Effects of Variations in Risk on Demand and Measures of Business Cycle Comovements

– three essays about the business cycle.

by

John AA. Hassler

Filosofie Licentiat

Submitted to the Department of Economics in partial fulfillment of the requirements for the Degree of Ph. D. in Economics at the Massachusetts Institute of Technology

May 1994

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Abstract

The thesis consists of three separate papers. The first paper (chapter II) addresses the question how to measure and quantify the degree of comovement between different business cycle variables. In the second paper (chapter III) the effect of variations in risk on the demand for durables is analyzed within an irreversible investment model. The last paper (chapter IV) investigates whether there is evidence of a negative relation between risk and aggregate demand for durables in aggregate U.S. data.

In chapter II a frequency band specific measure of the degree of linear comovement is defined. The distribution of the measure is simulated using Monte Carlo methods. The sensitivity of the distributions to the characteristics of the underlying processes is shown to be substantially reduced if low frequencies are excluded from the analysis. It is argued that "business cycle facts" established using filters that include low frequencies, for example the Hodrick-Prescott filter, may be dominated by the behavior at low frequencies. Explicit treatment of different frequency bands may then be preferable. The measure of linear comovement is applied to Swedish and foreign macro time series spanning the period 1861 to 1988. A substantial degree of comovement between Swedish consumption related variables and foreign GDP is found while Swedish GDP and manufacturing production is clearly less covariant with foreign GDP.

An irreversible investment model is constructed in chapter III. It is designed to represent the situation when the consumer only can adjust her durables stock after paying some discrete transaction cost. In the model the agents know that risk, defined as the instantaneous probability of a zero mean income shock, varies stochastically. Specifically, the economy shifts stochastically between one high risk state and one low risk state. The model contrasts with earlier models in which risk is taken as a fixed parameter and comparative statics is used to find the effects of variations in of risk. To explicitly model a stochastic level of risk implies that we can analyze if the expected length of a high risk period is important for the effects on demand. It is found that a short high risk period decreases the temptation to adjust. Adjustments are thus delayed more the shorter the high risk period is expected to be.

In chapter IV I use the assumption from chapter III that the risk fluctuates stochastically between two levels. These risk levels, the transition probabilities and the relation between risk

and return are estimated with a maximum likelihood method. Thereafter the likelihood that the U.S. economy is in the high risk state is calculated for each month from 1959 and 1992. These likelihoods are used to predict the relative level of durables demand. Durables demand is found to be strongly sensitive to financial volatility. No clear relation between relative durables demand and volatility of non-durables consumption is, however, found.

Thesis Advisor: Olivier J. Blanchard
Professor of Economics

Acknowledgments

In 1985 I decided that I might need some university education. Without to much reflection I chose the closest university, which happened to be Stockholm School of Economics. The distance to Stockholm University was about 2 miles longer. I could at that time certainly not imagine that I would end up at a university several thousand miles away from Sveavägen 76.

During the 8 years since I started at Stockholm School of Economics I have accumulated a great debt. Not so much financially, but mostly to those who have supported me and persuaded me to go on. First and foremost I thank my wife Sussi – without you everything would have been much harder.

My advisors at the Institute for International Economic Studies, Torsten Persson and Lars E. O. Svensson where instrumental in my decision to go to U.S. To be honest, the decision was maybe more theirs than mine. When writing this thesis I continuously received invaluable (economists cannot value everything) advice and support from Lars and Torsten. At IIES, Nils Gottfries, Petter Lundvik, Peter Sellin, Paul Söderlind and Anders Warne also provided inspiration and important comments to ideas and drafts. IIES is certainly an excellent place to be for a graduate student and researcher.

The time at MIT was the toughest in my life. Fortunately, the working climate at MIT is cooperative and supportive. Without that I don't think I would have been able to stand the pressure. When studying, cooperation is often more efficient than cut-throat competition. At MIT I spent more time discussing economics and solving problem sets with Jose-Vicente "Sevi" Rodriguez-Mora and Jesus-Guillermo Llorente-Alvaréz than I spent reading books and articles, and I think I learnt at least as much. I owe a lot to you and all other student friends at MIT.

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Chapter I
Introduction

The business cycle, with its recurrent upswings and downswings of economic prosperity and optimism, is one the most extensively examined areas of economic research. The economics profession has nevertheless so far not been able to give a satisfactory explanation of business cycles. Somewhat embarrassingly, we have to admit that we do not even agree on their existence.

In a small and fairly open economy like Sweden's, it is not farfetched to believe that the impulses behind the business cycle to a large extent originate from abroad. This has also since long been a popular idea among political leaders. It is always convenient to blame a disturbing and apparently unsolvable problem on an anonymous foreign influence. Economists have also long held this view. "It has often been said that Sweden has neither business cycles nor price fluctuations of her own – both are imported." (Erik Lundberg, 1953).[1]

Assume that the mechanism that potentially transmits the foreign influence to the domestic economy is fairly simple and stable. We could then hope that business cycles in for example Sweden swing together with foreign business cycles. A test of the hypothesis of substantial foreign influence on the business cycle could then be performed using standard regression analysis. A high degree of comovement could be taken as a sign in favor of the foreign influence hypothesis. A problem does, however, show up immediately; business cycles are typically thought of something different from the long swings in the growth of production and productivity that also seem to exist. Some periods, a decade or longer, are more prosperous than others. If we suspect that such fluctuations are generated by a different mechanism than the business cycle we may want to study business cycle fluctuations separately from these longer swings.

In chapter II I propose a measure of comovement such that we can explicitly specify the frequencies we are interested in. I show that this measure is more well-behaved than others in the sense that its distribution is less sensitive to the characteristics of the underlying stochastic process. We often have very limited information of the underlying stochastic

[1] Lundberg, Erik, (1957), *Business Cycles and Economic Policy*, George Allen & Unvin ltd., London.

process generating fluctuations in macro variables. Such insensitivity thus makes reliable inference easier.

The chapter is concluded with an application to Swedish data. I show that the foreign influence on the Swedish business cycle appears to be much smaller than commonly appreciated.

If the foreign influence is low, what could then be of importance? Variations in technology has since a while been a standard answer. I believe, however, that this is less plausible as a main cause of business cycles. The aggregate production function, i.e., the relation between input and output, is likely to be one of the more short run stable macroeconomic relation we have in the economy.[2]

In chapter III and IV I will examine another potential source of economic fluctuations – variations in the degree of uncertainty. Since long it has been known that risk affects behavior. One line of theoretical studies going back at least to Leland (1968)[3] is concerned with how risk affects savings through the precautionary savings motive. If risk goes up we may expect people to save more to have a buffer against bad outcomes.

Already Leland considers stochastic risk by bringing up the issue of covariance between risk and income. He makes the important conjecture that if income risk is stochastic and positively correlated with income, a positive shock to income leads to an increase in precautionary savings that partly offsets the increase in consumption. Not until almost a quarter of a century later Caballero (1990)[4] is the first to prove Leland's conjecture within a model where consumers know that risk is stochastic. However reasonable this argument seems, the idea that precautionary savings vary a lot has not received much empirical support.

Variations in risk may, however, have important effects on demand also in total absence of precautionary savings. This may happen if durables can be traded only at some transaction cost. It is not difficult to understand that risk in conjunction with irreversibility severely complicates the problem of the consumer. When an individual buys a car she has to acknowledge the risk that her income may change so much that it is optimal to sell the car

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[2] I owe this formulation to Assar Lindbeck.

[3] Leland, Hayne E., (1968), "Saving and Uncertainty: The Precautionary Demand for Saving", *Quarterly Journal of Economics*, No. 3.

[4] Caballero, Ricardo J., (1990),"Consumption Puzzles and Precautionary Savings", *Journal of Monetary Economics* 25.

and buy a cheaper or more expensive one. With transaction costs this means losing money. What seemed to be optimal turned out to be sub-optimal. The difference in utility can be thought of as a value of information or a value of waiting.[5] It should be noted that the value of waiting is not due to risk aversion. Also a risk neutral investor would have to consider the degree of risk when trying to make an optimal irreversible investment.

It is typical for consumer durables (cars, dish-washers, but also for example, houses) that purchases involve irreversibility.[6] It is costly to change the consumption plan. If we have to sell a used durable, it can often only can be done at a price lower than what can be accounted for by depreciation. Often the irreversibility is due to informational imperfections, for example lemons problems or costly information gathering. Informational problems are also often the explanation for why the irreversibility problem not always is solved by leasing arrangements.

The irreversibility problem does obviously not occur when buying non-durables. We never have to resell non-durables since we may continuously change the stream of purchases to follow our consumption. Even if there exist transaction costs, the irreversibility mechanism should thus not affect non-durables demand. This makes it possible to empirically distinguish between the irreversibility mechanism and precautionary savings.

In chapter III I will construct a theoretical model where consumers have to pay a transaction cost in order to purchase a durable. They are continuously faced with a probability of receiving a wealth shock. If they do, they will want upgrade or downgrade the durable and thus have to pay the transaction cost. In contrast to earlier models, the consumers in my model know that the probability of being hit by the wealth shock sometimes is higher than at others. The value of waiting will then fluctuate and I will show that this may have substantial effects on the demand for durables. The formulation of the model allows me to study how the effects of a shift to high risk depend on the expected length of the high risk period. I will show that a short expected high risk period will increase the incentive to wait before buying

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[5] The expression "Value of Waiting" was coined, I believe, in the important paper by McDonald and Siegel (1986), "The Value of Waiting to Invest", *The Quarterly Journal of Economics*, November 1986.

[6] The term irreversibility is taken to mean that there is some cost (finite or infinite) associated with reversing an investment.

the durable more than if the high risk period was expected to be long. Within previous models this question could not be addressed.

Chapter IV concludes the thesis. The aim here is to investigate whether there is a relation between risk and durables purchases. First I estimate a model where wealth follows a random walk with a volatility that depends on a two-state Markow process. I find clear evidence for extended periods of higher volatility of both non-durables consumption and a U.S. stock index (S&P500). I show that periods of high volatility of the stock market significantly tend to coincide with periods of low demand for durables.

Chapter II*
Are They Swinging Together?

A measure of linear comovement with an application to Swedish and foreign business cycles.

1. Introduction

This chapter attempts to contribute to a descriptive research agenda, inspired by the seminal paper "Understanding Business Cycles" by Lucas (1977), but is also part of a longer tradition involving, for example, Burns and Mitchell (1946). The goal of this agenda is the establishment of empirical regularities or "stylized facts" about business cycles. Description is a necessary part of the research process. In itself it certainly does not lead us all the way to the goal of research – understanding. It may, however, help us clarify what we need to understand.

One of the most problematic pitfalls in the strive to establish business cycle facts is the sensitivity of these facts to the statistical methods used to find them. A recent illustration of this is Canova (1991). Using a set of US real macroeconomic time series, Canova shows the that linear statistics typically used to establish "stylized facts" are highly sensitive to the filtering method used. One answer to this problem has been to try to find facts that are robust to variations in the statistical methods. It seems more reasonable to try to understand why the differences occur and then choose a suitable method.

Measures of comovement are typically included as a part of business cycle facts. Typically, they involve cross-correlations at some pre-specified number of leads and lags. Sometimes coherence functions are estimated. Given assumptions about exogeneity, measures of comovement may be given a causal interpretation. In this chapter I will present and discuss the properties of a measure of linear comovement that, to my knowledge, not have been used before in economics. In addition to this measure's own value, an examination of its properties

* Torsten Persson, Lars E. O. Svensson, Nils Gottfries and Anders Warne at IIES, Sune Karlsson and Anders Vredin at the Stockholm School of Economics, and Andrew Bernard and Guillermo Llorente-Alvarez at MIT have provided extensive and helpful comments and suggestions to this chapter. I also thank seminar participants at Stockholm University, University of Helsinki and Stockholm School of Economics.

may help us understand why different measures will behave differently when applied to macro time series.

The mere concept business cycles requires some form of stationarity. A cycle is necessarily something that fluctuates around a mean. This mean is obviously changing over time for many macro variables. The more irregularly it is changing and the higher its variance, the less relevant is the concept of business cycles. In the past it has been common to assume that the mean displays the least irregular change possible, namely a time trend. A newer variant of essentially the same idea is the Hodrick-Prescott filter[1] (called HP-filter below), where the trend is allowed to have a data dependent degree of curvature. Neither of these assumptions will be used in this chapter.

My operational definition of business cycles will be fluctuations with a period of three to eight years.[2] At the end of this chapter I will examine a data set with a time span of around 128 years, which is a long sample by economic standards. The number of business cycles in the sample is, nevertheless, only between 128/8=16 and 128/3≅43. I will thus have to make a strong stability assumption to use this sample in order to describe fine details of the mechanism that generates business cycles.

An important and early contribution to the idea of distinguishing between linear dependence on different frequencies for economic variables is Engle (1974). He shows how to estimate linear regressions on different frequencies. In effect he is using a band pass filter and then estimates a regression on the filtered data. A problem with his method is that a process with zero spectral density over a frequency range with positive measure is linearly deterministic. Standard asymptotic inference can thus not be applied.[3]

More recent studies where frequency domain statistics are used are Englund, Persson and Svensson (1992), Hassler, Lundvik, Persson and Söderlind (1992) and Stock and Watson (1990). They all examine macro time series using statistics in both the time and the frequency domain. The preferred filters in these papers are variants of band pass filters.

I believe it is fair to say that spectral methods involving the explicit treatment of differ-

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[1] Outside the field of economics this filter is known as the Whittaker-Henderson Type A filter (Hodrick and Prescott 1980).

[2] This range of periods has been used by, for example Englund et al (1990), Hassler et al (1992) and is also approximately in line with the NBER definition of business cycles as fluctuations with a period of 3-5 years.

[3] See e.g. Whittle (1983) p. 26 for a formal proposition.

ent frequencies have not been widely used. Within the real business cycle tradition Hodrick-Prescott filtering has instead become popular. At least since Nelson and Kang (1981) it has been known that inappropriate detrending can cause "spurious periodicity" in the examined time series. Later Harvey and Jaeger (1991) and Cogley and Nason (1992) show that this phenomenon may arise if Hodrick-Prescott filters are used on random walks.

In this chapter I show that frequency specific measures of linear comovements should be used in the attempt to establish facts about the business cycle. The proposed measure is easily interpreted in the time domain as a decomposition of the standard regression R^2. It does not, however, require any assumptions about the structure of the linear dependence between the processes. I will show that the distribution of this measure is likely to be less sensitive than other measures to the characteristics of the underlying data generating process. This means that the consequence of inappropriate detrending is less severe. I also show that without explicit treatment of different frequency bands, the behavior at low frequencies is likely to dominate time domain statistics of macroeconomic time series.

In Section 2 I define and simulate the distribution of the proposed frequency specific measure of linear comovement. Section 3 discusses the question of how to make the apparently non-stationary time series we deal with in economics stationary. Section 4 uses the proposed measure on a Swedish data set and Section 5 concludes the chapter.

2. Frequency Domain Comovement Measures

When studying the linear relationship between two processes we are typically assuming a model of the form

$$\begin{aligned} \mathbf{Y}_t &= \sum_{s=-\infty}^{s=+\infty} \mathbf{X}_{t+s}\beta_s + \varepsilon_t \quad x_t \perp \varepsilon_s, \forall s,t \\ &= \mathbf{B}(L)\mathbf{X}_t + \varepsilon_t \end{aligned} \tag{2.1}$$

where $\mathbf{B}(L)$ denotes a potentially double-sided and infinite lag-polynomial. Both $\mathbf{Y}_t$ and $\mathbf{X}_t$ may be multidimensional although below I will consider the LHS variable to be a scalar. Without any knowledge or assumptions about the structure of $\mathbf{B}(L)$, the task of estimating the parameters of this model is formidable. Assume we are only interested in the *degree* of linear comovement between **Y** and **X**. This could be the case if want to examine to what extent one process influences another, for example how much the Swedish business cycle covaries with a foreign counterpart.[4] In this case spectral analysis offers a method of estimating this without making assumptions about $\mathbf{B}(L)$. This contrasts with the common procedure in business cycle studies where cross-correlations at pre-specified leads and lags are reported and interpreted as measures of comovement.

To simplify, consider first the case where both **Y** and **X** are univariate. Following Priestley (1986), the spectral densities of Y and X, denoted $h_y(\omega)$ and $h_x(\omega)$, are related by

$$h_y(\omega) = \left|\mathrm{B}(e^{-i\omega})\right|^2 h_x(\omega) + h_\varepsilon(\omega). \tag{2.2}$$

We also have

$$\mathrm{B}(e^{-i\omega}) = \frac{h_{yx}(\omega)}{h_x(\omega)}, \tag{2.3}$$

where $h_{yx}(\omega)$ denotes the cross spectral density between Y and X, and

$$\sigma_y^2 = \int_{-\pi}^{\pi} h_y(\omega)d\omega \qquad \forall y. \tag{2.4}$$

Using (2.1)-(2.4) we have

[4] Given that we find significant degrees of covariation we may want to estimate the structure of $\mathbf{B}(L)$.

$$\begin{aligned}
\sigma_{\varepsilon}^{2} &= \int_{-\pi}^{\pi} h_{\varepsilon}(\omega)d\omega \\
&= \int_{-\pi}^{\pi}\left[h_{y}(\omega) - \left|\mathrm{B}(e^{-i\omega})\right|^{2} h_{x}(\omega)\right]d\omega \\
&= \int_{-\pi}^{\pi}\left[1 - \frac{\left|h_{yx}(\omega)\right|^{2}}{h_{x}(\omega)h_{y}(\omega)}\right]h_{y}(\omega)d\omega \\
&= \int_{-\pi}^{\pi}\left[1 - \left|w_{yx}(\omega)\right|^{2}\right]h_{y}(\omega)d\omega
\end{aligned} \tag{2.5}$$

where $\left|w_{yx}(\omega)\right|^{2}$ denotes the squared coherence between X and Y.

From (2.4) we can easily get an expression for the standard measure of linear relationship – R^2 of regression (2.1).

$$\begin{aligned}
R^{2} &= 1 - \frac{\sigma_{\varepsilon}^{2}}{\sigma_{y}^{2}} \\
&= 1 - \frac{\int_{-\pi}^{\pi}\left[1 - \left|w_{yx}(\omega)\right|^{2}\right]h_{y}(\omega)d\omega}{\sigma_{y}^{2}} \\
&= \frac{\int_{-\pi}^{\pi}\left|w_{yx}(\omega)\right|^{2} h_{yy}(\omega)d\omega}{\sigma_{y}^{2}}
\end{aligned} \tag{2.6}$$

Note that the interpretation of this R^2 is the amount of the variation in Y that can be accounted for in a regression on X *given that we are using the correct model as defined in equation* (2.1). Correct means that we are including all (possibly infinitely many) positive and negative lags of X that have a true regression coefficient different from zero.

The usefulness of (2.6) is that we may use it to decompose R^2 into frequency specific components. Assume that we are interested in the linear relationship at some frequency set Ω. Let R_{Ω}^{2} denote the degree of linear covariation at the frequency set of interest. This is then defined as

$$R_{\Omega}^{2} = \frac{\int_{\omega\in\Omega} \left|w_{yx}(\omega)\right|^{2} h_{y}(\omega)d\omega}{\int_{\omega\in\Omega} h_{y}(\omega)d\omega}. \tag{2.7}$$

Also a multivariate regression R^2 may be decomposed. In this case we only replace the univariate coherence $w_{yx}(\omega)$ with its multivariate counterpart $w_{y.x_1,\ldots,x_q}(\omega)$ and then use (2.6) or (2.7). The multivariate coherence is defined as

$$w^{2}{}_{y.x_1,\ldots,x_q}(\omega) = \frac{\mathbf{h}_{yx}(\omega)\mathbf{h}_{xx}(\omega)^{-1}\mathbf{h}_{yx}(\omega)^{*}}{h_{y}(\omega)} \tag{2.8}$$

where * denotes the complex conjugate of the transpose. Note here the close analogy to the corresponding time domain measures. Replacing the spectral density vectors and matrices with covariance vectors and matrices we get the regression R^2.

An important feature of coherence is its invariance under linear transformations. The coherence at frequency ω measures the correlation between spectral components of frequency ω in the spectral representation of different processes. Since linear transformation will affect these components proportionally, it is intuitive that this correlation measure is unaffected.[5] If a linear transformation cancels a frequency, the invariance result is, however, not true for that frequency. The coherence at that frequency is analogous to the correlation between two zero variance processes and is not well defined. If for example, a first difference filter is applied to some stationary series, the zero frequency is canceled. The coherence at frequency zero is thus not invariant to the first difference filter. The coherences at all other frequencies are, however, unaffected.

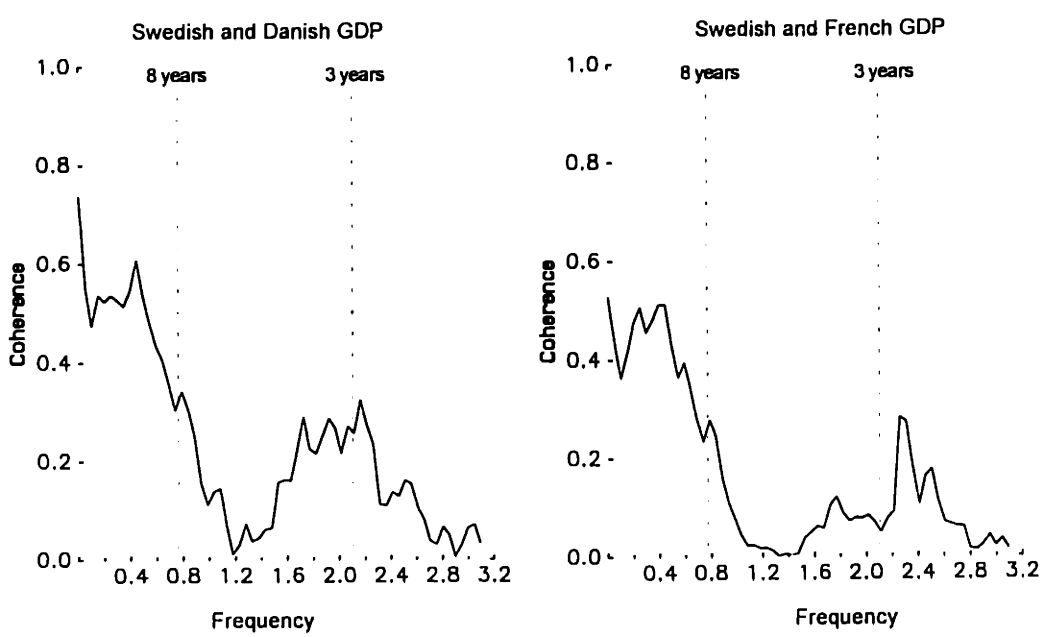
It will be useful to note from (2.7) that R_{Ω}^{2} is a weighted average of the coherences where the weights are the spectral densities $h_y(\omega)$. Since the latter clearly are affected by linear transformations, R_{Ω}^{2} will not be invariant. The procedure of first filtering two processes with a common linear filter and then studying the degree of linear relationship between the filtered processes is equivalent to manipulating the weights in (2.7). By weighing the coherences differently we may vary the degree of linear comovement, often within wide ranges. For any $\alpha \in [0,1]$ and $\Omega \in [-\pi,\pi]$, if the coherence in the interval Ω takes the value α

[5] See Priestley (1986), page 661, for a proof.

we can get an R^2_Ω arbitrarily close to α by appropriately choosing a weighing function (or equivalently; the right linear filter).[6]

The following illustrates the point that we may manipulate the degree of linear co-movement. Take the time series for (the differenced logarithm of) Swedish and French GDP from 1870 to 1988. The estimated coherence functions are depicted in Figure 1. The frequencies corresponding to periods between 3 and 8 years are indicated by vertical dotted lines. We see that in both cases there is a low frequency region where the coherence is fairly high. In the frequency range between 1.2 and 1.6, on the other hand, the coherence is very low. We may thus get very different degrees of linear dependence if we apply linear filters that pick out the low frequencies and the frequencies in the range (1.2, 1.6), respectively.

Figure 1 Coherence Functions

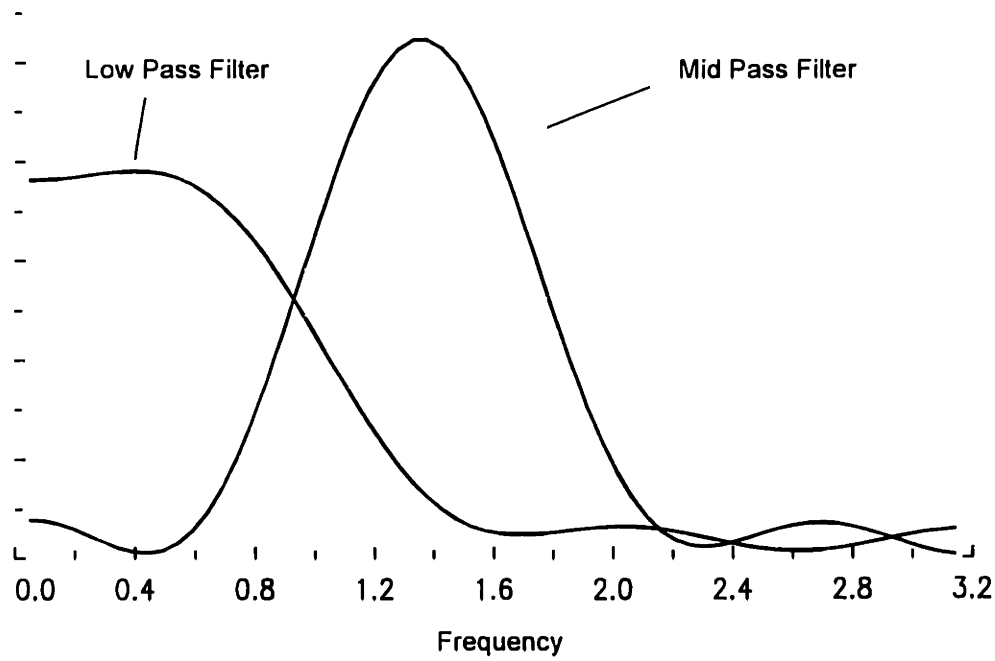


Filters with very peaked transfer functions typically have long lag structures. But also with rather short lag structures we may get sufficiently peaked transfer functions to illustrate the point. In Figure 2, I have graphed the transfer functions of two quite simple filters that are given by

[6] Here we also get an intuitive explanation for the fact that a spectrum with zero density over a range of frequencies with positive measure is equivalent to the process being deterministic. In this case we may apply a linear filter such that the process has exactly zero variance.

$$y_t^f = y_t + .9y_{t-1} + .6y_{t-2} + .2y_{t-3} - .2y_{t-5} \qquad \text{Low pass filter}$$
$$y_t^f = y_t + .3y_{t-1} - .8y_{t-2} - .6y_{t-3} + .3y_{t-4} + .6y_{t-5} \qquad \text{Mid pass filter.} \tag{2.9}$$

Figure 2 Transfer Functions of Two Simple Filters



The low pass filter picks out low frequency fluctuations while the mid pass picks out frequencies in the mid range. Another filter that may tend to pick out low frequencies is the Hodrick-Prescott filter. As is shown in Cogley and Nason (1992), the spectrum of a random walk detrended with a Hodrick-Prescott filter has a pronounced peak in the low frequency region.[7] The exact location of the peak and its relative height depends on the smoothing coefficient, higher λ implies higher peaks at lower frequencies. With $\lambda = 1600$ and yearly data, the peak is at periods of 30 years. Figure 3 shows the theoretical spectrum of a random walk detrended with a HP-filter with $\lambda = 1600$, 40 and 1 respectively.[8] Note also that if we let λ go to infinity the HP-filter becomes identical to removing a (log)linear time trend, implying (heuristically) that the latter will give us a process with an extreme peak in the low frequency region.

[7] For enough persistent series that are not random walks, we also get a peak in the low frequency region, although it is less pronounced than for the random walk.

[8] Or, equivalently, the transfer function of the filter $\{(1-L)^{-1}$ (HP-filter)$\}$. The transfer functions of the HP-filter are computed as the Fast Fourier transform of the filter weights of the 64th observation in a series of 128. In the figures the spectra are normalized to the interval [0,1].

Figure 3 Spectra of Random Walks Detrended with HP-filters

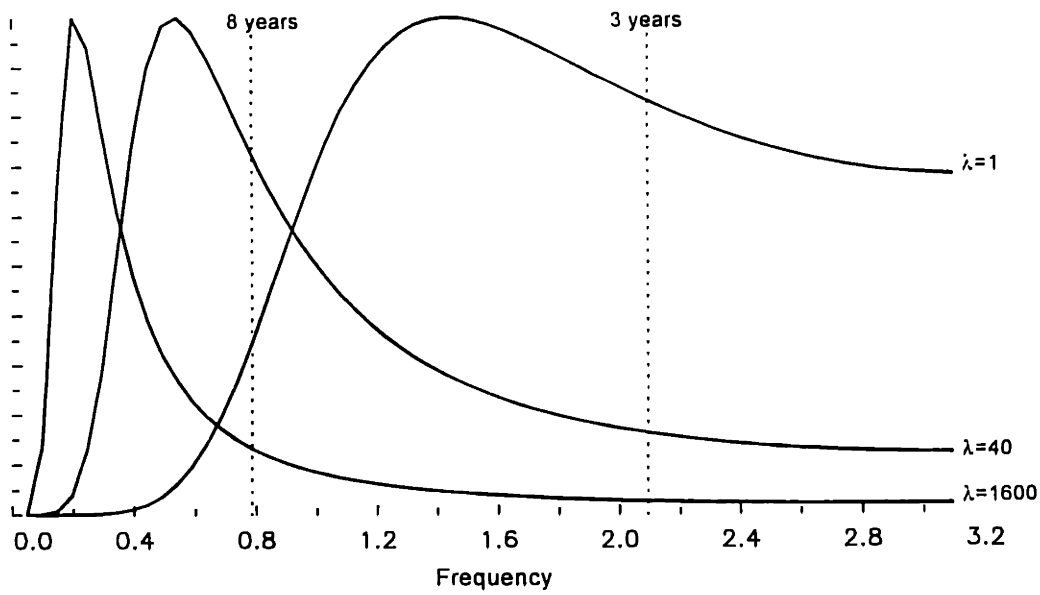


Table 1 Manipulated Linear Comovements

	Sweden and Denmark		Sweden and France	
	R^2	Corr.†	R^2	Corr.†
HP-Filter λ = 1600	0.51	0.64	0.42	0.64
Low Pass Filter	0.44	0.59	0.36	0.54
HP-Filter λ = 40	0.33	0.48	0.28	0.43
Mid Pass Filter	0.18	0.30	0.08	0.14
HP-Filter λ = 1	0.17	0.31	0.08	0.03

† Contemporaneous Correlation

Using these filters on Swedish, French and Danish GDP we would suspect that the low pass filter and the HP-filters with high λ's would pick out the high coherence ranges and thus give us high degrees of linear comovement. The mid pass filter and the HP-filters with low λ 's would, on the other hand, give low degrees of comovements. This conjecture is confirmed in Table 1. There we see that the degree of linear comovement, as measured by R^2_Ω, where Ω is taken to be all frequencies, is much higher for the low pass filter than for the mid pass filter.[9] Furthermore, the high λ filter behaves as a low pass filter while the converse is true for low λ's. In this case the structural relations between the processes seem to be quite simple,

[9] The series are first differenced before the low pass and the mid pass filters are applied.

we can thus see the same thing by looking only at the contemporaneous correlations. Attempts to establish stylized facts about the business cycle that are robust to different filters is here likely to fail. If we are interested in describing business cycles, we must use methods that describe the time series' statistical properties *at business cycle frequencies*.

2.1. Estimation

A problem arises in spectral analysis since we have to estimate the *functions* $w_{y,x}(\omega)$ and $h_y(\omega)$ where the number of different ω's in general is infinite. However, since most of the processes economists are interested in have continuous spectra[10], we may approximate the integral in (2.7) with a finite sum.

Let the ω's be evenly spaced[11] on the interval Ω. We can then estimate R^2_Ω by

$$\hat{R}^2_\Omega = \frac{\sum_{\omega\in\Omega} \left|\hat{w}_{y,x}(\omega)\right|^2 \hat{h}_y(\omega)}{\sum_{\omega\in\Omega} \hat{h}_y(\omega)} \tag{2.10}$$

where ^ indicate estimates.

The procedure of estimating the different spectral functions involves choosing a window. When estimating the coherence, the window is necessary even to get an estimate. The spectral components of a process are random variables that take some values for a particular realization of the whole process. The coherence at frequency ω is the correlation between these components of different processes at frequency ω in *repeated* realizations. In time series analysis we only have one realization of the process and thus also only one realization of the spectral component at each frequency. It is clearly impossible to estimate a correlation from just one observation. To circumvent this problem we may use the continuity of the spectral densities in conjunction with the (asymptotic) inter-frequency independence. This allows us to approximate repeated realizations at one frequency by observations at neighboring frequencies for one realization.

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[10] For example all ARMA processes have continuos spectra. Harmonic processes (finite sums of sine waves where each frequency contributes a strictly positive amount to the variance), on the other hand, have discrete spectra.

[11] If the ω's are unevenly spaced we use a weighted average where the weights are the relative lengths between the midpoints between the ω's.

The value of the spectral function at frequency ω is thus estimated as some average of neighboring "raw" values of the periodogram.[12] An unweighted average corresponds to a rectangular window, which is moved over the whole range of frequencies. Choosing a too wide window clearly leads to difficulties in detecting narrow peaks or valleys in the spectrum, while a too narrow window gives a high variance of the estimates. In this chapter I have used variants of a rectangular spectral window (also called Daniell window).

Standard spectral functions are only defined for stationary processes. In macroeconomics it is often clear that the studied time series may be non-stationary. Polynomial time trends can be removed and random walks can be made stationary by differencing. The problem is, however, that we are often uncertain as to the correct number of differencing to do. If we are studying coherences at strictly positive frequencies, over-differencing (differencing an I(k) process more than k times) is not necessarily a serious problem. This because the difference filter is linear and the coherence, as noted above, is invariant to differencing. Under-differencing, i.e. the application of spectral methods to non-stationary series, on the other hand, is certainly dubious.

For independent processes $w_{yx}(\omega)$ is zero for all ω so R^2_Ω, being a weighted average of $w_{yx}(\omega)$, is clearly zero for all Ω. The weights, i.e. the spectral densities, will of course be affected by the filter. Less equal weights may increase the variance of $\hat{R}^2_\Omega$ between independent processes. This because the $\hat{w}_{yx}(\omega)$'s are asymptotically iid for independent processes. Differencing increase the weights of high frequencies and reduce low frequency weights. Over-differencing could thus lead to reduced power in tests of zero $\hat{R}^2_\Omega$ if the differencing makes the spectra more peaked and thus the weights less equal. If, on the other hand, we have a persistent but stationary process, differencing may make the spectra flatter. In this case differencing may potentially reduces the variance of $\hat{R}^2_\Omega$. This conjecture is supported in simulations below.

In the estimation of $\mathbf{h}_{yx}(\omega)$ and $\mathbf{h}_{xx}(\omega)$ I have used bivariate methods to estimate each element separately. In all cases I have used the Fast Fourier Transform and written the programs in GAUSS.

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[12] This implies a somewhat disturbing similarity between the estimated coherence and $\hat{R}^2_\Omega$ – both are averages of neighboring cross spectral densities. A difference is, however, that $\hat{R}^2_\Omega$ has stochastic weights while the coherence weights are given by the window.

2.2. Distribution of bivariate $\hat{R}^2_\Omega$

It should be possible to calculate at least an asymptotic distribution of $\hat{R}^2_\Omega$. I have, however, chosen to simulate the distribution using Monte Carlo draws. The purpose of this chapter is to propose a measure of comovement between relatively short[13] time series of macro variables. The asymptotic distributions may thus not be very helpful for inference.

The distribution of the estimates in small samples depends on the structure of the underlying processes. I have thus performed simulations using different processes. I have included processes with most of their spectral power in the low frequency region, which is typical for macro time series. I will define business cycle frequencies as those corresponding to periods between 3 and 8 years.

Using GAUSS I have constructed pairs of series of 128 independent (pseudo) random variables for which I have calculated $\hat{R}^2_\Omega$. I let the frequency band Ω be all frequencies, business cycle frequencies and frequencies corresponding to periods over eight years. The results are presented in Figure 1.1-6 and in Table 2. The former show relative sample frequencies in the categories $\hat{R}^2_\Omega = \{[0,0.01),[0.01,0.02), \ldots,[0.99,1.00]\}$ from 25 000 draws of each process. In Table 2 I report the mean and the 90'th, 95'th and 99'th percentiles from 10 000 draws.

For two processes, white noise and AR(1), I have compensated for differencing when calculating $\hat{R}^2_\Omega$. Assume that the output from a process is differenced and used to estimate a spectrum. To get an estimate of the spectrum of the original, undifferenced process we can multiply the spectrum of the differenced process with the inverse of the transfer function for the first difference filter. Using obvious notation

$$\begin{aligned} h_y(\omega) &= \left|(1-e^{-i\omega})\right|^{-2} h_{\Delta y}(\omega) \quad \omega \neq 0 \\ &= 2(1-\cos(\omega))^{-1} h_{\Delta y}(\omega). \end{aligned} \tag{2.11}$$

When I compensate for differencing I thus first estimate the spectrum. Then I multiply it with $2(1-\cos(\omega))^{-1}$ before using it as the weighing function in (2.10). Since $2(1-\cos(\omega))^{-1}$ is undefined for $\omega=0$, the zero frequency is excluded from the calculations.

For all results reported in Figure 4.1-6 and Table 2 I have used a rectangular (Daniell)

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[13] In what appears to be a typical application Priestley gives an example with 17702 observations (Priestley, 1986, page 544).

window including 15 frequencies. Figure 4 and Table 2 show that the properties of the data generating process strongly influence the distribution of $\hat{R}^2_\Omega$. We see that for all Ω the distribution of $\hat{R}^2_\Omega$ is narrowest for the uncompensated white noise (Panel 4.1) and widest for the random walk (Panel 4.3). As discussed above it is intuitive that the more equal the weights $\hat{h}_y(\omega)$ in (2.10), the less wide is the distribution. The flat spectrum of a white noise process generates equal weights.[14] The finding that the white noise process generates the least dispersed distribution of $\hat{R}^2_\Omega$ is thus not surprising.

Table 2 Empirical Distribution of Bivariate

	All frequencies				Business Cycles				Long Periods			
128 observations		Percentiles				Percentiles				Percentiles		
	mean	90%	95%	99%	mean	90%	95%	99%	mean	90%	95%	99%
White noise	0.071	0.106	0.120	0.147	0.066	0.114	0.134	0.180	0.086	0.162	0.197	0.263
White noise*	0.120	0.256	0.317	0.431	0.067	0.117	0.140	0.188	0.122	0.265	0.329	0.446
Random walk	0.433	0.734	0.793	0.873	0.181	0.404	0.501	0.637	0.445	0.752	0.809	0.885
Random walk**	0.338	0.626	0.693	0.804	0.091	0.160	0.199	0.283	0.351	0.649	0.715	0.826
AR(1) with correlation 0.9	0.173	0.343	0.405	0.526	0.075	0.126	0.147	0.197	0.186	0.376	0.441	0.572
First difference of AR(1)*	0.123	0.254	0.310	0.425	0.067	0.116	0.137	0.185	0.127	0.266	0.327	0.448

*Compensated for differencing
**Detrended with a linear time trend

Over-differencing does not seem to cause any problems in the estimation. The mean is lower and the distribution seems slightly more peaked in the case where we take first differences of an AR(1) process and compensate for the differencing. This supports the conjecture done above. The distribution of $\hat{R}^2_\Omega$ for the random walk is extremely wide. We thus clearly want to use the first difference filter if we are unsure of whether the process is I(1) or I(0) with a high degree of persistence.

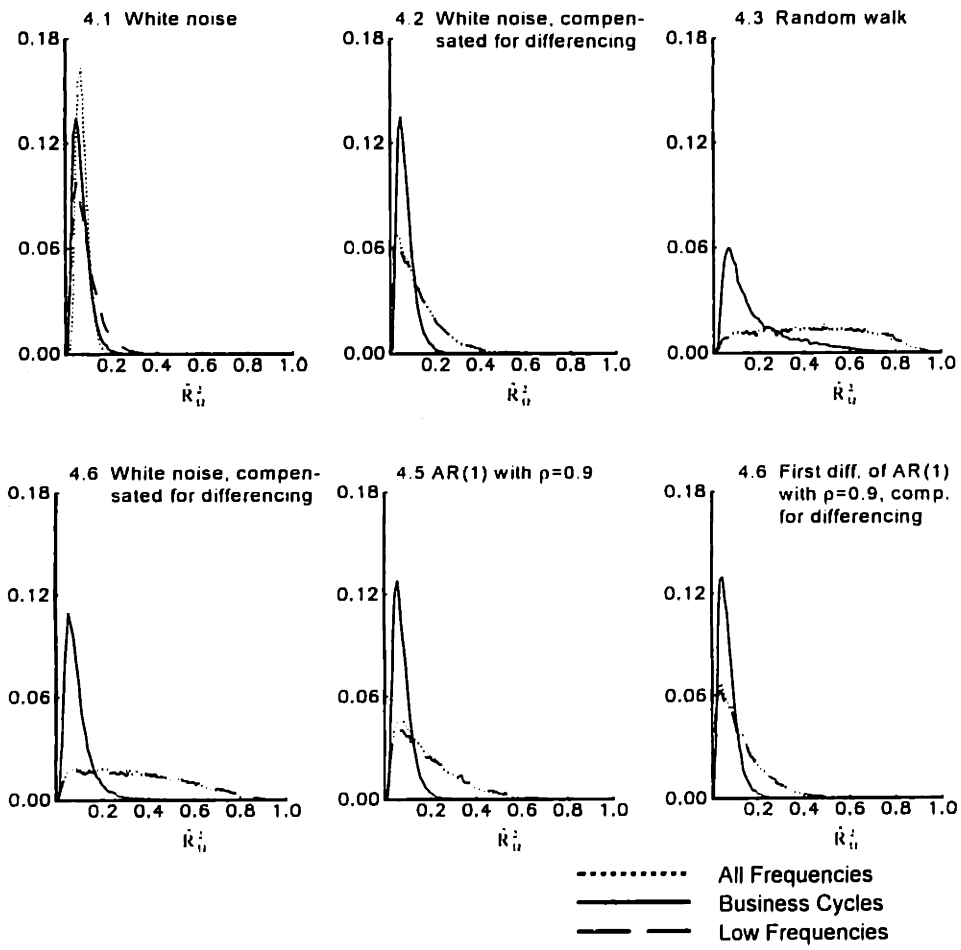
We also see that the problems with a wide distribution seem to be most severe if Ω contains low frequencies. For Ω being business cycle frequencies the distribution of $\hat{R}^2_\Omega$ is very similar for all the processes except the random walk. For low frequencies the distribution, on the other hand, changes a lot when we change the properties of the data generating process. For the random walk and the random walk minus a linear trend the distribution is not far from uniform both for all frequencies and low frequencies. This is a serious problem since

[14] The weights are stochastic but have the same mean.

we often cannot reject first order integration in macro time series.

It is important to note that the influence from the low frequencies clearly dominates that from business cycle frequencies. In Figure 4 the distribution of $\hat{R}^2_\Omega$ for Ω being all frequencies is indistinguishable from that for Ω being only low frequencies for all processes except uncompensated white noise.

Figure 4 Simulated Distributions of $\hat{R}^2_\Omega$ for 128 Observations



If we want to study business cycles these results show that it is dangerous not to filter out low frequency fluctuations. Statistical inference may be very inaccurate if low frequencies are included and we miss-specify the characteristics of the underlying processes. A reasonable conjecture is that also other methods of estimating the linear relationship between processes will show the same strong sensitivity to the properties of the underlying processes unless we filter out the low frequencies. Unless we have a strong a priori reason to include low frequencies when studying business cycles, it seems to be safer to study business cycle

frequencies and lower frequencies separately. The HP-filter is susceptible to this critique. Although it dampens low frequencies, filtered macroeconomic time series may, as discussed above, be dominated by low frequencies.

2.2.1. Modification of Daniell Window

The Daniell window includes frequencies outside of Ω in the estimation of spectral measures for $\omega \in \Omega$ if ω is close to the endpoints of Ω. In finite samples this may cause a bias in the estimation of R^2_Ω due to the influence from frequencies outside of Ω. This bias could be reduced if we use a window that only "average" over $\omega \in \Omega$ in the estimation. Unless Ω is too small, this should be feasible. If a priori we believe, or at least don't exclude, that the linear relationship between two processes is qualitatively different at different frequencies, this procedure should reduce the risk of the inference for $\omega \in \Omega$ being contaminated by relationships at other frequencies. In Table 3 I show the result of the same simulation as in Table 2 using a modified Daniell window modified in accordance with

$$\hat{h}(\omega_s) = \frac{\sum_k \lambda_{s,k} I(\omega_k)}{\sum_k \lambda_{s,k}}, \qquad \forall \omega_s \in \Omega$$
$$\lambda_{s,k} = \begin{cases} 1, & \text{if } 2|s-k| \le M \text{ and } \omega_k \in \Omega \\ 0, & \text{otherwise} \end{cases} \tag{2.12}$$

where $I(\omega_k)$ denotes the periodogram ordinate for frequency ω_k. I have set M to 7 so I take an average over the 15 frequencies surrounding that which is estimated, but exclude those of these that are outside of Ω.

One effect of using the modified window is that it reduces the power in tests of zero $\hat{R}^2_\Omega$. This is seen in Table 3 and is explained by the fact that the modification of the Daniell window leads to a reduction in the number of periodogram ordinates involved in the estimation of $\hat{R}^2_\Omega$. On the other hand, if we have positive coherence in the complement of (and close to) Ω, nominal significance levels in tests of zero $\hat{R}^2_\Omega$ based on simulations like that in Table 2 will be overstated in all small samples. This since the positive coherence outside Ω causes an upward bias in $\hat{R}^2_\Omega$ if $R^2_\Omega = 0$.

Table 3 Empirical Distribution of Bivariate $\hat{R}^2_\Omega$

Modified Daniell Window, 128 observations	Business Cycles			
		Percentiles		
	mean	90%	95%	99%
White noise	0.080	0.140	0.165	0.225
White noise, compensated for differencing	0.080	0.143	0.172	0.231
Random walk	0.191	0.426	0.518	0.662
Random walk detrended with a linear time trend	0.100	0.183	0.225	0.326
AR(1) with correlation 0.9	0.085	0.150	0.180	0.246
First difference of AR(1) compensated for differencing	0.080	0.140	0.166	0.222

2.2.2. Introducing Dependence

If we want to quantify the degree of linear comovements it will be of interest to test hypothesis that $\hat{R}^2_\Omega \geq k > 0$. It is likely that the small sample distribution of $\hat{R}^2_\Omega$ in the case where $R^2_\Omega > 0$ will depend on $\mathbf{B}(L)$. Unfortunately this implies that it is necessary to make assumptions about $\mathbf{B}(L)$ in order to make inferences about R^2_Ω. To get some idea about the distribution of $\hat{R}^2_\Omega$ I have anyhow done simulations using the simplest possible linear dependence. In Table 4 and in Figure I report the result from simulations where the underlying processes are constructed so that[15]

$$\begin{cases} Y_t & = \beta X_t + \varepsilon_t, X_t \perp \varepsilon_t, \forall t \\ \beta^2 & = \{0.15, .30, \ldots, .75\} \\ h_y(\omega) & = h_x(\omega), \forall \omega \end{cases} \tag{2.13}$$

implying

[15] The third line of equation (14) implies that $h_\varepsilon(\omega) = (1-\beta^2)h_x(\omega)$, so the variance of ε depends on β.

$$
\begin{aligned}
R_{\Omega}^{2} &= 1 - \frac{\int_{\omega\in\Omega} h_y(\omega) - \beta^2 h_x(\omega)d\omega}{\int_{\omega\in\Omega} h_y(\omega)d\omega} \\
&= 1 - \frac{(1-\beta^2)\int_{\omega\in\Omega} h_y(\omega)d\omega}{\int_{\omega\in\Omega} h_y(\omega)d\omega} \qquad (2.14) \\
&= \beta^2, \quad \forall\Omega.
\end{aligned}
$$

As in the case with independent processes we find an upward bias in the estimates of R_{Ω}^{2}. The bias decreases with R_{Ω}^{2} while the distance between the mean and the higher percentiles is largest for intermediate values of R_{Ω}^{2}. In Table 4 we can also see that the power in tests of zero $\hat{R}_{\Omega}^{2}$ is reasonably high already for a true value of 0.15.

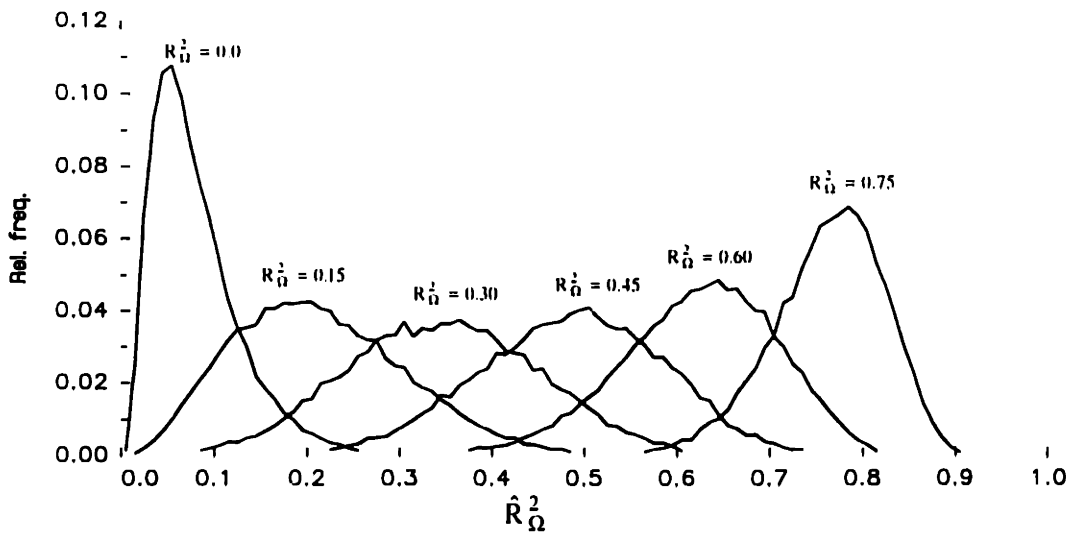
Table 4 Empirical Distribution of Bivariate $\hat{R}_{\Omega}^{2}$

White noise, compensated for differencing, 128 observations	Business Cycles						
		Percentiles			Prob. of rejection in tests of $R_{\Omega}^{2} = 0$		
Degree of Dependence, R^2	mean	90%	95%	99%	90%	95%	99%
0.00	0.080	0.143	0.172	0.231	-	-	-
0.15	0.214	0.341	0.380	0.455	0.758	0.643	0.398
0.30	0.348	0.487	0.522	0.595	0.978	0.957	0.863
0.45	0.485	0.609	0.639	0.692	1.000	1.000	0.992
0.60	0.624	0.729	0.754	0.795	1.000	1.000	1.000
0.75	0.763	0.837	0.852	0.875	1.000	1.000	1.000

2.3. Distribution of Multivariate $\hat{R}_{\Omega}^{2}$

As was shown above the bivariate $\hat{R}_{\Omega}^{2}$ has a multivariate analog. In Table 5 I show the result from a simulation with three RHS variables. Table 6 and Figure 5 shows the result of multivariate simulations with different degrees of dependence. As in Table 3 and 4, I use a modified Daniell window with 15 included frequencies. For computational reasons the number of draws is, however, in all multivariate simulations reduced to 1000 per series.

Figure 5 Simulated Distributions of $\hat{R}^2_\Omega$, White Noise compensated for differencing, 3 RHS Variables



Qualitatively, the simulation results are similar to those in the bivariate case. Not surprisingly is the upward bias higher and the distribution slightly less peaked in the multivariate case.

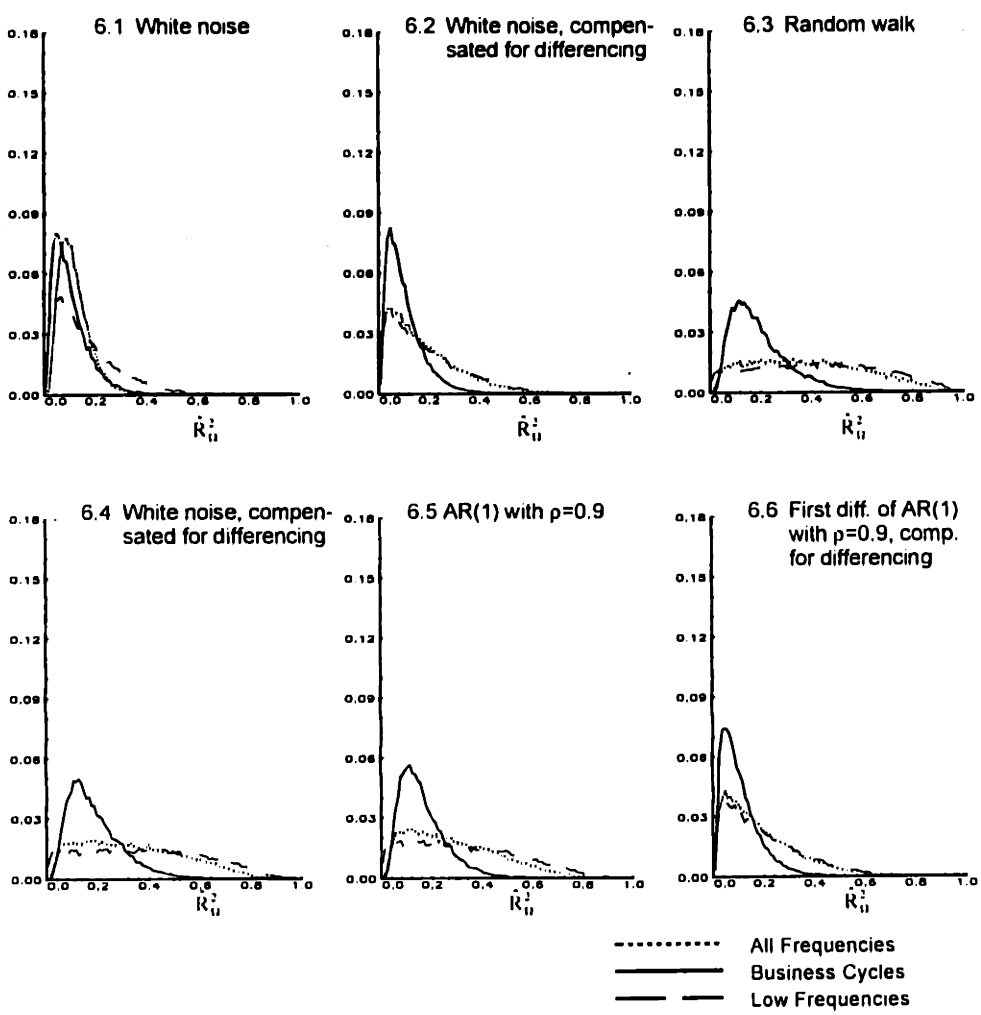
Table 5 Empirical Distribution of Multivariate $\hat{R}^2_\Omega$, 3 RHS Variables

		Percentiles		
Modified Daniell Window, 128 observations	mean	90%	95%	99%
White noise	0.235	0.323	0.351	0.410
White noise, compensated for differencing	0.237	0.330	0.364	0.424
Random walk	0.414	0.645	0.718	0.784
Random walk detrended with a linear time trend	0.319	0.464	0.520	0.623
AR(1) with correlation 0.9	0.251	0.351	0.382	0.450
First difference of AR(1) with correlation 0.9	0.235	0.321	0.352	0.404
First difference of AR(1), compensated for differencing	0.244	0.342	0.367	0.437

Table 6 Empirical Distribution of Multivariate $\hat{R}^2_\Omega$

White noise, compensated for differencing, 128 observations	2 RHS variables				3 RHS variables			
		Percentiles				Percentiles		
Degree of Dependence, R^2	mean	90%	95%	99%	mean	90%	95%	99%
0	0.165	0.251	0.286	0.342	0.237	0.330	0.364	0.424
0.15	0.285	0.410	0.462	0.517	0.349	0.477	0.511	0.570
0.30	0.403	0.536	0.569	0.618	0.458	0.576	0.620	0.672
0.45	0.534	0.649	0.683	0.744	0.575	0.690	0.714	0.753
0.60	0.658	0.760	0.780	0.821	0.691	0.782	0.805	0.836
0.75	0.787	0.855	0.871	0.896	0.802	0.865	0.880	0.901

Figure 6 Simulated Distributions of $\hat{R}^2_\Omega$, 40 Observations



2.4. Shorter Samples

As was noted above, a sample of 128 observations is short compared to standard applications of spectral theory. In an economic setting we would anyhow like to be able to use sample sizes considerably smaller than 128. I have made simulations with 40 observations and show the result in Figure 6.[16]

We see that the qualitative appearance of the distributions in Figure 6 is similar to those in Figure 4, except that the dispersion naturally is larger. The result that the distributions tend to be more peaked and less dependent on the process characteristics for business cycle frequencies remains valid with shorter samples. It should thus be possible to use $\hat{R}^2_\Omega$, at least bivariate, also in samples as short as 40. The power in tests of zero $\hat{R}^2_\Omega$ will, however, certainly be quite low.

2.5. Summary of Section 2

The following points summarize the results and ideas in this section:

- linear comovements may be different at different frequencies, both in terms of degree and structure.
- for typical macro series, statistics of comovement that include all frequencies can be expected to be dominated by the comovement at low frequencies.
- we may define easily interpreted measures of comovement for a specific range of frequencies, these may be thought of as decomposed regression R^2.
- the sensitivity of the distribution of such measures to the characteristics of the studied process is substantially reduced if we exclude the low frequencies.
- the establishment of "business cycle facts" should not (at least not without an explicit reason) be done with methods that tend to give statistics dominated by the behavior at low frequencies.

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[16] Since I am using a Fast Fourier Transform which requires sample lengths that can be written as 2^n with n being an integer, the 40 observations are extended with 24 zeros.

3. Detrending and Definition of Business Cycles

In the application of spectral methods, one of the more difficult problems is how to detrend or make stationary the clearly non-stationary macro data. The concept of business cycles (implicitly) assumes that data can be decomposed in one non-stationary component and one component that is without a trend in at least its first moment.[17] We can then write

$$Y_t = Y_t^l + Y_t^s \tag{3.1}$$

where Y_t^l denotes the (log of the) trending component and Y_t^s is the (log of the) non-trending component. A priori, it is not unlikely that these two processes are generated by somewhat different mechanisms. An example would be that the non-stationary component is generated by some mechanism producing variations in long run growth rates. An endogenous growth models could, for example, be used to model this mechanism. Monetary shocks in combination with sticky prices could, on the other hand, produce stationary fluctuations. The distinction between the two mechanisms may be useful if they have different characteristics and if the links between them can be identified.

As is shown in Quah (1992), there is in general no unique decomposition of a I(1) series in a stationary and a non-stationary component. By assuming different characteristics of the non-stationary component we can make its share of the sample variance of differenced data arbitrarily small. Heuristically, we may split the spectrum of the differenced data in two parts. As long as we set the spectral power at frequency zero to zero for one of the parts, this part is the spectrum of a stationary process. By giving more of the power at other frequencies to the stationary part we increase its share of the variance and vice versa.

If the goal of the decomposition is to study the (supposedly) different mechanisms generating Y_t^l and Y_t^s we would like to know (or assume) some characteristics of the data-generating mechanisms to do the right detrending.[18]

Another approach is to define as business cycles, fluctuations within some frequency

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[17] We often make the additional assumption that the non-trending component is stationary. This is necessary for the application of many statistical methods including spectral analysis. Clearly this assumption may be too strong. In an evolutionary society, the higher moments of the business cycle may very well be changing over time, without business cycles looses its meaning as a concept.

[18] One possible identifying assumption, which not will be used here, is that the trend is common to two or more variables. All stationary idiosyncrasy can then be attributed to the non-trending component.

band.[19] These fluctuations may then potentially come from both of the components Y_t^l and Y_t^s. The specification of business cycles as fluctuations within a specific frequency band could be thought of just as a definition of the area of interest.

A better justification for the frequency range definition is an assumption that the two components ΔY_t^l and Y_t^s have most of their spectral power in different frequency bands. In this chapter I have chosen to define business cycle frequencies as frequencies corresponding to periods between three and eight years. Fluctuations with higher frequencies are not included on the grounds that they may represent measurement errors stemming from erroneous "shuffling" between adjacent years. With more than one observation per year higher frequency fluctuations may also come from seasonals, not typically thought of as business cycles.

Before applying spectral methods the series have to be made stationary. Below I have done this by taking first-differences of the logarithms and subtracting the sample mean. The first-difference filter will make the series stationary unless they are integrated of a higher order than one. Some evidence against integration of order higher than one for real macro variables have been presented in Jones (1992).[20] A first difference filter also makes a spectrum with "the typical spectral shape"[21] flatter and thus easier to estimate.

The spectrum of the business cycle component is estimated as

$$\hat{h}_{Y^s}(\omega) = \left|(1 - e^{-i\omega})\right|^{-2} \hat{h}_{\Delta Y}(\omega), \ \forall \omega \in \Omega, \ 0 \notin \Omega \tag{3.2}$$

where Ω is the set of frequencies corresponding to periods between 3 and 8 years. Clearly this estimate will be contaminated by the trending component unless

$$h_{\varepsilon}(\omega) << h_{\Delta Y_t^s}(\omega), \qquad \forall \omega \in \Omega.^{22} \tag{3.3}$$

(3.3) implies that long run growth is generated by a process with a fairly high degree of

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[19] A good example of this is Englund et al (1990).

[20] Differencing once more and correcting for the effects on Y_t^s by multiplying the spectra with $\left|(1 - e^{-i\omega})\right|^{-4}$ would probably not change the results.

[21] Granger (1966) coined this expression. Note, however, the result by Nelson and Kang (1981) that inappropriately detrended time series typically posses "the typical spectrum" and that this is an artifact of the detrending procedure rather than the underlying data generating process.

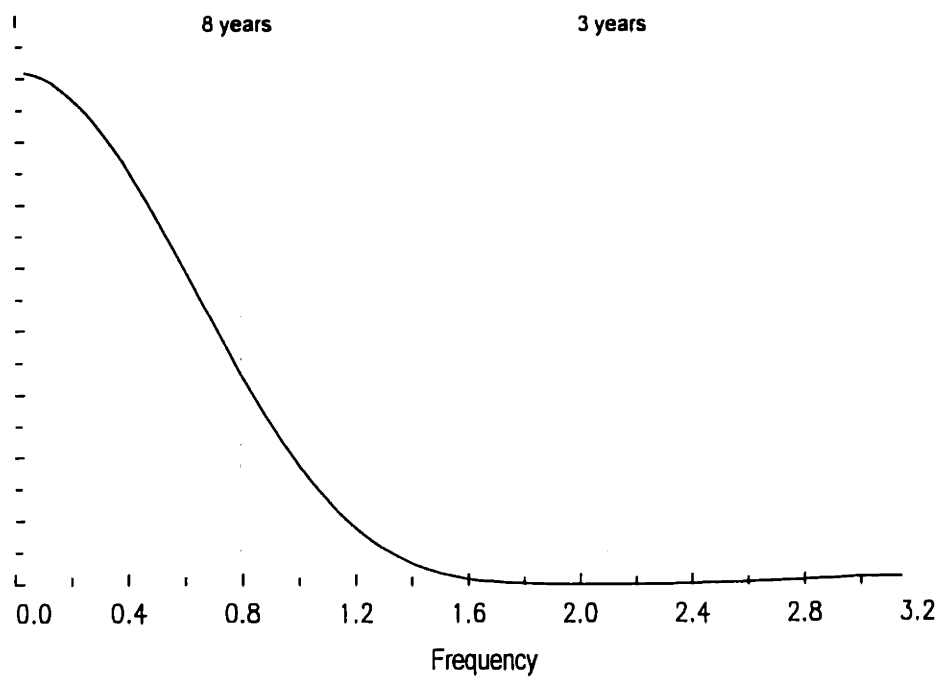
[22] For some applications it may be enough that $\int_{\omega \in \Omega} h_{\varepsilon}(\omega) << \int_{\omega \in \Omega} h_{\Delta Y_t^s}(\omega)$

persistence in its growth rates – a smooth trend. It should be possible to generate such a process within, for example, a model where growth is generated by the slow diffusion of inventions to new markets.[23] Consider the case when technological innovations, denoted as ζ_t, are i.i.d. white noise. New technology is adopted through the economy according to a standard S-shaped diffusion mechanism. Assume for example that the maximum impact on productivity comes after two years and the diffusion process is completed after four years. We may then represent the trend as

$$y_t^l = \alpha + y_{t-1}^l + \frac{\zeta_t}{3} + \frac{\zeta_{t-1}}{2} + \frac{\zeta_{t-2}}{1} + \frac{\zeta_{t-3}}{2} + \frac{\zeta_{t-4}}{3}. \tag{3.4}$$

In figure 7 I show that first differences of this trend has most of its spectral power outside business cycle frequencies. The same will be true also for other S-shaped diffusion patterns. For a linear time trend (3.3) is, of course, trivially true.

Figure 7 Spectrum of First Differences of a Diffusion Process



If, as a description of the reality, the inequalities in (3.3) should be reversed, business cycle students are fighting windmills, which of course is a possibility. Then most of the fluctuations in the series, also in the business cycle band, are fluctuations in the trending

[23] For a recent study on diffusion see Lippi and Reichlin (1993).

component. Here, a separate study of the behavior at business cycles is unlikely to be fruitful. Neither can the possibility that the two spectral densities are of the same order of magnitude be excluded. In this chapter I have taken a conservative approach and have not removed anything from ΔY within Ω when estimating the spectrum for the non-trending component. This is a notable difference from the procedure of trying to remove a random walk from the original series.

4. Application to Swedish Data

4.1. Data Set

The time series used in this chapter are taken from a data base put together at The Institute for International Economic Studies at Stockholm University. The Swedish series span the period from 1861 to 1988, except for unemployment that start at 1911. The foreign series go back to 1870. Definitions and a full description of the sources are given in Appendix 1. For a discussion of the sources and their quality, see Hassler et al (1992).

4.2. Results

4.2.1. Bivariate relations

In Table 7 I report bivariate business cycle $\hat{R}^2_\Omega$ between Swedish and foreign macro variables.[24] Black cells represent strong significance (at 99%) and shaded weak (at 90%). In Table 3 I showed that $\hat{R}^2_\Omega$ has a slightly wider distribution for differenced random walks than for differenced AR(1) processes with autocorrelations of 0.9. The 90% and 99% significance levels for the random walk case are 0.143 and 0.231 and for the AR(1) 0.140 and 0.222. To be cautious, the former levels have been used.

The overall impression is one of low degrees of stable, linear relationships between Swedish production related variables and foreign business cycles. The $\hat{R}^2_\Omega$'s between Swedish GDP and most foreign GDP series, as well as with the index of foreign demand, are low. Only with Norwegian GDP do we get a strongly significant $\hat{R}^2_\Omega$. This is also the only estimate of comovement between Swedish GDP and foreign variables that is higher than mean simulation value for a true R^2_Ω of 15% (as given by Table 4). None of the $\hat{R}^2_\Omega$ between manufacturing production and the foreign variables is significantly different from zero.

There is, on the other hand, evidence for substantial linear comovements between many of the Swedish macro variables and business cycle fluctuations in Denmark and Norway, especially Norway. All the Swedish variables I used, except manufacturing produc-

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[24] A modified Daniel window with a width of 15 frequencies is used in all what follows. As noted in the previous section, the times series are logged and differenced and the spectra compensated for the differencing before $\hat{R}^2_\Omega$ is computed.

tion, have a $\hat{R}^2_\Omega$ with Norwegian GDP that is at least weakly significant. The consumption related measures, private consumption and imports and also exports, have $\hat{R}^2_\Omega$ significantly larger than 15%. For Denmark six of the variables are significantly comoving, although only one (private consumption) at a degree significantly larger than 15%.

Table 7 Business Cycle $\hat{R}^2_\Omega$

	Y	C	G	I	X	M	MAN	H	U*	FY	UKY	DKY	NOY	GEY	FRY	USY
Y	1.000		0.213			0.150			0.180	0.095	0.122	0.168		0.113	0.058	0.153
C		1.000					0.126			0.111	0.083			0.211	0.106	0.204
G			1.000	0.070	0.174	0.100	0.043			0.087	0.120	0.086	0.205	0.145		0.180
I				1.000	0.075		0.046			0.132	0.147	0.167	0.220	0.081	0.069	0.104
X					1.000		0.139	0.163		0.079	0.113	0.186				0.049
M						1.000	0.053	0.175		0.151	0.053					0.093
MAN							1.000	0.176	0.122	0.085	0.061	0.042	0.134	0.046	0.044	0.132
H								1.000				0.088	0.170	0.075	0.071	
U*									1.000	0.187		0.173		0.101	0.127	
FY										1.000		0.205	0.036		0.056	
UKY											1.000	0.055	0.034	0.105	0.045	
DKY												1.000	0.185			0.126
NOY													1.000		0.160	0.123
GEY														1.000		0.188
FRY															1.000	0.103
USY																1.000

All variables are in real terms.
Black cells represent significance at 99% and gray at 90%.

Y= GDP
C=Private consumption
G=Public consumption
I=Gross capital formation
X=Exports
I=Imports
MAN=Value added in manufacturing and mining
H=Hours worked in MAN
U=Unemployment
FY=Index for foreign demand
UKY=GDP in Great Britain
DKY=GDP in Denmark
NOY=GDP in Norway
GEY=GDP in Germany
FRY=GDP in France
USY=GDP in USA
* Only 78 observations

It is interesting to note the high degree of comovement that hours worked and unemployment show with GDP in US and Great Britain and also with foreign demand. This is puzzling since Swedish GDP as well as manufacturing production show very low degrees of comovement with these foreign variables. Hours worked in manufacturing and mining is also more covariant with foreign demand than with manufacturing and mining production itself.

Exports and imports show the highest degrees of comovement with the foreign variables. They are both at least weakly significant for Denmark, Norway, Germany and France. The degrees of comovement with Great Britain, U.S. and foreign demand are on the other hand low for both exports and imports.

4.2.2. Multivariate relations

The degree of comovement with a univariate index of foreign demand may be a poor substitute for a broader, multivariate analysis. In Table 8 I report the multivariate $\hat{R}^2_\Omega$ using the same Swedish macro variables as in Table 7 as LHS variables but using two sets of multivariate variables as RHS (Scandinavian and non-Scandinavian countries). Black and gray cells have the same interpretation as in Table 7.

One of the motivations for the separation of linear comovement at different frequencies is, of course, that it may be different at different frequencies. To see whether this is the case I have in Table 8 included columns for $\hat{R}^2_\Omega$ where I let Ω be all frequencies below the business cycle band, i.e., periods longer than 8 years. I have not compensated for the differencing. This since it is assumed that we are interested in the degree of linear comovement between the innovations to the non-stationary component. To illustrate the behavior of the HP-filter I have also calculated the degree of linear comovement between HP-filtered data, using three different values for λ. After filtering the series, the degree of linear comovement is calculated, in the case of the HP-filter including all frequencies. As was shown in Section 2 above, the distribution of $\hat{R}^2_\Omega$ is much more sensitive to the characteristics of the underlying process when low frequencies are included. I have thus not indicated significance levels for low frequencies and for the HP-filtered data in Table 8.

Also in the multivariate analysis we find a low degree of comovement at business cycle frequencies for the production related variables. For manufacturing production $\hat{R}^2_\Omega$ is approximately equal to the mean value in the simulations, *given no linear comovement at all.* For GDP, the comovement is significant but not impressingly large. For the second set of foreign variables, it is equal to the average simulation value given a dependence of 15%. The degree of comovement with the Nordic countries appears to be somewhat higher, but not much so. The covariation between investments and Danish and Norwegian GDP is weakly significant.

Exports and imports are clearly covariant with the foreign business cycle. The degree of comovement is high and strongly significant $\hat{R}^2_\Omega$ for exports and the Nordic countries. It is higher than the 99'th percentile given 15% linear dependence and also close to the mean simulation value for 45% dependence. For imports the degree of comovement is even higher. It is significantly higher than 30% and also higher than the mean simulation value for 45%. The same pattern is observed when we use the second set of countries as independent variables.

Table 8 Multivariate $\hat{R}^2_\Omega$

	Low Frequencies[1]		HP-filtered data[1]						Business Cycles[2]	
			λ=1600		λ=40		λ=1			
	Denm. Norway	Germ. Britain US	Denm. Norway	Germ. Britain US	Denm. Norway	Germ. Britain US	Denm. Norway	Germ. Britain US	Denm. Norway	Germ. Britain US
GDP	0.595	0.641	0.557	0.565	0.436	0.446	0.350	0.297	0.353	0.342
Consumption	0.601	0.476	0.581	0.444	0.612	0.438	0.608	0.456	0.555	0.474
Gov. Cons.	0.268	0.583	0.232	0.585	0.226	0.593	0.290	0.455	0.285	0.499
Investments	0.244	0.193	0.242	0.196	0.285	0.224	0.258	0.258	0.312	0.258
Exports	0.639	0.688	0.642	0.673	0.576	0.578	0.497	0.505	0.519	0.535
Imports	0.771	0.616	0.672	0.648	0.612	0.571	0.500	0.580	0.565	0.619
Manuf. prod.	0.306	0.561	0.275	0.476	0.207	0.415	0.158	0.235	0.158	0.258
Hours worked	0.228	0.541	0.180	0.580	0.254	0.590	0.296	0.438	0.211	0.524
Unemployment	0.483	0.716	0.440	0.704	0.318	0.662	0.410	0.620	0.312	0.670

1. No significance levels indicated.
2. Black cells represent significance at 99% and gray at 90%.

Unemployment and hours worked are also strongly covariant with business cycle fluctuations of GDP in the non-Nordic countries. This is surprising given that hours worked refers to hours worked in manufacturing and mining. On the other hand we see in Table 7 that the bivariate degree of comovement on business cycle frequencies between hours worked and manufacturing production is low. Business cycle fluctuations in manufacturing and mining will thus to a large extent be reflected as changes in measured labor productivity.

Also private and government consumption have a significant degree of comovement with the foreign variables. Especially private consumption and GDP in the Nordic countries are strongly covariant. From Table 7 we find that $\hat{R}^2_\Omega$ between GDP and private consumption

is 0.449. This is significantly different from zero. Nevertheless it is low enough to allow for a clearly higher degree of comovement between private consumption and the foreign variables than what we see for GDP. This finding is in accordance with Lundvik (1992) who, within a real business cycle model, also finds a higher degree of foreign influence on consumption than on production.

In the preceding paragraphs we noted that the production related measures are less linearly covariant with the foreign variables than the consumption related are. This is, however, only true for business cycle frequencies. In contrast to the business cycle component, there is a high degree of comovement between the non-stationary (low frequency) component of GDP and the foreign variables. For low frequencies the degree of comovement of GDP with the foreign variables is in the same order of magnitude as that of private consumption and the trade variables.[25]

As discussed above and in Cogley and Nason (1992) the application of the Hodrick-Prescott filter with a sufficiently high λ to difference stationary series produce filtered series possessing spectra with a peak in the low frequency region. The use of such filter on difference stationary (and probably also on highly persistent) series would thus produce measures of linear comovement dominated by the coherence at low frequencies. In Table 8 we clearly see that the results for HP-filtered data are very close to those for the low frequencies when a λ of 1600 is used. In particular the lower degree of comovement for production than for consumption related variables with foreign GDP is not found in this case.

For lower values of λ the results come closer to those for business cycle. For $\lambda = 1$ most of the low frequencies are removed, while high frequencies are amplified relative to business cycle frequencies. In this case the results do not substantially change the measures of comovement. This, however, could only be checked for by actually removing their influence, using $\hat{R}^2_{\Omega}$.

The possibility of measuring the degree of linear comovement in subsamples is limited by the small sample size. I have anyway performed one test to check that the results at least

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[25] This difference is not an artifact of that only the business cycle component is compensated for differencing. The difference remains if the business cycle component is not compensated for differencing. Since the compensating function $\left|1 - e^{-i\omega}\right|^{-2}$ is relatively flat in the business cycle frequency band, we wouldn't expect the results to be very dependent on the compensation.

not are driven by the last decades when Sweden, in several respects, became more open than previously. This is to calculate $\hat{R}^2_\Omega$ for the period from 1861 to 1958, i.e., censuring the last 30 observations. The results are given in Table 9. There we see that the qualitative results are not changed much, when the last three decades are deleted. All $\hat{R}^2_\Omega$ are somewhat higher but their approximate ranking remains, including the higher degrees of comovement for consumption related measures than for production.

Table 9 Multivariate $\hat{R}^2_\Omega$

	Business Cycles[1]	
Last 30 observations censured	Denm. Norway	Germ. Britain US
GDP	0.470	0.454
Consumption	0.764	0.597
Gov. Cons.	0.388	0.544
Investments	0.327	0.307
Exports	0.588	0.635
Imports	0.593	0.698
Manuf. prod.	0.179	0.299
Hours worked	0.305	0.551

1. No significance levels indicated.

5. Some Interpretations and Remarks

That the establishment of stylized facts should is an important task for the students of business cycles is not a controversial statement. More debatable are the questions: what characteristics should be measured and which statistical methods should be used to use describe them? The aim of this chapter has been to convince the reader that $\hat{R}^2_\Omega$ should be included as a standard measure of comovement, taken for granted that comovement is an important characteristic to measure. Lastly I will make some evaluations and interpretations of the results.

An important finding is that business cycles in the production related measures, GDP, value added in manufacturing and mining and investments, have had fairly low degrees of comovement with the GDP of important trading partners of Sweden. Only Norwegian and to some extent Danish GDP seems to swing together with Swedish GDP. Because of the small size of Denmark and Norway this comovement may, however, reflect dependence where causality runs in the opposite direction, so that Norway and Denmark are dependent on Sweden rather than the opposite. Another possible explanation is that the Nordic countries have been hit by shocks common to them but not to (or as important for) other countries.

The low degree of covariation between Swedish production and non-Nordic GDP is certainly not indisputable evidence for the Swedish production business cycle having a large degree of international independence. The low degree of comovement could be explained by:

- unstable or non-linear transmission mechanisms.
- wrong variables. Considering the composition of Swedish exports, foreign investment demand may be a better variable than foreign GDP as a proxy for the foreign influence on Sweden.

The first point is, however, at least partly dealt with. The geographical patterns of Swedish trade have changed over time; U.S. and Germany have replaced Great Britain as the most important non-Nordic trading partner. The index for foreign demand is constructed to take into account such long term and low frequency changes. In this case this instability would thus have been controlled for by using the index for foreign demand that then should show a high $\hat{R}^2_\Omega$. In fact, the point estimate of $\hat{R}^2_\Omega$ between foreign demand and Swedish GDP of 0.095, is only marginally bigger than the average value in the simulations given independence. By censuring the last 30 observations I have also checked that the results are not

due to the last period being extreme.

It seems likely that the mechanism generating the foreign influence on Swedish business cycles involves (or even mainly goes via) Swedish trade. We find a substantial degree of comovement between both exports and imports and the foreign variables. This gives evidence for an alternative interpretation of the data, namely that the foreign influence on Swedish trade is strong but that the link between Swedish trade and GDP is rather weak. In the data only a third of the Swedish business cycle can be linearly attributed to fluctuations in exports. Other domestic factors seem to be responsible for the lion's share of business cycle fluctuations in Swedish GDP.

The difference in the degree of comovement with the foreign variables that we find between production and consumption related variables is also interesting. Consumption seems to be more dependent on the foreign variables than production. A possible explanation comes from the finding in Table 8 that the degree of comovement between Swedish and foreign GDP is highest at frequencies below the business cycle band. Assume that the non-stationary component of GDP has a high degree of international comovement, while the business cycle component is less covariant. Changes in life-time income of the agents will be driven mostly by innovations in the non-stationary component. Assume also that the consumers can (imperfectly) distinguish the non-stationary and the business cycle components. If the non-stationary component has non-negligible spectral density also at business cycle frequencies we may then expect consumption to be more covariant than production at business cycle frequencies.

Hours worked in manufacturing and mining appears to have a substantial degree of comovement with business cycles in the non-Nordic countries. This variable thus shows the same characteristics as the consumption related rather than the production related variables. In fact, business cycles in hours worked in manufacturing and mining have a clearly higher degree of comovement with private consumption than with value added in manufacturing and mining. This finding supports the hypothesis that shocks to labor supply are more important than shocks to labor demand as a driving force behind fluctuations in hours worked. This interpretation is also in line with the finding in Hassler et al (1992) of a negative contemporaneous correlation between hours worked and real wages.

The idea that it may be fruitful to model the economy as consisting of two distinct but

interrelated components, one generating long run, non-stationary growth and one generating stationary business cycles, gets some preliminary support by the finding that the relations with foreign variables are different at different frequencies. This difference also warrants caution when using statistics that are some kind of weighted average of all frequencies. This may be important when the Hodrick-Prescott filter is used since it may produce statistics dominated by the behavior at low frequencies. Given an intelligent choice of λ and that our studied processes do not have certain high frequency behavior, linear comovement business cycle statistics on HP-filtered data may come quite close to $\hat{R}^2_\Omega$. On the other hand, this may often not be the case. If we want to establish business cycle stylized facts about linear relations between macro time series, the use of $\hat{R}^2_\Omega$, including only those frequencies we are interested in, thus seems preferable.

Appendix 1 Data Sources and Definitions

GDP (Y)

Gross Domestic Product in 1985 fixed factor values. From 1950 GDP at current factor values is deflated by the implicit deflator for GDP at market values.

Sources:

1861-1949	Krantz and Nilsson (1975), Table 3.1
1950-1979	Statistics Sweden, N10 SM8901, Appendix 2-3, Tables 3.1 and 3.2
1980-1988	Statistics Sweden, N10 SM8901, Tables H:1 and H:2

Private Consumption (C)

Private consumption of goods and services at fixed 1985 prices.

Sources:

1861-1949	Krantz and Nilsson (1975), Table 2.3.1.
1950-1979	Statistics Sweden, N10 SM8901, Appendix 2-3, Table 3:2
1980-1988	Statistics Sweden, N10 SM8901, H2.

Public Consumption (G)

Public consumption of goods and services, state and municipalities deflated with the GDP deflator.

Sources:

1861-1949	Krantz and Nilsson (1975), Table 1.1.
1950-1979	Statistics Sweden, N10 SM8901, Appendix 2-3, Table 3:1
1980-1988	Statistics Sweden, N10 SM8901, H1.

Investments (I)

Gross domestic capital formation in fixed 1985 prices.

Sources:

1861-1949	Krantz and Nilsson (1975), Table 2.2.1.
1950-1979	Statistics Sweden, N10 SM8901, Appendix 2-3, Table 3:2, row 5.
1980-1988	Statistics Sweden, N10 SM8901, Table H2.

Exports (X)

Exports of goods and services in fixed 1985 prices.

Sources:

1861-1949	Johansson (1967), Table 50 col.14
1950-1979	Statistics Sweden, N10 SM 8901, Appendix 2-3, Table 3:2
1980-1988	Statistics Sweden, N10 SM 8901, Table H2

Imports (M)

Imports of goods and services in fixed 1985 prices.

Sources:

1861-1949	Johansson (1967), Table 52 col.14
1950-1979	Statistics Sweden, N10 SM 8901, Appendix 2-3, Table 3:2
1980-1988	Statistics Sweden, N10 SM 8901, Table H2

Manufacturing Production (Yman)

Value added in manufacturing and mining in fixed 1985 producer prices.

Sources:

1861-1949	Krantz and Nilsson (1975), Table 3.2.1
1950-1962	Statistics Sweden, SM N1975:98, Table 2A
1963-1969	Statistics Sweden, SM N1981:2.5, Appendix 4, Table 2A
1970-1988	Statistics Sweden, N10 SM8901, Table H:4

Worked Hours in Manufacturing (H)

Total number of worked hours by employees in manufacturing and mining. Between 1870 and 1949 it is constructed by dividing total wage bill and wage rate. Between 1950 and 1959 hours by

entrepreneurs ("företagare") are included.

Source for wage bill:
1870-1949 Jungenfelt (1966), p. 122 col. 5.
Sources for wage rate:
1861-1913 Bagge et al. (1935), Table 26, Column "Social Science Institute"
1914-1919 Bagge et al. (1935), Table 26, Column "Social Board"
1920-1929 Statistics Sweden, Lönestatistisk årsbok 1929, Table 8
1930-1934 Statistics Sweden, Lönestatistisk årsbok 1938, Table 24
1935-1938 Statistics Sweden, Lönestatistisk årsbok 1940, Table 28
1939-1984 Statistics Sweden, Löner 1984, 58 Table D
Sources for worked hours:
1950-1959 Statistics Sweden, Nr N1975:98, Appendix 5, Table 98
1960-1962 Statistics Sweden, Nr N1975:98, Appendix 5, Table 92
1963-1969 Statistics Sweden, N1978:8.4, Appendix 5, Table 92
1970-1979 Statistics Sweden, N10 SM8601, Appendix 5, Table 5:8
1980-1988 Statistics Sweden, N10 SM8901, Appendix 5, Table 5:8

Unemployment (U)

Yearly average of number of unemployed reported to unemployment insurance. ("Arbetslöshetskassor")

Sources:
1911-1956 Silenstam (1970)
1956-1969 Historical Statistics, Arbetsmarknadsstyrelsen, Swedish Labor Market Board.
1970-1988 Arbetsmarknadsstyrelsen, Swedish Labor Market Board.

Foreign Demand (Fy)

Aggregated weighted GDP in Great Britain, Denmark, Norway, (West) Germany, France and United States. Weights calculated as follows: Unfiltered weights are defined as:

$$\alpha_i = \frac{w_i X}{Y_i P_x}$$

w_i denotes the share of Swedish exports going to country i, X is nominal export prices, Y_i real GDP in country i, and Px is export prices. For some years without observations on w_i, linear interpolations are made. A Hodrick-Prescott filter with lambda = 6400 is then applied to $\alpha_{i\,i}$ and the trend component $\overline{\alpha}_i$ (normalized to sum to one) is used as the weight so that:

$$\mathrm{Fy} = \sum_i \overline{\alpha}_i Y_i$$

Source for
1871-1964 Ohlsson (1969), Table B:9
1965-1988 Statistisk Årsbok, Statistics Sweden

Output Great Britain (Uk)

GDP of Great Britain in fixed values.
Sources:
1870-1949 Maddison (1982), Table A6-A8
1950-1985 Summers and Heston (1988)
1986-1988 OECD, National Accounts

Output Denmark (Dk)

As for Uk.

Output Norway (No)

As for Uk.

Output Germany (Ge)
As for Uk.

Output France (Fr)
As for Uk.

Output United States (Us)
As for Uk.

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Chapter III*
Variations in Risk as a Cause of Fluctuations in Demand
– a theoretical model

1. Introduction

To live is to take risks. Today this truism has its mirror image in the commonplace use of stochastic variables in theoretical models where people's choices are shown to be affected by risk. A direct implication is that fluctuations in risk in principle could cause or contribute to fluctuations in economic activity – business cycles. The effects of shocks to perceived income risk could potentially be dramatic. In an inspiring paper Christina Romer (1990) suggests that the stock market collapse of 1929 caused a radical increase in consumers' uncertainty that lead them to delay purchases of durables. The consecutive fall in aggregate demand was large enough to be a critical factor leading to the Great Depression.

Irreversibility is a mechanism that could be responsible for a large sensitivity of aggregate demand to risk.[1] At least since Merton's path-breaking work some twenty years ago (e.g., Merton 1971), it has been known that the degree of risk is critical in the decision if and when to make an irreversible investment. During the last decade there has been a surge in the interest for irreversibility models, although most work has concerned firm behavior. The literature on irreversibilities has, however, taken the degree of risk as a constant parameter. If the effect of variations in risk is investigated, comparative statics is used. This may be appropriate if we are concerned with structural shifts in risk. If, on the other hand, we are interested in recurrent variations in risk in a business cycle context, comparative statics is unsatisfactory.

In this chapter I will construct a model in which a consumer wants to keep a continuously depreciating durable stock close to a target level that depends on current wealth. At the

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[1] An excellent introduction to and overview of the literature on irreversible investments is Pindyck (1990).

target level the instantaneous flow of utility is maximized. If the consumer adjusts the stock he has to pay an adjustment cost. This prevents the consumer from continuously keeping the durable stock at the target level. Without any friction, the durable stock would always be kept at the target level, which thus often is called the frictionless level.

Most work on irreversibility has assumed a continuous stochastic process for wealth, typically Brownian motion. Such processes may not be a very good description of how consumers' wealth evolves if we believe that shocks as unemployment, promotion or changes in the tax system are important. Instead of a continuous process I will use a Poisson model for wealth; the consumer is infrequently hit by a relatively large wealth shock. Risk is defined as the instantaneous probability that the agent is hit by a wealth shock. The risk level is in itself a stationary stochastic process that is known by the agents.

Optimal behavior in the model can be described as an *S-s* rule; the consumer sets a maximum positive and a maximum negative deviation of the durable stock from the target level. These deviations are called the upper and lower trigger levels. If the durable reach a trigger level, the consumer adjusts his durable stock to a level called the return point.

A shift to higher risk causes the agents to revise their *S-s* bands. The lower trigger point falls and the fall is larger the shorter the expected length of the high risk period. The return point, on the other hand, is less affected the shorter the expected high risk period. A shift to higher risk thus increases the difference between the lower trigger and the return point, and this increase is larger the shorter the expected length of the high risk period. The intuitive explanation for these findings is not that a short high risk period increases the value of waiting for risk to fall back to a lower level. Instead I will show that it is due to a qualitative difference between the problem of optimally choosing the triggers and the return point.

The decision about the return point is forward looking since it affects the flow of utility at least until the next adjustment. Assume that high risk periods are expected to be short relative to low risk periods and relative to the expected time between adjustments. Just after an adjustment risk is then expected to be low for most of the time until next adjustment regardless of whether risk is currently high or low. The optimal choice of return point is then insensitive to the current level of risk due to this decision's forward looking characteristics.

The decision about the triggers is, on the other hand, determined by an instantaneous arbitrage condition where the instantaneous value of waiting for new information is equalized

to the instantaneous gain from adjusting the stock of the durable. The more the return point falls when risk increases, the closer it comes to the target level, i.e., the point where *instantaneous* utility is maximized. The more the return point falls the larger is the immediate temptation to adjust. A short expected high risk period gives a small fall in the return point and the immediate temptation to adjust is thus smaller than for a long high risk period. The agent will thus wait longer before adjusting when risk is high if the high risk period is expected to be short.

A shift to high risk may cause a period of depressed markets for durables that is more severe the shorter the expected length of the high risk period. Models using comparative static results will underestimate the negative effects on demand of a risk shift since they implicitly assume the shift to be permanent.

In section 2, I present the model and provide some interpretations and motivations. The section ends with a solution to the model. In section 3, I apply ranges of parameter values to study the quantitative features of the model. Section 4 discusses likely consequences of relaxing some of the assumptions in the model and section 5 concludes the chapter.

2. The Model

2.1. The Formal Model

Consider an individual who solves the following problem

$$\min_{\{I_t,K_t\}_0^\infty}\left\{E_0\left[\int_0^\infty e^{-rt}\left(\frac{K_t^{\,2}}{2}+I_t c\right)dt\right]\right\}, \tag{2.1}$$

subject to

$$\begin{aligned} &K_0 = K,\ s_0 = s,\\ &I_t \in \{0,\Delta(0)\}, \end{aligned} \tag{2.2}$$

$$K_{t+dt} = \begin{cases} K_t - \delta dt + \varepsilon, & \text{with probability} \dfrac{\lambda(s_t)}{2} dt \text{ if } I_t = 0,\\ K_t - \delta dt, & \text{with probability} 1 - \lambda(s_t) dt \text{ if } I_t = 0,\\ K_t - \delta dt - \varepsilon, & \text{with probability} \dfrac{\lambda(s_t)}{2} dt \text{ if } I_t = 0,\\ \text{free}, & \text{if } I_t = \Delta(0), \end{cases} \tag{2.3}$$

$$\begin{aligned} &s_{t+dt} = \begin{cases} s_t, & \text{with probability } 1-\gamma(s_t),\\ \bar{s}_t, & \text{with probability } \gamma(s_t), \end{cases}\\ &s_t \in S = \{0,1\} \text{ and}\\ &\bar{s}_t = \overline{s_t \cap S}. \end{aligned} \tag{2.4}$$

K represents the log of the deviation of a capital stock from its frictionless optimum to be defined below. c is a transaction cost that has to be taken any time K_t is adjusted. When I_t is set to zero K_t is not adjusted and depreciates at rate δ. If I_t is set to $\Delta(0)$, the durable stock can be adjusted freely. $\Delta(\cdot)$ is the Dirac δ-function, which has a point mass of unity at zero and is zero everywhere else. This means that $I_t\,dt$ is equal to unity if the agent adjusts K_t and zero otherwise.

With instantaneous probability (arrival rate) $\lambda(s_t)$, K_t is hit by a zero mean shock with amplitude $\pm\varepsilon$. s_t indicates the state of the economy and takes the value 0 or 1. s_t follows a first order Markov process with transition probability $\gamma(s_t)$.

2.2. Model Interpretation and Motivation

The model can be thought of as representing the partial problem of controlling the (log of the) stock of a consumer durable, κ. The consumer must optimally balance the cost of frequent adjustments against being away from a frictionless target stock. To consider this problem in isolation we abstract from potential simultaneities. They could, for example, be due to a non-additive utility function where the marginal utilities of other consumer goods depend on the stock of the durable.

Take any expectation of the distribution of future flows of income. For any such expectation we can in principal calculate a level of κ that is optimal if no transaction costs existed: the frictionless durable stock. Assume that the expectation of future income only depends on current wealth and the current state. Let $\kappa^*(w_t, s_t)$ denote the frictionless durable stock where w_t is the wealth of the individual and s_t represents the current state of the economy.

By being away from $\kappa^*(w_t, s_t)$ the individual suffers a flow of utility loss. Define this flow as

$$C\left(\kappa, \kappa^*(w_t, s_t)\right) \tag{2.5}$$

with the natural properties

$$\begin{aligned} &C\left(\kappa^*(w_t, s_t), \kappa^*(w_t, s_t)\right) = 0, \\ &C_\kappa\left(\kappa^*(w_t, s_t), \kappa^*(w_t, s_t)\right) = 0, \\ &C_{\kappa\kappa}\left(\kappa^*(w_t, s_t), \kappa^*(w_t, s_t)\right) > 0. \end{aligned} \tag{2.6}$$

Suppose that the flow of disutility is a function of the relative deviation from $\kappa^*(w_t, s_t)$. (2.5) can then be written

$$C\left(\kappa - \kappa^*(w_t, s_t)\right). \tag{2.7}$$

Defining $K \equiv \kappa - \kappa^*$ and using (2.6), (2.7) has a second order Taylor approximation

$$\begin{aligned} C(K) &\approx C(0) + C_K(0)K + \frac{1}{2}C_{KK}(0)K^2 \\ &= \frac{1}{2}C_{KK}(0)K^2. \end{aligned} \tag{2.8}$$

Assume that the transaction cost is a constant c, measured in the same units as $C(K)$.[2] After normalizing C_{KK} to unity we then arrive at the instantaneous objective function in (2.1).

The shock ε in (2.3) represents a wealth shock. I want to focus on the effect of variations in risk so I let the shocks have an expected value of zero. The log of wealth is assumed to follow a Poisson process[3]

$$w_{t+dt} = \begin{cases} w_t - \varepsilon, & \text{with probability} \dfrac{\lambda(s_t)}{2} dt, \\ w_t, & \text{with probability} 1 - \lambda(s_t) dt \text{ and} \\ w_t + \varepsilon, & \text{with probability} \dfrac{\lambda(s_t)}{2} dt. \end{cases} \tag{2.9}$$

This means that with an instantaneous probability that depends on the current state, a wealth shock hits the agent. If this happens wealth increases or falls ε with equal probability. Risk is assumed to be higher in state 1 so $\lambda(1)>\lambda(0)$, i.e., the probability that the agent is hit by a wealth shock the next instant is higher in state 1. The current level of risk change is constant until the state shifts. By allowing the state shift probabilities, $\gamma(s_t)$, to be different in the two states, I allow the high risk and low risk periods to have different expected lengths.

Now I will make some simplifying assumptions about how the frictionless target stock depends on wealth and risk. First I set the wealth elasticity to unity, i.e.,

$$\frac{\partial \kappa^*(w_t, s_t)}{\partial w_t} = 1. \tag{2.10}$$

A wealth shock of ε then increases $\kappa^*(w_t, s_t)$ and reduces K by the same amount ε, i.e., $dW=\varepsilon \Rightarrow d\kappa^*=\varepsilon \Rightarrow dK = -\varepsilon$.

In reality we would expect that the frictionless target stock of durables depends on risk and thus on the state s. An obvious channel is variations in precautionary savings. A state shift from 0 to 1 increases the volatility of wealth, which raises precautionary savings if the

[2] If the marginal utility of wealth is decreasing this implies that the transaction costs in monetary terms is increasing in wealth. This would, for example, be the case if the transaction costs represents search time and the monetary value of time also increase in wealth.

[3] In section 4.1 ε is assumed to have a normal distribution.

individual is prudent. This will in general directly affect all types of consumption including the services from the durable.[4]

In this chapter I will disregard the precautionary savings effect. The aim of this paper is to study the irreversibility effect. I want to see if variations in perceived risk can have quantitatively non-negligible effects on demand that *do not* stem from wealth or precautionary savings effects but rather from the particularities of irreversible investments. To do this I will assume that κ^* is unaffected by variations in risk. Together with the previous assumption of a unitary wealth elasticity we then have

$$\kappa^*(w_t, s_t) = \omega_\kappa + w_t \tag{2.11}$$

where ω_κ is the log of the share of wealth that would be held in the form of durables in absence of irreversibilities. This assumption is not meant to be realistic, but is done in order to isolate the irreversibility effect. The previous assumptions and (2.11) imply that if κ depreciates at rate δ, K will follow the stochastic process described in (2.3).

2.3. Model Solution

To solve the model I will first assume that ε is so large that in the event of a shock, it is always optimal to adjust K_t immediately. As will be shown below, if this is not always satisfied, the problem becomes much more difficult to solve analytically. In appendix 3, I will present an algorithm for solving the problem numerically for more general assumptions about the shock. Numerical solutions for the case when ε has a normal distribution will follow in section 4.1. The large shock assumption has an important (and unrealistic) implication in this model: an increase in the risk due to an increase in the amplitude of the shock has no effects on optimal behavior. The important risk parameter is instead the arrival rate, i.e., the instantaneous probability of a shock.

The only state variables in this problem are K_t and s_t, and the optimal policy will consist of two time-independent but state-dependent bands for K_t. Inside these bands no adjustment will be done so K is left to depreciate on its own. When a band limit (also called trigger point) is reached, K is adjusted to a return point. Due to that the transaction cost is a fixed constant the return point is independent of the level of K immediately before adjustment.

[4] In a frictionless world with perfect capital markets durable services are not different from non-durables and should thus not behave differently.

Both the trigger points and the return point will however depend on the current state. If a shift in the state occurs, the new band is applied and if K is below the new lower limit, it is adjusted to the new return point.

Define the lower and upper band trigger points as $\underline{U}_s$ and $\overline{U}_s$, and define the return point as a_s. Note that the large shock assumption and negative depreciation imply that K is never above the higher of the return points for a strictly positive time interval. Only a wealth shock could move the agent above the upper return point and then K is by assumption immediately adjusted to the current return point.

For an optimal control we can define the associated total cost function $V(K_t, s_t)$. It gives the expected discounted sum of future costs given optimal behavior. $V(K_t, s_t)$ has a recursive structure that depends on K_t. For some values of K, a shift in s will lead to an immediate adjustment. As is intuitive and will be shown to be correct later, if the current state is high risk and K is close to but not equal to the lower adjustment point, a shift to low risk will reduce the value of waiting and cause immediate adjustment. For an interior region this will not happen. Inside this region $V(K,s)$ has the following recursive structure:

$$\begin{aligned} V(K_t,s) &= \frac{K_t^2}{2}dt + (1-rdt)\mathrm{E}_t\left(V(K_{t+dt}, s_{t+dt})\right) \\ &= \frac{K_t^2}{2}dt + (1-rdt)\left(\begin{array}{l} \overbrace{V(K_t,s)}^{1} - \overbrace{\delta dt V_K(K_t,s)}^{2} + \\ \overbrace{\lambda(s)dt\left(V(a_s,s)+c-V(K_t,s)\right)}^{3} + \\ \underbrace{\gamma(s)dt\left(V(K_t,\bar{s})-V(K_t,s)\right)}_{4} \end{array}\right). \end{aligned} \tag{2.12}$$

The second numbered term within the large brackets in (2.12) is due to the depreciation of K. The third term comes from the possibility of a wealth shock, in which case K by assumption immediately is adjusted to a_s. The probability of a wealth shock during the time period dt is $\lambda(s)dt$. The fourth term, lastly, is due to the possibility of a state shift. A state shift occurs during dt with probability $\gamma(s)dt$ and causes expected total costs to shift from $V(K_t, s)$ to $V(K_t, \bar{s})$.

Setting $dt^2=0$ and dividing by dt we get the following system of differential equations:

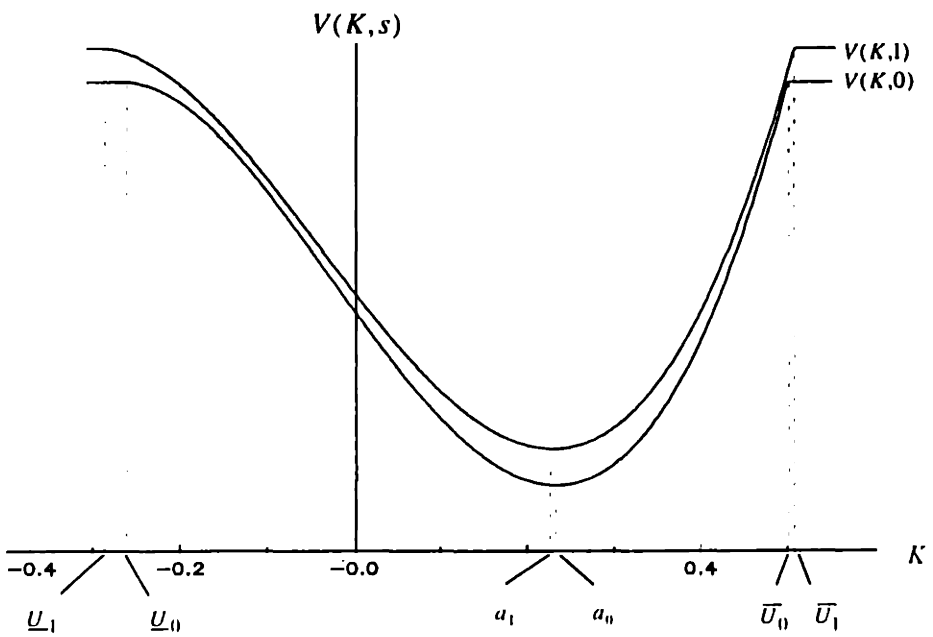
$$(r+\lambda(s)+\gamma(s))V(K,s) - \gamma(s)V(K,\bar{s}) + \delta V_K(K,s) = \frac{K^2}{2} + \lambda(s)\left(V(a_s,s)+c\right). \tag{2.13}$$

If K is equal to $\underline{U}_0$ it will immediately be adjusted if the state is 0. Guess that $\underline{U}_1 < \underline{U}_0$.[5] If K is in the region $[\underline{U}_1, \underline{U}_0]$, a switch from state 1 to zero then leads to immediate adjustment. The function $V(K,0)$ is thus constant for $K \in [\underline{U}_1, \underline{U}_0]$ and the system of differential equations degenerates to:

$$
\begin{aligned}
V(K_t,0) &= V(\underline{U}_0,0), \\
V(K_t,1) &= \frac{K_t^2}{2}dt + (1-rdt)\mathrm{E}_t\left(V(K_{t+dt}, s_{t+dt})\right), \\
&= \frac{K_t^2}{2}dt + (1-rdt)\begin{pmatrix} V(K_t,1) - \delta dt V_K(K_t,1) + \\ \lambda(1)dt\left(V(\overline{U}_1,1) + c - V(K_t,1)\right) + \\ \gamma(1)dt\left(V(\underline{U}_0,0) - V(K_t,1)\right) \end{pmatrix}.
\end{aligned}
\tag{2.14}
$$

For some parameter values it could be the case that $a_s > \overline{U}_{\bar{s}}$. If this is the case, a switch of state could induce downwards adjustments and we get a degenerate system of differential equations also above the lower of the two upper triggers. For all parameter values I have used below, however, this is not the case, i.e., $a_s < \overline{U}_{\bar{s}}, s = \{0,1\}$.

Figure 1 Total Cost Function



The problem is now to find the solutions to the systems of differential equations. This has been done algebraically and the steps are shown in appendix 1. Lastly we have to solve

[5] That this is true is verified when a full solution is found.

for the unknown band limits and the integration constants using numerical methods. Numerical results are presented in section 3. The total cost function has a graphical appearance like in figure 1.

2.4. Interpretations of Optimal Adjustment

From the principles of optimality we know that when K hits the lower trigger $\underline{U}_s$, the agent must be indifferent between switching now and waiting one instant and then doing what is optimal. Note that if s has shifted, it may be optimal to wait further before adjusting. The condition that the expected instantaneous cost of waiting equals the expected instantaneous cost of adjusting if $K = \underline{U}_s$ can be written in the following way.

Proposition 1

$$\frac{\underline{U}_s^2 - a_s^2}{2} = rc + \lambda(s)c + \gamma(s)\left(c + V(a_s, \bar{s}) - V(\underline{U}_s, \bar{s})\right) \tag{2.15}$$

where the L.H.S. can be interpreted as the instantaneous cost of waiting[6] and the R.H.S. as the instantaneous value of waiting. The value of waiting consists of three non-negative terms.

Proof in appendix 2

It is helpful to disentangle the parts of the value of waiting. The first term of the R.H.S. of (2.15) is the direct cost reduction by delaying adjustment. The second term comes from the possibility of a wealth shock during the next instant. If the shock occurs the agent has saved one adjustment cost c by not adjusting now. The third term comes from the possibility of the state s switching. If s switches to $\bar{s}$ during the next instant and the agent waits until after that instant he can do what is optimal then. This creates an option value of waiting that must be non-negative. This option value can be split up in two conceptually different parts. The first is the option to wait further before an adjustment is done if the state should shift. The other is the possibility to take advantage of the information that a state shift has taken place when adjusting and adjust to what is optimal in the new state. These two parts can be shown explicitly by using the value matching condition in (A.15) to rewrite the third term of (2.15) as

[6] Or the temptation to adjust, to use the phrasing in the introduction.

$$\gamma(s)\left(\overbrace{\left(V(\underline{U}_{\bar{s}},\bar{s})-V(\underline{U}_{s},\bar{s})\right)}^{1}+\overbrace{\left(V(a_{s},\bar{s})-V(a_{\bar{s}},\bar{s})\right)}^{2}\right) \tag{2.16}$$

where the first term is the option to wait further and the second to change return point in case of a state shift.

3. Quantitative Results

In the following sections I will present quantitative results from the model. I will focus on how the behavior is affected by state shifts and on how this depends on how much risk increases ($\lambda(1)$-$\lambda(0)$) and the length of the high risk periods ($\gamma(1)^{-1}$). For the other parameters I have chosen the values given in table 1.

Table 1 Parameters

Parameter	% per year
$\lambda(0)$	1
r	5
δ	10
$\gamma(0)$	40

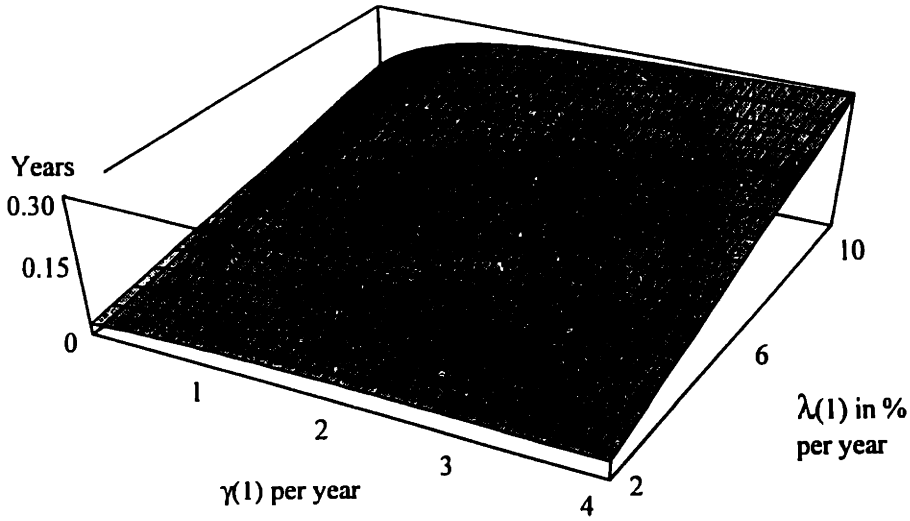
Note that the value for $\gamma(0)$ implies that a low risk period on average is 2.5 years long. In addition I have chosen c so that in state 0 the agents let there durables depreciate 5 years until they upgrade. This implies that c will take different values for different combinations of $\lambda(1)$ and $\gamma(1)$ and is done to facilitate comparison. For a given combination of parameters, c is always the same in the two states.

3.1 Shifts in Bandwidth and Trigger Points

The large shock assumption implies, as already noted, that K almost always will be in the region between the lower of the two lower trigger points and the higher of the return points. Only at the zero measure points in time when a wealth shock hits is K outside this region. Immediately after the shock K is adjusted back to the return point. The relevant band width to study is then $(\underline{U}_s, a_s)$.

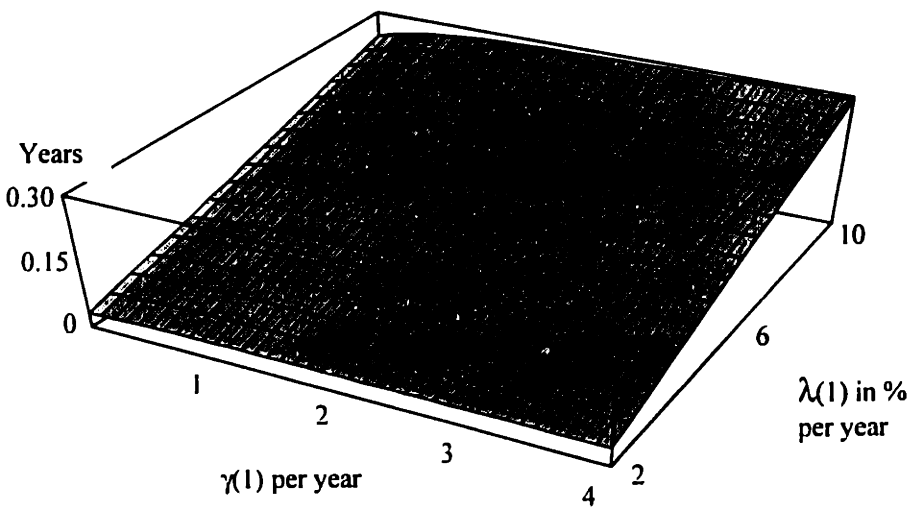
In figure 2 I show the change in bandwidth, $(a_1 - \underline{U}_1) - (a_0 - \underline{U}_0)$, for different values of $\lambda(1)$and $\gamma(1)$. I let $\lambda(1)$ vary between 2 and 10% per year and $\gamma(1)$ between 0.1 and 4 per year. The values for $\gamma(1)$ imply that the average high risk periods are between 10 and 0.25 years. Figure 2 shows that the widening is increasing in both $\lambda(1)$ and $\gamma(1)$. If the expected length of the high risk period is short, the band widens more for a given increase in risk. For $\lambda(1)$=10% per year, the band widening is between 2.0 and 3.8 months, i.e., between 3.4 and 6.3% of the bandwidth in state 0.

Figure 2 Change in Bandwidth After Shift to High Risk



While the increase is almost linear in $\lambda(1)$, the relationship between the length of the delay period and $\gamma(1)$ is non-linear. The figure suggests that the marginal effect of raising $\gamma(1)$ is strong when $\gamma(1)$ is fairly low, but decreases towards zero as $\gamma(1)$ increases. We should note here that analysis based on comparative statics on λ is equivalent to letting $\gamma(1)$ be 0. The result from such analysis thus underestimates the effects of a sudden increase in risk if the high risk period is expected to be temporary.

Figure 3 Fall in Lower Trigger Point after a Shift to High risk

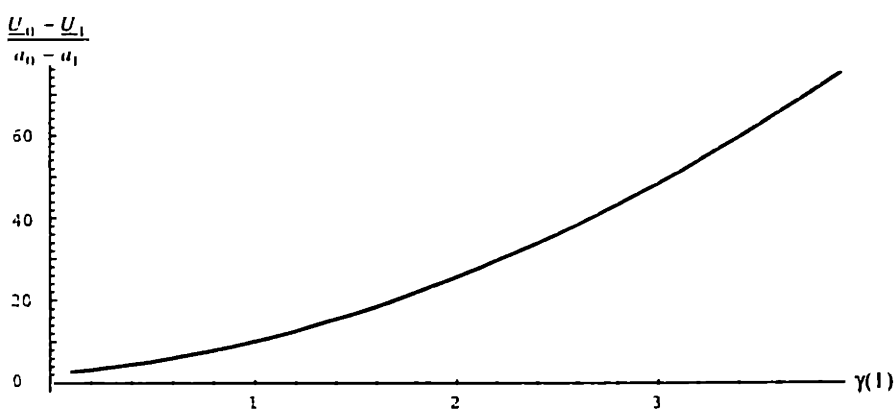


In figure 3 I show the fall in the lower trigger point after a shift to the high risk state. The fall increases in both $\lambda(1)$ and $\gamma(1)$. Also here is the effect almost linear in $\lambda(1)$ but con-

cave in $\gamma(1)$. For $\lambda(1) = 10\%$ the delay is between 3.2 and 3.8 months, i.e., 5.3 to 6.4% of the bandwidth in state 0.

We have thus seen that shorter high risk periods cause the lower trigger to fall more and the band to widens more. The bandwidth is more sensitive to the length of the high risk periods than the lower trigger is. This implies that the fall in the return point *increases* with the expected length of the high risk period. This is illustrated in figure 4 where I plot the ratio $(\underline{U}_0 - \underline{U}_1)/(a_0 - a_1)$, which increases in $\gamma(1)$. When the expected duration of the high risk period decreases, $\underline{U}_0 - \underline{U}_1$ increases while $a_0 - a_1$ decreases. In the figure we see that the fall in the lower trigger is much bigger than the fall in the return point except when the high risk period is expected to be very long.

Figure 4 Ratio of Changes in Lower Trigger and Return Point



3.2. Aggregate Effects

The stochastic process for K and the optimal S-s policy generates an ever-changing distribution of agents over their S-s bands. If the state changes permanently the distribution eventually settles down to a stationary distribution. Using this we may calculate an average K. Define the stationary density for a permanent state s as $f_s(\cdot)$. For a small time interval dt and for any K inside the S-s band stationarity implies that

$$\begin{aligned} & f_s(K) = f_s(K + \delta dt)(1 - \lambda(s)dt) \\ & \Rightarrow \frac{f_s(K + \delta dt) - f_s(K)}{dt} = f_s(K + \delta dt)\lambda(s). \end{aligned} \tag{3.1}$$

Letting $dt \to 0$ we get

$$\frac{f_s'(K)}{f_s(K)} = \frac{\lambda(s)}{\delta}$$
$$\Rightarrow f_s(K) = c_1 e^{\lambda(s)k/\delta}, \tag{3.2}$$
$$\Rightarrow F_s(K) \equiv \int_{\underline{U}_s}^{K} f_s(\kappa)d\kappa = \frac{\delta c_1 e^{\lambda(s)k/\delta}}{\lambda(s)} + c_2$$

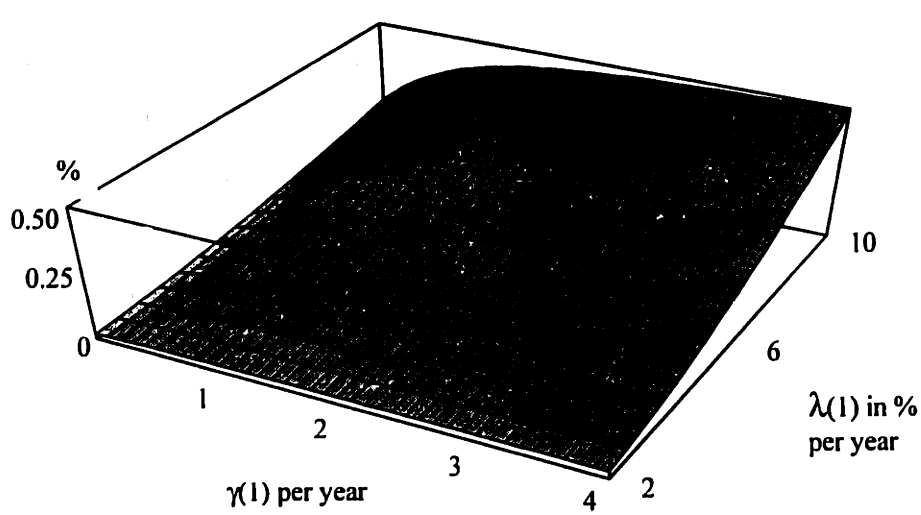
where the integration constants c_1 and c_2 are determined from $F_s(\underline{U}_s) = 0$ and $F_s(a_s) = 1$.

The long run change in average K, $\Delta\overline{K}$, after a permanent shift to the high risk state may then be calculated as

$$\Delta\overline{K} = \int_{\underline{U}_1}^{a_1} \kappa f_1(\kappa)d\kappa - \int_{\underline{U}_0}^{a_0} \kappa f_0(\kappa)d\kappa. \tag{3.3}$$

$\Delta\overline{K}$ is plotted in figure 5. Here we see that average K is *higher* in state 1 with up to 0.5%. The increased risk causes the distribution to be more concentrated at higher levels of K. This more than offsets the fall in the lower trigger and the return point.

Figure 5 Long Run Change in Average *K* after shift to High Risk



The effect of increased risk on the flow of purchases in *S-s* models can generally be separated into two parts. First, increased risk leads to a band widening. Second, increased risk cause agents to hit the trigger points more frequently for a given width of the band. Which effect is going to dominate, both in the long and short run, is generally ambiguous. In the present model these two effects are easily disentangled. The flow of consumers hitting the

lower trigger per unit of time equals the density at the lower trigger times the depreciation rate. In addition $\lambda(s)$ consumer per unit of time are hit by wealth shocks and adjust. The long run change in the share of consumers adjusting per unit of time can then be calculated from

$$\left(f_1(\underline{U}_1) - f_0(\underline{U}_0)\right)\delta + (\lambda(1) - \lambda(0)) \tag{3.4}$$

where the first term is due to band widening and the second to more frequent shocks.

(3.4) can also be used to find the change in average flow of adjustments per unit of time when the state shifts stochastically but not permanently. Define $f(\cdot|s_t = S_t)$ as the density conditional on s_t=S_t. The difference in the average share of consumers adjusting per unit of time between the two states is then

$$\left(f(\underline{U}_1|s_t = 1) - f(\underline{U}_0|s_t = 0)\right)\delta + (\lambda(1) - \lambda(0)) \tag{3.5}$$

We may likewise use other conditional distributions, for example $f(\cdot|s_t = 1, s_{t-1} = 0)$ to find the change in purchases one unit of time after a shift to the high risk state. The conditional densities may be complicated to calculate algebraically. They are, however, easy to find by simulation.

Right after a shift to the high risk state the density at the lower trigger is zero. By using this and (3.5) we can calculate a necessary and sufficient condition for a shift to high risk on average causing an initial fall in the flow of adjustments.

Proposition 2

The aggregate flow of adjustments on average initially fall after a shift to the high risk state iff

$$\lambda(1) < f(\underline{U}_0|s_{t+1} = 1, s_t = 0)\delta + \lambda(0). \tag{3.6}$$

Proof follows directly from (3.5).

In *Proposition 2* we see that the initial effect of a shift to high risk on average may be to increase adjustments if $\lambda(1)$ is large enough. By noting that the average density at the lower trigger generally is lower than the inverse of the bandwidth we can calculated the following approximation. If $\lambda(1) > (a_0 - \underline{U}_0)^{-1}\delta + \lambda(0)$ the flow of adjustments increases on average after a shift to higher risk. Set for example the bandwidth to 50%, depreciation to 10% and the arrival rate of shocks in state 0 to 1% per year. An increase in the arrival rate of shocks to more than 21% will then on average necessarily immediately increase the flow of adjustments.

We can also note that if the average length of a high risk period is short, i.e., $\gamma(1)$ is high, then this will tend to lead to a low value of $f(\underline{U}_1,1)$. This since the density at the new lower trigger is zero just after a state shift. Short expected high risk periods will thus work in the direction of causing average purchases to be lower in the high risk state.

3.3. Quantification of the Value of Waiting

An important finding in the previous section was that the length of the delay period is longer if the high risk period is short. A reasonable intuitive interpretation of this could be that it pays to wait for a state shift if the high risk period is expected to be fairly short. This interpretation, however, turns out to be incorrect. Below I will show that the value of waiting for a state shift is much larger in state 0 than in state 1.

In *Proposition 1* I showed that the instantaneous value of waiting can be expressed as the sum of three non-negative terms;

$$rc + \lambda(s)c + \gamma(s)\big(c + V(a_s,\bar{s}) - V(\underline{U}_s,\bar{s})\big) \qquad (2.15)'$$

where the first term is the value of delaying due to the discount factor. The second term is due to the possibility of a wealth shock and the third the value of waiting for a state shift.

Using the numerical results from the previous section and setting $\lambda(1)$ to 10% per year, we can quantify the different parts of the value of waiting. First, we should note that the two first terms are proportional to c. Assuming that r and λ have the same order of magnitude, these two terms also have the same order of magnitude. Clearly the first is independent of the state s while the latter increases when the state shifts to a state with higher λ.

In figure 6 I have graphed $c^{-1}\gamma(0)\big(c + V(a_0,1) - V(\underline{U}_0,1)\big)$ as a function of $\gamma(1)$. This is the third term of the value of waiting in state 0, normalized with c so it can be interpreted as a rate of c. The normalization facilitates comparison with the other terms. In state 1 the value of waiting for a state shift is at least ten times smaller than in state 0, and thus clearly negligible. The reason for the substantial state dependence is that the value of waiting for a state shift is dominated by the value of having the option to wait further before adjusting if the state should shift. This option value is, however, as discussed above, only of value in state 0, when a state shift makes it optimal to wait further before adjusting.

Figure 6 Third Term of the Value of Waiting in State 0

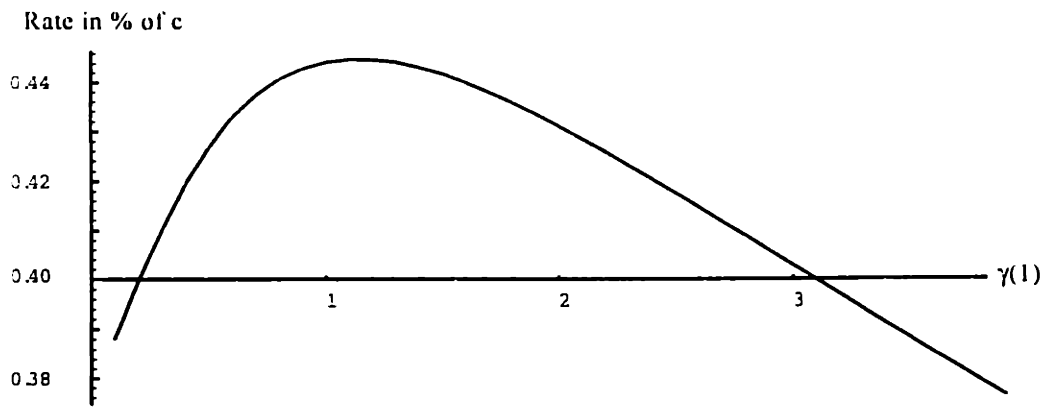
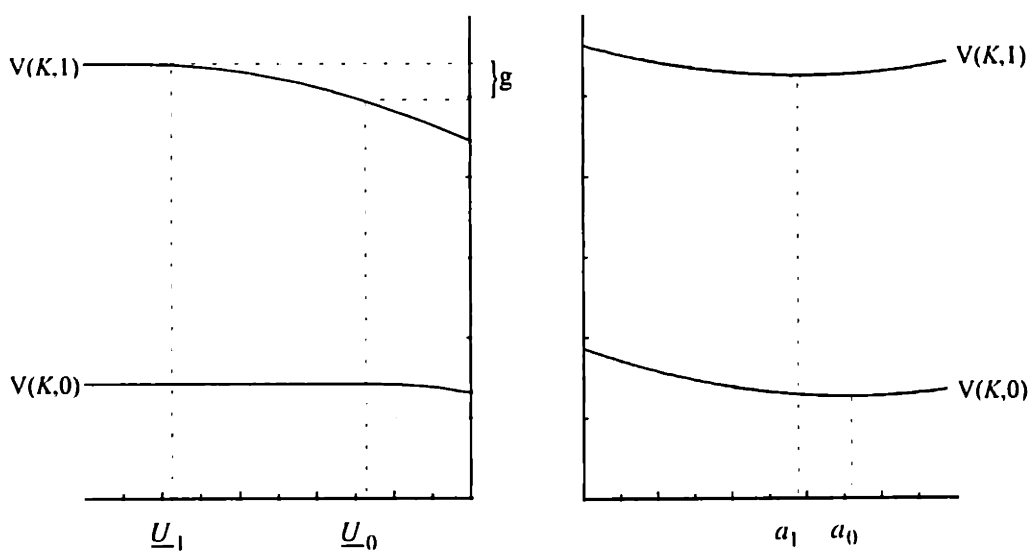


Figure 7 The Parts of the Third Term of the Value of Waiting



The value of waiting given a shift in the state is depicted in figure 7, which is a magnification of figure 1 around the lower trigger and the return point. The value of having the option to wait in case of a shift from state 0 to 1 equals $V(\underline{U}_1,1)-V(\underline{U}_0,1)$ and can be seen in the left panel as the distance g. In case of a shift from state 1 to 0, it equals $V(\underline{U}_0,0)-V(\underline{U}_1,0)$ which is zero. In the right panel of figure 7, which is drawn in the same scale as the left, we see that the total cost functions are fairly flat around return points. We also know that the difference between the return points in the two states is small unless $\gamma(1)$ is very low. This implies that the option value of being able to use the new information that the state has changed when choosing which point to adjust to is small.

For large $\gamma(1)$ the value of waiting in state zero coming from the possibility of a shift in the state becomes small. The difference in instantaneous value of waiting in the two states then mostly comes from the second term in (2.15)'. This leads to the following results.

Results

i. As the expected length of the high risk period becomes short, the difference in the instantaneous value of waiting between the two state at their respective lower triggers is $\geq(\lambda(1)-\lambda(0))c$.

ii. For tested parameter values and $\gamma(1)$ not too small the difference in instantaneous value of waiting between the two states at their respective lower trigger is approximately equal to $(\lambda(1)-\lambda(0))c$.

iii.

$$\lim_{\gamma(1)\to\infty} V(K,1)=V(K,0)$$

Proofs of result i and iii in appendix, result ii follows from the numerical results.

Using *result ii* and (2.15) we may compute a linear approximation to the changes of the band limits around the limits in state 0 as

$$(\underline{U}_1-\underline{U}_0)\underline{U}_0+(a_0-a_1)a_0\approx(\lambda(1)-\lambda(0))c. \tag{3.7}$$

In section 3.1 we noted that the difference in return points decreased in $\gamma(1)$. From (3.7) then follows that as $\gamma(1)$ increases the difference in lower trigger points must increase. That the return points become closer as the expected length of the high risk period decreases can be understood in the following way. The lower trigger point is determined from equalizing instantaneous flows of costs and benefits. The decision to choose the return point is, on the other hand, a forward looking decision in the following sense. Define T as the stochastic time until next optimal time of adjustment. We can then write the problem to choose the return point as follows

$$\min_{a_{s_t}} E_{T,t}\left[\int_0^T\left(a_{s_t}-\delta\tau\right)^2 e^{-r\tau}d\tau+e^{-rT}V\left(a_{s_t}-\delta T,s_T\right)\right] \tag{3.8}$$

where $E_{T,t}$ denotes the expectation over all T conditional on s_t and a_s. Clearly (3.8) is a forward looking decision. The flow of costs every instant until T depends on a_s and also T depends on a_s. In light of *result iii* it is reasonable to conjecture that as $\gamma(1)$ increases the

problem (3.8) becomes less state dependent in the sense that the expectations over T in state 1 converge to the expectations in state 0. The solutions to problem (3.8) for the two states should then also converge.

4. Some Extensions

4.1. Smaller Shocks

To be able to solve the model I have assumed that shocks to wealth have an amplitude large enough to cause immediate adjustment. Looking at figure 1 we see that this implies that a negative shock must be larger than $\max_s\{\overline{U}_s - \underline{U}_s\}$, and positive shocks have to be larger than $\max_s\{a_s - \underline{U}_s\}$.

Now assume that the wealth shock has a normal distribution with a standard deviation σ. The differential equation for $V(K,s)$ then becomes

$$\begin{aligned} V(K_t,s) &= \frac{K_t^2}{2}dt + (1-rdt)\mathrm{E}_t\left(V(K_{t+dt},s_{t+dt})\right) \\ &= \frac{K_t^2}{2}dt + (1-rdt)\begin{pmatrix} V(K_t,s) - \delta dt V_K(K_t,s) + \\ \lambda(s)dt\left(\int_{-\infty}^{\infty} V(K_t+\varepsilon,s)d\Phi(\varepsilon,\sigma) - V(K_t,s)\right) + \\ \gamma(s)dt\left(V(K_t,\bar{s}) - V(K_t,s)\right) \end{pmatrix}. \end{aligned} \tag{4.1}$$

where $\Phi(\cdot,\sigma)$ denotes the normal distribution function with standard deviation σ. Setting dt^2 to zero and dividing by dt, (4.1) can be rewritten as

$$\begin{aligned} &(r+\lambda(s)+\gamma(s))V(K,s) - \gamma(s)V(K,\bar{s}) + \delta V_K(K,s) \\ &= \frac{K^2}{2} + \lambda(s)\int_{-\infty}^{\infty} V(K+\varepsilon,s)d\Phi(\varepsilon,\sigma) \end{aligned} \tag{4.2}$$

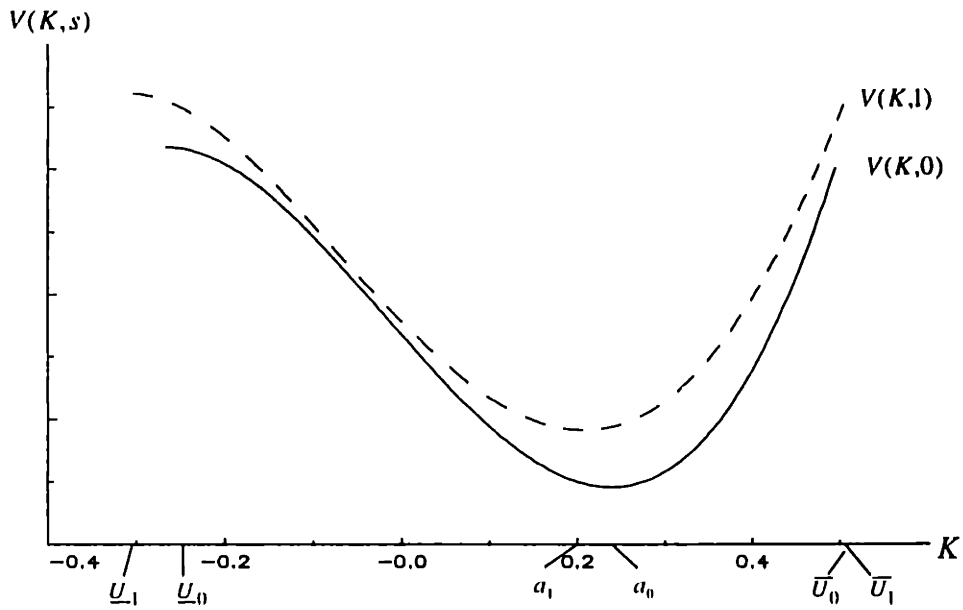
This is a differential equation with (infinitely many) leads and lags. This function can be solved numerically. The solving procedure is described in appendix 3. I use the same values as in table 1 and set σ to 20%, $\gamma(1)$ to 40% and $\lambda(1)$ to 10% per year.[7] I find that now lower trigger point falls by 1.3 months (2.2% of the bandwidth $[a_0, \underline{U}_0]$) and the return point with 0.7 months (1.2%). Approximately the same figures are found for $\gamma(1) = 2$.

The fall in the lower trigger point is smaller than for the same parameters in the large shock model. But here the standard deviation of the wealth shocks is much lower than in the large shock case. In the latter $(a_1 - \underline{U}_1) \approx 0.8$. This implies that the shock in this model im-

[7] As in the large shock case, the transaction costs c is chosen to get $a_0 - \underline{U}_0 \approx 0.5$.

plicitly is assumed to have four times as high standard deviation as in the case with normal shocks. A fall of the lower trigger point by 1.3 months occurs for $\lambda(1) = 4\%$ in the large shock model.

Figure 8 Cost Functions for Normal Shocks



By increasing the arrival rate to $\lambda(1)$ to 40% per we get same instantaneous standard deviation as for $\lambda(1) = 10\%$ per year in the large shock model. The cost functions for the normal shock case and $\lambda(1) = 40\%$ per year are shown in figure 8. Here the lower trigger falls by almost 5 months and the return point by 4 months after a state shift. Both these effects are larger than in the large shock case.

In the previous section we found that the delay effect was approximately linear in the arrival rate of shocks. This appears to be true also in the normal shock case. By setting $\lambda(1)$ to 20% per year the delay effect is 2.4 months.

The conclusion of the numerical results in this section is thus that quantitatively similar results are achieved if we relax the large shock assumption.

4.2. Other Instantaneous Cost Functions

I have used an even (symmetric around zero) quadratic specification of the instantaneous cost associated with a deviation from κ^*. Using a general form for the instantaneous cost, $C(K)$, (2.15) now reads

$$C(\underline{U}_s) - C(a_s) = rc + \lambda(s)c + \gamma(s)\big(c + V(a_s, \bar{s}) - V(\underline{U}_s, \bar{s})\big). \tag{4.3}$$

Assume that the dominating effect of a shift from $\lambda(0)$ to $\lambda(1)$ as in the previous cases comes from the second term of the R.H.S. The value of waiting then increases almost linearly in λ. The effect that will have on optimal behavior clearly depends on the slope of the instantaneous cost function at the triggers and the return point. It is possible to think of a function with extremely steep slopes. An example is a capital good that is almost necessary for human survival and has the property that it works fine until it breaks down. When it does break down it is of no use at all.[8] In this case the increased value of waiting has a very small effect on delaying purchases of new capital goods. The slope of the instantaneous cost function is of course an empirical question. However, it seems safe to say that there exist many capital goods with characteristics such that a delay in the replacement of the capital good of a few months has less detrimental consequences than in this example.

I showed above that for the symmetric quadratic case and $\gamma(1)$ not too small, the change in the value of waiting is approximately equal to c times the change in risk. I also showed that most of the change in bandwidth comes from a shift in the lower trigger. If this is true in general we may approximate the effects on the inaction band by differentiating (4.3) . We then get that

$$\underline{U}_1 - \underline{U}_0 \approx \frac{(\lambda(1) - \lambda(0))c}{C'(\underline{U}_0)}. \tag{4.4}$$

To get comparability between different instantaneous cost functions we should normalize with their values at $\underline{U}_0$. Doing this we may calculate the normalized slope as

$$\frac{C'(\underline{U}_0)}{C(\underline{U}_0)}. \tag{4.5}$$

Using (4.4) and (4.5), we find that a linear instantaneous cost function (e.g., $\mu|K - K^*|$) would give approximately twice as large effects as the quadratic, while a cubic (e.g., $\mu\left|(K - K^*)^3\right|$) would give $2/3$ of the effects. The qualitative behavior of the model does thus not seem very sensitive to the exact specification of the instantaneous cost function.

To get some feel for the relative slope of the quadratic cost function I have compared it to the Cobb-Douglas utility function. Define the following instantaneous cost functions

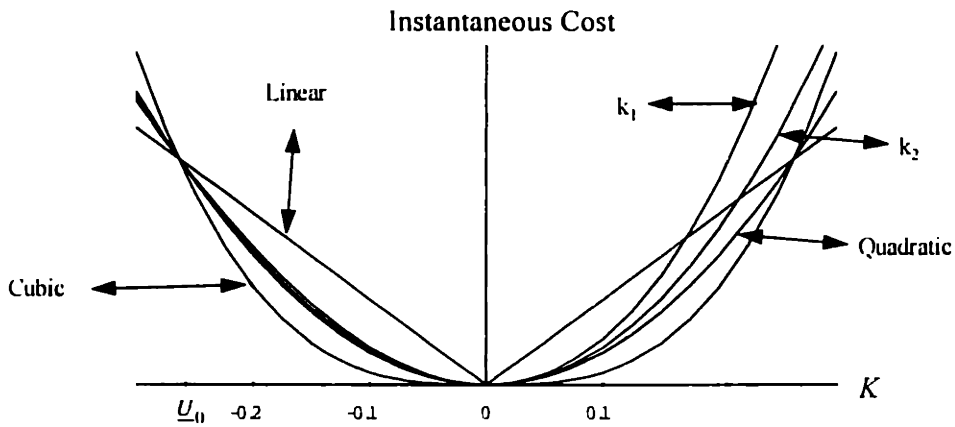
---

[8] In this case it may, however, seem more appropriate to view this as a form of non-linear depreciation.

$$C_1(K) \equiv (K^*)^{1/2}(2-K^*)^{1/2} - K^{1/2}(2-K)^{1/2}, \qquad K^* = 1$$
$$C_2(K) \equiv (K^*)^{1/10}(10-K^*)^{9/10} - K^{1/10}(10-K)^{9/10}, \qquad K^* = 1 \tag{4.6}$$

where K denotes levels while italics K, as above denotes logs. Normalizing these functions with their values at $\underline{U}_0$ as in equation (4.5)[9] and plotting them together with the linear, quadratic and cubic cost functions, we get figure 9.

Figure 9 Other Instantaneous Cost Functions



The relative slopes of the quadratic, k_1, k_2, the cubic, and the linear instantaneous cost functions at $\underline{U}_0$ are {-7.6, -7.2, -6.7, -11.4, -3.8} respectively. In figure 9 we see that the quadratic specification looks quite the same as the Cobb-Douglas specifications in terms of the slope around $\underline{U}_0$. My conjecture is thus that the quantitative results discussed above are fairly robust to the specification of the instantaneous cost function.

4.3. More states

In the model the stochastic behavior of risk is represented by two states, low and high risk. Certainly, we would like to know whether this is crucial for the results. Formally the model could handle any finite number of states. Each state, however, generates one extra differential equation. Addition of states also complicates finding the regions where the system of differential equations degenerates because of state switch induced adjustments as in (2.14). More states are thus practically difficult to handle. On the other hand we may draw

[9] To calculate $\underline{U}_0$ and a_0 I have used the same parameters as in table 1 with $\lambda(1)$ set to 10% and $\gamma(1)$ to 0.4.

some conclusions by once again looking at (2.15). With more states the value of waiting can be written

$$rc + \lambda(s)c + \sum_{t \neq s} \gamma_{s,t}\left(c - \left(V(\underline{U}_s, t) - V(a_s, t)\right)\right), \tag{4.7}$$

where $\gamma_{s,t}$ denotes the instantaneous probability of a shift from state s to t.

Let the added states in (4.7) represent intermediate levels of risk. We may then conjecture that the value of waiting continues to be dominated by the first two terms in (4.7). The value of waiting is then approximately linear in the current level of risk. A shock to the volatility of wealth should then have approximately the same effect as in the two-state model. A difference would of course be that the fluctuations of demand could be smoother with more states. A shift from high to low risk could occur through a succession of smaller shifts.

5. Concluding Comments

Using a model and some numerical examples I have shown that increased risk of large wealth shocks may have quantitatively important effects on purchases of consumer durables. An increase in the arrival rate of wealth shocks from 1 to 10% per year causes the lower trigger point to fall by 5 to 6%. This implies that purchases of durables, except what is induced by wealth shocks, cease for 3 to 4 months in the model. These large effects arise although the increased risk is assumed to have no impact on the expected income of the consumers nor on their willingness to engage in precautionary savings.

We must, however, acknowledge that a tenfold increase in risk is a very large increase. The shocks are also assumed to be large in amplitude; it is assumed that the shock is big enough to always cause immediate adjustment. If I reduce the standard deviation of the shock to 20% of wealth, which still is a very large shock I need to increase the arrival rate to 40% per year to get delay effects of comparable size. It seems unreasonable that increases in volatility of that magnitude could be found in aggregate data. Some idiosyncratic risks, like unemployment risk, may, however, very well be very volatile. A conclusion from the numerical results is thus that variations in risk may be of substantial importance, but only if the level of individual risk is very volatile.

Risk is resolved continuously in the model. This implies that increased risk immediately leads to increased arrival rates of wealth shocks. This tends to offset the negative impact on purchases caused by the fall in the lower trigger. A shift to a very high arrival rate of wealth shocks may even immediately increase purchases.

An important finding is that the effects of a shock to risk increase if the high risk period is expected to be short. The intuitive explanation is that shorter expected high risk periods reduce the immediate temptation to adjust by increasing distance between the return point and the frictionless target. This result also implies that the standard procedure of using comparative statics on the risk parameter underestimates the effects on demand if the high risk period is expected to be temporary.

A similar loophole is to look at the long run effects by increases in risk that would result if the shift to high risk is permanent. A shift to high risk may shift the lower trigger point downwards considerably. This will have a strong negative impact on purchases, but the effect will diminish over time. If high risk periods on average are short the behavior of pur-

chases in high risk periods is more likely to be dominated by this negative transitory effect. Even if a permanent shift to risk eventually causes the rate of adjustments to go up, average purchases in high risk periods may be lower if the level of risk shifts recurrently.

I have attempted to describe a possible mechanism capable of generating non-negligible variations in demand. The ultimate aim is to integrate irreversibility effects in a more general model of the business cycle. In such a model we need to specify supply. In principle there is nothing in my model that makes it specific to demand, rather its origin in inventory and option theory makes it as useful in supply and production. To construct a general equilibrium (or disequilibrium model) is, however, well beyond the scope of this chapter.

A last issue I want to mention is the question of where the variations in risk come from. An obvious candidate is unemployment risk, and also political instability, potential technical breakthroughs, devaluations or weather changes may be important. As is suggested by these examples, risk is composed of many different parts of which some should be endogenous to a business cycle model while others can be thought of as exogenous. Unemployment risk should, for example, be endogenous to a general business cycle model. If risk is believed to be mostly endogenous to the economic system, the model in this chapter should be thought of as describing a mechanism that may amplify other exogenous fluctuations – or as a building block of an inherently unstable economic system vulnerable to oscillations between different sun-spot equilibria with different levels of risk.

Appendix 1

The solutions to the homogeneous part of (2.13) are

$$\begin{bmatrix} V^h(K,1) \\ V^h(K,0) \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{\rho_1 K} \\ c_2 e^{\rho_2 K} \end{bmatrix}, \tag{A.1}$$

where $c_{1,2}$ are the integration constants and $[v_i,1]'$ is the eigenvector corresponding to the eigenvalue ρ_i of the matrix

$$\frac{1}{\delta}\begin{bmatrix} -(r+\lambda(1)+\gamma(1)) & \gamma(1) \\ \gamma(0) & -(r+\lambda(0)+\gamma(0)) \end{bmatrix}. \tag{A.2}$$

To get a particular solution I guess a quadratic functional form

$$\begin{bmatrix} V^p(K,1) \\ V^p(K,0) \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1}{2} & \beta_1 \\ \frac{\alpha_0}{2} & \beta_0 \end{bmatrix} \begin{bmatrix} K^2 \\ K \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix}. \tag{A.3}$$

Using the method of undetermined coefficients I find that

$$\alpha_s = \frac{1}{\Delta}(r+\lambda(\bar{s})+\gamma(\bar{s})+\gamma(s)),$$

where (A.4)

$$\Delta = (r+\lambda(1)+\gamma(1))(r+\lambda(0)+\gamma(0)) - \gamma(0)\gamma(1),$$

$$\beta_s = -\frac{1}{\Delta}\left((r+\lambda(\bar{s})+\gamma(\bar{s}))\alpha_s + \gamma(s)\delta\alpha_{\bar{s}}\right), \tag{A.5}$$

and

$$\theta_s = \frac{1}{\Delta}\begin{pmatrix} (\lambda(s)(c+V(a_s,s)) - \delta\beta_s)(r+\lambda(\bar{s})+\gamma(\bar{s})) + \\ \gamma(s)(\lambda(\bar{s})(c+V(a_{\bar{s}},\bar{s})) - \delta\beta_{\bar{s}}) \end{pmatrix}. \tag{A.6}$$

That this is a solution can be checked by substituting (A.3) into (2.13). To find $[V(a_1,1),V(a_0,0)]$, I solve

$$\begin{bmatrix} V(a_1,1) \\ V(a_0,0) \end{bmatrix} = \begin{bmatrix} \frac{\alpha_1}{2}a_1^2 \\ \frac{\alpha_0}{2}a_0^2 \end{bmatrix} + \begin{bmatrix} \beta_1 a_1 \\ \beta_0 a_0 \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} v_1 c_1 e^{\rho_2 a_1} \\ c_1 e^{\rho_2 a_0} \end{bmatrix} + \begin{bmatrix} v_2 c_1 e^{\rho_1 a_1} \\ c_1 e^{\rho_2 a_0} \end{bmatrix}, \tag{A.7}$$

giving

$$V(\overline{U}_1,1)=\frac{\Delta\begin{bmatrix}(\Delta-\lambda(0)(r+\lambda(1)+\gamma(1))\left(.5\alpha_1\overline{U}_1^2+\beta_1\overline{U}_1+v_1c_1e^{\rho_1\overline{U}_1}+v_2c_2e^{\rho_2\overline{U}_1}+\phi_1\right)+\\ \lambda(0)\gamma(1)\left(.5\alpha_0\overline{U}_0^2+\beta_0\overline{U}_0+c_1e^{\rho_1\overline{U}_0}+c_2e^{\rho_2\overline{U}_0}+\phi_0\right)\end{bmatrix}}{(\Delta-\lambda(1)(r+\lambda(0)+\gamma(0))(\Delta-\lambda(0)(r+\lambda(1)+\gamma(1))-\lambda(0)\lambda(1)\gamma(0)\gamma(1)},$$

and (A.8)

$$V(\overline{U}_0,0)=\frac{\Delta\begin{bmatrix}(\Delta-\lambda(1)(r+\lambda(0)+\gamma(0))\left(.5\alpha_0\overline{U}_0^2+\beta_0\overline{U}_0+c_1e^{\rho_1\overline{U}_0}+c_2e^{\rho_2\overline{U}_0}+\phi_0\right)+\\ \lambda(1)\gamma(0)\left(.5\alpha_1\overline{U}_1^2+\beta_1\overline{U}1_1+v_1c_1e^{\rho_1\overline{U}_1}+v_2c_2e^{\rho_2\overline{U}_1}+\phi_1\right)\end{bmatrix}}{(\Delta-\lambda(1)(r+\lambda(0)+\gamma(0))(\Delta-\lambda(0)(r+\lambda(1)+\gamma(1))-\lambda(0)\lambda(1)\gamma(0)\gamma(1)},$$

where

$$\phi_s=\frac{1}{\Delta}\left((r+\lambda(\overline{s})+\gamma(\overline{s}))(\lambda(s)c-\delta\beta_s)+\gamma(s)(\lambda(\overline{s})c-\delta\beta_{\overline{s}})\right). \tag{A.9}$$

Ultimately this gives

$$\begin{bmatrix}V(K,1)\\ V(K,0)\end{bmatrix}=\begin{bmatrix}\frac{\alpha_1}{2} & \beta_1\\ \frac{\alpha_0}{2} & \beta_0\end{bmatrix}\begin{bmatrix}K^2\\ K\end{bmatrix}+\begin{bmatrix}v_1 & v_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}c_1e^{\rho_1K}\\ c_2e^{\rho_2K}\end{bmatrix}+\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}+$$
$$+\frac{1}{\Delta}\begin{bmatrix}\lambda(1)(r+\lambda(0)+\gamma(0)) & \lambda(0)\gamma(1)\\ \lambda(1)\gamma(0) & \lambda(0)(r+\lambda(1)+\gamma(1))\end{bmatrix}\begin{bmatrix}V(\overline{U}_1,1)\\ V(\overline{U}_0,0)\end{bmatrix}. \tag{A.10}$$

In the region between $\underline{U}_1$ and $\underline{U}_0$ the differential equation governing the total cost functions will no longer be given by (2.13). Here $V(K_t,0)$ is a constant because for all $K<\underline{U}_0$, K will immediately be adjusted if the current state is 0.[10] The total cost function will here instead be given by

$$\begin{aligned}V(K_t,0)&=V(\underline{U}_0,0),\\ V(K_t,1)&=\frac{K_t^2}{2}dt+(1-rdt)\mathrm{E}_t\left(V(K_{t+dt},s_{t+dt})\right),\\ &=\frac{K_t^2}{2}dt+(1-rdt)\begin{pmatrix}V(K_t,1)-\delta dtV_K(K_t,1)+\\ \lambda(1)dt\left(V(\overline{U}_1,1)+c-V(K_t,1)\right)+\\ \gamma(1)dt\left(V(\underline{U}_0,0)-V(K_t,1)\right)\end{pmatrix}.\end{aligned} \tag{A.11}$$

[10] Given that $\lambda(0)<\lambda(1)$, I have guessed that $\underline{U}_1<\underline{U}_0$, which is verified when a full solution is found.

giving a degenerate system of differential equations. By setting $dt^2=0$ and collecting terms I get for $s=1$,

$$\begin{aligned}&(r+\lambda(1)+\gamma(1))V(K,1)+\delta V_K(K,1)=\\&=\frac{K^2}{2}+\lambda(1)\big(V(a_1,1)+c\big)+\gamma(1)\big(V(\underline{U}_0,0)\big).\end{aligned} \tag{A.12}$$

Using the same solution technique as above,

$$\begin{aligned}V(K,1)=&\frac{K^2}{2(r+\lambda(1)+\gamma(1))}-\frac{\delta K}{(r+\lambda(1)+\gamma(1))^2}+c_0e^{\frac{-(r+\lambda(1)+\gamma(1))K}{\delta}}\\&+\frac{\left(\lambda(1)\big(c+V(a_1,1)+\gamma(1)V(\underline{U}_0,0)+\left(\frac{\delta}{(r+\lambda(1)+\gamma(1))}\right)^2\right)}{(r+\lambda(1)+\gamma(1))}.\end{aligned} \tag{A.13}$$

Above the upper triggers the total cost function is horizontal. I have guessed (and verified) that for reasonable parameter values

$$V(a_{\bar{s}},s)-V(a_s,s)<c\Rightarrow a_s<\overline{U}_{\bar{s}},\ \ s=\{1,2\}. \tag{A.14}$$

The economic meaning of (A.14) is that a switch in state never in itself induces down-grading of K. Down-grading due to income shocks or that when eventually K is adjusted, a lower K is chosen than would have been chosen in a low risk state, may and will of course happen. Whether (A.14) is true or not has to be checked when a preliminary full solution is found. If it is not true we have to construct a degenerate system of differential equations with one equation being a constant also in the upper region. With the parameter values I have chosen $V(a_{\bar{s}},s)-V(a_s,s)$ is never bigger than 0.103% of c, so (A.14) is far from being violated.

To summarize; $V(K,0)$ is given by equation (A.10) for $K\in\left[\underline{U}_0,\max(a_1,a_0)\right]$, and is constant and equal to $V(\underline{U}_0,0)$ for $K\le\underline{U}_0$. $V(K,1)$ is given by equation (A.10) for $K\in\left[\underline{U}_0,\max(a_1,a_0)\right]$, by (A.13) for $K\in[\underline{U}_1,\underline{U}_0]$, and is constant and equal to $V(\underline{U}_1,1)$ for $K\le\underline{U}_1$.

To find the unknown integration constants and band limits we solve the following non-linear system:

$$
\begin{array}{ll}
\text{Smooth pasting} & \begin{cases} V_K(a_1,1)=0 \\ V_K(a_0,0)=0 \\ V_K(\underline{U}_1,1)=0 \\ V_K(\underline{U}_0,0)=0 \end{cases} \\
\text{Value matching} & \begin{cases} V(a_1,1)+c=V(\underline{U}_1,1) \\ V(a_0,0)+c=V(\underline{U}_0,0) \end{cases} \\
\text{Continuity} & \{V(K,1)\in C^0 \\
\text{Unknowns} \quad = & \{a_1,a_0,\underline{U}_1,\underline{U}_0,c_0,c_1,c_2\}.
\end{array}
\tag{A.15}
$$

The continuity equation in (A.24) is needed because the differential equation changes functional form at $\underline{U}_0$ when $V(K,1)$ becomes a constant. Continuity is assured at all points except $V(\underline{U}_0,1)$ so this equation pins down one constant.

It is not possible to find an algebraic solution to this system, so it has to be solved with numeric methods.

Appendix 2

Proof of *proposition 1*

Set the expected cost of adjusting equal to the expected cost of waiting one instant. Use (2.12) the smooth pasting condition that $V_K(\underline{U}_s, s) = 0$ and set dt^2=0; then

$$\begin{aligned} V(\overline{U}_s, s) + c = \frac{\underline{U}_s^2}{2} dt + (1 - rdt)\big((V(\underline{U}_s, s)\big) \\ +\lambda(s)dt\big(V(a_s, s) + c - V(\underline{U}_s, s)\big) \\ +\gamma(s)dt\big(V(\underline{U}_s, \overline{s}) - V(\underline{U}_s, s)\big). \end{aligned} \tag{A.16}$$

The L.H.S. represents the expected total cost of immediate adjustment and the R.H.S. the expected total cost if one waits dt and then does what is optimal. Expanding the L.H.S. with the use of the Bellman equation (2.14) I get

$$\begin{aligned} V(a_s, s) + c = c + \frac{a_s^2}{2} dt + (1 - rdt)V(a_s, s) + \\ \lambda(s)dtc + \gamma(s)dt\big(V(a_s, \overline{s}) - V(a_s, s)\big). \end{aligned} \tag{A.17}$$

By collecting terms, dividing by dt and using (A.16), (A.17) and the value matching condition[11] in (A.15) we have

$$\begin{aligned} \frac{\underline{U}_s^2 - \overline{U}_s^2}{2} &= rc + \lambda(s)\big(V(\underline{U}_s, s) - V(\overline{U}_s, s)\big) \\ &+\gamma(s)\Big(\big(V(\overline{U}_s, \overline{s}) - V(\overline{U}_s, s)\big) - \big(V(\underline{U}_s, \overline{s}) - V(\underline{U}_s, s)\big)\Big) \\ &= rc + \lambda(s)c + \gamma(s)\big(c + V(\overline{U}_s, \overline{s}) - V(\underline{U}_s, \overline{s})\big) \end{aligned} \tag{A.18}$$

Q.E.D.

The first two terms of the value of waiting are obviously positive and the last can be rewritten as

$$\gamma(s)\Big(c - \big(V(\underline{U}_s, \overline{s}) - V(a_s, \overline{s})\big)\Big) \geq 0. \tag{A.19}$$

The inequality is easy to see: $V(K,s) - V(L,s) \leq c \; \forall K, L$ with equality only for $\{K \leq \underline{U}_s \text{ or } K \geq \underline{U}_s\}$ and $L = a_s$. In less formal words, the minimum of the expected cost

[11] The value matching condition simply means that we adjust K when the gain in expected total cost from adjusting is equal to the transaction cost c.

function can always be reached if the adjustment cost c is paid. The expected cost can thus never be higher than this minimum plus c.

Q.E.D.

Proof of *result i.*

Use the definition of the value of waiting in *proposition 1* to get

$$\begin{aligned}&rc+\lambda(1)c+\gamma(1)\big(c+V(a_1,0)-V(\underline{U}_1,1)\big)\\&-\Big(rc+\lambda(0)c+\gamma(0)\big(c+V(a_0,1)-V(\underline{U}_0,1)\big)\Big)=\\&(\lambda(1)-\lambda(0))c+\gamma(1)\big(c+V(a_1,0)-V(\underline{U}_1,1)\big)\\&-\gamma(0)\big(c+V(a_0,1)-V(\underline{U}_0,1)\big).\end{aligned} \tag{A.20}$$

Take limits of (A.21) and use *result iii*, the non-negativity of the parts of the value of waiting and the value matching conditions in (A.24);

$$\begin{aligned}&\lim_{\gamma(1)\to\infty}\begin{bmatrix}(\lambda(1)-\lambda(0))c+\gamma(1)\big(c+V(a_1,0)-V(\underline{U}_1,1)\big)-\\ \gamma(0)\big(c+V(a_0,1)-V(\underline{U}_0,1)\big)\end{bmatrix}=\\&(\lambda(1)-\lambda(0))c+\lim_{\gamma(1)\to\infty}\gamma(1)\overbrace{\big(c+V(a_1,0)-V(\underline{U}_1,1)\big)}^{\geq 0}-\\&\lim_{\gamma(1)\to\infty}\gamma(0)\overbrace{\big(c+V(a_0,0)-V(\underline{U}_0,0)\big)}^{0}\geq\\&(\lambda(1)-\lambda(0))c.\end{aligned} \tag{A.21}$$

Q.E.D.

Proof of *result iii.* For any small dt write the total cost function in state 1 as

$$V(K_t,1)=\frac{K_t^2}{2}dt+(1-rdt)\begin{bmatrix}p(s_{t+dt}=0|s_t=1)E_t\big(V(K_{t+dt},0)|s_{t+dt}=0\big)+\\(1-p(s_{t+dt}=0|s_t=1))E_t\big(V(K_{t+dt},1)|s_{t+dt}=1\big)\end{bmatrix}. \tag{A.22}$$

Now as the transition probability from state 1 becomes high we have

$$\lim_{\gamma(1)\to\infty}p(s_{t+dt}=0|s_t=1)=1. \tag{A.23}$$

Now let $dt\to 0$ at a slower rate than $\gamma(1)\to\infty$ so that $\gamma(1)dt\to\infty$. Then (A.23) is satisfied for all dt and we have

$$\lim_{\substack{\gamma(1)\to\infty,\\ dt\to 0}} V(K_t,1) = \lim_{dt\to 0} \frac{K_t^2}{2}dt + (1-rdt)E_t\left(V(K_{t+dt},0)\middle|s_{t+dt}=0\right) = V(K_t,0). \quad \text{(A.24)}$$

Q.E.D.

Appendix 3

Without leads and lags a differential equation can be numerically solved recursively by computing the derivatives at some initial point. These are used to forecast the value of the function and its derivatives at a higher value and the procedure is then repeated. With terminal conditions the procedure is repeated for different initial values until the solution is sufficiently close to the terminal conditions. This so called shooting method cannot be used on (4.2) since we need knowledge of $V(K_0,s)$ for values of K also different from K_0 to be able to compute $V_K(K_0,s)$. Instead I have used the band matrix method[12]. Here an interval for K is chosen and divided into subdivisions. As in other numerical methods we approximate derivatives with difference quotients. If the differential equation is linear we then get a linear equation system that can be easily solved.

Since ε can take all value in $\mathbf{R}$ we need information of the value of $V(K,s)$ for all values of $K \in \mathbf{R}$ to be able to compute any $V_K(K',s)$. Fortunately we know that

$$V(K,s) \equiv V(\underline{U}_s,s) \equiv V(a_s,s), \qquad \forall K \notin [\underline{U}_s, a_s] \tag{A.25}$$

The band over which we solve the differential equation system is thus chosen to be $[\min_s[\underline{U}_s], \max_s[U_s]]$. As before we first guess that $\underline{U}_1 < \underline{U}_0$ and that $\overline{U}_1 \geq \overline{U}_0$ and verify that later. The interval $[\underline{U}_1, \overline{U}_1]$ is divided into n_1 subintervals of length

$$\Delta K = \frac{\overline{U}_1 - \underline{U}_1}{n_1}. \tag{A.26}$$

Now approximate differentials with first differences. Also approximate

$$\int_{\varepsilon-\frac{\Delta K}{2}}^{\varepsilon+\frac{\Delta K}{2}} V(K+\varepsilon,s)d\Phi(\varepsilon,\sigma) \approx \left(\Phi\left(\varepsilon+\frac{\Delta K}{2},\sigma\right)-\Phi\left(\varepsilon-\frac{\Delta K}{2},\sigma\right)\right)V(K+\varepsilon,s), \tag{A.27}$$

and define

$$\begin{aligned}
&\Phi(n) \equiv \Phi((n+0.5)\Delta K,\sigma),\\
&\phi(n) \equiv \Phi((n+0.5)\Delta K,\sigma) - \Phi((n-0.5)\Delta K,\sigma),\\
&\psi(s) \equiv (r+\lambda(s)+\gamma(s)).
\end{aligned} \tag{A.28}$$

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[12] See for example Dahlquist and Björk (1974).

For any $n \in \{0,1,\ldots,n_1\}$ and for s=1 we can then write the system (4.2) as the linear equation

$$\psi(1)V(\underline{U}_1+n\Delta K,1)+\gamma(1)V(\underline{U}_1+n\Delta K,0)-\delta\left(\frac{V(\underline{U}_1+n\Delta K,1)-V(\underline{U}_1+(n-1)\Delta K,1)}{\Delta K}\right)=$$
$$\left(\frac{\underline{U}_1+n\Delta K}{2}\right)^2+\lambda(1)\left(\sum_{s=1}^{n_1-1}\phi(s-n)V(\underline{U}_1+s\Delta K,1)+\Phi(-n)V(\underline{U}_1,1)+(1-\Phi(n_1-n))V(a_1,1)\right). \tag{A.29}$$

Doing the same for s=0 over the interval $[\underline{U}_0,\overline{U}_0]$ we get n_1+n_0+2 linear equations in the n_1+1 unknown $V(K,1)$ between $[\underline{U}_1,\overline{U}_1]$ and n_0+2 unknown $V(K,0)$ between $[\underline{U}_0,\overline{U}_0]$.

Now approximate the smooth pasting conditions with centered differences around the lower triggers, i.e., set

$$\left(\frac{V(\underline{U}_s+\Delta K,s)-V(\underline{U}_s-\Delta K,s)}{2\Delta K}\right)=0. \tag{A.30}$$

This gives two more equations and unknowns. By collecting the unknowns on the L.H.S. and stacking all the equations we get the linear system

$$\mathbf{AV}=\mathbf{K} \tag{A.31}$$

The vector **V** contain the unknown values of $V(K,s)$ and is solved for by

$$\mathbf{V}=\mathbf{A}^{-1}\mathbf{K} \tag{A.32}$$

We still have six unknown values; the four triggers and the two return points. These are determined by the following six equations

$$\begin{aligned}
&V(\underline{U}_0,0)-V(a_0,0)=c,\\
&V(\underline{U}_1,1)-V(a_1,1)=c,\\
&V(\overline{U}_0,0)-V(a_0,0)=c,\\
&V(\overline{U}_1,1)-V(a_1,1)=c,\\
&a_0=\arg\min_K[V(K,0)],\text{ and}\\
&a_1=\arg\min_K[V(K,1)].
\end{aligned} \tag{A.33}$$

The first four equations are the value matching equations and the last two variants of the remaining smooth pasting conditions. The (approximate) solution (A.33) have been found by guessing ranges for the values and then by performing a grid search. Since the

method described above uses discrete spaces for the unknowns, standard optimization methods like Newton-Raphson cannot be used. We also have to look for solutions to (A.33) in steps of ΔK as defined in (A.26). Decreasing the step length increases the size of the system (A.32). I have divided the interval $\min_s[\underline{U}_s], \max_s[\overline{U}_s]$ into 250 steps giving a step length of approximately 0.32%. The system (A.32) then has approximately 500 equations which is the largest I could conveniently solve.

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Chapter IV*
Variations in Risk as a Cause of Fluctuations in Demand
– some empirical results

1. Introduction

Are variations in risk important for business cycle fluctuations in aggregate economic activity? Most non-economists would probably answer this question with a clear yes, giving the intuition that high risk makes both producers and consumers hesitate to undertake investments. Economists, however, appear to be less sure. Caballero (1991) shows that the sign of the investment-uncertainty relationship is different in different specifications of irreversible investment models. Irreversibility is here taken to mean that it is costly, but not necessarily impossible, to reverse an investment. Empirically there is surprisingly little work done, although Schwert (1989), for example, shows evidence of a negative relationship between financial volatility and the level of aggregate economic activity.

Recent theoretical work on irreversibilities has mainly been concerned with firm behavior. Irreversible investment models have been used to explain for example the sluggish behavior of employment and investments in Europe.[1] Irreversibilities due to transaction costs are, however, endemic to the whole economy; consumers buying a house or a dishwasher also face a typical irreversible investment problem. If a consumer is hit by a wealth shock she may regret the purchase since it is costly to reverse. In this chapter I will study the effects of variations in risk on the aggregate demand for household durables. I will show that variations in risk affect durables purchases more than what can be accounted for by variations in precautionary savings.

Mankiw (1982) noted that the Hall random-walk hypothesis for non-durables consumption (Hall, 1978) also applies to the services of durables. Mankiw assumes that the

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* In particular Andrew Bernard, Olivier Blanchard, Ricardo Caballero, Jesus Guillermo Llorente-Alvaréz and Jose-Vicente Rodriguez-Mora at MIT, Harald Lang, Torsten Persson, Lars E. O. Svensson, Peter Sellin and Paul Söderlind at IIES, Danny Quah at London School of Economics and Torbjörn Becker at Stockholm School of Economics have provided extensive and helpful comments and suggestions to this chapter.

[1] See for example Bentolila and Bertola (1987) and Bean (1990).

services of durables are linear in the stocks. The stocks of durables should then also evolve as random walks, i.e.,

$$e_t = e_{t-1} + \tilde{\varepsilon}_t, \tag{1.1}$$

where e is the stock of durables and $\tilde{\varepsilon}_t$ is an innovation that is independent of all variables known at t-1. Knowing the stochastic behavior of stocks, durables purchases can easily be derived. Letting d denote durables purchases, we have

$$d_t = e_t - (1-\delta)e_{t-1}, \tag{1.2}$$

where δ is the rate of depreciation per unit of time. Taking first differences of purchases we get a moving average process;

$$\begin{aligned} \Delta d_t &= \left(e_t - (1-\delta)e_{t-1}\right) - \left(e_{t-1} - (1-\delta)e_{t-2}\right) \\ &= \Delta e_t - (1-\delta)\Delta e_{t-1} \\ &= \tilde{\varepsilon}_t - (1-\delta)\tilde{\varepsilon}_{t-1}. \end{aligned} \tag{1.3}$$

I will show that a measure of risk enters significantly into different variants of (1.3) even if the specification allows for slow adjustment to wealth shocks. To show this I make the stylized assumption that the degree of risk stochastically switches between two levels. These levels are empirically estimated together with the parameters of the switching process. Both for the S&P500 stock market index and for non-durables consumption, I find evidence of extended periods when volatility is higher than during the rest of the time.

When the economy enters a period of high financial risk, durables purchases are estimated to fall by approximately 3% and car purchases by around 7%. The effect builds over time if risk continues to be high; after half a year the initial negative effect has more than doubled. High volatility of non-durables consumption, however, seems to have a much weaker impact on durables purchases. Some of the empirical findings point in the direction of irreversible investment models. As will be seen, however, some questions still remain.

The chapter is organized in the following way. In section 2, I estimate a stochastic model for the level of risk based on work by Hamilton (1988). I show that risk is significantly higher during some periods of time than during others. In section 3, I derive a regression model based on equation (1.3). In section 4, I present the empirical results and finally in section 5, I discuss them within the framework of irreversible investment models.

2. Estimation of Changes in Risk

Let me define risk as the volatility of the innovations to household wealth. In chapter III I assumed that the level of risk is determined by a two-state stochastic process. This stylized assumption will also be used here. The economy is assumed to switch stochastically between two states, $s_t = 0$ or 1. When $s_t = 1$ risk is assumed to be higher. The current state of the economy, s_t, is not observable. By specifying a stochastic process for wealth it is, however, possible to compute the probability that the economy is in a particular state, conditional on a series of realized wealth innovations. In this section I will estimate such a model and compute a series of state probabilities. In the following section I will investigate if these state probabilities significantly enter a standard model for durables purchases based on (1.3).

2.1 Stochastic State Model

The process for aggregate wealth, $\tilde{w}_t$, is assumed to be a generalized random walk. I allow for state shifts to be associated with shifts in volatility as well as with shifts in level and drift;

$$\ln \tilde{w}_t = \ln \tilde{w}_{t-1} + \mu_0 + (\mu_1 - \mu_0)s_t + \mu_2(s_t - s_{t-1}) + (\lambda_0 + (\lambda_1 - \lambda_0)s_t)\vartheta_t \qquad (2.1)$$

where ϑ is a sequence of i.i.d. normal innovations. The drift of the log of wealth is thus μ_0 in state 0 and μ_1 in state 1. If the state shifts at t from 0 to 1 we also get a level shift of μ_2 and of $-\mu_2$ if the state shifts from 1 to 0. The standard deviation of the wealth innovations in absence of state shifts is λ_0 in state 0 and λ_1 in state 1.

As in chapter III, the state is assumed to follow a first-order Markov process;

$$\begin{aligned} P(s_t = 0 | s_{t-1} = 0) &= 1 - \gamma_0, \\ P(s_t = 1 | s_{t-1} = 0) &= \gamma_0, \\ P(s_t = 1 | s_{t-1} = 1) &= 1 - \gamma_1 \text{ and} \\ P(s_t = 0 | s_{t-1} = 1) &= \gamma_1. \end{aligned} \qquad (2.2)$$

Let capital letters denote realized values of the stochastic variables, which are written in small letters. The current state, S_t, is not observed by the econometrician but by observing $\{W_s\}_1^t$ we may draw inference about the likelihood $P\left(s_t = S_t \middle| \{W\}_1^t\right)$. Hamilton (1988) shows how to do this in a similar setup. Hamilton's method is recursive; given $P\left(s_{t-1} = S_{t-1} \middle| \{W\}_1^{t-1}\right)$ and W_t, we may compute $P\left(s_t = S_t \middle| \{W\}_1^t\right)$. We may also "back-cast"

and estimate $P\left(s_{t-k} = S_{t-k} \middle| \{W\}_1^t\right)$ for $k \geq 1$. The "back-casting" may improve the accuracy in the state estimation by using more information. The parameters of the model are estimated by maximizing the likelihood function implied by (2.1) and (2.2) . The details of this state estimation are given in appendix 1.

In the preceding paragraphs I have assumed that aggregate wealth is observable. This is, however, not the case. I have used two alternative ways around this problem. The first is to assume that current consumption of non-durables is proportional to wealth but with different proportionality factors in the two states. The difference is assumed to represent shifts in the share of wealth used for precautionary savings. Denoting spending on non-durables as c we can the write

$$c_t = \omega_c \tilde{w}_t \phi^{s_t}, \tag{2.3}$$

where ω_c is the state 0 share of wealth used for non-durables consumption per unit of time. $\omega_c \phi$ is the corresponding share in state 1. If the consumers are prudent and thus save more in the high risk state, ϕ is less than unity. Now define precautionary savings adjusted wealth, $w_t \equiv \tilde{w}_t \phi^{s_t}$ and note that consumption is proportional to this redefined wealth measure

$$c_t = \omega_c w_t. \tag{2.4}$$

The process for w_t is easily derived

$$\begin{aligned} \ln w_t &= \ln w_{t-1} + \mu_0 + (\mu_1 - \mu_0)s_t + (\mu_2 + \ln\phi)(s_t - s_{t-1}) + \left(\lambda_0 + (\lambda_1 - \lambda_0)s_t\right)\vartheta_t \\ &= \ln w_{t-1} + \mu_0 + (\mu_1 - \mu_0)s_t + \tilde{\mu}(s_t - s_{t-1}) + \left(\lambda_0 + (\lambda_1 - \lambda_0)s_t\right)\vartheta_t. \end{aligned} \tag{2.5}$$

We may then substitute c_t for w_t in (2.5) and estimate the parameters of the model. μ_2 and ϕ are, however, clearly not separately identified. A shift of state will lead to a shift in adjusted wealth and thus also in c also when holding non-adjusted wealth constant if $\phi \neq 1$. This effect will be picked up by the coefficient $\tilde{\mu}_2$ in the second line of equation (2.5), which will reflect both non-adjusted wealth shifts and shifts in precautionary savings associated with state shifts.

The second way is to use a stock market index to substitute for wealth in (2.1). There is evidence of a positive relationship between financial and macroeconomic volatility. Schwert (1989) reports that financial volatility significantly helps to predict future volatility in industrial production for the period 1891 to 1987. An important advantage of using the stock market instead of consumption is that the latter contains more volatile seasonal fluctuations.

Changing patterns of seasonal fluctuations (or changing methods for seasonal adjustment) may then erroneously be interpreted as variations in risk. This problem should be smaller when using stock market volatility as a basis for the state estimation. We may also suspect that the stock market index is less constrained by friction than household consumption. Especially high frequency shocks may then be more visible at the stock market.

The typical household does not own much public stock and a large portion of its wealth is expected future labor income, its human capital.[2] Despite this fact I argue that shifts in the volatility of the stock market are likely to be good indicators of shifts in the volatility of household wealth. Fluctuations in risk may be due to variations in the volatility of a stochastic trend common to both firm values and household wealth, for example technology shocks. In this case the variances of household wealth and the stock market, as well as their levels, have a positive correlation.

A positive correlation between the level of the stock market and household wealth is, however, not necessary for their volatilities to be positively correlated. Another potential source of volatility is variations in the share of total income going to labor. Such share variations would tend to give negative correlations between the value of firms and human capital. Nevertheless, increased share volatility will increase the volatility of the stock market as well as of human capital. We may also think of cases in which the levels are uncorrelated while the variances are positively correlated.

It is certainly possible that no association exists between stock market and household wealth volatility. This, however, seems to be an unlikely knife edge case. A maintained hypothesis is thus that periods of high stock market volatility tend to coincide with periods of high volatility of household wealth.[3]

2.2. State Estimation Results

Using 396 monthly observations of the S&P 500 stock market index and nominal consumption of non-durables, covering the period from January 1959 to January 1992, I have

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[2] For an early discussion of this, see Roll (1977).

[3] Note that it is this dating of high risk periods which is the aim of the state estimation discussed in this section. An estimation of the *levels* of volatility of household wealth requires more information than what I need for my purpose.

estimated the parameter of the model. The estimates together with estimated standard errors are given in table 1.[4]

Table 1 State Model Parameters

	S&P 500			Non-Durables Consumption		
	Estimated value %/month	Asymptotic St. Dev.	t-value ≠ from 0	Estimated value %/month	Asymptotic St. Dev.	t-value ≠ from 0
γ_0	3.53	4.67	-	9.01	12.19	.
γ_1	13.53	1.39	-	24.75	23.82	.
μ_0	0.91	0.18	4.94	0.51	0.07	7.63
$(\mu_1-\mu_0)$	-1.91	0.71	-2.71	0.12	0.17	0.72
μ_2, $\tilde{\mu}_2$†	-5.24	1.21	-4.35	-0.22	0.53	-0.41
λ_0	1.81	0.10	17.69	0.46	0.08	6.08
$(\lambda_1-\lambda_0)$	1.49	0.36	4.14	0.28	0.14	1.99

† For the consumption based probabilities we estimate $\tilde{\mu}_2 \equiv \mu_2 + \ln\phi$, see equation (2.5)

In table 1 we see that there is clear evidence of the existence of periods of higher financial volatility. The point estimate is that the standard deviation of the S&P 500 index increases with 1.49 percentage points from 1.81 to 3.30% per month in the high risk state. The relative increase in volatility is over 80% and it is strongly significant; the *t*-value for a null hypothesis of equal risk in both states is 4.14.

We also see that high-risk periods seem to be associated with lower than average stock returns; the mean growth rate is estimated to be negative in state 1. A shift to higher (lower) volatility is on average also associated with negative (positive) shocks to the value of the stock market index. This result is also reported in the finance literature, e.g., by French et al (1987). That study finds evidence for a negative relation between unexpected increases in stock market volatility and realized stock returns. The mechanism may be that an unexpected increase in volatility increases the expected future volatility since volatility is persistent. This increases the discount rate for future cash flows which reduces stock prices. The effect of a shift to the high risk state would thus be an immediate fall in stock prices, i.e., $\mu_2 < 0$.

[4] The covariance matrix of the parameters is estimated as the inverse of the estimated likelihood Hessian.

French et al (1987) find weak evidence for a positive association between predicted volatility and ex ante returns. Haugen et al (1991) find that monthly returns are lower (higher) during the second four weak period subsequent to a fall (increase) in volatility. In the high risk state expected volatility is high, which thus should increase μ_1. On the other hand, as long as the state does not shift back, realized volatility is higher than expected, tending to mitigate the positive effect on μ_1. The negative sign of $(\mu_1\text{-}\mu_0)$ is anyway somewhat surprising.

Also non-durables consumption appears to be more volatile during some periods although the marginal significance level is lower than for the S&P500 index. Volatility is estimated to be 0.28 percentage points higher in state 1, which corresponds to a relative increase of 61%. Above we found that shifts to higher risk are associated with a fall in the S&P500 index. If this is due to that higher risk cause higher discount rates, the same mechanism should reduce household wealth inclusive of human capital. On the other hand, state shifts may affect the expected future cash flows of firms and households differently. Shifts to higher volatility of household wealth are thus not necessarily associated with negative wealth shocks. In table 1 we see that μ_1 and $\tilde{\mu}_2$ both are close to zero. The drift and the level of non-durables consumption thus appear to be insensitive to state shifts.

We also find that the probability of a state switch is higher when the risk is high. This implies that periods of high risk on average are shorter than periods of low risk. Using S&P 500 the expected time to next state shift is approximately $1/\gamma_0 \approx 28.3$ months in state 0 while it is $1/\gamma_1 \approx 7.4$ months in state 1. For non-durables consumption the corresponding lengths are 11.10 and 4.04 months. The shorter high risk periods imply that the economy is in the low risk state most of the time. The unconditional probability of state 1 is 0.207 for S&P500 and 0.267 for non-durables consumption. The relatively shorter high risk periods may be of significant importance. In chapter III I showed that the effect on durables purchases of a shift to higher risk is larger the shorter the high risk period is expected to last.

Using the estimated parameters I calculate the series of conditional likelihoods $\left\{p\left(s_t = 0 \middle| \{W\}_1^{t+6}\right)\right\}_{t=1}^{T-6}$. The conditional likelihoods using S&P 500 are plotted in Figure 1. For 105 of the 396 S&P500 observations the conditional likelihood of state 1 is higher than the unconditional likelihood, which is depicted as the horizontal line in the graph. For non-durables the analogous event occurs 133 times.

In Figure 1 we see that the longest periods of high likelihoods of the high risk state occur around the years 1970, 1974 and 1982. We also have periods of high likelihoods of the high risk state in 1962, 1966, 1980, 1987 and 1990. The two latter appear to be due to the stock market crashes. A large fall in the stock market will increase the likelihood of state 1, both since μ_2 is negative and since a higher variance increases the likelihood of large realizations. It also seems reasonable that the degree of uncertainty was high immediately after these stock market crashes. After these two periods we see that the likelihood of state 0 recovers quickly.

Figure 1 Probability of High Risk State, S&P 500

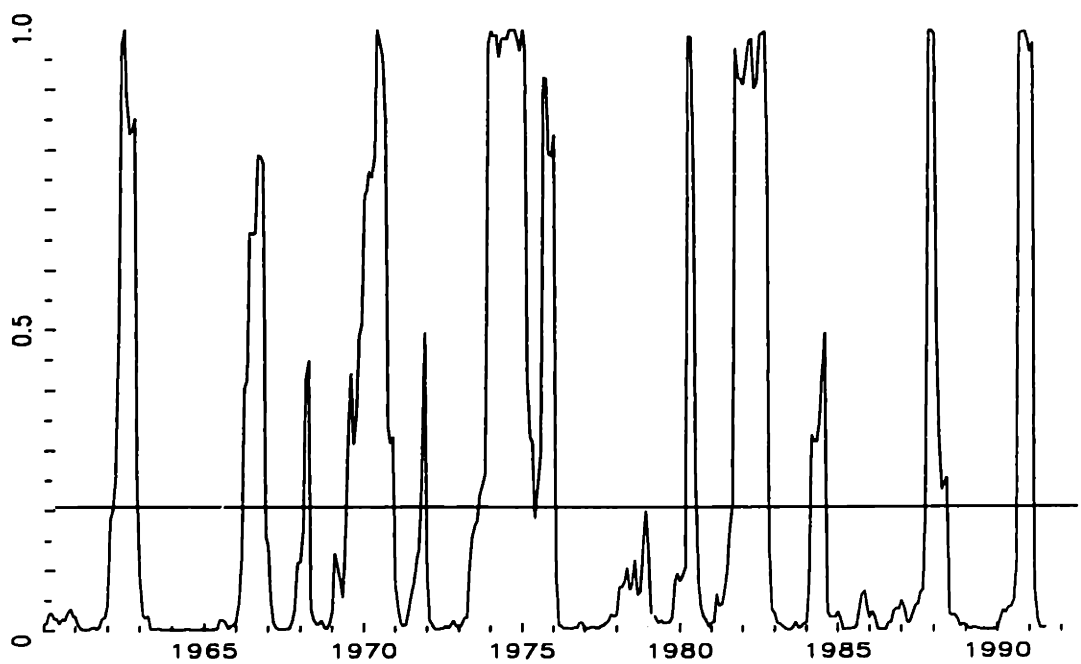
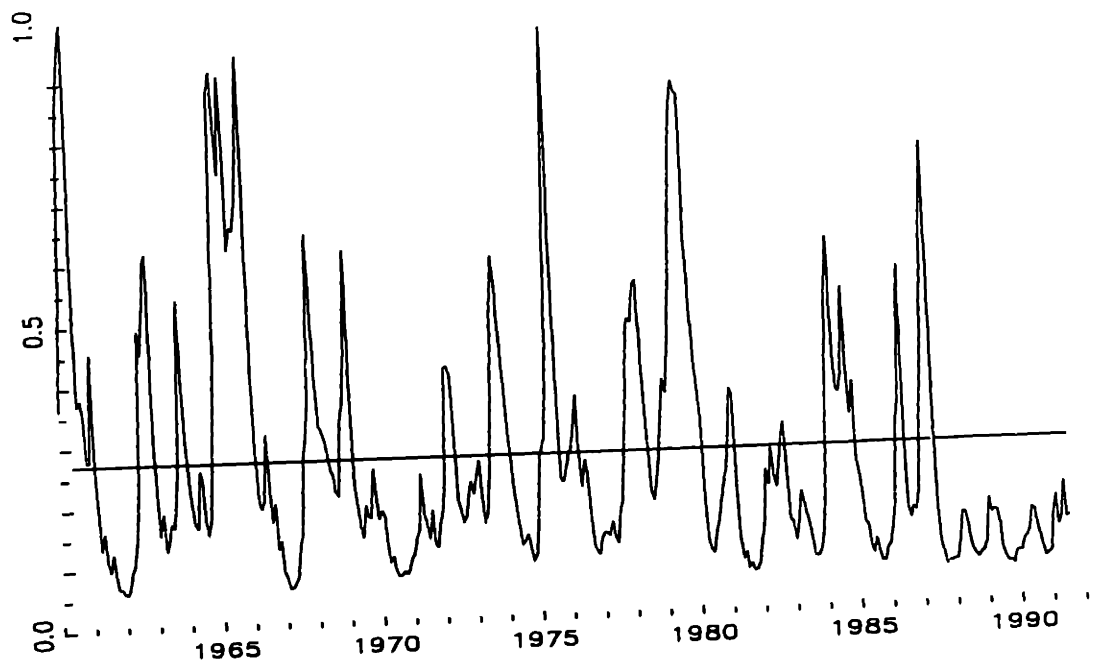


Figure 1 indicates a relationship between high risk periods, as indicated by state 1, and recessions. Such a relationship is also reported in the finance literature. Schwert (1989) shows that volatility generally has been higher during months that have NBER classified as recession. For the period 1859 to 1987 the standard deviation of monthly stock returns was 61 % higher during recession and between 1953 and 1987 68% higher.

Figure 2 depicts the probabilities of the high risk state based on the volatility of non-durables consumption. There we see a different pattern than in Figure 1. The probabilities are on average closer to the unconditional probability (the horizontal line). It appears more difficult to distinguish two distinct states in the stochastic development of non-durables con-

sumption compared to the S&P500 index. The two state process thus appear to be a fairly bad description of the data generating process for non-durables consumption.

Figure 2 Probability of High Risk State, Non-Durables Consumption



We also see that the peaks in 1975, 1980 and 1987 coincide only approximately with troughs for the S&P500 based state probabilities. The sample correlation between the two series of estimated state probabilities is even negative, -0.15. The duration of periods of low probability of state 0 is also shorter for the consumption based state probabilities. It is also difficult to see any relation between the likelihoods in Figure 2 and recessions.

3. Derivation of a Regression Model

Assume provisionally that the stock of durables behaves like non-durables consumption;

$$e_t = \omega_e w_t, \tag{3.1}$$

where ω_e is the share of durables in wealth net of precautionary savings. The innovations to precautionary savings adjusted wealth and durable stocks will then be identical up a proportionality factor ω_e. Using this and substituting from (1.3) and we get

$$\begin{aligned} d_t &= d_{t-1} + \Delta e_t - (1-\delta)\Delta e_{t-1} \\ &= d_{t-1} + \omega_e \Delta w_t - \omega_e (1-\delta)\Delta w_{t-1}. \end{aligned} \tag{3.2}$$

It has been noted in the literature that the adjustment of the durable stock to wealth shocks is much slower than implied by the equations above. An interpretation of this is that (3.1) is a relation between wealth and a target stock of durables. Due to friction or adjustment lags, target and actual stocks may temporarily diverge after a wealth shock. Caballero (1990b) finds that inclusion of more lagged wealth shocks increases the explanatory power of the equation. To allow for a slower adjustment mechanism I thus include more lagged wealth shocks.

A regression on (3.2) is likely to suffer from heteroscedasticity problems.[5] One remedy is to divide the regression by a scaling factor, e.g., wealth. There are two additional benefits of scaling with wealth. The first is that we may interpret coefficients on additional regressors as the effect on the ratio of durables purchases to wealth. The second is due to that the relative price of durables may be correlated with the state, i.e., the risk level. If individual utility functions are (close to) Cobb-Douglas the expenditure share in nominal terms would be (approximately) independent of such relative price changes. By normalizing with wealth and using nominal values we thus (partly) control for such nominal price changes.

In the previous section I showed that the return on the S&P500 index is correlated with the state. Also the interest rates that consumers face may then be correlated with the state and this may affect durables purchases. I will thus include a real interest rate r in the regression. We then arrive at a regression of the following form

[5] The variance of wealth shocks are growing over time. Unless also the variance of regression residuals from (3.2) grows, relative residual variance will go to zero and regression R^2 to one.

$$\frac{d_t}{w_{t-1}} = \alpha_0 \frac{d_{t-1}}{w_{t-1}} + \sum_{s=1}^{6} \alpha_{1,s} \frac{\Delta w_{t+1-s}}{w_{t-1}} + \alpha_2 r_t + v_t \tag{3.3}$$

where α_0 according to (3.2) should equal unity. Current wealth, w_t, could alternatively have been used as a scaling factor. All results reported below will, however, be robust to such a change of specification.

We expect the residuals from regression (3.3) to be negatively correlated with state shifts if durables purchases are more sensitive to variations in the level of risk than what can be accounted for by variations in precautionary savings. An immediate problem is that wealth shocks are not directly observable. Here I will use the same assumption as in (2.4).[6] Substituting from that equation we get

$$\frac{d_t}{c_{t-1}} = \alpha_0 \frac{d_{t-1}}{c_{t-1}} + \sum_{s=1}^{6} \alpha_{1,s} \frac{\Delta c_{t+1-s}}{c_{t-1}} + \alpha_2 r_t + v_t. \tag{3.4}$$

Regression (3.4) yields relatively high R^2 and Durbin-Watson statistics slightly above 2. The coefficient α_0 is not significantly different from 1. Specific regression results are reported in appendix 2.

To test for durables specific effects of risk I may investigate if state shifts are significant in regression (3.4). A potential complication is that the state is not directly observable by the econometrician. Instead we have to use estimated state probabilities. The consumers, on the other hand, are assumed to know the true state of the economy. We will then incorporate an extra error in the regression by including the estimated state probabilities instead of the true states. In appendix 3 I, however, show that this is not going to cause an inconsistent estimate of the effect of state shifts, provided that we fix α_0 to unity and move d_{t-1} to the right hand side of the regression. An intuitive explanation for this is the following; the calculated probability of state 1 equals the expected state conditional on observed wealth changes. The difference between the actual state and the conditional expected state is then by construction orthogonal to the series of wealth changes. The added regression error is thus uncorrelated with the regressors and OLS can be used. To test for state effects I thus run

[6] I will also report some results from using innovations to the stock market index as a measure of wealth shocks. Also the precautionary savings effect that is common to durables and non-durables will then cause a negative relation between the residuals and the state.

$$\frac{d_t - d_{t-1}}{c_{t-1}} = \sum_{s=1}^{6} \alpha_{1,s} \frac{\Delta c_{t+1-s}}{c_{t-1}} + \alpha_2 r_t + \alpha_3 \Delta P_t + \upsilon_t \tag{3.5}$$

where ΔP_t is the change in the probability of state 1 computed according to the algorithm discussed in the previous section. If durables purchases are more sensitive to variations in risk than can be accounted for by shifts in precautionary savings we expect α_3 to be negative.

4. Results

In this section I report the result from regressions (3.4) and (3.5). The time series for durables, non-durables and interest rates are taken from Citibase and cover monthly observations between 1959:01 and 1992:01. I have used nominal spending on durables and new cars to represent D and food and non-durables to represent C. 1 year T-bill rates minus realized CPI inflation one year ahead represents r. The spread between borrowing and lending rates may be correlated with risk. I have thus checked that the results are robust to using real house mortgage rates to represent r. Results from all four combinations of D and C will be presented for each of the two series of state probabilities.

4.1. S&P 500 State Probabilities

Before running the regressions with state probabilities included I will graphically illustrate how the residuals from regression (3.4) are associated with the state probabilities. The regression is written so that the residuals represent unexplained *changes* in durables purchases. The simple correlations between the residuals and the change in S&P 500 based state probabilities are -0.125, -0.112, -0.132, -0.123 for the combinations durables and non-durables, cars and non-durables, durables and food, and cars and food.[7]

The residuals and ΔP_t contain too much high frequency volatility for a plot to be revealing, which can be seen in the plot in appendix 2. By integrating[8] the residuals I get unexplained accumulated changes in purchases. The integrated residuals can thus be interpreted as unexplained *levels* of purchases. By plotting these against the state I can depict the association between the unexplained level of purchases and the level of the state probability. This is done in Figure 3.[9] The integrated residuals are more highly correlated with the level of the

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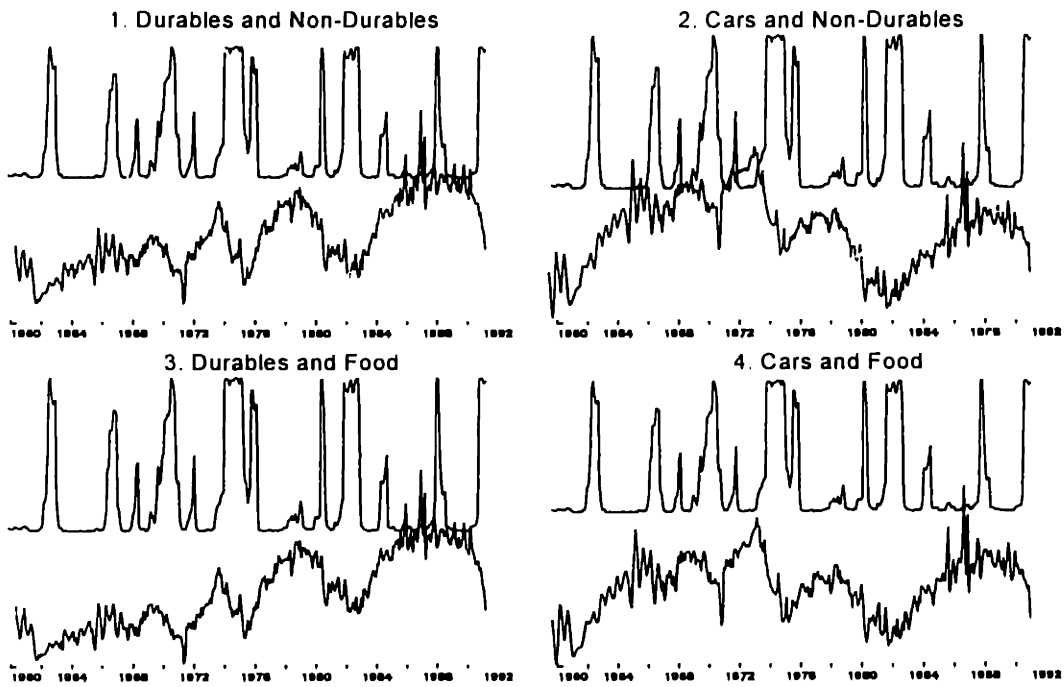
[7] If I use changes in the S&P500 index to represent wealth shocks the correlations are closer to zero; -0.087 and -0.091 for durables and cars respectively.

[8] The integrated residuals υ_t are defined as $\upsilon_t \equiv \sum_{\tau=1}^{t} \upsilon_\tau - \bar{\upsilon}$ where $\bar{\upsilon}$ is the sample average of the residuals. Since the regression contains no constant $\bar{\upsilon}$ is not necessarily zero..

[9] The scale on the vertical axis contains no information since the residuals have been scaled and a constant have been added to facilitate reading.

state probability than the residuals are with the change in the state probability: the correlations are -0.274, -0.154, -0.219 and -0.221.[10]

Figure 3 Residuals from regression (3.4) vs. P_t



We may also be interested in how the unexplained levels of durables purchases at business cycle frequencies are related to the state probabilities. In Figure 4 I show the integrated residuals from (3.4) after applying a bandpass filter. Frequencies corresponding to periods longer than 8 years and shorter than 3 years are removed. Here we see a strong and clear negative relationship between the state and the filtered residuals. The correlations are -0.414, -0.385, -0.396 and -0.394.

In table 2 I report the results from the regression (3.5) using S&P 500 based state probabilities. As already noted, α_3, should be interpreted as the immediate fall in relative durables purchases after a state shift that cannot be explained by changes in precautionary savings adjusted wealth (as measured by changes in non-durables consumption). I also report $\hat{\alpha}_3$ normalized with the average value of durables to non-durables purchases. This normalized

[10] For the S&P500 index representing wealth shock the correlations are -0.177 and -0.194 for durables and cars respectively.

value should then be interpreted as the *percentage* unexplained fall in durables purchases that follows immediately after a shift to the high risk state.

For the two last lines in the table I have used changes in the S&P500 index to represent C in (3.5). Here the interpretation of α_3 is the fall in the ratio of durables purchases to wealth associated with state shifts, inclusive of the effect due to variations in precautionary savings.

Figure 4 Business Cycle Residuals from regression (3.4) vs. P_t

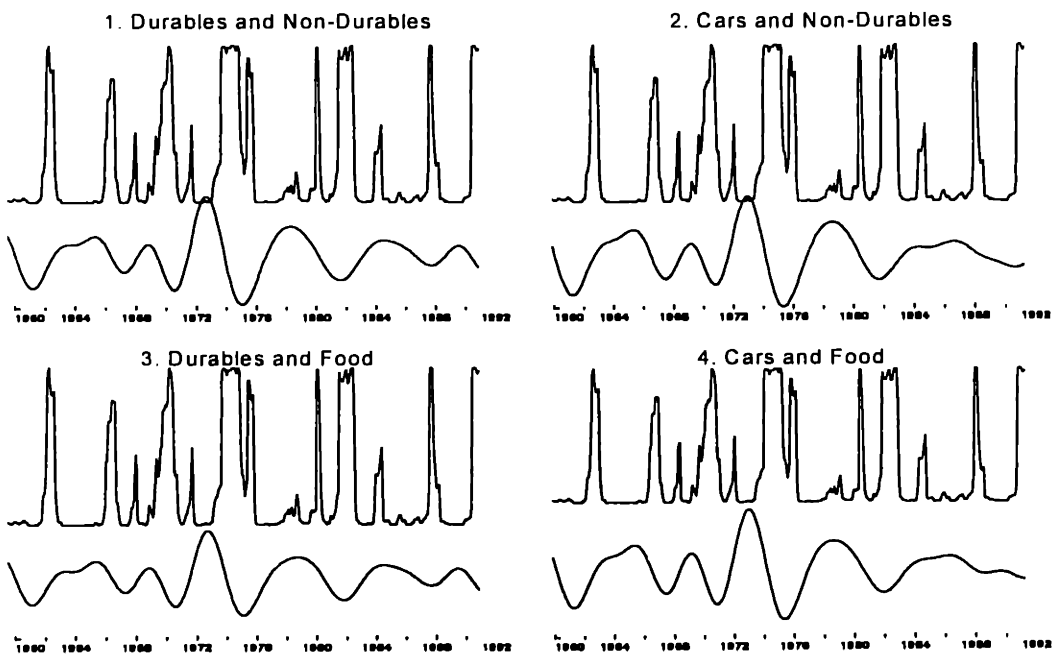


Table 2 Effect of S&P500 Based State Probabilities

Variables	$\hat{\alpha}_3$*100	t-stat†	$\hat{\alpha}_3$*100/avg(D/C)
D=Durables, C=Non-Durables	-0.975	-2.48	-2.86
D=Durables, C=Food	-2.071	-2.63	-3.11
D=Cars, C=Non-Durables	-0.653	-2.23	-7.34
D=Cars, C=Food	-1.491	-2.45	-8.57
D=Durables, C=S&P500	-4.902	-2.30	-3.74
D=Cars,C=S&P500	-3.671	-2.43	-11.1

† Ordinary t-statistics. Newey-West standard deviations computed with four lags always gave stronger significance.

In the table we see a very clear pattern; relative purchases of durables fall by around 3% after a shift to the high risk state. For relative purchases of cars the negative effect is even stronger; the fall is estimated to around 7-8%. In all cases the effect is clearly significant. We

also see that the effect is somewhat stronger when the S&P500 index is used as wealth measure. This is expected since we here also include a precautionary savings effect.

The results in table 2 change only negligibly if current non-durables consumption or food is used as the scaling factor or if house mortgage rates are used for interest rates. $\hat{\alpha}_3$ is almost unchanged but the marginal significance generally increases slightly.

4.2. Consumption Based State Probabilities

Figure 5 shows the integrated residuals from regression (3.4) plotted against the state probabilities computed from the observations of non-durables consumption. Here we get a mixed picture. It is hard to detect a negative relationship in the graphs. Before integrating, the residuals have the following correlations with ΔP_t; -0.022, -0.021, -0.056, -0.039. The correlations between the integrated residuals in Figure 5 and P_t are -0.081, 0.034, -0.095, -0.016. In Figure 6 non-business cycle frequencies, defined as above, have been removed from the residuals. Here correlations are even positive; 0.235, 0.230, 0.248, 0.237.

Figure 5 Residuals from regression (3.4) vs. P_t

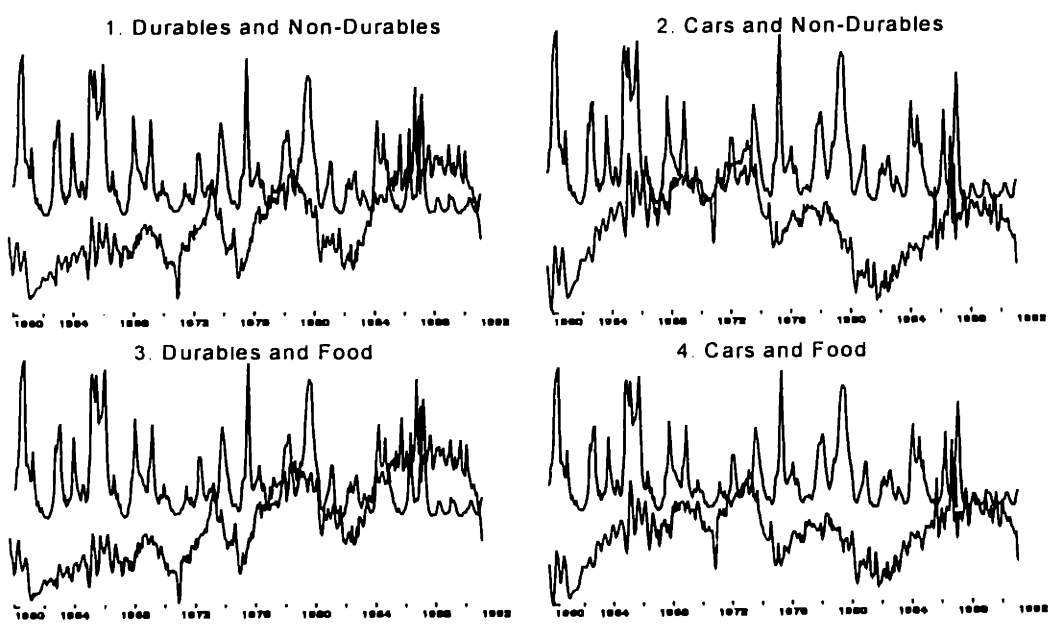
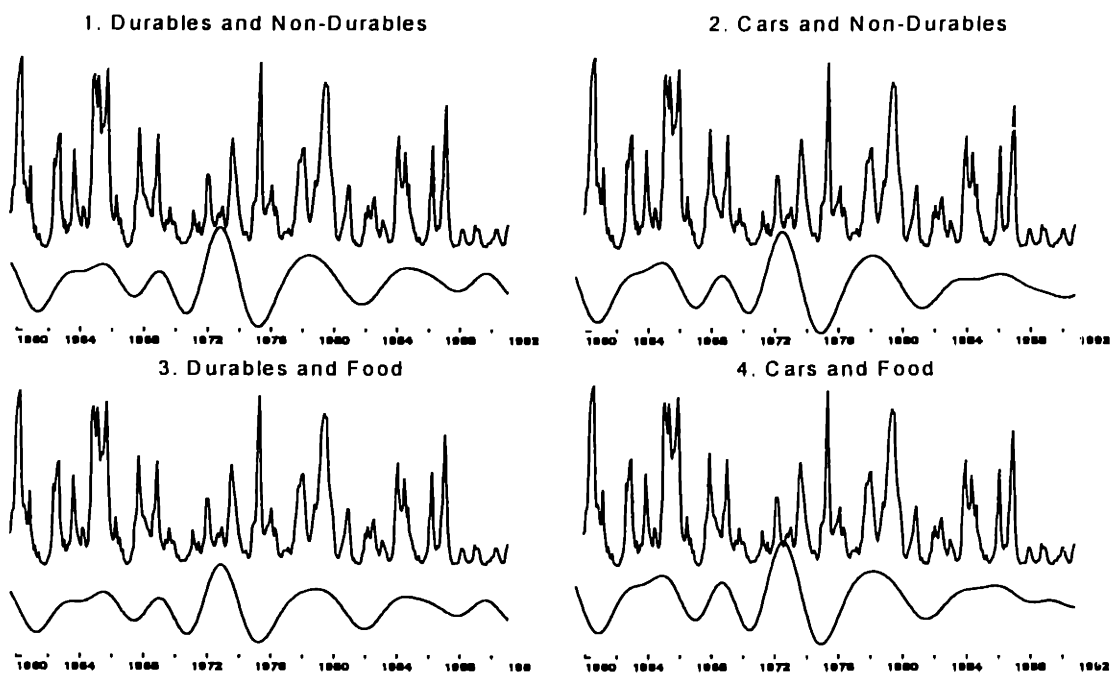


Table 3 Effect of Consumption Based State Probabilities

Variables	$\hat{\alpha}_3$*100	t-stat	$\hat{\alpha}_3$·100/avg(*D/C*)
D=Durables, *C*=Non-Durables	-0.344	-0.52	-0.98
D=Durables, *C*=Food	-1.508	-1.18	-2.27
D=Cars, *C*=Non-Durables	-0.319	-0.65	-3.58
D=Cars, *C*=Food	-0.882	-0.94	-5.07
D=Durables, *C*=S&P500	-2.371	-0.90	-1.81
D=Cars,*C*=S&P500	-1.611	-0.86	-4.89

Table 3 shows the result from regression (3.5) with consumption based state probabilities included. The effect is in all regressions estimated to be negative but not significant.

Figure 6 Business Cycle Residuals from regression (3.4) vs. P_t



4.3 Dynamic Effects

In the previous sub-sections I have estimated the immediate effect of a state shift on relative durables purchases. We would also like to know about dynamic and long run effects. These can be estimated by including lagged values of the differenced state probabilities in the regressions;

$$\frac{d_t - d_{t-1}}{c_{t-1}} = \sum_{s=1}^{6} \alpha_{1,s} \frac{\Delta c_{t+1-s}}{c_{t-1}} + \alpha_2 r_t + \sum_{s=1}^{5} \alpha_{3,s} \Delta P_{t+1-s} + \upsilon_t \qquad (4.1)$$

Table 4 shows the coefficients on ΔP_t lagged 0 to 5 months when S&P500 based state probabilities are used.[11] In column 3 I show the normalized coefficients as described above. By summing all 6 coefficients we get an estimate of the accumulated fall in the level of relative durables purchases 6 months after a shift to the high risk state (given that the state has not shifted back). These coefficient sums are given in the last table row.

Table 4 Dynamic Effects by S&P500 Based State Probabilities

	D=Durables, C=Non-Durables			*D=Durables, C= Food*			*D=Cars C=Non-Durables*			*D=Cars, C= Food*		
	1.	2.	3.	1.	2.	3.	1.	2.	3.	1.	2.	3.
ΔP_t	-0.895	-2.24	-2.62	-1.913	-2.38	-2.88	-0.600	-2.01	-6.72	-1.305	-2.22	-7.51
ΔP_{t-1}	-0.633	-1.63	-1.86	-1.184	-1.51	-1.78	-0.347	-1.19	-3.88	-0.633	-1.10	-3.64
ΔP_{t-2}	0.109	0.28	0.32	0.188	0.24	0.28	0.003	0.01	0.04	-0.018	-0.03	-0.10
ΔP_{t-3}	-0.064	-0.17	-0.19	-0.359	-0.46	-0.54	0.150	0.52	1.68	0.201	0.35	1.16
ΔP_{t-4}	0.041	0.11	0.12	0.175	0.22	0.26	-0.081	-0.28	-0.91	-0.112	-0.20	-0.65
ΔP_{t-5}	-0.730	-1.91	-2.14	-1.310	-1.69	-1.97	-0.477	-1.67	-5.35	-0.839	-1.48	-4.83
Sum	-2.172		-6.37	-4.403		-6.63	-1.352		-15.14	2.706		-15.57

1. $\hat{a}_{3,s} \cdot 100$
2. OLS t-statistic.
3. Regression coefficient ($\cdot 100$) normalized with the average value of *D/C*.

The previous result that the coefficient on contemporaneous ΔP_t is negative and significant does not change when the lags are included. We also find that the negative effect tends to accumulate over time. 6 months after a shift to the high risk state, relative purchases of durables are estimated to have fallen by 6-7% (column 3, last row). For cars the fall is estimated to 15-16%. In addition to a fall in purchases the first and second month after a state shift, also the last coefficient is negative for the variable combinations in table 4.[12] No coefficient on lagged state shifts, however, has a t-statistic with an absolute value larger than 2.

[11] I back-cast 6 months when estimating the state probabilities. As described in appendix 2 I may then include P_t lagged up to 6 months without getting an errors in variable problem.

[12] This is not an artifact of the back-casting procedure. Also if no back-casting is used I find this pattern.

In table 5 I show the coefficients on ΔP_t lagged 0 to 5 months for consumption based state probabilities. Only the coefficient on the last lag is significant. It is also positive, which is surprising.

Table 5 Dynamic Effect by Consumption Based State Probabilities

	D=Durables, C=Non-Durables			*D=Durables, C= Food*			*D=Cars C=Non-Durables*			*D=Cars, C= Food*		
	1.	2.	3.	1.	2.	3.	1.	2.	3.	1.	2.	3.
ΔP_t	0.095	0.13	0.28	-1.077	-0.78	-1.62	-0.013	-0.02	-0.15	-0.468	-0.46	-2.69
ΔP_{t-1}	-0.280	-0.36	-0.82	0.838	0.57	1.26	-0.153	-0.26	-1.71	0.254	0.23	1.46
ΔP_{t-2}	0.131	0.17	0.39	0.071	0.05	0.11	0.169	0.29	1.89	0.390	0.35	2.24
ΔP_{t-3}	-0.097	-0.12	-0.29	0.403	0.27	0.61	-0.135	-0.23	-1.52	0.103	0.09	0.60
ΔP_{t-4}	-0.152	-0.19	-0.45	-0.710	-0.48	-1.07	0.264	0.45	2.96	0.110	0.10	0.63
ΔP_{t-5}	2.049	2.84	6.01	4.083	2.91	6.14	1.345	2.50	15.07	2.595	2.53	14.94
Sum	1.746		5.12	3.608		5.43	1.477		16.54	2.984		17.18

1. $\hat{a}_{3,s} \cdot 100$
2. OLS t-statistic.
3. Regression coefficient ($\cdot 100$) normalized with the average value of *D/C*.

5. Discussion

The following points summarize the main empirical findings in this chapter:

- Financial volatility is significantly higher during extended periods of time. The average length of the high risk periods is 7.4 months while the average low risk period is 28 months.
- Durables relative to non-durables purchases are substantially lower in periods of high financial volatility. The effect is larger than what can directly be accounted for by variations in precautionary savings as measured by the fall in non-durables purchases.
- The fall is substantially larger for cars than for durables.
- The negative relation between financial volatility and durables purchases is stronger at business cycle frequencies.
- The fall in durables purchases after a shift to the high risk state appears to accumulate over time, although only the immediate effect is statistically significant.
- A shift to higher volatility of non-durables purchases is not associated with a corresponding shift in relative durables purchases.

In the introduction I noted that the level of risk is a critical parameter in irreversible investment models. This was also one of the results in chapter III where I constructed a model where stochastic fluctuations in the level of risk may give large effects on durables purchases. The mechanism at work is the following: consumer durables can only be traded at a discrete transaction cost, representing for example a search cost. Reversing the purchase of a durable is thus costly and this creates an option value of waiting with the investment. A shift to higher risk increases the probability that the investor wants to reverse the investment shortly after it has been done. The option value of waiting is then increasing in the level of risk. A shift to high risk then causes delayment of purchases. This mechanism is not affecting non-durables purchases.

It is tempting to interpret the findings as evidence in favor of an irreversible investment model. The second point above indicates that durables purchases are indeed more sensitive to variations in risk than non-durables as predicted by the model. The third result is that cars appear to be more sensitive than the broader aggregate of durables. It seems reasonable that the transaction costs are relatively large for cars. A car purchase also represents a large investment for a typical household. Within an irreversible investment model we may there-

fore expect car purchases to be more sensitive to risk than the average durable. The fourth point above indicates that irreversibility effect may be of importance in a business cycle context.

We thus get some support for the hypothesis that an irreversible investment model as in chapter III can help to explain variations in durables purchases. The other results, on the other hand, indicates that this conclusion may be somewhat premature; there are still questions not answered.

First, within the irreversible investment model in chapter III we expect a discrete jump in the level of risk to have a short-run effect on durables purchases that overshoots the medium and long run effect. The reason is that the negative effect of an increase in risk is due to that people delay up-grading of their durables for a while. But, eventually, depreciation makes further waiting too costly and purchases pick up again. Similarly, if risk jumps back to a low level we immediately get many people who upgrade their durables, but this increase in purchases is (partly) temporary.

The third point above thus appears to be at odds with the dynamic behavior of purchases in an irreversible investment model. A potential explanation is, however, the simplification introduced by only allowing two levels of risk. A more realistic model would allow risk to change more smoothly. A way to do this is to estimate the level of risk using ARCH and GARCH models from the finance literature. Here the level of risk is estimated in a way that allows a smooth transition from low to high risk.

It may also be unrealistic to assume that all consumers know the true state of the economy. It may be that also consumers have to make an inference about the level of risk they face. If they, furthermore, do not have access to the same information it may be that they react with different time-lags to a shift in risk. This could produce a dynamic pattern of the type observed in data. A model constructed along these lines could be a fruitful exercise.

Secondly, the irreversibility interpretation above relies on the assumption that periods of high financial volatility tend to coincide with periods of high volatility of household wealth. But if household wealth is unusually volatile also non-durables consumption should be unusually volatile. This appears not to be the case. State probabilities based on the S&P500 index are negatively correlated with state probabilities based on non-durables con-

sumption. Furthermore, state probabilities based on non-durables consumption cannot significantly help predict durables purchases.

The assumed two state process with extended periods of high volatility appears to be a bad description of the data generation process for non-durables consumption. It could be the case that innovations to aggregate non-durables purchases to a non-negligible extent are driven by liquidity variation among liquidity constrained households. A shift in aggregate non-durables purchases is then not always caused by shifts in household wealth. The standard remedy in this situation is to use an observable variable, e.g., as bank assets or income, to distinguish between households with different likelihood of being liquidity constrained. Disregarding data availability problems, this method should work also here.

Lastly, the differences in aggregate volatility between the two states that I find in section 2 are fairly small. Both financial and consumption volatility is less than twice as large in the high risk state as in the low risk state. In the model in chapter III a doubling of the arrival rate of wealth shocks from 1 to 2% per year had relatively negligible effects. In order for the model to be able to generate non-negligible fluctuations in purchases of durables we must assume that individual risk levels are much more volatile than aggregate risk levels are. This may be the case. Some idiosyncratic risk may be very important for the individual household without affecting aggregate volatility. An example of this is unemployment. Further work should be directed at measuring how volatile idiosyncratic risk is. More direct measures of individual risk, e.g., flows into unemployment, could also be included in regression models for durables purchases.

Despite the remaining question marks it appears that variations in risk have substantial impact on the demand for durables. Durables purchases have a significant negative relation to the level of financial volatility. The size of the effect is large enough to be non-negligible and thus warrants further investigation along the lines suggested above.

Appendix 1

Assume provisionally that we know the parameters of the model, i.e., $\{\mu_0, \mu_1, \mu_2, \lambda_0, \lambda_1, \gamma_0, \gamma_1\}$. Assume also that we know $P\left(s_{t-1} = S_{t-1} \middle| \{W\}_1^{t-1}\right)$. For each of the four pairs of $\{S_t, S_{t-1}\}$ the following 5 steps are followed. In each step the R.H.S. variables are known from assumptions or previous steps.

Step 1.

$$P\left(s_t = S_t, s_{t-1} = S_{t-1} \middle| \{W\}_1^{t-1}\right) = P(s_t = S_t | S_{t-1}) P\left(s_{t-1} = S_{t-1} \middle| \{W\}_1^{t-1}\right) \qquad \text{(A.1)}$$

where $P(s_t = S_t | S_{t-1})$ is known from the transition probabilities in (2.2). We then calculate the joint conditional density of $\{W_t\text{-}W_{t-1}, S_t, S_{t-1}\}$.

Step 2.

$$\begin{aligned} & f\left((W_t - W_{t-1}), S_t, S_{t-1} \middle| \{W\}_1^{t-1}\right) = \\ & \Phi\left[(W_t - W_{t-1}) - (\mu_0 + (\mu_1 - \mu_0)S_t) - \mu_2 \Delta S_t, (\lambda_0 + (\lambda_1 - \lambda_0)S_t)\right] \times \\ & P\left(s_t = S_t, s_{t-1} = S_{t-1} \middle| \{W\}_1^{t-1}\right) \end{aligned} \qquad \text{(A.2)}$$

where $\Phi[.,\sigma]$ denotes the normal density with zero mean and standard deviation σ. The next step is to calculate the marginal density of $W_t\text{-}W_{t-1}$.

Step 3.

$$f\left(W_t - W_{t-1} \middle| \{W\}_1^{t-1}\right) = \sum_{S_t=0}^{1} \sum_{S_{t-1}=0}^{1} f\left(W_t - W_{t-1}, S_t, S_{t-1} \middle| \{W\}_1^{t-1}\right). \qquad \text{(A.3)}$$

Now using the results from *Step 2* and *3*, we have:

Step 4.

$$p\left(s_t = S_t, s_{t-1} = S_{t-1} \middle| \{W\}_1^{t}\right) = \frac{f\left(W_t - W_{t-1}, S_t, S_{t-1} \middle| \{W\}_1^{t-1}\right)}{f\left(W_t - W_{t-1} \middle| \{W\}_1^{t-1}\right)} \qquad \text{(A.4)}$$

and

Step 5

$$p\left(s_t = S_t \middle| \{W\}_1^t\right) = \sum_{S_{t-1}=0}^{1} p\left(s_t = S_t, s_{t-1} = S_{t-1} \middle| \{W\}_1^t\right), \text{ and}$$
$$p\left(s_{t-1} = S_{t-1} \middle| \{W\}_1^t\right) = \sum_{S_t=0}^{1} p\left(s_t = S_t, s_{t-1} = S_{t-1} \middle| \{W\}_1^t\right). \tag{A.5}$$

By using a longer history of states to start *Step 1* with, e.g., $\{s_t, s_{t-1}, \ldots, s_{t-k}\}$, we may "back-cast" further and calculate $p\left(s_{t-k} = S_{t-k} \middle| \{W\}_1^t\right)$.

As already noted, the procedure above requires knowledge of the parameters $\{\mu_0, \mu_1, \mu_2, \lambda_0, \lambda_1, \gamma_0, \gamma_1\}$. Following the suggestion by Hamilton (1988) I have estimated them by maximizing the log-likelihood function:

$$L\left(\mu_0, \mu_1, \mu_2, \lambda_0, \lambda_1, \gamma_0, \gamma_1 \middle| \{W\}_1^T\right) = \sum_{t=2}^{T} f\left(W_t - W_{t-1} \middle| \{W\}_1^{t-1}\right) \tag{A.6}$$

where the R.H.S. of (A.6) is taken from *Step 3* above.

The recursive structure of the method also requires an initial set of likelihoods $P(s_1 = S_1 | W_1)$. For t prior to the first observation of W_t I have approximated the conditional likelihoods with the unconditional, i.e., I have set

$$P(s_1 = 0 | W_1) = P(s_1 = 0) = \frac{\gamma_1}{\gamma_0 + \gamma_1}$$
$$P(s_1 = 1 | W_1) = P(s_1 = 1) = \frac{\gamma_0}{\gamma_0 + \gamma_1}. \tag{A.7}$$

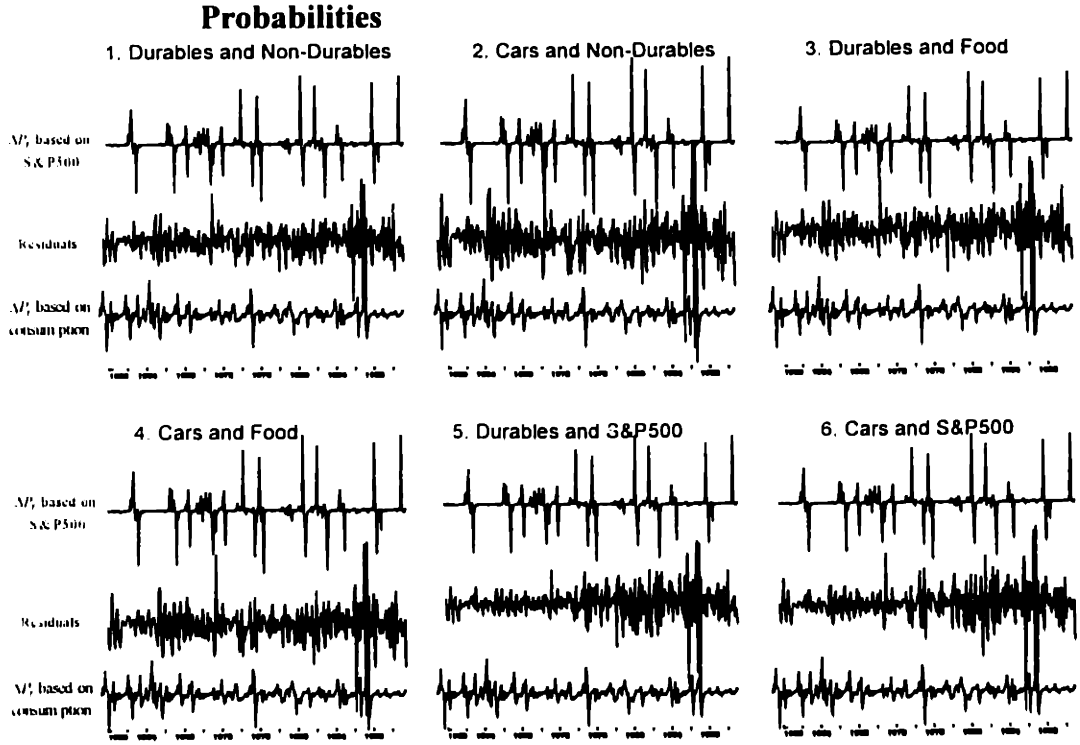
Appendix 2

Table 6 Results from Regression (3.4)†

	D=Durables, *C*=Non-Durables		*D*=Cars, *C*=Non-Durables		*D*=Durables, *C*=Food		*D*=Cars, *C*=Food		*D*=Durables, *C*=S&P500		*D*=Cars, *C*=S&P500	
		t-stat*		t-stat		t-stat		t-stat		t-stat		t-stat
α_0*	1.003	0.71	0.987	-1.28	1.005	1.30	0.992	-0.81	1.004	2.56	0.998	-0.46
α_{10}	0.293	3.65	0.104	1.76	0.061	0.41	-0.061	-0.56	0.157	2.12	0.078	1.49
α_{11}	0.007	0.08	0.044	0.72	-0.107	-0.66	-0.046	-0.40	0.044	0.59	0.055	1.06
α_{12}	-0.010	-0.12	0.019	0.32	0.027	0.17	0.084	0.73	0.001	0.02	-0.015	-0.27
α_{13}	-0.100	-1.19	-0.027	-0.45	0.068	0.42	0.155	1.37	0.073	0.98	0.023	0.43
α_{14}	-0.046	-0.55	0.015	0.24	0.032	0.19	0.123	1.06	0.013	0.17	0.012	0.22
α_{15}	-0.024	-0.28	0.033	0.53	0.014	0.09	0.073	0.62	0.043	0.58	0.034	0.64
α_{16}	0.039	0.48	0.043	0.72	-0.053	-0.35	-0.030	-0.27	0.063	0.86	0.047	0.91
α_2	0.001	0.75	0.000	-0.01	0.003	1.44	0.001	0.49	0.008	1.85	0.002	0.72
R^2	0.923		0.681		0.941		0.662		0.991		0.392	
D-W	2.349		2.147		2.356		2.163		2.412		2.316	
Degrees of Freedom	368		368		368		368		368		368	

† $$\frac{d_t}{c_{t-1}} = \alpha_0 \frac{d_{t-1}}{c_{t-1}} + \sum_{s=1}^{6} \alpha_{1,s} \frac{\Delta c_{t+1-s}}{c_{t-1}} + \alpha_2 r_t + \upsilon_t,$$ * t-statistic for H_0: $\alpha_0=1$

Figure 7 Residuals from Regression (3.4) and Changes in State Probabilities



Appendix 3

Given that the model for wealth in (2.1) and (2.2) is true and that we know the models parameters the Hamilton procedure gives estimates with the following property;

$$E\left(s_t \middle| \{W\}_1^{t+k}\right) = P\left(s_t = 1 \middle| \{W\}_1^{t+k}\right) \tag{A.8}$$

where E is the expectations operator, $P(\cdot)$ represents the Hamilton probability and $k \geq 0$ is the number of back-casting steps. Then the difference between the true state and the probability of state 1 is independent of, and in particular uncorrelated with W and ΔW, i.e., for

$$\begin{aligned} &\varepsilon_t \equiv s_t - P\left(s_t = 1 \middle| \{W\}_1^{t+k}\right), \\ &E\left(\varepsilon_t \middle| \{W\}_1^{t+l}\right) = E\left(\varepsilon_t P\left(s_t = 1 \middle| \{W\}_1^{t+l}\right)\right) = 0 \ \forall l \leq k. \end{aligned} \tag{A.9}$$

By rearranging (3.3) and substituting from (A.9) we then get the following regression

$$\begin{aligned} \frac{d_t - d_{t-1}}{w_{t-1}} &= \sum_{s=1}^{L} \alpha_{1,s} \frac{\Delta w_{t+1-s}}{w_{t-1}} + \alpha_2 r_t + \alpha_3 \Delta s_t + v_t \\ &= \sum_{s=1}^{L} \alpha_{1,s} \frac{\Delta w_{t+1-s}}{w_{t-1}} + \alpha_2 r_t + \alpha_3 \Delta P_t + v_t + \alpha_3 (\varepsilon_t - \varepsilon_{t-1}) \end{aligned} \tag{A.10}$$

where the extra error term, $\alpha_3(\varepsilon_t - \varepsilon_{t-1})$, is uncorrelated with the regressors. By moving d_{t-1} to the R.H.S. of the regression we eliminate the errors in variable problem caused by the correlation between d_{t-1} and ε_{t-1}. ε is of course not normal, it is for example bounded between -1 and 1, so small sample t-statistics are not correct. Errors in the measurement of w and r will also cause a standard errors in variable problem.

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