\Gamma}^{J'}\Gamma^0\tilde{\Gamma}^\mu\Gamma^J\epsilon\right)\\ &- v^0K^mK_m - \beta d\alpha_0\phi_0K^m(\nu_m\Lambda\epsilon) + \frac{\beta^2d^2}{4}v^0\sum_I(\alpha_I)^2\phi_I\phi^I.\end{aligned} \tag{C.5}$$

Next, we rewrite this expression as a quadratic form:

$$\begin{aligned}\mathcal{L}^{\rm b} &= A^\mu\left(-\delta_\mu^\nu\nabla^2 + \nabla^\nu\nabla_\mu - \left(\epsilon\Gamma_\mu{}^{\mu'\nu'\nu0}\epsilon\right)\nabla_{\mu'}\nabla_{\nu'} - 2\beta(d-3)\left(\epsilon\Lambda\Gamma_\mu{}^{\mu'\nu0}\epsilon\right)\nabla_{\mu'}\right)A_\nu - [A_\mu,\phi_0^{\rm cl}][A^\mu,\phi_0^{\rm cl}]\\ &+ \phi^J\left(-\nabla^2\delta_J^{J'} - 2\beta(d-1)\left(\epsilon\Lambda\Gamma_J{}^{\mu J'0}\epsilon\right)\nabla_\mu - \beta d\alpha_{J'}\left(\epsilon\Lambda\tilde{\Gamma}^{J'}\Gamma^0\tilde{\Gamma}^\mu\Gamma_J\epsilon\right)\nabla_\mu + \frac{\beta^2d^2}{4}(\alpha_J)^2\delta_J^{J'}\right)\phi_{J'}\\ &- [\phi_0^{\rm cl},\phi_J][\phi_0^{\rm cl},\phi^J] + \phi_0\left(\nabla^2 - \frac{\beta^2d^2}{4}\alpha_0^2\right)\phi_0 - 4\beta(d-2)A_\nu\nabla_\mu\phi_J\left(\epsilon\Lambda\Gamma^{\nu\mu}\Gamma^{J0}\epsilon\right)\\ &+ \beta d\alpha_J\,\phi_J\nabla_\mu A_\nu\left(\epsilon\Lambda\tilde{\Gamma}^J\Gamma^0\Gamma^{\mu\nu}\epsilon\right) - K^mK_m - \beta d\alpha_0\phi_0K^m(\nu_m\Lambda\epsilon),\end{aligned} \tag{C.6}$$

where we have used the Lorenz gauge condition and the relation $\tilde{\Gamma}_\mu\Gamma^{\mu\nu\mu'} = (d-2)\Gamma^{\nu\mu'}$. Now, note that the third term in the first row vanishes, and that for the second term, we can exchange the order of the covariant derivatives to get a term which is zero due to the Lorenz gauge condition and another one which contains a Ricci tensor, which on spheres is proportional to a Kronecker delta. Furthermore, we can combine the two terms which are proportional to $\nabla_\mu \phi_{J'}$ into one, and finally get:

$$\begin{aligned}\mathcal{L}^{\mathrm{b}} &= A^\mu \left(-\delta_\mu^\nu \nabla^2 + 4\beta^2(d-1)\delta_\mu^\nu - 2\beta(d-3)\left(\epsilon\Lambda\Gamma_\mu{}^{\mu'\nu 0}\epsilon\right)\nabla_{\mu'}\right)A_\nu - [A_\mu, \phi_0^{\mathrm{cl}}][A^\mu, \phi_0^{\mathrm{cl}}] \\ &+ \phi^J\left(-\nabla^2\delta_J^{J'} + \beta\left(-2(d-1) + d\alpha_{J'}\right)\left(\epsilon\Lambda\Gamma^{J'}\Gamma_J\Gamma^{\mu 0}\epsilon\right)\nabla_\mu + \frac{\beta^2 d^2}{4}(\alpha_J)^2\delta_J^{J'}\right)\phi_{J'} \\ &- [\phi_0^{\mathrm{cl}}, \phi_J][\phi_0^{\mathrm{cl}}, \phi^J] + \phi_0\left(\nabla^2 - \frac{\beta^2 d^2}{4}\alpha_0^2\right)\phi_0 + \beta\left(-4(d-2) + d\alpha_J\right)A_\nu\nabla_\mu\phi_J\left(\epsilon\Lambda\Gamma^{\nu\mu}\Gamma^{J0}\epsilon\right) \\ &- K^m K_m - \beta d\alpha_0\phi_0 K^m(\nu_m\Lambda\epsilon).\end{aligned} \tag{C.7}$$

This general result includes both the vector multiplet and the hypermultiplet bosons. We now specialize to the vector multiplet.

Vector multiplet

The vector multiplet contains the vector field A_μ and the scalar fields ϕ_0, ϕ_i, where the index i takes values $i = d+1, \cdots D$ and $D = 5$ for eight supersymmetries and $D = 3$ for four supersymmetries. We use

$$\alpha_0 = \frac{4(d-3)}{d}, \quad \alpha_i = \frac{4}{d}, \quad \text{for } i = d+1, \ldots, D. \tag{C.8}$$

We also combine μ and i indices into $\tilde{M} = \{\mu, i\}$ to write the bosonic part of the vector multiplet Lagrangian from equation (C.7) in the following compact form:

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}_{\mathrm{v.m}} &= A^{\tilde{M}}\,\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}\,A_{\tilde{N}} - [A_{\tilde{M}}, \phi_0^{\mathrm{cl}}][A^{\tilde{M}}, \phi_0^{\mathrm{cl}}] \\ &- K^m K_m - 4\beta(d-3)\phi_0 K^m(\nu_m\Lambda\epsilon) + \phi_0\left(\nabla^2 - 4\beta^2(d-3)^2\right)\phi_0.\end{aligned} \tag{C.9}$$

The operator $\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}$ is defined as follows:

$$\mathcal{O}_{\tilde{M}}{}^{\tilde{N}} = -\delta_{\tilde{M}}^{\tilde{N}}\nabla^2 + \alpha_{\tilde{M}}{}^{\tilde{N}} - 2\beta(d-3)\epsilon\Gamma_{\tilde{M}}{}^{\nu\tilde{N}89}\epsilon\nabla_\nu. \tag{C.10}$$

and $\alpha_{\tilde{M}}{}^{\tilde{N}}$ is the diagonal matrix given by:

$$\alpha_{\tilde{M}}{}^{\tilde{N}} = 4\beta^2\begin{pmatrix}(d-1)\,\delta_\mu^\nu & 0 \\ 0 & \delta_i^j\end{pmatrix}. \tag{C.11}$$

Hyper/chiral multiplet

The scalars $\phi^I, I = D+1, \dots, 9$ are part of the hypermultiplet. For eight supersymmetries we get a single hypermultiplet and for four supersymmetries we get three hypermultiplets by reduction of 10-d theory. We first focus on four supersymmetries:

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}_{\mathrm{c.m}} &= \phi^J \left(-\nabla^2 \delta_J^{J'} + \beta\left(-2(d-1) + d\alpha_{J'}\right) \left(\epsilon\Lambda\Gamma^{J'}\Gamma_J\Gamma^{\mu 0}\epsilon\right) \nabla_\mu + \frac{\beta^2 d^2}{4}\alpha_J^2 \delta_J^{J'} \right) \phi_{J'} \\ &- [\phi_0^{\mathrm{cl}}, \phi_J][\phi_0^{\mathrm{cl}}, \phi^J].\end{aligned} \tag{C.12}$$

For four supersymmetries, the values of α_I are given in (6.20). The Lagrangian of equation (C.12) splits up in three decoupled parts which take the form:

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}_{\mathrm{c.m}} = \sum_{\ell=1}^{3} &\phi^{I_\ell} \left(-\nabla^2 \delta_{I_\ell}^{L_\ell} - 2\beta(1 - 2i\sigma_{(\ell)}\mu_\ell) \left(\epsilon\Lambda\Gamma^{J_\ell}\Gamma_{I_\ell}\Gamma^{\mu 0}\epsilon\right) \nabla_\mu + \beta^2(d - 2 + 2i\sigma_{(\ell)}\mu_\ell)^2 \delta_{I_\ell}{}^{J_\ell} \right) \phi_{J_\ell} \\ &- [\phi_0^{\mathrm{cl}}, \phi_{I_\ell}][\phi_0^{\mathrm{cl}}, \phi^{I_\ell}].\end{aligned} \tag{C.13}$$

This can be simplified by noting that

$$\epsilon\Lambda\Gamma^{76\mu 0}\epsilon = v^\mu, \qquad \epsilon\Lambda\Gamma^{98\mu 0}\epsilon = -v^\mu, \qquad \epsilon\Lambda\Gamma^{54\mu 0}\epsilon = v^\mu. \tag{C.14}$$

This gives the following form of the chiral multiplet Lagrangian.

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}_{\mathrm{c.m}} &= \sum_{\ell=1}^{3} \left[\phi_{I_\ell} \left(-\nabla^2 + \beta^2(d - 2 + 2i\sigma_{(\ell)}\mu_\ell)^2 \right) \phi^{I_\ell} - [\phi_0^{\mathrm{cl}}, \phi_{I_\ell}][\phi_0^{\mathrm{cl}} \phi^{I_\ell}] \right] \\ &\quad + 4\beta \left(2i\mu_\ell - \sigma_{(\ell)}\right) \phi_{2\ell+2} v^\mu \nabla_\mu \phi_{2\ell+3}.\end{aligned} \tag{C.15}$$

For the case of eight supersymmetries the Lagrangian for hypermultiplet bosons can be obtained from the above expressions by ignoring ϕ_4, ϕ_5 and setting $\mu_2 = \mu_3$:

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}_{\mathrm{h.m}} = &\sum_{i=6}^{9} \left[\phi_i \left(-\nabla^2 + \beta^2(d - 2 + 2i\sigma_i\mu)^2 \right) \phi_i - [\phi_0^{\mathrm{cl}}, \phi_i][\phi_0^{\mathrm{cl}}, \phi_i] \right] \\ &+ 4\beta \left(2i\mu - 1\right) \phi_6 v^\mu \nabla_\mu \phi_7 + 4\beta \left(2i\mu + 1\right) \phi_8 v^\mu \nabla_\mu \phi_9.\end{aligned} \tag{C.16}$$

C.2 Fermionic part

The fermionic part of the fluctuations around the fixed point locus is given by:

$$\mathcal{L}^{\mathrm{f}} = \Psi\delta_\epsilon\left(\overline{\delta_\epsilon\Psi}\right) = \Psi\Gamma^0\delta_\epsilon^2\Psi - \Psi\Gamma^0\left[2\Gamma^{M'0}\delta_\epsilon F_{M'0}\epsilon + \alpha_0\Gamma^{\mu 0}\delta_\epsilon\phi_0\nabla_\mu\epsilon + 2\delta_\epsilon K^m\nu_m\right]. \tag{C.17}$$

We focus on the first term, involving two variations of the fermion.

$$\begin{aligned}\delta_\epsilon^2\Psi &= \left(\epsilon\Gamma^N D^M\Psi\right)\Gamma_{MN}\epsilon - \beta\left(\epsilon\Lambda\tilde{\Gamma}^\mu\Gamma^N\Psi\right)\Gamma_{\mu N}\epsilon - \frac{\alpha_I\beta d}{2}\left(\epsilon\Gamma_I\Psi\right)\tilde{\Gamma}^I\Lambda\epsilon \\ &+ \left(\epsilon \not{D}\Psi\right)\epsilon - \frac{1}{2}v^M\tilde{\Gamma}_M\not{D}\Psi + \Delta K^m\nu_m.\end{aligned} \tag{C.18}$$

This expression can be brought into the desired form by using triality and other identities. Using triality the first term in eq. (C.18)

$$-\left(\epsilon\not{D}\Psi\right)\epsilon - \left(\epsilon\Gamma^N\epsilon\right)D_N\Psi + \frac{1}{2}\left(\epsilon\Gamma^N\epsilon\right)\tilde{\Gamma}_N\not{D}\Psi. \tag{C.19}$$

Using triality, the second term in eq. (C.18) becomes

$$\begin{aligned}&\beta d\left(\epsilon\Lambda\Psi\right)\epsilon + \frac{1}{2}\beta\left(\epsilon\Lambda\Gamma_{MN}\epsilon\right)\Gamma^{MN}\Psi - \frac{1}{2}\beta\left(\epsilon\Lambda\Gamma_{IJ}\epsilon\right)\Gamma^{IJ}\Psi \\ &+ \frac{1}{2}\beta\left(\epsilon\Lambda\Gamma_{\mu\nu}\epsilon\right)\Gamma^{\mu\nu}\Psi - 2\beta\left(\epsilon\Gamma^\mu\Psi\right)\tilde{\Gamma}_\mu\Lambda\epsilon + d\beta\left(\epsilon\Gamma^N\Psi\right)\tilde{\Gamma}_N\Lambda\epsilon.\end{aligned} \tag{C.20}$$

The second term in the above expression can be simplified using the following Fierz identity [5, 73]:

$$-\frac{1}{2}\left(\tilde{\epsilon}\Gamma_{MN}\epsilon\right)\Gamma^{MN}\Psi - 4\left(\Psi\tilde{\epsilon}\right)\epsilon + 2\left(\epsilon\Gamma_N\Psi\right)\tilde{\Gamma}^N\tilde{\epsilon} = 0, \tag{C.21}$$

Combining all these pieces we get

$$\begin{aligned}\delta_\epsilon^2\Psi &= -v^N D_N\Psi - \frac{1}{4}\nabla_{[\mu}v_{\nu]}\Gamma^{\mu\nu}\Psi - \frac{1}{2}\beta\left(\epsilon\Lambda\Gamma_{IJ}\epsilon\right)\Gamma^{IJ}\Psi + \beta\left(2 - \frac{\alpha_I d}{2}\right)\left(\epsilon\Gamma_I\Psi\right)\tilde{\Gamma}^I\Lambda\epsilon \\ &+ \beta(d-4)\left(\left(\epsilon\Lambda\Psi\right)\epsilon + \left(\epsilon\Gamma_N\Psi\right)\tilde{\Gamma}^N\Lambda\epsilon\right) + \Delta K^m\nu_m.\end{aligned} \tag{C.22}$$

Let's focus on the second term in (C.17) and simplify all three terms appearing there. The first one is

$$-2\left(\Psi\Gamma^0\Gamma^{M'0}\epsilon\right)\delta_\epsilon F_{M'0} = 2\left(\Psi\Gamma^0\epsilon\right)\left(\epsilon\not{\nabla}\Psi\right) - 2\beta d\left(\Psi\Gamma^0\epsilon\right)\left(\epsilon\Lambda\Psi\right) + 2\left(\Psi\Gamma^{M'}\epsilon\right)\left(\epsilon\Gamma_{M'}D_0\Psi\right). \tag{C.23}$$

The second term is:

$$- \alpha_0 \Psi \Gamma^0 \Gamma^{\mu 0} \delta_\epsilon \phi_0 \nabla_\mu \epsilon = \alpha_0 d\beta \left(\Psi \Gamma^0 \epsilon\right) \left(\epsilon \Lambda \Psi\right). \tag{C.24}$$

The third term is:

$$\begin{aligned} -2 \left(\Psi \Gamma^0 \nu_m\right) \delta_\epsilon K^m = &-2 \left(\Psi \Gamma^0 \epsilon\right) \left(\epsilon \not{\nabla} \Psi\right) + 2 \left(\Psi \Gamma^0 \epsilon\right) \left(\epsilon \Gamma_0 D_0 \Psi\right) + v^M \left(\Psi \Gamma^0 \tilde{\Gamma}_M \not{D} \Psi\right) \\ &- 2 \left(\Psi \Gamma^0 \nu_m\right) \Delta K^m. \end{aligned} \tag{C.25}$$

Collecting all the terms, we get:

$$v^M \left(\Psi \Gamma^0 \tilde{\Gamma}_M \not{D} \Psi\right) + d\beta \left(\alpha_0 - 2\right) \left(\Psi \Gamma^0 \epsilon\right) \left(\epsilon \Lambda \Psi\right) + 2 \left(\Psi \Gamma^M \epsilon\right) \left(\epsilon \Gamma_M D_0 \Psi\right) - 2 \left(\Psi \Gamma^0 \nu_m\right) \Delta K^m. \tag{C.26}$$

The first term in the above expression can be rewritten using the identity $\tilde{\Gamma}^M \Gamma^N = g^{MN} + \Gamma^{MN}$, and the third one can be manipulated using the triality identity. The result is

$$\begin{aligned} &\left(\Psi \not{\nabla} \Psi\right) + v^\mu \left(\Psi \Gamma^0 \Gamma_{\mu\nu} \nabla^\nu \Psi\right) + \sum_{I=d+1}^{9} v^I \left(\Psi \Gamma^0 \Gamma_{I\nu} \nabla^\nu \Psi\right) + v^\mu \left(\Psi \Gamma^0 \nabla_\mu \Psi\right) \\ &+2 \left(\Psi \Gamma^0 D_0 \Psi\right) + d\beta \left(\alpha_0 - 2\right) \left(\Psi \Gamma^0 \epsilon\right) \left(\epsilon \Lambda \Psi\right) - 2 \left(\Psi \Gamma^0 \nu_m\right) \Delta K^m. \end{aligned} \tag{C.27}$$

Using integration by parts, the second and third terms can be modified to give

$$\begin{aligned} &\left(\Psi \not{\nabla} \Psi\right) - \beta \left(\epsilon \tilde{\Gamma}^{\mu\nu} \Lambda \epsilon\right) \left(\Psi \Gamma^0 \Gamma_{\mu\nu} \Psi\right) - \beta \left(\epsilon \tilde{\Gamma}^{J\nu} \Lambda \epsilon\right) \left(\Psi \Gamma^0 \Gamma_{J\nu} \Psi\right) + v^\mu \left(\Psi \Gamma^0 \nabla_\mu \Psi\right) \\ &+2 \left(\Psi \Gamma^0 D_0 \Psi\right) + d\beta \left(\alpha_0 - 2\right) \left(\Psi \Gamma^0 \epsilon\right) \left(\epsilon \Lambda \Psi\right) - 2 \left(\Psi \Gamma^0 \nu_m\right) \Delta K^m. \end{aligned} \tag{C.28}$$

Now, combining this with the result for $\Psi \Gamma^0 \delta_\epsilon^2 \Psi$, we get the complete expression for the fermionic part

$$\begin{aligned} \mathcal{L}^{\mathrm{f}} = &\left(\Psi \not{\nabla} \Psi\right) + \left(\Psi \Gamma^0 D_0 \Psi\right) + \beta \left(3d - 16\right) \left(\Psi \Gamma^0 \epsilon\right) \left(\epsilon \Lambda \Psi\right) - \frac{1}{2} \beta \left(\epsilon \tilde{\Gamma}^{MN} \Lambda \epsilon\right) \left(\Psi \Gamma^0 \Gamma_{MN} \Psi\right) \\ &+ \beta \left(2 - \frac{\alpha_I d}{2}\right) \left(\Psi \Gamma^0 \tilde{\Gamma}^I \Lambda \epsilon\right) \left(\epsilon \Gamma_I \Psi\right) + \beta (d - 4) \left(\Psi \Gamma^0 \tilde{\Gamma}^N \Lambda \epsilon\right) \left(\epsilon \Gamma_N \Psi\right) \\ &- \left(\Psi \Gamma^0 \nu_m\right) \Delta K^m. \end{aligned} \tag{C.29}$$

The terms on the second line can be modified by using the following identity:

$$\tilde{\Gamma}^N \Lambda \epsilon \left(\epsilon \Gamma_N \Psi\right) - 2 \tilde{\Gamma}^A \Lambda \epsilon \left(\epsilon \Gamma_A \Psi\right) = \frac{1}{2} v^N \tilde{\Lambda} \Gamma_N \Psi \tag{C.30}$$

So the quadratic part becomes

$$\begin{aligned}\mathcal{L}^{\mathrm{f}} &= \left(\Psi\not{D}\Psi\right) + \left(\Psi\Gamma^0 D_0\Psi\right) + \beta\,(3d-16)\left(\Psi\Gamma^0\epsilon\right)\left(\epsilon\Lambda\Psi\right) - \frac{1}{2}\beta\left(\epsilon\tilde{\Gamma}^{MN}\Lambda\epsilon\right)\left(\Psi\Gamma^0\Gamma_{MN}\Psi\right)\\ &+ \beta\mathcal{C}_I\left(\Psi\Gamma^0\tilde{\Gamma}^I\Lambda\epsilon\right)\left(\epsilon\Gamma_I\Psi\right) + \beta\frac{d-4}{2}v^N\left(\Psi\Gamma^0\tilde{\Lambda}\Gamma_N\Psi\right) - \left(\Psi\Gamma^0\nu_m\right)\Delta K^m,\end{aligned} \tag{C.31}$$

where the coefficient $\mathcal{C}_I$ which appear in the first term in second line is given by:

$$\mathcal{C}_A \;=\; 2d-6-\frac{\alpha_A d}{2}\;, \qquad \mathcal{C}_i \;=\; 2-\frac{\alpha_i d}{2}. \tag{C.32}$$

We now specialize to vector and hypermultiplets separately.

Vector multiplet

The vector multiplet fermions have same eigenvalues under the projection operators Γ,Γ' as the Killing spinor. We denote the vector multiplet fermion by ψ. For a fermion in the vector multiplet, the first term on the second line of eq. (C.31) does not contribute. It is easy to verify that for this term, either $\mathcal{C}_I$ vanishes or $(\epsilon\Gamma_I\psi) = 0$. Further, for the last term in eq. (C.31)), we take pure spinors $\nu^m, m = 1,2,\cdots D-2$ to have the same eigenvalues under projection operators as the Killing spinor and the vector multiplet fermion, while the rest of the pure spinors have the same eigenvalues as the hypermultiplet fermions. We use

$$\Delta K^m \;=\; \beta\,(d-4)\,\nu^m\Lambda\psi, \qquad \text{for} \qquad m=1,2,\cdots D-2, \tag{C.33}$$

to write:

$$\begin{aligned}-\left(\psi\Gamma^0\nu_m\right)\Delta K^m \;&=\; -\beta\,(d-4)\sum_{m=1}^{m=D-2}\left(\psi\Gamma^0\nu_m\right)\left(\nu^m\Lambda\psi\right),\\ &= -\beta\,(d-4)\sum_{m=1}^{7}\left(\psi\Gamma^0\nu_m\right)\left(\nu^m\Lambda\psi\right)\;,\\ &= \beta\,(d-4)\left(\psi\Gamma^0\epsilon\right)\left(\epsilon\Lambda\psi\right) - \frac{1}{2}\beta\,(d-4)\,v^N\left(\psi\Gamma^0\tilde{\Gamma}_N\Lambda\psi\right),\end{aligned} \tag{C.34}$$

where in the second equality we have used that for rest of the pure spinors $\left(\psi\Gamma^0\nu^m\right) = 0$. The last equality follows by using completeness relation of pure spinors and Killing spinor. Next we use that:

$$v^N\left(\psi\Gamma^0\tilde{\Lambda}\Gamma_N\psi\right) - v^N\left(\psi\Gamma^0\tilde{\Gamma}_N\Lambda\psi\right) \;=\; -2v^{\tilde{N}}\left(\psi\Gamma^0\tilde{\Gamma}_{\tilde{N}}\Lambda\psi\right). \tag{C.35}$$

Using all this information in equation (C.31), we get the quadratic Lagrangian for vector multiplet fermions to be:

$$\begin{aligned}\mathcal{L}^{\mathrm{f}}_{\mathrm{v.m}} &= (\psi \not{\nabla} \psi) + \left(\psi \Gamma^0 D_0 \psi\right) + 4\beta\,(d-5)\left(\psi\Gamma^0\epsilon\right)(\epsilon\Lambda\psi) - \frac{1}{2}\beta\left(\epsilon\tilde{\Gamma}^{MN}\Lambda\epsilon\right)\left(\psi\Gamma^0\Gamma_{MN}\psi\right) \\ &\quad - \beta\,(d-4)\,v^{\tilde{N}}\left(\psi\Gamma^0\tilde{\Gamma}_{\tilde{N}}\Lambda\psi\right).\end{aligned} \tag{C.36}$$

We use a few relations to simplify the Lagrangian further. First we can use the Fierz identity quoted in eq. (C.21) and triality to bring the third term above in the desired form:

$$4\beta(d-5)\left(\psi\Gamma^0\epsilon\right)(\epsilon\Lambda\psi) = -\frac{1}{4}\beta(d-5)\left(\psi\Gamma^0\Gamma^{MN}\psi\right)(\epsilon\Lambda\Gamma_{MN}\epsilon) - \frac{1}{2}\beta(d-5)v_N\left(\psi\Gamma^0\tilde{\Lambda}\Gamma^N\psi\right). \tag{C.37}$$

Secondly, we rewrite the last term of this equation as

$$v_N\left(\psi\Gamma^0\tilde{\Lambda}\Gamma^N\psi\right) = (\psi\Lambda\psi) - v^{\tilde{M}}\left(\psi\Gamma^0\tilde{\Gamma}_{\tilde{M}}\Lambda\psi\right). \tag{C.38}$$

Further, we note that for the vector multiplet fermions, we have:

$$\begin{aligned}-\frac{1}{2}(d-4)\beta\left(\epsilon\tilde{\Gamma}^{AB}\Lambda\epsilon\right)\left(\psi\Gamma^0\Gamma_{AB}\psi\right) &= \beta(d-4)\,(\psi\Lambda\psi) \\ \left(\epsilon\tilde{\Gamma}^{MN}\Lambda\epsilon\right)\left(\psi\Gamma^0\Gamma_{MN}\psi\right) &= \left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\left(\psi\Gamma^0\Gamma_{\tilde{M}\tilde{N}}\psi\right) - (9-D)\,(\psi\Lambda\psi).\end{aligned} \tag{C.39}$$

Combining these results in the general Lagrangian eq. (C.36) we get finally get the following expression for Lagrangian of vector multiplet fermions:

$$\begin{aligned}\mathcal{L}^{\mathrm{f}}_{\mathrm{v.m}} &= (\psi \not{\nabla} \psi) - \frac{1}{2}(d-3)\beta v^{\tilde{M}}\left(\psi\Gamma^0\tilde{\Gamma}_{\tilde{M}}\Lambda\psi\right) + v^0\left(\psi\Gamma^0 D_0\psi\right) \\ &\quad - \frac{1}{4}(d-3)\beta\left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\left(\psi\Gamma^0\Gamma_{\tilde{M}\tilde{N}}\psi\right) + m_\psi\,(\psi\Lambda\psi).\end{aligned} \tag{C.40}$$

Here $m_\psi = \frac{d-1}{2}$ for eight supersymmetries and $m_\psi = (d-2)$ for four supersymmetries.

Hyper/chiral multiplet

We treat eight and four supersymmetries separately. For eight supersymmetries, we have a single fermion in the hypermultiplet. We denote it as $\chi = -\Gamma\chi$. For the hypermultiplet fermion $(\epsilon\Lambda\chi) = 0$. We have $\mathcal{C}_6 = \mathcal{C}_7 = -\mathcal{C}_8 = -\mathcal{C}_9 = -(d-4+2i\mu)$. Also using that $\epsilon\Gamma_M\chi = 0$ for $M = 0, \tilde{M}$, we see that the first term on second line of eq. (C.31) can be written as

$$- \beta \mathcal{C}_6 \left(\chi \Gamma^0 \tilde{\Lambda} \Gamma^N \epsilon\right) \left(\epsilon \Gamma_N \chi\right) \;=\; \beta \frac{\mathcal{C}_6}{2} v^N \left(\chi \Gamma^0 \tilde{\Lambda} \Gamma^N \chi\right). \tag{C.41}$$

Using this, we get the following expression for the hypermultiplet fermion's Lagrangian

$$\begin{aligned} \mathcal{L}^{\mathrm{f}} = &\left(\chi \nabla\!\!\!\!/\, \chi\right) + \left(\chi \Gamma^0 D_0 \chi\right) - \frac{1}{2}\beta \left(\epsilon \tilde{\Gamma}^{MN} \Lambda \epsilon\right) \left(\chi \Gamma^0 \Gamma_{MN} \chi\right) \\ &- i\mu\beta v^N \left(\chi \Gamma^0 \tilde{\Lambda} \Gamma_N \chi\right) - \left(\chi \Gamma^0 \nu_m\right) \Delta K^m. \end{aligned} \tag{C.42}$$

It is easy to verify that the contribution of third term in eq. (C.42) is

$$\left(\epsilon \tilde{\Gamma}^{MN} \Lambda \epsilon\right) \left(\chi \Gamma^0 \Gamma_{MN} \chi\right) \;=\; \left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right) \left(\chi \Gamma^0 \Gamma_{\tilde{M}\tilde{N}} \chi\right). \tag{C.43}$$

The last term in eq. (C.42) gets contributions from

$$\Delta K^m \;=\; -2i\mu \nu^m \Lambda \chi, \quad \text{for} \quad m \;=\; 4,5,6,7. \tag{C.44}$$

Using the completeness property for pure spinors and that $\epsilon \Lambda \chi \;=\; 0$, we get:

$$- \left(\chi \Gamma^0 \nu^m\right) \Delta K^m \;=\; i\mu\beta v^N \left(\chi \Gamma^0 \tilde{\Gamma}_N \Lambda \chi\right). \tag{C.45}$$

This and the second to last term in eq. (C.42) can be combined using the identity eq. (C.38). After all the simplifications, we obtain the following form for the Lagrangian of the hypermultiplet fermion with eight supersymmetries:

$$\mathcal{L}^{\mathrm{f}}_{\mathrm{h.m}} = \left(\chi \nabla\!\!\!\!/\, \chi\right) + \left(\chi \Gamma^0 D_0 \chi\right) - \frac{1}{2}\beta \left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right) \left(\chi \Gamma^0 \Gamma_{\tilde{M}\tilde{N}} \chi\right) + 2i\mu\beta v^{\tilde{N}} \left(\chi \Gamma^0 \tilde{\Gamma}_{\tilde{N}} \Lambda \chi\right). \tag{C.46}$$

The chiral multiplet fermionic part with four can be obtained by similar computation:

$$\begin{aligned} \mathcal{L}^{\mathrm{f}}_{\mathrm{h.m}} = &\sum_{\ell=1}^{3} \left(\chi_\ell \nabla\!\!\!\!/\, \chi_\ell\right) + \left(\chi_\ell \Gamma^0 [\phi_0^{\mathrm{cl}}, \chi_\ell]\right) - \frac{1}{2}\beta \left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right) \left(\chi_\ell \Gamma^0 \Gamma_{\tilde{M}\tilde{N}} \chi_\ell\right) \\ &+ \sigma_{(\ell)} \beta \left(2i\mu_\ell v^{\tilde{N}} \left(\chi_i \Gamma^0 \tilde{\Gamma}_{\tilde{N}} \Lambda \chi_\ell\right) + \chi_i \Lambda \chi_\ell\right). \end{aligned} \tag{C.47}$$

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$\mathcal{D}$

Degeneracy of harmonics on S^d

The spectrum of the Laplacian and the degeneracy of symmetric traceless tensors on S^d is given in [69]. We summarize the results for scalar and divergence-less vectors here for completeness. Scalar harmonics are labelled by the eigenvalues of the Laplacian on the sphere, with the eigenvalues and the degeneracy given by:

$$\nabla^2 Y_m^k = -4\beta^2 k\,(k+d-1)\,Y_m^k, \qquad \mathcal{D}_k\,(d,0) = \frac{(2k+d-1)\,\Gamma\,(k+d-1)}{\Gamma\,(d)\,\Gamma\,(k+1)}. \tag{D.1}$$

Divergence-less vector harmonics are also labelled by eigenvalues of the Laplacian on the sphere which are different than scalars. Their degeneracy is given by:

$$\begin{aligned} \nabla^2 A_\mu^k &= -4\beta^2\,(k\,(k+d-1)-1)\,A_\mu^k, \qquad \nabla\cdot A^k = 0, \\ \mathcal{D}_k\,(d,1) &= \frac{k\,(k+d-1)\,(2k+d-1)\,\Gamma\,(k+d-2)}{\Gamma\,(d-1)\,\Gamma\,(k+2)}. \end{aligned} \tag{D.2}$$

Spinor harmonics on S^d are labelled by the eigenvalues of the Dirac operator. We summarize results of [70] here:

$$\nabla\!\!\!/\,\psi_\pm^k = \pm i\left(k+\frac{d}{2}\right)\psi_\pm^k, \qquad \mathcal{D}_k\,(d,+) = \mathcal{D}_k\,(d,-) = \frac{2^{\lfloor \frac{d}{2}\rfloor}\Gamma\,(k+d)}{\Gamma\,(d)\,\Gamma\,(k+1)}. \tag{D.3}$$

An important degeneracy factor that appears in the computation of the one-loop determinant is the number of spherical harmonics $Y_{\pm k}^k$. Since the spin is labelled by the Cartan generator along the direction of the vector field $v^{\tilde{M}}$, the degeneracy is different for the case of eight and four supersymmetries. We derive this degeneracy for the case of eight supersymmetries now.

Consider an S^d parameterized as follows:

$$|z_i|^2 + x_j^2 = 1, \tag{D.4}$$

where $z_i \in \mathbb{C}$, $x_j \in \mathbb{R}$ and the indices i, j range in $i = 1, \ldots, \frac{d-k'+1}{2}$ and $j = 1, \ldots k'$. We consider the vector field $v^{\tilde{M}}$, which acts on the sphere coordinates as

$$z_i \to z_i e^{i\phi}, \qquad x_j \to x_j. \tag{D.5}$$

The fixed point locus of this vector field is given by the equation $z_i = 0, i = 1, \ldots k$, which, when substituted in equation (D.4), leaves a $(k'-1)$-sphere fixed. For example, in the case of eight supersymmetries we have

- S^5: $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ has a fixed S^{-1},
- S^4: $|z_1|^2 + |z_2|^2 + x_1^2 = 1$ has a fixed S^0 (two points on the poles),
- S^3: $|z_1|^2 + x_1^2 + x_2^2 = 1$ has a fixed S^1,
- S^2: $x_1^2 + x_2^2 + x_3^2 = 1$ has a fixed S^2.

So in the case of eight supersymmetries the action of the vector field leaves an S^{4-d} fixed. In this parametrization, the scalar spherical harmonics Y_m^k can be written as polynomials in the variables z_i, $\bar{z}_i$ and x_j. To construct a spherical harmonic of level k and "charge" m, we assign charge $+1$ to z_i, -1 to $\bar{z}_i$ and 0 to x_j. Thus, the top spherical harmonics can be written as:

$$Y_k^k \sim z_{i_1} z_{i_2} \ldots z_{i_k}, \tag{D.6}$$

with the degeneracy given by:

$$N_{k,d} = \binom{k + \frac{d-k'+1}{2} - 1}{k} = \frac{\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-2)}. \tag{D.7}$$

In the case of four supersymmetries, $v^{\tilde{M}}$ leaves an S^{2-d} fixed, so the degeneracy of the top level harmonics is:

$$n_{k,d} = \frac{\Gamma(k+d-1)}{\Gamma(k+1)\Gamma(d-1)}. \tag{D.8}$$

D.1 Vanishing of top spinor modes

Certain elements of the basis for spinor harmonics vanish identically for $m = \pm k$. Here we will demonstrate explicitly that for $m = k$

$$\mathcal{X}_+^2 = \Gamma^{\tilde{M}} \hat{\nabla}_{\tilde{M}} Y_{+k}^k \eta_+ = \Gamma^\mu \nabla_\mu Y_{+k}^k \eta_+ + 2i\beta k Y_{+k}^k \eta_+ = 0. \tag{D.9}$$

We will take the top and the bottom modes of the scalar spherical harmonics to be given by:

$$Y_k^k = z^k \qquad \text{and} \qquad Y_{-k}^k = \bar{z}^k, \tag{D.10}$$

where

$$z = 2\beta \frac{x^1 + ix^2}{1 + x^2\beta^2}. \tag{D.11}$$

We will also use the relation between the gamma matrices with the flat and curved indices, given by:

$$\Gamma^\mu = (1 + x^2\beta^2)\Gamma^{\hat{\mu}}. \tag{D.12}$$

Now, the first term in equation (D.9) becomes:

$$\Gamma^\mu \nabla_\mu Y_k^k \eta_+ = \left[2\beta k(\Gamma^1 + i\Gamma^2) z^{k-1} - 2\beta^2 k x \cdot \Gamma z^k\right] \eta_+, \tag{D.13}$$

whose first term can be expanded to:

$$(\Gamma^1 + i\Gamma^2)\eta_+ = \Gamma_0 \left(\tilde{\Gamma}_6 + i\tilde{\Gamma}_7\right) \left[(\Gamma_1 + i\Gamma_2) + \beta(\Gamma_1 + i\Gamma_2) x \cdot \tilde{\Gamma}\Lambda\right] \frac{\epsilon_s}{\sqrt{1 + \beta^2 x^2}}. \tag{D.14}$$

Note however that:

$$\begin{aligned}(\tilde{\Gamma}_6 + i\tilde{\Gamma}_7)(\Gamma_1 + i\Gamma_2)\epsilon_s &= \left[\tilde{\Gamma}_6\Gamma_1 - \tilde{\Gamma}_7\Gamma_2 + i(\tilde{\Gamma}_6\Gamma_2 + \tilde{\Gamma}_7\Gamma_1)\right] \epsilon_s \\ &= \left(\Gamma_{61} - \Gamma_{72}\Gamma^{1267} + i(\Gamma_{62} + \Gamma_{71}\Gamma^{1267})\right] \epsilon_s \\ &= 0,\end{aligned} \tag{D.15}$$

where we have used that $\Gamma^{1267}\epsilon_s = +\epsilon_s$. This result implies that:

$$(\tilde{\Gamma}_6 + i\tilde{\Gamma}_7)(\Gamma_1 + i\Gamma_2)\tilde{\Gamma}_M \Lambda \epsilon_s = 0, \quad \text{for } M \neq 1, 2. \tag{D.16}$$

Thus:

$$\begin{aligned}(\Gamma^1 + i\Gamma^2)\eta_+ &= \Gamma_0 \left(\tilde{\Gamma}_6 + i\tilde{\Gamma}_7\right) \beta \left[(x^1 + ix^2)\Lambda\epsilon_s + i(x^1 + ix^2)\Gamma^0\epsilon_s\right] \frac{1}{\sqrt{1 + \beta^2 x^2}} \\ &= -2i\beta(x^1 + ix^2)(\Gamma_6 + i\Gamma_7)\frac{\epsilon_s}{\sqrt{1 + \beta^2 x^2}}.\end{aligned} \tag{D.17}$$

Let's proceed to second term of equation (D.13):

$$x \cdot \Gamma\eta_{+} = x \cdot \Gamma\,\tilde{\Gamma}_0(\Gamma_6 + i\Gamma_7)(1 + \beta x \cdot \tilde{\Gamma}\Lambda)\frac{\epsilon_s}{\sqrt{1+\beta^2x^2}}$$
$$= -\frac{1}{\beta}\Gamma_0(\tilde{\Gamma}_6 + i\tilde{\Gamma}_7)\Lambda(\beta x \cdot \tilde{\Gamma}\Lambda - \beta^2x^2)\frac{\epsilon_s}{\sqrt{1+\beta^2x^2}} \tag{D.18}$$
$$= +\frac{1}{\beta}\,i(\Gamma_6 + i\Gamma_7)(\beta x \cdot \tilde{\Gamma}\Lambda - \beta^2x^2)\frac{\epsilon_s}{\sqrt{1+\beta^2x^2}}.$$

Combining our results for the two terms of (D.13), we get:

$$k\left[-2i\beta(1+x^2\beta^2)(\Gamma_6 + i\Gamma_7) - 2i\beta(\Gamma_6 + i\Gamma_7)(\beta x \cdot \tilde{\Gamma}\Lambda - \beta^2x^2)\right]\frac{\epsilon_s}{\sqrt{1+\beta^2x^2}} = -2i\beta k(\Gamma_6 + i\Gamma_7)\epsilon \tag{D.19}$$

which finally implies that

$$\Gamma^{\tilde{M}}\hat{\nabla}_{\tilde{M}}Y^k_k\eta_{+} = 0. \tag{D.20}$$

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