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<th>Shahsavari, Setareh, and Gareth H. McKinley. “Mobility and Pore-Scale Fluid Dynamics of Rate-Dependent Yield-Stress Fluids Flowing through Fibrous Porous Media.” Journal of Non-Newtonian Fluid Mechanics 235 (September 2016): 76–82.</th>
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<td>As Published</td>
<td><a href="http://dx.doi.org/10.1016/J.JNNFM.2016.07.006">http://dx.doi.org/10.1016/J.JNNFM.2016.07.006</a></td>
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<td>Elsevier BV</td>
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<td>Version</td>
<td>Author’s final manuscript</td>
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Mobility and Pore-Scale Fluid Dynamics of Rate-Dependent Yield-Stress Fluids Flowing through Fibrous Porous Media

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Short Communication
Published in the Journal of Non-Newtonian Fluid Mechanics
July 26, 2016

Abstract
The steady flow of viscoplastic fluids through fibrous porous media is studied numerically and theoretically. We consider fluids with a plastic yield stress and a rate-dependent viscosity that can be described by the Herschel-Bulkley model. We first investigate the pore-scale flow characteristics through numerical simulations for flow transverse to a square array of fibers with comprehensive parametric studies to independently analyze the effects of the rheological properties of the fluid and the geometrical characteristics of the fibrous medium. Our numerical simulations show that the critical Bingham number at which the flow transitions from a fully-yielded regime to locally unyielded regions depends on the medium porosity. We develop a scaling model for describing the bulk characteristics of the flow, taking into account the coupled effects of the medium porosity and the fluid rheology. This model enables us to accurately predict the pressure-drop-velocity relationship over a wide range of Bingham numbers, power-law indices, and porosities with a formulation that can be applied to a square or a hexagonal array of fibers. The ultimate result of our scaling analysis is a generalized form of Darcy’s law for Herschel-Bulkley fluids with the mobility coefficient provided as a function of the system parameters. Based on this model, we construct a modified Bingham number rescaled with a suitable porosity function, which incorporates all the rheological and pore-scale parameters that are required to determine the dominant flow regime.

1 Introduction
Flows of complex fluids through porous media are encountered in many industrial processes such as polymer processing, filtration and enhanced oil recovery operations [1]. Understanding the flow characteristics in these
processes is challenging due to the complexity of the fluid behavior in the porous micro-structure. One of the most important objectives in studying porous media flows is to find the relationship between the macroscopic imposed pressure drop and the flow rate [2]. For Newtonian fluids, this relationship is described by Darcy’s law, which shows a linear proportionality between the flow rate and the pressure drop across a porous medium. Complex fluids do not typically obey Darcy’s law due to nonlinearities in their rheological response to an applied stress. However, it is possible to develop modified forms of Darcy’s law for generalized Newtonian fluids [3]. This involves evaluating an effective or characteristic fluid viscosity \( \eta_{eff} \), which depends on the constants in the relevant constitutive model as well as the flow rate and the microstructural characteristics of the porous medium. Consequently, the modified Darcy law for the flow of generalized Newtonian fluids through porous media is of the form

\[
\frac{\Delta p}{L} = \frac{\eta_{eff} U}{\kappa}
\]  

(1)

where \( \Delta p/L \) is the pressure drop per unit length of the porous medium, \( U \) is the superficial or apparent velocity of the fluid, and \( \kappa \) is the permeability of the porous medium.

There are several studies on the flow of generalized Newtonian fluids through porous media, many of which consider the flow of shear-thinning fluids (without a yield stress). Examples of these studies include experimental investigations of the flow through packed spheres [4–6] and Hele-Shaw cells (designed to capture relevant characteristics of a porous medium) [7, 8], as well as numerical studies [9–12] and pore-network modeling [13, 14]. However, many complex fluids such as filled polymer melts, foams, and emulsions also exhibit a yield stress; therefore, a critical imposed pressure drop typically needs to be reached before the material can flow through the porous medium. Among the earliest studies that considered a critical pressure gradient flow through porous media are the work by Gheorghitza [15] and by Entov [16], in which theoretical study of Bingham fluids was presented. Chevalier and Talon [17] recently used a lattice-Boltzmann scheme to numerically study the two-dimensional flow of Bingham fluids through porous media.

Inelastic materials with a yield stress and a rate-dependent viscosity are often described by the Herschel-Bulkley (HB) model [18], which captures both salient features and we use this model in the present study. Park [19] experimentally investigated steady flow of HB fluids through packed beds of spheres and reported large deviations from semi-empirical models developed for Newtonian fluids. Al-Fariss and Pinder [20] were the first to present an empirical model for describing the macroscopic flow behavior of yield stress fluids in porous media in the form of a modified Darcy’s law. Using the HB model, they modified the Blake-Carman-Kozeny equation to describe the flow of waxy oils through beds of packed spheres. Liu and Masliyah [21] studied HB fluids in ducts of arbitrary cross-sections and correlated the pressure drop to the flow rate using three fitting constants.
for each duct shape. Then using a volume averaging technique, they developed a suitable bundle-of-capillaries model. More recent studies on the flow of HB fluids include the work by de Castro et al. [22], who also used a bundle-of-capillaries model and matched the numerical simulations with experimental results for the flow of a xanthan gum solution in order to deduce the pore size distribution in a porous medium. It has been argued that capillary (bundle-of-tubes) models cannot describe the flow of HB fluids through porous media for a wide range of shear-thinning indices [23]. Balhoff [24] and Sochi [25] have used pore-scale network modeling to study the flow of yield stress fluids through porous media. A recent experimental study of HB fluids in porous media was presented by Chevalier et al. [26]. They measured the pressure drop vs. flow rate for the flow of a yield stress fluid (water-in-oil emulsion) through packed glass beads and proposed an empirical relationship for a medium porosity of 33%.

Most of the literature on the flows of complex fluids through porous media discuss the flow through packed beds of spherical particles. There are fewer studies that consider complex fluids flow through fibrous porous media. The anisotropic characteristics of fibrous media leads to a different functional form for the pressure drop dependence on the medium porosity compared to that of packed spheres. Moreover, studies on packed spheres typically only cover a narrow range of porosities close to those expected for random close packing; thus they cannot be applied for arbitrary fibrous media, where a much wider porosity range is accessible. Bleyer and Coussot [27] have investigated the flow of HB fluids in fibrous media using numerical simulations of flow through an ordered array of cylinders. Our numerical approach in this work is similar to that of Bleyer and Coussot. However, the focus of their study was on the velocity fields and demonstration of low sensitivity of velocity fields to the fluid rheology; while it is known that the rheological properties significantly affect the fluid mobility, which is the focus of our work here.

In this Short Communication, we investigate the flow of purely viscous rate-dependent yield stress fluids through fibrous media by means of numerical simulations and a scaling analysis. By comparing the two approaches, we develop an effective viscosity function from our scaling model, which can be used in the generalized Darcy’s law for steady fully-developed flow of HB fluids through fibrous media.

2 Rate-dependent mobility

In flows of complex fluids through porous media, the fluid effective viscosity, \( \eta_{\text{eff}} \), is a function of the medium porosity because the pore-scale shear rate in the fluid between the adjacent fibers or spheres also varies with the volume fraction of solid packing. Hence, the effective viscosity can be combined with the medium permeability, \( \kappa \), (which is also a function of the porosity) to yield a single coefficient that relates the pressure drop across the bed to the superficial velocity in the modified Darcy’s law (equation (1)). This single coefficient is called the
Fluid mobility (with units of [m²Pa⁻¹s⁻¹]) and is defined as [28]:

\[ M \equiv \frac{\kappa(\varepsilon, d)}{\eta_{\text{eff}}(\varepsilon, \dot{\gamma}_{\text{eff}}, \text{fluid rheology})} \]  

(2)

where \( \varepsilon \) is the porosity and \( \dot{\gamma}_{\text{eff}} \) is the effective shear rate, which is a function of the flow rate and the microstructure of the medium. We consider a regularized bi-viscosity formulation of the Herschel-Bulkley (HB) model, in which the viscosity can be written in the following form

\[ \eta(\dot{\gamma}) = \begin{cases} 
\eta_r & \sigma < \sigma_y \\
\frac{\sigma_y}{\frac{m}{n}} + m\dot{\gamma}^{n-1} & \sigma \geq \sigma_y 
\end{cases} \]  

(3)

where \( \sigma_y, n, \) and \( m \) denote the fluid yield stress, the power-law exponent, and the consistency of the fluid, respectively. The characteristic shear rate \( \dot{\gamma} \) is defined as \( \dot{\gamma} = \sqrt{\frac{1}{2} \left( \dot{\gamma} : \dot{\gamma} \right)} \) where \( \dot{\gamma} = \nabla u + \nabla u^T \). The theoretical viscosity for a true HB fluid at stresses below the yield/critical stress is \( \eta_r \to \infty \). In the regularized model, below the yield stress, the viscosity is taken to be constant at a very large value denoted by \( \eta_r \), and the crossover takes place at a critical shear rate \( \dot{\gamma}_c \) given by \( \sigma_y / \dot{\gamma}_c + m\dot{\gamma}_c^{n-1} = \eta_r \).

Four dimensionless groups that determine the fluid mobility are the Reynolds number, the Bingham number (i.e. the ratio of the fluid yield stress to a characteristic viscous stress [29]), the medium porosity \( \varepsilon \), and the power-law exponent \( n \). The Bingham number for a rate-dependent HB fluid can be defined as

\[ \text{Bn} \equiv \frac{\sigma_y}{m(U/d)^n} \]  

(4)

where \( d \) is the characteristic length scale of the flow. For a fibrous medium, the natural choice for this length scale is the average fiber diameter. We also use a generalized Reynolds number defined as \( \text{Re} = \rho U^2 - n d^n / m \) to take into account the rate-dependent inertial effects in the fluid. In the present communication, we study the low Reynolds number regime (where the fluid inertia is negligible), i.e. \( \text{Re} \ll 1 \) and we investigate the coupled and inter-dependent effects of the three other dimensionless numbers (Bn, \( \varepsilon \), and \( n \)) on the flow of HB fluids transverse to a periodic array of fibers.
3 Numerical Analysis

The conservation of mass and momentum for the steady-state flow of generalized Newtonian fluids are given by equations (5) and (6) respectively.

\[
\nabla \cdot \mathbf{u} = 0 \quad (5)
\]

\[
\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla \cdot (\eta(\dot{\gamma}) \{ \nabla \mathbf{u} + \nabla \mathbf{u}^T \}) \quad (6)
\]

We numerically solve these equations for transverse flow through aligned fibrous media, building on the approach that we adopted for steady slow of Carreau fluids [30]. The idealized domain that we consider here for representing a fibrous medium consists of a periodic array of cylinders in a square arrangement as shown in Figure (1). It is also straightforward to extend this analysis to hexagonal arrangements [30]. We assume steady

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**Figure 1:** Top: 3D schematic of flow transverse to an array of fibers. Middle: 2D schematic of the flow domain. Bottom: unit cell representing the periodic array. Boundary conditions assumed for the left and right faces of the unit cell are periodic; those for the top and bottom faces are symmetric.
two-dimensional flow and take advantage of the symmetry and periodicity in the system to solve the equations in a representative unit cell as shown in Figure 1 (bottom). COMSOL Multiphysics 4.3a is used to build the numerical model and solve the equations using a finite-element technique. Depending on the relative spacing $s/d$, varying numbers of mesh elements ($10^4 - 10^5$) were used to achieve mesh-independent results. In the unit cell shown in Figure 1, the imposed boundary conditions consist of zero slip velocity on the cylinder wall and symmetry on the top and bottom boundaries. The left boundary is periodically paired with the right boundary with a finite pressure difference $\Delta p$.

We use the Herschel-Bulkley model (equation (3)) for calculating the rate dependent viscosity in the momentum equation. For regularization of the discontinuity in the constitutive model, we have used a bi-viscosity model similar to that proposed by O’Donovan and Tanner [31]. Although numerically we do not need any regularization, we have used this regularization model to define a straightforward criterion for yield based on the local shear rate, i.e., $\dot{\gamma} > \sigma_y/\eta_r$. For optimum description of the topology of the yield surfaces (which separate the yielded and unyielded zones), the regularization viscosity $\eta_r$ should be higher than $10^3 \eta_f$, where $\eta_f$ is the effective viscosity in the flow domain in the absence of a yield stress [29]. The viscosity $\eta_f$ is calculated from equation (1) using simulations for $Bn = 0$. In our numerical model, we have used $\eta_r = 10^8 \eta_f$ although the selection of this regularization parameter for $\eta_r > 10^3 \eta_f$ does not affect our results. According to Burgos et al. [29], the bulk properties of the flow do not depend on the regularization model. Nevertheless, we have verified that we obtain the same results in terms of the local and bulk parameters using a different regularization model such as Papanastasiou [29].

Figure 2 shows contours of the dimensionless velocity distribution from numerical simulations of the flow transverse to the fiber array. In this figure we compare the velocity fields for shear-thinning fluids with and without a yield stress. In these simulations the power-law exponent is kept constant at $n = 0.5$. For the case of no yield stress ($Bn = 0$, figures 2 (a-c)), the velocity is always maximum on the horizontal symmetry line centered between two adjacent fibers. This is not always the case when the fluid has a yield stress (corresponding to a non-zero Bingham number). In fact, there is a transition in the flow regime for yield stress fluids as the medium porosity $\varepsilon$ is increased. Figure 2 (f) shows that when the relative spacing of the fibers is large (corresponding to the high porosity limit) and the fluid has a yield stress, the fibers do not interact hydrodynamically with each other (i.e., each fiber affect the flow as an isolated fiber). Much of the fluid is unyielded and the maximum velocity occurs somewhere close to the fibers. The dashed lines in the bottom figures show the yield surface separating yielded zones from the unyielded zones (determined where the local viscosity is equal to the regularization viscosity). It is noteworthy to distinguish two different types of unyielded zones: stationary domains (with zero local velocity) vs. moving plugs (where the local velocity is equal to the apparent velocity, i.e. $|\mathbf{u}|/U = 1$). In the high porosity limit, the volume fraction of the unyielded moving zone is close to unity and a large portion
of the fluid moves as a solid plug. Only very close to the cylinders is the local stress high enough to result in yielding and local flow. By contrast, at lower porosities, there are extended stationary plugs of unyielded fluid between the cylinders (where the velocity is zero).

**Figure 2:** Contours of the dimensionless velocity distribution from numerical simulation of the flow transverse to fibers: comparison of shear-thinning fluids with and without a yield stress. In all simulations the power-law exponent is $n = 0.5$. (a) porosity $\varepsilon = 0.50$, Bingham number $Bn = 0$, (b) $\varepsilon = 0.80$, $Bn = 0$, (c) $\varepsilon = 0.99$, $Bn = 0$, (d) $\varepsilon = 0.50$, $Bn = 10$, (e) $\varepsilon = 0.80$, $Bn = 10$, (f) $\varepsilon = 0.99$, $Bn = 10$. The dashed lines in the lower series of figures show the yield surface.

**Figure 3:** Dimensionless pressure drop as a function of the Bingham numbers for various medium porosities. The lines are drawn to connect the numerical data points.
on-dimensional mobility \( \tilde{M} = M \left( \frac{\sigma_y U_d}{\rho d^n} + \frac{mU^n}{d^{n+1}} \right) \) as a function of porosity: (a) numerical simulations for various flow Bingham numbers and power-law exponents. The generalized Reynolds number in all simulations is \( Re = \rho U^2 n U d^n / m = 10^{-3} \). The lines connecting the numerical data points are drawn to guide the eye. (b) Comparison of the numerical simulations for \( Bn = 1 \) and \( n = 0.8 \) with the empirical model by Al-Fariss and Pinder \cite{20}.

Figure 3 shows the results of numerical simulations in terms of the dimensionless pressure drop as a function of the Bingham number for various medium porosities and a power-law exponent of \( n = 0.5 \). The computed pressure drop per unit length is non-dimensionalized by a characteristic viscous pressure drop per fiber diameter \( (mU^n / d^{n+1}) \). This figure shows that there are two independent effects that contribute to the total pressure drop: the (plastic) yield stress and the (viscous) stress arising from viscous dissipation during flow. To understand this, it is helpful to consider how the pressure drop changes as the yield stress is increased in a given
geometry (fixed $\varepsilon$) and for a given set of power-law parameters (fixed $m$ and $n$). For low Bingham numbers the viscous terms dominate; therefore, increasing the fluid yield stress (or the Bingham number) does not change the pressure drop significantly. However, above a critical Bingham number, which depends on the medium porosity, the pressure drop associated with overcoming the yield stress and inducing a plastic flow dominates. In this regime, the pressure drop increases linearly with the fluid yield stress (and with the Bingham number). As the porosity decreases it is clear that the effects of viscous shearing dissipation (at fixed superficial velocity and fixed rheology) become increasingly important. For example when the porosity decreases by a factor of 2.6, the dimensionless pressure drop at low Bingham number increases by a factor of 100 and the crossover value of Bingham number at which the yield stress now dominates also increases by approximately the same orders of magnitude.

Figure 4(a) shows the results of our numerical simulations in terms of the non-dimensional mobility $\tilde{M} = M \left( \frac{\sigma_y}{U d} + \frac{m}{d^{n-1}} \right)$ plotted as a function of porosity for various Bingham numbers and power-law exponents. In the limit of $\varepsilon \to 1$, where the velocity field resulting from the presence of each fiber is independent of the fiber spacing, the pressure drop per length of a unit cell, $\Delta p/s$, scales with $\sigma_y/d + mU^n/d^{n+1}$ and all the data converge to a single value using this characteristic pressure drop for non-dimensionalizing the mobility. In Figure 4(b), we compare the results for $Bn = 1$ and $n = 0.8$ with the empirical relationship developed by Al-Fariss and Pinder [20]

$$\frac{\Delta p}{L} = \sqrt{\frac{\varepsilon}{2\kappa}} \sigma_y + \frac{m}{4\kappa} \left( 3 + \frac{1}{n} \right)^n \left( \frac{50\kappa\varepsilon}{3} \right)^{\frac{1-n}{n}} U^n$$

(7)

which is an extension of the Blake-Carman-Kozeny empirical model, for which the permeability of the porous medium is $\kappa = \frac{1}{150} d^2 \varepsilon^3 / (1 - \varepsilon)^2$ [20]. This empirical model was developed for flow through packed spheres and although it provides a good description of the numerical data for intermediate porosities, it is not able to predict the pressure drop in fibrous media over the full range of porosities. As we show in Figure 4(b), for a porosity of $\varepsilon = 0.25$, there is an order of magnitude difference between the numerically computed value of the pressure drop and the prediction of equation (7).

4 Scaling Analysis

In our previous work, we have developed a scaling model that enabled us to predict the pressure-drop-velocity relationship for the flow of Carreau fluids in fibrous media [30]. Here we extend the scaling approach presented in [30] to fluids with a yield stress that can be described by the Herschel-Bulkley model. Considering the dependency of the effective viscosity in equation (2) on the effective shear rate, we first find a suitable scale for this parameter. If the characteristic pore velocity is $U_p$, the scale for the effective shear rate in
the pore-scale is $\dot{\gamma} = U_p/\delta$ where $\delta$ is a characteristic length scale over which the velocity varies in the narrow gap between the cylinders. For the transverse flow through a fibrous medium with a fiber diameter of $d$ and a fiber spacing of $s$, this length scale is $\delta = (s - d)/2$ and the relation between the pore velocity and the superficial velocity $U$ is therefore, $U_p = U s / (s - d)$ from a simple mass conservation in the unit cell of Figure 1. Combining these scaling estimates, we obtain the following expression for the effective shear rate:

$$\dot{\gamma}_{\text{eff}} = \frac{2U}{d} \frac{s/d}{(s/d - 1)^2}$$

(8)

If we non-dimensionalize the shear rate with the superficial velocity and the fiber diameter such that $\dot{\gamma}^* = \dot{\gamma}_{\text{eff}} \left( \frac{U}{d} \right)^{-1}$, then the effective dimensionless shear rate in a fibrous medium can be expressed as a function of the porosity by substituting for the relative fiber spacing $s/d$ in terms of the porosity using $\varepsilon = 1 - ad^2/s^2$:

$$\dot{\gamma}^*(\varepsilon) = 2\sqrt{a(1 - \varepsilon)} \left( \sqrt{a} - \sqrt{1 - \varepsilon} \right)^2$$

(9)

where $a = \pi/4$ for a square packing and $a = \pi/(2\sqrt{3})$ for a hexagonal packing of fibers.

By substituting this effective shear rate in HB model, equation (3), we obtain the effective value of the viscosity:

$$\eta_{\text{eff}} = \frac{\sigma_y d}{U \dot{\gamma}^*(\varepsilon)} + m \left[ \frac{U}{d} \frac{\dot{\gamma}^*(\varepsilon)}{d} \right]^{n-1},$$

(10)

which when combined with the generalized Darcy’s law, equation (1), yields the following expression for the pressure drop per unit length for the flow of HB fluids through a fibrous bed:

$$\frac{\Delta p}{L} = \kappa(\varepsilon, d) \dot{\gamma}^*(\varepsilon) \left[ \frac{\sigma_y + m \left( \frac{U \dot{\gamma}^*(\varepsilon)}{d} \right)^n}{d} \right].$$

(11)

Here, the permeability of the fibrous medium, $\kappa(\varepsilon, d)$, can be calculated from the analytical solution for the flow of Newtonian fluids through fibrous media that we presented in [30] or from the following semi-empirical expression developed by Tamayol and Bahrami [32], which gives the dimensionless permeability $\kappa^*(\varepsilon)$ as a function of the porosity and the fiber arrangement.

$$\kappa^*(\varepsilon) \equiv \frac{\kappa(\varepsilon, d)}{d^2} = 0.16 \frac{a \left( 1 - \sqrt{(1 - \varepsilon)/a} \right)^3}{(1 - \varepsilon) \sqrt{\varepsilon}}$$

(12)

From equation (11), we can factorize out the term $m(U \dot{\gamma}^*/d)^n$ to obtain

$$\frac{\Delta p}{L} = \frac{m U^n \dot{\gamma}^{n-1}}{d^{n-1} \kappa} (Bn^* + 1),$$

(13)
where we introduce a rescaled Bingham number, defined as

$$Bn^* \equiv \frac{Bn}{\dot{\gamma}^n} = \frac{\sigma_y}{m(U/d)^n \dot{\gamma}^n}.$$  \hfill (14)

This rescaled Bingham number includes all the parameters that need to be considered to determine the dominant flow regime. For $Bn^* \ll 1$, viscous stresses in the fluid dominate while for $Bn^* \gg 1$, the plastic yield stress becomes increasingly important.

Now if we non-dimensionalize the pressure drop using a group that captures the rheology of the fluid ($m$, $n$, $U$, and $d$) and a group that includes the appropriate scaling functions of the medium porosity ($\dot{\gamma}^*$ and $\kappa^*$), we obtain

$$\Delta p^* \equiv \frac{\Delta p}{\frac{d^n \kappa^*}{\dot{\gamma}^n}} \left( \frac{n+1}{mU^n} \right)^{\frac{\kappa^*}{\dot{\gamma}^n}}.$$  \hfill (15)

and ultimately we can write the following simple form of the scaling model given in equation (11):

$$\Delta p^* = 1 + Bn^*.$$  \hfill (16)

When represented in this rescaled dimensionless form, all of the numerical simulation data shown in Figure 3 can be superimposed to produce a master curve as shown in Figure 5. The solid line shows the scaling model prediction, equation (16). The scaling model predicts the numerical results shown in Figure 5, with a maximum error of 7% and 11% for porosities of $\varepsilon = 0.25$ and $\varepsilon = 0.5$, respectively. For higher porosities, the error becomes
Figure 6: All of the numerical simulation data shown in Figure 4(a) can be superimposed by rescaling the dimensionless mobility with the effective viscosity calculated from the scaling model, equations (9-10), divided by the squared fiber diameter. The solid line shows the permeability of the fibrous medium predicted from numerical simulations for a Newtonian fluid.

larger at high \( Bn^* \) since our assumption of defining a single characteristic shear rate for the entire pore-space becomes less accurate. Nevertheless, at \( \varepsilon = 0.8 \) and \( Bn^* = 100 \), the error is less than 21%.

Finally we use our scaling theory to collapse all of the data for the mobility shown in Figure 4(a). First we non-dimensionalize the mobility using our model for the effective viscosity defined in equation (10) at the effective shear rate defined in equation (8). This results in a dimensionless mobility function:

\[
M^* = M \frac{\eta_{\text{eff}}}{d^2}.
\] (17)

Figure 6 shows that all of the numerical simulation data for the mobilities shown in Figure 4(a) can be collapsed using the effective viscosity \( \eta_{\text{eff}} \) calculated from the scaling model (equations (9-10)). The solid line shows the permeability of the fibrous medium predicted from numerical simulations for a Newtonian fluid. Since the rheological parameters characterizing the constitutive response of the fluid are now all included in the effective viscosity, the only remaining parameter controlling the pressure drop is the medium porosity. So the only computation we need to perform is an evaluation of the permeability \( \kappa^*(\varepsilon) \) for a Newtonian fluid through the same geometry (or alternatively using equation (12) with \( a = \pi/4 \)). Again the largest deviation of our numerical simulations from this curve occurs for the high porosity limit in the high Bingham number regime. For \( \varepsilon < 0.9 \), the root mean squared error between the rescaled numerical data shown in Figure 6 and equation (12) for \( Bn = 1 \) and \( Bn = 100 \) is 17% and 21% respectively.
5 Conclusions and Discussion

We have studied numerically the steady flow of viscoplastic yield stress fluids through homogeneous fibrous media and developed a scaling model for predicting the effective fluid viscosity in the porous medium as a function of the apparent flow velocity, the porous medium characteristics, and the rheological properties of the fluid. The generalized Darcy’s law derived here has a similar form to that derived by Al-Fariss and Pinder [20] and by Bleyer and Coussot [27], leading to the following general expression for the pressure drop (written in a form consistent with [27]),

\[ \frac{\Delta p}{L} = \alpha \left( \frac{\sigma_y}{d} \right) + \beta_n \left( \frac{mU^n}{a^{n+1}} \right). \]

Al-Fariss and Pinder provide the coefficients as a function of the porosity (see equation 7 in our manuscript) from an empirical model for packed spheres, which does not accurately predict the mobility in fibrous media when the tortuosity becomes important (especially at low porosities). Bleyer and Coussot provide these coefficient in form of integrals of the local shear rate for general porous media, but the local shear rate is not known without detailed numerical simulation. Our scaling model has the benefit of providing explicit expressions for the coefficients \( \alpha \) and \( \beta_n \) for transverse flow through fibrous media as a function of porosity and fiber arrangement:

\[
\alpha = \frac{\sqrt{\varepsilon}}{0.32(\sqrt{a/(1-\varepsilon)} - \sqrt{a})},
\]

\[
\beta_n = \frac{2^n \sqrt{a^n \varepsilon (1-\varepsilon)^{n+1}}}{0.32 (\sqrt{a} - \sqrt{1-\varepsilon})^{2n+1}},
\]

where the effect of the arrangement of fibers is included in the geometric parameter \( a \), with \( a = \pi/4 \) for square packing and \( a = \pi/(2\sqrt{3}) \) for hexagonal packing. It is important to note that the coefficients in this generalized Darcy’s law are independent of the material consistency and yield stress (which is consistent with the numerical simulations of Bleyer and Coussot [27]). As discussed in [27], the coefficient \( \alpha \), which determines the contribution of the yield stress to the total pressure drop is a function of the structure of the porous medium and the coefficient \( \beta_n \), which is the factor for the viscous stress contribution depends on both geometry and the shear-thinning index, \( n \). For a porosity of \( \varepsilon = 0.3 \) in the range of random close packing, we obtain \( \alpha = 12.9 \) (assuming hexagonal arrangement of fibers) and for the same porosity, considering a simple Bingham fluid \( (n = 1) \), we calculate \( \beta_1 = 443 \). These values are in good agreement with the values estimated in [27] \( \approx 16 \) and \( \approx 500 \) respectively. The respective values for \( \alpha \) and \( \beta_1 \) for the model by Al-Fariss and Pinder are found to be 20 and 2700; but as discussed earlier, they consider a packed bed of spheres, which does not provide a good description of the numerical simulations for low porosity fibrous media.
The numerical simulations show a pronounced transition in the pore-scale velocity field as the bed porosity increases, from a regime of strongly hydrodynamically interacting fibers to isolated non-interacting fibers. The characteristics of the flow are therefore, influenced by a coupled effect of the Bingham number of the fluid and the medium porosity. For instance, the material can be fully yielded even at high Bingham numbers (\(Bn > 1\)), in a low porosity medium, as has been observed experimentally by Chevalier et al. [33]. To capture this coupling, we have developed a modified or rescaled Bingham number (\(Bn^*(\varepsilon)\)) from our scaling model (equation (4)), which includes the nonlinear effects of the medium porosity in addition to the rheological parameters. For a rescaled Bingham number of \(Bn^* < 1\), we have a fully-yielded viscous flow regime, while \(Bn^* > 1\) indicates a regime in which a large fraction of the fluid volume in the pore space is unyielded (and moving as a plug) so that the pressure drop across the fiber bed is dominated by the magnitude of the fluid yield stress. This rescaling leads to a particularly simple expression for the dimensionless pressure drop (as defined in equation (16)).

The scaling model that we have developed here is in a good agreement with the numerical simulations over a wide range of porosities, shear-thinning indices, and Bingham numbers (\(\varepsilon \lesssim 0.9\) or \(Bn \lesssim 100\)). The resulting expression for the pressure-drop-flow-rate relationship (equation (11)), can be used as a modified Darcy’s law for the flow of Herschel-Bulkley fluids through fibrous media without requiring any additional fitting parameters other than those model constants required for describing the fluid rheology.

References


