An Arithmetic To Algebra Transition: Using metaphors to overcome arithmetic barriers to understanding of mathematical problems involving letters

by
Richard C. Carter

Submitted to the Department of Brain and Cognitive Sciences in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Psychology and Education

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ABSTRACT
An experimental study was carried out in which sixth grade students were introduced to the use of letters in mathematical problems through a curriculum based on the metaphor of a bag of marbles standing for an unknown value. Students understanding of letters was tested before and after the treatment using a four level framework of the understanding of letters developed by Katherine Hart. The hypothesis was that the marble bag curriculum would move students to Hart's level three performance; understanding letters as "specific unknowns". Students in the experimental group did improve significantly more than a control group, but did not reach the level of understanding letters as specific unknowns. These results led to the identification of incongruities in Hart's framework and the proposal of a new framework based on "an arithmetic frame of reference." The arithmetic frame of reference is used to resolves some contradictions in Hart's analysis, to identify alternative sources of difficulty of problems with letters, and identify changes that must occur for students to solve problems at different levels.

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Dedication

I would like to dedicate this thesis to my parents, George and Shirley, for the gifts of learning I received from them and to my children, Abby and Adam, for the learning that I hope will grow out of work such as this.

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Biographical Sketch

Richard Carter received his BA from Vassar College in 1970, and his Med from Lesley College in 1980. He is currently a scientist with the Educational Technologies group at Bolt Beranek and Newman. Previously he directed the Lesley College Educational Computer Lab and taught in Lesley's graduate program in Computers in Education. With BBN's Educational Technologies group he has been involved in developing educational software, and materials for teacher enhancement. His current research focuses on exploring ways to help teachers use technology to support an inquiry approach to the teaching of mathematics. Mr. Carter is currently principle investigator for the National Science Foundation funded project Empowering Teachers - Mathematical Inquiry Through Technology.

Selected Publications by Richard Carter

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Part 1:
Using the Marble Bags Metaphor for Letters.

Introduction: Need And Problem Statement

It is an often bemoaned fact that high school students in the United States do not learn algebra very well (Freudenthal, 1973; Wagner, 1984; Demana, 1988). For many students, in fact, algebra is the "end" of mathematics. A common experience is summed up by one student who said, "I did fine in math until I hit that X!"[1] Teachers also feel a lack of success with teaching algebra. One teacher described his ten years of teaching algebra as trying to "stuff equations into the skulls of students for whom it meant absolutely nothing"[2].

Why do many students who may do quite well in math up to algebra have such a difficult time with it? Several authors (Davis,1978; Matz,1980; Kieran, 1982) suggest that there are a series of ways of dealing with numbers, operations, and mathematical expressions introduced in algebra that are fundamentally different from the ways students have been used to dealing with numbers in standard elementary school arithmetic.

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[1]Reported to me by a colleague, John Richards. He heard it when he was interviewing undergraduates about their experiences with mathematics.

[2]Phil Lewis, a veteran high school algebra teacher, at Lincoln Sudbury High School outside of Boston.

1. One of the most agreed upon differences is that elementary students have a very different idea of equality (the meaning of the equals sign) from the one used in algebra. Marilyn Matz (1980) and Bob Davis (1978) both characterize elementary students' understanding as a "process-results" idea of the equals sign (see also Herscovics (1979) and Ginsburg (1977)). That is, the equals sign sets up a situation where students expect some process to be represented on the left side and a result of that process to be on the right side (their view of an "equation" is that you need to carry out the process on the left to get a numerical result on the right).

Behr et al. (1976) and Kieran (1981a) describe children's interpretation of the equals sign as a "do something" signal. For most middle school students the meaning of the equals sign is that some operation needs to be done to produce some result (e.g. 3+5=?). When Behr asked sixth graders about the meaning of equations such as 3=3 and 4+5=3+6 they typically responded by transforming the problems into standard arithmetic form (e.g. 1+2=3 or 4+5=9) so that the equals sign took on the "do something" role. He suggests, and Kieran's work confirms, that this view of the equals sign follows children into the upper grades. Kieran (1980, 1981) describes older students who had difficulty in defining the equals sign in terms other than "the answer" and in interpreting expressions without an equals sign (e.g. when asked about 3*a*+5*a* a typical response was; "*a* means nothing...there is no equal sign with a number after it.")

From experience with elementary arithmetic students learn a kind of frame: *# operation # = result.* They view the equals sign as just a signal to do the operation on the left to get a result on the right. And they have difficulty when they are confronted with problems that do not fit this operational frame.

One of the first places this arises in traditional algebra courses[3] is work with equations. Kieran (1982) gives a series of

[3]It should be noted that there are a number of researchers and math educators who are questioning the centrality of equations as the basic object of algebra and are suggesting alternatives such as functions (Schwartz 1993, Fey, 1991).

examples of students' misinterpretations of equations that can be traced to a misunderstanding of the nature of the equals sign.

To succeed in a traditional first year algebra course, an equation needs to be viewed quite differently. First of all, an equation in which the process is on the right is as legitimate as one where the "process" is on the left, and, in fact, either side of an equality can be made up of a variety of "processes", operations, or terms.

In algebra the equals sign can also be used in a variety of different contexts. For example, Matz (1979) differentiates tautologies from what she calls "constraint equations." In a tautology the equals sign is used to show that one expression has been transformed into another e.g. 2(x+3)=2x+6. In a tautology any number can replace the letter. In a constraint equation the equals sign is only valid for specific values of the letter (e.g. $3x+4=4x$).

2. A second difference between arithmetic and algebra: algebra uses letters and arithmetic does not. Both the idea of unspecified values and the use of letters is new to most elementary students. Hart and her associates (1981) have described a variety of meanings students give to letters. And both Hart's group and Kieran (1981) have documented a variety of difficulties in Algebra that arise from non-canonical interpretations of letters in mathematical expressions.

3. A third agreed upon area of difference is the syntax of mathematical expressions. Algebra involves a set of complex syntax structures that, although allowable in arithmetic, are rarely used before the introduction of algebra. Beyond the use of letters Matz (1980) outlines some aspects of the complex notation that is new to most students. The syntax includes parentheses, the fraction bar, the multiple use of concatenation (e.g. $3X$, $4(X+3)$, 4 3/4), the multiple use of the plus and minus signs (e.g. $2(3x)$, $3(x-4)$, -5), and issues of order of operations (e.g. $23+3(x+5)$). These are all new and potentially conflicting ideas when viewed from the perspective of the notation of elementary arithmetic. All of these notational differences have been shown to create difficulties for many students (Matz 1980). Students' broad

ranging misinterpretations of concatenation are particularly well documented (Kuchemann,1981; Wagner, 1981; Chalouh, 1983). Kieran (1981) has also done some work on the order of operations, and Herscovics (1980) has touched on similar difficulties encountered by his students. Although some of these problems can be dealt with independently from the use of letters[4], most of them really only become issues in the curriculum once letters are introduced.

4. In addition, Matz points out that pre-algebra students most often experience math as the application of some well defined (and predefined) step by step procedure to get a numerical result. For most elementary school students math is made up of a series of recipe like algorithms. In Matz's words: "adjusting to algebra involves having to compose a plan for solving a problem as well as carry it out. Furthermore, plans are often pieced together on the fly: the next steps in a plan are at times determined only after seeing the results of the previous step" (Matz, 1980 p. 145)[5].

In elementary school, solving math problems involves a recipe like matching of an algorithm to the problem, whereas in traditional algebra it is at times a case of figuring out how to apply an old rule to a new expression with no exact match[6].

The following list summarizes the new ideas that students must confront:

1. a new view of the meaning of the equals sign (and seeing that it can have several meanings depending on the context),

[4] An example is that Elementary students often interpret operations in a left to right sequence, ignoring the order of operations conventions of mathematics.

[5]Please note that here and in the following discussions the term "algebra" refers to the traditional equation based approach to algebra that focuses on symbol manipulation and solving for an unknown.

[6]I believe that an important step in improving students' ability to deal with algebra and mathematics in general will be to move away from an elementary school curriculum and pedagogy that leaves them feeling that arithmetic is simply a process of memorizing and applying recipe like procedures.

2. the idea of a variable in its many levels of complexity,

3. a new set of rules for interpreting mathematical expressions,

4. the need for a new approach to solving mathematical problems.

What becomes clear from this brief overview of the transition from arithmetic to algebra is the multifaceted nature of the change. The change is not based on one or even a few ideas. The change involves a whole set of transformations in what it means "to do mathematics" and as such it is not surprising that students experience difficulties when entering the world of algebra. One important route to making headway in trying to help students understand algebra is developing a much better understanding of the nature of the conceptual changes that must occur in all these areas. A thorough investigation of students' views in each of these ideas is needed. This should include investigations of how each of these ideas changes as students' mathematical understanding develops. In areas where the change seems to be difficult some experimentation needs to be done to find what techniques might be useful in supporting student development. In my experience it is experimentation with new approaches that often leads to useful reformulations of the nature of student difficulty. This thesis focuses on an experiment with an alternative approach (using "concrete metaphors") to the introduction of letters in a traditional algebra context.

I originally chose the introduction of letters as the focus of this study because of previous groundwork done on children's interpretation of letters (Hart 1981) and because of the central role that letters seem to take in the introduction to traditional algebra. It is the introduction into the curriculum of letters standing for unknown values (often misnamed "variables") that seems to precipitate the need for the new syntactic and manipulation rules outlined above as well as the new definition of the equals sign.

Previous researchers (see below) have suggested the change from arithmetic with numbers to solving problems that involve letters is simply a matter of applying the familiar rules and ideas of arithmetic to these new entities. In the course of this work it became clear that when sixth grade students encounter standard problems involving letters there is a set of fundamental changes in the way they view, experience, and organize their thinking about solving symbolic problems that needs to occur. They need to develop a new frame of reference for thinking about solving these problems. In the first part of the thesis I will report on the results of providing sixth grade students with an alternative curriculum for introducing problems with specific unknowns and in the second part I will propose a new way of looking at what happens when students confront dealing with problems that include letters based on the idea of an "arithmetic frame of reference."

Before exploring previous research on student understanding of letters in mathematical problems some clarification of language is necessary. Letters can be used in several ways in mathematics. The major differentiation relevant for this study is between letters used as specific unknowns and letters used as true variables. Both Hart and Harper (see below) make this differentiation, but use different terms and contexts to talk about them. Using letters as **specific unknowns** refers to using letters in problems where the letter stands for a specific, but unknown number (e.g. $3x+7=25$). The goal in most of these problems is to find the value of the letter ($x=9$ in this example). Another use of letters is in problems where it is expected that the value of the letter can and will vary (e.g. If $3x+4y=24$ for what values of x will y be greater than 1?). In this case we will talk about the letter being used as a **variable**.

Previous Research on Student Understanding of Letters

How do students who have not had any contact with letters in their arithmetic experience interpret letters when they are introduced in a mathematical context? Chalouh (1983) undertook a series of experiments with students who had never used letters in arithmetic. He presented them with expressions containing letters and asked them what they thought they might mean. His results show that students who had not been introduced to letters did not naturally treat them as if they stood for unknown numbers. Instead students interpret them in a variety of ways that were already familiar to them (e.g. headings on an outline).

How do students who have been exposed to letters in their mathematics classes interpret them? Eon Harper (1987), in his historical analysis of the idea of variables in mathematics, traces the development of the use of letters from the time when they were first introduced into mathematics by Diophantus (circa 250 A.D.) to the development of their more modern meaning by Vieta (seventeenth century). In Diophantus' time letters in mathematics were thought of as having only specific referents (in our terminology this is using a letter as a specific unknown). In contrast, Vieta viewed letters as something that could stand for any number and could be manipulated as an independent system not tied to any real world referent (in our terminology using a letter as a variable). Harper describes the difference in these two meanings of letters as follows:

> "At the turn of the 16th century Vieta introduced into mathematics the symbolic number concept. This innovation gave rise to two distinct language systems - arithmetic with letter appendages (in which the letter is interpreted as a classical unknown awaiting the discovery of a numerical content) and symbolic formalism (in which the letter is used as a symbolic number having the status of a mathematical object in its own right)" (Harper,1979 p. 238).

To clarify this difference he gives the following example from Hilbert:

a statement such as
$a+1=1+a$
can have two distinct meanings:
(a) it can be a "hypothetical judgement' about a particular (unknown) counting number '*a*'; and

(b) it can be a 'general' statement to the effect that all counting numbers commute with '1'.[7]

The focus of Harper's research is to see whether a shift similar to the movement from Diophantus to Vieta occurs for students: a movement from seeing letters as standing for specific referents to seeing them as standing for any number and able to be manipulated as part of a more general system. For Harper it is this shift to symbolic formalism, as he calls it, that is truly algebraic. The implication is that there is no significant change in view until students can see letters as a basis for "general statements." For Harper, early algebra, where letters are first introduced, is not a significant part of this development. He describes it as "arithmetic with letter appendages" (Harper, 1979 p 238). The implication is that you are just tying letters into or attaching them onto the students' arithmetic system with, from his perspective, no significant change. The letters are only "awaiting the discovery of their numerical content." Here Harper is making a differentiation between using a letter as a place holder for a specific number (a specific unknown), which, from his perspective, does not involve an significant cognitive change, and using letters to make generalizations about the number system (e.g. a+1=1+a for any number, a variable), which he believes does involve a significant cognitive change.

Katherine Hart headed a large research project in England on students' understanding of mathematics. While Harper explored the match between the historical development of algebra and the development of current students' understanding, Hart's group took a different approach and tried to look at how students actually interpreted letters in a broad range of problems. Through its early research the group came to the conclusion that students do interpret letters in quite different ways. Instead of Harper's two ways Hart came up with six different meanings that students can assign to letters in mathematical expressions. From these

[7] Note that a third perspective is also possible: the = sign is an interrogative asking under what conditions two functions are equal and a+1=1+a is just a special case with a tautological answer (Schwartz - personal communication).

differing interpretations Hart and her colleagues identified four levels of understanding of letters.

Level Four: "letter variables"

The highest level (which they call "letter as variable") involves the ability to describe or determine a relationship between two variables. (e.g. being able to determine, given the equation $5b + 6r = 90$, such things as "as *b* increases, *r* decreases"). At this level "a letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such values" (Hart, 1981 p. 104). This highest level has certain parallels to Harper's idea of arbitrary number in that you are looking not at the specific values of the letters, but at the relationships between them regardless of their value.

Level Three: "letter as generalized number"

Hart's next level involves an understanding that a letter may take on many different values. Here is a problem that Hart claims demands this level of understanding:

Problem #16:

16. What can you say about *C* if $C+D=10$ and *C* is less than *D*?
[answer: C < 5 or, if limited to counting numbers, as all students did, C could equal 0,1,2,3,or 4]

Although *C* could have any value less than 5 many students answer with just a single value. The idea is that, in order to answer this question appropriately, students must understand that in some algebraic expressions a letter can take on several possible values[8]. This idea of multiple values is also implicit in the idea of a letter used in a function.

[8] See the section on level three problems in part 2 for a look at how some students' responses to this question changed when they were interviewed.

Level Two: Letter as "Specific Unknown"

Hart's group goes on to suggest that in order to reach these upper levels of understanding, a student must first understand a letter as what she calls a "specific unknown." Hart explains that to understand letters as "specific unknowns," students must first be able to see a letter in a mathematical context (e.g. an expression, equation, or diagram) as standing for a number and to treat that letter as if it were a number. In order to treat a letter as a number, one thing students must be willing to do is to operate on it (to add, subtract, multiply, or divide it) just as they would with a number. To quote Hart: "Children (must be able to) regard a letter as a specific but unknown number, and can operate on it directly" (Hart, 1981 p. 104). Hart suggests that at least this level of understanding is necessary for students to meaningfully participate in algebraic work.

Level One: Non-algebraic Interpretations of letters

Hart's group also found that many students interpret and deal with letters in ways they describe as non-algebraic. The last three interpretations of letters she identifies fall into this category. For example, some students simply ignore letters, while others treat them as if they were numbers by simply assigning them a value even when it is not appropriate. Finally, in some problems students treat letters as though they were labels for objects (e.g. 3*b* might be thought of as 3 bananas).

It should be noted that work by Lockhead and Clement (Clement, 1982) suggest that the problem of treating letters as labels or objects may extend well beyond the elementary school students that Hart studied. Lockhead found that even students at the college level responded to algebra word problems by using letters in a label-like way (e.g. given the task of writing an equation that showed that in a particular college there were 6 times as many students as professors many college level subjects wrote 6s=p).

Another point worth noting here is that although Harper's and

Hart's groups approach the study of letters in quite different ways, both suggest that in the early stages letters are really part of, or an add on to, arithmetic. Harper calls them arithmetic appendages and Hart suggests that the task of understanding letters as specific unknowns is really just learning to treat them as one treats numbers.

To summarize: the work of these researchers suggests that students must make a transition from seeing letters as non-mathematical entities (e.g. outline headings or labels), to treating them as though they have mathematical significance (e.g. they stand for numbers), to treating them as mathematical entities on which one can do operations (specific unknowns in Hart's terms), and eventually to treating them as full variables that can take on different values and be used to express generalizations.

Hart's Analysis: How Different Problems Involving Letters Demand Different Levels of Meaning

In this study I focus on the early parts of this transition: moving from the first exposure to letters to seeing them as operable mathematical entities. As a first step in trying to understand this transition we will examine some sets of problems that Hart uses to explain the different early interpretations students can give to letters. Hart argues that not every problem involving letters demands an algebraic (specific unknown) level of understanding. In her analysis a number of problems that involve using letters to stand for numbers can be solved without demanding that a student directly and specifically deal with the letter as if it were a number.

Hart's first criterion for algebraic understanding (treating a letter as a *specific unknown) is that in order to solve a problem students* must be willing and able to operate on the letter itself as they would a number. In Hart's examples we can compare problems that she suggests can be solved without such an interpretation with those that Hart argues demand a "specific unknown" interpretation of letters. For example, Hart (1981) explains that the following problem does not demand a "specific unknown" interpretation of letters because students do not need to do arithmetic operations on or with the letters to solve it:

Problem #5a:

If $a + b = 43$
$a + b + 2 = \ldots$
[answer: 45]

Although problem #5a may seem to involve two "unknowns", it can be solved by a simple matching procedure which focuses on the +2. Students need only note that the left hand side of the second equation differs from the first only by the +2 and they can then simply add the 2 onto the 43. Essentially it is possible to solve this, she suggests, by eliminating the letters from the calculation.

She then compares this problem with the following:

Problem #5c:

If $e + f = 8$
$e + f + g = \ldots.$
[answer: 8 + g]

In order to solve this, she suggests, students must operate with "*g*" to produce the answer 8+*g*. In the previous problem, #5a, all they had to do was add 2 onto 43 while here they have to add *g* onto 8. Hart seems to suggest that in this problem they have to be able to treat "*g*" as if it were just like the 2 that they add onto 43, and, it is this, treating *g* as if it were a number, that so many of the students found impossible to do. Evidence for the difference between these two problems is found in the fact that 97% of 14 year olds answered the first problem correctly, while only 41% of the same group answered the second problem correctly.

Another example can be seen in the following sequence:
First a problem that uses a letter, but demands little more than number fact knowledge:

Problem #6a:

What can you say about *a* if
$a + 5 = 8$
[answer: a=3]

Here we have a letter (an unknown) in the problem, but, Hart found, most students solve this type of problem by simply remembering a number fact or counting from 5 to 8. To solve it in this way no operation need be done with or on the letter.

Next a problem that only demands giving the letter a numerical value, what Hart calls directly evaluating the letter:

Question #11b:

What can you say about *m* if:
$m = 3n + 1$
$n = 4$
[answer: m=13]

Hart notes that empirically this problem is more difficult than the previous one, but, because the value of *n* is given ($n=4$), the value of *m* can be calculated directly and again no operation needs to be done with the letter itself as a letter.

Finally a related problem in this sequence that she feels demands the letter itself be operated on:

Problem #14:

What can you say about *r* if:
$r = s + t$ and
$r + s + t = 30$
[answer: $r=15$]

Hart suggests that in order to solve this students must be willing to replace $s+t$ in the second equation with r from the first. In order to do this students must be able to think in terms of operating on the letters themselves as if they were numbers (the value of r is equal to the value of $s+t$).

Again students' performance suggests that this last problem is much more difficult than the previous ones (correct responses for the above three problems for 14 year olds were 92%, 62%, and 35%).

A third sequence shows further examples of the differences Hart suggests exist between the need to interpret letters as "specific unknowns" and situations that can be solved without that understanding.

> Problem #13d[9]:
> **Simplify:**
> **$2a + 5b + a =$**
> **[answer: $3a+5b$]**

Hart describes how this problem can be solved by thinking of the letters as standing for objects (as labels or a kind of shorthand for the names of objects) such as *a* for apples and *b* for bananas (or even as objects in their own right: 2 "*a*"'s plus 5 "*b*"'s). In solving this a student might say 2 apples plus 5 bananas plus 1 apple.

The next two questions are more difficult and do not fit so easily with the "letters as objects" interpretations:

> Problem #13h:
>
> **$3a - b + a =$**
> **[answer: $4a-b$]**
>
> Problem #13e:
>
> **$(a - b) + b =$**
> **[answer: a]**

[9]The example problem in this set was 3a + a = 4a

Hart points out that the idea of 3 apples (or *a*'s) take away one banana (or one *b*) makes little sense to these students. These problems were again significantly more difficult than the previous one that allowed an object interpretation of the letters. To make sense out of these last two, according to Hart, students must be able to interpret the letters as standing for unknown numbers rather than objects.

Another problem set that Hart feels illustrates the difference between problems that demand the use of the idea of a "specific unknown" and those that do not:

Problem #4a:

Add 4 onto
$n + 5$
[answer: $n + 9$](68%)

As in some of the above problems Hart suggests that this question can be answered by simply ignoring the letter. Note the subtle difference between this and the following problem:

Problem #4b:

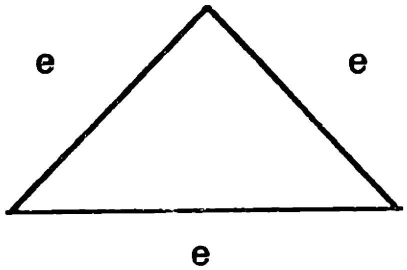
Add 4 onto
$3n$
[answer: $3n + 4$](36%)

Hart suggests that students who answer this problem $7n$ (or just 7) are unwilling to treat the *n* as an unknown number and feel that they must do some kind of operation and come up with a numerical result.

A final comparison of problems that do not demand students operate with letters with problems that do demand operations with letters:

Problem #9a:

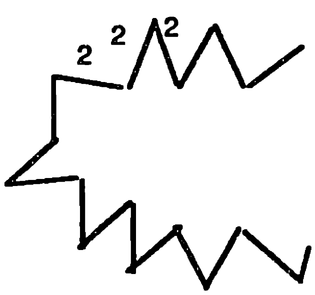
What can we write for the perimeter of each of these shapes?



P = ...
[answer: 3*e*] (94%)

Hart suggests that this problem can be solved by interpreting the *e*'s as labels for the sides rather than as unknown values representing the length of the side. Many students who got this right got the following problem wrong.

Problem #9d:



Part of this figure is not drawn.
There are *n* sides altogether, all (sic) of length 2.
P = ...
[answer: 2*n* or *n*2] (38%)

Hart points out that the form of the answers to #9a and #9d are exactly the same, but that in #9d you cannot use *n* as a label for the sides. The letter *n* is explicitly described as a number and it must be operated on to solve the problem. Hart reports that a sizable number of students gave straight numerical answers suggesting that they were unwilling to deal with *n* as a letter that could be operated with (multiply the length of each side (2) by *n* (the number of sides)) when its value was not known[10].

In summary there are many problems involving letters that can be solved without a student necessarily understanding Hart's idea of a specific unknown. To understand letters at the level of a specific unknown a student must see that a letter stands for an unknown number and that arithmetic operations can be performed on or with the letter as they can with numbers. In some problems the status of the letters can be ignored entirely (#5a), in some students can simply replace the letter with a number and never have to operate with the letter at all (#11b), and in some problems the letters can be treated simply as labels or objects (#13d). At the same time there are some problems that Hart argues do demand that students interpret letters as specific unknowns (e.g. #5c, #14, #13d, #4b, #9d).

Hart (1981) argues that in order to deal with algebra effectively students must at least be able to interpret or understand letters as specific unknowns. She also points out that not all problems that use letters demand that students interpret the letters as specific unknowns. But, she argues, it is possible to create problems that do demand that the letters be interpreted as specific unknowns. In order to evaluate students' ability to deal with letters, using problems like the ones described above, Hart's group developed the Chelsea Algebra Test (Brown 1984). It is made up of some 23 questions (See appendix C). Hart's group has tried the test in

[10] Another way to view the difficulty of #9d is in terms of the extensive and intensive quantities. Schwartz (personal communication) has suggested that students limited repeated addition definition of multiplication (ignoring issues of intensive and extensive quantities) allows them to do #9a while getting in the way of doing #9d. The difference in difficulty between the problems stems from the fact that the extensive quantity in #9a is known by value (3), whereas in #9d the extensive quantity (n) is not known by value, but only by name. Students need to know the value of an extensive quantity to think in terms of doing repeated addition.

England and their analysis of the performance data suggests that in most cases problems that demand students operate with letters are in fact more difficult than the ones she argues do not demand the use of the specific unknown meaning of letters. She and her co-researchers set up a scoring method for assigning levels based on performance on specific problems. Based on a statistical analysis of difficulty they made three clusterings of problems which they labeled level one, level two, and level three. They argue that problems which, in their analysis, demand that students interpret letters as specific unknowns are at their level three.

How do students perform on the test?

The youngest group they analyzed were 13 year olds (8th graders) and only 17% of them achieved level three, where all of the problems demanded interpreting letters as specific unknowns. This may not be surprising as the material in most text books for that age deal only informally with letters used to represent unknowns. The most surprising data is from the older students: of the 15 year olds (10th graders), who have all been formally introduced to work with letters as specific unknowns, only 40% achieve level three. The implication is that more than half of the students who were taking algebra were unable to solve problems that demanded they treat or interpret letters as specific unknowns.

From Hart's data it appears that students can understand the idea of a letter standing for a specific number and most can solve problems where this is all that is demanded of them (where they can easily determine the numerical value of the letter). But in problems that demand that the letter be treated as a specific unknown many students interpret letters inappropriately. For example they seem to ignore them, treat them as labels, or give them an inappropriate value.

Textbooks and the Introduction of Letters

How do text books treat the introduction of letters? Here is a typical example of how beginning algebra text books first introduce letter variables: "a variable is a symbol that is used to represent one or more numbers" (Dolciani, 1989; Jacobs, 1982 is quite similar). This appears to be a relatively simple idea. Students are then introduced to evaluating expressions. They are given an expression with a letter in it (e.g. $3x+7$), a value for the letter ($x=4$) and are asked to evaluate the expression (e.g. if $x=4$ what is the value of $3x+7$). Then, within the first 50 pages, students are typically asked to operate with the letters in a variety of contexts. For example, solve equations in which they must operate on a letter e.g. find the value of *f* if $(51+g)16=80f+16g$. In this example they must do things like multiply *g* by 16 or subtract $16g$. They are also asked to write expressions that involve operating with letters e.g. "write an expression for "the number of minutes in s seconds" (Dolciani, 1989). They are also asked to simplify expressions such as $-3v+(-1/2\ t)+2\ 1/2\ v+(-1/2\ v)$ (Dolciani, 1981). All of these tasks involve doing operations on or with letters. There is a very rapid transition from having students evaluate expressions (where they are given a value for a letter) to solving problems that involve what Hart describes as operating with or on the letters.

Harts' group's data suggests that more than half of the students who study this kind of material have not reached a level where they understand letters well enough to successfully negotiate problems that demand that letters be treated as numbers and can be operated upon. Given this finding and the standard introductory algebra content it is not surprising that many students have trouble with algebra.

Summary And Statement Of Research Focus

In trying to understand student difficulty in making the transition from arithmetic to algebra previous research on students' understanding of letters has laid some important groundwork. For example, Hart's and Harper's work on identifying the variety of different understandings students bring to problems that involve letters, and their preliminary analyses of the kinds of understandings demanded by different problems.

In my view both Hart and Harper inappropriately suggest that significant cognitive change only occurs after students can treat letters as specific unknowns. They both seem to suggest that the first steps in understanding the role of letters in mathematical problems involves simply applying arithmetic understandings to problems with letters. Based on the experimental work described in the first part of the thesis I have come to see that these first steps are much more complex and it is exactly the application of an arithmetic perspective that creates much of the difficulty students experience with letters. I also believe it is because of this short sighted view that none of the previous researchers addressed the question of what might help students make some of the early transitions from demanding that letters be replaced by numbers or treated as objects to a more flexible view of letters as independent mathematical entities that one can operate with mathematically.

In this thesis I will address both of these issues: What approaches might help students develop a more successful understanding of letters, and why are some problems that involve letters so difficult for students? In trying to analyze student difficulty in the second part of the thesis I will focus particularly on the role of the arithmetic understandings students bring to problems that involve letters.

A New Approach to Introducing Letters in Mathematics Classes

-The Role of Prior Knowledge

The work of Hart's group suggests that part of the problem with students learning algebra may be the failure of standard algebra curriculum materials to embrace the perspective that a "variable" can have many meanings to students. Katherine Hart comments:

> "The blanket use of the term variable in generalized arithmetic is a common practice which has served to obscure both the meaning of the term itself and the very real differences in meaning that can be given to letters (by students)" (Hart, 1981. p. 110).

My first question is how might curricular materials take account of students' differing perspectives? To answer this we need to explore what it is that might interfere with students treating letters as specific unknowns and consider what might help them maintain letters as standing for unknown quantities in these situations. I take a constructivist perspective here. Student learning is highly dependent on the ideas that they already have in their heads (Driver, 1978), new learnings are built from these current understandings. In introducing new ideas to students we must be cognizant of and take account of the ideas that they already hold.

In science education there has been a focus on the role of previous knowledge in learning and an interest in the role of analogies in helping students come to understand new ideas (Brown and Clement, 1989. Clement, 1994). Analogies are seen as a way of connecting current knowledge with new knowledge. In their paper David Brown and John Clement note that:

> "Students' prior knowledge has been increasingly recognized as playing a crucial role in learning. According to this view, prior knowledge determines the meanings derived from instruction, and teaching which does not build on existing knowledge and understanding will fail to produce meaningful learning" (Brown and Clement, 1989 p. 2).

In the case of letters in mathematical expressions it seems clear from Hart's data that many students have failed to derive appropriate meaning from their mathematics instruction about letters. Is it their prior knowledge that is interfering with an appropriate understanding of letters?

Brown and Clement have found in their own work with physics students that prior knowledge is often a source of difficulty:

> "Within the past decade and a half there has been an increasing awareness of the detrimental effects (to school learning) of some of students' prior knowledge. Students come to class with preconceptions which inhibit the acquisition of content knowledge and are often quite resistant to remediations." (Brown 1989, p. 4)

-Using an Analogy as a Bridge

If we begin with the hypothesis that students' arithmetic knowledge inhibits their understanding of letters what can be done?

Brown and Clement continue:

> "The use of analogy is often viewed as one of the primary means of drawing on students' existing knowledge. By activating relevant prior knowledge which is already understood by the learner, the analogy helps give meaning to incoming information" (Brown, 1989 p. 2)[11].

They have been testing an analogical teaching strategy which attempts to build on students' existing valid intuitions. The instructor suggests a case which she views as analogic which will appeal to the students' intuitions. They call such a situation an <u>anchoring example.</u> The situation to be explained by analogy is

[11] For a literature review and other references see Brown, 1989.

called the target, and the better understood analogous situation is called the base or anchor. The instructor then attempts to find a "bridging analogy" (or series of bridging analogies) conceptually intermediate between the target and the anchor. This can often be done by transforming the anchor to make it conceptually closer to the target.

How might we apply these ideas to students learning to treat letters as mathematical entities?

Brown suggests beginning with a metaphor that appeals to students' intuitions. If one of the spontaneous intuitive misinterpretations that students have is to see letters as labels or objects (Hart, 1981) it might be helpful to try starting from a model that uses an object, but an object that through a metaphor combines the idea of an object <u>and</u> an unknown quantity in one image.

<u>The Marble Bag Metaphor</u>

As our metaphor for letters we chose a bag with an unknown number of marbles in it. The idea of a marble bag with an unknown number of marbles in it is the anchoring metaphor. We needed to create an anchor that would appeal to student intuitions and that could be transformed into the target which in this case was letters (objects that represent an unknown value and can be operated on mathematically to generate mixed expressions).

The idea of a bag of marbles is intuitively clear to most children and they can differentiate between a known number of marbles outside the bag and an unknown number inside the bag. And, through a guess-the-number-game (described below), they can operate with the bags and the marbles. As a basis for moving toward standard symbolization students can easily pictorially represent the bags and marbles, expressions made up of them, and operations done with them (see below). Once students become familiar with these elements and manipulating them this anchor can be transformed to make it conceptually closer to the target. Bags can be represented by the letter *X* (see below), sets of marbles can be represented by numbers, and operations can be written explicitly. Pictorial representations of the results of actions with the bags and marbles can be transformed step by step into a mathematical record of the actions (x, $x+2$, $2x+4$....). Once this is done (and all students were able to do this kind of work) the students are working directly with letters and with writing expressions to represent mathematical operations on letters. **My hypothesis is that this experience can be transferred to other problems that use letters, but do not specifically refer to or derive from the marble bag metaphor.**

If students' previous ideas and practices in arithmetic interfere with their understanding of letters, a second aspect of the marble bag activities is that it gives letters a context of their own (which is different from the arithmetic context) where the letters can take on or maintain their own meaning. My hypothesis is twofold, first, that intuitions derived from standard arithmetic (e.g. that letters are not relevant to doing arithmetic, that operations can only be done with numbers, that all answers must be numeric) negatively influence students' responses to problems with letters and second, that the use of the marble bags ideas can provide a context in which solving problems with letters will not be swamped by those intuitions. The idea is to give students some alternate experiences (alternate to their standard arithmetic experience) to which they can refer when they see a letter. The image presented in most pre-algebra and algebra books is a letter stands for a specific number, and this interpretation works fine as long as you are just replacing the letters with given values. But, when you have problems that do not allow you to simply replace a letter with a number and transform the problem into a simple arithmetic form, when you are presented with a problem in which you must actually operate with the letter itself the replacement view of letters does not work. This replacement view does not provide any context in which students can operate on an expression without having to immediately turn it into numbers.

Our approach is to introduce the analogy of a marble bag in a context in which the idea of something standing for an unknown value can develop apart from the arithmetic context that may be overwhelming the introduction of letters. What would characterize this context?

It would be:

1. a context independent of the standard arithmetic context (because putting letters in situations that invoke that context may result in inappropriate interpretations),

2. a context in which an unknown value cannot so easily be immediately resolved into a number,

3. a context filled with problems where you operate on an unknown as natural and necessary steps in the problem,

4. a context which provides an alternative to the numbers only based operations of arithmetic. In arithmetic an operation like subtract means take two numbers and do a calculation; either retrieve a memorized fact or carry out some numerically based algorithm. A common student response to a problem that involves multiplying 3 times *a* is, "how can I multiply them when I do not know the value of *a*" (Matz, 1979. p. 37, see also Davis, 1978). What we need is an alternative model for doing operations with letters and so we look for a model that will allow us to provide a different picture of what doing an operation means.

We created such a context by introducing our marble bag metaphor through a guess the number game where students drew a picture of a marble bag to represent an unknown quantity and made diagrams of bags and marbles to represent the results of operating on the unknown quantity.

Our goal was to provide students with some "transitional objects" that would help them make the bridge between the concreteness of standard arithmetic (in general you can connect ideas of arithmetic to operations with concrete objects) and the abstractions of traditional algebra (an unknown is an abstract and unknown entity, in traditional algebra you must deal with ideas like multiplying a number by another when you do not know what one or both of the numbers are).

As mentioned above our major metaphor for specific unknowns was a marble bag - a bag filled with some unknown number of marbles. We proposed activities where students solved problems using actual bags filled with marbles (actually washers). We had them draw pictures of the operations they did and their results using collections of objects (marble bags and marbles), and slowly made the transition to using standard symbols (letters, mathematical symbols, and the syntax of algebra expressions). We moved back and forth between marble bag imagery and the symbolism of algebra. When students got confused using algebraic syntax we often referred back to the imagery of marble bags and marbles.

Below is a sample lesson taken from Preliminary Teachers Guide to the Logo Algebra Project (Carter, 1987) in which the idea of marble bags is introduced:

Logo Algebra Project
Module One: Equations: from marbles to algebra

Background to the module:

In this module students will be introduced to the algebraic manipulation of equations. This is done through a very concrete representation - pictures of marbles and bags of marbles. The bags stand for unknowns, they are closed and the number of marbles in a bag is not known. We begin with a guess my number trick. Students are taught how to do the trick by the manipulation of pictures of marbles and marble bags, and the drawing of marble bags and marbles to represent problems presented verbally. To solve the problems students must manipulate these drawings. Students generate their own marble bag stories and practice representing them and manipulating them with diagrams. Eventually standard algebraic notation is introduced as a shorthand way of recording the marble stories.

FOCAL IDEAS FOR THIS MODULE:

1. The idea of an unknown (introduced in Module one)
2. The idea of manipulating an unknown.
3. How to discover the value of an unknown.
4. Reversibility of operations (solving by undoing).
5. Recording mathematical ideas using a shorthand notation, first using pictorial images (marbles and marble bags), and then using algebraic notation.

OUTLINE OF STEPS IN THIS MODULE:

1. Do a guess your number trick with the students,
   show marble bag diagrams as a way to do the trick.

2. Students practice and make up their own number tricks using marble bag notation.
   Computer is used for making up number trick stories and for recording them via bag drawings. The computer will also be used for testing out their marble bag number tricks.

3. Algebraic notation is introduced as a quick way of writing down bag stories.
   a. introducing *X*, representing multiplication by concatenation ($3X$), and using parentheses to keep a record of actual steps in a series of operations: if an expression is written as: $3(x+2)$, you can tell that 2 was added to x and then the result was multiplied by 3, whereas if it is written as $3\ x+2$ without the parentheses you might think that x was multiplied by three and then two was added to that. This is motivated as an aid in the debugging of erroneous recordings of marble bag stories.
   b. exercises in moving from one representation to another.
   c. Using bag drawings to test algebraic equivalencies and solve equations.

Introductory Activities:

1. Begin by introducing students to the idea of "marble" stories and diagrams.

This is done by first playing a game with them. It goes something like this:

"I want to try to guess the day of the month of your birthday. Think of the number of the day of the month of your birthday (e.g. if I was born on the fifth of January, my number would be 5).

Do you have your number?"

"Now add 4 to it.... OK? " [e.g. of their calculation using 5 as their original number: 5+4=9]

(as you tell the story write down a marble bag representation (see below) on a pad of paper that they can't see)

"Now double that number...... OK?" [9*2=18]

(Note: material in square brackets ([]) on the right side of the text is a record of calculations that students or you might do.)

"Now subtract 6 from the number you have now. OK?" [18-6=12]

"Now tell me the number you have."
(student tells the result) [12]
(you do a quick calculation: [e.g. (12 -2)/2 = 5])

You then tell them what their birthday day number is. [5]

Do this again with another child if you wish.

Once students get the idea of what is going on, the next step is to teach <u>them</u> how to do it. You can approach this by simply doing the trick again, but this time making your drawings on the board and explaining what you are doing each time.

(What you will be doing is constructing a marble bag picture of the results of the operations you ask them to do on their number.)
(below "•" will stand for a marble and "&" will stand for a picture of a marble bag)

"Once again I want to try to guess a number you think of so think of the number between 1 and 20........

Do you have your number?"

(To begin draw:
& (actually draw a picture of a small bag)
explain that this is a bag full of marbles, the same number as the student is thinking of. We draw it as a closed bag because we do not know how many marbles are in the bag. The number of marbles represents the student's number. Again it is a bag with marbles in it, but you do not know how many.)

"Now add 4 to your number.... OK? " [e.g. of their calculation [using 5 as their original number: 5+4=9]

(you add a new line to your drawing with four additional marbles (•'s) point out that this now represents their current result:)

&••••

"Now double that number...... OK?" [9*2=18]

(the next line of your drawing will look like:
&••••
&••••
explain that you are doubling the number of marbles)

"Now subtract 6 from the number you have now, OK?" [18-6=12]

(Now you cross out six marbles:
&••ØØ
&ØØØØ)

"Now tell me the number you have."
(student tells the result) [12]

Write:
&&•• =12

and explain that this tells you that 2 bags and 2 marbles makes 12 (or whatever their result was).

Tell them the goal is now to get figure out how many marbles there are in one bag (the original number - point it out at the start of your diagram). See if you can get them to tell you how to do it: subtract two for the two remaining marbles on the left side of your diagram (see above) and subtract 2 from their result (in this case leaving 10) and then divide by 2 for the two marble bags. The result is that one bag must equal 5 marbles)

A next step might be to have them keep diagrams for a story that you tell and see if they can guess your number. If necessary you can do this in a step by step fashion: first posing the next step (e.g. now add 3 marbles), then having them try to record it, then putting it up on the board so they can check their work.
(end of excerpt from Preliminary Teacher's Guide)

In the above activities students are representing and manipulating a specific unknown using marbles and marble bags instead of a letter and algebraic notation. In other lessons students make the transition to standard algebra notation in a series of bridging analogy steps (e.g. using the letter *X* is introduced as a shorthand way to draw a picture of a bag). An outline of the complete curriculum can be found in appendix A.

Why might this approach be useful?

First, this marble bag guess-the-number motif begins independent of a standard arithmetic context. There are no operation or equals signs. The idea is to introduce an unknown without invoking the world of arithmetic, so that the unknown can remain an unknown and will not have to be turned into the numbers demanded by arithmetic. Secondly, since part of the problem (as described by Hart) is that students cannot operate on an unknown, we set up a context where the arithmetic idea that doing an operation means doing a calculation and getting a numerical result is not invoked. The marble bags activities of drawing pictures sets up an alternative model of doing an operation. At the beginning an idea like add

4 onto an unknown value is represented by simply drawing four more marbles in your diagram. For multiplying an unknown we start with familiar vocabulary like "double" and apply the operation to a picture of a marble bag and a few marbles. In this context doubling just means reproducing the picture a second time. Also taking away an unknown (a bag) simply involved removing (perhaps crossing out) a bag from a sketch of bags and marbles. The idea was to provide an alternative model for what doing an operation means based on familiar intuitions. Our hypothesis was that this would make it easier to integrate operations on and with letters into an arithmetic situation as we moved to standard notation through a series of bridging analogies. Students' work with letters would no longer be overwhelmed by the rules of arithmetic because they would have an alternative model and set of experiences to draw on.

Initially the marble bag imagery is an attempt to connect with students' intuitions. The metaphor speaks to one of the interpretations that Hart found students had of letters. Hart suggests that some students interpreted letters as standing for objects (3*a* meaning 3 apples). In this case 3*x* can be thought of as 3 bags, but is also a representation of the quantity of marbles in the bag. The idea was to include the ideas students seemed to naturally bring to the situation and add onto that the idea of a letter standing for a quantity (the quantity of marbles contained in the bag). My questions are; first, can students operate with letters in this context and second, can this experience be transferred to other problems that use letters, but do not specifically refer to or derive from the marble bag metaphor?

<u>SUMMARY</u>

Many students have trouble learning algebra. A central aspect of learning algebra has to do with using letters in mathematical problems. Hart's research group has found that less than half of the students they tested who were studying algebra had an understanding of letters they felt was necessary to do algebraic work. Hart's group found that students understand and interpret letters in different ways, many of which do not fit understandings demanded by algebraic work (interpreting letters a "specific unknowns"). Current algebra curriculum is almost totally symbol manipulation oriented and takes no account of students' differing interpretations of letters. Our goal is to help students move to an interpretations of letters that will help them master problems in algebra like those on the Chelsea Algebra test. The work of Brown and Clement suggests a good place to begin is the perspective students bring to work with letters that derives from their work with arithmetic. Based on Brown and Clement's work I hypothesize that a curriculum focused on materials that use familiar analogies (e.g. marble bags) and that provide a transition between manipulating and representing objects and dealing with purely symbolic materials will help students develop their understanding of letters[12].

[12]A difficulty with trying to develop new curriculum is there has been little work done that has attempted to understand students' interpretations of letters in a broader framework and to analyze the changes in students' interpretation of letters as they develop mathematically. Hart's work provides a useful beginning, but more work needs to be done aimed at deepening our understanding of how students interpret letters and how that understanding develops. Part 1 of this thesis focuses on the findings from an experiment with the Marble Bags curriculum. Based on the results of this work in part 2 I will propose and examine an alternative view of how student understanding of arithmetic effects their understanding of letters in traditional algebra problems.

Methods and Procedures

Choice of Experimental Group

In order to explore the effect of the marble bag imagery we[13] chose a 6th grade class. We did this because we wanted students who had not been exposed to the use of letters in their mathematics classes. We didn't want memorized rules and algorithms about letters to interfere. We wanted to explore what happens when students are first exposed to the ideas of letters through the use of the marble bags metaphors.

We chose a school (Hanscom Middle School in Bedford, MA) that services an Air Force Base because it provided a varied rather than a homogeneous population of students. The group we worked with was a 6th grade "computer" class that is a randomly chosen group constrained slightly by scheduling[14].

Choice of the Control Group

The control group was a heterogeneously grouped 6th grade math class tested and interviewed the following year (time constraints did not allow for all the necessary interviewing with a control group the same year as the experimental group and a computer class was not available the following year). The experimental group did not do any activities with letters in their math classes. The control group class was introduced to letters through problems with formulas such as rate times time equals distance ($r \times t = d$), and through solving ratio and proportion problems (e.g. if $3/n = 4/10$ what is the value of n?). This means that the control group had a standard introduction to letters that for most students occurs somewhere in the middle school years and the experimental group did not. The control group's teacher explained that she only

[13] "We" refers to the Logo Algebra Research Group which included Wally Feurzeig and John Richards (BBN), Nancy Roberts and Frank Davis (Lesley College), and myself.

[14] E.g. Because of scheduling our group excluded some students who were taking foreign language classes.

introduced letters in the sixth grade if the class got through the standard curriculum before the end of the year. It happened that this was true for the control group, but not for the experimental group. No data comparing the general or mathematical performance of the two groups was available. All that can be said here is the fact that the control group worked with letters in their math class and the experimental group did not makes their experiences more parallel than if the control group had had no work with letters at all.

Treatment

From January to June (approximately 20 weeks) the experimental group students participated in a "computer class" one of whose modules was the marble bags curriculum outlined above. The class met 4 times a week, each class was approximately 40 minutes in length. Appendix B contains a week by week outline of the material covered. The work based on the marble bag metaphors went on for 8 weeks.

Instrumentation

Katherine Hart's Chelsea Algebra test (Brown, 1984) was used to explore and evaluate students' understanding of letters. Hart's test involves a broad range of standard and non-standard problems that use letters in a variety of contexts (evaluating expressions, simplifying expressions, interpreting diagrams, writing formulas, solving equations, etc.). Hart's levels analysis and scoring system provided one way to evaluate the effect of the work with the marble bags metaphor.

From a statistical analysis of test results Hart's group derived several performance levels (Hart, 1981). They did this by grouping items on the test. The grouping was based on an analysis of level of difficulty so that success on a higher level predicts success on all lower levels. Mathematical coherence and similarity of like age group results were also used as criteria for forming the groups of test items. On the algebra test they formed four levels to keep it consistent with many other tests they had developed. In Hart's view "groups can be regarded as representing different levels of understanding of generalized arithmetic" (Hart, 1981 p. 113) (generalized arithmetic is an english term equivalent to the american term algebra)[15].

As Hart (1981) describes, level one involves problems that can be solved on a purely numerical basis. Where an expression needs to be written the amount can be determined by counting and/or treating the letter as an object.

Students who did not answer the level one problems correctly were assigned level 0.

Level two problems were more difficult than the level one problems. Some of them (3 out of 7) involved the need to write expressions with a "+" sign in them, one involved multiplying two letters, two involved evaluating equations with 2 letters in them, and the final one was a word problem that involved transferring a method from an example to a problem. Hart claims that none of these problems demand that students operate with letters.

Based on the Hart group analysis students who only succeeded at levels one or two but not level three did not understand letters as specific unknowns ("The items in levels one and two can all be solved without having to operate on letters as unknowns." (Hart, 1981 p. 116). Level three problems were the ones that demanded that students be able to treat and operate on letters as specific unknowns. Hart also specified a fourth level which, because none of the students reached it, will not be dealt with in this study. Hart's level four problems demanded that letters be treated as specific unknowns, but the problems were more complex and some demanded that letters be treated as true variables (the ability to understand and specify the relationship between two variables in an equation).

Students were assigned levels (0-4) based on the the highest level

[15] See appendix C and D for test items and level criteria used in this study.

at which they answered a criteria number of questions correctly e.g. students who answered 5 out of the 7 level two problems were said to have achieved level 2. See appendix E for specific test items and criteria used to determine each level.

Because of time constraints and because I was sure none of our sixth grade students would be at level four I cut out many of the fourth level questions from the original test (none of the students achieved level three in the pre-test and on the post test only one student answered one of the level four questions that remained on the test correctly).

The Chelsea Algebra test provided a tool to measure the change in student understanding of letters based on the most coherent analysis I could find at the beginning of the study.

Administration of the Test

Hart's procedure was followed in giving the test. The test began with a brief introduction in which the students tried some sample problems involving letters. Then the teacher went over the sample problems with the group explaining that letters can stand for numbers and that when a letter and a number are put together (e.g. $3a$) it means to multiply. These sample problems mostly involved substituting a number for a letter in an expression. e.g. "If $a=5$ what does $4+a$ equal? If $a=3$ what does $4a$ equal?". A copy of the practice items, practice item instructions, and the test itself are all included in Appendix D.

A second source of data is a set of paper and pencil test questions devised by an external evaluator for the Logo Algebra project (Roberts, 1987). The test consisted of two parts. Part one was a series of 12 questions that asked students to do marble bag based tasks to see if they had mastered the marble bags curriculum. A second set of 8 questions were standard linear algebra equations from simple (($5x+13=23$) to relatively complex (e.g. $4(x+10)/2 - 10 = 70$) in which students were asked to solve for x. The purpose here was to see how well the students could transfer the skills they had learned with marble bags to a standard algebra (non marble bag) situation. (See appendix H for a complete list of the questions used and the results).

Procedure

The students were given the 14 question subset of Hart's paper and pencil Chelsea Algebra test as a pre-test (pre-test P&P) before they began the class. The class started and during the next 2 months each student was interviewed and problems that he or she missed on the pre-test were gone over (Pre-test Interview). The interviewer probed to try to determine how the student was interpreting the problem and tried to see how various prompts or differing formats might effect a students' answer (e.g. If a student evaluated a letter inappropriately the interviewer might probe by saying: "You said that n=4. What could you write if we did not know the value of n," or the interviewer might suggest an alternate answer: "some students gave the answer 8+g to that problem, would that be correct?").

In the middle of the semester, at the end of the marble bags unit, the students took the external evaluator's paper and pencil marble bags test described above.

At the end of the semester students were again given the paper and pencil version of the test (Post P&P) with some additional questions that were attempts to make marble bag metaphor versions of the test questions (see appendix D), and, lastly, the students were subsequently interviewed one final time where they were given the problems they had missed again and the interviewer again probed to find the basis for the answers (Post-test Interview).

The following year the same testing and interview procedures were carried out with a similarly heterogeneously grouped 6th grade math class. They also had a computer class during the semester, but it did not involve any work with letters. As mentioned above they did some work with letters in their math class that the experimental group did not. Also, the control group did not take the external evaluator's marble bag problems test.

Experimental Design

To summarize, the experimental design is rather simple and straightforward. The experimental group was given a pre-test and interview, a treatment, and then a post test followed by an interview. The control group also was given a pre test and interview, was not given the the treatment experience, and did have a post test and interview. The question is, assuming the two groups were similar and performed similarly on the pre-test, whether the experimental group performed significantly better on the post test than the control group. At a more general level the question is what evidence of change in understanding of "letters" can be found.

The specific hypothesis is that the treatment, experience with a curriculum that involved using the metaphor of a marble bag for the abstraction of a specific unknown, will improve students' performance on a more generalized test of their ability to treat letters as specific unknowns and will improve it significantly more than a control group that did not have the treatment.

Results

Comparing Pre-Test Performances: no difference

The first thing we need to establish is that the two groups are indeed similar in their performance on the pre-test. Below is a table of control and experimental group Hart levels performance on the paper and pencil based pre test:

TABLE 1
Control and Experimental Group Pre-Test Levels

	Control n=19	Experimental n=16
Started at level 0:	1 (5.2%)	2 (12.5%)
Started at level 1:	14(74%)	11(69%)
Started at level 2:	4 (21%)	3 (18.75%)
Started at level 3:	0	0

(See Appendix E for test items and criteria used to determine levels)

To compare the two groups statistically I used a chi-squared test. The null hypothesis is that the two groups were similar in their levels performance on the test. A non-significant chi-squared (p=.747) indicates that statistically there is no significant difference between the performance of the two groups. This leads us to accept the null hypothesis that the two groups are statistically similar in their level performance on the pre test.

Comparing Post-Test Performances: a significant difference

Having established that the two groups are not statistically significantly different on their performance on the pre-test we can now ask if there was a significant difference in their achievement on the post test. Did the experimental group that worked with the marble bags curriculum change their performance in significantly larger numbers than the control group as my hypothesis suggests? The table below summarizes the changes that occurred based on Hart's levels.

TABLE 2
Summary of results of level based performance on pre and post tests

	Control (n=19)	Experimental (n=16)
Level 0 to Level 1:	1 (5%)	2 (13%)
Stayed at level 1:	11(58%)	3 (19%)
Level 1 to Level 2:	3 (16%)	7 (44%)
Stayed at Level 2:	3 (16%)	1 (6%)
Level 1 to Level 3:	0 (0%)	1 (6%)
Level 2 to Level 3:	1(5%)	2 (13%)

We can now organize this information to compare the number of students who changed levels in each group. This is done in the following two tables.

TABLE 3
Comparison of number of students who made level changes from the Pre to the Post Test.

Change	Control (n=19)	Experimental (n=16)
up one level:	5	11
up 2 levels:	0	1
Total	5	12
Percent change	26%	75%

TABLE 4
Summary of Improvement in Control and Experimental Groups

Change	Control	Experimental	
Improved	5	12	(improved means moved up at least 1 level)
No Change	14	4	

If we again take as the null hypothesis that the two groups are the same, a significant Fisher's exact test (p=.0067) leads us to reject the null hypothesis and say there is a significant difference between the control and the experimental groups on the number of students who moved up at least one of Hart's levels.

Is the Difference Based on Control Group Movement to Level 3?

What is the basis of this difference? Is it movement to level three, the level that Hart argues students need to reach to prove they understand letters as specific unknowns?

-Achieving Hart's Level 3: a difference, but not reaching significance

As can be seen in table 5 no students in either group achieved level three in the pre test. In the post test 3 of the 16 students in the experimental group achieved level 3 and only 1 member of the 19 control group students achieved level 3. There is a difference, but using a Fisher exact test it does not reach significance.

TABLE 5
Number of subjects who achieved level 3 performance on the chelsea algebra test

	pre test level 3	post test level 3
Experimental Group (n=16)	0	3
Control Group (N=19)	0	1

	Exp grp n=16	Ctrl Grp n=19
did not achieve level 3	13	18
did achieve level 3	3	1

The null hypothesis is that the two groups are the same in terms of achieving level three. A non significant Fisher's exact result (p=0.312) leads us to accept this null hypothesis. Although the experimental group performance is better the difference in achieving level three is not statistically significant.

-Overall Number Correct on Level 3 : different, but again not significant

Another way to look at this is to examine the change in the overall number correct on level three questions. Table 6 details the changes for each group.

TABLE 6

Number Correct for Level three questions on the Pre test and the Post test

	pre test	post test	change
Control group (n=19)			
TOTAL:	10	33	23
per student:	.53 per student	1.74 per	1.21 per
Experimental group: (n=16)			
TOTALS:	13	46	33
per student:	.81 per student	2.88 per	2.06 per

Although there is a difference here a repeated measures analysis of variance reveals a non statistically significant difference within subject group effect (p=.117). Although there is a difference between the groups it does not reach statistical significance.

-Modified level 3 Criteria: a significant difference

The level criteria can be seen as a rather coarse measure of performance. Students must get 5 out of 8 problems in the level three set to achieve level three status. If we look under this cut off at the performance of the experimental group we discover that there were a number of students who almost reached the level three criteria; 3 students got 4 out of the 5 necessary to qualify as level 3. If we open the comparison slightly and include those who almost made level three (got 4 out of 8 instead of 5 out of 8, and therefore missed achieving level three by one question) the comparison becomes 6 from the experimental group vs. 1 from the control group. None of the control group students who failed to reach level three got 4 of the level three questions on the post test (See Appendix F for listing of individual student scores). These figures are summarized in table 7.

TABLE 7

Number of subjects who almost achieved level three performance on the chelsea algebra test (achieved at least 4/8 instead of 5/8)

	pre test level 3	post test level 3
Experimental Group (n=16)	0	6
Control Group (N=19)	0	1

	Exp grp n=16	Ctrl Grp n=19
less than 4/8 on level 3	10	18
almost achieved level 3 (4/8 or better)	6	1

If we apply the same Fisher exact test to these results the difference between the two groups does reach significance (p= 0.0318)

Analysis of Experimental Group Changes

What do these results show? As predicted for both groups the pre-test scores are quite small. The experimental group, the better of the two, got only 0.81 problems out of 8 right on average. But, the improvements are quite small as well; on average an improvement of only 2.06 problems for the experimental group (the better of the two).

The experimental group did perform slightly better than the control group on level three problems and if we measure this difference in terms of who got 4 out of 8 correct on the post test we see a significant difference between the two groups. The problem with this is that an improvement of 2 problems out of 8 on average (or the experimental group's increased improvement of a little less than 1 problem over the control group's improvement) is not the kind of level 3 change I was hoping for.

It seems that the work with the marble bag curriculum did not transform these sixth graders into students who had an understanding of letters as specific unknowns as defined by Hart's algebra test criteria.

But, the fact remains that there was a significant difference in performance on the Chelsea Algebra test between the experimental and the control groups. The next task is to analyze and evaluate the significant changes that did occur in the experimental group.

-Student Performance on Marble Bags Problems

The original hypothesis is, given the marble bag experience, students will learn to deal with letters in the marble bag context and then will be able to transfer that understanding over to a standard algebra context as represented by the problems on the Chelsea algebra test. A first question, as we examine the changes that did occur, is, did students reach level three performance within the context of the marble bags work itself? Did the students master the marble bags metaphor and were they able to apply it to level three problems that used marble bags?

We have two sources of data on this question. First assessment tests given by an external evaluation team to all students during the project which included a series of marble bag questions and secondly a set of marble bag questions that were part of this study's post-test built to mirror the content of the Hart questions, but given in the context of marble bag problems.

Report of External Evaluator:

The work reported here was part of a large National Science Foundation algebra curriculum project (Roberts, 1987). One aspect of the evaluation of the project was a series of pre and post tests give by external evaluators. One of the tests addresses my current question of student mastery of manipulation of letters in the marble bag context and is described as follows in the project's final report;

> "The evaluation test was designed to test algebraic skills acquired in the context of manipulating the computer microworld of marble bags. The test was structured in two parts: Part 1 measured the ability to translate algebraic problems into and from marble bag notation. Part 2 measured the ability to solve algebraic equations written in traditional notation" (Roberts, 1987 p. 43).

How did the students perform on this paper and pencil test? Table 8 contains the results of the post test given to the experimental group at the end of the semester (there was no control group for this evaluation).

TABLE 8
Summary of Results of Marble Bags Post Test

	Part 1: marble bags	part 2:Equations
N=16		
Mean=	14.8(89.4%)	5.0(71.4%)
*S=	1.6(9.5%)	1.2(17.3%)
*MPS=	16.5	7

*(S stands for standard deviation and MPS stands for maximum possible score)
(Roberts, 1987. For more detail see appendix H)

The evaluators report; "the very strong performance of all children on part 1 of the test...clearly show the children's ability to handle algebraic concepts in the context of the marble bags microworld" (p. 44) and "the children also performed well on the part of this context dependent test that was written in traditional form (part 2)." (see Appendix H for example problems and results from the test.)

Are the "algebraic concepts" referred to in the report parallel to the concepts about letters that are demanded by the level three questions in the Hart test? This is clearly the case. For example, on the Hart test question #15b (a level three question) asks students to write the expression k-3 to represent the number of diagonals of a K sided figure. On the Logo-Algebra test there are several questions that ask students to write down a marble bag story in algebraic form (e.g. think of a number, double it, add four marbles, triple the total, take away the original number). In response to these questions students wrote expressions such as x, $2x$, $2x+4$, $3(2x+4)$. On the Hart test another level three question asks students to add 4 onto the expression $3n$ (#4c). On the Logo Algebra test there are several places where students are asked to generate algebraic expressions and then write algebraic expressions for operations such as "add four marbles" or "subtract 10". On all these Logo-Algebra questions the experimental groups' range was 94% to 100% correct. On the Hart test in another level three question students are asked to simplify the following expression $3a-b+a$ (#13). On the Logo Algebra test students are asked to solve $4(7x+3)-2x=64$. In order to do this students tried to simplify $4(7x+3)-2x$. 94 percent of the students (all but one) got this problem correct. This problem was in part two of the test, the section that gave no marble bag context (note that in the curriculum students had been through a sequence where they first solved problems like these with marble bag diagrams and then made a transition to solving them when they were presented in standard algebraic format). All of these problems involve operating on letters.[16]

These results provide strong evidence that students were able to deal with letters as specific unknowns when they were presented in the context of marble bag problems and even when they were

[16]One might ask why students were able to do these problems, but had trouble with the ones that I argue have similar content on the Hart test. As mentioned, the first two examples (writing expressions) were problems given directly in a marble bag context and the third (the simplification) was a type of problem that the students had solved in both marble bag and algebraic contexts as part of the curriculum. It seems that as long as problems were given in a marble bag context or were problems that students had solved in a marble bag context they were able to do them.

presented in standard algebra context if students had worked on solving both marble bag and algebraic versions of the problem in the curriculum (see footnote above).

Marble Bag Problems Added to the Hart Test:

A second source of data on the question of student understanding of letters as specific unknowns in the context of marble bags can be found in some additional problems added to the post test in this study. Of the 8 problems that Hart suggests are indicators of interpreting letters as "specific unknowns" I was able to create 3 parallel problems using the marble bag metaphor (#4c,#5c, and #9d. See appendix D for examples and appendix G for a more detailed chart of the results). These problems were included on the post paper and pencil (P&P) test. Table 9 below summarizes the results.

TABLE 9

Comparison of responses to marble bag and standard versions of three level three questions on the Post paper and pencil test

Post P&P:

	same:√	same: x	marbles only√	standard only √
4c.	6	1	8	1
5c.	5	4	5	2
9d.	6	10	0	0

Key:

same:√ student was correct on both standard and marble bag version

same:x student was incorrect on both standard and marble bag version

marbles only√ student got marble bag version correct and standard wrong.

standard only √ student got standard version correct and marble bag version wrong.

To make sure that the order of presentation of the problems (marble bags problems first or standard problems first) did not effect the performance of the students I split the group in half and gave half a test with the marble bag problems first and half tests with the standard problems first. The similarity of the results of the two groups show that there were no major differences in the performance of these cohorts of the experimental group. (see appendix I.)

On the post test for problem #4c nine students got the standard version wrong and of those eight got the marble bags version correct (this includes one student who misinterpreted the problem and immediately corrected it when it was explained to him in the post interview). Only one student got the standard correct and the marble bag version wrong. For #5c nine got the standard version wrong and 5 of those got the marble bags version correct. While only 2 students got the standard version correct and the marble bags version wrong.

For problem #9d no student who got the standard version wrong got the marble bag version correct. This example used the marble bag metaphor in a new way: to refer to weight rather than referring simply to a number of items. But, three students on the post interview were able to solve #9d correctly when the marble bag metaphor was introduced verbally

Students were able to solve problems that involved treating letters as specific unknowns in the marble bag context and for two of the problems when given the same problem in the two contexts of those students who could not do the standard version many found it easier to do the problem in the marble bag context. The reverse was a rare occurrence.

Statistically we can ask whether the students performed significantly better on the marble bag questions than on the standard versions. Because the performance of the experimental group is being compared on these two questions I will use a binomial McNemar to compare the significance of the difference in their performance. For the McNemar I set up the following table:

TABLE 10

Comparison of performance on question 4c; marble bag vs. standard Question (N=16)

Standard	Marble Bags Right	Marble Bags Wrong
Right	6	1
Wrong	8	1

The table shows that of the 16 students eight got the standard version wrong and the marble bags version right and only one got the marble bags version wrong and the standard version right. The null hypotheses is that there is no significant difference between performance on the standard and marble bag questions. A significant one tailed result (p=.001) leads us to reject the null hypothesis. The performances are statistically significantly different. The experimental group performed significantly better on the marble bag version of question 4c.

TABLE 11

Comparison of performance on question 5c; marble bag vs. standard Question (N=16)

Standard	Marble Bags Right	Marble Bags Wrong
Right	5	2
Wrong	5	4

For question #5c again the null hypotheses is that there is no significant difference between performance on the standard and marble bag questions. A non significant one tailed result (p= .113) leads us to accept the null hypothesis in this case. Although the experimental group performed better on the marble bag version of #5c it did not quite reach significance.

One might ask whether there was more improvement on problem #5c standard than there was on problem #4c and whether this might account for the lack of significance in the differences in performance on marble bag and and standard version of problem #5c. In fact the improvement on both standard versions of the question was exactly the same as the following table shows.

TABLE 12

Percent Change for Experimental Group for Problems 4c and 5c Columns indicate number of correct answers out of total students:

EXPERIMENTAL GROUP
N=16

	PRE	POST	%Chng
4c.	2	8	+38%
5c.	2	8	+38%

For problem #9d there was not any difference in performance between the marble bags version and the standard version. Students either got them both right or both wrong. The added difficulty introduced by the new use of marbles (weight) that may have made this problem much more difficult has already been mentioned

This data gives us some, but not overwhelming evidence that students did better on the marble bag versions than on the standard versions.

As a final check we also need to make sure that the marble bag questions were not simply easier in general than the standard problems. I want to attribute the success of the experimental group on the marble bag questions to the experience with the marble bags curriculum. To make this claim I need to show that the marble bag questions are not just easier in general. I will check this by comparing the performance of the control and experimental groups on the marble bag questions. The null hypotheses will be that they are not different.

The following tables makes this comparison:

TABLE 13

Comparing Control and Experimental Groups on
Number of correct vs. incorrect marble bag answers (4c,5c,9d) on Post Paper and Pencil test. (Control group n=19, Experimental group n=16)

Performance on marble bag version of problem 4c: comparison of control and experimental groups:

	RIGHT	WRONG
Control	3	16
Experim.	14	2

Performance on marble bag version of problem 5c: comparison of control and experimental groups:

	RIGHT	WRONG
Control	3	16
Experim	10	6

Performance on marble bag version of problem 9d: comparison of control and experimental groups:

	RIGHT	WRONG
Control	5	14
Experim	6	10

The null hypotheses is that there is no significant difference between performance of the two groups on the marble bag questions.

The difference in performance for problems #4c and #5c is obviously significant and the lack of difference on problem #9d is also obvious. The experimental group performed significantly better on the marble bag questions than did the control group on problems #4c and #5c. I attribute the lack of difference in problem #9d to the fact that in this problem the marble bags were used in a new way (as mentioned above this example used the marble bag metaphor to refer to weight rather than referring simply to a number of items). This new use may be the basis of the experimental groups' difficulty with the marble bag version of problem #9c.

A Problem of Transfer:

I have now shown that many students were able to do problems that demand that they treat letters as specific unknowns in the marble bag context but that that ability seemed to have little impact on their performance on Hart's level three problems in the post test. One implication is that although students learned to deal with letters as specific unknowns in the marble bag context they did not transfer this ability very robustly, based on Hart's standard, to an unfamiliar standard algebra context. From this perspective the problem might seem to be one of transfer. The students were able to operate on letters in the marble bag context, but were unable to transfer this ability to a standard context that had not been tied to marble bag activities.

-What is the Source of the Significant Difference?

Part of the goal of this work is to better understand how students come to understand letters in a mathematical context. If experimental students could solve level three type problems in the marble bag context, but did not move to level three on the Chelsea Algebra test in significant numbers what changes, if any, can we see? In fact, the experimental group did do significantly better than the control group in terms of more general change in Hart levels on the tests. If this change is not based on movement to level three, where does it come from? As Table 2 shows the major difference comes from movement from level one to level two.

Table 14 shows the difference between changes from level one to level two in the experimental and control groups.

TABLE 14
Comparing movement from level 1 to level 2 in Control and Experimental Groups.

CATEGORY	GROUP	
	Control n=19	Experimental n=16
started at level one	14(74%)	11(69%)
stayed at level one	11(58%)	3(19%)
moved to level two	3(16%)	7(44%)

(Note: 1 student in the experimental group went from level one to level three)

Of the students who started at level one (the majority, almost 70%, of each group) for the control group only 16% moved up to level two whereas for the experimental group 44% moved up.

So here, based on Hart's levels, we do have a strong difference between the experimental and control groups. The significant difference between the experimental and control groups is based on the experimental group's movement from Hart's level one to level two.

-Hart's Analysis and the Need for an Alternative View

What does Hart have to say about the difference in these two levels? First empirically level two problems are significantly harder (Hart 1981 p. 7), but Hart argues that the difference is not based on a better understanding of letters as specific unknowns ("children at this level (two) could not cope with specific unknowns" (p. 113)). In fact the basis of the difference based on Hart's analysis is not entirely clear; the only clear characterization that Hart gives is that they are more complex

(Hart, 1981 p. 113).

I wish to explore a different way of characterizing these students that will help us understand the significance of the movement from level one to level two. Hart's characterization is descriptive - she describes the interpretations and misinterpretations of letters that students make (ignore, evaluate, treat as objects, treat as specific unknowns), but does not give any analysis of the basis of these various interpretations. She implies that at first letters are not treated as though they were numbers and to understand them as specific unknowns they must be treated as numbers in that students must be able to do operations with them. The difficulty is this characterization does not give us any general structure within which to interpret what is going on or to think about how change might happen.

In part 2 I will propose such a framework: a framework for looking at students' various interpretation of letters, a framework that can help us look at and understand students' misinterpretations, and a framework that will help us examine the nature of the changes necessary to interpret letters differently

An Arithmetic To Algebra Transition:
Using metaphors to overcome arithmetic barriers to understanding mathematical problems involving letters

Part 2

An exploration of "the arithmetic frame of reference" as a barrier to student understanding of problems involving letters

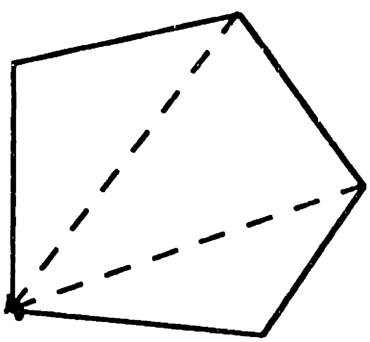
Introduction To Part 2

The work with the marble bags curriculum only partially fulfilled my hopes for significant change in student performance, but it did provide a lot of data about letter-naive sixth grade students and how they deal with letters in mathematical problems. In addition the work with marble bags did have a significant effect on these students, increasing their ability to solve Hart's level two problems. I now want to turn to some problematic aspects of Hart's analysis. Hart sorts the problems on the Chelsea Algebra test into three levels of empirical difficulty. These levels are based on a statistical analysis of student performance. In this scheme mastery of a higher level implies mastery of all lower level problems. e.g. students who achieved level two mastered level one problems, but were not able to master those in level three. Given these empirical levels of difficulty Hart then develops an explanation of the difference between level three and levels one and two based on the different interpretations of letters that she has identified (being able to treat a letter like a number vs. a variety of other interpretations). As I analyzed student work on the test I found two difficulties with Hart's analysis.

First, as mentioned in the first part of this paper, Hart does not provide a coherent explanation of why level two problems are harder than those in level one. In fact, Hart's analysis does not give us any consistent framework to explain why each level is more difficult than the previous level. Second, as I looked more closely at the problems in Hart's empirical levels, I uncovered some inconsistencies in her analysis of students' difficulties with the level three problems.

Hart suggests that only level three problems demand that students be able to operate with letters, treat them as though they were numbers, and do mathematical operations with them. An example is provided by problem #15b:

15.

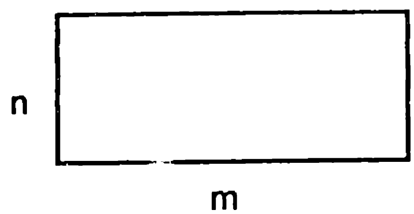


In a shape like this you can work out the number of diagonals by taking away 3 from the number of sides.
So, a shape with 5 sides has 2 diagonals:
(a) a shape with 57 sides has .... diagonals.
(b) a shape with *k* sides has .... diagonals.
[Correct answer: *k*-3]

Problem #15b, a level three problem, asks how many diagonals a shape with *k* sides will have and the correct answer demands that students write *k*-3 as an answer. Hart suggests that in writing *k*-3 students must operate on the letter *k* as though it were a number.

Hart argues that all the level three problems have this character and that level one and two problems do not. The difficulty here is that there are a few level two problems which also seem to demand that students operate with letters. For example problem #7c. presents a rectangle labeled *n* and *m* and asks students for its area:

Problem #7c:

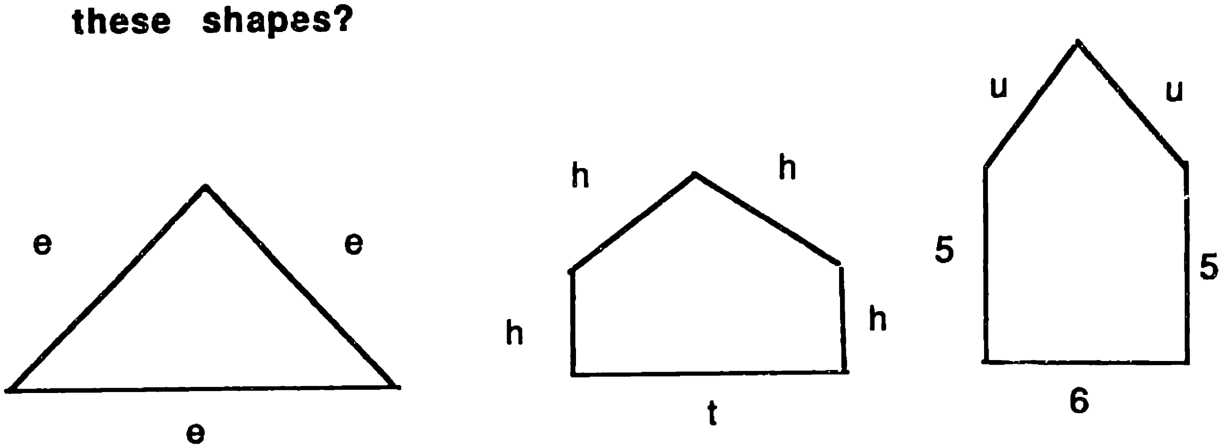


A=..................................
[Correct: *nm*]

To answer this problem correctly students must write an expression to represent multiplying *n* times *m*. From the perspective of operating with letters this seems quite parallel to problem #15b which is a level three question.

The same seems to be true of problems #9b and c:

9. What can we write for the perimeter of each of these shapes?



a). P = ... **b). P = ...** **c). P = ..**
[Correct: $3e$ $4h+t$ $2u+16$, $2 \times u+2 \times 5+1 \times 6$]

Once again in order to answer #9b and #9c correctly I argue you have to operate with the letters *h,t* (multiplying *h* times 4 and adding *t*), and *u* (multiplying *u* by 2 and adding 16 to it). Hart, in contrast, suggests that students interpret these letters as labels or objects, but interview data suggests that this is not the case (see section on level two problems below (page 77)). And, even if this were the case, students must operate with letters as mathematical entities (that is generate expressions).

Those apparent inconsistencies put in question Hart's argument that students' difficulty with level three questions is based on their inability to operate with letters. This difficulty coupled with Hart's lack of clarity about the differences between level one and level two suggests the need for an alternative analysis that will:

1. Provide a more comprehensive way to look at the increasing difficulty of the problems in Hart's different levels.
2. Provide an analysis that helps us differentiate problems in level one and from those in level two.
3. Clear up the inconsistency in Hart's claim that only level three problems demand that students operate with letters when it seems that several of the level two problems demand that students operate with letters.

Frames Of Reference

In response to these difficulties I have developed an alternative analysis. I have found a productive way to look at these questions is in terms of something that has been called a "frame of reference" (Herscovics, 1985; Minsky, 1985).

Applying this concept to the students' responses on the Chelsea Algebra test we can view the students in this study as approaching the mathematics problems within a frame of reference. In general students bring to problems a way of looking at the mathematical environment, a set of expectations about the entities they will encounter (e.g. numbers, operations, and the equals sign), and the rules for how they interact. This amounts to a set of expected patterns.

My use of the idea of a frame of reference grows out of a constructivist perspective (von Glasersfeld, 1991). Stated simply, I believe we cannot "know" what is in students' heads. All we can do is to build models that attempt to account for the data we collect and that, hopefully, will help us to make accurate predictions. I see the idea of a frame of reference as an imperfect, but evolving functional tool for describing behavior, a tool, that in the long run, can be used to inform those who wish to help children learn more effectively.

In part two I will outline the content of what I call an "arithmetic" frame of reference and look at how it can be used to understand and explain student performance on the problems of the Chelsea Algebra test. This alternative analysis needs further experimentation, investigation, and refinement. However, based on the data I have, I present it as a series of hypotheses and suggested probes that can form the basis of productive future work.

THE ARITHMETIC FRA : OF REFERENCE

Based on their performance of the pre-test most of the sixth grade students in this study can be characterized as "letter-naive" and can be seen as bringing what I will call an Arithmetic Fram of Reference to their first contact with the problems with letters o the Chelsea Algebra test. Hart identifies three ways that students inappropriately interpret or deal with letters (ignore them, inappropriately replace them with numbers, treat them as object). What she does not do is to put these interpretations into a coherent framework that suggest how they might function for these students in terms of some broader understanding of mathematics. The idea of the arithmetic frame takes the interpretations identified by Hart and some of the difficulties identified by other researchers (e.g. interpretation of the equals sign as a "do it" operation) and attempts to place them in a larger and more coherent framework.

<u>The Arithmetic Frame Template</u>

The standard form of the archetypal arithmetic problem is 2+3=.... or more abstractly: *a operation b = c*. In this standard problem *a* and *b* are numbers, the operation can be add (+),subtract (-), multiply (x),or divide (/), and *c*, the answer, is also a number. The value of *c* is derived by doing a calculation with *a* and *b* that is specified by the operation. The important characteristics of this are; two numbers on the left with an operation between them, and an equals sign. As Behr's and Kieran's research shows, for most young students, the equals sign is a symbol whose only meaning is carry-out-the-operation-to-the-left-of-the-equals-sign-to-produce-a-number-which-is-to-be-written-on-the-right.

If this is a general description of the template or perspective which these students bring to math problems, how might we expect them to react to problems with letters in them? The question raised here is: what do students do when they run into something that does not fit their expectations: the entities and rules with which they are familiar. From a Piagetian perspective, we can describe what happens as students trying to assimilate the new material to familiar patterns. They try to make the problems fit the arithmetic frame. For some problems it is relatively easy and mathematically appropriate to assimilate problems with letters in them to the arithmetic frame, and these problems, I predict, the students would answer correctly. Those that are not so easy to assimilate may either be left out or in some cases inappropriately transformed to fit the arithmetic frame, producing predictably incorrect answers. One difficulty with Hart's focus on students' ability to operate with letters is that it provides us with only a very coarse metric for predicting comparative difficulty between problems. The idea of an arithmetic frame of reference suggests that the further a problem departs from the expected arithmetic frame format the more difficult it will be.

Arithmetic Frame Behaviors: Responses to Letters

Data from this study supports the idea that when students are able to easily and appropriately transform problems with letters into a form that fits their $a+b=c$ arithmetic frame expectations they can solve them easily and correctly.

For example, during a five minute set of practice problems given as an introduction to the test the students were told that letters often stood for numbers and they were exposed to problems where they were given expressions with letters in them (e.g. $n+5$) and then were given values for the letters and were asked to evaluate the expressions (e.g. For $n+5$, if $n=1$ what is its value? If $n=5$ what is its value, etc?). Given this introduction students were able to solve some problems with letters in them if they were given numerical values to substitute for the letters and could thus turn the problem into a standard arithmetic frame problem. For example in problem #5a students had no trouble substituting 43 for $a+b$ and coming up with the answer 45. (see problem #5a below).

Problem #5a:
5a. if $a+b=43$
$a+b+2=$.............

Students were generally quite successful at solving these arithmetic frame format substitution problems. One place they ran into difficulty was when a problem had letters in it and there was no legitimate way to substitute a value for the letter. There were a variety of responses to this situation. One response was to simply leave the problem out. When asked about problems to which they had responded by simply not putting down any answer, students often said that the problem could not be done or described it as impossible. Another response was to inappropriately assimilate (or deform) the problems to fit their arithmetic frame expectations. Some students errors can be seen as attempts to transform a problem with letters in it to fit the arithmetic frame. There are a variety of ways they do this which include inappropriately giving a letter a numerical value and several variations on treating a letter as an object. Below are some examples of these responses:

(1) One response to problems with letters in them is to simply ignore the letter or treat it as if it had no value.

RL's response to an interviewer's probes about his answers to problem #2 provides an example of treating a letter as if it had no value:

Problem #2:

2. Write down the smallest and the largest of these:

$n + 1,\ n + 4,\ n - 3,\ n,\ n - 7$

<u>Smallest</u> <u>Largest</u>

....

RL's Responses: *n* *n*+4

RL had written ***n*** for the smallest and ***n*+4** for the largest. In the interview he responded to an interview probe about the value of ***n*-3** as follows (R precedes student responses and I precedes the Interviewer's responses):

R: So this is just plain 3, you don't know what n is
(*R is saying the value of n-3 is 3*)
I: you don't know what n is that's just plain 3?
R: Yeah.
I: Why is that?
R: Cause you don't know what *n* is.
I: Why does that make it just plain 3?
R: Cause if you don't know what n is you can't get another number to subtract it by.
I: Unhuh.
R: See the same thing with that one. (referring to *n*-7)
I: You mean they don't count some how?
R: Unhuh.

R's point seems to be that because he does not know the value of *n* it has no value[17].

For some students this is based on the idea that letters are not arithmetic entities (only numbers are) and therefore when they have not been given a value they are irrelevant to arithmetic. The result is that for some students letters are simply ignored (as one students said, "the letter doesn't mean anything").

(2). A second response is to inappropriately give a letter a numerical value. Some students clearly appropriated the rule about a letter standing for a number, and instead of ignoring the letter tended to over generalize the idea of substituting a numerical value for a letter. They would often turn a problem with letters into one that fit the arithmetic frame by inappropriately giving a letter a numerical value which would then allow them to carry out an arithmetic operation. The thinking seems to be something like: if letters in fact do stand for numbers in order to solve the problem you must find or generate a value for the letter and replace the letter with the number it stands for (this turns the problem into something that will fit in the arithmetic frame).

[17]This vignette also reveals another arithmetic frame related behavior, ignoring a minus sign in an expression where there is a letter, a minus sign, and then a number because a minus sign without a number in front of it has no meaning (e.g. interpreting *n*-3 to mean 3 as R did above).

This was the most common erroneous interpretation of letters among the subjects in this study.

A typical example is problem #5c:

> **5(c).** If $E+F=8$
> $E+F+G=$.....
> [Correct answer: $8 + G$]

Many students assumed that the letters in #5c all equalled 4 and wrote 12 as an answer. Some others used the following interpretation: if a letter must have a value, determine it by the letter's alphabetic position. In problem #5c (above) some students gave G the value of 7 (G is the seventh letter of the alphabet) to make the total 15.

In the previous examples students found ways to avoid using or considering letters in their answers. Did these arithmetic frame students ever include a letter as part of an answer? In some cases they did. As Hart discovered, one way students interpret letters is to treat them as standing for objects. Every first grader has encountered questions like: "How much is 3 apples and 2 apples?" Whether through workbook pictures in kindergarten and first grade or "word problems" at the end of a textbook chapter in later grades, solving problems by adding up the number of objects is a familiar task for all elementary school students. Because of this one natural way of interpreting a problem such as 3a+2a is to think of it as 3 apples plus 2 apples. Once thought of in this way it is easy to give the answer of 5a, meaning 5 apples. If letters can be given a non-numerical role, such as standing for an object, then it seems to be relatively easy to think of encompassing a problem with letters within the arithmetic frame because once letters are interpreted as objects the problem just involves a standard numerical computation and a numerical result. Problems that can be interpreted this way fit right into the arithmetic frame expectations of students who have been solving similar sorts of "word" problems since first grade. Two problems on the test had this single-letter-which-can-stand-for-an-object aspect: #9a and #13a. The answer to each of these is a simple concatenated number and letter. To our eyes this may look like a mathematical expression, but to these students, I suggest, it is really more akin to an English sentence about some number of objects (e.g. three *e*'s). To differentiate this type of answer from answers that involve writing an arithmetic operation as part of the answer (e.g. $4h+t$) I will call these concatenated answers "simple expressions." This will be elaborated further when I analyze the level two problems below. For the level one problems that only demanded simple expressions for answers more than half of the sixteen students in the experimental group responded with correct answers (#9a: 13, #13a: 9). The other students just tended to leave them out[18].

(3) A third inappropriate response, that often occurs when problems do not to fit the simple single-letter-equals-object interpretation, is to separate out the letters and the numbers in an expression and operate with the numbers independently from the letters. There were a number of problems that fit this pattern (e.g. problem #13d which asks students to simplify $2a+5b+a$ or #13b: $2a+5b$). A characteristic response to problems of this type was to pick out the numbers in the problem, do an arithmetic operation with them, write down the numerical result, and then append the letters (e.g. for $2a+5b+a$ one might get answers such as $7ab$).

In this context it is important to note that we cannot simply say that the students treat letters differently from numbers because, in fact, when you do operations in mathematics letters do have a special status that is different from numbers. When you operate on two numbers you get a numerical result, but when you operate with letters you generate an expression as a result. What students with an arithmetic frame tend to do in problems where they do not simply ignore the letters, treat them as objects, or give them numerical values is to separate out operations with numbers and make them inappropriately independent of operations involving

[18]Note that there were a number of problems (e.g. #13b and #13h) which, although they could be interpreted in terms of letters as objects, did not fit the single-letter-stands-for-an-object format. Students had more difficulty with these.

letters. They radically separate numbers and letters rather than treating them as different, but deeply related. In the arithmetic frame you only operate with numbers and you get a numerical result. In some situations students in the arithmetic frame will include letters in their answers but when doing the problem will operate with the numbers. They seem to set the letters aside while operating with the numbers and then simply append the letters to their numerical answer.

We can see this in a typical response to problem #13b where students are asked to simplify expressions where possible (this one cannot be simplified).

$2a+5b =$ ....

Some students wrote $7ab$ as an answer. They would combine the numbers (doing an operation) and then combine the letters by concatenating or leaving the plus sign between them (other answers of this type to #13b are: $10b+a$, $7a+b$, and $9a,b$).

Separating numbers and letters - operating with the numbers, and including letters in an answer - can, in some instances, yield correct answers to problems with letters in them. In a problem like #4b, which asks students to add 4 onto $n+5$, just doing the operation with the numbers and including the letter in the result ($n+9$) is appropriate. But the limitation and the arithmetic frame character of this view is revealed when these students try to do a problem like #4f. #4f asks them to multiply $n+5$ by 4 and they respond with $n+20$. (Note: although this pattern of correct response to #4b and an incorrect response to #4f showed up later in the study, on the pre-test none of the students were able to answer #4b with an expression. A willingness to include letters in an answer seems to come later. At first arithmetic frame based responses such as giving a letter an inappropriate numerical value, or simply leaving the problem out seem to stand in the way of using letters in answers.)[19]

[19] The issues of change raised by these different responses (replacing a letter with a number vs. willingness to include a letter as part of an answer) is discussed below as part of the movement from level one to level two.

Arithmetic Frame Beliefs

I have described the arithmetic frame in terms of a set of expected formats for problems and answers and a set of behaviors. The arithmetic frame can also be characterized by a set of beliefs. Some of the responses outlined above can be seen as deriving from the following arithmetic frame beliefs.

1. In order to do a problem you must do an operation and get a numerical result. This belief can be seen driving the responses described in (2) and (3) above.
2. For some, though not all students, there is a belief that an expression with a letter in it cannot be an answer, it is still an unfinished problem (see differentiation of level one and level two responses described below).
3. Although many students can tell you that a letter stands for a number, many of them also seem to behave as though, for them, letters are not arithmetic entities; you can't do arithmetic with or on them if they do not have a value. (see (3) above)

The Arithmetic Frame in Action

Here is an example of a student whose behavior seems driven by the arithmetic frame.

This is a dialogue from an interview taken after the pre-test. It focuses on question #4 in which students are asked to add 4 onto various quantities:

> **4 added to *n*** can be written as ***n* + 4.**
> add 4 onto each of these:
> (a) 8 (b) n+5 (c) 3n

JK had given the following typical answers:

> (a)12, (b) 9 , and (c)12

The interviewer asks her to read the problem aloud. After JK reads the problem the interviewer points to her answer to #4b (9) and asks:

I: So here you said...
J: "I changed the *n* into a 4"

[*Here we see belief number 2: you must change a letter into a number. She has also interpreted the question, which asked the students to add 4 onto the expressions, as meaning replace the n with 4. In doing this she has turned it into an arithmetic frame problem.*]

I: In fact you don't know the value of *n*. If you had this thing that said *n*+5 and you wanted to add 4 more onto that...
J: "that would be *n*+9"

[*Here, once heavily prompted, she comes up with the correct answer, and, based on the prompt, she seems to move out of the pure arithmetic frame. She has generated an expression, but the limitation of this view and her need to do a numerical operation is revealed in her response to the next problem:*]

I: and what would ....if you had 3*n* and you had to add 4 onto that?
(She writes 7*n*).

[*again she solves it as a simple arithmetic problem with the n appended onto it. Her arithmetic frame of doing an operation with the numbers is dominant here.*]

I: I see you added 4 onto the 3, but what about the *n*? what does 3*n* mean?
J: "it means *n* times 3"
I: *n* times 3....
J: "but you don't know the value of *n* so you can't multiply."

[*based on her arithmetic frame she wants to carry out a multiplication operation*]

I: So if you had something like 3*n* and you wanted to add on 4...so what you did was to add 4 onto..
J: "the 3"
I: would that add 4 onto the whole thing?

[*The implication is that she has not seen the single entity aspect of 4n. She has focused on it being mathematically two separate things (which, we might note, they would be if 4 and n were added (n+4) instead of being multiplied).*]

J: "no"
I: how could you write it so you added 4 onto the whole thing?
J: "could put 5 and then *p*, cause you'd add 2 letters onto *n* and 2 numbers onto 3"

For JK we see the separation of letters and numbers as inappropriately different kinds of entities. 3*n* does not seem to be a number like entity and in order to "add 4" she must do an arithmetic operation and get a result. Here it seems that "add" can only mean "**do or carry out** an operation". JK is able to articulate that 3*n* means 3 times *n* which she would most likely also acknowledge as an unknown value. Her first interpretation of the problem was to replace *n* with 4. Once she is told that the problem does not mean replace *n* with 4 and she is asked how she would add 4 onto 3*n* she writes 7*n*. She has found another alternative for actually doing an arithmetic operation and coming up with a result. When asked about the *n* she explains the meaning of 3*n* accurately, but then focuses on the fact that you cannot multiply 3 times *n* because you do not know the value of *n*. Again she is focused on the need to do an arithmetic operation. Her resolution is quite imaginative. If she must add onto "the whole thing" of 3*n* then she will do an operation on both elements adding 2 onto 3 and 2 steps in the alphabet onto *n* (to get *p*). In each case her whole focus is on carrying out an arithmetic operation. One might argue that part of her difficulty is a mathematical one of not knowing how to add onto a multiplicative term. One might ask what she would do if asked to add 4 onto 76x354 without doing the multiplication first. Although the interviewer did not ask this question of JK it was asked of several other subjects and they often either added the 4 onto one of the terms or added 2 onto each term. This kind of behavior might suggest that the problem is not with letters per se as much as it is with a lack of knowledge of the underlying arithmetic (how you add a number to a multiplicative term), but the fact still remains that JK did not add 2 onto 3 and *n*. While she added 2 onto the 3 she added <u>two letters</u> onto the *n*. She treats the 3 and the *n* as two different kinds of entities. Although, in fact, when doing operations letters <u>are</u> treated differently than numbers, JK does not yet seem to understand the way letters are used to represent unknown quantities, at least in this context.

<u>Summary</u>

Both Hart and Harper suggest that learning to deal with letters as specific unknowns simply involves applying the rules of arithmetic to letters, treating letters as one treats numbers. This view does not help explain the many and varied difficulties students have with the transition from numbers-only arithmetic to dealing with problems that involve letters. Students do need to learn to operate with letters, and, as Hart has shown, many students start out with inappropriate interpretations of letters. The conflicts between the arithmetic of numbers and the use of letters and the nature of the changes necessary to deal appropriately with letters are not made clear by Hart's and Harper's formulations. In order to find effective ways to help students make the transition to dealing with letters we need a finer grain analysis and the idea of an arithmetic frame of reference is a preliminary attempt to do just that. The arithmetic frame analysis gives us a tool for looking at the problems on the test and how students responded to them. The idea is that students approach the problems on the test from within a frame of reference: they bring to problems a way of looking at the mathematical environment, a set of ideas, a set of expectations, a set of definitions, a set of expected patterns. For most standard arithmetic problems the arithmetic frame rules serve students well. The question is what happens when they are confronted with problems with letters in them that do not fit the rules and patterns of the arithmetic frame. Quite simply they try to interpret them, and at times actually deform them, to fit the rules and patterns with which they are familiar. One way of looking at level one and two problems is that students can solve level one problems even though they involve letters because the solution processes conform to the rules and assumptions of an arithmetic frame and level two problems are harder because they demand that you take steps away from the arithmetic frame in order to answer them correctly. In the next sections I will analyze level one, level two, and level three problems from this perspective.

The Arithmetic Frame And Level One Problems

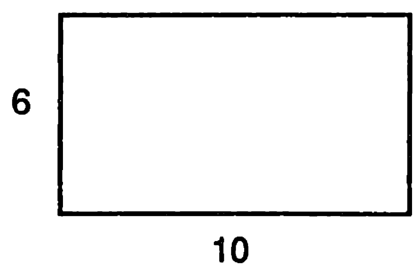
In this section I will show that an arithmetic frame analysis explains why level one problems, even though many of them involve letters, were easy for the students to solve because they all conform to arithmetic frame patterns. Below is a list of the level one problems.

Level One problems:
(Correct answers are in square brackets ([]) following each problem.)

5a. $a+b=43$
$a+b+2=$........
[Correct answer: 45]

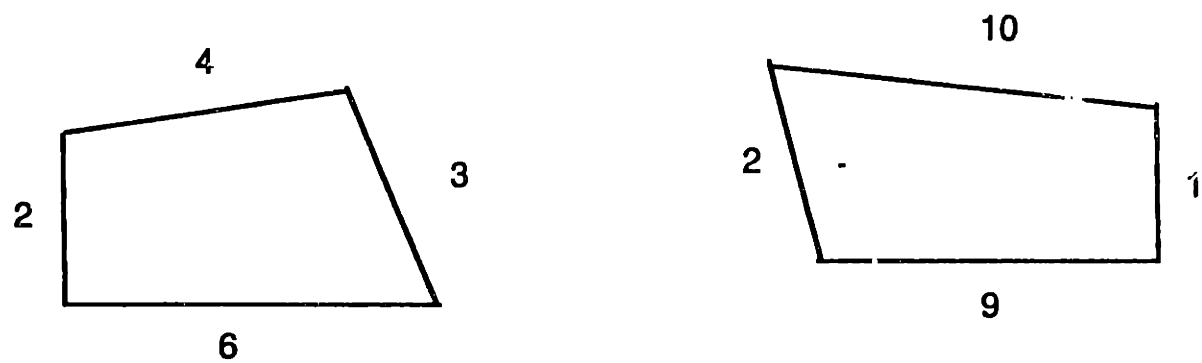
6a. What can you say about a if $5+a=8$ ....
[Correct answer: 3]

7b.



A=.................................
[Correct answer: 60]

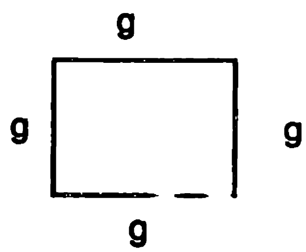
8.



The perimeter of this shape is equal to 6 + 3 + 4 + 2, which equals 15.

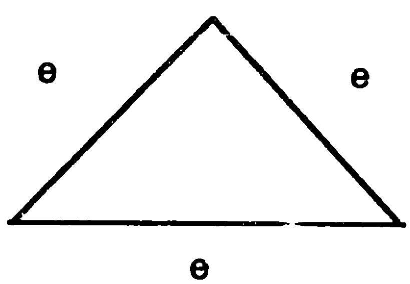
Work out the perimeter of this shape. P = ...
[Correct answer: 22]

9.



This square has sides of length g.
So, for its perimeter, we can write $P = 4g$

What can we write for the perimeter of each of these shapes?
9a.



P=......................
[Correct answer: $3e$]

13a. $2a+5a$=....................
[Correct answer: $7a$]

Level One Problems and The Arithmetic Frame

First, four out of the six level one problems (#5a, #6a, #7b, #8) have purely numerical answers and answers to the last two (#9a, #13a) are derived by appending a letter onto a simple count and simple addition. The answers and the activities are numerical in nature, simple arithmetic calculations can be used to solve each one. They can each be easily mapped onto a problem form with numbers and operations on the left and and a numerical answer on the right. Four of the problems do involve letters, but upon closer examination each one is easily reduced to a problem in the arithmetic frame format.

Analysis of Level One Problems

Problem #6a:

if $5+a=8$ $\quad a=$....

[Correct answer: $a=3$]

Problem #6a is simply a missing addend problem familiar to all students of arithmetic.

Problem #5a:

if $a+b=43$

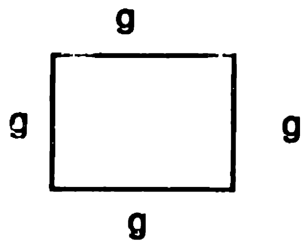
$a+b+2=$............

[Correct answer: 45]

Although problem #5a may appear to be complex because it uses multiple letters it can be easily reduced through simple matching to a standard arithmetic problem 43+2. The key here is replacing $a+b$ with 43. And in this case replacing $a+b$ with 43 is appropriate. This kind of simple replacement that leads to a standard arithmetic format seemed to present little problem to the students. Only two students in the experimental group got this wrong.

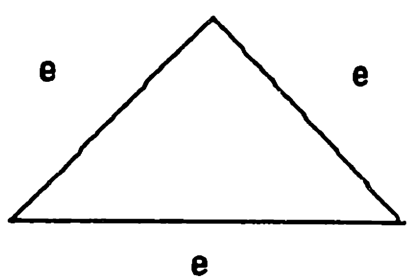
Problem #9a:

9.



This square has sides of length g.
So, for its perimeter, we can write $P = 4g$

What can we write for the perimeter of each of these shapes?



P = ...
[Correct answer: $3e$]

As can be seen above, problem #9a asks for the perimeter of a triangle with each side labeled with a letter "e". Solving this problem involves simply copying the immediately previous example problem (one is told that the perimeter of a square with each side labeled with the letter g is $4g$) and simply counting the sides. Again this presented no significant difficulty to the students.

Problem #13a:

#13a $2a+5a=$....................

[Correct answer: $7a$]

Problem #13a can also be solved by simply adding the numbers 2 and 5. The letters can be interpreted as objects e.g. 2 apples plus 5 apples = 7 apples or $7a$. It reduces to an arithmetic frame problem of 2+5 with letters attached (this is reminiscent of Harper's description of early algebra work as arithmetic with letter appendages).

Level One Summary

In all of the above cases the arithmetic operation one has to do is clear and executable and the result includes a number that is the result of the operation. Three of the six problems (#6a,#7b, and #8) are simply familiar numerical arithmetic problems. The other three (#5a,#9a,and #13a) do involve letters, but can be easily reduced to arithmetic problems through replacement (#5a) or by treating the letters as objects (#9a,#13a). Based on this analysis all of the level one problems are either directly solvable in the arithmetic frame or can be easily transformed into the arithmetic frame by direct one step replacement or by interpreting letters as objects. This is what makes level one problems easier than the level two problems each of which makes a significant departure from the arithmetic frame.

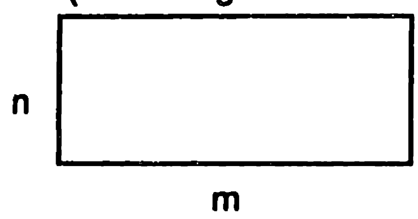
The Arithmetic Frame And Level Two Problems

We have seen how some problems with letters can be appropriately answered by students within the arithmetic frame of reference because they can be easily interpreted within the constraints of the arithmetic frame. This seems to be the case with all of the level one problems. This even includes some problems where letters are part of the answer when the letters can be easily interpreted as labels for objects and the answer is a concatenated letter and number. The next question is whether the level two problems, which are empirically more difficult, can be analyzed in terms of their breaking the rules and constraints of the arithmetic frame. Below is a list of Level two problems.

Level Two problems

(Correct answers follow problems in square ([]) brackets)

7c. (A rectangle labeled *n* and *m*. What is its area.......................)

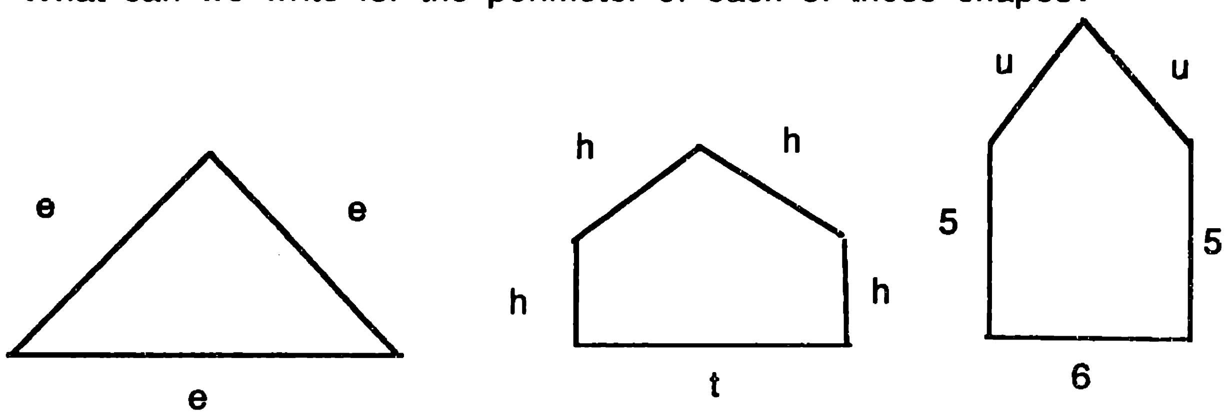


A=.................................

[Correct answer: *nm*]

9.

What can we write for the perimeter of each of these shapes?



9a). P = ... 9b). P = ... 9c). P = ...

[Correct answers: $4h+t$ $2u+16$, $2\times u+2\times 5+1\times 6$]

11a. what can you say about u if $u=v+3$ and $v=1$?.............

[Correct answer: $u=4$]

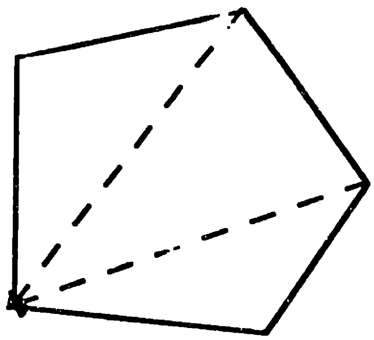
11b. What can you say about m if $m=3n+1$ and $n=4$?..

[Correct answer: $m=13$]

13d. $2a+5b+a=$.................

[Correct answer: $3a+5b$]

15.



In a shape like this you can work out the number of diagonals by **taking away 3** from the number of sides.
So, a shape with 5 sides has 2 diagonals:

15(a) a shape with 57 sides has .... diagonals.

[Correct answer: 54]

15(b) a shape with k sides has .... diagonals.

Level Two Problems and the Arithmetic Frame

The level two problems fall into two main categories. The first category is problems that break arithmetic frame rules by demanding what I describe as "full expressions", and the second is problems that disrupt the left to right order of the arithmetic frame.

Full vs. Simple Expressions

In contrast to the level one problems, four out of the seven level two problems involve breaking my hypothesized arithmetic frame rule about having a numerical result for an answer. These problems (#7c, #9b&c, #13d) demand that students generate unexecuted or "full" expressions as answers.

Although the answers to some of the level one problems do involve letters (e.g. #3a and #13a) and, in a formal sense, can be seen as demanding that students write expressions, they are fundamentally different from level two problems that ask students to do things like record the perimeter of a shape as $4h+t$ or $16+2u$. The answers to the level one problems that use letters involve simply appending the results of counting or addition to a letter. Perhaps students find this acceptable because the level one answers generated by counting or simple arithmetic (like $3e$ or $7a$) give the feeling of a computed result. One could imagine answering a standard arithmetic problem with words like "three e's" or "seven a's." Some of the level two problems, on the other hand, involve generating an expression with a "+" in it that, to arithmetic frame students, may imply that the computation is not yet finished. "Sixteen plus two *u*" does not sound at all like a familiar answer to an arithmetic problem. I shall indicate this difference by calling expressions that may seem like arithmetic answers "simple expressions" (e.g. $3e$) and expressions that include at least two terms separated by a "+" or a "-" "full expressions" (e.g. $4h+t$).

The answers to a number of the level two problems (#9b&c, #13d) depart from the arithmetic frame in that they demand full expressions.

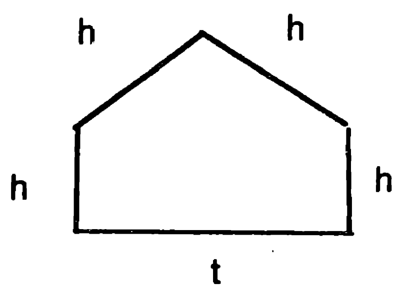
To summarize: my hypothesis is that students in the arithmetic frame can solve problems with letters as long as the solution process involves something that they can interpret as a numerical computation and the result can be interpreted as a "numerical answer" (a number or simple expression e.g. $3e$). When arithmetic frame students confront problems that do not allow them to do an $a+b=c$ type computation or which demand a full expression for an answer they have difficulty. Their response is most often to either leave the problems out or to inappropriately transform the problem to a form that better fits the arithmetic frame. I will now examine each of the level two problems and see how they fit with these ideas.

<u>Analysis of Level Two Problems that demand full expressions as answers</u>[20]

Level two problem #9b involves writing an expression for the perimeter of a 5 sided shape whose sides are labeled with letters

Problem #9b:

#9b. What can we write for the perimeter of each of these shapes?...............



P=.............................

[Correct answer: $4h+t$]

[20] The answer to the first level two problem (#7c) demands an expression as an answer (nm), but I will leave it until later because it has some anomalous characteristics which will make more sense once the other problems have been analyzed.

Katherine Hart found that the level two problems, including #9b, were significantly more difficult than those she categorizes as level one. The performance of our students on #9a and #9b provides support for this claim. On the level one problem #9a 14 students wrote 3*e* or an alternate simple expression and 2 left it out. For #9b there is a dramatic change in performance. As opposed to the 14 students who wrote concatenated expressions for #9a, only 3 students wrote full appropriate expressions as an answer to #9b.

To test the significance of this difference we can compare the number of students who got both problems right, one right and one wrong, and both wrong. Two children got both #9a and #9b correct, 3 got both wrong, whereas, in contrast, 11 got #9a right and #9b wrong. None showed the reverse pattern ($p<.01$, McNemar 1-tailed).

What is the nature of the difficulty of #9b? How does #9b differ from level one problems? In #9b, although you can count up the *h*'s (4) as in some of the level one problems, you have a second letter (*t*), and, more significantly from our perspective, in order to answer the problem correctly you have to write an expression that includes a "+" sign (what I have called a full expression).

How does this conflict with the arithmetic frame?
From the perspective of the arithmetic frame $4h+t$ cannot be an answer, and, as some students said; "its still a problem." It should be noted that none of the answers of the level one problems have this form.

What did the students do? The most common answer (7 students) was to make what we might call a list of the letters (e.g. 4*h*, *t*). I have identified several ways that students in the arithmetic frame can deal with letters. One way to handle letters is to ignore them, another is to give them values, and a third way is to simply leave the problem out saying it cannot be done. Students responded to #9b in a way that does not seem to fit any of these categories.

How did students derive the $4h,t$ answers and how can I argue that these answers derive from an arithmetic frame perspective?

I suggest that students took the model of their answer to #9a and simply added t as a second member of the list of things they had to combine (expressing the idea of four h's and a t). This suggests that they may view the $3e$ (and 4 h), the most common answers to #9a (and #9b), as a kind of list meaning 3 of the e's (or 4 of the h's). They invented their own way of recording what they would do without writing it as something that broke the arithmetic frame constraints (in the arithmetic frame a full expression cannot be an answer, the addition sign means there is still an operation to do and you cannot add things whose values you do not know.). All but two of the other students gave other arithmetic frame based responses such as simply ignoring the second letter (3), or leaving it out (3).

Given this data one might still ask why the students responded by making lists rather than simply replacing the letters with numbers and doing a straightforward computation as they did in many other problems, particularly those categorized by Hart as level three. This issue will be discussed in more detail when I consider the level three problems. From this perspective one can say students tend to inappropriately replace letters with numbers when it is the only way they can do an actual computation[21]. When they can do something that feels like a computation (in this case counting up the sides) they are more comfortable including letters in their answer. Secondly problem #9b is based on what I will call an external referent (computing the perimeter of a pictured geometric figure) as opposed to simply being a computational problem with no referent. It seems that it is easier to include letters in the answers to problems that refer to an external referent, perhaps because it is easier to think of the letters as objects when there is an external referent. As we will see, in contrast to some level two problems, a number of the level three problems, which were more

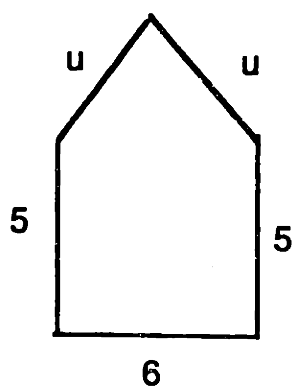
[21] It would be interesting to see what students would do if presented with a problem similar to #9b in which they could not count up anything. e.g. a shape in which the sides were all labeled with different letters and perhaps one with a number. My current conjecture, that doing a computation of some kind is a central part of problem solving in the arithmetic frame, would suggest that a problem in this form would be more difficult.

difficult for students, have this no-external-referent characteristic (e.g. #4c, #5c). I will return to this issue in more detail when we consider the nature of the level three problems in the next section.

Problem #9c is of the same type as #9b (see below),

Problem #9c:

#9c. What can we write for the perimeter of each of these shapes?



P=.............................
[correct answer; $2u+16$]

The only difference is that, instead of just counting instances of a letter, you do an addition operation (5+5+6). But you are still left with the 2 sides labeled *u;* the correct answer ($2u+16$) demands that you write an expression that includes numbers, a letter and an operation. Again, this is not an answer in the arithmetic frame. As in #9b a majority of the students responded with arithmetic frame answers. On the pre-test twelve of the sixteen students either left it out (5), answered with some kind of list (5), or did not write full expressions (2). Of the four remaining students one got it correct and three wrote inappropriate full expressions.

Student Understanding of #9b&c

Evidence from the interviews suggests that on both #9b&c the students knew full well that solving the problem involved adding the lengths of the sides together. Here are responses from interviews with level one students which are typical:
(I stands for Interviewer and the other initial stands for the student)

CV left out all of #9 on the pre paper and pencil test. During the interview, when presented with #9, at first he said "I don't get it." But after he reread the example problem aloud he suggested 3*e* as an answer for #9a:

I: And why is this one 3*e*? (referring to his answer to #9a)
C: There are 3 sides and each one is labeled *e*.
I: And what does *e* stand for?
C: The length of a side.

After RL wrote "4*h* 1*t*" on the pre-interview test the following dialogue occurred:

I: How would you figure the perimeter? (referring to #9a)
R: Add all these together.
I: This one? (referring to #9b)
R: Add all the *h*'s together, and there is 4*h*'s and 1 *t*.
...
I: What about this one? (referring to #9c)
R: I did 5 plus 5 is ten plus 6 is 16 then there's...(he corrects it) there's 2 *u*'s.

CJ left #9b out both on the paper and pencil pre-test and when he tried problems for the interview, but when the interviewer went over the example problem (4*g* for a square) the following dialogue occurred:

I: Now what about this one? (referring to #9b)
C: is that a 1? (*referring to the lower case t that was used to label one of the sides*)

I: That's a *t*.
C: four times *h* plus *t*?
I: OK, next one?
C: 2 times *u* plus ...22?"
I: OK....how did you get 22?
C: 6 plus 5 is 11 and 6 plus 5 is 11 and that makes 22.

All of these students seem to understand that perimeter involves adding up the values of the lengths of the sides. The problem seems to be in accepting a full expression as a legitimate way to express an answer to a problem that involves unknown values.

This analysis provides a specification for when letters are and are not acceptable in the arithmetic frame. It seems that as long as the letters can be interpreted as representing an object or value that is being counted up they are acceptable within the arithmetic frame and can be expressed as a simple concatenation (e.g. 3*e* in #9a). The difficulty of level two problems documented here seems to arise when unknown values must be expressed as part of a written full expression. For the perimeter problems on the test students, with minimal scaffolding, seem to be able to articulate that they want to add up the values of the sides. They seem to know what to do, but, because the format of the answer does not fit the arithmetic frame, they record it in an inappropriate way. The data seems to confirm the hypothesis that arithmetic frame students run into difficulty when the answer to a problem demands a full expression.

<u>Analysis of Level Two Problems where disruption of left to right order seems to play a role</u>

Problems #11a and b are the first problems where the disruption of left to right step by step solution seems to create difficulties.

Problem **#11a**:
What can you say about u if $u=v+3$ and $v=1$?.............
[correct: $u=4$]

Although students performed better on problem #11a than on any other level two problem, six students did get it wrong. If we compare the performance of students who got #11a wrong with their performance on #5a, a parallel level one problem (see below), we find that all those who got #11a wrong got #5a correct.

Problem #5a.

$$a+b=43$$

$$a+b+2=\ldots\ldots$$

[Correct answer: 45]

What makes #11a harder than the level one problems? It certainly does not share the demand for a full expression as an answer with problems #9b&c. In fact one could argue that it is a lot like the level one problem #5a because all you have to do is replace v with 1 as you replaced $a+b$ with 42 in #5a. One might argue that #11a should be easier than #5a because you are only dealing with replacement of one letter (v) instead of the two ($a+b$) of problem #5a. But, there are several subtle differences here. First in #5a the form was exactly $a+b=c$ with a and b given and c being blank. Note that in problem #11a this format is reversed $u=v+3$ or $c=a+b$ and in this case c is not a blank space to be filled in but a letter to be evaluated. While #5a can be solved left to right you cannot simply work left to right to solve #11a. #11a may be harder than the level one questions for some students because its format does not fit the arithmetic frame. In #9b&c the issue was the format of the answer (not being interpretable as a numerical result). In this case it is the format of the problem itself which does not fit the arithmetic frame model of $a+b=\ldots\ldots$ In the arithmetic frame standard problems are worked left to right[22] .

As mentioned above, support for this focus on the difference between #5a and #11a comes from the fact that all of the students who got #11a wrong got problem #5a correct.

---

[22]Note that the level one problem #6a (the missing addend problem $5+a=8$) can be thought of as $a+b=c$ or 5 plus what equals 8, this maintains the left to right format and is also a format with which students are readily familiar. In problem #11a, because the format is reversed ($u=v+3$), the left to right structure of the arithmetic frame cannot be maintained.

In this analysis it is the format of the presentation in #11a that is the major source of the difficulty. This is a testable hypothesis that should be explored. For example, this hypothesis suggests if #11a was presented in a different format it would be easier to do. e.g. if it was presented as:

If $v=1$ and $v+3=u$
What is the value of u...........

(note that I have also changed the order of the equations to provide the value of v first to facilitate the left to right format of the arithmetic frame)

Based on my arithmetic frame analysis this alternate version of #11a would be at the same level of difficulty as the level one substitution problem #5a.

This focus on format raises a series of other questions. For those students who have difficulty:

•What aspect of the change in format from the arithmetic frame is most significant?
-Is it just having the result of an operation on the left?
-Is it that the value of v is given after the equation?
-Is it the presence of two letters?
- Could having multiple letters and letters on both sides of the equals sign be a significant part of the difficulty?
- Is the interpretation of the equals sign an important issue here? (If the equals sign is seen as a "do it" symbol and the problem is seen in terms of left to right steps, the equals sign, in this format, may not make much sense to arithmetic frame students).

• Could most of these students solve a purely numerical arithmetic problem whose order has been changed such as .....= 7 + 8? Solving such a problem would be more difficult because it does not fit the arithmetic frame pattern of operation on the left and numerical answer on the right ($a+b=c$). If it turns out that these sixth grade students <u>could</u> solve a reversed addition problem easily[23]

[23] Although I have not had a chance to formally test this, discussion with several elementary

one would have to ask what the interaction is between introducing letters into a problem and changing its format.

• Is it simply a question of never having had experience with this format? Is lack of familiarity the problem and is the new format something which, although it might initially cause difficulty, could be easily mastered? Evidence against the suggestion that it is just a question of familiarity is the fact that there was no significant change in the number of students who got this problem right from the pre to the post test.

Although collecting further data on these questions is beyond the scope of this study, this sort of work will either confirm or suggest modifications (and in any event add important detail) to the idea of an arithmetic frame of reference.

Student Responses

How did students who got #11a wrong respond and how might their answers make sense in terms of arithmetic frame issues? Two students simply left this problem out presumably deciding the difference in format made the problem too difficult to deal with. Two students gave answers that might specifically reflect a deforming of the problem to fit the a+b=c arithmetic frame (if the problem was deformed to $u+v=3$, then 2 would be the correct answer for u. One student did this. Another seemed to treat the problem as meaning $u=v$ ignoring the +3 on the right, he answered $u=1$)

A third pattern is revealed by two students who suggested that the letters could have any value. This response suggests that they somehow ignored the $v=1$ statement and only focused on $u=v+3$. This response might be seen as a variation of saying the problem cannot be done because you cannot determine a value for the letters, or as a more sophisticated view that a letter can take on several values (in Hart's analysis this is an interpretation that is characteristic of students who performed at level 3).

school teachers has led me to believe that most sixth grade students can solve problems in this form.

The interview data was inadequate to confirm the underlying basis of these answers to problem #11.

Although it should be kept in mind that 10 out of the 16 experimental group students got this problem correct in the pre-test, if my hypothesis about the importance of format in the arithmetic frame is correct, we are left with two questions. First, what is it about the arithmetic frame of the students who got this wrong that made this problem difficult and second, how must the arithmetic frame of those students be modified to answer this problem correctly? One obvious possibility is that for some students the standard arithmetic frame has an inflexible $a+b=c$ structure, and it must be modified to allow problems of the form $c=a+b$ to be interpreted appropriately. Questions for future investigation should include; what is the nature of the internal frame structure and exactly what changes might be involved (e. g. does it have to do with the ability to put off doing the computation, the ability to see the similarity between $a+b=c$ and $c=a+b$, a change in view of the equals sign...)?

An alternate way to frame these questions: How do students who got #11a and/or #11b correct perceive or experience them differently from those who did not get them correct? And can these differences be productively described in terms of an internal frame of reference? And, as students learn to deal with problems like #11a & b, is the idea of a change in their frame of reference a useful descriptive tool that can inform instruction?

Problem **#11b**.

> What can you say about m if $m=3n+1$ and $n=4$?....................
> [Correct answer: $m=13$]

The analysis of #11a also applies to #11b. As with #11a, one can solve #11b by replacing n with 4 and doing a simple calculation and from that perspective it might seem like it also ought to be a level one problem, but, as with #11a the format of the problem ($c=a+b$) is different from the standard arithmetic frame.
Although more students answered #11b incorrectly than #11a (8 of

the 16 for #11b vs. 6 for #11a) the difference is not significant.

Although #11a and #11b are not significantly different in level of difficulty, one interesting difference is that #11b involves concatenation (3*n*) and #11a does not. To solve #11b correctly students must remember that concatenation means multiply. The response of some students to #11b reveals a common pattern in dealing with concatenation. They had been told in the practice exercise before the test that when a number and a letter are put together it means multiply and in the interviews most students could easily articulate that putting a number and a letter together means multiply (e.g. if asked "What does 3*n* mean," most students would readily reply "multiply."). But, as in this case, a number of students often seemed to forget this definition in the midst of a problem. All four of the students who answered #11a correctly but made errors on #11b gave answers that could result from misinterpreting or ignoring the concatenation in some way (e.g. they added instead of multiplying) and those who were challenged in the interview (two of the four students were asked "what does 3*n* mean?") changed their answers to the correct interpretation.

We can see all these confusions in terms of distance from the standard arithmetic frame format. The results of #11a and #11b support the argument that the more a problem departs from the standard arithmetic frame format the more students will have difficulty with the problem.

<u>Summary</u>

Student responses to problems #11a&b support the idea that for some level one students changing the format of the problem to a $c=a+b$ form makes it more difficult. The number of students who have difficulty increases with the addition of the new syntax of concatenation. The details of just what produces the difficulty; if and how the frame of a student who has difficulty with these problems is different from the frame of students who do not have difficulty, and how the frame needs to change to accommodate problems of this form are questions for further study. Such study should both help clarify the usefulness of the idea of a frame of reference and deepen our understanding of how students come to solve these kinds of problems.[24].

Problem **#13d**

The format of #13d, as that of #11a&b, does not fit the left to right step by step $a+b=c$ arithmetic frame structure.

13d. $2a+5b+a=$.................
[correct answer: $3a+5b$]

Is #13d harder than level one problems? Almost twice as many students got #13d wrong as the related level one problem #13a (9 students answered #13d incorrectly vs 5 incorrect answers for #13a):

#13a. $2a+5a=$ ....
[Correct answer: $7a$]

Why is #13d harder? First it requires a full expression where #13a does not. Secondly, although this problem is different from #11a and b in that it does not involve substitution or a numerical result, its format and order, like that of #11a and b, may be a second source of its difficulty. In problem #13d students cannot follow the standard arithmetic frame form: number operation number. Nor does it fit the procedural path of calculate and get a numerical result that characterizes the arithmetic frame.

The first step in the problem, going left to right, is $2a+5b$. Some arithmetic frame students get stuck here because they cannot do any computation. This may explain the high incidence of simply

[24]It should be noted that one problem with my arithmetic frame interpretation of problem #11a is that the two students who gave answers that we suggested could be attributed to deforming #11a to fit the arithmetic frame (see above), did not deform #11b in a similar way, in fact, they got #11b correct. This raises questions about how they interpreted #11a and #11b; e.g. it could be that for some students the concatenation in #11b was its most salient feature and led them to simply do the substitution and multiplication as the first step. This is the kind of analysis that will help focus future interview work with students.

leaving the problem out as an arithmetic frame response to level two problems. Almost half of the students simply left out #13d. A second pattern of response to problems that do not fit the arithmetic frame is to deform the problem to fit the frame and these students deformed the problem using an approach I have identified with the arithmetic frame: add up and/or count the instances of letters (2+5+1), in this case ignoring the *b* as different from the *a*, leaving it out, and generating an arithmetic frame answer (e.g. 8*a*). A third pattern is also represented in the answers to #13d: doing a computation (in this case multiplication) with the available numbers (2 and 5) and sticking the result onto the letters (10*a*+*b*).

Problem #13d provides an example of how the introduction of letters into arithmetic demands new rules and procedures, and new ways of doing things (in this case being able to put off execution and associate terms).[25]

A question for further investigation is just how students who are stuck at having to go left to right in a step by step computation fashion learn to see a multi-term problem as a separable set of terms in which like terms can be combined out of order. An important part of this will be exploring the role an object interpretation of letters might have in that transition. The marble bag metaphor that includes the idea of letters as objects (the marble bag) with letters as numbers (the number of marbles in the bag) may be one avenue of productive investigation.

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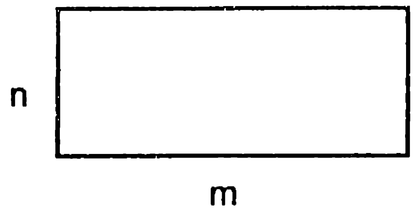
[25]One should note that in actuality the introduction of letters is not the first place this problem arises, the introduction of multi-step problems (e.g. 4+5x6) and the rules of the order of operations presents a similar challenge to the arithmetic frame. Student difficulty with order of operations issues is well known among middle school mathematics teachers. The introduction of letters creates an additional context for this departure from the arithmetic frame.

An AnomalousProblem #7c

7. What are the areas of these shapes........................

#7c.



A=..................................

[Correct answer: *nm*]

Although problem #7c is like some of the other level two problems in that its answer is neither a number nor a number and letter concatenated and letters are used to label the sides of a geometric diagram, it also has some characteristics that do not fit with the other level two problems. Before considering those differences I will compare its difficulty to that of similar level one problems.

Was problem #7c harder than similar level one problems for our students? The immediately previous problem on the test (#7b) is classed by Hart as a level one problem. It is a purely numeric version of #7c with no letters. #7b was answered correctly by 13 students. Problem #7c was answered correctly by only 4 students. Of the 12 who got it wrong 4 wrote inappropriate full expressions, 4 gave standard arithmetic frame responses of evaluating the letters and giving numerical answers, 1 answered zero, and 3 left it out.

Why is #7c harder? First, the answer is neither numerical nor a simple expression based on adding or counting up instances of a letter. With these characteristics problem #7c obviously does not fit the arithmetic frame characterization of the level one problems.

But, as noted above, it does not quite fit my level two characterization of problems that allow some computation, but demand a full expression (multiple terms with an operation sign) as an answer. In contrast to the other level two problems, this problem does not allow you to do <u>any</u> computation. You can neither count nor add nor multiply in a standard arithmetic way and you have no numerals at all in the problem or the answer as all the other level two problems do. The form of the answer is also something new. It neither fits the definition of a simple expression (e.g. 3*a*) nor a full expression (with an explicit operation sign).

Based on an arithmetic frame analysis one might expect the problem to be more difficult than the other level two problems because there is no numerical operation that can be done (a described in the next section the lack of numerical computation is a characteristic of level three problems).. But problem #7c was not harder. #7c had the same difficulty level as most of the other level two questions. In terms of number of incorrect answers it fell right in the middle of level two responses[26]. A possible hypothesis for why it was not more difficult is that the very lack of numbers does not invoke the computational demands of the arithmetic frame and makes it easier for some students to generate a letters-only expression[27]. The question of the difficulty of (and the effect of working on) problems with just letters and no numbers vs. working with problems that involve numbers <u>and</u> letters is another topic worth further study. I will return to this problem and its significance when I analyze the level three problems below.

Problem **#15a**

The last level two problem is #15a, a purely numerical problem. Based on my analysis its status as a level two problem is a bit mysterious. In many ways it fits the arithmetic frame: two

[26]And if you include the four students who wrote full expressions, but used an inappropriate operation (e.g. $n+m$), as breaking out of the arithmetic frame, the problem appears even easier within the set of level two answers.

[27]One other possibility is that students are familiar with a formula for the area of a rectangle in the form $A=LW$. In his analysis of the use of variables Usiskin (1988) lists this as one of the common uses that students are familiar with. This familiarity may make it easier for them to accept nm as an answer to this problem.

numbers are presented and the example problem models subtracting them. Why it is more difficult than the level one numerical problems is an interesting question. One possible answer is that it is more difficult because students must construct their own equation (must generate the a-b format), but because there are no letters involved I will not pursue it here. One thing that the categorization of #15a as a level two question makes clear is that there are multiple sources of difficulty, and that the difficulty of some of these problems may not depend solely on the introduction of letters. Other sources of difficulty may include the issue of format and, in this case, the need for students to generate their own equation. Although this study focuses on letters and the arithmetic frame, future development of the idea of an arithmetic frame will need to explore a broader range of parameters

Summary Of Level Two And The Arithmetic Frame

My argument is that we can make sense of why the level two problems are harder than level one problems if we look at them in terms of how good a fit they make to an internalized model students have of what a math problem is supposed to look like and what solving a math problem is supposed to involve. From my analysis of level one problems I have found several constraints imposed by the arithmetic frame:

1. Problems must produce a numerical result, a result that is derived from some numerical calculation (including counting). Letters can be part of an answer if they can be interpreted as objects that are being counted up and take the form of a number concatenated with a letter (e.g. 3*a* is interpreted as meaning 3 *a*'s and the 3 must be able to be derived by counting or simple addition).

2. For some students problems must either have the form $a+b=c$ or be easily reducible to that form.

3. Problem must be able to be done left to right with a calculation at each step.

I have argued that problems that contradict one or more of these constraints are more difficult than the level one problems and an analysis of the level two problems seems to bear this out.

Changes in the Arithmetic Frame at Level Two

The changes in the standard arithmetic frame suggested by this analysis of level two problems are:

1. A willingness to accept or generate an answer that includes a explicit operation in it (a plus or a minus). Students must accept the idea that a number is not the only legitimate answer to a mathematical problem, that a full expression (e.g. one with a plus sign in it) can also be an answer. At level two this seems to develop in some contexts (e.g. #7c&d,#9b&c,#13d), but not in others (e.g. #4, #5, #9d). I will look more carefully at these differences in the next section (level three).

2. To solve problems like #11a and b students must not be limited to dealing only with problems of the form $a+b=c$. They must not be limited to a view of problems that says "do an operation on the left to get a number on the right". Students must develop a willingness to allow and carry out operations on either side of the "=" sign.

3. In addition students must not be limited to linear left to right carrying out of operations. They must develop the ability to recognize and combine like terms in an addition context (associativity) (#13d is an example of this).

Note that each level two problem, except #7c, still allows the student to actually do some kind of numerical computation (in #13d ($3a+2b+a$) it is the counting up of objects described above). In #7c there are no numbers at all so the idea of doing a computation may simply not be invoked.

Level two questions are not the most difficult in the portion of the

Hart algebra test I used. There is another set of questions referred to by Hart as level three. In the pre-test none of the students reached level three. In fact fully half the students did not get any of the eight level three questions. The most any student got was 3 of the 8 and only two of the sixteen students did this well. Hart asserts that the reason the level three problems are more difficult is that they demand that students operate with letters as they would with numbers. As I have already pointed out this analysis does not seem to be consistent (e.g. problem #7c). In the next section I will examine the level three questions from the perspective of the arithmetic frame of reference.

The Arithmetic Frame And Level Three Problems

Based on Hart's statistical analysis all of the level three problems are harder than the level two questions. Below are the level three problems.

Level Three Problems

Level three problems are in bold:
Correct answers are in square brackets ([]) following each problem.

4. 4 added to n can be written as $n + 4$. add 4 onto each of these:

(c) $3n$

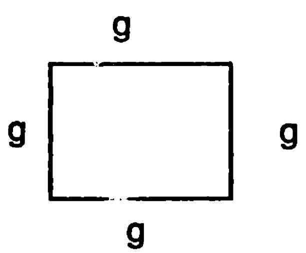
[correct answers: **(c) $3n+4$**]

5. **(c). $E+F=8$**

$E+F+G=$.....

[Correct answer: $8+G$]

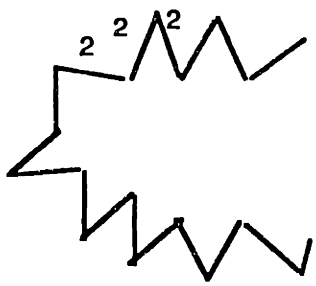
9.



This square has sides of length g.
So, for its perimeter, we can write $P = 4g$

What can we write for the perimeter of each of these shapes?

9(d)



Part of this figure is not drawn.
There are *n* sides altogether, all of length 2.
P = ...
[correct answer: 2*n*]

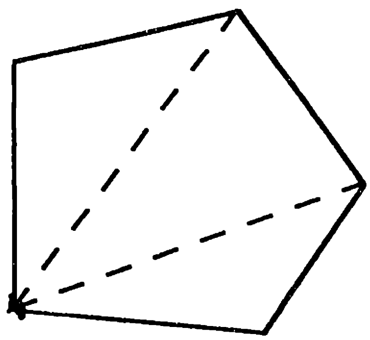
13. $a + 3a$ can be written more simply as $4a$.
Write these more simply, where possible:

13 (b) 2*A*+5*B*=...............
[Correct answer: 2*A*+5*B*]

13(h). 3*A*-*B*+*A*=..................
[Correct answer: 4*A*-*B*]

14. *R*=*S*+*T*, *R*+*S*+*T*=30, *R*=.................
[Correct answer: *R*=15]

15.



In a shape like this you can work out the number of diagonals by taking away 3 from the number of sides.
So, a shape with 5 sides has 2 diagonals:

15(b) a shape with *k* sides has .... diagonals.
[Correct answer: *k*-3]

16. What can you say about *C* if *C*+*D*=10 and *C* is less than *D*?....................
[Correct answer: *C*=1,2,3, or 4]

Level Three Problems and the Arithmetic Frame

Why are level three questions harder than those of level one or two? I will begin with a brief review of Hart's position. Hart claims that all level three questions demand students be able to operate on or with letters as they would with numbers, and level one and two questions do not demand students operate with letters. The problem is Hart's definition does not differentiate a majority of the level two problems from the level three problems. An example is problem #7c which, in Hart's empirical analysis of difficulty, is classed as a level two problem. In order to generate the correct answer to problem #7c you must multiply *n* times *m* and write *nm*. This seems to be a clear instance of having to operate with a letter. A second example of this difficulty are the level two problems #9b&c. In problem #9c students must write an expression to represent the perimeter of a 5 sided shape labeled 5,5,6,*u*, and *u*. To answer correctly they must write something like 16+2*u* or 5+5+6+*u*+*u* and many students who were not at level three did just that. These answers also seem to involve operating on letters (much like the level three problem #4c in which you must add 4 onto 3*n* and answer 3*n*+4). Many students were successful in writing appropriate expressions for these level two problems, but not for #4c. Problem #9b is the same except the 5 sided figure is labeled *h,h,h,h,t* and students must write an expression such as 4*h*+*t*. Again this seems a clear example of operating with letters, yet, in terms of difficulty, it is not a level three problem. All three of these problems (#7c, #9b, and #9c) were much easier for students to do than the level three problems. Unless we can explain why #7c and #9b&c are special exceptions they would seem to be strong counter-examples to Hart's explanation of why the level three problems are more difficult. Is there an alternative way to explain the difficulty of level three problems? Looking at the problems from an arithmetic frame perspective can resolve some of the difficulties with Hart's "operating with letters" characterization.

The level three problems <u>are</u> statistically more difficult than either the level one or level two problems. The story of students dealing with level three problems is complex. When viewed from a frames perspective it seems to involve combinations of features of the problems. In this study I was not able to go through enough iterations to definitively isolate the complex of factors, but through the following analysis of the level three problems I have teased out a set of hypotheses that I believe are worth further exploration.

Problem Analysis of #4c: What are level three characteristics?

Level three problems have a number of characteristics. I will begin with a detailed analysis of problem #4c which includes the largest number of those characteristics then look at how the other level three problems reflect and, in some cases, expand the set of issues it raises.

Problem #4c:

4. 4 added to *n* can be written as ***n* + 4.**
Add 4 onto each of these:
(a) 8 (b) n+5 **(c) 3*n***
[The correct answer to #4c: 3n+4]

How is this problem similar to and different from the level two problems?

Like the answers to many of the level two problems, the answer to this problem does not fit in the arithmetic frame. As in several of the level two problems the answer to #4c is not a number, students need to write a full expression to generate the correct answer. If students must write expressions for both this and many of the level two problems how is this different and why is it harder?

One difference is that in the level two problems that involved numerals students had to do some kind of numerical calculation (either addition or counting) as part of the process of generating the correct answer (e.g. #9a&b, #11a&b, #13d), in problem #4c generating a correct answer does **not** involve doing **any** actual

computation with numbers.

Not only is there no calculation in the correct solution process, but in this problem a second difference is that the **expectations** of a problem that will fit the **arithmetic frame** are **strongly invoked**. In the instructions the students are told explicitly to add and there are two numbers (4 and 3) presented in the problem. This situation makes it relatively easy for students to follow their newly learned arithmetic frame compatible rule that "a letter must stand for a number" by (inappropriately) replacing *n* with 4. Many of the level two problems did not have this characteristic of explicitly telling students to do an operation, providing two numbers, and having a letter in the problem. As mentioned earlier, one way of looking at the function of the "letters stand for numbers" rule for arithmetic frame students is that it allows them to turn problems with letters in them into numerical problems that fit the arithmetic frame.

Many of the problems on the test begin with a simple model problem to introduce students to the problem set. This is often a simple numerical problem that fits easily within the arithmetic frame. Problem set #4 and most level two problems follow this pattern, but a third difference between many of the level two problems and problem #4c is that for the level two problems the previous problems in the set (e.g #9a, #13a) often directly model the kind of calculation needed to solve the level two problem (#9b&c, #13d). #9a models counting up the number of instances of a letter. Counting up is the operation demanded by #9b&9c. The same pattern holds for #13a and #13d. In problem #4, on the other hand, although the model problem (adding *n* onto 4 produces *n*+4) models the process for #4c the first problem in the set (#4a) does have you add two numbers and get a numerical result (8+4=12). In #4c you cannot do a numerical calculation and you get an expression rather than a numerical result. In #4, as opposed to #9 and #13, doing an arithmetic frame problem is modeled in previous problems. One can think of it as #4a invoking some aspects of the arithmetic frame, invocations that do not occur for #9b&c or #13d.

A fourth element that makes #4c different from some of the level two problems (#7 and #9) is that the letters in those level two problems stand for the lengths of lines in geometric diagrams. The letter in #4c has no referent at all. It may be that these students feel less compelled to turn a letter into a number when it specifically refers to something one can think of as having a specific value (like the length of a line in a diagram) and that in situations where the letter does not have an identifiable referent students find it harder to generate a full expression. It could be that it is easier to think of a letter as having a value when it is used to label a line in a figure and there is less need therefore to give it an actual value[28]. This is a testable hypothesis and needs to be explored with appropriate problems. For example, one might use a problem that is like #4c, but is based on a geometric model such as:

#4c variation:

Each of the marked segments in the line below is *n* inches long:

|*n*|*n*|*n*|

the length of this line can be written as 3*n* inches

Write an expression for a line that is 4 inches longer............................

I would hypothesize that this version of #4c would be easier for students than the original because in the graphically based version the *n* has an identifiable concrete referent. I would also argue that this version would not invoke the arithmetic frame expectations as strongly because in the original you are presented with a number that can be thought of as replacing a single letter (the *n* in 3*n*) whereas in the alternative version the 3 *n*'s are explicitly displayed. These are ideas that need to be explored through interviewing students about how they solve different versions of #4c.

The above discussions lead me to the following conjectures:

[28] If this is the case it might help explain why #7c (based as it is on a diagram) is easier than some of the level three problems.

• problem #4c, and perhaps other level three problems, are more difficult because, for the first time, students must accept that it is legitimate to generate an answer to a problem **without actually doing any numerical calculation** (as they do for most level two problems).
• Level three problems are more difficult because they present a context in which students must understand that it is acceptable to not do any numerical calculation <u>even when the problem explicitly tells you to add a numerical value</u>.
• The idea of providing a numerical answer may be reinforced by previous problems modeling numerical computation in the solution process.
• Another possibly significant characteristic is the abstract non-referential use of letters (for many of the level three problems letters are not explicitly tied to something that can be experienced as an object like the sides of a geometric figure).

<u>Problem #7c - a seeming anomaly revisited</u>

Given this analysis of problem #4c we are now ready to re-examine the level two problem #7c and see if an arithmetic frame analysis can help clarify its anomalous character. #4c is a level three question and #7c is a level two question. This means, based on Hart's statistical analysis of her data, #7c is easier for students to answer correctly than #4c and that students who got #4c correct tended to also get #7c correct, but not the reverse.

The first question to raise is how did our sixth grade students fair with these two problems. In the pre-test all the students responded to #4c with numerical arithmetic frame answers. If we look at #7c from the perspective of correct/incorrect we see that 12 of the 16 students gave incorrect answers on the pretest, but if we look at it from a frames perspective we discover that actually 8 students wrote expressions and thus were able to break out of the arithmetic frame in answering the question (four students interpreted the problem in terms of addition rather than multiplication and wrote expressions that added the values of n and m). From a frames perspective fully half the students on the pre-test were able to move outside the arithmetic frame in responding to #7c whereas none of them were able to do so for problem #4c.

Given that #7c was easier, our problem is that it also seems to have a number of characteristics that fit a level three characterization:
1. The arithmetic frame is invoked by previous problems (#7a&b are pure numerical computation problems)
2. It involves operations with letters (*n* times *m*).
3. There is no computation involved.

Since both Hart's statistical analysis and the data from this study tell us that #7c is easier than #4c and #7c has the above non-arithmetic frame/level three characteristics the difference in difficulty raises questions for both Hart's and an arithmetic frame analysis. If #7c involves operating with letters and has these non-arithmetic frame characteristics why is it less difficult?

One difference between #4c and #7c is #4c involves addition and #7c involves multiplication. One might question whether this might provide a basis for the difference in difficulty. Collis (1978) has suggested that a term like *nm* might look more like a number to students than $3n+4$ and therefore might be more acceptable as an answer. One piece of evidence that suggests that this difference may not be significant as a broad generalization is that several students responded to #7c as though it were a perimeter problem and wrote $n+m$ as an answer (4 of the sixteen in the pre-test). One way to check the conjecture that multiplication problems that result in concatenated letter answers (e.g. *nm*) are easier than addition problems with letter based answers (e.g. $n + m$) would be to devise a question that involved addition with just letters (e.g. a perimeter problem instead of area such as a triangle with sides labeled *l*, *m* and *n*). Collis's explanation suggests that the letters only multiplication problem would be easier than the letters only addition problem because the form of the answer would more easily fit the arithmetic frame. Although it may at first appear that #9b and #9c provide this kind of problem, it is important to note that in both #9b and #9c there is counting up of letters which may be interpreted by students as a kind of numerical calculation. It is my hypothesis that this component of numerical computation makes them easier than the level three problems.

If my conjecture about previous problems invoking the arithmetic frame (e.g. #4a, #4b) is true we would expect the first problems in #7, which are purely arithmetic, to invoke the arithmetic frame as I argue it did for problem #4c. Although four students did give numerical answers to problem #7c all 16 gave numerical answers to problem #4c. Why was #4c so much more powerful in stimulating numerical answers? Although correct answers to both #4c and #7c involve no numerical computation one difference between the two problems is that #7c does not <u>support</u> any numerical calculation, in #7c there are no numbers to calculate with, the problem only presents the letters *n* and *m*. In contrast problem #4c presents two numbers and one is told explicitly to add. **The lack of numbers in #7c may make it easier for students to come up with an answer that does not fit the arithmetic frame.** It may be that having numbers in the problem is a feature that invokes the arithmetic frame and the lack of numbers is a characteristic that allows students to break out of the arithmetic frame. One way of looking at this is problem #4c, which, because it has numbers in it, is easier for students to deform into a problem that fits the arithmetic frame than problem #7c. In problem #4c the need to do an operation and get a numerical result is easily fulfilled by replacing *n* with 4. Neither problem #7c nor any of the other level two problems had this characteristic.

One topic for future investigation is the interaction between the characteristics of previous or model problems and the characteristics of the problem itself. One thing to test will be problems which provide non-arithmetic frame models in previous problems, but still include invocation of the arithmetic frame through having numbers and explicit reference to operations in the problem itself (e.g. present #4c in a context that does not invoke the arithmetic frame). In related work Booth (1984) found that for a problem that involved numbers, and did not specifically suggest an operation, using different model problems did have a significant impact on student responses. Interpreted from a frames perspective the broader question is what aspects of a problem (e.g. the previous problem, numbers present in the problem, the explicit invocation of an arithmetic operation) are most powerful in invoking an arithmetic frame interpretation by students and if and how do those aspects interact?

The second difficulty mentioned above is the fact that #7c involves operations with letters (*nm*). Hart tells us that level three problems involve operating with letters and solving #7c involves operating with letters. It seems that Hart's own data shows that level two students can operate with letters; in #7c they successfully express the idea that *n* is multiplied by *m* by writing *nm*.

Using the arithmetic frame how can we recharacterize level three problems to resolve this conflict? For level two students who can write *nm* the difficulty seems to arise when they must add the number 4 onto a letter or expression with a letter in it. I now believe the significant issue, for students who can already respond to problem #7c with "*nm*", is **the ability to accept that writing "+4" is a legitimate way to express "doing" an operation that involves a number and accepting and understanding that there are situations where you cannot (or where it is legitimate to not) actually computationally add the 4.** The salient feature of the arithmetic frame is the need to do a computation with numbers. One way of thinking about this is that for these students, who are able to express *n* times *m* as *nm*, it is the definition of "add 4" (or add a number) that needs to change or be expanded to cover situations where you cannot do an actual arithmetic calculation. My hypothesis is that students who can answer #7c with *nm* can generate expressions with letters, but not in all contexts. Many of these students were at a stage where problems that involve letters and numbers and do not involve any numerical computation invoke the arithmetic frame in a way that leads them to inappropriately replace a letter with a number and thus avoid generating an expression that involves both numbers and letters.

The above analysis leads to a more general critique of Hart's characterization. Hart argues that moving into level three involves learning to treat letters just as you do numbers. She describes children who were unable to answer a level three question as being unable "to use a letter as a numerical entity in its own right" (Hart, 1981 p. 117). The implication is that this process should be a simple transfer or mapping of doing what you do with numbers to letters. In fact, it is not at all a question of simple transfer. When letters are introduced into problems with numbers you have to add a whole new set of entities (expressions) and rules for doing operations. In the arithmetic frame when one adds a number one simply does a computation and comes up with a numerical result. In the arithmetic frame if one was told to add 4 onto 8 and one responded with "8+4" it would be seen as just restating the problem not answering it, yet in problem #4c that is exactly what one is expected to do. The problem is that level two students still do not see writing "+4" as "doing" the operation add four. In #4c adding 4 means generating a new expression in place of doing a numerical calculation. In a sense **students definition of adding needs to expand in steps: first, at level two, to simply being able to write an expression as part of an answer (but still an answer that was generated by doing some calculation), and then, at level three, to include writing an expression without doing any computation.**

Another way to articulate this continuum of difficulty is to say that problems that involve two numbers are easy to solve (they follow the rules of the arithmetic frame), problems that involve just letters do not fit the arithmetic frame at all and so it is easier for students to generate expressions without computation as answers, but problems that involve letters and numbers engage the ideas of the arithmetic frame (solving a problem involves doing a numerical calculation) and are more readily deformed to fit into the arithmetic frame.

Summary of Development towards Solving Level three problems

From a rules perspective: at levels one and two students feel a need to do a numerical operation. For them solving a problem means doing a numerical operation. The first step (level two) away from a pure

arithmetic frame is being able to write a full expression (they can accept something that is not simply a numerical answer), but only if they have done some kind of computation (including counting)[29].
It is a further step **to be able to simply write an expression and do no calculation particularly if there are numbers involved (#4c).**

Table 15 presents one way of combining this analysis with the analysis of level two development.

TABLE 15
Characteristics of Problems at different levels of Difficulty Identified by Hart's Analysis

Kinds of problems that students can solve at each level:

	Simple Calc	Comp&Exp	Non Left To Right Comp	No Comp&Full Exp
Level 1:	√			
Level 2:	√	√	√	
Level 3:	√	√	√	√

Key: Simple Calc (simple calculation): problem involves simple arithmetic calculation only.

Comp&Exp (Computation and Expression): problem involves some kind of calculation (e.g. counting or addition) and the generation of a simple expression (e.g. $3n$)

Non Left To Right Comp: computation cannot be done in a step by step way moving left to right in the problem.

No Comp&Full Exp (No Computation and Full Expression): the problem does not involve any numerical computation and demands a full expression as an answer

Because it is currently an unexplored conjecture there is one additional factor not included in this chart: it may be that problems that use letters to refer to lengths of lines are easier to generate expressions from than problems where the referent of the letter is unspecified.

In my analysis the change from level one to level two to level three

[29] Problem #7c does not fit this characterization, but I have suggested that its level two status may derive from two characteristics: in contrast to problems like #4c it does not involve numbers at all and it is based on a diagram where the letters refer to lengths.

is **a multi-dimensional context-dependent gradual step by step modification of the arithmetic frame rather than simply Hart's level three development** of the ability to operate with letters as one would with numbers. Hart's characterization hides the steps, the importance of context (e.g. computation in the problem, the letter's referent, presence of numbers), and the multi-dimensional character of that development.

In the long run (beyond the scope of this study) these conjectures need to be explored empirically by devising a broader range of tasks that differentiate the different types of problems and conditions suggested. In the short run we can look at the other level three problems in the test and see how they fit the characteristics I have outlined as a source of increased difficulty. The characteristics of level three problems identified by my arithmetic frame analysis are:

1). Arithmetic frame expectations are explicitly set by previous problems (e.g. they model doing numerical computation) and by the structure of the problem itself (e.g. having two numbers or a mix of numbers and letters in the problem, told to explicitly add, a number readily available to replace a letter).
2). No numerical calculation is involved in generating the answer.
3). The arithmetic frame idea of replacing a number with a letter is supported by providing a number in the question (e.g 4) and a letter (e.g. *n*) in the problem.
4). Letters do not have a concrete referent such as the length of a line in a geometric diagram.

<u>The Other Level Three Problems</u>

Now I will turn to an analysis of the other level three problems to see how they support and extend the findings for problem #4c. For each problem I will consider how it has characteristics like those I have identified in #4c, how the students performed on the problem, and then how the arithmetic frame must change in order for students to answer the problem correctly. Where appropriate I will also consider questions for further investigation raised by specific problems.

A Problem Similar To #4c: #5c

The level three problem that shares the most characteristics with problem #4c is from the next problem set on the test, #5c:

Problem #5:

5\. (a). $A+B=43$,

$A+B+2=$........[Correct answer: 45]

(b). N-246=762

N-247=..........[Correct answer: 761]

(c). $E+F=8$

$E+F+G=$.....[Correct answer: 8+G]

Level Three Characteristics

1\. In problem #5c, as in #4c, the arithmetic frame is strongly invoked by previous problems that model a numeric answer and involve replacement.

Modeling a Numeric Answer:

Like the first problem in problem set #4, problem #5a ($a+b=43$ $a+b+2=$....) (although it does have letters in it) is a numerical problem with a numerical answer. #5b also has a numerical answer. The first part of problem #5c itself models doing a calculation of some kind and getting a numerical result ($E+F=8$) (note that here you <u>are</u> adding two letters together and getting a numerical result).

Modeling Replacement of a letter with a number:
The idea of replacing a letter with a number is modeled in both #5a and the first part of #5c.

These aspects of the problem reinforce the expectation that an answer should be a number. The format and structure of these problems invokes the arithmetic frame. The result is that students seem to assume that G must have a value and invent inappropriate ways to give it one. This is true even for students who generated correct full expressions as answers to level two problems.

2. #5c shares a second characteristic with #4c: there is no numerical calculation to do in the problem.

3. And finally, the letters in #5c do not refer to parts of a geometric diagram.

Other similarities to #4c include; the answer to #5c is a full expression that includes both a letter and a number and #5c involves addition.

Student Performance:

In the pre-test most students either wrote numbers (11) or left it out (3). Only 2 students gave the correct answer. These results are very similar to those of #4c.

Changing the Arithmetic Frame:

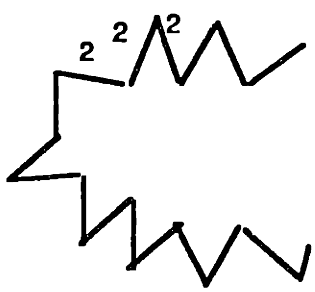
In my analysis, in order to answer #5c correctly, arithmetic frame students must change their frame for interpreting and solving problems. This change can be described as moving away from "the only way to solve a problem is to do numerical calculations," to the idea that **there are other legitimate solution activities/operations, some of which may not involve numerical calculations**. In considering #4c I said that students need to change their definition of what an idea like "add 4" might mean. A more general way of stating this to include both #4c and #5c is: to answer many of the level three questions students must **expand their definition of legitimate mathematical operations**. In the new context of these problems they must learn to differentiate the mathematically legitimate operations from operations that are not legitimate. In this new context students need to learn in which situations it is legitimate to directly substitute values, to determine a value by manipulating the expression (simplification or other operations) and then substitute, and in which situations the only legitimate thing they can do is substitute or simplify without a resulting computation or numerical answer. Problem #5c is an example of substitution with no computation. In the expression $e+f+g$ all you can do is replace $e+f$ with 8 (with the result of $8+g$) and in the arithmetic frame this does not qualify as an operation.

Another way to think of this is to say that some students have a model of a math problem that says one has to do an operation to solve a problem. To deal with #5c students have to move from the idea that the operation must involve a numerical calculation (a level two understanding) to the idea that, in this context, replacement (e.g. 8 for $e+f$) is a legitimate operation, and, as in this case, you do not have to do a numerical calculation even when there are numbers involved. In this case students need to add replacement to their repertoire of legitimate and adequate mathematical operations.

Level Three Problems That Involve Geometric Diagrams

Although #4c and #5c do not involve geometric diagrams some of the level three problems do. The first of these is in problem set #9:

#9(d):



Part of this figure is not drawn.
There are *n* sides altogether, all of length 2.
P = ...
[Correct answer: 2*n*]

Level Three Characteristics:

What characteristics does #9d share with the earlier level three problems that might make it more difficult than the level two problems?

1. The arithmetic frame is invoked by the presence of several numbers and a letter in the problem, and by previous problems all of which involve some sort of arithmetic computation in the generation of their answers.
2. There is no numerical calculation to do in the problem.
3. Although this problem is based on a geometric figure, as some of the level two problems are, in #9d there is no concrete referent for the letter *n* in the problem (*n* stands for the number of sides which is not specified).

Comparison with level two problems:

One way to look at the difficulty of #9d in more detail is to compare it with level two problems.

#9d and #7c:
First a comparison with the level two problem #7c, a problem that also focuses on writing an expression about an aspect of a geometric figure (its area whereas #9d involves perimeter). #7c is a problem that clearly involves operating with letters (in order to get the correct answer you must multiply *n* times *m*). If the problems are not different based on the fact that you have to operate with letters in each case why is #9d harder? Three differences that might contribute to the increased difficulty are:

A. The fact that #7c has no numbers at all whereas #9d mixes numbers and letters (the idea is that the mixing invokes the arithmetic frame: there are numbers to operate with and so students feel pushed to do a calculation).
B. The fact that #7c follows the context/pattern of the earlier problems in problem set #7 (in earlier problems you multiply one side times the other) and #9d does not (the earlier problems involve adding up the lengths of the sides while in #9d you must multiply the length of one side by *n*).
C. In #7c the letters are used to represent and specifically label a visually available concrete object (length of the side) whereas in #9d the letter (*n*) stands for the number of sides and there is no

specific visual representation of the number of sides (the drawing is incomplete and *n* does not label anything).

Point 1 in the list of level three characteristics suggests previous problems as a potentially important variable. In problem #9 the previous problems all involve counting the number of sides so one might argue that the precedent for counting (or in this case estimating) the number of sides has been set or invoked by the previous problems. For problem #7c the previous problems all involved multiplying the lengths of the two sides together and getting a numerical result. In the study referenced above Booth (1984) specifically experimented with changing the previous problems for #9d to make them match the process of #9d (modeling multiplication and not doing a computation instead of modeling counting and addition) and found that changing the previous problems changed student responses to #9d significantly. These points suggest that invoking aspects of the arithmetic frame can have a significant effect on student response to problems. These points also raise questions about at what point or under what circumstances the arithmetic frame can be inappropriately invoked. This also leads to another question: if the arithmetic frame can be invoked in some situations, could explicitly asking students to consider alternate frames (e.g. not doing a computation) be a powerful way to help them develop a more flexible approach to problems such as these? These are all questions worth future investigation.

In reference to point B above it is interesting to note that many more students inappropriately gave a letter a value in #9d than did so in #7c (nine vs. four). This supports the idea that the mixing of letters and numbers in a problem may more strongly invoke the need to do a computation. It seems that in problem #9d, which has numbers and letters, students were more apt to see their job as estimating the number of sides, whereas in #7c, which was a letters only problem, they were less apt to estimate the length of the sides. This leads to the following hypothesis: if problem #7c were changed to include a letter and a number it would be more difficult and would produce a different set of responses.

#9d and #9b & c:

The issue of why problem #9d is harder than the two level two problems #9b&c is a bit more complex. One might argue that #9d should be simpler than #9b&c because it only involves one letter and you only have to write a simple expression for it. But, in fact, it is much more difficult. This raises a number of questions: under what circumstances do students inappropriately give a letter a value (as they did in #9d), what seems to lead them to do this and when are they willing to simply write expressions with letters in them instead (as they did in #9b&c)? In this case we must explore what differentiates #9b&c from #9d. First, in #9b&c students generate expressions, but also do some kind of computation. In #9d they must answer with an expression and can do no computation. Second, the letters in #9b&c stand for lengths and in #9d the letter stands for the number of sides (which are not specifically represented). The resulting difference is that in #9b&c students are asked to do an addition of the form $a+a+c$ (a being a letter and c being a letter (#9b) or a number (#9c)) and in #9d we are asked to add "2" n number of times. Is this difference the basis for the difference in student response? Is it that in #9b&c (and #9a as well) you are counting up the number of n's - an action that has a familiar arithmetic quality to it, whereas in #9d you are being asked to do something different enough that students reject it and assume that they are supposed to count up the number of 2's. In #9a you are saying $3n$'s (or e's) and here you are saying $\underline{n}$ 2's. Does the position or function of the unknown (a label for a side vs. being a counter for the number of sides) make a big difference? These are also questions that warrant future exploration.

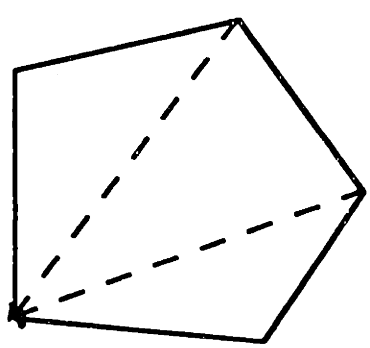
Changing the Arithmetic Frame:

How must students' arithmetic frame change in order to answer #9d correctly? As with #4c, in order to answer #9d correctly students must accept that it is possible to generate an answer without doing any numerical calculation. Students may experience a letter that stands for the length of a side in a diagram quite differently from the way they experience a letter used to represent an unknown number of sides (or as a counter). If this is the case students who can comfortably count up the number of sides and record the results as a full expression would need to learn to think of the counting they have just been doing as an unspecified value, and represent that value with a letter. Beyond this the role of context in making #9d difficult needs further exploration. Is part of the difficulty having to switch gears and see that problem #9d is asking students to do something unfamiliar and different from the work of the previous problems? How much of the difficulty comes from the role of the letter as counter and how much comes from having to switch from one way of seeing it to another? Booth's work (see above) does suggest that an important aspect of the difficulty may be the change in how the letter is used within a problem set.

More questions for future study: would it be useful to help students be explicit about the role of the letter in terms of what it represents? In this case would it help students to explore how using a letter to represent the length of a side of a shape is different from using it to represent the number of sides? If so, it would also be useful to explore how students perceive these differences.

Another level three problem that involves a geometric diagram is problem #15b:

15.



In a shape like this you can work out the number of

diagonals by **taking away 3** from the number of sides.
So, a shape with 5 sides has 2 diagonals:
15(a) a shape with 57 sides has ....diagonals.
15(b) a shape with *k* sides has .... diagonals.
[Correct answer to 15b: k-3]

Level Three Characteristics:

Why is #15b harder than the level two questions?
As in all the level three questions the arithmetic frame is invoked by the example and previous problem. Both problems involve doing an arithmetic operation (5-3 and 57-3) and have numerical answers. Secondly, although there are letters and numbers in the problem, there is no numerical computation needed to get an answer, and the answer demanded is a full expression. Finally there is no concrete referent for the letter (*k*) in the problem. One can think of the letter *k* in this problem having certain parallels to *n* in problem #9d. Although the problem is not multiplicative, *k* can be thought of as a counter representing the number of sides a figure has just as *n* was in problem #9d. Another way to describe how this problem departs from the arithmetic frame is to say that students must generate an expression without doing any computation and there is nothing in the presented problem itself that resembles the form $a+b=c$. In this case the students must generate the left side themselves.

Student Performance:

Student performance on the pre-test supports the idea that most of the students viewed this problem through an arithmetic frame. 12 of the 16 students gave standard arithmetic frame responses, they either answered with a number (7) or left it out (5). Three other students wrote single letters thus avoiding writing a full expression. Only one student answered it correctly.

Changing the Arithmetic Frame:

How must students' frames change in order to answer problem #15b correctly? First they must make the level two modification to have a full expression as an answer to a problem. Second, as in the other level three questions, in order to generate the answer *k*-3, they must be willing to generate an answer to a problem without doing an explicit calculation even though a number (3) is involved.

One point to make here is that it is not simply an issue of treating a letter like a number, as Hart implies, because you cannot actually do a calculation with the letter as you can with numbers. The idea is that you must be willing to express a hypothetical operation (what you would have to do if you knew the value of *k*) that you cannot carry out at the moment and accept the symbolic representation of that as an answer. The arithmetic frame expects a computation and a number as an answer. A phenomenon that is suggestive of the role of the arithmetic frame was students' response to a prompt that was used occasionally in the interview. When students did not know what to do with this problem (e.g. "I can't do it") the interviewer often asked them what they would have to do to solve it. A number of the students responded by first articulating that they would have to subtract 3 from *K* and then, when asked how they might write that down, wrote some version of *k*-3. JK's response on the pre-test interview is a typical example: she begins by explaining how she solved #15a: and then explains her answer to #15b which was the letter *l*. She explains that she subtracted two letters from *k* to get *l* (noting that she made a mistake thinking it was 2 and not 3). She finishes up saying, "see you don't know how much *k* equals." The interviewer responds: " So what would you have to do?" and she answers, "that would be *K* minus 3...equals *N*" and she writes down *K*-3=*N*. This was a typical response. For a number of the students once the interviewer focused on the idea of actually doing a calculation they seemed much more comfortable expressing the idea in terms of *k*-3. When they could think of it in terms of doing an actual computation (in arithmetic frame terms) they seemed to find the idea of *k*-3 much more acceptable. In this case the difficulty may not be as much the ability to operate with a letter, as it is in their frame based need to carry out a computation.

Looking at it from a frame perspective we might say that in the arithmetic frame there are only actual computations. One way of thinking about problem #15b is that here we must deal with computations that you would be able to do at some later point and record those computations as an answer[30].

Level Three Problems Where Step-By-Step-Left-To-Right-Order Plays A Role

In analyzing some of the level two problems I said that expecting problems to be done in a step by step left to right procedure with a computation at each step was a characteristic of the arithmetic frame. There are two level three non-geometric figure based problems where this characteristic comes into play.

The first of these is #13h:

13(h). $3A-B+A=$.............
[correct answer: $4A-B$]

Level Three Characteristics:

Why is this a level three problem? There is a computation one can do as there was in the level two problem #13d. This is an interesting problem because in form and structure it is exactly like the level two problem #13d ($3a+b+a$) except that the second term is subtracted instead of added. If we look at the #13 problem set in terms of levels of difficulty we can see a sequence that goes as follows:

Level 1:
13a. $2a+5a=$....................
[Correct answer: $7a$]

Level 2:
13d. $2a+5b+a=$.................

[30] I believe it is also important to note that this characterization is a very different idea from K as a representation of <u>any</u> number, the idea that *K* stands for and could take on any value within the range of the problem.

[Correct answer: $3a+5b$]

Level 3:
13h. $3a-b+a=$..........
[Correct answer: $4a-b$]

An arithmetic frame analysis that includes the issue of left to right step by step solution procedure provides one explanation for why the level two and three problems are more difficult than the level one problem. One way students interpret problems so they stay in the arithmetic frame is to interpret letters as objects. This may help students to answer level two questions such as #13d that do not support a left to right step by step procedure. This suggests that one step out of the arithmetic frame is to first think of the letters as labels for objects and then imagine putting together the objects, say, apples and bananas without concern for the order of their presentation. But, this object interpretation leads to difficulties with problems like #13h. Here Hart's analysis does provide an explanation of why #13h is harder even for students who have been able to answer #13d correctly. Even if students are able to break out of the need for a left to right step by step solution procedure at level two through an object interpretation of letters, they will need to change that view of letters to answer #13h correctly. Hart suggests that the empirical difficulty of this problem represents a break down of a label/object interpretation of letters. She suggests that students who see or interpret letters as labels for objects or objects themselves can handle problems like $3a+b+a$ because they can interpret it as something like three apples plus one banana plus one apple and can then combine the like objects. In a problem like #13h this interpretation breaks down because the idea of three apples take away one banana plus one apple does not make sense. Problem #13h takes a further step away from the arithmetic frame than that of #13d by involving an operation that is "impossible" to imagine under an object interpretation. Hart argues that it is only if one interprets the letters as standing for numbers and can operate with them that one can make sense out of this problem. Combining the issue of left to right step by step operations (one element of the arithmetic frame) and Hart's interpretation provide an explanation for some of the differential levels of difficulty in problem set #13.

The next level three question where order plays a role is problem #14:

14. What can you say about *R* if $R=S+T$,
and $R+S+T=30$
[correct answer: R=15]

Level Three Characteristics:

Although this question has a numerical answer (15), the solution process does not follow any step by step left to right process. Perhaps the nature of its structure can best be considered by comparing it to question #5a:

5. (a). $a+b=43$,
$a+b+2=$.......

Question #5a was answered correctly by 14 out of the 16 students in the experimental pre-test group whereas problem #14 was only answered correctly by 2 of those. Both problems involve two equations, multiple letters, and demand substitution. Why is #14 so much more difficult? Problem #5a demands only that a number replace two letters (which immediately creates an arithmetic frame problem) whereas #14 demands that letters be substituted for letters, and even when this is done one is left with an expression and no immediate numerical computation to do. In addition there are differences that are somewhat more subtle. #5a is a two step problem, but both parts have an arithmetic frame $a+b=c$ format. In contrast, in #14, the first equation is presented in $c=a+b$ form ($r=s+t$) and the requested answer is not the result of a computation on the left side of the equals sign. In #5a the substitution is the first step and, as mentioned above, this immediately sets up a numerical computation, whereas in #14 the substitution occurs in the second half of the second equation and does not lead immediately to any computation. Problem #5a presents two numbers to operate with and problem #14 does not. For students who have a rule that problems must involve numerical computation, in problem #14 there must seem to be nothing they can do that will bring them quickly to a numerical computation. Problem #14 simply does not fit the arithmetic frame and fully half of the experimental group left it out.

Changing the Arithmetic Frame:

What changes in the arithmetic frame are necessary to solve this problem? This problem involves substitution; students can solve it by substituting R for $S+T$. Arithmetic frame students are capable of substitution as evidenced by their ability to handle problem #5a, but in #5a the substitution led directly to an arithmetic frame format whereas in this problem the result of substitution still looks like a problem with letters ($R+R=30$). Although I have not had a chance to test it experimentally, through conversations with sixth grade math teachers, I have been convinced that most arithmetic frame students could solve $R+R=30$ or $2R=30$ (just as they solved problem #6a: What can you say about a if $a + 5 = 8$.). But the difficulty is that in the arithmetic frame the expectation is that any substitution should provide you with an arithmetic frame problem that leads directly to a numerical operation. To test this hypothesis one might give the students a problem such as:

> What can you say about R if $R=S+T$ and $15+S+T=30$,

to see if a single substitution that leads to a problem that more clearly fits the arithmetic frame ($15+R=30$) is easier for them to do or if the problem of substituting one letter for a letter sum is an important source of the difficulty.

In addition one might ask as a prompt:

"Some kids said since $R=S+T$ you can replace $S+T$ with R in the second equation. Are they right and would that help?"

to see if pushing them to do the first substitution would help. If these are easier it would lend credence to the idea that the arithmetic frame is limiting students ability to do two step substitution problems. If this is true arithmetic frame students must free themselves from the need to do (or see) numerical computations as an outcome of any step they take and accept that some problems may involve one or more steps that involve, from their arithmetic frame perspective, "non-numerical operations" before they get to a numerical (or other) answer.

<u>Further Steps Away From The Arithmetic Frame</u>

There are two other level three problems whose difficulty arises from further steps away from the arithmetic frame.

The first of these is #13b:

> **13. $a+3a$ can be written more simply as $4a$.**
> **Write these more simply where possible.**
>
> **13b. $2a+5b$ = ... [correct answer: $2a+5b$]**

Level Three Characteristics:

In a number of ways this is a standard level three problem. First, the arithmetic frame is invoked by prior problems because even though they use expressions the example problem ($3a+a=4a$) and the immediately previous problem (#13a: $2a+5a=$...) model being able to do an arithmetic operation. Secondly, in this problem, as in problem #4c, the student is confronted with two numbers and, as in #4c, #5c, and #9d, the problem does not allow the student to do any numerical calculation. This contradicts a basic tenant of the arithmetic frame (solving a problem involves doing some kind of calculation) even if the frame has been modified to allow expressions as answers (level two). Finally the letters in this problem do not have any concrete referents such as lengths in a geometric diagram.

But, in addition to these characteristics, problem #13b takes one step further away from the arithmetic frame because not only is there no numerical computation you can do, but there is also no legitimate operation you can do at all. In problems #4c and #9d even though you do not do an arithmetic computation you at least generate a new expression. In this problem there is no operation or transformation that you can do at all.

Student Performance:

As one might expect students had difficulty responding to this problem. Of the 15 who did not answer it correctly in the pre-test, 7 students did inappropriate operations with the numbers; 2 replaced the letters with numbers, and 6 left it out. Thus 9 students can be seen as directly deforming the problem to fit the arithmetic frame.

Changing the Arithmetic Frame:

What is necessary to understand in order to do this problem successfully? How must an arithmetic frame, even one modified to handle level two questions, change? This problem is different from almost every other problem on the test. In every other problem you have to make some mathematical transformation in order to generate an answer. In this problem you simply do nothing. From a frames perspective one must be willing to accept that you can generate an answer without doing any numerical computation when numbers are involved in the problem (Here you have a 5 and a 2 and a plus sign and in the last problem you added them together). This is an example of the arithmetic frame being strongly invoked. But, in addition, you must also accept that in some situations you cannot do any operations or transformations at all.

Hart's analysis suggests that the difficulty with level three problems arises from the inability of students to interpret and treat letters as if they were numbers. One question in this problem is whether part of the difficulty is based on student understanding of the underlying arithmetic. Could at least part of the difficulty be traced to a lack of understanding or experience with similar arithmetic situations? For example, how would these students respond to numbers only variations of this problem if they were told not do do the multiplication? For example, what would happen if you gave them 2x94 + 3x94 and asked them if there was any way they could simplify it before they did the actual multiplication? I would predict that many of them would have no idea of what to do. Does the difficulty with #13b reside in the use of letters, in the underlying arithmetic, or in some combination? Results from this kind of investigation might give us better insight into the difficulties students have with problems like #13b.

We also need to get a better understanding of how students who answer the problem correctly are thinking about it. In future work one might ask students who got it right the same numerical questions and in addition challenge them by asking why they couldn't add the two coefficient numbers together. This might reveal the level or basis of their thinking. For example, do they make an argument based on an understanding of multiplicative terms and when they can be added (the need for common terms), or is their thinking based on an adding apples and bananas metaphor (the "you can't add apples and oranges" idea), or are some students who get it right specifically using some other metaphor (like marble bags) as the basis for their thinking?

The last level three problem is number 16:

> **16. What can you say about *C* if *C*+*D*=10 and *C* is less than *D*? ....**
> [the correct answer includes the idea that C can have more than one value. e.g. C can be 0,1,2,3, or 4]

Why is this a level three problem? From the perspective of the arithmetic frame problem #16 seems like a standard arithmetic problem: find a pair of numbers that will add up to 10. Students find a single solution and think they are done. The *a*+*b*=*c* structure of the problems suggests a simple arithmetic frame response e.g. 4+6=10 and you are done. Ten out of the sixteen students responded this way (one got it right and five left it out). What students do not seem to consider when they answer with a single value for *C* is that the question is asking for more than one solution. This does not mean that they cannot think of a letter as having a variety of values. Note that in the interviews many of the students (8 out of the 16) acknowledged that *C* could have several values. e.g. In the pre-interview JK said that *C* equalled 4, but when asked if that was the only value she responded, "It doesn't have to be because that could be 6 plus 4 or 4 plus 6," and JG, when explaining his answer of 2 said, "2+8, but it could be 1+9" and when the interviewer asked, "Could it be anything else," he responded, "It could be 1, 2, 3, or 4."

As in #15b I argue that rather than an inability to think of letters in a particular way (in this case as having multiple values), students just did not think that expressing the idea that *C* could have multiple values was a significant part of the answer. In this problem, as in many of the other level three problems, I find that focusing on how the students' arithmetic frame determines how the problem and its answer are experienced is more productive than Hart's operating with letters view (or in this case a multiple values view) of level three problems. In this case we discover that many of the students who answered the problem incorrectly can, in fact, think of *C* as having multiple values. It is just their arithmetic frame view of what an answer is that produces the incorrect results we see on the paper and pencil test.

Summary Of Level Three Problems

Level three problems are significantly harder than those of level two. Hart argues that level three problems are harder because they demand that students operate with letters. I argue that this cannot be the complete story because several level two problems, which are empirically easier, also demand that students operate with letters. Looking at these problems in terms of an arithmetic frame of reference helps us focus attention on at least three features that may be important sources of differential difficulty. First, a number of level three problems not only demand expressions for answers (as many level two problems do), but, in addition, they do not allow students to do any numerical computation as they generate the answer. Second, a number of these problems have features which seem to produce an expectation that the answer will be numeric or will involve some kind of computation. I have described this as invoking the arithmetic frame of reference. Finally the letters in many of the level three problems do not refer to things that are concretely represented in diagrams (such as lengths of sides of figures) whereas many of the level two problems are based on letters representing lengths in geometric diagrams. The basic idea is that the level three problems depart from the arithmetic frame expectations in a variety of ways more strongly than level two problems and/or invoke the expectations of the arithmetic frame more strongly than the level two problems. How might we expect these differences to effect student performance? Problems that strongly invoke the arithmetic frame would lead students to deform the problem to fit the arithmetic frame resulting in more inappropriate numerical answers.

How did the responses of students to level two problems differ from their responses to level three questions? With level two problems students wrote significantly more inappropriate expressions, and, as predicted for the level three problems, students significantly more often replaced letters inappropriately with numbers and gave numerical answers. (see Table 16)

TABLE 16

Comparison of form of answers for experimental group pre-test results for level two and level three (Including only problems that demand an expression as an answer).

level two:

	expressions				numbers	left out	(other)
	total	full	simp	list			
7c:	8	8			5	3	
9b:	13	3	3	7		3	
9c:	10	5	2	3		6	
13d:	8	5	3	_	__	7	(inapp # calc 1)
Totals:	39	(21	8	10)	5	19	1

(out of 64 responses)

level 3:

	expressions				numbers	left out	other
	total	full	simp	list			
4c:					16		
5c:	2	2			9	4	
9d:					9	7	
13b:	7	3	3	1	1	6	(inapp #calc 2)
13h:	6	4	2		1	9	
15b:	1	1	_	__	7	5	(a letter 3)
totals:	16	(10	5	1)	43	31	5

(out of 95 responses)

The following table compares the percentage of expression and number answers to the problems at each level.

TABLE 17

Percentage of types of answers to problems whose answers involve expressions at levels 2 and 3.
(n = 16).

	expressions	numbers
Level 2	48% (31)	8% (5)
level 3	16% (15)	45% (43)

There is a statistically significant difference between the students' use of numbers and expressions in their responses to the two levels of problems (using a two-factor analysis of variance p=.00 for difference between levels). These data are consistent with the hypothesis that the arithmetic frame need to do a calculation and the features of the level three problems that led to expectations of arithmetic frame processes is the probable basis for the significantly lower percentage of expressions (16% vs. 48%) and the significantly higher percentages of inappropriate numerical answers (45% vs 8%) we find in the level three problems.

Summary Of Levels And Changes In The Arithmetic Frame Necessary To Solve Problems At Different Levels

The hypothesized arithmetic frame is organized around the following assumptions or expectations:

1. Math problems will produce a numerical result, a result that is derived from some numerical calculation (including counting).

2. Generally math problems have the following format: "number" "operation" "number" = ... To solve these problems you begin on the left with a number, the number is followed by an operation sign which tells you what to do with the first number and the number that follows the operation sign (e.g. 3+5). If there is another operation sign you continue to do operations with the derived and written values (e.g. 3+5+4...). Once you hit an equals sign you are done and you can write the result to the right of the equals sign. The standard format for a math problem in the arithmetic frame is number, operation, number, equals sign, and a place to write the result.

Problems that do not have this format are to be reduced to the arithmetic frame format. For example, 3+....=5 is solved by asking 3 plus what will make 5.

Letters have no place in this frame. There are two ways letters can be introduced into this frame. First, they may be thought of as standing for a number and one's job is to find out that number. As long as a student can easily replace a letter with a number and reduce the problem to an arithmetic frame format they generally will have no trouble with problems with letters.

Second, in some cases letters may be interpreted as objects. e.g. a problem like $3a+2a$ can be interpreted within the arithmetic frame if the student thinks of the a as standing for an apple. This reduces the problem to 3 apples plus 2 apples which makes 5 apples or $3a+2a=5a$, a familiar problem to any elementary school child.

Difficulties arise when students cannot transform a problem with letters into the arithmetic frame either because of the form of the problem, because of the nature of the solution process, or the form of the answer that is demanded.

Problems that involve letters and do not conform to the *a+b=c* format are more difficult. Problems with letters that cannot simply be interpreted as adding up objects and/or demand an expression as an answer are also more difficult.

Level two problems are more difficult because they do not follow some of these assumptions and expectations. For example a number of them are problems that demand a full expression with a "+" sign in it as an answer. Some of the level two problems do not conform to the *a+b=c* format and the expectation that problems will be solved left to right with a computation at each step. Level two problems include one or more of these factors.

In order to solve level two problems students must in some cases accept that even though they do a computation they may get a full expression as an answer and that a full expression can be an appropriate answer in some cases. In other problems students must deal with a form that does not fit the step by step a+b=c solution process. They must be able to deal with problems of the form *c=a+b*, and problems that do not allow a computation at each step as they move through the problem. The exact source of the difficulty of these variations on the arithmetic frame (does it center around one thing such as the interpretation of the equals sign, or the focus on doing computations at each step, or is it different issues in different problems) and what aspects of the frame change as students become able to solve them are questions that need further investigation.

Level three problems are more difficult than those of level two. They typically include strongly invoked arithmetic frame expectations and the lack of any computation in the solution process. In addition most of the level three problems present letters with no specific referent (such as the length of a line in a diagram) for their value. The arithmetic frame is invoked by having previous problems that have numerical answers and involve computations, while the level three problems themselves involve numbers and letters but do not allow any computation in the solution process and demand a full expression for an answer.

To answer level three questions correctly students must accept that it is legitimate to generate an answer to a problem without actually doing any numerical calculation. Students must accept that doing a numerical operation (e.g. adding 4) can be expressed by simply creating an expression and that that can be a satisfactory answer in some contexts. One way to view this is to say that students must expand their definition of legitimate mathematical operations. For example, in the case of problem #5c, they need to add replacement to their repertoire of legitimate and adequate mathematical operations. Hart's level three characterization that students must learn to operate with letters the way they do with numbers, although useful in some situations, is too simple and does not reveal with sufficient detail the nature of these students' difficulties. The arithmetic frame analysis leads to a much richer picture of these students' thinking. It gives us a tool for understanding the differential difficulty of the three empirical levels of problems that Hart identifies. The common issue is, on the one hand, how far a problem departs from the format and expectations of the arithmetic frame and on the other hand how much the context, elements, and format of the problem invoke the expectations of the arithmetic frame. Both aspects seem to contribute to how difficult a problem is for these students.

Using The Arithmetic Frame to Predict Within Level Problem Difficulty

Hart does not take up the question of differential difficulty of problems within her levels. To explore whether an arithmetic frame analysis can help us predict and explain why some problems within a level are more difficult than others we must first look at the problems within a level to determine if the arithmetic frame would predict some to be more difficult than others. Second, we must examine student performance to see if differences in difficulty we can identify do fit with the arithmetic frame analysis.

For example, among the six level one problems two (#9a and #13a) demand that students write simple expressions with letters in them (#3*e* and #7*a*), the other four problems have numerical answers. The arithmetic frame rule of arithmetic problems having numerical results suggests that these two problems would be harder than the other four. Table 18 shows the level one problems ordered by number of correct answers for each given by students on the pre-test. The problems that demand a letter in the answer are at the bottom of the list.

TABLE 18
Number Correct for Level One Problems
Combined data from experimental and control groups (n=35)

Level 1 problems:

Problem:	Exp (n=16)	Ctrl(n=19)	Totals
6a.	15	19	34
8.	15	18	33
5a.	14	16	30
7b.	13	16	29
(The following two problems demand expressions in their answers.)			
9a.	13	16	29
13a.	9	10	19

Within level one the two problems that demanded simple expressions with letters in them were generally answered incorrectly by more students than the questions that demanded numerical answers. To test the significance of these differences we can compare student performance on the two groups of questions (level one problems with numerical answers vs. problems demanding expressions as answers) with a Wilcoxon sign rank test. Table 19 compares student performance on the two sets of problems.

TABLE 19

Comparing Student Performance on Level One Problems with Numerical and Expression Answers

Combined data from experimental and control groups (n=35)

Student	Numerical % correct 5a,6a,7b,8	Expressions % correct 9a,13a	Change in % correct (from expressions to numerical)
SA	100	50	+50
DB	100	50	+50
MB	75	50	+25
KC	100	100	0
JD	25	100	-75
LD	100	100	0
MD	100	50	+50
JG	100	50	+50
JJ	100	100	0
CJ	75	50	+25
TJ	50	100	-50
JK	100	0	+100
RL	100	100	0
TO	75	50	+25
JS	100	100	0
CV	100	50	+50
RB	50	50	0
CC	100	100	0
BC	100	100	0
DC	100	0	100
LD	100	100	0
RF	100	100	0
TG	100	100	0
AG	100	100	0
CH	100	100	0
AH	75	50	25
KH	75	50	25
SJ	100	50	50
BM	50	100	-50

JN	100	100	0
SO	100	50	50
LP	100	50	50
TS	100	100	0
HT	75	100	-25
KW	75	0	75

The null hypothesis is that there is not a significant difference between the number of correct responses in the two sets of problems. Only two of the students got a smaller percent of the numerical problems correct than the expression problems, and the one tailed result of the Wilcoxon test is significant (p=.012) leading us to reject the null hypothesis. These students did significantly better on the level one problems with numerical answers. The results conform to the prediction of the arithmetic frame analysis.

Does this analysis hold up for level two problems? Table 20 shows the level two problems also ordered by number of correct answers for each given on the pre-test. The problems that demand a letter in the answer are once again at the bottom of the list.

TABLE 20

Level Two Problems Ordered by Number of Correct Answers

Data from experimental and control groups (n=35)

Level 2 problems:

Problem	Number correct Exp(n=16)	Ctrl(n=19)	Total
#11a.	10	15	25
#15a.	9	16	25
#11b.	8	11	19
(The above problems all demand numerical answers)			
(The following problems all demand expressions as answers)			
#7c.	4	5	9
#9c.	3	4	7
#9b.	2	4	6
#13d.	5	1	6

Once again we can compare the problems based on the types of answers they demand. Problems #7c, #9b, #9c, and #13d all demand expressions as answers whereas problems #11a, #11b and #15a all have numerical answers. As with the level one problems, and as the arithmetic frame analysis predicts, all the numerical answer problems are answered correctly by more students than any of the problems that demand expressions (see table 20). To test the significance of these differences we can again compare student performance on the two groups of questions (level two problems with numerical answers vs. problems demanding expressions as answers) with a Wilcoxon sign rank test. Table 21 compares student performance on the two sets of problems.

TABLE 21
Comparison of Performance on Level Two Questions;
#11a, #11b, #15a (numerical answers)
vs
#7c, #8b, #9c, #13d (expressions as answers)
Combined data from experimental and control groups (n=35)

Student	Numerical % correct 11a,11b,15a	Expressions % correct 7c,8b,9c,13d	Change in % correct (from exprs. to numerical)
SA	100	25	+75
DB	100	25	+75
MB	67	0	+67
KC	100	100	0
JD	67	0	+67
LD	33	0	+33
MD	0	25	-25
JG	100	50	+50
JJ	67	50	+17
CJ	33	0	+33
TJ	33	0	+33
JK	67	0	+67
RL	0	50	+50
TO	0	25	+25

JS	67	25	+42
CV	33	0	+33
RB	67	0	67
CC	67	75	- 8
BC	0	25	-25
DC	0	0	0
LD	67	0	67
RF	100	0	100
TG	100	50	50
AG	67	75	- 8
CH	33	50	-17
AH	100	0	100
KH	100	0	100
SJ	100	0	100
BM	67	25	42
JN	100	0	100
SO	100	25	75
LP	67	25	42
TS	67	0	67
H T	100	0	100
KW	100	0	100

Once again our null hypothesis is that the groups are not different. In all but two case students got a higher percentage of the numerical problems correct. The Wilcoxon result (p=.0000) leads us to reject the null hypothesis. There is a significant difference between student performance on level two numerical and expression problems in the direction predicted by the arithmetic frame analysis.

These data[31] provide further support for the idea that the arithmetic frame analysis can be a useful tool for predicting and explaining the differential difficulty both within and across levels for the problems on the Chelsea Algebra Test.

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[31] I did not include level three data because there were no significant differences between responses to level three problems. The average difference between the number of correct answers for the eight level three problems was 1.1.

FUTURE RESEARCH

As I have described in the above analysis an arithmetic frame perspective also opens up a number of new avenues for investigation. The arithmetic frame seems to have considerable power to explain the difficulties the sixth grade students of this study had with the problems on the Chelsea Algebra Test. Further research will be necessary to clarify a number of hypotheses and questions that this investigation has raised. Suggestions for such research follow.

In addition the results of this research may assist in the rethinking of the teaching of algebra that is currently going on in the United States. In the past algebra has not generally been taught until the ninth grade. There is a movement currently going on in many communities to teach algebra to all eighth grade students. There is also a controversy about what the content of this first algebra course should be. I believe that whatever the resolution about the content of algebra, developing successful learning environments will demand a careful consideration of the most appropriate and effective ways to help students, who have an arithmetic frame of reference, move towards a mathematical understanding of the use of letters. As I review questions for further research I will also include what implications this research might eventually have for teaching algebraic ideas to middle school students a central part of which will be helping them move out of the arithmetic frame of reference.

1. What aspects of problems with letters and their context contribute to their level of difficulty?

What aspects of a problem context (e.g. the previous problem, numbers present in the problem, the explicit invocation of an arithmetic operation) are most powerful in inappropriately invoking an arithmetic frame interpretation by students? Do these aspects interact and if so how? And what characteristics support or allow students to move out of rigid arithmetic frame interpretation of problems involving letters?

e.g. What is the role of having vs. not having numbers in a problem and in an answer? For example does this factor account for why #7c is easier than #4c? Is it the lack of numbers in the problem and/or format of the answer (nm) that make #7c easier than #4c? What role do each of these features play in making it more or less difficult to produce appropriate answers to problems? If it is easier to work with problems that do not have numbers, would this be an effective bridge to solving problems that involve both numbers and letters?

Booth's work (Booth, 1984) suggests that exploring contexts that invoke the arithmetic frame and those that do not might be of value in thinking about how to help students break out of arithmetic frame dominated interpretations.

2. What is the role of problem format in the arithmetic frame?

What specific aspects of changing the arithmetic frame problem format contribute to making problems more difficult? Does it have to do with an inflexible left to right $a+b=c$ template? Does it have to do with student understanding of the meaning of the equals sign? What helps students learn to deal with these alternate formats? Is it simple familiarity with alternate formats or are there more fundamental understandings (such as the meaning of the equals sign) that must change? We need further investigation of the nature of student expectations (the structure of their arithmetic frame) and exactly what changes might be involved as students learn to solve new problems. Does the change derive from something like the ability to put off doing the computation, from developing the ability to see the similarity between one format (e.g. $a+b=c$) and another (e.g. $c=a+b$), or some other source? Is it based on something like a change in view of the equals sign? Or is it some combination of these? Answering this question might help one think about how to focus the curriculum aimed at helping students learn to solve problems that involve letters.

3. Under what circumstances do students inappropriately give a letter a value?

There were some problems that involved letters to which students responded by simply inappropriately giving the letters numerical values (e.g. #4c, #5c, #9d), and there were other problems where they did this much less often or never (e.g. #7c, #9b&c, #13). What makes the difference? As one might expect, it appears as though there may be a number of factors that determine just how students respond. One hypothesis that needs further investigation is that problems that involve numbers and letters are more apt to invoke inappropriate replacement than problems that just involve letters (e.g. if #7c was presented in a numbers and letters version instead of just letters would students be more apt to replace the letter with a number?). If this turns out to be the case doing problems with letters only might provide a good way to start helping students get comfortable writing full expressions.

Another related hypothesis is that the mathematical role of the letter may make a difference. Letters used to represent the unknown length of a line in a diagram (e.g. #7c, #9b&c) may be easier to deal with than letters that represent an unknown number of sides of a figure that are not represented concretely in a diagram (e.g. #9d, #15b). This hypothesis needs further elaboration and investigation. If it turns out to be true, it too, could have important implications for the sequencing of work with students, or at least indicate some of the specific ideas that students need to confront as they make a transition out of the arithmetic frame. Booth's work suggests the idea that one of the things students may need to develop is the flexibility to deal with situations in which the way letters are used changes within a problem set. For example, in problem set #9 letters are first used to represent lengths of sides of geometric figures and then, in #9d, the function of the letter is changed, it is used to represent the number of sides a figure has. This also could use further investigation and may turn out to have useful implications for the focus of teaching.

A final variation on this question is raised by problem #16. Under what circumstances do students appropriately give letters a value, but fail to recognize that the situation would allow more than one value for the letter. In problem #16 most students gave letters a single value when the correct response was that the letter could take on a number of different values (1, 2, 3, or 4)[32]. In the interviews it became clear that many students, when asked, could easily identify that the letters could take on a variety of values. It seemed that doing so was just not part of their experience, or, one could say, was not part of their arithmetic frame. It is interesting to note that some of the problems on the test involved diagrams the lengths of whose sides were labeled with letters. The status of those letters can be seen as somewhat ambiguous (Do they represent single values, or multiple possible values?). When asked about this in the interviews most students thought of them as having a single unknown value, an attribute of arithmetic frame problems. Learning to think in terms of the possibility of multiple values is an important step out of the arithmetic frame. The whole idea of functions is based on multiple values. Some researchers (e.g. Fey 1991, Schwartz 1993) argue that the introduction of letters should begin with functions where letters can take on multiple values. One argument is that creating situations where multiple values are needed will provide a meaningful basis for the abstraction of letters in the first place. One question is how to present and represent problems so that students can think in these terms. The recent advent of computer programs with student controlled dynamic graphic capabilities seems a perfect tool for this kind of activity. With current computer programs one can display a rectangle and the student can dynamically change the lengths of its sides. This could allow students to directly experience the need to represent an idea with something other than a specific number. In this context a letter becomes an appropriate abstraction. Based on this an alternate approach to the

[32]This needs to be differentiated from situations where a letter can take on any value, often described as tautologies, or situations where the value of the letter is not relevant (e.g. simplification problems). These are two additional areas for future investigations.

introduction of letters could begin with functions derived from dynamic diagrams that students can manipulate. This might create a context that would help students move quickly beyond the arithmetic frame.

4. What is the role of diagrams where letters are used to label lengths?

Growing out of the last question is the hypothesis that when letters are used to label something that has a concrete referent (like the length of a line in a geometric drawing) it its easier for students to maintain them as letters than when letters are used without any specific referent at all. Many of the level two problems used diagrams (#7c, #9b&c) and a number of the level three problems did not (#4c, #5c, #13b&h, #14). And in the level three problems that did use diagrams the letter did not refer to a specific length (#9d, #15b). Further investigation could involve creating diagram based versions of level three questions to see if they are easier for students to deal with and non-diagram based versions of level two questions to see if they are harder.

5. Problems with arithmetic vs problems with letters. Where is the source of difficulty?

One question I was not able to explore in depth was the possibility that some of the difficulties that students had with these problems are not just based on the students' interpretation of letters, but arises from their lack of understanding of number based arithmetic that is only brought to the fore by problems with letters in them. For example, is the difficulty with #13b ($3a+5b=$....) based on a lack of understanding of the use of letters or does it derive from a lack of understanding of the nature of multiplication and rules of simplification (e.g. 5x34+3x34=8x34, but 5x34+3x35 does not equal 8 of anything.). If we are able to identify some specific misunderstandings of arithmetic one question is whether working with students on some of these underlying ideas of arithmetic might help them deal with some parallel problems that involve letters[33].

6. Ability to accept substitutions (one step vs. two step)

One hypothesis is that arithmetic frame students can learn to carry out substitutions without any difficulty and, based on an analysis of problem #14, perhaps even complex ones like replacing a letter sum with a single letter. Under this hypothesis it is the arithmetic frame and its numerical computations requirement that limits students ability to deal with some problems. It may be that problems involving more than one transformation, each of which a student may be able to do if presented singly (e.g. in #14: substitution, r for $s+t$, followed by missing factor problem: $2r=30$), are particularly difficult. This is an issue that should be explored and, if true, might lead to teaching based on identifying single operations acceptable under the arithmetic frame and then working specifically on supporting students in combining them in sequence.

7. Components of the frame and their importance

Another question for further research is a clearer specification of the source of difficulty of problems that do not fit the arithmetic frame. In some level two problems students must deal with a form that does not fit the step by step $a+b=c$ solution process. To be successful they must be able to deal with problems of the form $c=a+b$, and problems that do not allow a computation at each step as they move through the problem. The exact source of the difficulty of these variations on the arithmetic frame is not yet clear. Does it center around one thing (such as the interpretation of the equals sign, or the focus on doing computations at each step) or is it different issues in different problems. What aspects of the frame change as students become able to solve these problems is also a question that needs further investigation.

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[33]Some curriculum has been development with this as a focus by The Educational Testing Service under the name Algebridge.

Another area of specific future focus will be exploring the parameters of the conjecture that levels of difficulty are determined by how closely the problem conforms to the arithmetic frame. Here is an example of how one might pursue this using problem set #9. #9a is a level one problem. This is so because you can do a computation and get a numerical result and think of the letters as objects. #9b is a level two problem because you can still do a computation, but you must produce a full expression (which does not fit the arithmetic frame). My hypothesis is that one of the things that makes level three problems harder is they do not allow you to do a numerical computation. To explore this idea I would like to add some new problems to problem set #9. One that included a shape where each side was labeled by a different letter and one that included different letters and one number. In either case the student could neither count nor do any other computation. My analysis predicts that these problems will be harder than the other level two questions in problem set #9. A further question is whether the letters-only version would be similar in difficulty to #7c (a level two problem). Would the no-numbers-at-all aspect make it easier? And, what would be the effect of including a number? Would it invoke the arithmetic frame more strongly and lead more students to do some inappropriate computation? My current conjecture, that doing a computation of some kind is a central part of problem solving in the arithmetic frame, predicts that a problem in this form would be more difficult. It also predicts that changing the problem in this way would make it, in terms of difficulty, a level three problem because no numerical computation would be able to be done.

8. Taking steps away from the arithmetic frame: how it happens, what can support it?

A question for further investigation is just how students who at one point can only solve problems that go left to right in a step by step computation fashion learn to see a multi-term problem as a separable set of terms in which like terms can be combined out of order. An important part of this will be exploring the role an object interpretation of letters might have in that transition.

9. Further work that does not focus on letters:

It seemed clear that some of the difficulty experienced by students with these problems did not just derive from their lack of understanding of letters. We need to explore a broader developmental perspective about the arithmetic frame. What does not fit it? How does the frame expand and evolve as students experience new ideas in mathematics? We need to move beyond just problems with letters to situations such as problem #15a. Did the difficulty of #15a derive from students having to construct their own numerical equations (because they were not given something in the $a+b=c$ format). For students who are in the arithmetic frame are problems where they must generate their own format more difficult and is this an area where more experience would be useful? Another area that did not involve letters was problems that involved the order of operations and departed from the "left-to-right, step-by-step, with a numerical computation at each step" aspect of the arithmetic frame. One question for future investigation is whether focus on areas such as these in purely numerical contexts with explicit identification of how they contradict the arithmetic frame might be useful in helping the students deal with problems with letters that gave them difficulty on the Chelsea Algebra test.

FINAL REMARKS

This study began with an alternative set of metaphors for introducing young students to the use of letters in mathematics and used Hart's analysis as a framework for examining student performance and understanding. I found that experience with the marble bag metaphor was useful in moving student performance from Hart's level one to level two. But, as my work progressed, I found a number of anomalies in Hart's analysis and a number of questions it did not address adequately. I developed an alternative way of looking at how students interpret problems with letters in them: the arithmetic frame. Analyzing student performance from the perspective of an arithmetic frame has helped resolve some of the anomalies in Hart's analysis and provided a more effective tool for understanding potential sources of problem difficulty. Using the idea of an arithmetic frame has also drawn my attention away from Hart's focus on treating letters like numbers and onto the students' need to expand their definitions of what an operation is or can be (e.g. you do not have to do a numerical computation to carry out an operation) and how it can be represented (e.g. "......+4" is a perfectly adequate way to represent adding four in some situations). The question of the power of this new perspective and how it might eventually inform effective curriculum development will need to await future experimental work.

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APPENDICES

A. General Outline of the Marble Bags Curriculum.

B. Week by week outline of Marble Bags Curriculum.

C. Complete Chelsea Algebra Test from which test questions were drawn.

D. Pre and Post Test questions (selected from the Chelsea Algebra test) and additional post-test marble bag questions matched to Chelsea Algebra test questions.

E. Pre and Post Test questions sorted by Hart's levels.

F. Number Correct for Level three questions on the Pre and Post tests, including per student scores.

G. Comparison of responses to marble bag and standard versions of three level three questions on the Post paper and pencil test

H. Marble Bags Unit evaluation test given by outside evaluators and results.

I. Effect of Position of Marble Bag Questions on Post Test

APPENDIX A

Logo Algebra project
CURRICULUM MODULES OUTLINE

MODULES:

1. Solving Equations: Number Games and Marble Bag Icons

A. Introduce the idea of marble bag stories, notation and manipulation.

Do a guess your birthday trick with the students, show marble bag diagrams as a way to do the trick.

B. Macintosh is introduced and students do exercises using the marble bags lab going from stories in english to marble bag icon representations and then finding the computer's original number by further manipulations.

C. Students practice and make up their own number tricks using marble bag notation. (homework assignment to make up a story, try it out, and write it down in english)

Marble bag lab is used for reviewing making up number trick stories and for recording them via bag drawings. The computer will also be used for testing out their marble bag number tricks.

D. Balance software is introduced and is used for solving marble bag story problems. (Finding out how many marbles (or coins) are in a bag by using a balance.)

E. Algebraic notation is introduced as a quick way of writing down bag stories.

a. introducing X, multiplication by concatenation (3X).

b. exercises in moving from one representation to another. (computer exercises in moving from icons to algebra and from algebra to english)

F. Solving marble bag problems using algebraic notation for manipulations is practiced using the balance software.

G. Introduction of use of parentheses as one way of recording multiplication, and getting rid of ambiguity. Followed by fraction bar as a way of notating division.

a. first via icons 2 (&&••)

b. using parentheses to make one-liner records, using these to debug secret keeper's arithmetic errors.

c. practice exercises with the marble bag lab in moving from english to algebra one liners (makestory).

d. exercises in simplifying parenthesized expressions.

e. work with expression writer program

(Using bag drawings to test algebraic equivalencies and solve equations.)

2. <u>Introduction to Variables and Equations</u>

Begins with writing Logo procedures to generate GOSSIP (random sentences). (introducing PRINT, OUTPUT, SENTENCE (with parens), PICK, defining procedures, inputs.) Includes introduction to function machine images.

These lead into writing simple interactive quizzes for other students.(introduction to Logo interaction: READLIST, IF tests.)

Next is writing arithmetic quizzes for younger students (includes Logo arithmetic operations, RANDOM).

Next step is beginning "algebra" quizzes. Student written program generates equation of a standard type and replaces one of the numbers with an "x". Asks user to solve it. e.g. 2X+5=11

This is then extended to encourage the creation of programs that will tell the user why his answer is wrong, or how to get the answer. (developing the algorithm via undoing).

3. <u>Introduction to Functions</u>

Writing procedures for generating and decoding secret codes as an introduction to functions (performed on letters) or a unit on dealing with storm data (calculating the distance of a storm from a city as a function of the time between a lightning flash and the sound of thunder (distance = 1/5 * time) - extended to include adding the distance of the observation point (weather station) from the city (distance = 1/5* times + distance of weather station from the city))

Writing Logo Guess my number function machines, and extending it to Sequence finder procedures.

Introduction to Black Box function problems where students must figure out how to find the missing input, or output, or function. Focus on using an inverse or undoing approach. Use of graphing to analyze both code and weather station problems.

APPENDIX B

Week by week outline of Content of Marble Bags Curriculum
Spring 1987

Module ONE

Week 1

ALGEBRA:

-Students are introduced to equalities and inequalities at the concrete level by using a balance scale.
-Introduction to an unknown by using a bag of washers.
-Operating on an equality to find the value of an unknown.

ACTIVITIES:

-Introduction of the balance scale as a model for simple arithmetic equalities and inequalities.
-Recording equalities based on balance work.
-Operating on a balance to find the value of an unknown (e.g. $3X = 12$, $5X + 3 = 18$).

COMPUTER SOFTWARE:

-"Introduction to Balance" - students manipulate screen balance adding and subtracting marble icons.
-"Marble Problems" - students balance the scale by adding and subtracting to each side.
-"Solve Coin Bags" - students are presented with problems having a bag with an unknown number of marbles in it. They must operate on the balance to try to figure out how many marbles are in the bag.

Week 2

ALGEBRA:

-Recording and solving equations with an unknown at the pictorial level.

ACTIVITIES:

-Introduction to Marble Bag Stories.
-Students record and solve stories as well as make up their own.

COMPUTER SOFTWARE:

-"Tell A Story" - students make up and solve marble bag stories using icons of bags and marbles. They print out finished stories to try out on their friends.

Week 3

ALGEBRA:

-Solving equations at the concrete level using a balance.
-Working with equations that involve simple division.
-Focusing on the most efficient strategy for solving simple linear equations in 2 steps.
-Exploring problems where the unknown equals zero and exploring problems where the unknowns cancel out.

ACTIVITIES:

-The balance scale is used as a way to solve marble bag story problems. Students determine how many marbles are in a bag by doing operations on both sides of the balance.
-Students practice solving marble bag problems using the balance software.
-Students make up and try out their own stories.
-Special stories are introduced: the secret number = 0, the unknown is cancelled out so that all stories result with the same number.
-Stories with division are introduced.
-Students are challenged to solve problems in the fewest possible steps using the balance software.
-Students try to make up problems that will take more than 3 steps to solve.

COMPUTER SOFTWARE:

-"Tell A Story" - students make up and solve marble bag stories using icons of bags and marbles.
-"Solve Coin Bag Problems" - marble bag problems are presented and solved using a picture of a balance; students can also enter their own problems.

Week 4

ALGEBRA:

-Introduction to standard algebra notation (X, concatenation e.g. 3X, 4X+6).
-Translating English number stories into algebra notation.
-Solving linear equations using standard notation (e.g. 3X+7=19).
-Simplifying expressions; collecting and combining terms in an equation.
-Parenthesized expressions are introduced via a package metaphor, an intuitive base for factoring and distribution (done at both the concrete and symbolic levels).

ACTIVITIES:

-Students try to record marble bag stories using algebra notation ("lazy mathematicians" replace the bag with an X).
-Students practice moving from English language stories to algebraic representation.
-Students use the balance to solve marble bag problems presented in algebra notation.
-Collecting and combining terms is introduced through "discombobulated" stories (e.g. X+1+X+4+X-2-X). Students simplify complex expressions and solve equations. They also make up their own for other students to simplify and solve.
-Parenthesized expressions are introduced. Students practice packaging and unpackaging expressions. Making up expressions that can be packaged in several different ways. Students use "marble bag calculators" with bag and washer chips.
-Students try to generate a variety of marble bag stories that will lead to a final expression of 2X+7.

COMPUTER SOFTWARE:

-"Solve Coin Bag Problems" - marble bag problems are presented and solved using a picture of a balance; students can also enter in their own problems.
-"English to Algebra" - the students must record algebraically a story presented in English.

Week 5

ALGEBRA:
-Working with parenthesized expressions: recording stories, simplifying, and solving equations.
-Order of operations.

ACTIVITIES:

-Students learn to record marble bag stories as "one-liners" using parenthesized expressions for multiplication.
-Students solve "one-liner" stories
-Students are confronted with the issue of order of operations.
-Given a set of numbers students write all the different expressions they can using parenthesized expressions.
-Given a set of numbers students try to write expressions that evaluate to a given goal. Students make up their own problems for other students.

COMPUTER SOFTWARE:

-"Expression Writer" - students make up and solve problems where they are given a set of numbers and operations and instructed to assemble them to evaluate to a given goal.

Week 6

ALGEBRA:

-Solving equations with parenthesized expressions and multiple instances of a variable (e.g. 2(2X+3)-X=27).
-Using the fraction bar as algebraic notation for recording division.
-Solving algebra problems with a variable on both sides of the equation (e.g. 3X+10=X+16).

ACTIVITIES:

-Students practice recording marble bag stories as algebra notation "one-liners".
-Students make up and solve "one-liner" problems.
-Division notation using the fraction bar is introduced and students solve division problems.
-Wise Guy stories with a variable on both sides of an equation are introduced.
The balance is used to solve them. Students practice making up and solving Wise Guy problems.
-Simplificaion and solving of more complex problems with parenthesized expressions, fraction bars, multiple terms, and variables on both sides.

COMPUTER SOFTWARE:

-"One-Liner" Problems - students must record a marble bag story as a one line expression using parenthesized expressions.
-"Solve" - a program in which students must solve equations by simplifying them step by step. The program checks each simplification step. Students can enter their own problems or have the computer generate problems of varying levels of complexity.
-"Solve Coin Bag Problems" - problems with X's on both sides of the balance.

Week 7

ALGEBRA:

-Review of solving equations with variable terms on both sides.
-Equivalence of X=1 and 1=X.

ACTIVITIES:

-The recording of Wise Guy problems in algebra notation is reviewed.
-The balance is used for solving Wise Guy problems.

COMPUTER SOFTWARE:

-"Solve Coin Bag Problems" - students solve problems with variable terms on both sides of the equation and make up their own Wise Guy problems.

Week 8

ALGEBRA:

-Review of the use of the fraction bar to notate division.
-Going from an algebraic notation to an English story.
-Introduction to checking the solution to an equation by plugging in a value for X.
-Review of translating from English story to algebraic notation, simplification, and strategies for solving equations.
-Introduction to problems where the division on one side of an equation can not be simplified.

ACTIVITIES:

-Generating stories for "one-liners" with fraction bars presented in algebra notation.
-Solving "one-liners" with fraction bars.
-Checking answers by plugging in the value of X (e.g. $\frac{2(4X+3)-2X}{3} = 6$ Does X=5?)

-Students generate solutions to problems some of which are correct and some of which are not. Other students must find the incorrect solutions.

COMPUTER SOFTWARE:

-"Solve" - students use it to solve equations.

APPENDIX C

Complete Chelsea Algebra test from which pre and post test questions were drawn

CHELSEA DIAGNOSTIC MATHEMATICS TESTS

ALGEBRA

Name:	Today's date:
School:	Class:
Date of birth:	

	Level 1	Level 2	Level 3	Level 4
Criterion	4/6	5/7	5/8	6/9
Pupil's score				
Levels passed (✓)				

Highest level attained

Comments:

NFER-NELSON

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2 Oxford Road East, Windsor, Berkshire SL4 1DF
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First published 1984
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[illegible]

Practice Item 1

1. What number does $a + 4$ stand for if $a = 2$ ....
if $a = 5$ ....

What number does $4a$ stand for if $a = 2$ ....
if $a = 5$ ....

Practice Item 2

2. **Fill in the gaps:**
Work down the page

$x \longrightarrow 3x$	$x \longrightarrow x + 3$	$x \longrightarrow 7x$	$x \longrightarrow x + 8$
$2 \longrightarrow 6$	$5 \longrightarrow 8$	$2 \longrightarrow$	$3 \longrightarrow$ ....
$5 \longrightarrow$	$4 \longrightarrow$		
	$n \longrightarrow$		

1. Fill in the gaps:

$x \longrightarrow x + 2$ $\qquad$ $x \longrightarrow 4x$

$6 \longrightarrow$ .... $\qquad$ $3 \longrightarrow$ ....

$r \longrightarrow$ ....

2. Write down the smallest and the largest of these: smallest largest

$n + 1$, $n + 4$, $n - 3$, n, $n - 7$ ....

3. Which is the larger, $2n$ or $n + 2$? ....

Explain: ....

4. 4 added to n can be written as $n + 4$. Add 4 onto each of these:

8 $\qquad$ $n + 5$ $\qquad$ $3n$

....

n multiplied by 4 can be written as $4n$. Multiply each of these by 4:

8 $\qquad$ $n + 5$ $\qquad$ $3n$

....

5. If $a + b = 43$

$a + b + 2 =$ ....

If $n - 246 = 762$

$n - 247 =$ ....

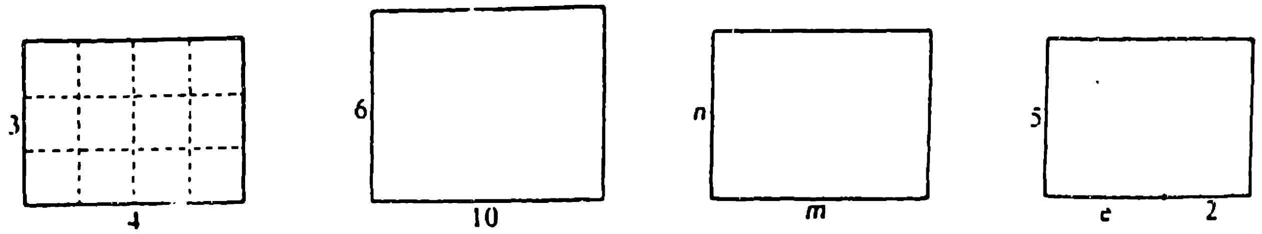
If $e + f = 8$

$e + f + g =$ ....

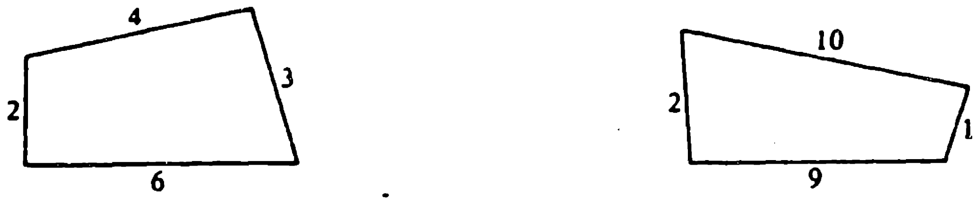
6. What can you say about a if $a + 5 = 8$ ....

What can you say about b if $b + 2$ is equal to $2b$ ....

7. What are the areas of these shapes?

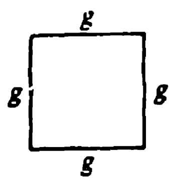


A = ... A = ... A = ... A = ...



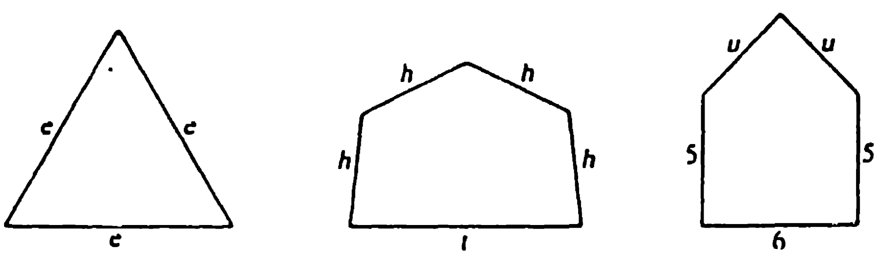
8. The perimeter of this shape is equal to 6 + 3 + 4 + 2, which equals 15.

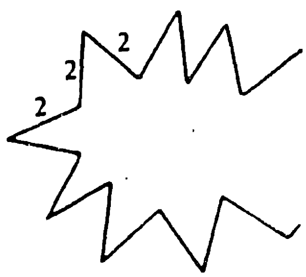
Work out the perimeter of this shape. P = ...



9. This square has sides of length g. So, for its perimeter, we can write $P = 4g$.

What can we write for the perimeter of each of these shapes?





Part of this figure is not drawn. There are n sides altogether, all of length 2.

P = ... P = ... P = ... P = ...

10. Cabbages cost 8 pence each and turnips cost 6 pence each.

If c stands for the number of cabbages bought
and t stands for the number of turnips bought,
what does $8c + 6t$ stand for? ....

What is the total number of vegetables bought? ....

11. What can you say about u if $u = v + 3$
and $v = 1$ ....

What can you say about m if $m = 3n + 1$
and $n = 4$ ....

12. If John has J marbles and Peter has P marbles, what could
you write for the number of marbles they have altogether? ....

13. $a + 3a$ can be written more simply as $4a$.

Write these more simply, where possible:

$2a + 5a =$ ....

$2a + 5b =$ ....

$(a + b) + a =$ ....

$2a + 5b + a =$ ....

$(a - b) + b =$ ....

$3a - (b + a) =$ ....

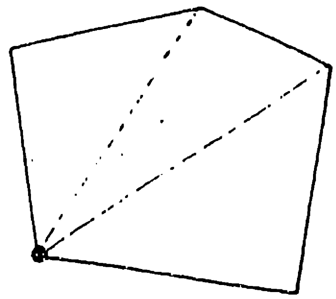
$a + 4 + a - 4 =$ ....

$3a - b + a =$ ....

$(a + b) + (a - b) =$ ....

14. What can you say about r if $r = s + t$
and $r + s + t = 30$ ....

15. In a shape like this you can work out the number of diagonals by **taking away 3 from the number of sides.**



So, a shape with 5 sides has 2 diagonals;

a shape with 57 sides has .... diagonals;

a shape with k sides has .... diagonals.

16. What can you say about c if $c + d = 10$ and c is less than d ....

17. Mary's basic wage is £20 per week.
She is also paid another £2 for each hour of overtime that she works.

If h stands for the number of hours of overtime that she works, and if W stands for her **total** wage (in £s), write down an equation connecting W and h: ....

What would Mary's total wage be if she worked 4 hours of overtime? ....

18. When are the following true – always, never, or sometimes?
Underline the correct answer:

$A + B + C = C + A + B$ Always Never Sometimes, when ....

$L + M + N = L + P + N$ Always Never Sometimes, when ....

19. $a = b + 3$. What happens to a if b is increased by 2? ....

$f = 3g + 1$. What happens to f if g is increased by 2? ....

20. Cakes cost c pence each and buns cost b pence each.
If I buy 4 cakes and 3 buns,
what does $4c + 3b$ stand for? ....

21. If this equation is true when $x = 6$, $(x + 1)^3 + x = 349$

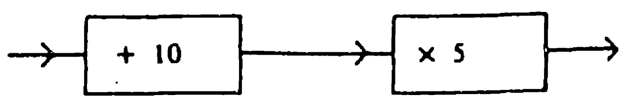
then what value of x will make this equation true? $(5x + 1)^3 + 5x = 349$

$x =$ ....

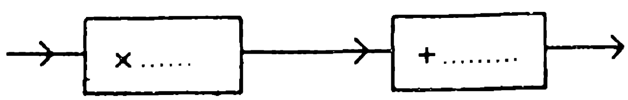
22. Blue pencils cost 5 pence each and red pencils cost 6 pence each.
I buy some blue and some red pencils and altogether it costs me 90 pence.

If b is the number of blue pencils bought, and
if r is the number of red pencils bought,
what can you write down about b and r? ....

23. You can feed any number into this machine:



Can you find another machine that has the same overall effect?



APPENDIX D

Pre and Post Test questions (selected from the Chelsea Algebra test) and additional post-test marble bag questions matched to Chelsea Algebra test questions.

Dietmar Küchemann

Contents

Using the Algebra Test

The objective of the Algebra test is to assess children's levels of understanding across a broad range of typical secondary school algebra tasks. These include substitution, simplifying expressions, and constructing, interpreting and solving equations. The assessment focuses on the different ways in which children use and interpret the letters in generalised arithmetic. In devising the test an attempt has been made to minimise the need for remembered techniques and conventions.

A short series of practice items precedes the main test. Its purpose is to remind children of certain conventions — for example, that $4a$ means $4 \times a$, and to show that letters can be used to represent numbers. The practice items should be completed before the test itself is attempted, and the answers should be discussed with the whole class.

Age of student to be tested:
The test was developed for use with children of all abilities within the second, third and fourth year of secondary school i.e. ages 12+ to 15+. It can also be used with older children.

Time required to do the test:
The series of practice items should be worked through for 2 or 3 minutes, and the answers discussed for another 5 minutes. The test itself should take about 30–35 minutes.

Equipment needed:
Pen (or pencil).

Administration of the test:
The administrator should first read the general instructions for the administration of the tests in section 2.

The Algebra test contains a small number of practice items which must be completed by the children before they attempt the test itself. After distributing the test booklets, the children's attention should be drawn to the practice items on page 8. They should be told that they will have two or three minutes to work through them, then the items will be discussed with the whole class. (The discussion should be used to remind the children briefly of certain algebraic conventions.)

Marking the test:
General instructions for marking all the tests are presented in section 2. The marking key for the Algebra test is on pages 14–15. To make sense of the codes used in the

Practice Item 1

1. What number does $a + 4$ stand for if $a = 2$ ....
   if $a = 5$ ....

   What number does $4a$ stand for if $a = 2$ ....
   if $a = 5$ ....

Practice Item 2

2. Fill in the gaps:
   Work down the page

$x \longrightarrow 3x$
$2 \longrightarrow 6$
$5 \longrightarrow$ ....

$x \longrightarrow x + 3$
$5 \longrightarrow 8$
$4 \longrightarrow$ ....
$n \longrightarrow$ ....

$x \longrightarrow 7x$
$2 \longrightarrow$ ....

$x \longrightarrow x + 8$
$3 \longrightarrow$ ....

1. Fill in the gaps:

$x \longrightarrow x + 2$ $x \longrightarrow 4x$

$6 \longrightarrow$ .... $3 \longrightarrow$ ....

$r \longrightarrow$ ....

2. Write down the smallest and the largest of these: smallest   largest

$n + 1$, $n + 4$, $n - 3$, n, $n - 7$ ....

3. Which is the larger, $2n$ or $n + 2$? ....

Explain: ....

4. 4 added to n can be written as $n + 4$.
Add 4 onto each of these:

8    $n + 5$ $3n$

....

n multiplied by 4 can be written as $4n$.
Multiply each of these by 4:

8    $n + 5$ $3n$

....

5. If $a + b = 43$ If $n - 246 = 762$ If $e + f = 8$

$a + b + 2 =$ .... $n - 247 =$ .... $e + f + g =$ ....

6. What can you say about a if $a + 5 = 8$ ....

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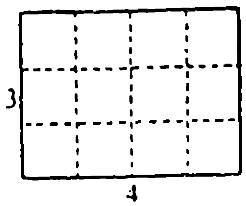
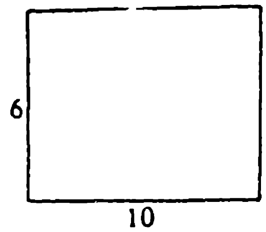
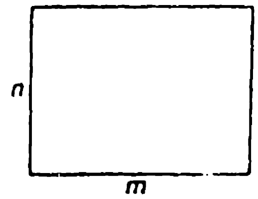
7. What are the areas of these shapes?



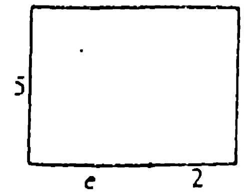
A = ...



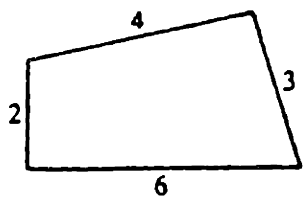
A = ...



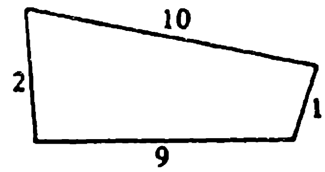
A = ...



A = ...

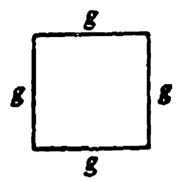


8. The perimeter of this shape is equal to 6 + 3 + 4 + 2, which equals 15.

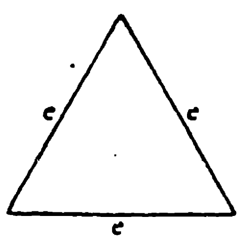


Work out the perimeter of this shape. P = ...

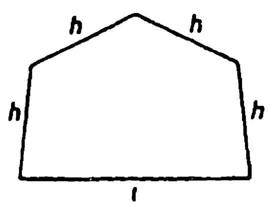
9. This square has sides of length g.
So, for its perimeter, we can write $P = 4g$.



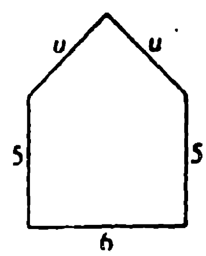
What can we write for the perimeter of each of these shapes?



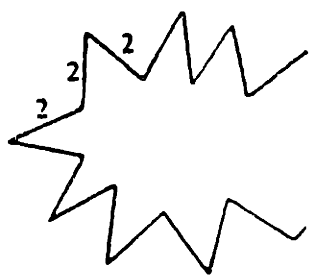
P = ...



P = ...



P = ...



Part of this figure is not drawn. There are n sides altogether, all of length 2.

P = ...

11. What can you say about u if $u = v + 3$ and $v = 1$ ....

What can you say about m if $m = 3n + 1$ and $n = 4$ ....

13. $a + 3a$ can be written more simply as $4a$.

Write these more simply, where possible:

$2a + 5a =$ ....

$2a + 5b =$ ....

$(a + b) + a =$ ....

$2a + 5b + a =$ ....

$(a - b) + b =$ ....

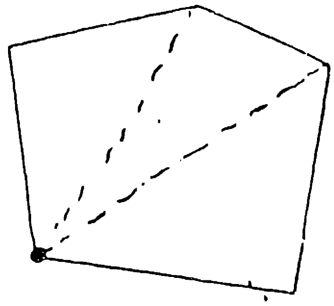
$3a - (b + a) =$ ....

$a + 4 + a - 4 =$ ....

$3a - b + a =$ ....

$(a + b) + (a - b) =$ ....

14. What can you say about r if $r = s + t$, and $r + s + t = 30$ ....

15. In a shape like this 
you can work out the number of diagonals by taking away 3 from the number of sides.

So, a shape with 5 sides has 2 diagonals;

a shape with 57 sides has .... diagonals;

a shape with k sides has .... diagonals.

16. What can you say about c if $c + d = 10$
and c is less than d ....

2.) Here are five simple marble bag stories see if you can figure out which one would give you the smallest result and which one the largest result:

Think of a number

a) start with your original number and
Add 1

b) start with your original number and
Add 4

c) start with your original number and
Take away 3

d) just your original number.

e). start with your original number and
Take away 7

Which story would give you the smallest result? ....

Which story would give you the largest result? ....

4.) If the first step in a marble bag story was:
Think of a number
you would write down:
X
and if the next step was to:
Add 4

You would write the one liner as:
X+4

Here are two marble bag stories written as one liners in algebra,
For each one what would you write if the next step in the story was:

Add 4

a) **X+5**

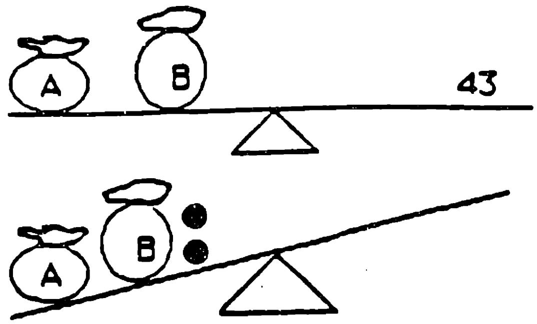
....

b) **3X**

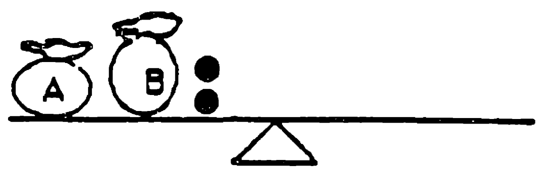
....

5.) Here are some marble bag balance problems. On the first balance there are 2 bags. Each one has a different number of marb[illegible] in it. Your job is to figure out what you could put on the right side of the balance to get it to balance.

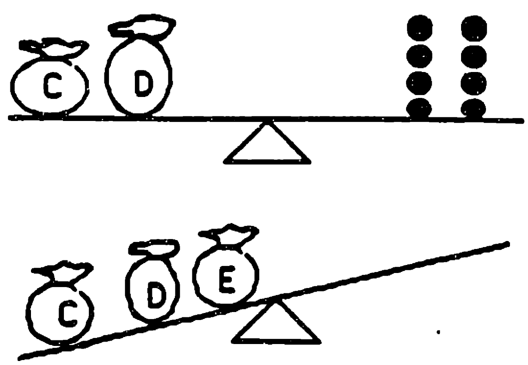
a)



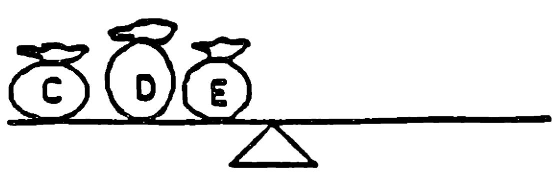
Show what would you put on the right hand side to balance the balance.



b) This problem uses three bags. (each with a different number of marbles in it.)



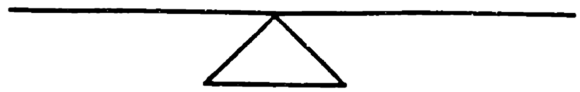
Show what you could use to make it balance.



6)

This is a wise guy marble bags story. Can you figure out how much is in the bag? You may use the balance picture if you wish.

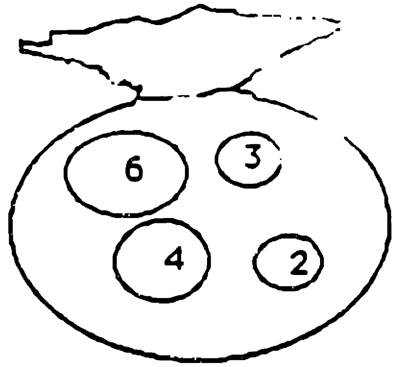
$$X + 2 = 2X$$



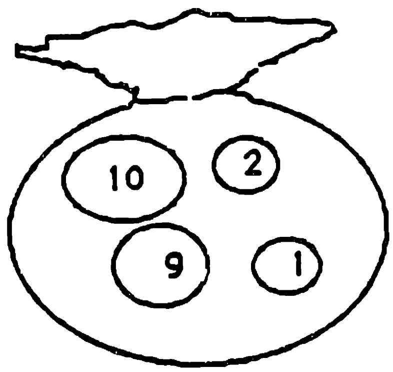
7e.) Write an algebra one liner for this story:

Think of a number.
Add 2
Multiply by 5

8.)

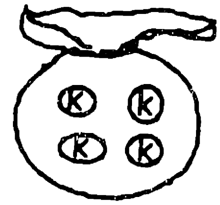


Each marble in the bag is labeled with its weight. The total weight of this bag is equal to 6 + 3 + 4 + 2, which equals 15.

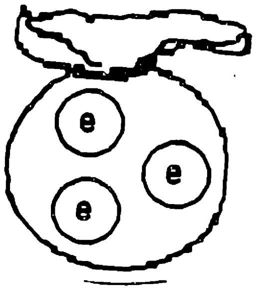


Work out the weight of this bag. W = ...

9.) This bag has four marbles in it and each marble weighs "k" pounds
So, for the total weight of the bag, we can write $W = 4k$.

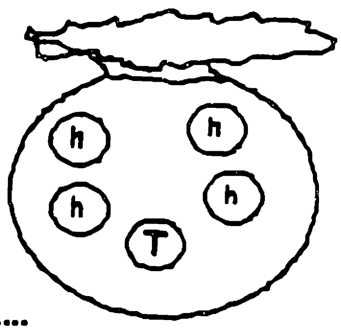


a.



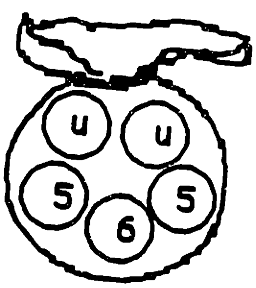
W = ...

b.



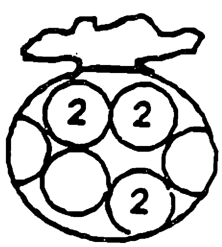
W = ...

c.



W = ...

d.



Here is a bag of marbles. Not all the marbles are shown. Altogether there are X marbles in the bag. Each one weighs 2 pounds. What can we write for the weight of this bag of marbles?

W = ...

APPENDIX E

Problems from the Chelsea Algebra Test by Level

Correct answers to problems are in square ([]) brackets.
Where multiple problems are presented as part of a set the level appropriate problem will be in **bold**.

Level 1 problems:

5a. a+b=43
a+b+2=........[45]

6a. What can you say about a if 5+a=8 [3]

7b. A picture of a rectangle with sides labeled 6 and 10 is displayed. What is its area?...................... [60]

8. A picture of a quadrilateral with sides labeled 2, 10, 1, 9 is displayed. What is its perimeter?.................... [22]

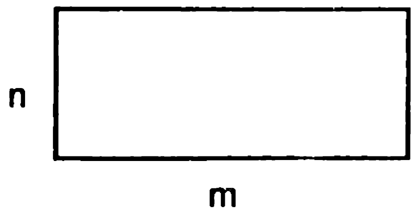
9a. A picture of an equilateral triangle with each side labeled e is displayed. What can we write for its perimeter?........................[3e]

13a. 2a+5a=..................... [7a]

The criterion for level one is getting 4/6 of the level one problems correct. (Students who got fewer than 4 of the level one questions were assigned level 0)

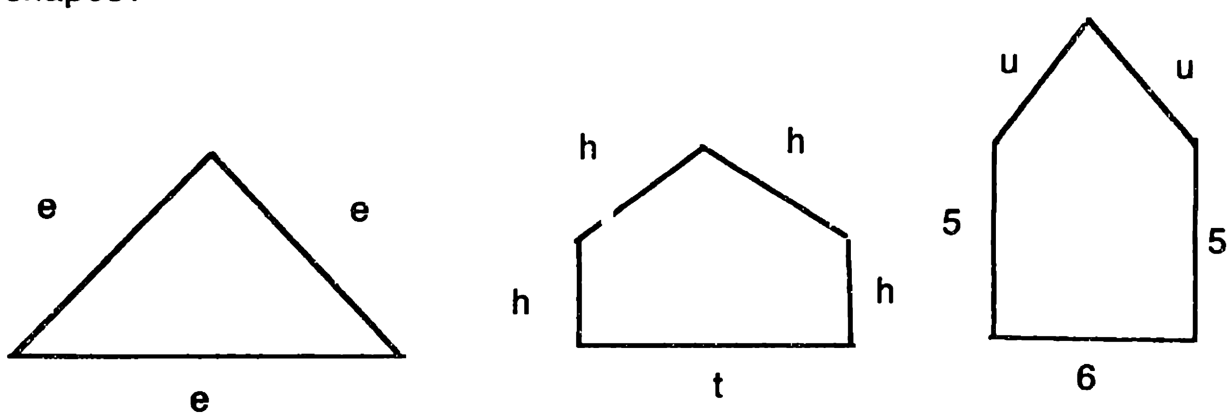
Level 2 problems:

7c. What is the area of this shape......................



A=................................
[Correct: nm]

9a,b&c. What can we write for the perimeter of each of these shapes?



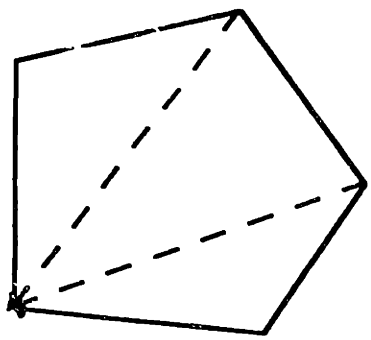
a). P = ... b). P = ...c). P = ...
[Correct: 3e 4h+t 2u+16, 2xu+2x5+1X6]

11a. what can you say about u if u=v+3 and v=1?..............[u=4]

11b. What can you say about m if m=3n+1 and n=4?.....................[m=13]

13. Simplify where possible....
13d. 2a+5b+a=..................[3a+5b]

15.



In a shape like this you can work out the number of diagonals by **taking away 3** from the number of sides.
So, a shape with 5 sides has 2 diagonals:

15(a) a shape with 57 sides has .... diagonals.
[54]

15(b) a shape with k sides has .... diagonals.

The criterion for level two is getting 5/7 of the level two problems correct.

Level 3 problems:

4. 4 added to n can be written as **n + 4.** add 4 onto each of these:

(a) 8 (b) n+5 **(c) 3n**

[3n+4]

5. (a). A+B=43,

A+B+2=........

(b). N-246=762

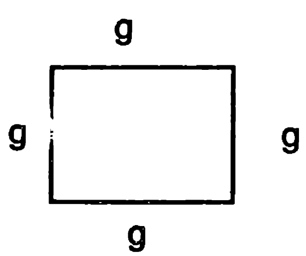
N-247=..........

(c). E+F=8

E+F+G=.....

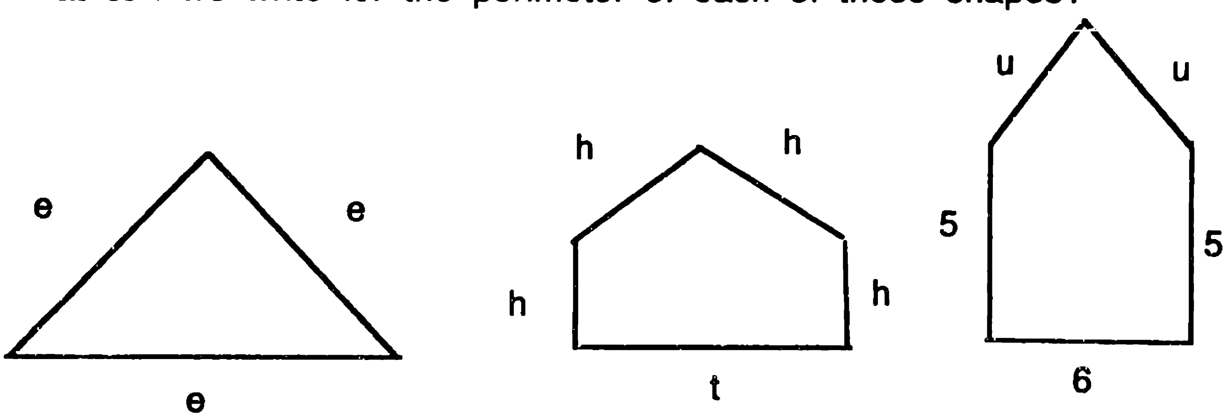
[8+G]

9.



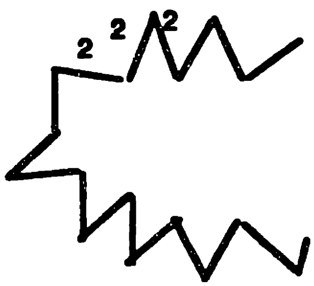
This square has sides of length g.
So, for its perimeter, we can write P = 4g

What can we write for the perimeter of each of these shapes?



P = ...   P = ...   P = ...

9(d)



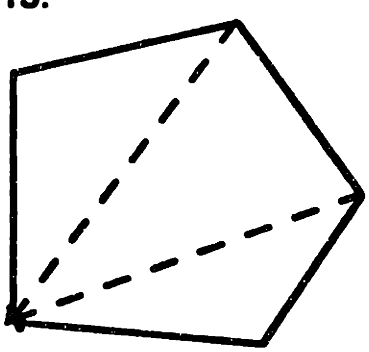
Part of this figure is not drawn.
There are n sides altogether, all of length 2.
P = ...
[2n]

13. a + 3a can be written more simply as 4a.
Write these more simply, where possible:

13 (b) 2A+5B=............
[2A+5B]

13(h). 3A-B+A=............
[4A-B]

15.



In a shape like this you can work out the number of diagonals by

taking away 3 from the number of sides.
So, a shape with 5 sides has 2 diagonals:
(a) a shape with 57 sides has .... diagonals.
15(b) a shape with k sides has .... diagonals.
[K-3]

14. What can you say about R if R=S+T and
R+S+T=30? ....
[R=15]

16. What can you say about C if C+D=10 and
C is less than D?
[the correct answer includes the idea that C can have more than one value. e.g. C can be 0,1,2,3, or 4]

The criterion for level three is getting 5/8 of the level three problems correct.

APPENDIX F

<u>Number Correct for Level three questions on the Pre and Post tests</u>

	pre test	post test	change
Control group			
TOTAL:	10	33	21
per student:	.53 per student	1.74 per	1.11 per
Experimental group:			
TOTALS:	13	46	33
per student:	.81 per student	2.88 per	2.06 per

<u>Per student listing of total number correct on level 3 questions</u>
<u>Experimental Group</u> (6th grade, n=16):

	pre test	post test	change
SA	2	3	+1
DB	1	5	+4
MB	0	1	+1
KC	1	8	+7
JD	0	1	+1
LD	0	4	+4
MD	0	1	+1
JG	3	4	+1
JJ	3	6	+3
CJ	0	0	0
TJ	0	1	+1
JK	1	2	+1
RL	0	0	0
TO	1	3	+2
JS	1	4	+3
CV	0	3	+3
TOTALS:	13	46	33
per student:	. 81	2.88	2.06

Control Group: (6th graders n=19)

	Pre Test	POST TEST	change
RB	0	3	+3
CC	0	1	+1
BC	0	3	+3
DC	0	0	0
LD	1	0	- 1
RF	0	2	+2
TG	3	5	+2
AG	3	3	0
CH	1	3	+2
AH	0	1	+1
KH	0	0	0
SJ	0	2	+2
EM	0	0	0
JN	1	2	+1
SO	1	2	+1
LP	0	2	+2
TS	0	0	0
HT	0	0	0
KW	0	2	±2
Control group			
TOTAL:	10	33	21
per student:	.53 per student	1.74 per	1.11 per
Experimental group:			
TOTALS:	13	46	33
per student:	.81 per student	2.88	2.06

APPENDIX G

Comparison of responses to marble bag and standard versions of three level three questions on the Post paper and pencil test

Summary of results:

Key:

same:√ means student got both standard and marble bag version correct

same:x means student got both standard and marble bag version wrong

marbles only√ student got marble bag version correct and standard wrong.

standard only √ student got standard version correct and marble bag version wrong.

Initials are student initials

Post P&P:

	same:√	same: x	marbles only√	standard only √
4c.	6	1	8	1
	SA,DB,KC,LD,JG,JJ	MD[3]	MB,JD,CJ,JK[4],RL,TO,JS,TJ	CV
5c.	5	4	5	2
	KC,LD,JJ,TO,JS	MD,CJ,TJ,RL	SA,MB,JD[2],JG,CV	DB,JK
9d.	6	10	0	0
	SA[1],DB[1],MB[1],KC,JG,JJ	JD,LD,MD[3],CJ,TJ,JK[5],RL,TO,JS[6],CV		

1. with good picture
2. misinterpreted Post P&P, but got it on post int.
3. M got them both wrong, but when I explained the problem to him in marble bags he fixed it immediately, when I tried to explain the standard one he did not understand until I introduced the marble bag metaphor.
4. J's standard could be careless error (3nx4)
5. J: standard: Px2 marbles: 12, but easily prompted in interview.
6. J: on post clinical marbles interview

Results: Of the 8 problems that Hart suggests are indicators of interpreting letters as "specific unknowns" I was able to create 3 parallel problems using the marble bag metaphor (#4c, #5c, and

#9d). These problems were included on the post P&P test. The group was made up of 16 students. For #4c 9 students got the standard version wrong and of those 7 got the marble bags version correct (8 if you count the student who misinterpreted the problem and immediately corrected it when it was explained to him in the post interview). For #5c 9 got the standard version wrong and 5 of those got the marble bags version correct.

For #9d no one who got the standard version wrong got the marble bag version correct but three students on the post interview were able to solve it when the marble bag metaphor was introduced verbally (this example used the marble bag metaphor in a new way: to refer to weight rather than referring simply to a number of items).

APPENDIX H

Marble Bags Unit evaluation test given by outside evaluators and results

The experimental group is referred to as the Logo-Algebra 2 group or Logo 2 in the following.

Evaluation Summary-Test Forms B & G

Overall Test Statistics: (Maximum possible score: 23.5)

Logo-Algebra 1	Logo-Algebra 2
N=19	N=16
Minimum=6.5 (27.7%)	Minimum=14.5(61.7%)
Maximum=23.5 (100%)	Maximum=22.5(95.7%)
Mean=18.2 (77.3%)	Mean=19.6(84%)
Standard Dev.=4.9 (20.9%)	Standard Dev.=2.3(9.9%)

Part 1

---

1. Write the following story using marble bag notation and algebraic notation.

Story	Marble Bag			Algebra	
		Logo 1	Logo 2	Logo 1	Logo 2
Think of a number	No Answer:	0	0	0	0
	Wrong Answer:	0	0	0	0
	Correct Answer:	19 (100%)	16(100%)	19(100%)	16(100%)
		Logo 1	Logo 2	Logo 1	Logo 2
Add two marbles	No Answer:	0	0	0	0
	Wrong Answer:	0	0	0	0
	Correct Answer:	19 (100%)	16(100%)	19(100%)	16(100%)
		Logo 1	Logo 2	Logo 1	Logo 2
Double the total	No Answer:	0	0	0	0
	Wrong Answer:	1 (5.3%)	0	2(10.5%)	0
	Correct Answer:	18 (94.7%)	16(100%)	17 (89.5%)	16(100%)
		Logo 1	Logo 2	Logo 1	Logo 2
Take away two marbles	No Answer:	0	0	0	0
	Wrong Answer:	1(5.3%)	0	3(15.8%)	0
	Correct Answer:	18(94.7%)	16(100%)	16(84.2%)	16(100%)

---

2. If the final result in the story in problem 1 is 14, how many marbles are in the bag? Use the space below to show or explain how you solve the problem.

	Logo 1	Logo 2
No Answer:	0	0
Wrong Answer:	6 (31.6%)	5(31.3%)
Correct Answer:	13 (68.4%)	11(68.8%)

3. Write the following story using marble bag notation and algebraic notation.

Story	Marble Bag			Algebra	
		Logo 1	Logo 2	Logo 1	Logo 2
Think of a number	No Answer:	0	0	0	0
	Wrong Answer:	0	0	1(5.3%)	0
	Correct Answer:	19 (100%)	16(100%)	18 (94.7%)	16(100%)
		Logo 1	Logo 2	Logo 1	Logo 2
Double it	No Answer:	0	0	0	0
	Wrong Answer:	0	0	0	0
	Correct Answer:	19 (100%)	16(100%)	19 (100%)	16(100%)
		Logo 1	Logo 2	Logo 1	Logo 2
Add four marbles	No Answer:	0	0	0	0
	Wrong Answer:	0	0	0	0
	Correct Answer:	19 (100%)	16(100%)	19(100%)	16(100%)
		Logo 1	Logo 2	Logo 1	Logo 2
Triple the total	No Answer:	0	0	0	0
	Wrong Answer:	3 (15.8%)	3(18.8%)	3(15.8%)	1(6.3%)
	Correct Answer:	16 (84.2%)	13(81.3%)	16(84.2%)	15(93.8%)
		Logo 1	Logo 2	Logo 1	Logo 2
Take away the original number	No Answer:	0	0	0	0
	Wrong Answer:	5 (26.3%)	3(18.8%)	3(15.8%)	1(6.3%)
	Correct Answer:	14 (73.7%)	13(81.3%)	15(84.2%)	15(93.8%)

4. If the final result in the story is 42, how many marbles are in a bag? Use the space below to show or explain how you solve this problem.

	Logo 1	Logo 2
No Answer:	0	0
Wrong Answer:	7(36.8%)	3(18.8%)
Correct Answer:	12(63.2%)	13(81.3%)

5. Write marble bag stories that can be represented by the following algebraic expressions.

Expression		Story Logo 1	Logo 2
A) 3X + 10	No Answer:	0	0
	Wrong Answer:	3(15.8%)	3(18.8%)
	Correct Answer:	16(84.2%)	13(81.3%)

Expression		Story Logo 1	Logo 2
B) 5(X + 7)	No Answer:	0	0
	Wrong Answer:	4(21.1%)	0
	Correct Answer:	15(78.9%)	16(100%)

6. Write an algebraic equation that represents the following situation on a balance. & stands for a marble bag and • stands for a marble.

	Logo 1	Logo 2
No Answer:	5 (26.3%)	0
Wrong Answer:	2(10.5%)	4(25%)
Correct Answer:	12(63.2%)	12(75%)

7. Explain what you would do to the balance in problem 6 to find the number of marbles in a bag.

	Logo 1	Logo 2
No Answer:	1(5.3%)	0
Wrong Answer:	2(10.5%)	2(12.5%)
Correct Answer:	16(84.2%)	14(87.5%)

8. Write an algebraic equation that represents the situation on the following balance.

	Logo 1	Logo 2
No Answer:	6 (31.6%)	0
Wrong Answer:	2(10.5%)	2(12.5%)
Correct Answer:	11(57.9%)	14(87.5%)

---

9. Explain what you would do to the balance to find the number of marbles in a bag.

	Logo 1	Logo 2
No Answer:	2(10.5%)	0
Wrong Answer:	4(21.1%)	4(25%)
Correct Answer:	13(68.4%)	12(75%)

---

10. Write the following stories as algebraic expressions (one-liners).

<u>Story</u>

A) Think of a number
Multiply by 6
Divide by 3.

(one-liner)
<u>Expression</u>

	Logo 1	Logo 2
No Answer:	0	0
Wrong Answer:	3(15.8%)	0
Correct Answer:	16(84.2%)	16(100%)

<u>Story</u>

B) Think of a number
Add 10
Multiply by 4
Divide by 2
Subtract 10.

(one-liner)
<u>Expression</u>

	Logo 1	Logo 2
No Answer:	0	0
Wrong Answer:	1(5.3%)	0
Correct Answer:	15(78.9%)	16(100%)

11. Write stories that can be represented by the following expressions.

Expression	Story	Logo 1	Logo 2
A) $\frac{4X}{2}$	No Answer:	0	0
	Wrong Answer:	1(5.3%)	0
	Correct Answer:	18(94.7%)	16(100%)

Expression	Story	Logo 1	Logo 2
B) $\frac{10(X+2)}{5} - 4$	No Answer:	1(5.3%)	0
	Wrong Answer:	2(10.5%)	1(6.3%)
	Correct Answer:	16(84.2%)	15(93.8%)

Part 1 Summary: (Maximum score=16.5)

Logo-Algebra 1	Logo-Algebra 2
N=19	N=16
Minimum=5.5 (33.3%)	Minimum=11.5(69.7%)
Maximum=16.5 (100%)	Maximum=16.5(100%)
Mean=13.3 (80.7%)	Mean=14.8(89.4%)
Standard Dev.=3.2 (19.2%)	Standard Dev.=1.6(9.5%)

Part 2

12. Solve the following algebraic equations for X.

			Logo 1	Logo 2
A)	$5X + 13 = 23$	No Answer:	0	0
		Wrong Answer:	1(5.3%)	4(25%)
		Correct Answer:	18(94.7%)	12(75%)

			Logo 1	Logo 2
B)	$3(X + 1) = 18$	No Answer:	0	0
		Wrong Answer:	4(21.1%)	1(6.3%)
		Correct Answer:	15(78.9%)	15(93.8%)

			Logo 1	Logo 2
C)	$4(7X + 3) - 2X = 64$	No Answer:	2(10.5%)	0
		Wrong Answer:	4(21.1%)	1(6.3%)
		Correct Answer:	13(68.4%)	15(93.8%)

D) $3X + 15 = X + 31$

	Logo 1	Logo 2
No Answer:	1(5.3%)	2(12.5%)
Wrong Answer:	7(36.8%)	3(18.8%)
Correct Answer:	11(57.9%)	11(68.8%)

E) $\frac{9X}{3} = 33$

	Logo 1	Logo 2
No Answer:	1(5.3%)	1(6.3%)
Wrong Answer:	3(15.8%)	2(12.5%)
Correct Answer:	15(78.9%)	13(81.3%)

F) $\frac{4(X+10)}{2} - 10 = 70$

	Logo 1	Logo 2
No Answer:	2(10.5%)	0
Wrong Answer:	3(15.8%)	5(31.3%)
Correct Answer:	14(73.7%)	11(68.8%)

G) $\frac{X+3}{5} = 4$

	Logo 1	Logo 2
No Answer:	5(26.3%)	5(31.3%)
Wrong Answer:	8(42.1%)	8(50%)
Correct Answer:	6(31.6%)	3(18.8%)

Part 2 Summary (Maximum score=7)

Logo-Algebra 1

N=19
Minimum=1 (14.3%)
Maximum=7 (100%)
Mean=4.8 (69.2%)
Standard Dev.=2.3(32.4%)

Logo-Algebra 2

N=16
Minimum=3(42.9%)
Maximum=7(100%)
Mean=5(71.4%)
Standard Dev.=1.2(17.3%)

Appendix I.

Effect of Position of Marble Bag Questions on Post Test

Experimental Group Students sorted by whether they did the the marble bag questions before or after the standard ones on the post test. And performance on both standard and marble bags versions of questions 4c, 5c, and 9d.

Students to Whom Marble Bags Problems Presented **Before** Standard Problems On Post Test (N=8)

	4c		5c		9d	
	St	Mrb	St	Mrb	St	Mrb
SA (pre)	√	√	X	√	√	√
DB (pre)	√	√	√	x	√	
JD (pre)	X	√	X	√	X	X
MD (pre)	X	X	X	X	X	X
JJ (pre)	√	√	√	√	√	√
TJ (pre)	X	√	X	X	X	X
JK (pre)	X	√	√	X	X	X
TO (pre)	X	√	√	√	X	X

Students to Whom Marble Bags Problems Were Presented **After** Standard Problems On Post Test (N=8)

	4c		5c		9d	
	St	Mrb	St	Mrb	St	Mrb
MB (post)	X	√	X	√	√	√
KC (post)	√	√	√	√	√	√
LD (post)	√	√	√	√	X	X
JG (post)	√	√	X	√	√	√
CJ (post)	X	√	X	X	X	X
RL(post)	X	√	X	X	X	X
JSh (post)	X	√	√	√	X	X
CV (post)	√	X	X	√	X	X

Summary

	Before	After
4c. both right:	3/8	3/8
marbles only:	4/8	4/8
standard only:	0	1/8
both wrong:	1/8	0/8

	Before	After
5c. both right:	2/8	3/8
marbles only:	2/8	3/8
standard only:	2/8	0/8
both wrong:	2/8	2/8

	Before	After
9d. both right:	3/8	3/8
marbles only:	0/8	0/8
standard only:	0/8	0/8
both wrong:	5/8	5/8

Analysis: The performance of the two groups was quite similar. The largest difference was a one student difference. This strongly supports the claim that the order of the marble bag and standard problems had no significant effect on the experimental group's performance on the test.