Homotopy Type of Intervals of the Second Higher Bruhat Orders

HOMOTOPY TYPE OF INTERVALS OF THE SECOND HIGHER BRUHAT ORDERS

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ABSTRACT. The higher Bruhat order is a poset generalizing the weak order on permutations. Another special case of this poset is an ordering on simple wiring diagrams. For this case, we prove that every interval is either contractible or homotopy equivalent to a sphere. This partially proves a conjecture due to Reiner. Our proof uses some tools developed by Felsner and Weil to study wiring diagrams.

1. Introduction

The weak order is a well-studied partial ordering on the permutations of $[n] = \{1, \ldots, n\}$. Permutations are ordered by their *inversion set*, which is a collection of 2-element subsets of $[n]$. Introduced by Manin and Schechtman [10], the higher Bruhat order $B(n,d)$ is a generalization of the weak order to "consistent" subsets of $\binom{[n]}{d+1} = \{I \subseteq [n] : |I| = d+1\}$, defined in Section 3. The posets $B(4,2)$ and $B(5,2)$ appear in Figure 1.

Many equivalent interpretations of these posets were established in [18, Theorem 4.1], [11], and [9]. They prove that $B(n,d)$ may be interpreted as the collection of "admissible" permutations of $\binom{[n]}{d}$ up to a suitable equivalence, the cubical tilings of a cyclic zonotope, or the single-element extensions of an alternating oriented matroid. A couple of these interpretations will be recalled in Sections 3 and 4, as necessary. The higher Bruhat orders have appeared in a wide variety of contexts, including higher categories and Zamolodchikov's tetrahedral equation [9], soliton solutions of the Kadomtsev−Petviashvili equation [5], and the multidimensional cube recurrence [8].

Rambau proved that $B(n,d)$ is homotopy equivalent to an $(n-d-2)$−dimensional sphere as an application of his Suspension Lemma [13]. In [14, Conjecture 6.9], Reiner conjectured that every interval is either contractible or homotopy equivalent to a sphere. For $d = 1$, the poset $B(n,d)$ is the weak order, for which this conjecture holds by work of Björner [1]. Our main result proves Reiner's conjecture for $d = 2$.

Theorem 1.1. Every interval of $B(n,2)$ is either contractible or homotopy equivalent to a sphere.

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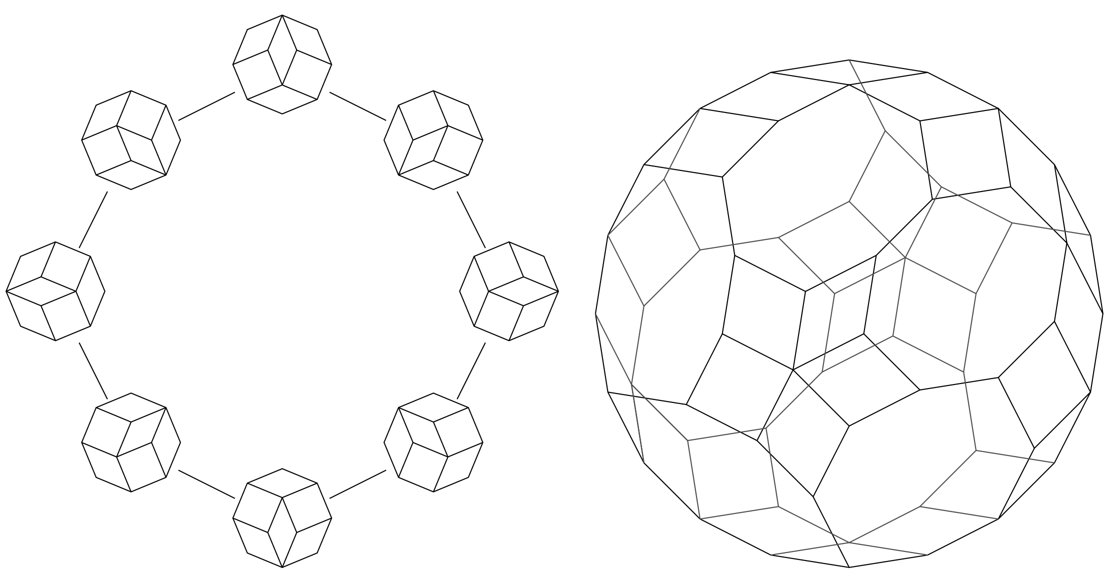

FIGURE 1. (left) $B(4,2)$ as a poset of rhombic tilings of a zonogon. (right) $B(5,2)$

The paper is organized as follows. Some topological preliminaries are given in Section 2. We prove some results on general higher Bruhat orders in Section 3. Wiring diagrams are defined in Section 4 along with other results specific to the second higher Bruhat orders. Finally, the proof of Theorem 1.1 is given in Section 5.

2. Poset Topology

We establish some notation and recall a few fundamental results on the topology of posets, following Björner [2].

We say a poset P is *bounded* if it has both a minimum and maximum element, denoted $\hat{0}$ and $\hat{1}$, respectively. If P is bounded, then the *proper part* P' is the subposet $P - \{\hat{0}, \hat{1}\}$. Given $x \leq y$, the *closed interval* $[x, y]$ (*open interval* (x, y)) is the set of $z \in P$ such that $x \leq z \leq y$ ($x < z < y$).

The *order complex* $\Delta(P)$ of a poset P is the abstract simplicial complex with vertex set P and simplices $\{x_0, \ldots, x_d\}$ where $x_0 < \cdots < x_d$ is a chain of P. We define the topology of a poset to be that of its order complex. One may reduce many results on the homotopy types of posets to the following lemma.

Lemma 2.1 ([16] Proposition 6.1)**.** If $x \in P$ such that $P_{<x}$ or $P_{>x}$ is contractible, then P is homotopy equivalent to $P - \{x\}$.

Let $x_1 < \cdots < x_N$ be a linear extension of P. By deleting elements x_i such that $P_{>x_i}$ is contractible in the order of the linear extension, we deduce the following result; see e.g. [6, Lemma 4].

Lemma 2.2. P is homotopy equivalent to the subposet

$$\{x \in P : \; P_{>x} \text{ is non-contractible } \}.$$

We let P_{nonc} be the subposet of P defined in Lemma 2.2. Let $\text{Int}(P)$ be the poset of closed intervals of P, ordered by inclusion. It is known that if P is bounded, then $\text{Int}(P)'$ is homeomorphic to the suspension of P'. This was originally proved by Walker [17, Theorem 6.1(c)] by specifying a "subdivision map" between geometric realizations of their order complexes. An alternative proof was given in [12, Lemma 3.3.10] by constructing the order complex of $\text{Int}(P)'$ from the suspension of P' by a sequence of edge-stellations.

For a bounded poset P, let $\text{Int}_{\text{nonc}}(P)$ be the poset of closed intervals $[x, y]$ for which (x, y) is non-contractible, ordered by inclusion. We note that if $x = y$ or y covers x, then $\Delta((x, y))$ is an empty complex, which is non-contractible.

Lemma 2.3. The poset $\text{Int}_{\text{nonc}}(P)'$ is homotopy equivalent to the suspension of P'.

Proof. From the above discussion, it suffices to show that $\text{Int}_{\text{nonc}}(P)'$ is homotopy equivalent to $\text{Int}(P)'$.

Let $I_1, \ldots, I_N$ be a linear extension of $\text{Int}(P)'$. For $i \geq 0$, let

$$Q_i = \{I_j : \; j \leq i \text{ or } I_j \text{ is non-contractible}\}.$$

Then $Q_N = \text{Int}(P)'$ and $Q_0 = \text{Int}_{\text{nonc}}(P)'$. If I_i is non-contractible then $Q_{i-1} = Q_i$.

Let $i \geq 0$ and assume I_i is contractible. Since $I_j \subseteq I_i$ implies $j \leq i$, the subposet $(Q_i)_{<I_i}$ is equal to $\text{Int}(I_i)'$. The latter is the suspension of a contractible complex, so it is contractible. Hence, $Q_{i-1} \simeq Q_i$. The result now follows by induction. □

3. Higher Bruhat orders

Fix $d, n \in \mathbb{N}$ such that $1 \leq d < n$. A subset X of $\binom{[n]}{d+1}$ is *closed* if for $I \in \binom{[n]}{d-1}$, $i, j, k \in [n] - I$, $i < j < k$,

$$\text{if } I \cup \{i, j\} \in X \text{ and } I \cup \{j, k\} \in X \text{ then } I \cup \{i, k\} \in X.$$

For $X \subseteq \binom{[n]}{d+1}$, we let $\overline{X}$ be the smallest closed set containing X. If X is a family of subsets of $[n]$ and $P \subseteq [n]$, the *restriction* $X|_P$ of X to P is the subfamily of subsets contained in P. A subset X of $\binom{[n]}{d+1}$ is *consistent* if X and $\binom{[n]}{d+1} - X$ are both closed. Equivalently, X is consistent if for any $(d+2)$-subset $P = \{i_0, \ldots, i_{d+1}\}$, $i_0 < \cdots < i_{d+1}$,

$$X|_P = \begin{cases} \{P \setminus i_{d+1}, \ldots, P \setminus i_t\} & \text{for some } t, \textit{ or} \\ \{P \setminus i_t, \ldots, P \setminus i_0\} & \text{for some } t. \end{cases}$$

We remark that while we use the notation $X - Y$ for the set of elements in X not in Y, we use the shorthand $X \setminus i$ to mean $X - \{i\}$.

The *higher Bruhat order* $B(n, d)$ is the poset of consistent subsets of $\binom{[n]}{d+1}$ ordered by *single-step inclusion*; that is, $X \leq Y$ if there exists a sequence of consistent subsets $X_0 \subseteq \cdots \subseteq X_t$ such that $X = X_0$, $Y = X_t$ and

$|X_i - X_{i-1}| = 1$ for all i. The same set ordered by ordinary inclusion is denoted $B_\subseteq(n,d)$. The posets $B(n,d)$ and $B_\subseteq(n,d)$ are both graded with rank function $X \mapsto |X|$ [18, Theorem 4.1(G)].

When $d = 1$, $B(n,1)$ is isomorphic to the weak order on the symmetric group, and the two orders $B(n,1)$ and $B_\subseteq(n,1)$ coincide. The weak order $B(n,1)$ is a lattice where the join of X and Y is $\overline{X \cup Y}$. If $d \geq 2$, the poset $B(n,d)$ may not be a lattice; in particular, $B(6,2)$ is not a lattice [18, Theorem 4.4]. Ziegler proved that $B(n,d) = B_\subseteq(n,d)$ when $n - d \leq 4$, but $B(8,3)$ is weaker than $B_\subseteq(8,3)$ [18, Theorem 4.5]. In fact, his example in $B(8,3)$ shows that $\overline{X \cup Y}$ need not be the join of $X, Y \in B(n,d)$ even if $\overline{X \cup Y}$ is consistent.

For $X, Y \in B(n,d)$, if $X \subseteq Y$ we define the *ascent set*

$$\mathrm{Asc}(X,Y) = \{I \in Y - X : \ X \cup \{I\} \in B(n,d)\}.$$

If $Y = \hat{1}$, we write $\mathrm{Asc}(X)$ for $\mathrm{Asc}(X,Y)$.

Lemma 3.1. Fix $X \in B(n,d)$. The ascent set $\mathrm{Asc}(X)$ decomposes as the disjoint union $\mathrm{Asc}(X) = A_1 \sqcup \cdots \sqcup A_N$ where

(1) $A_t = \{\{a_{t,1} < \cdots < a_{t,d+1}\}, \{a_{t,2} < \cdots < a_{t,d+2}\}, \ldots, \{a_{t,r_t} < \cdots < a_{t,d+r_t}\}\}$ (i.e. A_t is the set of contiguous intervals in the set $\{a_{t,1}, \ldots, a_{t,r_t+d}\} \subseteq [n]$), and
(2) if $I \in A_s,\ J \in A_t,\ s \neq t$ then $|I \cap J| < d$.

Proof. We first show that any ascent $I \in \mathrm{Asc}(X)$ shares d elements with at most two other ascents of X. Suppose $I, J \in \mathrm{Asc}(X)$ such that $|I \cap J| = d$ with $I < J$ in lexicographic order. The restriction $X|_{I \cup J}$ is an element of $B(|I \cup J|, d)$ with two ascents, so it must be the bottom element. Consequently, the set I (J) is the lex-minimal (lex-maximal) $(d+1)$-subset of $I \cup J$.

Now suppose $J' \in \mathrm{Asc}(X),\ J' \neq J$ such that $J' > I$ in lexicographic order and $|J' \cap I| = d$. Then J' is the lex-maximal $(d+1)$-subset of $I \cup J'$ by the above argument. But, $|J \cap J'| = d$ and J, J' are not at opposite ends of their $(d+1)$-packet, a contradiction.

We have now established that for any $I \in \mathrm{Asc}(X)$, there is at most one $J > I$ in lexicographic order for which $|I \cap J| = d$. By similar reasoning, there is at most one $L < I$ with $|I \cap L| = d$. Thus, $\mathrm{Asc}(X)$ decomposes into chains $I_1^t < I_2^t < \cdots < I_{m_t}^t$ where $|I_i^t \cap I_{i+1}^t| = d$ and all other intersections have cardinality strictly less than d. □

Lemma 3.2. If $X \in B(n,d)$, then $X \cap \overline{\mathrm{Asc}(X)} = \emptyset$.

Proof. The ascent set $\mathrm{Asc}(X)$ is a subset of $\binom{[n]}{d+1} - X$, and the latter set is closed. Hence, $X \cap \overline{\mathrm{Asc}(X)}$ is empty. □

4. The second higher Bruhat order

In this section, we give another interpretation of the second higher Bruhat order $B(n,2)$ as a poset of simple wiring diagrams. Most of our results about $B(n,2)$ will be proved by alternating between wiring diagrams and consistent sets.

A *wiring diagram* is a collection of *wires*, continuous piecewise linear curves $C_1, \ldots, C_n$ in $\mathbb{R}^2$, satisfying the following conditions.

- The projection of C_i onto the first coordinate is bijective.
- The wires are in order $C_1, \ldots, C_n$ top-to-bottom, sufficiently far to the right.
- Distinct wires C_i, C_j cross at a unique point.
- All crossings are transverse.

We shall further assume that the wiring diagram is *simple*, meaning there are no common intersections among three or more wires. In particular, each wire C_i determines a permutation $\pi_i = a_1 \cdots a_{n-1}$ of $[n] \setminus i$ where if $r < s$ then the first coordinate of $C_i \cap C_{a_r}$ is less than that of $C_i \cap C_{a_s}$. Two wiring diagrams are considered equivalent if they determine the same sequence of wire permutations $(\pi_i)_{i \in [n]}$.

A *pseudoline arrangement* is a collection of piecewise linear curves in the plane such that every pair of curves intersect transversely at a unique point. Our definition of wiring diagram is combinatorially equivalent to that of pseudoline arrangements. By this identification, simple wiring diagrams are in natural bijection with the rhombic tilings of a zonogon by the Bohne-Dress Theorem [4], as illustrated in Figure 2. We prefer to use the language of wiring diagrams in our proofs, but similar arguments could be made in terms of pseudoline arrangements or rhombic tilings.

For $1 \leq i < j < k \leq n$, if the crossing of C_i and C_k is below (above) C_j, then $\{i,j,k\}$ is an *inversion triple* (*non-inversion triple*). The map taking a wiring diagram to its set of inversion triples defines a bijection between

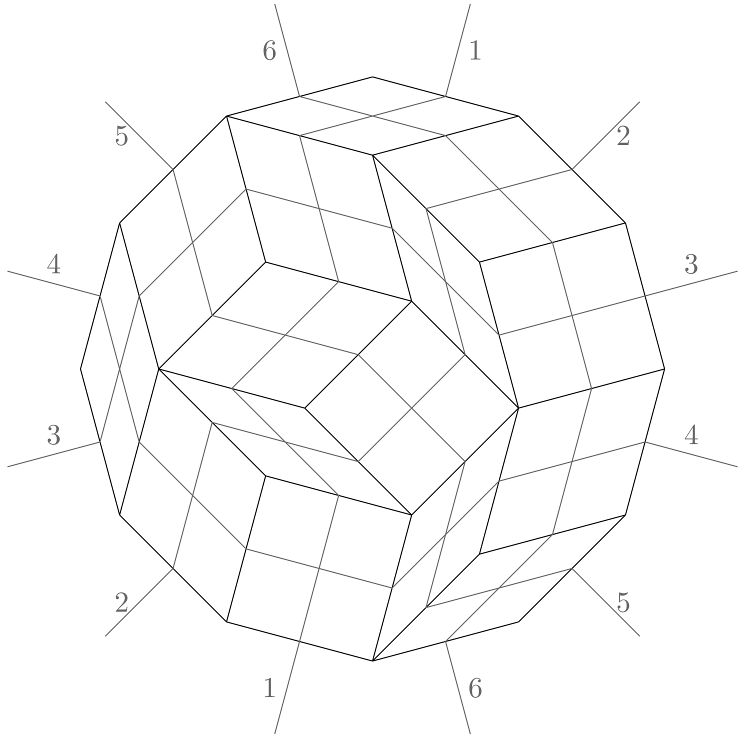

FIGURE 2. This is a simple pseudoline arrangement with corresponding rhombic tiling. The associated wiring diagram is obtained by "stretching" the curves to extend left-to-right without backtracking.

equivalence classes of simple wiring diagrams with n wires and consistent subsets of $\binom{[n]}{3}$. A *block* is a set of non-inversion triples of the form $\{\{i_j, i_{j+1}, i_{j+2}\} : j \in [m]\}$, where $1 \le i_1 < \cdots < i_{m+2} \le n$.

For distinct $i, j, k \in [n]$, the piece $S^j_{i,k}$ of C_j between $C_i \cap C_j$ and $C_k \cap C_j$ is called a *segment* of C_j. If $\{i, j, k\}$ is a non-inversion triple, $i < j < k$, then the *floor* of $\{i, j, k\}$ is the segment $S^j_{i,k}$. The *floor of a block* $\mathcal{I}$ is the union of the floors of elements of $\mathcal{I}$. A floor is *elementary* if its interior is not intersected by any other wire. The *height* of a non-inversion triple $\{i, j, k\}$, denoted $\mathrm{ht}(\{i, j, k\})$, is the number of wires that pass below the segment $S^j_{i,k}$. If a block $\mathcal{I}$ has an elementary floor, then all of its elements have the same height, which we denote $\mathrm{ht}(\mathcal{I})$.

Proposition 4.1 ([7])**.** Let W be a simple wiring diagram with inversion set X. Let $Y \in B(n, 2)$ such that $X \subsetneq Y$.

(1) There exists an element of $Y - X$ with an elementary floor in W. [7, Lemma 2.2]
(2) Among those elements of $Y - X$ with an elementary floor, if I is of maximum height, then $X \cup \{I\}$ is consistent. In particular, $\mathrm{Asc}(X, Y)$ is nonempty. [7, Lemma 2.3]

Corollary 4.2. The second higher Bruhat order $B(n, 2)$ is ordered by inclusion; that is, $B(n, 2) = B_{\subseteq}(n, 2)$ as posets.

Given $\mathcal{I} \subseteq \binom{[n]}{d+1}$, let $[\mathcal{I}]$ denote the union $\bigcup_{I \in \mathcal{I}} I$.

Lemma 4.3. Let $X \in B(n, 2)$ have wiring diagram W. If $\mathcal{I}$ is a block with an elementary floor in W, then $X \cup \overline{\mathcal{I}}$ is not consistent if and only if there exists a wire p intersecting the segments $S^{i_0}_{i_1, i_m}$ and $S^{i_m}_{i_0, i_{m-1}}$ where $[\mathcal{I}] = \{i_0 < \cdots < i_m\}$.

Proof. Let $\mathcal{I}$ be a block with an elementary floor in W, and let $p \in [n] - [\mathcal{I}]$. If $X \cup \overline{\mathcal{I}}$ is consistent, then for $i \in [\mathcal{I}]$ the words π_i have the elements of $[\mathcal{I}] \setminus i$ flipped with the other letters in the same relative order. Hence, $X \cup \overline{\mathcal{I}}$ is consistent if and only if every wire in $[n] - [\mathcal{I}]$ does not intersect any segment $S^k_{i,j}$ for $i, j, k \in [\mathcal{I}]$.

If $X \cup \overline{\mathcal{I}}$ is not consistent, then there exists a wire p intersecting some segment $S^k_{i,j}$ with $i, j, k \in [\mathcal{I}]$. As $\mathcal{I}$ has an elementary floor in W, p must intersect the segments $S^{i_0}_{i_1, i_m}$ and $S^{i_m}_{i_0, i_{m-1}}$ by planarity. □

To determine the homotopy type of intervals of $B(n, 2)$, we use a stronger version of Proposition 4.1(2).

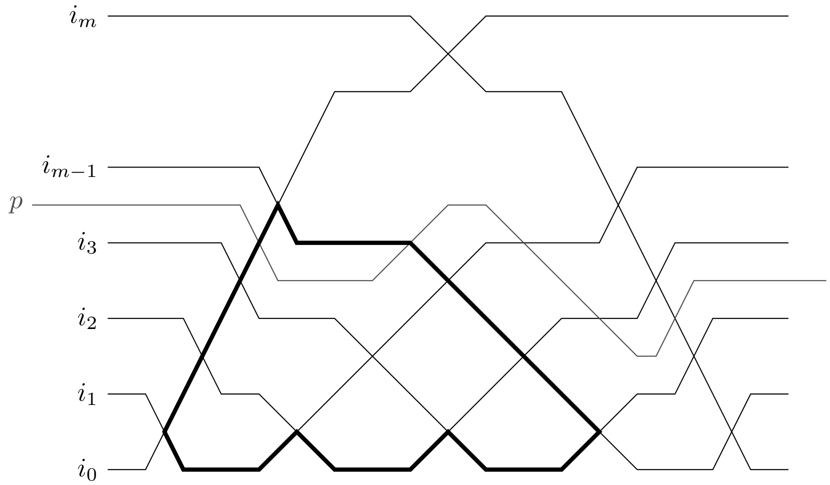

FIGURE 3. Wire p intersects $S^{i_0}_{i_1,i_{m-1}}$ so $X \cup \overline{\mathcal{I} \setminus \{i_{m-2}, i_{m-1}, i_m\}}$ is not consistent.

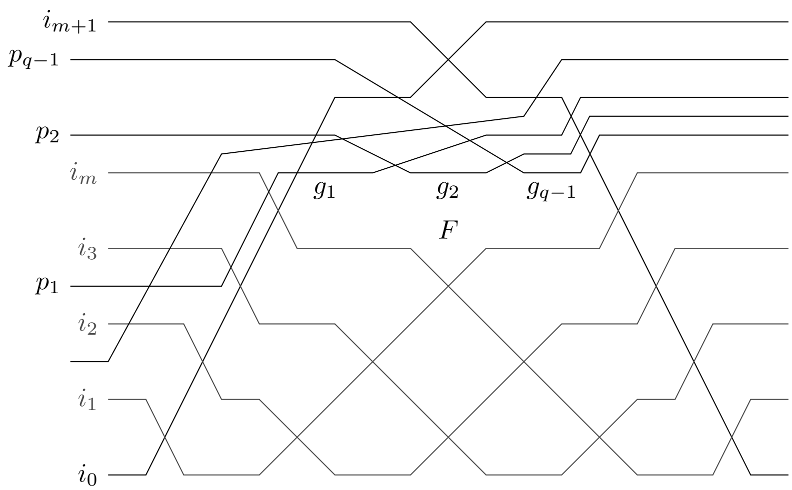

FIGURE 4. The region F, edges $g_1, \ldots, g_{q-1}$, and supporting wires $p_1, \ldots, p_{q-1}$ in the proof of Proposition 4.4.

Proposition 4.4. Let W be a simple wiring diagram with inversion set X. Let $Y \in B(n,2)$ such that $X \subseteq Y$. Among the blocks in $Y - X$ with an elementary floor, if $\mathcal{I}$ is of maximum height, then $X \cup \overline{\mathcal{I}}$ is consistent.

Proof. Let $\mathcal{I}$ be a difference block of $Y - X$ of maximum height with an elementary floor, and assume $X \cup \overline{\mathcal{I}}$ is not consistent. Replacing $\mathcal{I}$ by a smaller block, we may assume that $X \cup \overline{\mathcal{I}'}$ is consistent for every block $\mathcal{I}'$ that is a proper subset of $\mathcal{I}$. Let $[\mathcal{I}] = \{i_0, \ldots, i_m\}$ where $i_0 < \cdots < i_m$. By Lemma 4.3, there exists a wire p intersecting the segments $S^{i_0}_{i_1,i_m}$ and $S^{i_m}_{i_0,i_{m-1}}$. By the minimality of $\mathcal{I}$, every such wire intersects the subsegments $S^{i_0}_{i_{m-1},i_m}$ and $S^{i_m}_{i_0,i_1}$; see Figure 3.

Let P be the set of wires intersecting $S^{i_0}_{i_{m-1},i_m}$ and $S^{i_m}_{i_0,i_1}$. If $p \in P$, we claim that $i_0 < p < i_m$ and $\{i_0, p, i_m\}$ is a difference triple in $Y - X$. This follows by restriction of W to the wires $\{i_0, p, i_1, i_m\}$.

Let F denote the region above the wires $i_1, \ldots, i_{m-1}$, below i_0, i_{m+1} and below all of the wires in P. As shown in Figure 4, we label the upper edges of F by $g_0, g_1, \ldots, g_q$ which are supported by the wires $i_0 = p_0 < p_1 < \cdots < p_{q-1} < p_q = i_m$.

We show by induction that one of the g_j, $j \in [q-1]$, is the floor of a difference triple in $Y - X$. We are given that $\{i_0, p_j, i_m\}$ is in Y. Suppose $\{p_{j-1}, p_j, i_m\} \in Y$. Considering the quadruple $\{p_{j-1}, p_j, p_{j+1}, i_{m+1}\}$ either $\{p_{j-1}, p_j, p_{j+1}\} \in Y$ or $\{p_j, p_{j+1}, i_m\} \in Y$. The former case has an elementary floor g_j. Induction on j completes the argument.

Hence there exists a difference triple $\{p_{j-1}, p_j, p_{j+1}\}$ with an elementary floor. But this triple has height strictly greater than that of $\mathcal{I}$, a contradiction. $\square$

Lemma 4.5. Let $X \in B(n,2)$ and $\mathrm{Asc}(X) = A_1 \sqcup \cdots \sqcup A_N$ as in Lemma 3.1. Suppose $|[A_s] \cap [A_t]| \geq 2$ for some $s \neq t$ and assume $\mathrm{ht}(A_s) \leq \mathrm{ht}(A_t)$. Then

$$[A_s] \cap [A_t] = \{\min[A_t], \max[A_t]\}.$$

Consequently, $X \cup \overline{A_s}$ is not consistent.

Proof. Let s,t be distinct indices with $|[A_s] \cap [A_t]| \geq 2$ and $\mathrm{ht}(A_s) \leq \mathrm{ht}(A_t)$. Let $i,k \in [A_s] \cap [A_t]$ such that $i < k$. We let W denote a wiring diagram of X.

We first show that $\mathrm{ht}(A_s) < \mathrm{ht}(A_t)$. By Lemma 3.1(2) there exists $q \in [A_t] - [A_s]$ such that $i < q < k$. Let $[i,k] \cap [A_s] = \{i = j_0 < j_1 < \cdots < j_r < j_{r+1} = k\}$ and let e_α be the floor of $\{j_{\alpha-1}, j_\alpha, j_{\alpha+1}\}$ for $1 \leq \alpha \leq r$. Since $\{j_{\alpha-1}, j_\alpha, j_{\alpha+1}\}$ is an ascent, each e_α is a segment. Let $e = \bigcup_\alpha e_\alpha$ be the union of these segments. Then q does not intersect e_α. If q is above e_α, then $\mathrm{ht}(A_s) < \mathrm{ht}(\{i,q,k\}) = \mathrm{ht}(A_t)$ as desired. If q is below e_α, then $\mathrm{ht}(A_s) > \mathrm{ht}(\{i,q,l\}) = \mathrm{ht}(A_t)$, contrary to the hypothesis.

Let $p \in [A_t] - \{i,k\}$. It remains to show that $i < p < k$. From this, it follows that $[A_s] \cap [A_t]$ must intersect only at the two elements which lie at opposite ends of $[A_t]$.

Suppose to the contrary that $p < i$. Since $\{p,i,l\} \notin X$, $\pi_p(i) < \pi_p(l)$. If $\min[A_s] < i$ then the floor of $\{p,i,l\}$ includes the floor of an ascent I in A_s. By assumption on the height of A_t, this implies $I \in A_t$, a contradiction. If $\min[A_s] = i$ then the floor of $\{p,i,l\}$ includes the crossing $i \cap j$ where $j = \min([A_s] \setminus i)$. Consequently, $\mathrm{ht}(A_t) \leq \mathrm{ht}(A_s)$, a contradiction.

A symmetric argument shows that $p \not> l$, thus completing the proof. $\square$

The following proposition is the key to the proof of Theorem 1.1, as described in the introduction.

Proposition 4.6. Let W be a simple wiring diagram with inversion set X. Let $Y \in B(n,2)$ such that $X \subsetneq Y$, and let $\mathcal{I} \subseteq \mathrm{Asc}(X,Y)$ such that $X \cup \overline{\mathcal{I}}$ is consistent. If $I_0 \in \mathrm{Asc}(X,Y)$ is of maximum height in W, then $X \cup \overline{\mathcal{I} \cup \{I_0\}}$ is consistent.

Proof. We may assume that $I_0 \notin \mathcal{I}$, as the result is otherwise immediate.

From Lemma 4.5, we know that $\mathcal{I}$ uniquely decomposes as a union of blocks $\mathcal{I}_1, \ldots, \mathcal{I}_m$ such that $|[\mathcal{I}_s] \cap [\mathcal{I}_t]| \leq 1$ for all $s \neq t$. By this decomposition, if $s \neq t$, then no packet contains both a subset of $[\mathcal{I}_s]$ and of $[\mathcal{I}_t]$. Hence $X \cup \overline{\mathcal{I}_s}$ is consistent for all s.

Lemma 3.1 implies that $|I_0 \cap [\mathcal{I}_s]| = 2$ for 0, 1, or 2 blocks $\mathcal{I}_s$. We consider each of these cases in turn.

If $|I_0 \cap [\mathcal{I}_s]| \leq 1$ for all s, then $X \cup \overline{\mathcal{I} \cup \{I_0\}} = X \cup \overline{\mathcal{I}} \cup \overline{\{I_0\}}$ is consistent.

Suppose $|I_0 \cap [\mathcal{I}_s]| = 2$ for exactly one block $\mathcal{I}_s$. Then $X \cup \overline{\mathcal{I}_s \cup \{I_0\}}$ is consistent by Proposition 4.4. By Lemma 4.5, we deduce that $|[\mathcal{I}_s \cup \{I_0\}] \cap [\mathcal{I}_t]| \leq 1$ for $t \neq s$, so $X \cup \overline{\mathcal{I} \cup \{I_0\}}$ is consistent in this case.

Finally, assume that $|I_0 \cap [\mathcal{I}_s]| = 2$ and $|I_0 \cap [\mathcal{I}_t]| = 2$ for two blocks $\mathcal{I}_s, \mathcal{I}_t$. Then $X \cup \overline{\mathcal{I}_s \cup \{I_0\} \cup \mathcal{I}_t}$ is consistent by Proposition 4.4. Lemma 4.5 implies that $|[\mathcal{I}_s \cup \{I_0\} \cup \mathcal{I}_t] \cap [\mathcal{I}_u]| \leq 1$ if $u \neq s$ and $u \neq t$. Therefore, $X \cup \overline{\mathcal{I} \cup \{I_0\}}$ is consistent.

$\square$

5. Proof of Theorem 1.1

Let V, W be wiring diagrams such that W is simple and V is not simple. We say W is *incident* to V if V may be obtained by moving the wires of W to a more special position. More precisely, W is incident to V if the associated oriented matroid of V is a weak map image of the oriented matroid associated to W; see [3, Section 7.7] for background on weak maps. An interval (X,Y) of $B(n,2)$ is called *facial* if the closed interval $[X,Y]$ is the set of inversion sets of simple wiring diagrams incident to some fixed wiring diagram. For example, the wiring diagram obtained by stretching the curves in Figure 5 has two non-simple crossings that may be resolved in 16 ways, one of which is shown in Figure 2.

The following lemma follows from the proof of Lemma 4.5.

Lemma 5.1. Let $X, Y \in B(n,2)$ such that $X < Y$. Then (X,Y) is facial if and only if $Y = X \cup \overline{\mathrm{Asc}(X,Y)}$ and $X \cup \overline{\mathcal{I}} \in B(n,2)$ for $\mathcal{I} \subseteq \mathrm{Asc}(X,Y)$.

We now prove the main theorem of this paper, restated in a more detailed form.

Theorem 5.2. Let $X, Y \in B(n,2)$ such that $X < Y$. If (X,Y) is facial, then it is homotopy equivalent to a sphere of dimension $(|\mathrm{Asc}(X,Y)| - 2)$. Otherwise, (X,Y) is contractible.

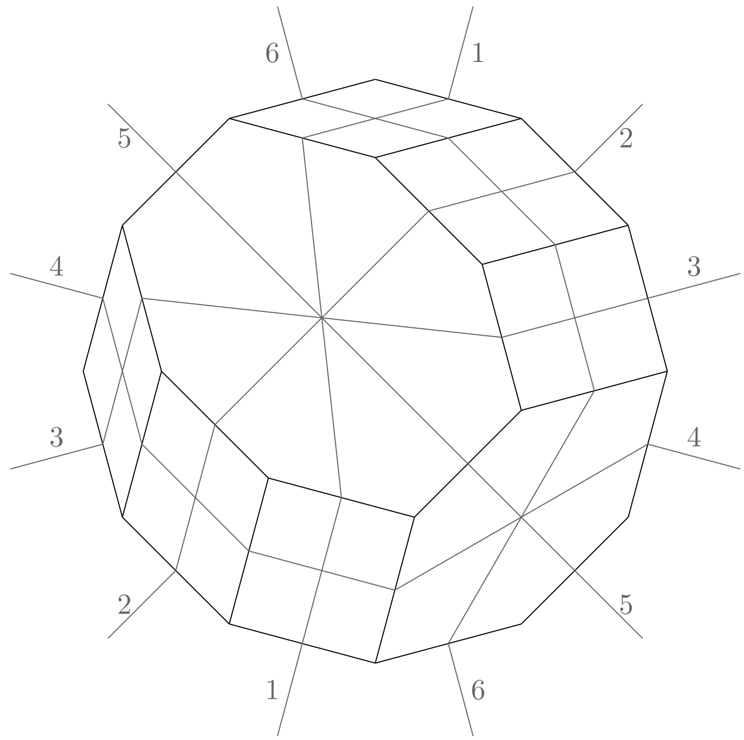

Figure 5. A non-simple pseudoline arrangement with corresponding zonogonal tiling.

Proof. Assume the statement holds for intervals (X, Z) with $X \leq Z < Y$. By Lemma 2.2,

$$(X, Y) \simeq (X, Y)_{nonc}$$

where $(X, Y)_{nonc} = \{Z \in (X, Y) \mid (X, Z) \text{ is not contractible}\}$. Since $B(n, 2)$ is ordered by inclusion, $X \cup \overline{\mathcal{I}} \leq Y$ whenever $\mathcal{I}$ is a subset of $\operatorname{Asc}(X, Y)$ such that $X \cup \overline{\mathcal{I}}$ is consistent. By the inductive hypothesis,

$$(X, Y)_{nonc} = \{Z \in (X, Y) \mid (X, Z) \text{ is facial}\}.$$

Suppose (X, Y) is facial. By Lemma 5.1, $Z \in (X, Y)_{nonc}$ if and only if $Z = X \cup \overline{\mathcal{I}}$ for some non-empty proper subset $\mathcal{I}$ of $\operatorname{Asc}(X, Y)$. Hence $(X, Y)_{nonc}$ is the face poset of the boundary of a simplex. Thus, (X, Y) is homotopy equivalent to a sphere of dimension $|\operatorname{Asc}(X, Y)| - 2$.

Now assume that (X, Y) is not facial. By Lemma 5.1, $(X, Y)_{nonc}$ is the face poset of a simplicial complex over $\operatorname{Asc}(X, Y)$. By Proposition 4.6, this simplicial complex has a cone point $I_0 \in \operatorname{Asc}(X, Y)$, so $(X, Y)_{nonc}$ is contractible. □

Let us point out one consequence of Theorem 5.2. We let $\omega(n, 2)$ denote the set of all wiring diagrams on n wires up to equivalence. This forms a poset under the incidence relation. Alternatively, $\omega(n, 2)$ may be viewed as the poset of facial intervals of $B(n, 2)$, ordered by inclusion. By Lemma 2.3 and Theorem 5.2, we deduce that the proper part $\omega(n, 2)'$ is homotopy equivalent to $\mathbb{S}^{n-3}$. This corollary was proved in [15] by other means.

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