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INVARIANCE TO REPRESENTATION OF INFORMATION

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ABSTRACT. Under weak assumptions on the solution concept, I construct an invariant selection across all finite type spaces, in which the types with identical information play the same action. Along the way, I establish an interesting lattice structure for finite type spaces and construct an equilibrium on the space of all finite types.

Keywords: invariant selection, invariance, equilibrium, universal type space

JEL Numbers: C72, C73.

1. INTRODUCTION

In Game Theory, incomplete information is modeled using a type in a Bayesian game a la Harsanyi (1967). Unfortunately, the representation is not unique: a given piece of information can be modeled using many distinct types, coming from distinct Bayesian games. It is desirable that the solution be invariant to alternative representations of the same information, in the sense that the types that represent the same information all take the same action according to the solutions to the games they come from. In this note, I explore the implications of such an invariance condition.

To be more precise, consider a researcher. Given any type t_i of any player i in any Bayesian game G , she thinks that the relevant information of t_i is $h_i(t_i, G)$. If there is another type t'_i from a game G' with $h_i(t_i, G) = h_i(t'_i, G')$, then she considers (t_i, G) and (t'_i, G') as two alternative representations of the same relevant information. Hence, she requires that types t_i and t'_i play the same action according to the solutions to games G and G' , respectively. If she selects a solution to each game satisfying her requirement, then she obtains an *invariant selection* from her solution concept. Such a selection is needed to study the solutions that are invariant to alternative representations of the same relevant information. Can she find such a selection? Can she ensure that her solutions to the games of her interest are part

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of an invariant selection without analyzing other hypothetical situations? I answer these questions affirmatively.

In the baseline model, I take h as the infinite hierarchy of beliefs about some parameters that are deemed relevant. I note that the same piece of information can be modeled by an uncountably many types, coming from Bayesian games with complicated interconnections that are difficult to foresee. Hence, invariance is a strong restriction on the selections of the solutions, requiring that the actions of all these types to be equal. Moreover, given the complicated interconnections between these games, constructing an invariant selection is a difficult task, which involves equalizing the actions of all these types and doing this for all games at the same time.

I consider all solution concepts with the following two basic properties. Given any game G , construct a new game G^h by representing each type t_i in G by its relevant information $h_i(t_i, G)$. The first property is that for every solution σ of G^h , the strategy profile $\sigma \circ h$, in which each t_i plays $\sigma_i(h_i(t_i, G))$, is a solution to G . The second property is that given any finite type space T and any invariant selection for its proper subspaces, there exists a solution on T that yields the selection when we restrict the solution to the proper subspaces of T . Both properties are exhibited by canonical solution concepts, such as Bayesian Nash equilibrium and rationalizability.¹

Given the generality above, invariance may put non-obvious restrictions on the selection of solutions across games that may appear to be unrelated. Nonetheless, I show that when finite type spaces are embedded in a universal type space using the mapping h , they exhibit an interesting and useful lattice structure. Using this structure, I construct an invariant selection in a straightforward manner. I further show that any invariant selection within a class of games can be extended to the set of all games with finite type spaces, maintaining the invariance condition throughout. Conceptually, this shows that the invariance requirement does not impose any additional restrictions on the solutions to individual games or on the selections for subfamilies. On a practical level, it ensures that if a researcher is only interested in behavior in a class of games, she can focus on constructing an invariant selection for that class without worrying about the invariance across all games. In particular, if she is interested only in a specific game, she can analyze the game in isolation without analyzing the other possible strategic situations. All she needs to do is to make sure that the types with identical relevant information play the same action in the specific game. In contrast, if she is interested in behavior across a family of games (as in the analysis of comparative

¹By rationalizability, I mean interim correlated rationalizability and interim independent rationalizability.

statics), then she needs to analyze invariant selections for the family, and invariance imposes many more conditions on such selections than on the solutions to individual games.

There is a one-to-one correspondence between the invariant equilibrium selections on a given class of games and the equilibria on the subspace of the universal type space in which these games are embedded. The above results on invariance selections then lead to equilibrium existence results on the universal type space. First, since there is an invariant equilibrium selection for all games with finite type spaces, there exists an equilibrium on the space of all finite types, which consists of the images of all types from all finite type spaces. This fills an important gap in the literature, in which very little is known on the existence of equilibria in prominent subspaces of the universal type space. Second, any equilibrium within a subspace can be extended to the space of all finite types. This result is quite useful in equilibrium analysis on such spaces. In such an analysis, one is often interested in the behavior of types within a small class. This result ensures that one can simply focus on the class without worrying about the construction of equilibrium on the entire space, which is often the main difficulty. Third, as a special case of the second result, an equilibrium of a game can be extended to the space of all finite types as long as the types with identical information play the same action in the equilibrium. This result is important for robustness analysis. In such an analysis, one considers an equilibrium in the universal type space and explores its sensitivity to information (see for example Weinstein and Yildiz, 2007, 2011). This result shows that such an analysis is not vacuous and the robustness of any equilibrium as above can be analyzed within this methodology. Börgers and Smith (2014) provide an important application of invariance in robust mechanism design.

I study invariance of solutions with respect to the alternative representations of incomplete information. More broadly, one wants the solution to be invariant to any alternative representation of the strategic situation. Within this larger context, a number of authors, such as Kohlberg and Mertens (1986) and Govindan and Wilson (2006, 2009, 2012), have studied other invariance conditions, such as invariance with respect to the introduction of mixed strategies as pure strategies and the “small worlds” condition, which requires that embedding a game into a larger game with additional players does not affect the solutions induced on the original game. The invariance condition here differs from the above conditions in two ways. First, it focuses on incomplete information. Second, it is a condition on how the solution changes across games rather than being a condition on the solution sets. Indeed, it does not at all restrict the set of solutions to any given game with no “redundant” types (i.e. multiple types with identical relevant information).

A solution concept and the notion of relevant information are inherently related. If a researcher strongly believes in a solution concept, she can use the solution concept to determine which information is strategically relevant. This is especially true when she does not have strong knowledge of players' concerns. For example, using equilibrium or rationalizability, one can discover that whether a seller has private information can be quite relevant in trade and that higher-order beliefs do matter in strategic situations. Unfortunately, solution concepts are often introduced as working hypotheses, using analogy and extrapolation from toy examples or existing theories.² The researcher may then have a better understanding of players' concerns and the relevance of a piece of information. In that case, she may use her knowledge of the situation to modify or refine the solution concept. In general, one expects a researcher to strike a balance between the solution concept and her knowledge of the situation. No matter where she strikes the balance, invariance helps the researcher in refining the set of selections from the solution concepts she considers and provides a better understanding of how players' behavior changes from one situation to another. To the extent that she uses her own knowledge of the situation, invariance helps further in refining her solutions to individual games.

It is known that some prominent solution concepts, such as Bayesian Nash equilibrium, may depend on the specific type space through which a belief hierarchy is modeled. For example, when there are redundant types, there may be additional equilibria in which distinct types with identical hierarchies play distinct actions. Many researchers proposed alternative ways to address this issue. For example, Ely and Peski (2006) proposed a broader class of belief hierarchies under which interim independent rationalizability is not affected by the presence of redundant types. Dekel, Fudenberg, and Morris (2007) introduced a weaker solution concept (namely, interim correlated rationalizability) that is immune to the above problem. In a way, the former authors take the solution concept as given and explore the information relevant for the solution concept, while the latter authors expand the set of solutions in order to account for some relevant information that may have been overlooked (Liu, 2009). Finally, noting that some equilibria may fail to extend to a larger type space, Friedenberg and Meier (2007) proposed that we should embrace the dependence of solutions on the type space: we should use the type space in order to model the context in which

²In contrast, rationalizability in complete information games and interim correlated rationalizability in Bayesian games are introduced as characterizations of a set of rationality assumptions. Recently, Battigalli et al (2011) argued that one must use a solution concept that can be expressible as such assumptions—about players' beliefs, information and actions. Once the researcher settles on such a solution concept, she will be in the previous case above.

the game is played.³ Such a dependence of the solution set on the type space is not a main concern in my paper, and my invariance condition is consistent with all of the above approaches, which offer alternative solution concepts and notions of relevant information. Of course, once a researcher settles on a particular notion of relevant information and a solution concept, invariance refines the solution concept and more importantly puts strong restrictions on how the solution varies from one game to another.

In the next section, I introduce invariance formally and explain its implications and my construction on a simple example. In Section 3, I demonstrate that finite type spaces exhibit a useful lattice structure when they are represented as subspaces of a universal type space. I study invariant selections in Section 4, invariant equilibrium selections in Section 5, and the equilibria on the universal type space in Section 6. I present two extensions in Section 7, one to countable type spaces and one to broader notions of relevant information. Section 8 concludes. The omitted proofs are in the Appendix.

2. INVARIANCE—DEFINITION AND EXAMPLES

Fix a set $N = \{1, 2, \dots, n\}$ of players i , a set $A = A_1 \times \dots \times A_n$ of action profiles a ,⁴ and a set Θ^* of payoff parameters θ . For each $i \in N$, fix also a utility function $u_i : \Theta^* \times A \rightarrow \mathbb{R}$. A finite type space is a triplet (Θ, T, κ) where $\Theta \subseteq \Theta^*$ is a finite set of parameters, $T = T_1 \times \dots \times T_n$ is a finite set of type profiles t , and κ_{t_i} is a probability distribution on $\Theta \times T_{-i}$, representing the belief of t_i , for each type $t_i \in T_i$. A Bayesian game is a list $G = (N, A, u, \Theta, T, \kappa)$. The set of all Bayesian games with varying finite type spaces is denoted by \mathcal{G} . Throughout \mathcal{G} , (N, A, u) is fixed for clarity.

For any game $G = (N, A, u, \Theta, T, \kappa) \in \mathcal{G}$ and any player i , a *strategy of i for G* is a mapping $\sigma_i : T_i \rightarrow \Delta(A_i)$, and a *strategy profile for G* is a list $(\sigma_1, \dots, \sigma_n)$ of strategies. A *solution concept* is any correspondence $\Sigma : \mathcal{G} \rightrightarrows \bigsqcup_T \prod_i \Delta(A_i)^{T_i}$ that picks a set $\Sigma(G)$ of strategy profiles for G at each game $G \in \mathcal{G}$, where $\bigsqcup_T \prod_i \Delta(A_i)^{T_i}$ is the set of all strategy profiles available within \mathcal{G} . Given any solution concept Σ and any $\mathcal{G}' \subseteq \mathcal{G}$, I write $\Sigma|_{\mathcal{G}'}$ for the restriction of Σ to \mathcal{G}' . By a *selection from Σ for \mathcal{G}'* , I mean a selection from $\Sigma|_{\mathcal{G}'}$, i.e., a

³Friedenberg and Meier also study the relation between the equilibrium sets of two type spaces when one is embedded in the other, showing that Bayesian Nash equilibrium satisfies the basic properties I assume for the solution concepts here.

⁴**Notation:** For any list X_1, \dots, X_n of sets, X denotes $X_1 \times \dots \times X_n$ and X_{-i} denotes $\prod_{j \neq i} X_j$. For any x_1, \dots, x_n , write $x = (x_1, \dots, x_n) \in X$, $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$, and $(x'_i, x_{-i}) = (x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)$. For any family of functions $f_j : X_j \rightarrow Y_j$, write $f(x) = (f_j(x_j))_{j \in N}$ and $f_{-i}(x_{-i}) = (f_j(x_j))_{j \neq i}$. In general, drop a subscript to denote the entire mapping that varies with that subscript; e.g., write κ_{t_i} for the belief of type t_i and κ for the belief function. The set of all probability distributions on a set X is denoted by $\Delta(X)$.

function that picks a solution $\sigma^G \in \Sigma(G)$ for each $G \in \mathcal{G}$:

$$\mathcal{G}' \ni G \mapsto \sigma^G \in \Sigma(G).$$

A selection for \mathcal{G} is simply called a *selection*.

2.1. Invariance. For every game $G \in \mathcal{G}$ and every type t_i of a player i in G , let $h_i(t_i, G)$ be the relevant information of t_i (according to a researcher). Given that all the relevant information is contained in $h_i(t_i, G)$, if $h_i(t_i, G) = h_i(t'_i, G')$ for some types t_i and t'_i from games G and G' , respectively, then (t_i, G) and (t'_i, G') are just alternative representations of the same information. One may then desire the solution to be independent of the representation, requiring that t_i and t'_i play the same action according to the solutions at G and G' , respectively. The next definition formalizes this idea.

Definition 1. Fix a solution concept Σ , a map h from domain \mathcal{G} to some abstract codomain and a set $\mathcal{G}' \subseteq \mathcal{G}$ of games. A selection $G \mapsto \sigma^G$ for \mathcal{G}' is said to be *h -invariant* if

$$(C) \quad h_i(t_i, G) = h_i(t'_i, G') \implies \sigma_i^G(t_i) = \sigma_i^{G'}(t'_i)$$

for all games $G, G' \in \mathcal{G}'$ and for all types t_i and t'_i in G and G' , respectively. Likewise, for any $G \in \mathcal{G}$, a strategy profile σ in G is said to be *h -invariant* if for all types t_i and t'_i in G ,

$$(C') \quad h_i(t_i, G) = h_i(t'_i, G) \implies \sigma_i(t_i) = \sigma_i(t'_i).$$

Invariance of selections is a condition on how the solution varies across games, rather than restricting the set of solutions at a given game. On the other hand, invariance of a strategy profile is a condition on the solution of a game, requiring that the types with identical relevant information play the same action. Of course, the latter is a special case of invariance of a selection for $\mathcal{G}' = \{G\}$.

For concreteness, in the baseline model, I consider the following notion of relevance (see Section 7 for more general notions). I take

$$\Theta^* = \Theta_R^* \times \Theta_{NR}^*,$$

so that the underlying parameters are of the form

$$\theta = (\theta_R, \theta_{NR}).$$

Here, θ_R is a vector of parameters that are deemed relevant while θ_{NR} is a vector of parameters that are deemed irrelevant. The relevant information of a type t_i in a game G is the hierarchy of beliefs about θ_R :

$$h_i(t_i, G) = (h_i^1(t_i, G), h_i^2(t_i, G), \dots)$$

T^0				T^1				T^2						
θ_0	t_2^0	θ_1	t_2^0	θ_0	t_2^1	θ_1	t_2^1	θ_0	t_2^1	θ_1	t_2^1			
t_1^0	1	t_1^0	0	t_1^1	0	t_1^1	1	t_1^1	1/2	t_1^1	1/2			
T^3						T^6								
θ_0	$t_2^{3,1}$	$t_2^{3,2}$		θ_1	$t_2^{3,1}$	$t_2^{3,2}$		θ_0	$t_2^{6,1}$	$t_2^{6,2}$		θ_1	$t_2^{6,1}$	$t_2^{6,2}$
$t_1^{3,1}$	1/2	0		$t_1^{3,1}$	0	0		$t_1^{6,1}$	1/6	1/12		$t_1^{6,1}$	1/12	1/6
$t_1^{3,2}$	0	0		$t_1^{3,2}$	0	1/2		$t_1^{6,2}$	1/12	1/6		$t_1^{6,2}$	1/6	1/12
T^4						T^5								
θ_0	$t_2^{4,1}$	$t_2^{4,2}$		θ_1	$t_2^{4,1}$	$t_2^{4,2}$		θ_0	$t_2^{5,1}$	$t_2^{5,2}$		θ_1	$t_2^{5,1}$	$t_2^{5,2}$
$t_1^{4,1}$	1, 1	0, 0		$t_1^{4,1}$	0, 0	0, 0		$t_1^{5,1}$	1, 1	0, 0		$t_1^{5,1}$	0, 0	0, 0
$t_1^{4,2}$	0, 0	1/2, 1/2		$t_1^{4,2}$	0, 0	1/2, 1/2		$t_1^{5,2}$	0, 0	0, 0		$t_1^{5,2}$	0, 0	1, 1
$t_1^{4,3}$	1/2, 0	0, 0		$t_1^{4,3}$	0, 0	1/2, 0		$t_1^{5,3}$	$\varepsilon, 0$	$\varepsilon^2, 0$		$t_1^{5,3}$	0, 0	$1 - \varepsilon - \varepsilon^2, 0$

TABLE 1. The type spaces T^0, \dots, T^6 . In T^0, T^1, T^2, T^3 and T^6 , the number in a cell (θ, t) indicates the ex-ante probability of (θ, t) . In T^4 and T^5 , the first and the second entries in a cell (θ, t) are the interim beliefs of types t_1 and t_2 , respectively; e.g., $t_1^{4,1}$ assigns probability 1 on $(\theta_0, t_2^{4,1})$.

where $h_i^1(t_i, G)$ is the first-order belief of type t_i about θ_R , $h_i^2(t_i, G)$ is the second-order belief of type t_i about $(\theta_R, h_{-i}^1(t_{-i}, G))$, and so on.

The set of all such belief hierarchies forms an infinite type space, called Θ_R^* -based universal type space (Mertens and Zamir, 1985, and Brandenburger and Dekel, 1993). The finite type spaces are mapped to a subspace $(\Theta^*, T^u, \kappa^u)$ of this universal type space. The uncountable set $T^u = T_1^u \times \dots \times T_n^u$ is called *the space of finite types*. Note that

$$(2.1) \quad T_i^u = \{h_i(t_i, G) \mid t_i \in T_i \text{ for some } G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}\}$$

for each $i \in N$. Note also that the beliefs are preserved by the embedding:

$$(2.2) \quad \kappa_{h_i^u(t_i, G)}^u(\theta, h_{-i}(t_{-i}, G)) = \sum_{t'_{-i} \in T_{-i} \cap h_{-i}^{-1}(h_{-i}(t_{-i}, G), G)} \kappa_{t_i}(\theta, t'_{-i})$$

for all i , all $G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}$, and all $(\theta, t_i, t_{-i}) \in \Theta \times T$.

An important special case arises when $|\Theta_{NR}^*| = 1$ so that all the parameters are deemed relevant. Even in this case, type spaces may contain some distributed information that is not contained in the hierarchies, and some Bayesian Nash equilibria may use such information (see Section 2.3 below). According to the notion considered in the baseline model, such information is deemed irrelevant, and the equilibria that use such information are discarded.

2.2. Examples. In the rest of this section, I illustrate the implications of invariance on a simple example, using the Bayesian Nash equilibrium as the solution concept. I fix $N = \{1, 2\}$, $A = \{a, b\} \times \{c, d\}$, $\Theta^* = \Theta_R^* = \{\theta_0, \theta_1\}$, and fix the utility functions as:

θ_0	c	d	θ_1	c	d
a	5, 1	0, 0	a	0, 1	5, 0
b	3, 0	3, 1	b	3, 0	3, 1

I analyze the games G_0, \dots, G_6 with type spaces T^0, \dots, T^6 , respectively, depicted in Table 1. The games G_0 and G_1 are complete information games in which it is common knowledge that $\theta = \theta_0$ and $\theta = \theta_1$, respectively. Their type spaces T^0 and T^1 are embedded in T^3 , rather trivially. In T^3 , the true state is common knowledge at the interim stage, but they are equally likely ex ante. The hierarchies of $t^{3,1}$ and $t^{3,2}$ are identical to those of t^0 and t^1 , respectively. On the other hand, G_2 is also a complete information game, in which the players find the states equally likely. Although the type space T^6 looks quite different, it also expresses the information that it is common knowledge that the players find the states equally likely. One can see this by observing that each type in T^6 assigns probability 1/2 on each state. Hence, t^2 , $t^{6,1}$ and $t^{6,2}$ all have the same relevant information.

The relationship between type spaces can be even more complex when there is no common prior. To see this, consider T^4 and T^5 . In T^4 , the relevant information of $t^{4,1}$ is identical to that of t^0 in T^0 : it is common knowledge that $\theta = \theta_0$. Likewise, the relevant information of $t^{4,2}$ is identical to that of t^2 in T^2 : it is common knowledge that the states are equally likely. Finally, according to type $t_1^{4,3}$, either $\theta = \theta_0$ and player 2 thinks that this fact is common knowledge, or $\theta = \theta_1$ and player 2 thinks that it is common knowledge that the states are equally likely. In T^5 , type $t_1^{5,3}$ is almost certain that $\theta = \theta_1$ and that player 2 believes that this is common knowledge, but the details of the beliefs refer to the hierarchies of t^0 and t^1 .

The Bayesian Nash equilibria of games in set $\mathcal{G}_1 = \{G_0, G_1, G_2, G_3, G_4, G_5\}$ are as follows. Game G_0 has three equilibria: $s^{0,1} = (a, c)$, $s^{0,2} = (b, d)$, and $s^{0,3}$, which is in mixed-strategies; G_1 has a unique (mixed) equilibrium σ^1 , and G_2 has a unique equilibrium $s^2 = (b, d)$. Accordingly each game G_m with $m \in \{3, 4, 5\}$ has three equilibria $s^{m,1}$, $s^{m,2}$, and $s^{m,3}$, where

$$(2.3) \quad s^{m,k}(t^{m,1}) = s^{0,k},$$

$$s^{3,k}(t^{3,2}) = s^{5,k}(t^{5,2}) = \sigma^1 \text{ and } s^{4,k}(t^{4,2}) = (b, d),$$

$$(2.4) \quad s_1^{4,1}(t_1^{4,3}) = s_1^{5,1}(t_1^{5,3}) = s_1^{4,3}(t_1^{4,3}) = a \text{ and } s_1^{4,2}(t_1^{4,3}) = s_1^{5,2}(t_1^{5,3}) = s_1^{5,3}(t_1^{5,3}) = b$$

for each $k \in \{1, 2, 3\}$. There are 81 possible selections for \mathcal{G}_1 , as one can mix and match the equilibria of the individual games arbitrarily.

Invariance puts strong restrictions on selections, however. Since t^0 , $t^{3,1}$, $t^{4,1}$, and $t^{5,1}$ all have identical relevant information, invariance requires that one must select the same outcome in games G_0 , G_3 , G_4 , and G_5 for these type profiles. Hence, by (2.3), if one selects $s^{0,k}$ in game G_0 , then he must also select the equilibria $s^{3,k}$, $s^{4,k}$, and $s^{5,k}$ in games G_3 , G_4 , and G_5 , respectively. Therefore, only 3 selections are invariant:

	G_0	G_1	G_2	G_3	G_4	G_5
Selection 1	$s^{0,1}$	σ^1	s^2	$s^{3,1}$	$s^{4,1}$	$s^{5,1}$
Selection 2	$s^{0,2}$	σ^1	s^2	$s^{3,2}$	$s^{4,2}$	$s^{5,2}$
Selection 3	$s^{0,3}$	σ^1	s^2	$s^{3,3}$	$s^{4,3}$	$s^{5,3}$

Some of the restrictions imposed by invariance may be unforeseen. For example, by (2.4), under an invariant selection, $t_1^{4,3}$ must play a whenever type $t_1^{5,3}$ does so, despite the apparent lack of connection between the hierarchies. Despite this, invariance does not restrict the solution set of a given game: every equilibrium of every game is selected by some invariant selection. The latter observation turns out to be true in general, as long as the games do not have redundant types, i.e., multiple types with identical belief hierarchies.

When there are redundant types, there may be non-invariant Bayesian Nash equilibria, expanding the equilibrium set. This well-known fact is illustrated by games G_2 and G_6 . Recall that according to all types in games G_2 and G_6 , it is common knowledge that the states are equally likely, all having the same relevant information. Therefore, G_6 has an equilibrium $s^{6,1}$ in which all types play according to the unique equilibrium of G_2 , yielding (b, c) as the outcome. Nevertheless, G_6 has also another equilibrium $s^{6,2}$:

$$s_1^{6,2}(t_1^{6,1}) = a; s_1^{6,2}(t_1^{6,2}) = b; s_2^{6,2}(t_2^{6,1}) = c; s_2^{6,2}(t_2^{6,2}) = d.$$

Note that $s^{6,2}$ prescribes different actions for type $t_i^{6,1}$ and $t_i^{6,2}$ although they have identical relevant information. Therefore, equilibrium $s^{6,2}$ is not invariant.

Although redundant types are not my main concern here, conceptually my approach to redundant types is closest to that of Liu (2009). He shows that any type space with redundant types can be represented as a type space without redundant types by introducing additional underlying parameters (i.e., by adding new dimensions to θ and obtaining a Θ_R^* larger than original Θ^*). Moreover, the set of Bayesian Nash equilibria are identical in the two type spaces. He then interprets redundant types as a consequence of the researcher's ignoring some of the parameters that are deemed relevant by the players.

2.3. Construction. To illustrate my construction, I include a new game G_7 with type space T^7 :

θ_0	$t_2^{7,1}$	$t_2^{7,2}$	θ_1	$t_2^{7,1}$	$t_2^{7,2}$
$t_1^{7,1}$	1/2	0	$t_1^{7,1}$	0	0
$t_1^{7,2}$	0	1/4	$t_1^{7,2}$	0	1/4

The set of equilibria for G_7 is $\{s^{7,1}, s^{7,2}, s^{7,3}\}$ where $s^{7,k}(t^{7,1}) = s^{0,k}$ and $s^{7,k}(t^{7,2}) = s^2$ for each k .

In my construction, I first embed the Bayesian games into the Θ_R^* -based universal type space, using h . In the ongoing example, this leads to a new set of games G_0^h, \dots, G_7^h , with type spaces $h(T^0, G_0), \dots, h(T^7, G_7)$, respectively. After constructing an invariant selection for G_0^h, \dots, G_7^h , I select equilibria of games G_0, \dots, G_7 by transforming the selected equilibrium s^m in each game G_m^h to an equilibrium of G_m by $s^m \circ h^{-1}$.

The embedding for G_6 maps many types to one, yielding $G_6^h = G_2^h$, and changing the solution set. For all the other games, the embedding mappings are isomorphisms, which rename the types without changing the solution in each game. Note however, that some types from distinct type spaces are mapped to the same hierarchy, yielding non-empty intersections of type spaces $h(T^k, G_k)$ and $h(T^l, G_l)$ for some k and l . These non-empty intersections make the construction non-trivial.

The following structure makes a straightforward construction possible: with the inclusion of the empty set, the set of all resulting type spaces from the embedding is a lattice under the set inclusion, and it is closed under an arbitrary number of intersections. For example, $h(T^4, G_4)$ and $h(T^5, G_5)$ have a non-empty intersection due to the identical hierarchies for $t^{4,1}$ and $t^{5,1}$, but

$$h(T^4, G_4) \cap h(T^5, G_5) = h(T^0, G_0)$$

is also a type space for another finite game, namely G_0 . The lattice structure is depicted by the graph in Figure 1, where a link between two games indicates that the type space of the game above contains the type space of the game below.

This structure allows one to rank finite type spaces according to the length of the longest chain of its subspaces under the strict set inclusion. For example, the games G_0^h, G_1^h , and G_2^h do not contain any proper subspace, and all have rank 1. The games G_3^h and G_7^h contain proper subspaces, as depicted in Figure 1, but they contain only games of rank 1. These games then have rank 2. Finally, games G_4^h and G_5^h have rank 3, containing proper subspaces that have at most rank 2. The structure also implies that, for each type, there is a unique minimal type space that contains the type. For example, the minimal type space

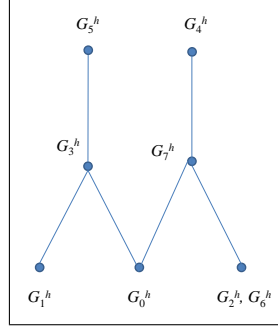


FIGURE 1. The lattice structure of the type spaces in the example.

for the hierarchy $h_1(t_1^{4,1}, G_4)$ is $h(T^0, G_0)$, while the minimal type space for $h_1(t_1^{4,3}, G_4)$ is $h(T^4, G_4)$.

The above structure leads to the following straightforward construction of an invariant selection from the embedded games, which leads to a selection across all games. I first consider the type spaces of the first rank. These type spaces do not have any subspace. In particular, they are disjoint because the intersection would be a subspace by the lattice structure. I select a solution for each of these type spaces. Since they are disjoint, the selection is invariant. For example, games G_0^h , G_1^h and G_2^h are of rank 1, and I first select an arbitrary equilibrium for each of these games, e.g., selecting $s^{0,1}$ for G_0^h , σ^1 for G_1^h and s^2 for G_2^h . Next, I consider the type spaces of the second rank, e.g., the games G_3^h and G_7^h in the ongoing example. These type spaces contain only subspaces of the first rank, for which the solution has been selected already. Each of them has a solution that extends the existing selection for the proper subspaces to the type space itself. For example, G_3^h has equilibrium $s^{3,1}$ that extends $s^{0,1}$ and s^2 to G_3^h , and G_7^h has equilibrium $s^{7,1}$ that extends $s^{0,1}$ and σ^1 to G_7^h . This property of Bayesian Nash equilibrium is assumed in the class of solution concepts considered here. I select such a solution from each of these type spaces, e.g., selecting $s^{3,1}$ and $s^{7,1}$ for G_3^h and G_7^h , respectively. Since these type spaces intersect each other, this could have led to a violation of invariance. That is, a type in the intersection could have played different actions according to the solutions of the distinct type spaces. This is not the case. Any such intersection is of the first rank, for which actions have been determined in the previous round. Hence, all of the selected solutions prescribe the same action for any type in the intersection. For example, the actions of types $h_1(t_1^{3,1}, G_3)$ and $h_1(t_1^{7,1}, G_7)$ are selected as a in round 1. Iterating this argument, I select a solution for every type space of third rank, fourth rank, and so on. For example, at round 3, I select equilibria $s^{4,1}$ and $s^{5,1}$ for games G_4^h and G_5^h , respectively, the equilibria that extend $s^{3,1}$ and $s^{7,1}$ to games G_4^h and G_5^h ,

respectively. Since each type space has a finite rank, this leads to a selection for every type space.

The construction here addresses two inherent difficulties. First, there are typically uncountably many type spaces by which the same piece of information can be modeled. Hence, in any construction, at some stage, one needs to select the solutions for uncountably many such type spaces simultaneously without violating the invariance condition. For example, the hierarchy $h_1(t_1^{4,1}, G_4)$ is contained in a continuum of games in which the beliefs of type $t_1^{4,3}$ vary, and one must select equilibria of all of these games simultaneously, making sure that the types with hierarchy $h_1(t_1^{4,1}, G_4)$ all play the same action. In the above construction, this is accomplished by fixing the action of any type t_i in any game G at the earliest round at which $h_i(t_i, G)$ is available, relying on the fact that the minimal type space for $h_i(t_i, G)$ is the only type space that contains $h_i(t_i, G)$ at that round. When the solution to G is selected at a later round, one may need to select solutions to uncountably many type spaces that contain $h_i(t_i, G)$, but the solutions will all assign the same fixed action to $h_i(t_i, G)$. For example, the actions of the types with hierarchy $h_1(t_1^{4,1}, G_4)$ are all selected as a at round 1 when $s^{0,1}$ is selected for G_0^h .

The second difficulty is that the space of all finite types cannot be partitioned into smaller subspaces: for any two types t_j and t'_j , one can construct a type t_i that puts positive probabilities on t_j and t'_j , in which case the types t_j and t'_j must be in the same subspace—along with t_i —in any partition. For example, because of type $t_1^{5,1}$, one cannot separate $h(t^0, G_0)$ and $h(t^1, G_1)$. This prevents one from using more straightforward techniques or the existing existence results. For example, if the space could be partitioned into countable subspaces, one could obtain an invariant selection in each subspace, by simply selecting a solution from each of the type spaces one-by-one in the order given by counting the type spaces within the subspace. This would lead to an invariant selection in the entire space. If that were the case, one could also use the existence result of Simon (2003) for finitely generated type spaces.

3. MODELS IN UNIVERSAL TYPE SPACE AND THE LATTICE STRUCTURE

In this section, I embed every type space in T^u as a subspace and show that the resulting set of subspaces exhibit a very useful lattice structure.

Models. A subset $T = T_1 \times \cdots \times T_n \subset T^u$ is said to be a *belief-closed subspace* if (Θ, T, κ^u) is a type space for some $\Theta \subseteq \Theta^*$. That is, for each $t_i \in T_i$, $\kappa_{t_i}^u(\Theta \times T_{-i}) = 1$. A belief-closed subspace T is said to be *finite* if Θ and T above are finite. Finite belief-closed subspaces are

simply called *models*. I will include the empty set to the set of models and write

$$M = \{T \subset T^u \mid T \text{ is a finite belief-closed subspace}\} \cup \{\emptyset\}.$$

Given any two models $T, T' \in M$, define the collage of T and T' as $T \vee T' \equiv (T_1 \cup T'_1) \times \dots \times (T_n \cup T'_n)$, which is clearly also a model.

Note that, by (2.2), the image $h(T, G)$ of any game G is a model:

$$(3.1) \quad h(T, G) \in M \quad (\forall G = (N, A, u, \Theta, T, \kappa) \in \mathcal{G}).$$

By definition, T^u is simply the collage of the images of games with finite type spaces, and it can be written as the collage of all models in T^u .

Lattice Structure of Models. I now show that models form a lattice under the set inclusion, exhibiting many useful properties. In particular, one can rank the models depending on how far they are removed from the empty set.

Proposition 1. *(M, \supseteq) is a lattice with $T \vee T' \in M$ and $T \wedge T' \equiv T \cap T' \in M$ for all $T, T' \in M$. Moreover, (M, \supseteq) is a complete meet-semilattice, i.e., for any $M' \subseteq M$, $\cap_{T \in M'} T \in M$.*

Proof. That $T \vee T' \in M$ is immediate. Fixing any $M' \subseteq M$, I will show that $\bar{T} \equiv \cap_{T \in M'} T \in M$. If $\bar{T} = \emptyset$, $\bar{T} \in M$ by definition. Assume that $\bar{T} \neq \emptyset$. By definition, for each $T \in M'$, there exists a finite set Θ^T , such that $\kappa_{t_i}^u(\Theta^T \times T_{-i}) = 1$ for each $t_i \in T_i$. Define $\bar{\Theta} = \cap_{T \in M'} \Theta^T$. In order to show that \bar{T} is a finite belief-closed subspace, it suffices to show that $\kappa_{t_i}^u(\bar{\Theta} \times \bar{T}_{-i}) = 1$ for every $i \in N$ and $t_i \in \bar{T}_i$. To this end, take any $t_i \in \bar{T}_i$ and (θ, t_{-i}) with $\kappa_{t_i}^u(\theta, t_{-i}) > 0$. Then, $(\theta, t_{-i}) \in \Theta^T \times T_{-i}$ for each $T \in M'$, as $t_i \in T_i$. Hence, $(\theta, t_{-i}) \in \cap_{T \in M'} (\Theta^T \times T_{-i}) = \bar{\Theta} \times \bar{T}_{-i}$. This shows that $\kappa_{t_i}^u(\theta', t'_{-i}) = 0$ for every $(\theta', t'_{-i}) \notin \bar{\Theta} \times \bar{T}_{-i}$. Since $\kappa_{t_i}^u$ has a finite support, this further shows that $\kappa_{t_i}^u(\bar{\Theta} \times \bar{T}_{-i}) = 1$. \square

Proposition 1 shows that M is a lattice under the set inclusion, with join \vee and meet \wedge defined as above. Moreover, it is a complete meet-semilattice, as it is closed under all intersections. It is not a complete lattice because infinite collages of finite models are not necessarily finite. In particular, $T^u \notin M$.

By (3.1), each type $t_i \in T_i^u$ is in a model $T \in M$, but usually there are uncountably many such models. Proposition 1 implies that there is a unique minimal model $T^{t_i} \in M$ that is included in all models that contain t_i . Here,

$$(3.2) \quad T^{t_i} = \bigcap_{T \in M, t_i \in T_i} T.$$

T^{t_i} is the minimal type space in which t_i can be expressed. Note that $T^{t_i} \neq \emptyset$ because $(t_i, t_{-i}) \in T^{t_i}$ for each $(\theta, t_{-i}) \in \text{supp} \kappa_{t_i}^u$.

I will next rank the models according to how far they are removed from the empty set. Define $R_0 = \{\emptyset\}$. Define R_1 as the set of models $T \in M \setminus R_0$ for which there is no model $T' \in M \setminus R_0$ with $T' \subsetneq T$. That is, T does not have any proper belief-closed subspace other than the empty set. Note that every finite model is either in R_1 or contains a subspace that is in R_1 . Proceeding in the same fashion, one can inductively define the sets R_k , $k = 1, 2, \dots$, by defining R_k as the set of models $T \in M$ such that (i) $T \notin R_{k'}$ for any $k' < k$, and (ii) for any model $T' \subsetneq T$, $T' \in R_{k'}$ for some $k' < k$. I will say that a model $T \in M$ has rank k if $T \in R_k$. The next lemma establishes some useful facts about the ranks of the models.

Lemma 1. *The following are true.*

- (1) *For every $T \in M$, $T \in R_k$ iff k is the largest integer for which there exist models $T^0, \dots, T^k \in M$ with $\emptyset = T^0 \subsetneq \dots \subsetneq T^k = T$.*
- (2) *Every $T \in M$ has a rank $k_T \leq |T|$.*
- (3) *For any $T, T' \in M$ with $T \subsetneq T'$, $k_T < k_{T'}$.*
- (4) *For any $M' \subseteq M$, $\bar{T} \equiv \bigcap_{T \in M'} T$ has rank $k_{\bar{T}}$ such that $k_{\bar{T}} \leq k_T$ for each $T \in M'$, with strict inequality whenever $T \neq \bar{T}$.*

The first three properties above are shared by any lattice formed by a family of finite sets under inclusion. The last property also relies on the fact that the meet is the same as the intersection operator and the lattice is closed under arbitrary intersections. The second and the fourth properties are the most crucial properties for this paper. The second property states that every model has a rank, and the fourth property states that the rank of an intersection is lower than the ranks of the intersecting models.

4. INVARIANT SELECTIONS

In this section, I show that there exists an invariant selection from any solution concept that satisfies two basic properties. I further show that any invariant selection in a subfamily can be extended to a larger family. In particular, invariance of a selection has only one implication for the solutions of a given game: the solution is invariant.

In order to state the properties sufficient for invariance, I introduce two auxiliary games. For any $T \in M$, I write $G^T = (N, A, \Theta, T, \kappa^u, u) \in \mathcal{G}$ for the Bayesian game with type space T . For any game $G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}$, I write $G^h = (N, A, \Theta, h(T, G), \kappa^u, u)$ for the image of G in the universal type space under h . The first property is that the solution set $\Sigma(G)$ of G includes all of the solutions to the image G^h of G in universal type space:

Assumption 1. For all $G \in \mathcal{G}$, $\Sigma(G) \supseteq \Sigma^h(G) \equiv \{\sigma \circ h(\cdot, G) \mid \sigma \in \Sigma(G^h)\}$.

This assumption corresponds to a well-known property of Bayesian Nash equilibrium: any Bayesian Nash equilibrium in a terminal space, such as the universal type space, induces a Bayesian Nash equilibrium in every game embedded in that space.⁵ Assumption 1 also holds for other canonical solution concepts, such as rationalizability. In general, it holds for any *expressible* solution concept (Battigalli et. al., 2011) if one requires the characterizing assumptions to be expressible with respect to the *relevant* information (by designating h_i as the information mapping χ_i in Battigalli et. al. (2011), and obtaining $\Sigma = \Sigma^h$).⁶

On the other hand, since the set of Bayesian Nash equilibria is affected by the existence of redundant types, Assumption 1 may not hold for equilibrium refinements based on external criteria such as efficiency and fairness. For example, in game G_6 of Section 2, equilibrium $s^{6,2}$ Pareto-dominates $s^{6,1}$, which corresponds to the unique equilibrium of G_6^h . Hence, Assumption 1 does not hold for the Pareto-dominant Bayesian Nash equilibrium, which selects $s^{6,2}$ in G_6 and $s^{6,1}$ in G_6^h . Incidentally, since $s^{6,2}$ is not invariant, this also shows that there does not exist an invariant selection from the Pareto-dominant Bayesian Nash equilibrium solution concept.

Note that the solutions of the form $\sigma \circ h(\cdot, G)$ do not use any irrelevant information, in the sense that each type t_i plays $\sigma_i(h_i(t_i, G))$. If an invariant selection selects $\sigma \in \Sigma(G^h)$ at G^h , it must also select $\sigma \circ h(\cdot, G)$ at G . Hence, it is necessary for invariant selection that *some* solution that does not use any irrelevant information is available at $\Sigma(G)$, i.e., $\Sigma(G) \cap \Sigma^h(G) \neq \emptyset$. Assumption 1 strengthens this necessary condition by requiring that *all* such solutions are available at $\Sigma(G)$. The necessary condition is not enough because invariance imposes many similar conditions, such as $\cap_{\hat{G}^h=G^h} \Sigma(\hat{G}) \neq \emptyset$. Assumption 1 ensures that all such conditions are satisfied.

The second property is a basic extension (and existence) property.

Assumption 2. For any $T \in M$ and any invariant selection $G^{T'} \mapsto \sigma^{T'}$ from Σ for games $G^{T'}$ with $T' \in 2^T \cap M \setminus \{T\}$, there exists $\sigma \in \Sigma(G^T)$ such that for every $t \in T' \subset T$, $\sigma(t) = \sigma^{T'}(t)$.

That is, given a finite type space, invariant solutions for its subspaces can be extended to the type space. When there is only one proper subspace T' , this reduces to the well-known extension property of Bayesian Nash equilibrium: any equilibrium of T' can be extended

⁵This is of course vacuously true when there is no equilibrium on the universal type space. Assumption 1 also vacuously holds when $\Sigma(G^h)$ is empty.

⁶See also Heifetz and Samet (1998) for the concept of an expressible assumption.

to T . Once again, canonical solution concepts, such as Bayesian Nash equilibrium and rationalizability, have this property. It is necessary for invariant selection that *some* invariant selections for the subspaces are extendable to T . Assumption 2 strengthens this necessary condition by requiring that *all* such selections are extendable to T .

Within the context of Bayesian Nash equilibrium, Friedenbergh and Meier (2007) provide a thorough analysis of both properties above. They show that while the "pull-back" property in Assumption 1 holds generally, the extension property in Assumption 2 may fail in infinite type spaces (see Sections 5 and 6 for more on the literature).

Toward constructing an invariant selection, I rank the games in \mathcal{G} as follows. Recall that for any game $G \in \mathcal{G}$ with type space T , $h(T, G) \in M$ by (3.1). Hence, by Lemma 1, $h(T, G)$ has some finite rank k , i.e., $h(T, G) \in R_k$ for some finite $k \geq 1$. A game G is said to be of rank k if $h(T, G)$ is of rank k ; G and G^h are of the same rank. Write \mathcal{G}^k for the set of all games $G \in \mathcal{G}$ of rank k , and write $\hat{\mathcal{G}}^k = \cup_{l \leq k} \mathcal{G}^l$. We are now ready to state and prove the main result:

Proposition 2. *Under Assumptions 1 and 2, there exists an h -invariant selection from Σ .*

Proof. Using induction on the rank k , the proof constructs an invariant selection $G \mapsto \sigma^G$ from Σ for \mathcal{G} , rank by rank. Take $k = 1$. For every model $T \in R_1$ (with rank $k = 1$), pick an arbitrary $\sigma^{G^T} \in \Sigma(G^T)$. Here, $\Sigma(G^T) \neq \emptyset$ by Assumption 2. For any other $G \in \mathcal{G}^1$, pick $\sigma^G = \sigma^{G^h} \circ h$. Note that, since $\sigma^{G^h} \in \Sigma(G^h)$, by Assumption 1, $\sigma^G \in \Sigma(G)$. Note also that this constructs a selection $G \mapsto \sigma^G$ from Σ for \mathcal{G}^1 . The selection is invariant because the games in \mathcal{G}^1 have disjoint images under h by the last part of Lemma 1.

Now consider any $k > 1$, and assume that an invariant selection $G \mapsto \sigma^G$ from Σ for $\hat{\mathcal{G}}^{k-1}$ has been constructed. Consider any model $T \in R_k$ (with rank k). By Part 3 of Lemma 1, each proper subspace T' of T is of rank $k-1$ or lower. Hence, by the inductive hypothesis, an invariant selection $G^{T'} \mapsto \sigma^{G^{T'}}$ from Σ for the class of games $G^{T'}$ with $T' \in 2^T \cap M \setminus \{T, \emptyset\}$ has been constructed already. Then, by Assumption 2, there exists $\sigma^{G^T} \in \Sigma(G^T)$ such that $\sigma^{G^T}(t) = \sigma^{G^{T'}}(t)$ for all $t \in T' \in 2^T \cap M \setminus \{T, \emptyset\}$. Pick σ^{G^T} as the solution at G^T , and repeat this for every $T \in R_k$. For all other games $G \in \mathcal{G}^k$ with rank k , pick $\sigma^G = \sigma^{G^h} \circ h$, where $\sigma^{G^h} \in \Sigma(G^h)$ by Assumption 1. (Note that $G^h = G^T$ for some $T \in R_k$.) This constructs a selection from Σ for $\hat{\mathcal{G}}^k$.

In order to complete the inductive construction, check that the selection $G \mapsto \sigma^G$ from Σ for $\hat{\mathcal{G}}^k$ is indeed invariant. To this end, take any distinct (t_i, G) and (t'_i, G') with $h_i(t_i, G) = h_i(t'_i, G')$ where $G = (N, A, \Theta, T, \kappa, u)$, $G' = (N, A, \Theta, T', \kappa, u) \in \hat{\mathcal{G}}^k$, $t_i \in T_i$ and $t'_i \in T'_i$. Note that, since the rank of $G^h \equiv G^{h(T, G)}$ is at most k , by construction, $\sigma_i^{G^{T''}}(t''_i) = \sigma_i^{G^h}(t''_i)$

for any $t'_i \in T''_i$ with $T'' \subseteq h(T, G)$. But, by definition, $h_i(t_i, G) \in T^{h_i(t_i, G)}$ and $T^{h_i(t_i, G)} \subseteq h(T, G)$, where $T^{h_i(t_i, G)}$ is the unique minimal model that contains $h_i(t_i, G)$. Therefore,

$$(4.1) \quad \sigma_i^{G^{T^{h_i(t_i, G)}}} (h_i(t_i, G)) = \sigma_i^{G^h} (h_i(t_i, G)).$$

Likewise, $\sigma_i^{G^{T^{h_i(t'_i, G')}}} (h_i(t'_i, G')) = \sigma_i^{G^h} (h_i(t'_i, G'))$. Therefore,

$$\begin{aligned} \sigma_i^G(t_i) &= \sigma_i^{G^h} (h_i(t_i, G)) \\ &= \sigma_i^{G^{T^{h_i(t_i, G)}}} (h_i(t_i, G)) \\ &= \sigma_i^{G^{T^{h_i(t'_i, G')}}} (h_i(t'_i, G')) \\ &= \sigma_i^{G^h} (h_i(t'_i, G')) = \sigma_i^{G'}(t'_i), \end{aligned}$$

where the first and the last equalities are by construction, the second equality is by (4.1), and the third equality holds simply because $h_i(t_i, G) = h_i(t'_i, G')$ and $T^{h_i(t_i, G)}$ is unique.

Since each game has a finite rank (i.e., $\mathcal{G} = \cup_k \mathcal{G}^k$), this construction picks a solution $\sigma^G \in \Sigma(G)$ for each $G \in \mathcal{G}$ at some round k . In order to check that the selection $G \mapsto \sigma^G$ for \mathcal{G} is invariant, note that for any $G, G' \in \mathcal{G}$, since $G, G' \in \hat{\mathcal{G}}^k$ for some k , $\sigma_i^G(t_i) = \sigma_i^{G'}(t'_i)$ whenever $h_i(t_i, G) = h_i(t'_i, G')$, as established in the previous paragraph. \square

Under Assumptions 1 and 2, Proposition 2 establishes that there exists an h -invariant selection from Σ , and its proof explicitly constructs such a selection. As discussed above, Assumptions 1 and 2 are strengthenings of basic necessary conditions for invariant selection. These are weak assumptions in that they hold for canonical solution concepts, such as Bayesian Nash equilibrium and rationalizability.

The straightforwardness of the construction in the proof of Proposition 2 may be misleading, as it finesses the following inherent difficulty. The same piece of relevant information can be modeled by types in uncountably many games. In order to construct an invariant selection, one then needs to select the solutions for uncountably many such games simultaneously and maintain invariance. In the construction such a difficult task is made possible by the lattice structure established in the previous section, as follows.

Recall from the previous section that for any type t_i in any game $G \in \mathcal{G}$, there exists a unique minimal type space $T^{h_i(t_i, G)} \in M$ in which the relevant information $h_i(t_i, G)$ of t_i can be modeled. This type space has the lowest rank $k_{T^{h_i(t_i, G)}}$ among the type spaces that can model $h_i(t_i, G)$. The action of all types t'_i from games G' with $h_i(t'_i, G') = h_i(t_i, G)$ is selected at round $k_{T^{h_i(t_i, G)}}$, which is the first time it is possible to express $h_i(t_i, G)$, using a solution for the minimal type space $T^{h_i(t_i, G)}$, which is the only model that contains $h_i(t_i, G)$ at that rank. Of course, many of these games have higher ranks than $k_{T^{h_i(t_i, G)}}$, and the

solutions to these games are selected in later rounds. In the construction, these selections respect the specification of the action for $h_i(t_i, G)$ that had been made at round $k_{Th_i(t_i, G)}$.

I will next explore the restrictions imposed by the invariance requirement. I will first establish that any invariant selection in a subfamily can be extended to all games. Conceptually, this establishes that the invariance requirement on a larger family of games does not impose any extra restriction on the subfamilies. Practically, it ensures that if one is only interested in the behavior in a class of games (e.g. in the solution of a particular game), she can focus on constructing an invariant selection for that class without worrying about the invariance across all games.

Proposition 3. *Under Assumptions 1 and 2, for any $\mathcal{G}' \subseteq \mathcal{G}$ and any h -invariant selection $G \mapsto \sigma^G$ from Σ for \mathcal{G}' , there exists an h -invariant selection $G \mapsto \hat{\sigma}^G$ from Σ for \mathcal{G} such that $\hat{\sigma}^G = \sigma^G$ for every $G \in \mathcal{G}'$.*

Proof. I will construct a refinement Σ' of Σ that satisfies Assumptions 1 and 2 and such that $\Sigma'(G) = \{\sigma^G\}$ for all $G \in \mathcal{G}'$. Then, by Proposition 2, there exists an invariant selection $G \mapsto \hat{\sigma}^G$ from Σ' for \mathcal{G} as in the proposition. Since Σ' is a refinement of Σ , $G \mapsto \hat{\sigma}^G$ is also a selection from Σ .

To define Σ' , note that, since $G \mapsto \sigma^G$ is h -invariant, $\sigma^G = \sigma^{G^h} \circ h$ for some solution $\sigma^{G^h} \in \Sigma(G^h)$ for each $G \in \mathcal{G}'$. Write $\bar{\mathcal{G}}' = \mathcal{G}' \cup \{G^h | G \in \mathcal{G}'\}$ and set

$$\Sigma'(G) = \begin{cases} \{\sigma^G\} & \text{if } G \in \bar{\mathcal{G}}', \\ \{\sigma^{G^T}|_{T'}\} & \text{if } G = G^{T'}, T' \subset T, G^T \in \bar{\mathcal{G}}', \\ \Sigma(G) & \text{otherwise,} \end{cases}$$

where $\sigma^{G^T}|_{T'}$ is the restriction of σ^{G^T} to T' . Since $G \mapsto \sigma^G$ is invariant, Σ' is well-defined.

To check Assumption 1, note that the equality $\Sigma'(G) = \{\sigma \circ h(\cdot, G) | \sigma \in \Sigma'(G^h)\}$ holds for $G \in \mathcal{G}'$ by construction and holds for $G \in \{G^T | T \in M\}$ by definition. For any $G \notin \mathcal{G}' \cup \{G^T | T \in M\}$,

$$\Sigma'(G) = \Sigma(G) \supseteq \{\sigma \circ h(\cdot, G) | \sigma \in \Sigma(G^h)\} \supseteq \{\sigma \circ h(\cdot, G) | \sigma \in \Sigma'(G^h)\},$$

where the first equality is by construction of Σ' , the next inclusion is by Assumption 1 for Σ , and the last inclusion is due to the fact that Σ' is a refinement of Σ .

To check Assumption 2, note first that when $G^T \in \bar{\mathcal{G}}'$, Assumption 2 holds for T and Σ' by construction. Now consider any $T \in M$ with $G^T \notin \bar{\mathcal{G}}'$ and any invariant selection $G^{T'} \mapsto \sigma^{T'}$ from Σ' for the class of games $G^{T'}$ with $T' \in 2^T \cap M \setminus \{T\}$. This is also an invariant selection from Σ , and $\Sigma(G^T) = \Sigma'(G^T)$. Hence, by Assumption 2 for Σ , there

exists $\sigma \in \Sigma(G^T) = \Sigma'(G^T)$ such that for every $t \in T' \subset T$, $\sigma(t) = \sigma^{T'}(t)$. Therefore, Assumption 2 holds for Σ' . \square

I will next characterize the implications of invariance to the solutions of a given game. As discussed before, invariance of a selection trivially implies that the solution is invariant, i.e., the types with identical relevant information play the same action. The next corollary establishes that this is the only implication of invariance of selections to the set of solutions of a given game. In the following, given a game G^* , a solution $\sigma \in \Sigma(G^*)$ is said to be *selected by* $G \mapsto \sigma^G$ if $\sigma^{G^*} = \sigma$.

Corollary 1. *Under Assumptions 1 and 2, for any $G \in \mathcal{G}$ and $\sigma \in \Sigma(G)$, σ is selected by an h -invariant selection from Σ if and only if σ is h -invariant.*

Proof. Necessity immediately follows from the definition of invariance (applied to the types in G). To prove sufficiency, take $\mathcal{G}' = \{G\}$ in Proposition 3 and note that σ is an invariant selection for $\{G\}$. \square

Under Assumptions 1 and 2, the above results establish that there is always an invariant selection and that invariance does not impose any additional restriction on the solutions for the subfamilies of games. In particular, a solution to a game is selected by an invariant selection if and only if types with identical relevant information play the same action according to the solution. Hence, if one is interested only in behavior in a game G , then she can analyze G in isolation by focusing on h -invariant solutions to G .

5. INVARIANT EQUILIBRIUM SELECTION

In this section, under the usual regularity conditions, I show that Bayesian Nash equilibrium satisfies the sufficient conditions for invariant selection, and hence the conclusions of the previous results are true for Bayesian Nash equilibrium: there exists an invariant equilibrium selection and an equilibrium is selected by an invariant equilibrium selection if and only if the equilibrium is invariant.

Given any game $G = (N, A, u, \Theta, T, \kappa)$, by a *Bayesian Nash equilibrium of G* , I mean any strategy profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ such that $\sigma_i^*(t_i) \in BR_{t_i}(\sigma_{-i}^*|G)$ for each $t_i \in T_i$, where $BR_{t_i}(\sigma_{-i}^*|G)$ denotes the set of all mixed best replies of type t_i to σ_{-i}^* in game G . I write $BNE(G)$ for the set of Bayesian Nash equilibria of G . By an *equilibrium selection*, I mean a selection from BNE . I will consider the following regularity condition.

Assumption 3. *The set Θ^* is compact. For each $i \in N$, action set A_i is a compact metric space, and each u_i is continuous in a , measurable in θ , and does not depend on θ_{NR} .*

Note that Assumption 3 holds in most games considered in Game Theory and its applications, including finite games. It is made to ensure that Nash equilibrium exists when types are considered as players. Under this weak assumption, the next result establishes that Bayesian Nash equilibrium satisfies the sufficient conditions for invariant selection.

Lemma 2. *Under Assumption 3, Assumptions 1 and 2 hold for BNE.*

Proof. The fact that Assumption 1 holds for BNE is well known (see Friedenberg and Meier (2007) for a general proof). To see that Assumption 2 also holds for BNE, observe that any invariant equilibrium selection $G^{T'} \mapsto \sigma^{T'}$ for games $G^{T'}$ with $T' \in 2^T \cap M \setminus \{T\}$ induces a Bayesian Nash equilibrium σ on the subspace $\hat{T} \equiv \bigvee_{T' \in 2^T \cap M \setminus \{T\}} T'$ (see Lemma 3 below). It is known that σ can be extended to T .⁷ \square

Hence, the conclusions of the previous section apply to equilibrium selection:

Proposition 4. *Under Assumption 3, the following are true.*

- (1) *For any $\mathcal{G}' \subseteq \mathcal{G}$ and any h -invariant equilibrium selection $G \mapsto \sigma^G$ for \mathcal{G}' , there exists an h -invariant equilibrium selection $G \mapsto \hat{\sigma}^G$ for \mathcal{G} such that $\hat{\sigma}^G = \sigma^G$ for every $G \in \mathcal{G}'$.*
- (2) *For any $G \in \mathcal{G}$ and any $\sigma \in BNE(G)$, σ is selected by an h -invariant equilibrium selection if and only if σ is h -invariant.*
- (3) *There exists an h -invariant equilibrium selection.*

Proof. Note that Part 1 implies both Part 2 (for $\mathcal{G}' = \{G\}$) and Part 3 (for $\mathcal{G}' = \emptyset$). Part 1 immediately follows from Lemma 2 and Proposition 3. \square

Under the usual regularity conditions, Proposition 4 establishes that any invariant equilibrium selection for a subset of games can be extended to all games with finite type spaces. In particular, there is an invariant equilibrium selection for all such games. It also implies that, beyond the basic restriction on the actions of types with identical information, invariance does not lead to any equilibrium refinement; it only restricts the way the solutions vary across games.

⁷One extension result is known as the Basic Lemma in the robustness literature (Kajii and Morris, 1997) and dates back to Monderer and Samet (1989). A version of the Basic Lemma states that if one embeds \hat{T} in T as a p -evident event, then any equilibrium σ on \hat{T} can be extended to T as $(1-p)$ -equilibrium (when the payoffs are restricted to be within $[0, 1]$); another version for complete information games states that any p -dominant equilibrium of \hat{T} can be extended to T . Here, I take $p = 1$. Friedenberg and Meier (2007) provides an extension result for $p = 1$ that holds under Assumption 3.

6. EQUILIBRIUM ON T^u

There is a one-to-one correspondence between the invariant equilibrium selections and the equilibria on T^u , the universal space of finite types. In this section, using this correspondence and the results of the previous section, I will show that there exists an equilibrium on T^u , and indeed, every equilibrium on its belief-closed subspaces can be extended to T^u .

For any player i , by *strategy*, I mean any function $\sigma_i : T_i^u \rightarrow \Delta(A_i)$. By a *Bayesian Nash equilibrium* on T^u , I mean any strategy profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ such that $\sigma_i^*(t_i) \in BR_{t_i}(\sigma_{-i}^*)$ for each $t_i \in T_i^u$ where $BR_{t_i}(\sigma_{-i}^*)$ is the set of best replies for t_i against σ_{-i}^* . The strategies and equilibria on subspaces are defined similarly. It is crucial here that I do not impose any measurability conditions on the strategies. Since strategies condition on players' types already, measurability is not needed for players' knowing their own actions. It is not needed for expectations either because the types' beliefs have finite supports, yielding well-defined beliefs on $\Theta^* \times A_{-i}$ for each t_i and each σ_{-i} . Consequently, $BR_{t_i}(\sigma_{-i}^*)$ is well-defined.

There is a one-to-one correspondence between invariant selections and the strategy profiles on T^u . Any h -invariant selection $G \mapsto \sigma^G$ for \mathcal{G}' yields a well-defined strategy profile σ^* on $T^{\mathcal{G}'} \equiv \bigvee_{G=(N,A,\Theta,T,\kappa,u) \in \mathcal{G}'} h(T, G)$, defined by

$$(6.1) \quad \sigma_i^*(h_i(t_i, G)) = \sigma_i^G(t_i) \quad (\forall G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}', i \in N, t_i \in T_i).$$

Conversely, for any strategy profile σ^* on $T^{\mathcal{G}'}$, (6.1) yields an h -invariant selection $G \mapsto \sigma^G$ for \mathcal{G}' . The following lemma establishes a similar one-to-one correspondence between the invariant equilibrium selections and the equilibria on the subspaces of T^u .

Lemma 3. *For any $\mathcal{G}' \subseteq \mathcal{G}$, an equilibrium selection $G \mapsto \sigma^G$ for \mathcal{G}' is h -invariant if and only if there exists a Bayesian Nash equilibrium σ^* on $T^{\mathcal{G}'}$ such that*

$$(6.2) \quad \sigma_i^*(h_i(t_i, G)) = \sigma_i^G(t_i) \quad (\forall G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}', i \in N, t_i \in T_i).$$

That is, invariant selections on \mathcal{G}' are precisely the selections obtained by restricting the equilibria on $T^{\mathcal{G}'}$ to its subspaces. Due to this correspondence, the previous results on invariant equilibrium selection immediately yield the following existence result for equilibrium on the space of all finite types. (See the appendix for a detailed proof.)

Proposition 5. *Under Assumption 3, the following are true.*

- (1) *For any $\mathcal{G}' \subseteq \mathcal{G}$ and any h -invariant equilibrium selection $G \mapsto \sigma^G$ for \mathcal{G}' , there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(h_i(t_i, G)) = \sigma_i^G(t_i)$ for every $G \in \mathcal{G}'$ and every type t_i in G .*

- (2) For any $M' \subseteq M$ and any Bayesian Nash equilibrium σ on $T^{M'}$, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma^* = \sigma$ on $T^{M'}$.
- (3) For any $G \in \mathcal{G}$ and any h -invariant $\sigma \in BNE(G)$, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(h_i(t_i, G)) = \sigma_i^G(t_i)$ for all t_i in G .
- (4) There exists a Bayesian Nash equilibrium on T^u .

Part 2 states that any equilibrium defined on any subspace can be extended to T^u , the space of all finite types. By Lemma 3, this is equivalent to stating that for any invariant selection for any family of games, there is an equilibrium on T^u that specifies the actions of the types in the family according to the selection—Part 1. This result is very useful in equilibrium analysis on T^u . In such an analysis, one often needs to specify a partial strategy for a given set of types and have an equilibrium on T^u in which the given types play according to the specification. For a suitably selected set of types, it is relatively easy to verify that the specified behavior of the given types is part of an equilibrium of the games that the types come from and that these equilibria form an invariant selection. On the other hand, specifying a nontrivial equilibrium on T^u is a prohibitively difficult task because of the complex interconnections between the types in T^u and between their best responses. The result stated in Parts 1 and 2 frees the researcher from the latter daunting task. Thanks to this result, she can focus on specifying the equilibrium behavior on the relevant games without worrying about whether the specified behavior is part of an equilibrium on T^u . (See Weinstein and Yildiz (2011) for such an application.)

As a special case (for $\mathcal{G}' = \{G\}$), this result implies that any invariant equilibrium of any game G can be extended to the space of all finite types, as in Part 3. That is, one can focus on the equilibria on T^u without ruling out any equilibria of games in which distinct types have distinct relevant information. For example, in robustness analysis for equilibria of these games, it suffices to analyze the sensitivity of equilibria on T^u to perturbations in T^u .

Finally, as another special case (for $\mathcal{G}' = \emptyset$), the result establishes existence of an equilibrium on the space of finite types. This fills an important gap in the literature, in which very little is known on the existence of equilibrium on the universal type space and on its prominent subspaces, such as T^u . Simon (2003) shows existence of equilibrium on spaces that can be partitioned into countable subspaces and in which the types have finite support. Unfortunately, one cannot partition T^u because given any two types in T^u , there is another type in T^u that puts positive probabilities on both of the given types. (This property is exhibited by most prominent subspaces of the universal type space, such as the spaces of all finite types with common prior and all countable types, and one cannot partition them either.)

The crucial modeling assumption here is that I do not require that the strategies are measurable, which is not necessary here. In a larger type space with types that have uncountable supports, one needs to impose a measurability restriction on the strategies. In that case, the above results may not be true. For example, in a particular class in which his existence result applies, Simon (2003) shows that all equilibria must be in non-measurable strategies (cf. Part 4). Under the measurability restriction, Friedenberg and Meier (2007) show that some equilibria of a given game may not be extendable to a larger space even if there is an equilibrium in the larger space (cf. Part 3).

Finally, note that if A is convex and u_i is concave in own action, the above results also apply to equilibria in pure strategies.

7. EXTENSIONS

The baseline model focuses on finite type spaces and assumes a specific class of h mappings. This section presents extensions to countable type spaces and to more general h mappings.

7.1. Countable Type Spaces. The analysis of countable type spaces is similar to that of finite type spaces. The only difficulty is that a countable type space may have an infinite rank; the results and the proofs extend to the case of countable type spaces with finite rank verbatim. With type spaces of infinite rank, one needs to consider countable ordinals and use transfinite induction for the construction, but the results are the same.

Let $\tilde{\mathcal{G}}$ be the set of games with countable type spaces, \tilde{T}^u be the space of countable type profiles $h(t, G)$ with $G \in \tilde{\mathcal{G}}$, and \tilde{M} be the set of countable models within \tilde{T}^u ; \tilde{M} consists of the images of games $\tilde{G} \in \tilde{\mathcal{G}}$ along with the empty set. Extend Assumptions 1 and 2 to $\tilde{\mathcal{G}}$, by replacing (\mathcal{G}, T^u, M) with $(\tilde{\mathcal{G}}, \tilde{T}^u, \tilde{M})$. One can use the existing analysis to show that \tilde{M} is a lattice under inclusion and complete under all intersections (cf. Proposition 1). One can then extend the existing results to countable type spaces as follows:

Proposition 6. *Under the extensions of Assumptions 1 and 2 to $\tilde{\mathcal{G}}$, for any $\tilde{\mathcal{G}}' \subseteq \tilde{\mathcal{G}}$ and any h -invariant selection $G \mapsto \sigma^G$ from Σ for $\tilde{\mathcal{G}}'$, there exists an h -invariant selection $G \mapsto \hat{\sigma}^G$ from Σ for $\tilde{\mathcal{G}}$ such that $\hat{\sigma}^G = \sigma^G$ for every $G \in \tilde{\mathcal{G}}'$. Under Assumption 3 (alone), the following are also true.*

- (1) *For any $\tilde{\mathcal{G}}' \subseteq \tilde{\mathcal{G}}$ and any h -invariant equilibrium selection $G \mapsto \sigma^G$ for $\tilde{\mathcal{G}}'$, there exists a Bayesian Nash equilibrium σ^* on \tilde{T}^u such that $\sigma_i^*(h_i(t_i, G)) = \sigma_i^G(t_i)$ for every $G \in \tilde{\mathcal{G}}'$ and every type t_i in \tilde{G} .*
- (2) *For any $\tilde{M}' \subseteq \tilde{M}$ and any Bayesian Nash equilibrium σ on $T^{\tilde{M}'}$, there exists a Bayesian Nash equilibrium σ^* on \tilde{T}^u such that $\sigma^* = \sigma$ on $T^{\tilde{M}'}$.*

- (3) For any $G \in \tilde{\mathcal{G}}$ and any h -invariant $\sigma \in BNE(G)$, there exists a Bayesian Nash equilibrium σ^* on \tilde{T}^u such that $\sigma_i^*(h_i(t_i, G)) = \sigma_i^G(t_i)$ for all t_i in G .
- (4) There exists a Bayesian Nash equilibrium on \tilde{T}^u .

7.2. General Notions of Relevance. The baseline model assumes that h is the Θ_R -based hierarchy of beliefs for some Θ_R with $\Theta = \Theta_R \times \Theta_{NR}$. Such a functional form is not needed. Indeed, the working paper shows that all of the results here are true so long as h embeds all games to a general type space $(\Theta^*, T^*, \kappa^*)$ while preserving beliefs. For example, h can be an X -based hierarchy of beliefs for any set X .⁸ In particular, h can be taken as the hierarchy of beliefs obtained by Ely and Peski (2006) and Sadzik (2008) for interim independent rationalizability and Bayesian Nash equilibrium, respectively. For another example, h can be the identity mapping.

Some important notions of relevant information do not lead to a belief-preserving embedding, however. For example, a prominent notion of relevant information is the first k orders of beliefs for some finite k , denoted by \hat{h}^k hereafter.⁹ One cannot have a belief function for $\hat{h}^k(T, G)$ that preserves the beliefs in the original type spaces. Indeed, since $\hat{h}_i^k(t_i, G)$ is the k th-order belief of type t_i , the belief of type $\hat{h}_i^k(t_i, G)$ on $(\theta, \hat{h}_{-i}^k(t_{-i}, G))$ is the $(k+1)$ st-order belief of type t_i , which can vary as one varies (t_i, G) while fixing the k th-order belief $\hat{h}_i^k(t_i, G)$. Note that one can define a type space for each type set $\hat{h}^k(T, G)$ by assigning some beliefs, but one cannot preserve the beliefs as in (2.2).

Without a belief-preserving embedding, one cannot obtain the results regarding Nash equilibrium. Nevertheless, one can still construct an invariant selection for other solution concepts as follows. Let h map each (t, G) to some set $T^* = T_1^* \times \dots \times T_1^*$.¹⁰ Let also M be the set of all images $h(T, G)$ with $G \in \mathcal{G}$ in T^* along with the empty set. Assume that a solution concept Σ is defined on M as well as \mathcal{G} , and in Assumptions 1 and 2 replace G^h with $h(T, G)$ and G^T with T .

Proposition 7. *There exists an h -invariant selection from Σ whenever (i) M is a lattice that is closed under arbitrary intersections, and (ii) Assumptions 1 and 2 hold.*

⁸Afterall, one can transform a Θ -based type space to a X -based type space, by computing the interim beliefs about $X \times T_{-i}$ for each type t_i . For the results about the Bayesian Nash equilibrium, one needs to include everything that is necessarily for computing the best responses in X .

⁹See for example, Rubinstein (1989), Weinstein and Yildiz (2007), Strzalecki (2014), Kets (2012). Cianciaruso and Germano (2011) provide a thorough analysis of quotient type spaces for a fixed k .

¹⁰Canonically, one can specify T^* by listing all true propositions from a set P of *relevant propositions* as in Aumann (1999). The key distinction is that P is restricted to a subset of propositions that are deemed relevant. For example, P consists of the propositions that does not refer to θ_{NR} in the baseline model, while P consists of the propositions that does not refer to higher-order beliefs in \hat{h}^k .

Proof. The proof is identical to the proof of Proposition 2. \square

For example, consider \hat{h}^k . Assumption (i) holds because M is obtained by projecting the models in the universal type space to the space of lower-order beliefs. Although rationalizability and Bayesian Nash equilibrium are not well-defined on M (Weinstein and Yildiz (2007)), the k th-order iterated dominance is well-defined on M and satisfies Assumptions 1 and 2 (Dekel, Fudenberg, and Morris (2007)). Level- k thinking is also well-defined on M .

8. CONCLUDING REMARKS

A piece of relevant information can be modeled through multiple types coming from various Bayesian games. In order to avoid solutions that depend on the choice of modeling the information, one may want to require that all of these types take the same action according to the solutions to these games. That is, the solutions to the Bayesian games form an invariant selection. In this paper, I construct an invariant selection for the space of all games with finite type spaces from arbitrary solution concepts that satisfy two basic conditions. I further show that any invariant selection within a subfamily can be extended to the family of all games with finite type spaces. Constructing such a selection is a difficult task because one needs to equalize the actions of uncountably many types from various games with complicated interconnections that are difficult to foresee. In order to construct such a selection, I first establish an interesting lattice structure for the finite type spaces within the universal type space, a structure that is clearly useful beyond the scope of this paper. It is this structure that enables me to construct an invariant selection in a straightforward manner without making any significant assumptions.

There is a one-to-one correspondence between the invariant equilibrium selections and the equilibria on the space of all finite types. Using this correspondence, I show that there exists an equilibrium in that space, filling an important gap in the literature, and show that indeed any equilibrium in any type space can be extended to entire space, which is a quite useful result in equilibrium analysis on the space of all finite types.

APPENDIX A. OMITTED PROOFS

Proof of Lemma 1. Part 1 immediately follows from the definition of R_k , and Part 3 immediately follows from Parts 1 and 2.

(Part 2) Take any $T \in M$ and integer K such that $T \notin R_k$ for any $k \leq K$. Then, by Part 1, there exist models $T^1, \dots, T^K \in M \setminus \{\emptyset\}$ with $T^1 \subsetneq \dots \subsetneq T^K \subsetneq T$. In particular, $|T| > K$. Therefore, for any $T \in M$, $T \in R_{k_T}$ for some $k_T \leq |T|$.

(Part 4) By Proposition 1, $\bar{T} \in M$. Hence, by Part 2, it has a rank $k_{\bar{T}}$. For any $T \in M'$, since $\bar{T} \subseteq T$, by Part 3, $k_{\bar{T}} \leq k_T$, with strict inequality whenever $T \neq \bar{T}$. \square

Proof of Lemma 3. A strategy profile σ^* on $T^{\mathcal{G}'}$ is well-defined by (6.2) iff the family σ^G is h -invariant. Moreover, one can show that, for any $G \in \mathcal{G}'$ and t_i in G , $BR_{t_i}(\sigma_{-i}^G|G) = BR_{h_i(t_i, G)}(\sigma_{-i}^*)$. Hence, by (6.2), $\sigma_i^G(t_i) \in BR_{t_i}(\sigma_{-i}^G)$ iff $\sigma_i^*(h_i(t_i, G)) \in BR_{h_i(t_i, G)}(\sigma_{-i}^*)$. Therefore, σ^G is an equilibrium of G for every $G \in \mathcal{G}'$ iff σ^* is a Bayesian Nash equilibrium on $T^{\mathcal{G}'}$. \square

Proof of Proposition 5. (Part 1) By Proposition 4, there exists an h -invariant equilibrium selection $G \mapsto \hat{\sigma}^G$ for \mathcal{G} such that $\hat{\sigma}^G = \sigma^G$ for every $G \in \mathcal{G}'$. Then, by Lemma 3, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(h_i(t_i, G)) = \hat{\sigma}_i^G(t_i)$ for every $G \in \mathcal{G}$ and every type t_i in G . When $G \in \mathcal{G}'$, $\sigma_i^*(h_i(t_i, G)) = \hat{\sigma}_i^G(t_i) = \sigma_i^G(t_i)$.

(Part 2) Take $\mathcal{G}' = \{G^T | T \in M'\}$. For each $T \in M'$, set $\sigma^{G^T} = \sigma|_T \in BNE(G^T)$, where $\sigma|_T$ is the restriction of σ to T . Then, by Lemma 3, $G^T \mapsto \sigma^{G^T}$ is an h -invariant equilibrium selection for \mathcal{G}' . Hence, by Part 1, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(t_i) = \sigma_i^*(h_i(t_i, G^T)) = \sigma_i^{G^T}(t_i) = \sigma_i(t_i)$ for all $t_i \in T_i$ and $T \in M$, where $h_i(t_i, G^T) = t_i$.

Finally, Part 1 implies both Part 3 (for $\mathcal{G}' = \{G\}$) and Part 4 (for $\mathcal{G}' = \emptyset$). \square

Proof of Proposition 6. Since \tilde{M} is a meet-complete lattice, one can rank countable models $T \in \tilde{M}$ using countable ordinals as follows. For any $T \in \tilde{M}$, say that T is of finite rank k if there exists the largest integer k for which there exist models $T^0, \dots, T^k \in \tilde{M}$ with $\emptyset = T^0 \subsetneq \dots \subsetneq T^k = T$ (as in the baseline model). If the length of such chains are unbounded, then T is of infinite rank. If all proper subspaces of T are of finite rank but there does not exist a bound k as above, then T is of rank ω . Define orders $\omega + 1, \omega + 2, \dots, 2\omega, 2\omega + 1, \dots$ similarly: $T \in \tilde{M}$ is of rank $m\omega + k$ if its highest ranked proper subspace is of rank $m\omega + k - 1$; it is of rank $(m + 1)\omega$ if it has subspaces of rank $m\omega + k$ for an unbounded set of integers k and all of its proper subspaces are of rank $m'\omega + k$ for some $m' \leq m$ and k .

The construction of an invariant selection is as in the baseline model. Constructing a solution to the type spaces of rank $m\omega + k$ mimics the construction for finite-ranked type spaces. Construction of solutions to the type spaces of rank $m\omega$ is as follows. Suppose that there exists an invariant selection σ for the class \mathcal{G}^m of all games with rank $m'\omega + k$ for all $m' < m$. Now, for each $T \in \tilde{T}^u$, by Assumption 2, there exists $\sigma^{G^T} \in \Sigma(G^T)$ such that $\sigma^{G^T}(t) = \sigma^{G^{T'}}(t)$ for all $t \in T' \in 2^T \cap M \setminus \{T, \emptyset\}$, where T' is necessarily of rank $m'\omega + k$ for some $m' < m$ and k and hence $G^{T'} \in \mathcal{G}^m$. Pick σ^{G^T} as the solution at G^T . As in the proof for finite ranks, since the intersections of disjoint type spaces are always of a lower rank, the solutions constructed this way yields an invariant selection for games with rank up to $m\omega$. \square

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