SOUND SCATTERING BY A CYLINDRICAL VORTEX

by

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ABSTRACT

The equations of motion appropriate for waves in moving and inhomogeneous media are developed from the basic equations of hydrodynamics with sufficient generality to allow application to all geometries and all reversible flows. Scalar and vector potentials are used to describe the sound particle velocity; the general equations are shown to reduce to two coupled equations in the potentials.

Particular applications of the equations of motion are made to two-dimensional cylindrical vortex flow. Two types of vortex flow are considered. One is the commonly known ideal vortex consisting of a rotational core and an induced irrotational flow field; the other is a "rotor" consisting only of a rotational region. The vector potential is approximately evaluated for each in terms of the scalar potential, and a single second order differential equation is developed to describe the effect of the motion on sound waves.

The differential equation appropriate for sound propagation through a rotor is recast into an integral equation. The latter is solved by means of a Born-Kirchoff approximation and a WKBJ trial function approximation. The calculations show that sound scattering from a rotor is important when the rotor circumference is greater than the sound wavelength multiplied by the square root of the ratio of the sound frequency to the rotor frequency, and when the rotors occur in large numbers. These calculations are in agreement with measurements made over large distances near the ground.

The refracted field caused by the irrotational part of an ideal vortex is calculated using an approximate WKBJ technique. It is shown to be antisymmetric in phase, and may explain the large phase fluctuations measured in the atmosphere.

Supervisor: K. U. Ingard, Assistant Professor of Physics

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I. Introduction

Problems dealing with the acoustics of moving inhomogeneous media have recently become of increased practical importance. Many experimental and theoretical investigations have consequently been initiated with the hope of gaining fuller fundamental and applied knowledge of this general subject. For example, the intense noise field associated with the advent of the jet and rocket engine has sparked investigation of the noise of aerodynamic origin.[29]* It may seem somewhat surprising that the subject of moving inhomogeneous media should have been neglected for so long in a science as old as acoustics. Actually many investigations of a fundamental nature were carried out in the last century and the early part of this century, dealing with such topics as aeolian harps, singing flames,[30] fog-horn sound propagation in the atmosphere,[31] etc. Since that time, however, there has developed a fuller understanding of the subject of non-radiative flow, that is, the subject of aerodynamics. The science and technology of aerodynamics has had two effects on the field of acoustics. It has produced powerful noise sources, so powerful that the problem now is to find means of quieting these sources, whereas the studies of the last century were usually oriented toward finding means of producing louder noise sources. It has also provided a better understanding of rotational flow, particularly turbulence, which is inherently present in most problems dealing with moving inhomogeneous media.

This thesis is oriented to the problem of sound propagation in the atmosphere. As pointed out above, the problem is an old one. But because of the louder noise

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*Numbered references are listed on pages xvi, and xvii.

sources produced by present technology, and the consequent annoyance heaped upon those who live in the environs, this subject has received new interest. It also has connection with several military problems; the detection of low-flying aircraft, and the analogous undersea problem, the detection of submerged vessels, have also sparked new interest in the subject.

More particularly, the prime objective of this thesis is to develop the theory of the propagation of sound in moving media under a condition in which the flow is rotational. While a considerable literature exists on studies of sound propagation in moving media, almost all the work deals with irrotational flow. The notable exceptions, however, are recent studies dealing with the propagation of sound through turbulent flow; they were carried out by M. J. Lighthill(27) in England, D. Blokhintzev(3) in Russia, and R. H. Kraichnan(32) in this country. The rotational flow to be considered in this thesis is ordered vortex motion. The reason for studying ordered vortex flow, rather than turbulent flow, is three-fold: First of all, for the particular application of interest, atmospheric propagation, there is reason to believe that large scale vortex motion forms an important part of the flow near the ground. Secondly, complete calculations of the effect of turbulent flow on sound propagation have succeded only in the case of isotropic turbulence, a condition known to be incorrect for the flow near the ground. (Some of the work of Lighthill is an exception. Under the restriction of high frequencies, he has succeded in obtaining an expression for the total power scattered out of plane waves for an arbitrary turbulent field, although he was unable to obtain the scattering pattern.) Finally, it appears desirable in a problem of this kind to proceed from the simplest configuration of the scattering process to the more complex; the simplest configuration of rotational flow is ordered vortex motion.

In particular, the scattering of plane waves of sound from two-dimensional time-independent circularly cylindrical vortices will be considered. This problem does not appear to have been solved before. Using geometrical techniques, R. B. Lindsay(26) has calculated the refraction of a plane wave only in the induced irrotational region of a vortex, but one of the important conclusions of the present study is that the rotational core of a vortex gives rise to far more important effects.

This investigation is naturally divided into two parts. The first part, contained in Chapter II, deals with the derivation of the appropriate equations of motion. The second part, contained in the remaining Chapters, deals with the application of the equations of motion to the scattering problem.

In Chapter II, the equations of motion are developed with sufficient generality to be applicable to all geometries and all steady reversible flows. The well-known laws of hydrodynamics and thermodynamics are used as the starting point. All pertinent variables are divided into two components. One component is associated with the net flow, and is taken to be independent of the presence of sound. The other component is associated with the sound. This well-known technique gives rise to a set of equations for the sound variables in terms of the net flow variables. Next, scalar and vector potentials are introduced for the sound particle velocity in order to reduce the equations to the form of inhomogeneous wave equations. This rather obvious step does not appear to have been taken before by other workers, and leads to a general formulation of the equations of motion which has not been given before. It is shown that in general the sound particle velocity is rotational, a fact which is not clearly recognized in most of the pertinent literature. To date, the best treatment of the equations of motion is given in the book by D. Blokhintzev[3]. Although he recognizes that the sound particle velocity is rotational in the special case of rotational net flow, he does not use a general vector potential, and his results are thus more specialized than that given here.

In Chapters III and IV, the scattering from vortex motion is obtained approximately by the use of various techniques similar to those familiar to several branches of physics. A Born-Kirchoff approximation is used to obtain the angle distribution of the scattered sound, a WKBJ solution improved by an integral equation iteration is used to obtain the scattering cross-section, and a WKBJ solution is used to compute the refraction by the induced velocity field of a vortex core. The results of these calculations are shown to be in agreement with measurements on sound propagated over ground, obtained by K. U. Ingard.[18]

On the basis of the calculations, it is shown in Chapter V that the conditions for measuring vortex scattering in the laboratory are unfavorable. This fact has been verified by an attempted underwater experiment. On the other hand, vortex scattering in the atmosphere is a measurable effect, and further experimental work involving extensive field tests is suggested.

II THE EQUATIONS OF MOTION

In the usual case, the acoustic equations are obtained by linearizing the combined equations of the conservation of mass, momentum, energy, and the equation of state. For a uniform, quiescent, isentropic, medium described by the ideal gas equation, the scalar wave equation may be obtained:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad (2.1)$$

where

- ψ is a scalar completely describing the field, usually the particle velocity potential, $\underline{u} = -\nabla\psi$
- $\underline{u}$ is the sound particle velocity,
- c is the adiabatic sound velocity $= \sqrt{\frac{\gamma p}{\rho}}$
- γ is the ratio of specific heats,
- p is the ambient pressure,
- ρ is the ambient density.

If the problem is complicated one step by the presence of viscosity, the field is no longer scalar, but becomes a vector field. The sound particle velocity is rotational or vortical, and by Helmholtz's theorem it can be written as:

$$\underline{u} = -\nabla\psi + \nabla \times \underline{A} \quad ; \quad \nabla \cdot \underline{A} = 0 \qquad (2.2)$$

where $\underline{A}$ is the vector potential.

The resulting field equations are then,[1]

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{1}{c^2}\left(\frac{4}{3}\nu + \eta\right)\nabla^2\frac{\partial\psi}{\partial t} = 0 \qquad (2.3)$$

$$\nabla^2\underline{A} = \frac{1}{\nu}\frac{\partial \underline{A}}{\partial t} \qquad (2.4)$$

where ν is the kinamatic shear viscosity $= \frac{\mu}{\rho}$,

μ is the shear viscosity coefficient,

η is the kinematic compressive viscosity $= \frac{\lambda}{\rho}$,

λ is the compressive viscosity coefficient.

Equation (2.3) may be interpreted as the wave equation with a damping term; equation (2.4) may be interpreted as a vector diffusion equation. If the viscosity terms are small (as indeed they usually are), then,

$$\frac{\partial \underline{A}}{\partial t} \simeq 0$$

except near solid boundaries, where $\nabla^2 \underline{A}$ may be large. Thus, to a good approximation, at large distances from boundaries the sound particle velocity is irrotational. Under these circumstances, the effect of viscosity on wave motion is exhibited only as a small damping term, which may be introduced by solving the wave equation (2.1) with complex wave numbers or frequencies.

In the same way, another complicating irreversible effect, heat conduction, may be disregarded at large distances from boundaries; its presence can also be taken into account by proper damping parameters. Inasmuch as this thesis is concerned primarily with sound phenomena related to the flow and the inhomogeneity of the medium, the effects of viscosity and heat conduction on the sound will be disregarded, with the knowledge that they may be accounted for in most physically realizable problems by simple damping parameters.

In this chapter, the equations of motion appropriate for the propagation of sound in an inhomogeneous moving medium are developed. Although the equations are derived from well known physical relations, they are developed in considerable detail, because the general results form the core of present and similar problems and do not appear to have been presented before in the literature.

The program to be followed is this: It is assumed that for each flow parameter (such as velocity, pressure, etc.) a sum of two quantities can be introduced into the equations of mass, momentum, energy, and state. One quantity of the sum describes that part of the total flow which is associated with the net motion of the medium; the other quantity describes that part of the total flow which is associated with the wave motion. The equations of mass, momentum, energy, and state for the net flow are separated out from the equations for the total flow, assuming that the wave flow does not alter the net flow. This process is essentially a first order perturbation, and results in a set of equations for the wave flow, containing parameters of the net flow. It is further assumed that viscosity and heat conduction effects are minor in the wave motion, as discussed above, and consequently are discarded. Hence, the resulting equations describe only the effect of the net flow and the concomitant inhomogeneity on the wave flow. Next the equations are linearized, and scalar and vector potentials are introduced for the velocity. Finally the equations are specialized for the present problem, in such a way that the d'Alembertian, or wave equation operator,

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad (2.5)$$

can be abstracted. The remaining terms may then be interpreted as the inhomogeneous terms that give rise to scattering or diffraction, and hence are of interest in this study.

A. EQUATIONS OF MASS, MOMENTUM, ENERGY AND STATE

The appropriate equations of fluid motion are well known.(2)(3) In the Eulerian reference frame they may be written as:

Conservation of Mass:

$$\frac{\partial D}{\partial t} + \nabla \cdot (D\underline{V}) = 0 \qquad (2.6)$$

Conservation of Momentum:

$$\frac{\partial \underline{V}}{\partial t} + (\nabla x \underline{V}) \ x \ \underline{V} + \frac{1}{2}\nabla(\underline{V} \ . \ \underline{V}) = - \frac{1}{D}\nabla P$$

$$+ \ \underline{g} + \nu\nabla^2\underline{V} + (\frac{\nu}{3} + \eta)\nabla\nabla \cdot \underline{V} \qquad (2.7)$$

Conservation of Energy:

$$T\frac{\partial S}{\partial t} + T\underline{V} \ . \nabla S = C_V K \nabla^2 T + \frac{Q}{D} \qquad (2.8)$$

Equation of State (Ideal Gas):

$$P = RDT \qquad (2.9)$$

where D is the density,
$\underline{V}$ is the velocity,
P is the pressure,
$\underline{g}$ is the body force per unit mass,
S is the entropy,
T is the temperature,
C_V is the specific heat at constant volume,
K is the coefficient of thermal conductivity,
Q is the dissipation function, i.e. the heat lost per unit volume by viscous forces (2),
R is the gas constant.

Note that if K, ν, and η are small, as they usually are, equation (2.8) is very nearly:

$$\frac{\partial S}{\partial t} + \underline{V} \cdot \nabla S = 0$$

or

$$\frac{dS}{dt} = 0$$

or

$$S = \text{constant.}$$

Consequently the fluid motion is isentropic in the limit of zero heat conductivity and viscosity, and very nearly isentropic for fluids with small heat conductivity and viscosity.

It is convenient to rewrite equation (2.9) in the variables P, D, and S, rather than P, D, and T,

$$\frac{P}{P_o} = \left(\frac{D}{D_o}\right)^{\gamma} e^{\frac{S - S_o}{C_v}} \qquad (2.9)'$$

where the subscript zero corresponds to any given state of the medium.

Let:

$$\begin{aligned} \underline{V} &= \underline{v} + \underline{u} && \text{velocity} \\ P &= p + \pi && \text{pressure} \\ D &= \rho + \delta && \text{density} \\ S &= s + \sigma && \text{entropy} \end{aligned} \qquad (2.10)$$

where $\underline{v}$, p, ρ, s are associated with the net flow, $\underline{u}$, π, δ, σ are associated with the wave flow.

Inserting relations (2.10) into equations (2.6), (2.7), and (2.8), results in the following set of equations for the wave flow variables:

$$\frac{\partial \delta}{\partial t} + \nabla . (\delta \underline{v}) + \nabla . (\rho \underline{u}) + \nabla . (\delta \underline{u}) = 0 \qquad (2.11)$$

$$\frac{\partial \underline{u}}{\partial t} + (\nabla x \underline{u}) \text{ x } \underline{v} + (\nabla x \underline{v}) \text{ x } \underline{u} + (\nabla x \underline{u}) \text{ x } \underline{u}$$

$$+ \nabla(\underline{v} . \underline{u}) + \frac{1}{2}\nabla(\underline{u} . \underline{u}) = - \frac{1}{\rho}\nabla \pi + \frac{\delta}{\rho^2}\nabla p + \frac{\delta}{\rho^2}\nabla \pi \qquad (2.12)$$

$$\frac{\partial \sigma}{\partial t} + \underline{v} . \nabla \sigma + \underline{u} . \nabla s + \underline{u} . \nabla \sigma = 0 \qquad (2.13)$$

$$\pi = \frac{\gamma p}{\rho}\delta + \frac{p}{C_v}\sigma = c^2 \delta + \frac{p}{C_v}\sigma \qquad (2.14)$$

In order to obtain the above set, the body force has been taken as zero, and the net flow variables are required to satisfy the original equations of mass, momentum, energy, and state:

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \underline{v}) = 0 \qquad (2.15)$$

$$\frac{\partial \underline{v}}{\partial t} + (\nabla x \underline{v}) \text{ x } \underline{v} + \frac{1}{2}\nabla(\underline{v} . \underline{v}) = - \frac{1}{\rho}\nabla p$$

$$+ \nu \nabla^2 \underline{v} + (\frac{\nu}{3} + \eta) \nabla \nabla . \underline{v} \qquad (2.16)$$

$$\frac{\partial s}{\partial t} + \underline{v} . \nabla s = \frac{C_v K}{T} \nabla^2 T + \frac{Q}{\rho T} \qquad (2.17)$$

$$\frac{p}{p_o} = (\frac{\rho}{\rho_o})^{\gamma} e^{\frac{s - s_o}{C_v}} \qquad (2.18)$$

In addition, it has been assumed that

$$\left(\frac{\delta}{\rho}\right)^2 \ll 1,$$

and

$$\left(\frac{\sigma}{C_v}\right)^2 \ll 1.$$

It is important to emphasize that equations (2.15) to (2.18) enforce a given net motion on the medium, independent of the presence of wave motion. Hence the reverse problem, that of computing the generation of net motion by wave motion cannot be attacked with this formulation. Furthermore, this formulation obviously excludes interaction effects between the wave and net motions. On the other hand, it is undoubtably true that the above formulation is an excellent approximation to most physically realizable situations of the type under study in this thesis.

The net flow equations will be specialized for steady motion, so that

$$\frac{\partial \rho}{\partial t} = \frac{\partial \underline{v}}{\partial t} = \frac{\partial s}{\partial t} = 0 \qquad (2.19)$$

In addition, it will be assumed that irreversible effects are small, thus giving rise to isentropic net flow. For steady motion, this can be expressed as:

$$\nabla s = 0 \qquad (2.20)$$

B. GENERAL EQUATIONS FOR THE WAVE FLOW

Equations (2.11) through (2.14) are differential relations for the wave flow variables. They are rewritten below using the condition (2.20) and omitting all terms containing the product of two wave flow variables. The resulting equations are thus the linearized isentropic equations of sound propogation in an inhomogeneous steadily moving isentropic gas:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot (\delta \underline{v}) + \nabla \cdot (\rho \underline{u}) = 0, \qquad (2.21)$$

$$\frac{\partial \underline{u}}{\partial t} + (\nabla \times \underline{u}) \times \underline{v} + (\nabla \times \underline{v}) \times \underline{u} + \nabla (\underline{v} \cdot \underline{u}) = - \frac{1}{\rho} \nabla \pi + \frac{\delta}{\rho^2} \nabla p \qquad (2.22)$$

$$\frac{\partial \sigma}{\partial t} + \underline{v} \cdot \nabla \sigma = \frac{d\sigma}{dt} = \text{constant} = 0 \qquad (2.23)$$

$$\pi = \frac{\gamma p}{\rho} \delta = c^2 \delta \qquad (2.24)$$

Strictly speaking, the equations given above are valid only in a region in which $\underline{v}$ is much greater than $\underline{u}$. If, however, the net flow field is such that through some region of the field $\underline{v}$ is either comparable to or less than $\underline{u}$ (as indeed it may be in vortex motion), then in that region the terms containing $\underline{v}$ are smaller than the omitted terms containing the product of the wave variables, and hence should be dropped. Consequently the amplitude of the wave defines the "size" of the moving medium, to the extent that it is possible to define a surface within which equations (2.21) through (2.24) are to be used, and exterior to which the same equations are to be used with, however, $\underline{u}$ replacing $\underline{v}$ in all terms. But if it is insisted that only linear wave phenomena are to be studied, then it is clear that in general the size-defining-surface recceeds to infinity, and that, strictly speaking, equations (2.21) through (2.24) may be used to derive results which will be valid only in the limit of zero wave amplitude. This limitation is similar to the limitation imposed by linearizing the equations without flow, and is therefore to be considered relatively unimportant for most practical purposes.

Equation (2.22) may be simplified by noting that ∇p may be written as:

$$\nabla p = \left.\frac{\partial p}{\partial \rho}\right|_s \nabla \rho = c^2 \nabla \rho \qquad (2.25)$$

by virtue of the isentropic condition and the gas law expressed by equations (2.20) and (2.18). Hence:

$$\frac{\partial \underline{u}}{\partial t} + (\nabla \times \underline{u}) \times \underline{v} + (\nabla \times \underline{v}) \times \underline{u} + \nabla (\underline{v} \cdot \underline{u}) = -\nabla\left(\frac{\pi}{\rho}\right) \qquad (2.22)'$$

To facilitate the combination of equations (2.21), (2.22)', and (2.24), it is convenient to introduce scalar and vector potentials for $\underline{u}$:

$$\underline{u} = -\nabla \psi + \nabla \times \underline{A}, \quad (\nabla \cdot \underline{A} \text{ is arbitrary}) \qquad (2.26)$$

It is clear that $\underline{u}$ can always be divided into a longitudinal and a transverse part. But the physical interpretation of the transverse part is not always clear. For example, in the case of waves in a viscous fluid discussed at the beginning of this chapter, the pressure is not affected by the transverse part of the velocity.[1] Consequently, from the point of view of a pressure measuring device, the transverse part would not be classed as wave motion. On the other hand, from the point of view of a velocity measuring device, the transverse part may be classed as wave motion if the measurement is made at one point in the field. But if the velocity measuring device traverses the field, the transverse part would be identified as a diffusion phenomenon, and not true wave motion. Analogously the physical interpretation of the transverse part of $\underline{u}$ will depend upon the method and the variable measured. $\underline{u}$ was defined earlier as that part of the total flow associated with wave motion; now its definition must be broadened to include that part of the total flow which is not associated with the net flow. Thus both the scalar and vector potentials defined by equation (2.26) will be carried along formally; the physical interpretation of $\underline{A}$ will be left for particular examples.

1. General Equations for ψ and $\underline{A}$.

According to equation (2.26), the vector momentum equation (2.22)' is:

$$-\nabla\frac{\partial\psi}{\partial t} + \nabla \times \frac{\partial \underline{A}}{\partial t} - \nabla(\underline{v}\cdot\nabla\psi) + \nabla(\underline{v}\cdot\nabla\times\underline{A}) \qquad (2.27)$$

$$+ \nabla\psi \times (\nabla\times\underline{v}) - (\nabla\times\underline{A})\times(\nabla\times\underline{v}) - \underline{v}\times(\nabla\times\nabla\times\underline{A}) = -\nabla\left(\frac{\pi}{\rho}\right)$$

let

$$\begin{aligned} \nabla\psi \times (\nabla\times\underline{v}) &= -\nabla\alpha_1 + \nabla\times\underline{B}_1, \\ (\nabla\times\underline{A})\times(\nabla\times\underline{v}) &= -\nabla\alpha_2 + \nabla\times\underline{B}_2, \\ \underline{v}\times(\nabla\times\nabla\times A) &= -\nabla\alpha_3 + \nabla\times\underline{B}_3, \\ \nabla\cdot\underline{B}_i \text{ is arbitrary, } i &= 1,\ 2,\ 3, \end{aligned} \qquad (2.28)$$

In this way, equation (2.27) may be written as the sum of two vectors, one of which is the sum of only longitudinal (curl-free) vectors and the other of which is the sum of only transverse (divergence-free) vectors. By Helmholtz's theorem,[4] any vector field may be uniquely separated in this manner. Consequently equation (2.27) yields two equations,

$$-\frac{\partial\psi}{\partial t} - \underline{v}\cdot\nabla\psi + \underline{v}\cdot(\nabla\times\underline{A}) - \alpha_1 + \alpha_2 + \alpha_3 + \frac{\pi}{\rho} = 0 \qquad (2.29)$$

$$\frac{\partial\underline{A}}{\partial t} + \underline{B}_1 - \underline{B}_2 - \underline{B}_3 = 0, \qquad (2.30)$$

and the auxiliary condition,

$$\frac{\partial}{\partial t}\nabla\cdot\underline{A} + \nabla\cdot\underline{B}_1 - \nabla\cdot\underline{B}_2 - \nabla\cdot\underline{B}_3 = 0 \qquad (2.30)'$$

The task now is to determine the α_i and $\underline{B}_i$. By operating on equations (2.28) with the divergence and the curl separately, it can readily be shown that

$$\nabla^2\alpha_1 = \nabla\psi\cdot(\nabla\times\nabla\times\underline{v})$$

$$\nabla^2\alpha_2 = (\nabla\times\underline{A})\cdot(\nabla\times\nabla\times\underline{v}) - (\nabla\times\underline{v})\cdot(\nabla\times\nabla\times\underline{A})$$

$$\nabla^2\alpha_3 = \underline{v}\cdot\nabla\times(\nabla\times\nabla\times\underline{A}) - (\nabla\times\nabla\times\underline{A})\cdot(\nabla\times\underline{v})$$

$$\nabla^2\underline{B}_1 = (\nabla\times\underline{v})\nabla^2\psi - (\nabla\times\underline{v}\cdot\nabla)\nabla\psi + (\nabla\psi\cdot\nabla)\nabla\times\underline{v} \qquad (2.31)$$

$$\nabla^2\underline{B}_2 = (\nabla\times\underline{A}\cdot\nabla)\nabla\times\underline{v} - (\nabla\times\underline{v}\cdot\nabla)\nabla\times\underline{A}$$

$$\nabla^2\underline{B}_3 = (\nabla\times\nabla\times\underline{A})(\nabla\cdot\underline{v}) - (\nabla\times\nabla\times A\cdot\nabla)\underline{v} + (\underline{v}\cdot\nabla)\nabla\times\nabla\times\underline{A}$$

$$\nabla\cdot\underline{B}_i = 0$$

Note that if all the $\underline{B}_i$ vanish as a consequence of the vanishing of the transverse parts of the vectors in equation (2.28), then, by virtue of equation (2.30), the vector potential $\underline{A}$ may be taken as zero, because $\underline{A}$ should be time-varying to be of interest in the wave motion.

The above equations for α_i and $\underline{B}_i$ are of the form of Poisson's equation;

$$\nabla^2\alpha_i = -\rho_i,$$

$$\nabla^2\underline{B}_i = -\underline{\rho}_i.$$

with solutions,[5] (provided that the vectors (2.28) go to zero at infinity)

$$\alpha_i(\underline{r}) = \frac{1}{4\pi}\int \rho_i\ (\underline{r}')\ G\ (\underline{r}/\underline{r}')\ dV' \qquad (2.32)$$

$$\underline{B}_i(\underline{r}) = \frac{1}{4\pi}\int \underline{\rho}_i\ (\underline{r}')\ G\ (\underline{r}/\underline{r}')\ dV' \qquad (2.33)$$

where the Green's function* is given by,

$$G(\underline{r}/\underline{r}') = \begin{cases} \frac{1}{R}, \text{three dimensions} \\ -\ 2\ \ln R, \text{two dimensions} \end{cases}$$

$$R = |\underline{r} - \underline{r}'|$$

Note that ρ_i equal to zero implies that α_i is equal to zero because then $\nabla\alpha_i$ may be written as $\nabla \times \underline{B}_i'$, and is included in $\nabla \times \underline{B}_i$. Similarly, $\underline{\rho}_i$ equal to zero implies that $\underline{B}_i$ is equal to zero, for then $\nabla \times \underline{B}_i$ may be written as $\nabla\alpha_i'$, and is included in $\nabla\alpha_i$. If a pair ρ_i and $\underline{\rho}_i$ are both zero, then it is best to retun to the original vector given by equations (2.28) to determine whether or not it is identically zero. Note also that the condition $\nabla \cdot \underline{B}_i$ equal to zero is satisfied by equation (2.33). This may be demonstrated as follows:

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*Equation (2.33) is valid for the rectangular coordinate system. In general, the Green function is a dyadic for other coordinate systems.

$$\nabla \cdot \underline{B}_i = \frac{1}{4\pi}\int \underline{\rho}_i(\underline{r}') \cdot \nabla G\,(\underline{r}/\underline{r}')\,dV',$$

$$= -\frac{1}{4\pi}\int \underline{\rho}_i(r') \cdot \nabla' G\,(\underline{r}/\underline{r}')\,dV'.$$

Now, it is obvious that

$$\nabla' \cdot \underline{B}_i = \frac{1}{4\pi}\int \nabla' \cdot [\underline{\rho}_i(\underline{r}')\,G\,(\underline{r}/\underline{r}')]\,dV' = 0.$$

By virtue of the transverse character of $\underline{\rho}_i$,

$$\nabla' \cdot \underline{\rho}_i(\underline{r}') = -\nabla' \cdot \nabla \times \nabla \times \underline{B}_i(\underline{r}') = 0.$$

Hence,

$$\int \underline{\rho}_i(\underline{r}') \cdot \nabla' G\,(\underline{r}/r')\,dV' = 0$$

Therefore,

$$\nabla \cdot \underline{B}_i = 0$$

The auxiliary condition (2.31) is now simplified to

$$\frac{\partial}{\partial t}\,\nabla \cdot \underline{A} = 0.$$

Which may as well be taken as

$$\nabla \cdot \underline{A} = 0. \qquad (2.34)$$

In summary, the appropriate equations in and $\underline{A}$ are collected below:

$$-\frac{\partial \psi}{\partial t} - \underline{v} \cdot \nabla \psi + \underline{v} \cdot \nabla \times \underline{A} - \alpha_1 + \alpha_2 + \alpha_3 + \frac{\pi}{\rho} = 0 \qquad (2.29)$$

$$\frac{\partial \underline{A}}{\partial t} + \underline{B}_1 + \underline{B}_2 - \underline{B}_3 = 0 \qquad (2.30)$$

$$\nabla \cdot \underline{A} = 0 \qquad (2.34)$$

$$\frac{\partial \delta}{\partial t} + \underline{v} \cdot \nabla \delta + \delta\,\nabla \cdot \underline{v} - \nabla \psi \cdot \nabla \rho + (\nabla \times \underline{A}) \cdot \nabla \rho - \rho\,\nabla^2 \psi = 0 \qquad (2.35)$$

$$\pi = c^2 \delta \tag{2.24}$$

where α_1 and $\underline{B}_1$ are solutions of equations (2.31). Equation (2.24) may be substituted in (2.29), and the result may be combined with (2.35). This process leads to a useful reduction of the five equations given above to the three equations given below:

$$\nabla^2\psi - \frac{1}{c^2}\frac{d^2\psi}{dt^2} = \frac{1}{c^2}\frac{d}{dt}\left(\alpha_1 - \alpha_2 - \alpha_3\right) - (\nabla\psi - \nabla \times \underline{A}) \cdot \nabla \ln\rho$$

$$- \frac{1}{c^2}\,\underline{v} \cdot \left(\frac{d\psi}{dt} - \underline{v} \cdot \nabla \times \underline{A} + \alpha_1 - \alpha_2 - \alpha_3\right)\nabla \ln c^2 \tag{2.36}$$

$$- \frac{1}{c^2}\,\underline{v} \cdot \left[\frac{d}{dt}(\nabla \times \underline{A}) + (\nabla \times A \cdot \nabla)\,\underline{v} + (\nabla \times A) \times (\nabla \times \underline{v})\right]$$

$$\frac{\partial \underline{A}}{\partial t} + \underline{B}_1 - \underline{B}_2 - \underline{B}_3 = 0, \tag{2.30}$$

$$\nabla \cdot \underline{A} = 0 \tag{2.34}$$

where $\frac{d}{dt}$ is the total time derivative. Explicitly,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla \tag{2.37}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\,\underline{v} \cdot \nabla\frac{\partial}{\partial t} + (\underline{v} \cdot \nabla)(\underline{v} \cdot \nabla) \tag{2.38}$$

Equations (2.36) (2.30), and (2.34) form the general result of this section. Because both α_1 and $\underline{B}_1$ depend upon ψ and $\underline{A}$, it is clear that in general the equations are "coupled"; the ease of applying these equations will depend to a large extent on particular geometries and net flows which lead to decoupling. To the author's knowledge, the general equations given above have not been presented elsewhere: they have the advantage of providing a unified basis for attacking many problems of the type under study in this thesis.

The most complete exposition to date on this subject has been given recently by D. Blokhintzev,[3] but his work is rather more specialized than the present treatment, for he treats only one special form of the vector potential. E. N. Parker[6] has also recently treated this problem, but his work appears to be incorrect, for he has omitted the transverse part of $\underline{u}$, even for rotational viscous flow.

In the remainder of this Section, the specialization of the general equations to irrotational net flow and zero net flow will be carried out. In Part D of this Chapter, further specialization will be made to the case of vortex motion.

Before going on, however, it is appropriate to discuss the effect of geometry in the general case of rotational (vortical) net flow.

In <u>one-dimensional problems</u>, $\underline{v}$ and $\underline{u}$ are parallel and are independent of the coordinates transverse to their direction. It is obvious that the curl of $\underline{v}$ must be zero; consequently rotational flow cannot exist in one-dimension and therefore this case is discussed under irrotational flow.

In <u>two dimensional problems</u>, $\underline{v}$ and $\underline{u}$ must lie in parallel planes and must be independent of the coordinate defining the planes. Consequently the curl of $\underline{v}$ is perpendicular to the planes of $\underline{v}$ and $\underline{u}$. This affords some simplification of the equations (2.31) for α_i and $\underline{B}_i$; terms containing the operators

$$(\nabla \times v \cdot \nabla)$$

$$(\nabla \times \nabla \times A \cdot \nabla)$$

are automatically zero, for they involve derivatives with respect to the coordinate defining the planes of $\underline{v}$ and $\underline{u}$. In general, however, this simplification is not of great help.

In three-dimensional problems, equations (2.31) are to be used as they stand, and no simplifications of α_i and $\underline{B}_i$ are possible in general.

Consequently the geometry of the situation does not appreciably ease the problem (except in the trivial one-dimensional case), and therefore the only hope is that the particular net flow under consideration will lead to important simplifications. It has already been pointed out that if all the $\underline{B}_i$ are zero, the vector potential $\underline{A}$ is zero, and, of course, tremendous simplification results. It is readily shown that if only $\underline{B}_2$ and $\underline{B}_3$ turn out to be zero, then $\underline{A}$ can be readily solved for in terms of ψ, giving rise to the simplification of a single equation for ψ. On the other hand, if only $\underline{B}_1$ turns out to be zero, then two decoupled equations result, one in ψ and $\underline{A}$, the other in $\underline{A}$ only. But, if the particular flow under consideration does not lead to such values, that is, if $\underline{B}_1$ and $\underline{B}_2$ or $\underline{B}_3$ are non-zero, then the differential equations for ψ and $\underline{A}$ are coupled (i.e. ψ and $\underline{A}$ appear in both equations) and a solution is extremely difficult. Some special flows are discussed below which lead to zero values of the $\underline{B}_i$, and hence to useful simplifications.

2. Irrotational Net Flow

In this case,

$$\nabla \times \underline{v} = 0.$$

Now it is well know that continuous flows of zero vorticity retain this characteristic if the motions are isentropic and the external forces are conservative or zero.[7] If in some part of the total flow field (say at infinity) the boundary condition on the wave flow is such that it is required to be irrotational, then the total flow field is irrotational, and must remain so throughout the field. Under these circumstances, the wave flow must remain irrotational throughout the field; i.e. $\nabla \times \underline{A}$ is zero. For purposes of generality, it is of interest to retain $\nabla \times \underline{A}$ in the formulation to allow for the possibility of the appearance of rotational wave flow as the consequence of a rotational boundary condition. Consequently equations (2.31) become:

$$
\begin{aligned}
\alpha_1 &= 0 \\
\alpha_2 &= 0 \\
\nabla^2 \alpha_3 &= \underline{v} \cdot \left[\nabla \times (\nabla \times \nabla \times \underline{A})\right] \\
\underline{B}_1 &= 0 \\
\underline{B}_2 &= 0 \\
\nabla^2 \underline{B}_3 &= (\nabla \times \nabla \times A)(\nabla \cdot \underline{v}) - (\nabla \times \nabla \times A \cdot \nabla)\underline{v} \\
&\quad + (\underline{v} \cdot \nabla)(\nabla \times \nabla \times A)
\end{aligned}
\tag{2.39}
$$

Thus $\underline{B}_1$ is zero for general irrotational net flow, and two decoupled equations result, one in ψ and $\underline{A}$, and the other in $\underline{A}$ only:

$$
\begin{aligned}
\nabla^2 \psi - \frac{1}{c^2}\frac{d^2\psi}{dt^2} = &- \frac{1}{c^2}\frac{d\alpha_3}{dt} - (\nabla\psi - \nabla \times \underline{A}) \cdot \nabla \ln\rho \\
&- \frac{1}{c^2}\underline{v} \cdot \left(\frac{d\psi}{dt} - \underline{v} \cdot \nabla \times \underline{A} - \alpha_3\right)\nabla \ln c^2 \\
&- \frac{1}{c^2}\underline{v} \cdot \left[\frac{d}{dt}(\nabla \times \underline{A}) + (\nabla \times A \cdot \nabla)\,\underline{v}\right]
\end{aligned}
\tag{2.40}
$$

$$\frac{\partial \underline{A}}{\partial t} = \underline{B}_3; \quad \nabla \cdot \underline{A} = 0. \tag{2.41}$$

It is appropriate to discuss the special forms of equations (2.40) and (2.41) and in various geometries.

In <u>one-dimensional problems</u>, $\underline{v}$ and $\underline{u}$ are parallel and are independent of coordinates transverse to their direction. It is clear therefore that $\nabla \times \underline{u}$ must be zero. Consequently the wave flow must be irrotational, and equations (2.40) and (2.41) reduce to

$$\nabla^2 \psi - \frac{1}{c^2}\frac{d^2\psi}{dt^2} = -\nabla\psi \cdot \nabla \ln\rho - \frac{1}{c^2}\underline{v} \cdot \frac{d\psi}{dt}\nabla \ln c^2 \tag{2.42}$$

(one dimensional irrotational flow)

or, if x is the coordinate in the direction of motion, the explicit form is:

$$\left[1 - \frac{v^2}{c^2}\right]\frac{\partial^2\psi}{\partial x^2} + \left[\frac{\partial \ln\rho}{\partial x} + \frac{v^2}{c^2}\frac{\partial \ln c^2}{\partial x} - \frac{v^2}{c^2}\frac{\partial \ln v}{\partial x}\right]\frac{\partial\psi}{\partial x}$$

$$- \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{v}{c^2}\frac{\partial \ln c^2}{\partial x}\frac{\partial\psi}{\partial t} - \frac{2v}{c^2}\frac{\partial^2\psi}{\partial x \partial t} = 0 \tag{2.42'}$$

where v is the speed of the net flow. This equation is appropriate for the study of oscillations in heated tubes (Rijke phenomena), oscillations in combustion processes (under some conditions), perturbation of the normal modes of rooms by temperature gradients, etc.

In <u>two-dimensional problems</u>, the general irrotational equations (2.39) for α_1 and $\underline{B}_1$ are simplified to some extent by the omission of the term

$$(\nabla \times \nabla \times \underline{A} \cdot \nabla)\,\underline{v}$$

However $\nabla^2 \underline{B}_3$ is in general non-zero. Thus $\nabla \times \underline{A}$ may be a function of the coordinates and time, and the wave motion may be rotational.

To demonstrate this, note that for two-dimensions, curl of equation (2.41) leads to a linear equation of first order in $\nabla \times \nabla \times \underline{A}$:

$$\frac{\partial}{\partial t} (\nabla \times \nabla \times \underline{A}) = \nabla \times \left[\underline{v} \times (\nabla \times \nabla \times \underline{A}) \right], \qquad (2.41)'$$

or

$$\frac{d}{dt} (\nabla \times \nabla \times \underline{A}) = - (\nabla \times \nabla \times \underline{A})(\nabla \cdot \underline{v}). \qquad (2.41)''$$

It is clear that a non-zero general solution of this equation (for $\nabla \times \nabla \times \underline{A}$) exists, and thus $\nabla \times \underline{A}$ may be non-zero in general. This is a somewhat surprising result, for at first it may be thought that rotational wave flow could not occur in conjunction with irrotational net flow. As a matter of fact, many authors commit the error of assuming $\underline{u}$ to be irrotational, *a priori*, when dealing with irrotational net flows.[3][6] In many cases of practical interest, however, particularly in problems dealing with unbounded media, it will be necessary to set $\nabla \times \underline{A}$ equal to zero. On the other hand, in certain problems dealing with bounded media, it appears that an effect like viscosity at the boundaries of a fluid (which is otherwise considered inviscid), may be treated approximately as a boundary condition which specifies a transverse component of $\underline{u}$. In this case, the wave flow is in general rotational, despite the fact that the net flow may be irrotational (outside of the boundary layer).

In three-dimensional irrotational net flow problems, the equations (2.39), (2.40), and (2.41) are to be used as they stand. The wave flow may be rotational in general, despite the irrotational character of the net flow, and the discussion given above for two-dimensional irrotational net flow applies for this case also.

3. Zero Net Flow

As a final example of the simplification of the general equations of motion, it is useful to consider the case of $\underline{v}$ identically zero. In this case, equations (2.31) require that the α_i and $\underline{B}_i$ are zero. Hence $\nabla \times \underline{A}$ must be zero, the wave motion is irrotational, and equation (2.36) reduces to

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\nabla\psi\cdot\nabla \ln\rho \qquad (2.43)$$

This equation describes the propagation of waves in a stationary medium whose density and temperature obey the adiabatic ideal gas relation. Note that this is a rigorous equation. The equations for $\underline{v}$ non-zero are obtained essentially from a first-order perturbation, but this equation can be derived directly from the linearized equations of mass, momentum, energy, and state.

C. Circular Cylindrical Vortex Motion

The net flow is required to satisfy the equations of mass, momentum, energy, and state, as set forth in equations (2.15), (2.16), (2.17), and (2.18). In addition, it is required to be steady and isentropic, as expressed by equations (2.19) and (2.20), and it is to be devoid of the action of body forces, i.e., g is equal to zero. The appropriate equations are:

$$\nabla \cdot (\rho \underline{v}) = 0 \qquad (2.44)$$

$$(\nabla \times \underline{v}) \times \underline{v} + \frac{1}{2} \nabla (\underline{v} \cdot \underline{v}) = - \frac{1}{\rho} \nabla p \qquad (2.45)$$

$$s = \text{constant} \qquad (2.46)$$

$$p = \text{constant } \rho^{\gamma} \qquad (2.47)$$

Before proceeding, it is worthwhile to elaborate on the assumption of steady isentropic net flow. In order to establish the net flow, it is obvious that viscosity must be present, and that in general the flow builds up and decays in a finite time. The condition of steady flow, however, may be expressed by requiring the characteristic time of the net flow to be large compared to the period of the wave flow, a condition which is often met in practice. But it is not obvious how to treat the gradients of entropy which accompany the build up and decay of the flow, except by the same intuitive argument. That is, if the net flow is sensibly time-independent, then the entropy gradients should be sensibly zero. This assumption can be tested by solving the net flow equations without excluding viscosity, and then proceeding to the limit of a large characteristic time. This proceedure, however, does not appear to be necessary for purposes of the present investigation.

Inspection of the equations for ψ and $\underline{A}$ reveal that the effect of the net flow enters through the terms $\underline{v}$, $\nabla \cdot \underline{v}$, $\nabla \times \underline{v}$, c^2, $\nabla \ln c^2$, and $\nabla \ln \rho$. The task of this section is to evaluate the last three terms for the particular net flows to be studied. By use of equation (2.47), it is readily shown, however, that

$$\nabla \ln \rho = \frac{1}{\gamma - 1} \nabla \ln c^2.$$

Hence it will only be necessary to determine c^2.

To describe the motion, standard circular cylindrical coordinates r, ϕ, and z, are used. The velocity $\underline{v}$ lies in the r, ϕ plane and is independent of z. Two types of vortex motion will be studied; rotor motion, which shall be defined shortly, and ideal vortex motion, which is commonly known. Both types of motion may be characterized by a velocity of the form

$$\underline{v} = f(r)\, \underline{i}_\phi$$

where $f(r)$ is some function of r only, and $\underline{i}_\phi$ is the unit vector for the ϕ coordinate. Obviously, the divergence of $\underline{v}$ is automatically zero, and by virtue of equation (2.44), $\nabla \rho$ must be normal to $\underline{v}$, i.e., in the r direction. Consequently the concomitant inhomogeneity of the medium is independent of the angle ϕ.

1. Rotor Motion

For purposes of this study, rotor motion may be defined by the following configuration of the velocity field:

$$\underline{v} = \Omega r\, \underline{i}_\phi, \quad r < a$$
$$\underline{v} = 0, \quad r > a \tag{2.49}$$

Thus rotor motion may be described as "rigid body" rotation of a cylinder of fluid of radius a, revolving at a frequency of Ω. The fluid outside the rotation cylinder is undisturbed. The vorticity of the medium is:

$$\nabla \times \underline{v} = 2\Omega \underline{i}_z, \quad r < a$$
$$\nabla \times \underline{v} = 0 \qquad r > a \tag{2.50}$$

Physically, this motion may be achieved in the laboratory using a cylinder made of a thin plastic sheet within which a stirring device is located. Rotor motion does not appear to be of importance in natural occurances, but it is the simplest two-dimensional rotational flow that can be studied, and may serve as an approximate model for physical cases in which the flow outside a region of vorticity is very small.

From equation (2.45), one may readily obtain,

$$\frac{dp}{dr} = \rho r \Omega^2,$$

for r less than a. Then by using (2.47) and integrating, the following equation results:

$$c^2 = \frac{\gamma p}{\rho} = \frac{(\gamma - 1)}{2} \Omega^2 r^2 + \text{constant}, \; r < a.$$

The constant may be evaluated by examining the conditions at the cylinder surface. There it is required that the pressure is continuous. Consequently the sound velocity may be written as:

$$c^2 = \begin{cases} c_o^2 \left[1 - \frac{\gamma - 1}{2} M^2 \left(1 - \frac{r^2}{a^2}\right)\right], & r < a \\ c_o^2, & r > a \end{cases} \tag{2.51}$$

where c_o is the quiescent sound velocity, and, where M is the maximum Mach number of the net flow. M is evaluated by taking the ratio of the speed at the cylinder surface to the quiescent sound velocity, c_o. Explicitly,

$$M = \frac{\Omega a}{c_o} \tag{2.52}$$

The relations (2.49) and (2.51) for rotor motion are illustrated in Figure 1.

2. Ideal Vortex Motion

Ideal vortex motion is characterized by the following velocity field:[8]

$$\begin{aligned} \underline{v} &= \Omega r \, \underline{i}_\phi, \quad r < a \\ \underline{v} &= \frac{\Omega a^2}{r}, \quad r > a \end{aligned} \tag{2.53}$$

Thus ideal vortex motion differs from rotor motion only in that there is an induced velocity field outside the rotating cylinder. The vorticity of the medium is nevertheless identical in both cases, i.e., for ideal vortex motion:

$$\begin{aligned} \nabla \times \underline{v} &= 2\Omega \underline{i}_z, \quad r < a \\ \nabla \times \underline{v} &= 0 \qquad\;\;, \quad r > a \end{aligned} \tag{2.54}$$

This motion may be achieved in the laboratory by rotating a cylinder made of a thin plastic sheet. However ideal vortex motion often occurs as a result of the interaction of a uniform flow with an object. For example, uniform flow past a cylindrical obstacle results in a trail of well defined (approximately) ideal vortices in the Reynolds number range of about 10^2 to 10^4. Or atmospheric wind often causes the shedding of vortices from objects like buildings or trees.

The pressure distribution in the core of the vortex (r less than a) is obviously identical to that of the rotor. Hence,

$$c^2 = \frac{\gamma-1}{2}\Omega^2 r^2 + \text{constant}, \; r < a.$$

In the irrotational region of the vortex (r greater than a), the differential equation for the pressure is, from equation (2.45):

$$\frac{dp}{dr} = \rho\Omega^2 a^4 \frac{1}{r^3}, \; r > a.$$

Consequently the sound velocity squared is:

$$c^2 = -\frac{\gamma-1}{2}\frac{\Omega^2 a^4}{r^2} + \text{constant}, \; r > a.$$

The constant in the irrotational region may be evaluated by requiring

$$c^2 \longrightarrow c_o^2, \quad r \rightarrow \infty.$$

Finally the constant in the core region may be evaluated by requiring pressure continuity. Thus,

$$c^2 = \begin{cases} c_o^2 \left[1 - \frac{\gamma-1}{2} M^2 \left(2 - \frac{r^2}{a^2}\right)\right], & r < a \\ c_o^2 \left[1 - \frac{\gamma-1}{2} M^2 \frac{a^2}{r^2}\right], & r > a \end{cases} \tag{2.55}$$

Relations (2.53) and (2.55) for ideal vortex motion are illustrated in Figure 2.

D. Equations for Diffraction and Scattering

The task in this section is to specialize the two-dimensional equations in ψ and $\underline{A}$ for circular cylindrical vortex motion. In so far as possible, the equations are to be reduced to the form of inhomogeneous wave equations; the inhomogeneous terms may then be regarded as source terms which give rise to scattered radiation or diffraction.

The rigorous equations will be developed from the results of Section B and C of this Chapter. It will be shown, however, that the rigorous equations for the core region are coupled in ψ and $\underline{A}$, and are not amenable to solution. Consequently an approximation scheme will be adopted which simplifies the equations considerably, and which, nevertheless, retains the essential elements of the problem. Both the approximate equations and the rigorous equations will be used as starting points for the actual calculation of the scattering and diffraction.

The net flow motion described in the last section was found to have the following general properties;

$$\underline{v} = f(r)\underline{i}_\phi$$

$$\nabla \cdot \underline{v} = 0$$

$$\underline{v} \cdot \nabla \ln c^2 = 0$$

$$\nabla \times \underline{v} = \begin{cases} 2\Omega \underline{i}_z, & r < a \\ 0, & r > a \end{cases}$$

$$\nabla \times \nabla \times \underline{v} = 0 ,$$

where f(r) is determined by equation (2.49) for rotor motion, and by equation (2.53) for ideal vortex motion. f(r) and $\nabla \times \underline{v}$ are both different for the core region (r less than a) and the exterior region (r greater than a) of the vortex motion. The specialization of the general equations in ψ and $\underline{A}$ will now be carried through for the two distinct regions of the vortex motion.

1. Exterior Region (r greater than a)

a. Rotor Motion

In this case f(r) and $\nabla \ln\rho$ are zero. The wave flow is irrotational and equation (2.43) for zero net flow reduces to the standard wave equation:

$$\nabla^2\psi - \frac{1}{c_o^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad (2.56)$$

where $\underline{u} = -\nabla\psi$. For reference purposes, it is convenient to determine the wave flow pressure. From equation (2.29):

$$\pi = \rho\frac{\partial\psi}{\partial t} \qquad (2.57)$$

b. Ideal Vortex Motion

In the exterior region of an ideal vortex, the net flow is irrotational. Hence equations (2.39), (2.40), and (2.41) are to be applied. Inasmuch as the wave motion at large distances from the vortex ($r \rightarrow \infty$) is required to be irrotational, $\nabla \times \underline{A}$ can be taken as zero. Hence equation (2.40) reduces to:

$$\nabla^2\psi - \frac{1}{c^2}\frac{d^2\psi}{dt^2} = -\nabla\psi \cdot \nabla \ln\rho$$

With the use of equations (2.38) and (2.53), this equation becomes:

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 2\frac{\Omega a^2}{c^2 r^2}\frac{\partial^2 \psi}{\partial \phi \partial t} + \frac{\Omega^2 a^4}{c^2 r^4}\frac{\partial^2 \psi}{\partial \phi^2} - \frac{1}{\gamma - 1}\frac{\partial \ln c^2}{\partial r}\frac{\partial \psi}{\partial r} \qquad (2.59)$$

where $$c^2 = c_o^2\left[1 - \frac{\gamma - 1}{2} M^2 \frac{a^2}{r^2}\right], \qquad (2.60)$$

and where $\underline{u} = -\nabla\psi$. From equation (2.29), the wave flow pressure is:

$$\pi = \rho\frac{\partial \psi}{\partial t} + \rho\frac{\Omega a^2}{r^2}\frac{\partial \psi}{\partial \phi} \qquad (2.61)$$

The equations (2.59) and (2.61) for the exterior region of ideal vortex motion are considerably more complicated than the corresponding equations (2.56) and (2.57) for rotor motion.

2. Core Region (r less than a)

a. Rotor Motion

In this region, the net flow is rotational. Thus equations (2.31), (2.36), (2.30), and (2.34), specialized for two-dimensional vortex motion, are to be used. Explicitly, equations (2.31) become:

$$\begin{aligned} \alpha_1 &= 0 \\ \nabla\alpha_2 &= (\nabla \times \underline{v}) \times (\nabla \times \underline{A}) \qquad (2.62) \\ \nabla^2\alpha_3 &= \underline{v} \cdot \left[\nabla \times (\nabla \times \nabla \times \underline{A})\right] - (\nabla \times \nabla \times \underline{A}) \cdot (\nabla \times \underline{v}) \end{aligned}$$

((2.62) continued on next page)

$$\nabla \times \underline{B}_1 = \nabla\psi \times (\nabla \times \underline{v})$$

$$\underline{B}_2 = 0$$

$$\nabla^2 \underline{B}_3 = (\underline{v} \cdot \nabla)(\nabla \times \nabla \times \underline{A})$$

Because $\underline{B}_1$ and $\underline{B}_3$ are non-zero, the differential equations for ψ and $\underline{A}$ are coupled. As expected, the wave flow is rotational, as may be seen from the curl of equation (2.30):

$$\frac{\partial}{\partial t} \nabla \times \underline{A} = \nabla \times \underline{B}_3 - \nabla \times \underline{B}_1 \qquad (2.63)$$

The curl of equation (2.63) takes on a simple form, and will be useful for discussion:

$$\frac{d}{dt} (\nabla \times \nabla \times A) = (\nabla \times \underline{v}) \nabla^2 \psi \qquad (2.64)$$

or,

$$\frac{d}{dt} (\nabla^2 \underline{A}) = -(\nabla \times \underline{v}) \nabla^2 \psi \quad , \qquad (2.64)'$$

Now, the vorticity of the wave flow is $\nabla \times \underline{u}$, or $\nabla \times \nabla \times \underline{A}$. Thus, for circular cylindrical vortex motion, the wave vorticity is parallel to the net vorticity, and varies in direct proportion to it and to the amplitude of the scalar potential ψ. If it is permissible to discard the "flow" part of the total derivative in equation (2.64), (later this shall be shown to be possible for many practical cases), then, for harmonic time dependence, a simple relation exists between the longitudinal and transverse parts of the wave flow velocity:

$$\nabla \times \underline{u} \simeq \frac{(\nabla \times \underline{v})}{\omega} (\nabla \cdot \underline{u})\, e^{-i\,\pi/2}, \qquad (2.65)$$

where ω is the frequency of the wave flow. Thus the transverse part is 90° out of phase with the longitudinal part, and the magnitude of the wave flow vorticity is the order of $2\Omega/\omega$ of the wave flow divergence. Hence the dimensionless ratio Ω/ω forms a useful measure of the importance of the effect of net flow vorticity on wave propagation.

In general equation (2.63), or equation (2.64), though relatively simple to interpret in an approximate manner, must be solved simultaneously with the other equation in ψ and $\underline{A}$ obtained from equation (2.36):

$$\nabla^2\psi - \frac{1}{c^2}\frac{d^2\psi}{dt^2} = -\frac{1}{c^2}\frac{d}{dt}(\alpha_2 + \alpha_3) \qquad (2.66)$$
$$- \frac{1}{\gamma-1}(\nabla\psi - \nabla \times \underline{A}) . \nabla \ln c^2$$
$$- \frac{1}{c^2}\underline{v} \cdot \left[\frac{d}{dt}(\nabla \times \underline{A}) + (\nabla \times \underline{A} \cdot \nabla)\underline{v} - \nabla\alpha_2\right]$$

where α_2 and α_3 are determined from equation (2.62), where

$$c^2 = c_o^2\left[1 - \frac{\gamma-1}{2} M^2 \left(1 - \frac{r^2}{a^2}\right)\right], \qquad (2.67)$$

and where $\underline{u} = -\nabla\psi + \nabla \times \underline{A}$. From equation (2.29), the wave flow pressure is:

$$\pi = \rho\frac{\partial\psi}{\partial t} + \rho\Omega\frac{\partial\psi}{\partial\phi} - \rho\Omega r \underline{i}_\phi \cdot \nabla \times \underline{A} - \rho(\alpha_2 + \alpha_3) \qquad (2.68)$$

In part 3 of this Section, approximations will be introduced to facilitate the solution of the equation pairs (2.63) and (2.66); general solution of the simultaneous equations as they now stand appears hopeless.

b. Ideal Vortex Motion

In the core region the net flow velocity and vorticity for ideal vortex motion is identical to that of rotor motion. Consequently all of the equations and all of the discussion given in (a) above are valid here, with one exception. For ideal vortex motion, the correct equation for the sound velocity is:

$$c^2 = c_o^2 \left[1 - \frac{\gamma - 1}{2} M^2 \left(2 - \frac{r^2}{a^2}\right) \right], \qquad (2.69)$$

Hence equation (2.69) must replace equation (2.67).

3. Approximations

As pointed out in Section B of this Chapter, if $\underline{B}_1$ and $\underline{B}_2$ or $\underline{B}_3$ are non-zero, then the differential equations for ψ and $\underline{A}$ are coupled. This is the situation encountered in the core region of vortex motion, as expressed by equations (2.63) and (2.66). But it was also pointed out that if both $\underline{B}_2$ and $\underline{B}_3$ are zero, then the two coupled equation can be reduced to a single equation for ψ. This would obviously represent an appreciable simplification of the core equations (2.63) and (2.66). Consequently it is appropriate to investigate the conditions under which $\underline{B}_3$ may be considered very much smaller than $\underline{B}_1$ ($\underline{B}_2$ is already zero). if $\underline{B}_3$ is zero, then

$$\frac{\partial}{\partial t} \nabla \times \underline{A} = - \nabla \times \underline{B}_1$$

or

$$\nabla \times \underline{A} = \int (\nabla \times \underline{v}) \times \nabla \psi \, dt. \qquad (2.70)$$

Hence $\nabla \times \underline{A}$ is determined by ψ. Insertion of equation (2.70) into equation (2.66) would then lead to a single equation for ψ. The error entering equation (2.70) by the fact that $\underline{B}_3$ is not zero may be expressed as the ratio of:

$$\frac{\nabla^2 \underline{B}_3}{\nabla^2 \underline{B}_1} = \frac{(\underline{v} \cdot \nabla)\,(\nabla \times \nabla \times \underline{A})}{(\nabla \times \underline{v})\,\nabla^2 \psi} \tag{2.71}$$

To establish the order of magnitude of the error, $\nabla \times \underline{A}$ will be calculated from the approximate equation (2.70), and ψ will be taken as a plane wave:

$$\psi = e^{ikr\cos\phi - i\omega t},$$

where $k = \omega/c$, is the propagation constant. It is readily determined that the error expressed by equation (2.71) is approximately:

$$\frac{\nabla^2 \underline{B}_3}{\nabla^2 \underline{B}_1} \simeq \frac{\Omega}{\omega} kr\sin\phi \; .$$

Hence, the root-mean-square error in the core region is approximately

$$\left.\frac{\nabla^2 \underline{B}_3}{\nabla^2 \underline{B}_1}\right|_{rms} \simeq \frac{1}{2}\frac{\Omega}{\omega} ka = \frac{1}{2} M.$$

It is important to note also that on the basis of equation (2.70), the magnitude of $\nabla \times \underline{A}$ compared to the magnitude of $\nabla\psi$ is of the order of $2\Omega/\omega$. Thus the use of the approximate equation (2.70) in equation (2.26) ($\underline{u} = -\nabla\psi + \nabla \times \underline{A}$) is correct to the order of Ω/ω; the error incurred in the wave velocity by the use of the approximate equation is therefore the order of

$$\frac{\Omega}{\omega} M.$$

In addition, with the use of the approximate equation (2.70), the wave flow pressure in the core region may be written as:

$$\pi = \rho \frac{\partial \psi}{\partial t} + \rho\Omega \frac{\partial \psi}{\partial \phi} \tag{2.72}$$

Despite the error the order of M in the approximate equa- for $\nabla \times \underline{A}$, the pressure as given in equation (2.72) is correct to the order M; the errors are the order of Ω^2/ω^2, M^2, and $\Omega/\omega\, M$.

Consequently it will be convenient to require that terms the order of

$$\frac{\Omega}{\omega} M,\ \frac{\Omega^2}{\omega^2},\ \text{and}\ M^2$$

are small compared to unity in the measurable quantities $\underline{u}$ and π. Then the approximate equation (2.70) for $\nabla \times \underline{A}$ may be used,* and the wave pressure equation may be simplified to equation (2.72). This requirement does not represent a serious restriction on the practical application of the results, for Ω is usually much less than ω, and M is rarely greater than 10^{-1}.

In summary, the appropriate equations are given below for both regions of the two types of vortex motion under study. They have been suitably reduced according to the approximation scheme:

$$\frac{\Omega^2}{\omega^2} << 1,$$

$$M^2 << 1, \qquad (2.72)$$

$$\frac{\Omega}{\omega} M << 1.$$

Rotor Motion:

$r > a$,

$$\nabla^2\psi - \frac{1}{c_o^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad (2.73)$$

((2.73) continued on next page)

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*Of course, $\nabla \times \underline{A}$ will have an error the order of M, but the quantities of interest, $\underline{u}$ and π, will have much smaller errors. Equation (2.70) for $\nabla \times \underline{A}$ was first introduced by A.M. Obukhov, as discussed in D. Blokhintzev's book.(3)

$$\underline{u} = - \nabla \psi$$

$$\pi = \rho_o \frac{\partial \psi}{\partial t} .$$

$r < a$,

$$\nabla^2 \psi - \frac{1}{c_o^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{2\Omega}{c_o^2} \frac{\partial^2 \psi}{\partial \phi \partial t}$$

$$\underline{u} = - \nabla \psi + \int (\nabla \times \underline{v}) \times \nabla \psi dt \qquad (2.74)$$

$$\pi = \rho_o \frac{\partial \psi}{\partial t} + \rho_o \Omega \frac{\partial \psi}{\partial \phi}$$

Ideal Vortex Motion

$r > a$

$$\nabla^2 \psi - \frac{1}{c_o^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{2 \Omega a^2}{c_o^2 r^2} \frac{\partial^2 \psi}{\partial \phi \partial t}$$

$$\underline{u} = - \nabla \psi \qquad (2.75)$$

$$\pi = \rho_o \frac{\partial \psi}{\partial t} + \rho_o \frac{\Omega a^2}{r^2} \frac{\partial \psi}{\partial \phi}$$

$r < a$

$$\nabla^2 \psi - \frac{1}{c_o^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{2\Omega}{c_o^2} \frac{\partial^2 \psi}{\partial \phi \partial t}$$

$$\underline{u} = - \nabla \psi + \int (\nabla \times \underline{v}) \times \nabla \psi dt \qquad (2.76)$$

$$\pi = \rho_o \frac{\partial \psi}{\partial t} + \rho_o \Omega \frac{\partial \psi}{\partial \phi}$$

It is important to note that the approximate equations(2.73) to (2.76) do not depend upon the inhomogeneity of the medium; inspection of equations (2.51) and (2.55) shows that the medium inhomogeneity is the order of M^2. Thus for scattering and diffraction, the net flow terms are more important than the concomitant inhomogeneity.

III. Scattering by Rotor Motion

A rotor causes a discontinuity in value and slope of the net flow velocity and vorticity of the medium. In a sense it is analogous to a "square-well" scatterer in quantum mechanical problems, for the vorticity is everywhere zero except within a circular cylinder, where it is a constant. Just as in the case of the square-well scatterer in quantum mechanics, it is useful to study the rotor despite the fact that it has few natural counterparts, for it represents as simple a case as can be studied, and can, nevertheless, lead to the essential physical properties to be expected from general vortical scattering.

The problem may be stated in the following way. Consider a rotor of radius a and angular frequency Ω. Circular cylindrical coordinates r, ϕ, z, are arranged such that the axis of the rotor coincides with the z axis, and the direction of rotation is in the positive ϕ sense, as shown in Figure 3. Thus the curl of the velocity in the core is directed in the positive z sense. The space outside of the core, or rotor, is designated as Region I; the core, or rotor, is designated as Region II. A wave in Region I is incident upon Region II, such that its original direction at $r = \infty$ is from $\phi = \pi$. The problem is to determine the total wave field at distances large compared with the radius of the rotor and the wavelength of the incident sound.

The incident wave will be required to be a plane, simple-harmonic time dependent wave. Without loss of generality, its amplitude may be assumed to be unity. Thus,

$$\psi_{in} = e^{ik_I r\cos\phi}\, e^{-i\omega t} \qquad (3.1)$$

where $k_I = \frac{\omega}{c_0}$ is the propagation constant in Region I. Inasmuch as the appropriate differential equations developed in Chapter II are linear, solutions may be sought which also have the time dependent factor

$$e^{-i\omega t}.$$

In what follows, it will be convenient to omit the time dependent factor, and replace the operators

$$\frac{\partial}{\partial t} \quad \text{and} \quad \int dt$$

by $-i\omega$ and i/ω respectively.

The total wave field may be expressed as the sum of the incident wave and a scattered wave. In certain scattering problems, the scattered wave may be found rigorously by summing the scattering from each mode. That is, eigenfunction expansions valid separately in the external medium and in the scatterer can be adjusted term by term according to continuity or matching conditions at the mutual surface, and the total scattering can be interpreted as a superposition of scattering from modes corresponding to each term. This method, when it can be applied, leads to results which are useful principally at low frequencies. At high frequencies, a large number of modes must be considered, and the evaluation of the sum becomes cumbersome. Inspection of the appropriate differential equations for the core region, however, shows that the equations are not separable. Thus exact determination of the eigenfunctions for the core is not possible. It is possible, however, to approximate the eigenfunctions by performing a perturbation on some closely related but exactly solvable problem. Hence, in principle, it would be possible to attack the present problem from the point of view of scattering by modes, but this does not appear to be a convenient approach.

Instead an approximate solution for the scattered wave using integral equation techniques will be sought. The wave equations valid separately in the external medium and in the scatterer can be recast into integral equations which implicitly contain the continuity and boundary conditions. These integral equations can be constructed for arbitrarily shaped scatterers. They are not, however, any easier to solve rigorously than the differential equations, but they do lend themselves more readily to approximate solution.

Briefly, the procedure to be followed is this: The approximate differential equations (2.73) and (2.74) for ψ are recast into integral equations using the fact that π and the normal components of $\underline{u}$ are continuous across the rotor surface. The integral equations are arranged in such a way that the total wave field outside the rotor is related to the integrals of the wave field and its derivatives inside and on the surface of the rotor. The equations are then specialized for distances large compared to the radius of the rotor and the wavelength.

One approximate solution for ψ is obtained by taking the wave field of the rotor, i.e., ψ in the volume integrals and in the surface integrals, to be the incident plane wave. This approximation is called the Born-Kirchoff (BK) approximation: it yields the scattered field as a function of ϕ. It is shown that for at least one point in the rotor (the center), the BK trial assumption of a plane wave is exact. It is therefore concluded that the approximation gives useful results, on the basis that the field within the rotor is probably not much different than the field at the center.

Another approximate solution for ψ is obtained by taking the wave field of the rotor to be that calculated by a WKBJ* iterative technique. In order to apply the WKBJ technique, the two-dimensional problem is reduced to an ensemble of one-dimensional problems; the coordinate perpendicular to the normal of the incident wave takes on the role of an "impact" parameter for straight-line rays whose phases are determined by the approximate differential equations (2.73) and (2.74). The resulting ray ensemble is used as a trial function in the two-dimensional integral equations. Because the WKBJ procedure is more accurate, it may be expected to yield more accurate results than the BK approximation.

Finally, the results obtained in this Chapter are compared with a statistical theory of scattering from turbulence developed by M. J. Lighthill.[27]

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*The letters stand for G. Wentzel, H. A. Kramers, L. Brillouin, and H. Jeffreys.

A. Integral Equation Formulation

1. Rigorous Integral Equations

Consider an infinitely-long cylinder of arbitrary cross section, as shown in Figure 4. $\underline{n}_I$ and $\underline{n}_{II}$ are outward pointing normal vectors (of magnitude unity) for Regions I and II respectively; $\underline{k}_0$ and $\underline{k}_1$ are propagation vectors, both of magnitude k_I, and are in the incident radiation direction and the measurement direction respectively.

The approximate differential equations (2.73) and (2.74) for the rotor may be written as:

$$(\nabla^2 + k_I^2 + W^2)\,\psi\,(\underline{r}) = 0 \qquad (3.2)$$

where

$$W^2(\underline{r}) = \begin{cases} i2\frac{\Omega}{\omega} k_I^2 \frac{\partial \ln \psi}{\partial \phi}, & r < a \\ 0, & r > a \end{cases}$$

Equation (3.2) is in the form of the time independent Schroedinger equation. The perturbation W^2, compared to k_I^2, is the order of magnitude of Ω/ω times the logarithmic derivative of ψ with respect to ϕ. Thus the relative perturbation is the order of or less than M, the maximum Mach number of the rotor, and hence equation (3.2) may be approached with the theory of small perturbations of Helmholtz's equation:

$$(\nabla^2 + k_I^2)\,\psi\,(\underline{r}) = 0$$

The boundary conditions on ψ for the mutual surface S of Regions I and II may be determined by requiring continuity of π and the normal component of $\underline{u}$. Hence:

$$\psi_I \Big|_S = \psi_{II} \Big|_S + i\frac{\Omega}{\omega} \frac{\partial \psi_{II}}{\partial \phi} \Big|_S , \qquad (3.3)$$

and

$$\underline{n}_{II} \cdot \nabla \psi_I = \underline{n}_{II} \cdot \nabla \psi_{II} - i2\frac{\Omega}{\omega} \underline{n}_{II} \cdot \underline{i}_z \times \nabla \psi_{II} ,$$

or

$$\frac{\partial \psi_I}{\partial r} \Big|_S = \frac{\partial \psi_{II}}{\partial r} \Big|_S + i2\frac{\Omega}{\omega} \frac{1}{a} \frac{\partial \psi_{II}}{\partial \phi} \Big|_S . \qquad (3.4)$$

Note that in consideration of equation (3.3), equation (3.4) may be rewritten as

$$\psi_I + \frac{a}{2} \frac{\partial \psi_I}{\partial r} = \psi_{II} + \frac{a}{2} \frac{\partial \psi_{II}}{\partial r} . \qquad (3.4)'$$

The boundary condition on the surface at infinity for Region I is that there be a source and sink distribution corresponding to the incident plane wave.

The inhomogeneous Helmholtz equation for the two-dimensional Green's function, $G(\underline{r}/\underline{r}')$, will be associated with equation (3.2). It is:

$$(\nabla_r^2 + k_I^2)\, G(\underline{r}/\underline{r}') = -\delta(\underline{r} - \underline{r}'). \quad (3.5)$$

$G(\underline{r}/\underline{r}')$ represents a wave at $\underline{r}$ due to a source at $\underline{r}'$ described by the two-dimensional Dirac delta function $\delta(\underline{r} - \underline{r}')$. It is invariant with respect to an interchange of $\underline{r}$ and $\underline{r}'$. The boundary condition on $G(\underline{r}/\underline{r}')$

appropriate for the present problem is that it be an outgoing cylindrical wave at infinity. The solution for free space is:

$$G(\underline{r}/\underline{r}') = \frac{i}{4} H_0^{(1)} (k_I R) \qquad (3.6)$$

where $$R = \left| \underline{r} - \underline{r}' \right| .$$

The volume perturbation term $W^2\psi$ may be considered the inhomogeneous part of a Helmholtz equation in ψ. The two regions of interest, I and II, within which ψ and $\nabla\psi$ are continuous are to be considered separately. Then, using Weber's two-dimensional analogue of Helmholtz's theorem concerning solutions of Helmholtz's equation,[9][10] the following set of integral equations result:

Region I (3.7)

$$e^{i\underline{k}_0 \cdot \underline{r}} + \int_S \left[\frac{\partial \psi_I(\underline{r}')}{\partial n_I} G(\underline{r}/\underline{r}') - \frac{\partial G(\underline{r}/\underline{r}')}{\partial n_I} \psi_I(\underline{r}') \right] dS' + \int_{V_I} W_I^2 (\underline{r}') \psi_I(\underline{r}') G(\underline{r}/\underline{r}') dV' = \begin{cases} \psi_I(\underline{r}), & \underline{r} \text{ in I} \\ 0, & \underline{r} \text{ in II} \end{cases}$$

Region II (3.8)

$$\int_S \left[\frac{\partial \psi_{II}(\underline{r}')}{\partial n_{II}} G(\underline{r}/\underline{r}') - \frac{\partial G(\underline{r}/\underline{r}')}{\partial n_{II}} \psi_{II}(\underline{r}') \right] dS' \; \int_{V_{II}} W_{II}^2 (\underline{r}') \psi_{II} (\underline{r}') G(\underline{r}/\underline{r}') dV' = \begin{cases} 0, & \underline{r} \text{ in I} \\ \psi_{II}(\underline{r}), & \underline{r} \text{ in II} \end{cases}$$

The term $e^{i\underline{k}_0 \cdot \underline{r}}$ results from evaluating the surface integral at infinity, and is the incident plane wave of unit amplitude. For the particular case of equation (3.2),

W_I^2 is zero and W_{II}^2 is identically W^2.

In order to join the two integral equations, the continuity conditions given in equations (3.3) and (3.4) are to be used. Consider $\underline{r}$ in Region I (outside the rotor). Then from equation (3.8):

$$\int \frac{\partial G}{\partial n_{II}} \psi_{II} \, dS' = \int W^2 \psi_{II} G dV' + \int \frac{\partial \psi_{II}}{\partial n_{II}} G dS' \ .$$

Application of the pressure continuity condition in the surface integral yields:

$$-\int \frac{\partial G}{\partial n_{II}} \psi_I \, dS' = \int W^2 \psi_{II} G dV' + \int \frac{\partial \psi_{II}}{\partial n_{II}} G dS'$$

$$+ \ i\frac{\Omega}{\omega} \int \frac{\partial G}{\partial n_{II}} \frac{\partial \psi_{II}}{\partial \phi'} \, dS' \ ,$$

where use has been made of the relation

$$\frac{\partial G}{\partial n_{II}} = - \frac{\partial G}{\partial n_I} \ .$$

The latter equation is then added to equation (3.7), with the result:

$$\psi_I = e^{i\underline{k}_0 \cdot \underline{r}} + \int \frac{\partial \psi_I}{\partial n_I} G dS' + \int W^2 \psi_{II} \, G dV'$$

$$+ \int \frac{\partial \psi_{II}}{\partial n_{II}} G dS' + i\frac{\Omega}{\omega} \int \frac{\partial G}{\partial n_{II}} \frac{\partial \psi_{II}}{\partial \phi'} \, dS' \ .$$

Finally, the continuity condition on the normal component of $\underline{u}$ may be used to obtain:

$$\psi_{I}(\underline{r}) = e^{i\underline{k}_0 \cdot \underline{r}} - i\frac{2}{a}\frac{\Omega}{\omega}\int\frac{\partial\psi_{II}}{\partial\phi'}\, G dS' + \int W^2\, \psi_{II} G dV'$$

$$+ \; i\frac{\Omega}{\omega}\int\frac{\partial G}{\partial n_{II}}\,\frac{\partial\psi_{II}}{\partial\phi'}\, dS',$$

or, if the explicit form for W^2 is employed,

$$\psi_{I}(\underline{r}) = e^{i\underline{k}_0 \cdot \underline{r}} - i\frac{2}{a}\frac{\Omega}{\omega}\int\frac{\partial\psi_{II}}{\partial\phi'}\, G dS' + i2\frac{\Omega}{\omega}\, k_I^{\,2}\int\frac{\partial\psi_{II}}{\partial\phi'}\, G dV'$$

$$i\,\frac{\Omega}{\omega}\int\frac{\partial\psi_{II}}{\partial\phi'}\,\frac{\partial G}{\partial r'}\; dS' \qquad (3.9)$$

The integral equation (3.9) is a rigorous result. It relates the wave field outside the rotor to the ϕ derivatives of the field inside and on the surface of the rotor. In this form, approximation to the "inside" function ψ_{II} yields the corresponding approximation to the "outside" function ψ_I directly.

Note that surface as well as volume integrals appear in the above integral equations. In quantum mechanical problems dealing with volume perturbations, ψ and $\frac{\partial\psi}{\partial r}$ are continuous, so that it is unnecessary to make a distinction between two regions, and only a volume integral appears.[11] In ordinary acoustical problems dealing with volume perturbations, $\frac{\partial\psi}{\partial r}$ is continuous, but ψ may be discontinuous (corresponding to a jump in density), so that it is necessary to distinguish between two regions of space, and both volume and surface integrals appear.[12] In the latter case, however, surface integrals containing $\frac{\partial G}{\partial r}$ do not appear; in the present case they do.

2. Far-Field Relations

Often the field at large distances from the rotor is desired. In that case the asymptotic form of $G(\underline{r}/\underline{r}')$ can be used and its argument approximated. For $k_I r$ large, the argument $k_I R$ approaches

$$k_I R \simeq k_I \; (r - r'\cos\phi') = k_I r - \underline{k}_I \cdot \underline{r}' \; ,$$

and is used in the phase of $G(\underline{r}/\underline{r}')$. In the amplitude of $G(\underline{r}/\underline{r}')$, $k_I R$ is taken as $k_I r$. Hence

$$G(\underline{r}/\underline{r}') \simeq \frac{i}{4}\sqrt{\frac{2}{\pi k_I i}}\,\frac{e^{ik_I r}}{\sqrt{r}}\; e^{-\underline{k}_1 \cdot \underline{r}'} \; .$$

Then equation (3.9) becomes

$$\psi_I(\underline{r}) = e^{i\underline{k}_0 \cdot \underline{r}} + \frac{i}{4}\sqrt{\frac{2}{\pi k_I i}}\,\frac{e^{ik_I r}}{\sqrt{r}}\; S(\underline{k}_1, \underline{k}_0) \qquad (3.10)$$

where

$$\begin{aligned} S(\underline{k}_1,\underline{k}_0) = &- i\frac{2}{a}\frac{\Omega}{\omega}\int \frac{\partial \psi_{II}}{\partial \phi'}\, e^{-i\underline{k}_1 \cdot \underline{r}'}\, dS' \\ &+ i2\frac{\Omega}{\omega}\, k_I^{\,2}\int \frac{\partial \psi_{II}}{\partial \phi'}\, e^{-i\underline{k}_1 \cdot \underline{r}'}\, dV' \qquad (3.11) \\ &+ \frac{\Omega}{\omega}\, k_I \int \frac{\partial \psi_{II}}{\partial \phi'}\, \cos\phi'\, e^{-i\underline{k}_1 \cdot \underline{r}'}\, dS' \end{aligned}$$

The second term on the right of equation (3.10) is recognized as the scattered radiation. $S(\underline{k}_1,\underline{k}_0)$ is the amplitude of the scattered cylindrical wave measured in a direction $\underline{k}_1$ due to a wave incident in a direction $\underline{k}_0$. Hence $S(\underline{k}_1,\underline{k}_0)$ may be interpreted as $S(\phi,o)$ if the incident wave is coming from $\phi = \pi$ (headed toward $\phi = o$).

In general, equation (3.11) for the far field is easier to integrate than equation (3.9), but it does not lend itself to an iterative procedure. The reason for this is that only the asymptotic value of ψ is determined by equation (3.11), and values of ψ on and near the scatterer are required for iteration. In many practical cases, however, only the far field is of interest, and even without iteration it is often possible to obtain a useful solution for ψ based on a first trial inside function. In Section B and C of this Chapter, approximate solutions are found for ψ based on two different inside trial functions; the trial functions used must be simple enough to afford integration and accurate enough to lead to useful results.

It can readily be shown that the scattered intensity per unit incident intensity is given by

$$I(r,\phi) = \frac{1}{8\pi k_I} \frac{1}{r} \left| S(\underline{k}_1,\underline{k}_0) \right|^2 , \qquad (3.12)$$

and that the scattering cross-section is given by

$$\sigma = \frac{1}{8\pi k_I} \int_0^{2\pi} \left| S(\underline{k}_1,\underline{k}_0) \right|^2 d\phi \quad . \qquad (3.13)$$

These relations are derived in Appendix I. There is also a theorem connecting the total cross-section (scattering plus absorption cross-sections) and the forward scattered amplitude:

$$\sigma_t = \sigma + \sigma_a = \frac{1}{k_I} \, \mathrm{Im} \left[S(\underline{k}_0,\underline{k}_0) \right] . \qquad (3.14)$$

Equation (3.14) is well known for ordinary scattering problems,[13] but it is not obvious that it should also hold in problems of the type under study here. Consequently the proof of equation (3.14) is given in Appendix I. Throughout this study absorption effects have been discarded;

thus σ is directly related to the imaginary part of the scattered wave in the incident wave direction.

B. Born-Kirchoff Approximation

One of the simplest inside trial functions is a plane wave equal to the incident plane wave. This trial function leads to a first "Born-Kirchoff" approximation. From an intuitive point of view, the smaller M and Ω/ω, the better the approximation will be.

The inside function is chosen to be a plane wave of propagation constant, k_I, which is equal to the propagation constant in the external medium. Thus,

$$\psi_{II} = e^{i\underline{k}_0 \cdot \underline{r}} . \tag{3.15}$$

Before proceeding with the calculation of the scattered amplitude, it is appropriate to investigate the error associated with the BK approximation. To do this, a scheme due to Schiff is used.[14] He calculates the field at $\underline{r} = 0$, and investigates its value compared to the assumed value of $e^{i\underline{k}_0 \cdot \underline{r}}$. To apply this scheme to the present problem, note that for $\underline{r}$ in Region II, equations (3.7) and (3.8) give

$$e^{ik_0 \cdot r} + \int \left[\frac{\partial \psi_I}{\partial n_I} G - \frac{\partial G}{\partial n_I} \psi_I \right] dS' = 0 ,$$

and

$$\psi_{II}(\underline{r}) = \int \left[\frac{\partial \psi_{II}}{\partial n_{II}} G - \frac{\partial G}{\partial n_{II}} \psi_{II} \right] dS' + \int W^2 \psi_{II} \, G dV' .$$

Explicit use of the value for W^2, and application of the continuity conditions in the surface integrals of these two equations leads to*

$$\psi_{II}(\underline{r}) = e^{i\underline{k}_0 \cdot \underline{r}} + i2\frac{\Omega}{\omega} k_I^2 \int \frac{\partial \psi_{II}}{\partial \phi'} G dV'$$
$$- i\frac{2}{a}\frac{\Omega}{\omega}\int \frac{\partial \psi_{II}}{\partial \phi'} G dS' + i\frac{\Omega}{\omega}\int \frac{\partial \psi_{II}}{\partial \phi'} \frac{\partial G}{\partial r'} dS' . \quad (3.16)$$

The integrals on the right of equation (3.16) are equal to the error incurred by assuming ψ_{II} to be the incoming plane wave. For $\underline{r} = 0$, the error is given by

$$- \frac{1}{4}\frac{\Omega}{\omega}\Big[2k_I^2 \int \frac{\partial \psi_{II}}{\partial \phi'} H_0^{(1)}(k_I r')\, dV'$$
$$- 2H_0^{(1)}(k_I a)\int \frac{\partial \psi_{II}}{\partial \phi'} d\phi' \quad (3.17)$$
$$- k_I\, a\, H_1^{(1)}(k_I a)\int \frac{\partial \psi_{II}}{\partial \phi'} d\phi' \Big].$$

In order to calculate the error, ψ_{II} will be taken as the incoming plane wave in the integrals of equation (3.17). Then, all of the integrals have angle parts which are of the form

$$\int_0^{2\pi} e^{ik_I r\cos\phi} \sin\phi d\phi .$$

---

* Equation (3.16) for ψ_{II} ($\underline{r}$ in II) is of the same form as equation (3.9) for ψ_I ($\underline{r}$ in I). At first sight this appears incorrect, for at $r = a$ this implies that $\psi_{II} = \psi_I$, in violation of the pressure continuity condition. Further inspection reveals, however, that the sigularity of $\frac{\partial G}{\partial r}\Big|_s$ calculated in II is the negative of the singularity of $\frac{\partial G}{\partial r}\Big|_s$ calculated in I; the surface integrals containing $\frac{\partial G}{\partial r}$ thus give rise to the discontinuity of ψ, as expected.

But these integrals are all zero, the error is therefore zero, and hence the incident plane wave is the exact solution for the center of the rotor. In addition, inspection of the continuity equations (3.3) for the surface of the rotor shows that the discontinuity in ψ is less than about M, if ψ_I and ψ_{II} are assumed to be the incident plane wave. Thus it seems reasonable that the error incurred by assuming ψ_{II} to be the incident plane wave is less than the order of M throughout the rotor, and tends toward zero at the center of the rotor. Inasmuch as the calculation of ψ_I involves integrals of ψ_{II} multiplied by Ω/ω (see equation (3.9)), the net error is the order of $\frac{\Omega}{\omega}$ M (or $\frac{\Omega^2}{\omega^2}$ or M^2), and the BK approximation is thus adequate for the present problem. In summary, the zero order solution of the problem is the incident plane wave, and is in error by the order of Ω/ω or M. The first order solution, the BK approximation, is obtained by iteration of the zero order solution, and is in error by the order of $\frac{\Omega^2}{\omega^2}$ or M^2 or $M\frac{\Omega}{\omega}$.

Now the scattered amplitude will be calculated. It may be written as

$$S(\underline{k}_1,\underline{k}_0) = -2\frac{\Omega}{\omega} k_I a \int_0^{2\pi} e^{i\underline{K}\cdot\underline{r}'} \sin\phi' d\phi' \qquad (3.18)$$
$$- i\frac{\Omega}{\omega}(k_I a)^2 \int_0^{2\pi} e^{i\underline{K}\cdot\underline{r}'} \sin\phi' \cos\phi' d\phi'$$
$$+ 2\frac{\Omega}{\omega} k_I^3 \int_0^a \int_0^{2\pi} e^{i\underline{K}\cdot\underline{r}'} \sin\phi' d\phi' (r')^2 dr',$$

where the vector $\underline{K}$ is given by,

$$\underline{K} = \underline{k}_0 - \underline{k}_1 .$$

Hence,

$$|\underline{K}| = 2k_I \sin(\phi/2),$$

$$\underline{K} \cdot \underline{r}' = 2k_I r' \sin(\phi/2)\cos(\underline{r}', \underline{K}).$$

If a change of variable is made to the angle coordinate θ, where $\theta = \phi' + \frac{1}{2}(\pi - \phi)$, then simple geometrical considerations show that $\cos(\underline{r}', \underline{K}) = \cos\theta$. That is, $\underline{K}$ defines a new coordinate line, such that $\underline{r}'$ is at an angle θ from $\underline{K}$. This is shown in Figure 5. The transformation to the angle θ will be used in each of the three integrals in equation (3.18); each integral will now be investigated in turn.

Consider the first integral of equation (3.18). It may be written as:

$$\sin(\phi/2)\int_{\frac{\pi}{2}-\frac{\phi}{2}}^{\frac{5\pi}{2}-\frac{\phi}{2}} e^{i2k_I a\sin(\phi/2)\cos\theta} \sin\theta \, d\theta$$

$$- \cos(\phi/2)\int_{\frac{\pi}{2}-\frac{\phi}{2}}^{\frac{5\pi}{2}-\frac{\phi}{2}} e^{i2k_I a\sin(\phi/2)\cos\theta} \cos\theta \, d\theta \qquad (3.19)$$

The integration is around a closed circle. Thus the limits given in equation (3.19) may be changed to 0 and 2π. The first integral of equation (3.19) is zero because the exponential term is an even function around $\theta = \pi$, while

$\sin\theta$ is an odd function around $\theta = \pi$. The second integral is proportional to an integral representation of the Bessel function of the first kind, of order unity.[15] Thus equation (3.19) reduces to

$$- 2\pi\ i\ \cos(\tfrac{\phi}{2})\ J_1\left(2k_I a\sin(\tfrac{\phi}{2})\right) . \qquad (3.20)$$

Similarly the second integral of equation (3.18) is found to be:

$$\pi\ \sin\phi\ J_2\left(2k_I a\sin(\tfrac{\phi}{2})\right) \qquad (3.21)$$

The angle part of the third integral is identical to the first integral; the radial integration is simple, and results in:

$$- \pi\ i\ \cos\phi/2\ \frac{a^2}{k_I\sin\phi/2}\ J_2(2k_I a\sin\phi/2). \qquad (3.22)$$

Thus the scattered amplitude may be written as:

$$S(\underline{k}_1,\underline{k}_0) = i\pi M k_I a\sin\phi\ \cdot \qquad (3.23)$$
$$\cdot \left[2\Lambda_1(Ka) - \tfrac{1}{2}(1 + \sin^2\phi/2)\ (k_I a)^2\ \Lambda_2(Ka)\right]$$

where

$$K = |\underline{K}| = 2k_I\sin\phi/2,$$

$$\Lambda_n(Ka) = \frac{n!}{(\frac{Ka}{2})^n}\ J_n(Ka),$$

$$\phi = \cos^{-1}\left[\frac{\underline{k}_0\cdot\underline{k}_1}{k_I^2}\right].$$

Several interesting physical features of this result will now be discussed.

First of all, S is an odd function of ϕ. Thus the phase of the scattered field undergoes a phase shift of 180° at $\phi = 0$ and π, and the lines of constant phase are asymmetrical with respect to the forward direction. On the other hand, the intensity of the scattered field, which is essentially $\psi\psi^*$, is symmetrical with respect to the forward direction. Consequently the lines of constant phase and the lines of constant intensity of the scattered field are behaviorally different. Acoustic ray theory predicts that in the presence of net flow, sound intensity is propagated with a velocity different from the velocity of phase propagation.(16) Thus it may be expected that the phase and intensity in the rotor are distorted differently, and that the resulting phase and intensity patterns external to the rotor are behaviorally different. This is borne out by wave theory using the BK approximation.

Because dissipative forces have been neglected, the scattering process described by equation (3.23) is elastic. Thus no net torque can act on the rotor, and the total angular momentum of the wave field around the origin must be zero. However because of the asymmetrical nature of the scattered field, the total field by virtue of interference effects is asymmetrical, and it may be expected that the local angular momentum density around the origin may be asymmetrical and non-zero. To see that this is the case, it is sufficient to note that if two observation points (r, ϕ) and $(r, -\phi)$ are considered, the angular momentum density for each is the equal and opposite of the other for symmetrical scattering, but unequal for asymmetrical scattering. To gain quantitative information, one may use as a measure of the angular momentum asymmetry the sum of the quantity at (r, ϕ) and at $(r, -\phi)$. If this sum is formed and integrated from $\phi = 0$ to π, then the result is a measure of the field angular momentum in an infinitesmal ring dr located at r, and upon r integration must yield zero. Thus it may be concluded that rotor scattering gives rise to asymmetrical angular momentum densities which oscillate about zero values as a function of r.

The scattered amplitude may be interpreted as a product of two factors. The first factor is the solution valid at low frequencies; the second factor is a directivity factor which alters the form of the scattered wave as a function of frequency. Thus for low frequencies,

$$S(\underline{k}_1,\underline{k}_0) \simeq i\pi\ M(k_I a)\sin\phi, \tag{3.24}$$

the scattering is proportional to $k_I a$, and the scattering pattern is a simple sine function. On the other hand, for high frequencies,

$$S(\underline{k}_1,\underline{k}_0) \simeq -\ i\pi\ M(k_I a)^3 \sin\phi(1 + \sin^2 \tfrac{\phi}{2})\ \Lambda_2(Ka), \tag{3.25}$$

the scattering is proportional to the cube of $k_I a$, and the radiation pattern is such that only one major peak exists; it occurs between $\phi = 0$ and the first zero of $\Lambda_2(Ka)$. In the intermediate range, the complete equation (3.23) must be used.

To illustrate its behavior, several plots have been made of $S(\underline{k}_1,\underline{k}_0)$ in the angular range of 0 to 180 degrees* for $k_I a$ between 0 and 8. In Figure 6 it is evident that

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*Only the range 0° to 180° is given in the figures. Of course, $S(\underline{k}_1,\underline{k}_0)$ for the range 0° to -180° is just the negative of the curves shown.

the scattering pattern for $k_I a$ less than $\frac{1}{2}$ is little distorted from a sine curve. As frequency is increased however, the peak of the scattered intensity moves forward from 90° because of the term $\Lambda_1(Ka)$. Near $k_I a$ equal to unity, term in $\Lambda_2(Ka)$ becomes important and distorts the curve through zero to negative values; the peak of the intensity therefore moves backward again. At $k_I a$ of 2, $S(\underline{k}_1, \underline{k}_0)$ is all negative (for this range of ϕ) and thus a phase shift of 180° from the low frequency field has been accomplished. For $k_I a$ greater than 2, the term in $\Lambda_2(Ka)$ predominates, and the peak of the scattered intensity moves forward again, as shown in Figure 7. The peak increases and the angle of the peak decreases continuously as $k_I a$ increases beyond 4.

It is of interest to investigate the order of the effect of rotor scattering on sound propagation by referring to particular examples. It is assumed that rotor motion serves as an approximate model for two-dimensional vortical effects. Suppose an underwater experiment* in which a sound beam is to traverse a single rotor. Assume the frequency of the sound to be 50 KC, and the radius of the rotor to be 60 CM (this about the "size" of the inhomogeneities measured in the ocean).[17] Now the maximum Mach number that can occur for rotor motion is probably about 5×10^{-3}, for this corresponds to a velocity of about 25 ft/sec, which is roughly the cavitation threshold at atmospheric pressure. Thus $k_I a$ is about 8, $\frac{\Omega}{\omega}$ is about 10^{-3}, $S(20^\circ, 0)$ is about 1/2 per unit incident amplitude, and I_s, the scattered intensity per unit incident intensity

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*Although the general equations developed in Chapter II refer specifically to an ideal gas, the final equations which are used in this Chapter are independent of the equation of state of the medium.

is about -45 decibels at a distance of 10 radii from the rotor. Thus the effect for a single rotor, even at the peak of the scattering pattern, is small.

On the other hand, consider a similar experiment in air. The diameter of the rotor is taken to be 10 ft, which is the order of magnitude of the size of inhomogeneities near the ground. The velocity is taken to be 25 ft/sec, or M about 3×10^{-2}, corresponding to usual gust velocities. Then for a frequency of 1000 cps, $k_I a$ is about 30, $\frac{\Omega}{\omega}$ is about 10^{-3}, $S(5^{\circ},0)$ is about 5 (at the peak) and I_s at a distance of 10 rotor radii is about -25 decibels. Hence for this case too the effect of single rotor scattering is small, although not as small as in the underwater case. It is clear however, thatat higher frequencies, the effect will become greater.

There is one particular case where this effect, though small, is very important. It is well known that wind causes a sound shadow zone; ray theory predicts the sound in the shadow zone to be zero, although measurements show that this is not quite the case.[18] The example cited above shows that for reasonable values of the velocity, sound may be scattered into the shadow zone to a non-negligible extent, the total effect depending upon frequency, velocity, rotor "size" (which roughly depends upon height), and the number and distribution of rotors.

In Figure 8, measurements pertaining to this phenomenon are shown. The measured wave attenuation in a shadow zone is plotted against frequency for a case in which both source and receiver are 10 ft from the ground. The upper curve refers to propagation against the wind, the middle curve to propagation 45° against the wind, and the lower curve at right angles to the wind. Because the axes of rotational motion are most likely to be normal to the wind direction, the upper curve fits more nearly into the category of the present theory. At frequencies up to about 1000 cps, the measured attenuation increases with frequency; from a ray theory point of view it is expected that the attenuation should increase without limit, i.e., the field should go to zero at high frequencies. Above 1000 cps, however, the measured attenuation decreases. This suggests that energy is being scattered into the shadow zone; Figure 8 shows therefore that the scattered field is about -25 decibels at 1000 cps, and increases with increasing frequency. Thus the calculation in the example above based on an approximate model of the vorticity, appears to be able to predict the order of magnitude of the measured effect remarkably well, provided that some reasonable assumptions are made regarding the rotor properties.

It may be concluded that scattering from a single rotor is a relatively unimportant effect, except in cases like scattering into a shadow zone. This is an important result. On the other hand, in many problems of interest vorticity is not concentrated in one rotor, but rather distributed throughout space in many rotors, usually in a random manner. Hence it is appropriate to investigate the effect of a distribution of rotors on sound propagation, where the radius, frequency, and the position of the rotor vary according to some prescribed statistical laws. The statistical analysis will not be carried out in this study, for the proper specification of the statistical properties of rotational flow is a problem in itself.

In one simple and very important case however, it is possible to give immediately the connection between the sound field and a statistical field of rotors: For a medium containing a random spatial distribution of rotors of a single size and rotation frequency, the effective propagation constant may be written as:[19]

$$(k')^2 = k_I^2 + N\,S(\underline{k}_0,\underline{k}_0) \qquad (3.26)$$

where N is the number density of the rotors. Thus the effective propagation constant is complex; the real and imaginary parts are approximately

$$\begin{aligned} \mathrm{Re}\,k' &\simeq k_I\left[1 + \frac{N}{2k_I^2}\,\mathrm{Re}\,S(\underline{k}_0,\underline{k}_0)\right], \\ \mathrm{Im}\,k' &\simeq \frac{N}{2k_I}\,\mathrm{Im}\,S(\underline{k}_0,\underline{k}_0) \quad . \end{aligned} \qquad (3.27)$$

The imaginary part obviously corresponds to attenuation in the medium, and is usually of great interest. With the use of equation (3.14), the attenuation in nepers per unit length may be written as

$$\mathrm{Im}\,k' \simeq \frac{N}{2}\,\sigma \quad . \qquad (3.28)$$

It is clear that analogues to equations (3.26) and (3.27) may be obtained for particular statistical laws of interest; in any case one approach to the statistical problem is to solve the single scattering problem, as evidenced by the appearance of the quantities $S(\underline{k}_0',\underline{k}_0)$ and σ in equations (3.26), (3.27) and (3.28). On the basis of these equations it can be concluded that scattering from a large number of rotors may be an important effect.

Note that the BK approximation does not properly predict σ, for from equation (3.23) the scattered amplitude is zero in the forward direction. This a common failing of the Born approximation used in ordinary quantum mechanical problems.[20] It may be traced back to the fact that the imaginary part of $S(\underline{k}_0, \underline{k}_0)$ is directly related to the assumed phase perturbation of the wave in the rotor, which in the BK approximation is taken as zero. Of course, equation (3.13) may be used to obtain σ by integrating the square of the scattered amplitude on a closed surface, but this is not a convenient procedure. Consequently in the following Section another approximation is considered which assumes the inside wave field to have a phase perturbation, and which leads directly to the scattering cross-section σ .

C. The WKBJ Approximation to the Trial Function

The WKBJ approximation is usually applied to one-dimensional problems. In order to adopt this technique to the present problem, it is imagined that the wave normals do not have components in a direction other than the incident wave direction $\underline{k}_0$. Then it is not hard to show that ψ may be written approximately as[21]

$$\psi = e^{ik_I x + \frac{i}{2k_I}\int^{x} W^2 dx'} \qquad (3.29)$$

where W^2 is now thought of as a function of x and y, rather than r and ϕ. In the present case, y plays the role of an "impact" parameter; it selects the one-dimensional wave field at a distance y from the forward axis. It is expected that equation (3.29) is a better inside trial function for the integral equation than the BK approximation, for it takes into approximate account the phase variation in the rotor caused by the perturbation W^2. Although it is incorrect by requiring the wave normals to be parallel to $\underline{k}_0$, this also was implicitly required in the BK approximation.

Actually equation (3.29) is not quite the ordinary WKBJ expression, but rather one that may be derived from it by expanding

$$\left[k_I^2 + W^2\right]^{\frac{1}{2}}$$

in inverse powers of k_I. It was pointed out in Section A, however, that W^2 compared to k_I^2 is the order of M; thus the expansion is valid. In addition, the usual condition for the validity of the WKBJ solution is[21]

$$\left| \frac{1}{2\left[k_I^2 + W^2\right]^{3/2}} \frac{dW^2}{dx} \right| << 1. \tag{3.30}$$

It will be shown shortly that this condition is fulfilled.

Explicitly, the inside function of equation (3.11) will be taken as

$$\psi_{II}(x,y) = e^{ik_I x + \frac{i}{2k_I}\int_{-b}^{x} W^2 (x', y)\, dx'} \tag{3.29}$$

in the volume integrals, and as

$$\psi_{II}\Big|_S = e^{ik_I x} , \quad \frac{3\pi}{2} > \phi > \frac{\pi}{2} \tag{3.31}$$

$$\psi_{II}\Big|_S = e^{ik_I x + \frac{i}{2k_I}\int_{-b}^{b} W^2(x',y)\, dx'} , \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}$$

in the surface integrals. where b is equal to

$$\left[a^2 - y^2\right]^{\frac{1}{2}} .$$

Thus on the "illuminated" surface of the rotor, the inside function is just the incident plane wave; on the "shadow" surface of the rotor it is the incident plane wave modified by a phase which is a function of y. Within the rotor, the inside function is the incident plane wave modified by a phase depending upon the depth of penetration and upon the impact distance y.

For the present purposes, it is more convenient to write the equation for $S(\underline{k}_1, \underline{k}_0)$ in terms of W^2, rather than in terms of $\frac{\partial \psi_{II}}{\partial \phi}$. Thus equation (3.11) becomes

$$S(\underline{k}_1,\underline{k}_0) = -\frac{1}{k_I{}^2 a}\int W^2\,\psi_{II}\, e^{-i\underline{k}_1\cdot\underline{r}'}\,dS'$$

$$+\int W^2\,\psi_{II}\, e^{-i\underline{k}_1\cdot\underline{r}'}\,dV' \qquad (3.32)$$

$$-\frac{i}{2k_I}\int W^2\,\psi_{II}\,\cos\phi' e^{-i\underline{k}_1\cdot\underline{r}'}\,dS'$$

$S(\underline{k}_1,\underline{k}_0)$ will be specialized for the forward direction inasmuch as it is of interest to use the WKBJ technique to determine the scattering cross-section σ. With the use of equations (3.29) and (3.31) for the inside function, the forward scattered amplitude is:

$$S(\underline{k}_0,\underline{k}_0) = -\frac{1}{k_I{}^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W^2 e^{\frac{i}{2k_I}\int_{-b}^{b} W^2 dx'}\,d\phi' - \frac{1}{k_I{}^2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} W^2 d\phi'$$

$$-\frac{ia}{2k_I}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W^2 e^{\frac{i}{2k_I}\int_{-b}^{b} W^2 dx'}\cos\phi'\,d\phi' \qquad (3.33)$$

$$-\frac{ia}{2k_I}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} W^2\cos\phi'\,d\phi' + \int W^2 e^{\frac{i}{2k_I}\int_{-b}^{x} W^2 dx'}\,dx\,dy.$$

On order to proceed further, W^2 should be determined. From its definition, W^2 may be written as

$$W^2 = i2\frac{\Omega}{\omega} k_I^2 \frac{\partial \ln \psi_{II}}{\partial \phi}$$

$$= \frac{\Omega}{\omega} \left[2k_I^3 r + k_I W^2 (r + a) \right] \sin\phi,$$

where the latter relation involves the use of equation (3.29). Hence

$$W^2(r,\phi) \doteq \frac{2\frac{\Omega}{\omega} k_I^3 r \sin\phi}{1 - \frac{\Omega}{\omega} k\ (r + a)\sin\phi} \tag{3.34}$$

$$W^2(x,y) = \frac{2\frac{\Omega}{\omega} k_I^3 y}{1 - \frac{\Omega}{\omega} k_I y - \frac{\Omega}{\omega} k_I a \frac{y}{\left[y^2 + x^2\right]^{\frac{1}{2}}}} \qquad (3.34)'$$

Inasmuch as the last two terms in the denominator of equation (3.34)' are less than M, W^2 may be taken as

$$W^2(x,y) \simeq 2\frac{\Omega}{\omega} k_I^3 y. \tag{3.35}$$

Thus W^2 is approximately independent of x, and the validity condition for the WKBJ technique given by equation (3.30) is fulfilled. Note that equation (3.35) implies that ψ_{II} may be considered an undistorted plane wave for purposes of calculating W^2. In other words, the zero-order approximation, the plane wave, is used to calculate W^2. Now equation (3.32) is to be used to imporve the first-order approximation.

The surface integrals of equation (3.33) will be considered first. Because W^2 is an odd function of ϕ, the second and fourth integrals are zero. The integral of W^2 in the phase part of ψ_{II} is

$$\frac{1}{2k_I}\int_{-b}^{b} W^2 dx' = 2\frac{\Omega}{\omega}(k_I a)^2 \sin\phi' \cos\phi' .$$

Thus the remaining surface integrals of $S(\underline{k}_0,\underline{k}_0)$ may be written as

$$-2\frac{\Omega}{\omega} k_I a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i2\frac{\Omega}{\omega}(k_I a)^2 \sin\phi' \cos\phi'} \sin\phi' d\phi'$$

(3.37)

and

$$- i\frac{\Omega}{\omega}(k_I a)^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i2\frac{\Omega}{\omega}(k_I a)^2 \sin\phi' \cos\phi'} \sin\phi' \cos\phi' d\phi'$$

In order to facilitate evaluation of these integrals, only their imaginary parts will be investigated, for their real parts are of no concern in the evaluation of the cross-section by means of equation (3.14). The imaginary part of equation (3.37) is zero because it is in the form of a product of an even and odd function:

$$\int_0^{2\pi} \cos(z\sin\beta)\sin\beta\, d\beta .$$

Hence only equation (3.36) remains of the surface integrals for the imaginary part of $S(\underline{k}_0,\underline{k}_0)$. Without trouble its imaginary part may be reduced to

$$-2M\int_0^{\pi} \cos\beta \sin(Mk_I a\sin 2\beta)\, d\beta$$

This integral may be put in the standard form of Lommel's functions by changing the integration variable from β to 2β.[22] It will be convenient, however, to use Anger's functions in place of Lommel's functions; in terms of Anger's functions,[22] the imaginary part of equation (3.36) is

$$- M\pi \left[\tilde{J}_{\frac{1}{2}}(z) - \tilde{J}_{-\frac{1}{2}}(z) \right] \tag{3.38}$$

where $\tilde{J}_{\nu}(z)$ is the Anger function of order ν , and where z is given by

$$z = Mk_I a = \frac{\Omega}{\omega} (k_I a)^2$$

The volume integral may be evaluated by the following method. Note that the exponential term is an exact differential:

$$\frac{d}{dx} \left[e^{\frac{i}{2k_I} \int_{-b}^{x} W^2 dx'} \right] = \frac{i}{2k_I} W^2 e^{\frac{i}{2k_I} \int_{-b}^{x} W^2 dx'}$$

Hence the volume integral is given by

$$\frac{2k_I}{i} \int_{-a}^{a} \left[e^{\frac{i}{2k_I} \int_{-b}^{x} W^2 dx'} \right]_{-b}^{b} dy$$

or

$$i2k_I \int_{-a}^{a} \left[1 - e^{i2\frac{\Omega}{\omega} k_I^2 y \sqrt{a^2 - y^2}} \right] dy \tag{3.39}$$

The imaginary part of equation (3.39) is just

$$2k_I \int_{-a}^{a} \left[1 - \cos\beta\right] dy \qquad (3.40)$$

where

$$\beta = 2\frac{\Omega}{\omega} k_I^{\,2} y \sqrt{a^2 - y^2}$$

By simple reductions, equation (3.40) may also be put in the form of Lommel's function or Anger's functions; expressed in terms of Anger's functions it is

$$4k_I a - k_I a\pi \left[\tilde{J}_{\frac{1}{2}}(z) + \tilde{J}_{-\frac{1}{2}}(z)\right] \qquad (3.41)$$

where

$$z = Mk_I a = \frac{\Omega}{\omega}(k_I a)^2$$

Hence from equations (3.14), (3.38), and (3.41) the scattering cross-section is given by

$$\frac{\sigma}{2a} = 2 - \frac{\pi}{2}\left[\tilde{J}_{\frac{1}{2}}(z) + \tilde{J}_{-\frac{1}{2}}(z)\right] - \frac{\Omega}{\omega}\frac{\pi}{2}\left[\tilde{J}_{\frac{1}{2}}(z) - \tilde{J}_{-\frac{1}{2}}(z)\right] \qquad (3.42)$$

Equation (3.42) is the formal result of this Section. The integral representation, the series expansion, the asymptotic expansion, and other related properties of Anger functions are given in Appendix II. It is shown there that in the limit of z zero,

$$\tilde{J}_{\frac{1}{2}}(z) + \tilde{J}_{-\frac{1}{2}}(z) \rightarrow \frac{4}{\pi}$$

$$\tilde{J}_{\frac{1}{2}}(z) - \tilde{J}_{-\frac{1}{2}}(z) \rightarrow 0 .$$

Therefore the scattering cross-section is zero in the limit of zero frequency or zero M. Furthermore, it is shown that in the limit of large z,

$$\tilde{J}_{\frac{1}{2}}(z) + \tilde{J}_{-\frac{1}{2}}(z) \rightarrow \sqrt{\frac{2}{\pi z}}\,(\sin z + \cos z)$$

$$\tilde{J}_{\frac{1}{2}}(z) - \tilde{J}_{-\frac{1}{2}}(z) \rightarrow \sqrt{\frac{2}{\pi z}}\,(\sin z - \cos z)$$

Hence, for high frequencies the scattering cross-section approaches twice the geometrical cross-section, to within small oscillating terms. Thus the scattering cross-section for rotor scattering is behaviorally similar to that obtained for ordinary scattering problems.

Note that by virtue of the smallness of $\frac{\Omega}{\omega}$, the term containing the sum of the Anger functions is much greater than the term containing the difference, except when the former is zero. For plotting purposes it is sufficient to use only the sum term; the resulting plot of $\frac{\sigma}{2a}$ versus $Mk_I a$ is shown in Figure 9. In order to plot this curve, a short table of Anger functions of order $\frac{1}{2}$ has been constructed and is included in Appendix II. The error committed by excluding the $\frac{\Omega}{\omega}$ term from $\frac{\sigma}{2a}$ may be obtained by consulting the table in Appendix II; it is almost always negligible, for practically speaking, $\frac{\Omega}{\omega}$ is usually the order of 10^{-2} or 10^{-3}.

Inspection of Figure 9 indicates that rotor scattering is indeed similar to ordinary scattering but with two important exceptions. In ordinary scattering, the cross-section becomes twice the geometrical cross-section at ka the order of unity, and the oscillations become small at ka the order of ten. In the present case, however, the scattering cross-section becomes twice the geometrical cross-section when $Mk_I a$ (or $\frac{\Omega}{\omega}(k_I a)^2$) is the order of unity, and the oscillations become small at $Mk_I a$ (or $\frac{\Omega}{\omega}(k_I a)^2$) the order of one hundred. Thus a plot of $\frac{\sigma}{2a}$ versus $\sqrt{\frac{\omega}{\Omega}}\, k_I a$ for rotor scattering would be nearly identical to a plot of $\frac{\sigma}{2a}$ versus ka for ordinary scattering, and the conceptual backround of ordinary scattering may be applied to rotor scattering (as far as total scattering is concerned). Thus the frequency scale for rotor scattering is contracted relative to ordinary scattering by a similarity parameter $\sqrt{\frac{\Omega}{\omega}}$; for example, rotor scattering becomes important when $k_I a$ is the order of or greater than $\sqrt{\frac{\omega}{\Omega}}$ (or $\frac{1}{M}$).

It is of interest to calculate the attenuation of a medium containing rotors by use of the scattering cross-section computed by the WKBJ technique. To do this, the example given in Section B will be used. Consider again a rotor in air of diameter 10 ft and of maximum Mach number 3×10^{-2}. At 1000 cps, the cross-section is about 4 ft per unit length. If the distance between rotors is assumed to be such that N is about 10^{-4} per ft^2,* then by equation (3.28), the attenuation is about $1\frac{1}{2}$ decibels per 1000 ft. This is just the order of magnitude of attenuation measured in the atmosphere[18] (outside of the shadow zone) for gently rolling, sparsely treed terrain.

It must be emphasized that the agreement between measurements and theory cited here and in the last Section is only as good as the estimates of the parameters a, M, and N,

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*The number of rotors in a field will depend upon the number of obstacles producing the vorticity and upon the decay of vorticity associated with each obstacle. For the problem of interest here, the Reynold's number is greater than 10^6; hence decay should be rapid and N should be more characteristic of the number of obstacles.

and the extent to which the net flow can be idealized as two-dimensional rotor motion. The net flow is no doubt more complicated than the assumed rotor motion, but it is expected nevertheless that the order of magnitude of the effects may still be predicted provided that the mean wind direction and the sound direction are parallel. The magnitude of the radius a may be predicted fairly well; most meteorological workers take the size of vortical disturbances to be either equal to the height above gound[23] or to the obstacle dimension projected normal to the mean wind.[24] In the case of atmospheric acoustics, distinction need not be made between the two possibilities for the measure of a ; the heights of interest and the obstacles of interest are both roughly 10 ft. The maximum velocity of vortical motion in the atmosphere is not well known, but it is reasonable to expect that it is the order of magnitude of the mean translational velocity, just as it is in the case of vortex shedding from cylinders or plates measured in the laboratory.[25] Finally the appropriate choice for N is not clear, but it appears reasonable to suppose that it is associated with the roughness of the terrain, to the extent that it is an approximate measure of the square of the number of obstacles per unit length in the sound path. These considerations have dictated the estimates given in this Section and in Section B; subject to the validity of these estimates it appears that the BK approximation and the WKBJ approximation yield useful results.

D. Comparison with Lighthill's Result

M. J. Lighthill has recently computed the scattering of plane waves of sound from statistical turbulence.[27] He makes no special assumptions on the nature of the turbulence, but for turbulence macroscales larger than the sound wavelength, he has succeeded in obtaining an approximate expression for the scattered power. In the present notation, his result is

$$P = 2k_I^{\,2} L \, \overline{M^2} \tag{3.43}$$

where P is the scattered power per unit incident intensity per unit volume,
L is the macroscale of the turbulence in the direction of sound propagation,
$\overline{M^2}$ is the mean of the square Mach number of the disturbance.

Thus the scattering cross-section for a volume V is

$$\sigma = 2k_I^{\,2} \int_V L \, \overline{M^2} dV \tag{3.44}$$

In order to compare his results with the results given in Section C, note that for rotor scattering, $\overline{M^2}$ and L may be given by:

$$\overline{M^2} = \tfrac{1}{2}M^2 \ ,$$

$$L \simeq 2a$$

The scattering cross-section for one rotor is then

$$\frac{\sigma}{2a} \simeq \pi M^2 (k_I a)^2 \qquad (3.45)$$

subject to the restriction that $k_I a > \pi$.

Under the same conditions, equation (3.42) of Section C reduces to

$$\frac{\sigma}{2a} \simeq \frac{2}{3.75} M^2 (k_I a)^2 + O\left[(Mk_I a)^4\right]$$

Thus Lighthill's result is equivalent in form to the present result provided that,

$$Mk_I a < 1$$

In other words, agreement is obtained for the following range of frequency:

$$\pi < k_I a < \frac{1}{M} \qquad (3.46)$$

In this range Lighthill's result is about a factor of 5 greater than that given in Section C; uncertainty as to the appropriate value of L and V, however, may easily account for the discrepancy.

Outside of the upper limit of the range given by equation (3.46), the behavior of the two results are quite different. The cross-section derived from Lighthill's statistical approach increases indefinitely; the cross-section based on the rotor model tends to the limit of twice the geometrical cross-section. Thus there appears to be an essential physical difference between scattering from a region of statistical vorticity and a region of ordered vorticity, at least for high frequencies. This difference may be thought of as arising from coherent phase effects in the field of the rotor which have no counterpart in the field of statistical vorticity.

These phase effects become important for $k_I a > 1/M$. As pointed at in the last Section, the total scattered energy becomes large enough to be of importance in this range also. Thus from the point of view of application to practical problems, it is of importance to determine whether or not the vortical motion is statistical, in order to determine whether to use equation (3.44) or (3.42) for σ. In many problems of practical interest, it is probable that neither the extreme of statistical nor ordered rotor motion is correct, but rather some mixture of both. In the case of sound propagation over ground, it is known that the disturbances are non-isotropic,[23] but it is not known whether the vorticity is more ordered than statistical, although it appears reasonable to assume so. It is clear that additional experimental information is required on the flow field before definite conclusions can be drawn.

IV Refraction by Ideal Vortex Motion

In this Chapter, the effect of ideal vortex motion on sound waves will be studied. From a wave point of view, the rotor, which was treated in the last Chapter, is fundamentally different from the ideal vortex. In the former case, the effect of the motion is confined to a finite region of space; in the latter case, the motion extends to infinity, giving rise to distortion of the wave over an infinite region of space. This is in a sense analogous to the situation in quantum mechanics regarding the fundamental difference between the scattering properties of a square-well potential and a Coulomb potential. Just as in quantum mechanics, different techniques are required to evaluate the effects of rotor motion and ideal vortex motion.

The problem may be stated exactly as it was in the last Chapter, except that in this case Region II (the region external to the core) undergoes a net motion. The present case is illustrated in Figure 10.

Solutions of the approximate equations (2.75) and (2.76) for ideal vortex motion will be sought which behave like

$$e^{i\underline{k}_0 \cdot \underline{r}}$$

at large distances from the vortex center. In Section A of this Chapter, it is shown that the integral equation approximation techniques used in Chapter III are inappropriate for this case. Instead, in Section B, a WKBJ procedure is used to determine the wave refraction; this procedure is shown to contain the essentials of a ray solution to the problem given by R. B. Lindsay,[26] as well as information relavent to the intensity distribution.

Finally, in Section C, the effect of the inhomogeniety terms omitted in the approximate differential equations is investigated by use of the WKBJ technique given in Section C of Chapter III.

A. Integral Equation Approach

The development in this Section follows very closely that of Section A in Chapter III. Consequently only those points that are substantially new are discussed. The approximate differential equations for the ideal vortex may be written as

$$(\nabla^2 + k_I^2 + W^2)\,\psi\,(\underline{r}) = 0 \qquad (4.1)$$

where

$$W^2(\underline{r}) = \begin{cases} i2\frac{\Omega}{\omega}\, k_I^2\ \frac{\partial \ln\psi}{\partial \phi}, & r < a \\ i2\frac{\Omega}{\omega}\, k_I^2\, \frac{a^2}{r^2}\ \frac{\partial \ln\psi}{\partial \phi}, & r > a \end{cases}$$

The continuity conditions at the core surface lead to the following conditions on ψ:

$$\psi_I\Big|_s + i\frac{\Omega}{\omega}\frac{\partial \psi_I}{\partial \phi}\Big|_s = \psi_{II}\Big|_s + i\frac{\Omega}{\omega}\frac{\partial \psi_{II}}{\partial \phi}\Big|_s \qquad (4.2)$$

$$\frac{\partial \psi_I}{\partial r}\Big|_s = \frac{\partial \psi_{II}}{\partial r}\Big|_s + i\frac{2}{a}\frac{\Omega}{\omega}\frac{\partial \psi_{II}}{\partial \phi}\Big|_s \qquad (4.3)$$

Use of equations (3.5), (3.6), (3.7), and (3.8), together with (4.1), (4.2), and (4.3), leads to an expression for ψ_I corresponding to the result in Chapter III:

$$\psi_I(\underline{r}) = e^{i\underline{k}_0 \cdot \underline{r}} - i\frac{2}{a}\frac{\Omega}{\omega}\int \frac{\partial \psi_{II}}{\partial \phi'} GdS'$$

$$+ \int (W_I{}^2 \psi_I + W_{II}{}^2 \psi_{II}) \, GdV' \qquad (4.4)$$

$$- \int (\psi_{II} - \psi_I) \frac{\partial G}{\partial r'} dS'$$

Thus the field outside the core cannot be written explicitly in terms of the field in and on the surface of the core. This fact would not be bad in itself, but the volume integral diverges* for any reasonable guess of ψ_I. The difficulty may be traced back through $W_I{}^2$ to the "flow" part of the total time derivative of ψ, i.e. to

$$W_I{}^2 = - \frac{2}{c^2} \underline{v} \cdot \nabla \frac{\partial^2 \psi_I}{\partial \phi \partial t} + O(M^2).$$

Thus the velocity field does not fall off rapidly enough to allow the use of equation (4.4) to improve a zero-order approximation such as a plane wave.

This difficulty suggests the possibility of modifying the r dependence of $\underline{v}$ outside the core such that the range of the net flow field is decreased. As a matter of fact, in most cases of physical interest the ideal velocity field will be screened or interfered with by neighboring vortex motion, by obstacles, and by viscosity.

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*A Green's function other than the free-space Green's function may be used in the above formulation. It may be defined separately for Regions I and II, with appropriate joining conditions at S, and it would lead to another form for the surface integrals of equation (4.4). However it must still represent an outgoing wave at infinity and hence the volume integral still leads to difficulty.

When the external net flow field is no longer ideal, however, it is no longer irrotational* and suitable modification must be made of equations(4.1), (4.2), and (4.3). Assuming these modifications to be made, the form of the external net flow field must be determined. At present however, there is no experimental or theoretical basis for choosing a form for v; rotor motion studied in the last Chapter appears to be as reasonable and simple a choice as can be made. As a matter of fact, it is expected that rotor scattering contains most of the qualitative aspects and many of the quantitative aspects of scattering from any screened velocity field. Consequently no attempt will be made to use equation (4.4) in conjunction with a screened velocity field. Rather, another method will be used which will allow the retention of the ideal velocity field, and which will yield at least an approximate idea of the effect of the irrotational part of ideal vortex motion.

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*The only appropriate net velocity field which gives rise to divergenceless and curl-less flow is the $\frac{1}{r}$ field of the ideal vortex.

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B. The WKBJ Solution

Before starting this calculation it is appropriate to note again that the core of the vortex is identical to the rotor treated in the last Chapter. It is therefore intuitively expected that the scattering from an ideal vortex is similar to the scattering from a rotor, except that the incident and scattered waves are refracted by the net flow field external to the core. The failure of simple trial functions in the integral equation technique suggests the use of simpler (and hence less accurate!) techniques to evaluate the refraction. One such technique that may be applied is the WKBJ method used earlier to calculate a trial function for the rotor; another, and less accurate technique, is the method of ray acoustics. Actually, WKBJ solutions are not expected to be very much better than ray solutions, in as much as the WKBJ method as applied here requires the two-dimensional problem to be broken up into an ensemble of one-dimensional problems. Nevertheless, it shall prove convenient to proceed along the lines of the WKBJ method.

In order to apply this method to the case of the ideal vortex, the appropriate differential equation (4.1) is written in rectangular coordinates:

$$\frac{\partial^2 \psi}{\partial x^2} + \left(k_I^{\,2} + W^2 + \frac{\partial^2 \ln\psi}{\partial y^2}\right)\psi = 0 \qquad (4.5)$$

Formally, the WKBJ solution for a wave originating at $x \to -\infty$ is:[21]

$$\psi(x;y) = \frac{A\, e^{\,i\int_{-\infty}^{x}\left[k_I^{\,2} + W^2 + \frac{\partial^2 \ln\psi}{\partial y^2}\right]dx}}{\left[k_I^{\,2} + W^2 + \frac{\partial^2 \ln\psi}{\partial y^2}\right]^{\frac{1}{4}}} \qquad (4.6)$$

This equation is to be interpreted as a wave depending on x, and traveling on a line y = constant. Strictly, equation (4.6) is an integral equation for ψ. W^2 and the y derivative term are small* however, so that it is assumed that approximations to ψ in W^2 and in the y derivative term will yield improved values of ψ.

W^2 and $\frac{\partial^2 \ln\psi}{\partial y^2}$ have simple physical interpretations. W^2 is essentially determined by the ϕ derivative of the phase of ψ, and may therefore be considered the coupling agent between waves in the incident x direction and the y direction. $\frac{\partial^2 \ln\psi}{\partial y^2}$ is essentially a measure of the extent to which W^2 has distorted the incident wave.

In addition to the usual validity condition for the WKBJ solution (see equation (3.30) in Section C of Chapter III), and to the condition that W^2 and $\frac{\partial^2 \ln\psi}{\partial y^2}$ are small, two other conditions at the core surface must be met. Consider the quantities

$$\Delta_1 = (\psi_I - \psi_{II})\Big|_s + i\frac{\Omega}{\omega}\left(\frac{\partial\psi_I}{\partial\phi} - \frac{\partial\psi_{II}}{\partial\phi}\right)\Big|_s \quad ,$$
$$\Delta_2 = \left(\frac{\partial\psi_I}{\partial r} - \frac{\partial\psi_{II}}{\partial r}\right)\Big|_s - i\frac{2}{a}\frac{\Omega}{\omega}\frac{\partial\psi_{II}}{\partial\phi}\Big|_s \quad . \qquad (4.7)$$

The quantities are zero when the joining conditions given in equations (4.2) and (4.3) are satisfied. But the requirement that ψ be a one-dimensional wave will make it impossible to satisfy the joining conditions exactly.

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*The y derivative term must go to zero as W^2 does to zero. Hence it is assumed that it is the order of magnitude of W^2.

The maximum values of $|\Delta_1|$ and $|\Delta_2|$ compared to $|\psi_I|$ and $\left|\frac{\partial\psi_I}{\partial r}\right|$ respectively will be taken as additional measures of the validity of the present approach; both ratios will be required to be small.

Just as in the case of the BK approximation used in Chapter III, the zero-order wave will be taken as a first guess for integral equation (4.6). Thus for,

$$\psi_I = \psi_{II} = \psi_0 = e^{ik_I x} ,$$

the "validity" parameters are:

$$\left(\frac{W_0}{k_I}\right)^2 = \begin{cases} 2\frac{\Omega}{\omega} k_I y & , r < a \\ 2\frac{\Omega}{\omega} k_I y \frac{a^2}{r^2} & , r > a \end{cases}$$

$$\left| \frac{1}{2k_I^3} \frac{dW_0^2}{dx} \right| \leq \begin{cases} 0 & , r < a \\ \frac{\Omega}{\omega}\left(\frac{a}{r}\right)^2 & , r > a \end{cases} \qquad (4.8)$$

$$\frac{1}{k_I^2} \frac{\partial^2 \ln \psi_0}{\partial y^2} = 0$$

$$\frac{|\Delta_1|}{|\psi_I|} = 0$$

$$\frac{|\Delta_2|}{\left|\frac{\partial \psi_I}{\partial r}\right|} \leq 2\frac{\Omega}{\omega}$$

Thus the zero-order solution is in error by no more than the order of $M(=\frac{\Omega}{\omega}k_I a)$ and $\frac{\Omega}{\omega}$, as expected. The first iterate ψ_1 may be calculated with the hope that error of ψ_1 will be smaller. It will be necessary to distinguish the field outside the core and its geometrical shadow, inside the geometrical shadow, and in the core. The wave functions will be designated ψ_1, ψ_1^s, and ψ_1^c respectively. The integrations involved are straightforward, and the results are:

$$\psi_1\ (x;y) = \left[1 + \frac{M}{2}\frac{ay}{r^2}\right]^{-1} e^{ik_I x}\, e^{i\theta(x;y)}$$

$$\psi_1^c(x;y) = \left[1 + \frac{M}{2}\frac{y}{a}\right]^{-1} e^{ik_I x}\, e^{i\theta(-\sqrt{a^2-y^2};y)}\, e^{i\beta(x;y)}$$

$$\psi_1^s(x;y) = \psi_1(x;y)\, e^{i\alpha(y)}$$

$$\theta(x;y) = Mk_I a\left[\tan^{-1}\frac{x}{y} \pm \frac{\pi}{2}\right] \qquad (4.9)$$

(use + for $y > 0$, - for $y < 0$)

$$\theta(x;0) = 0$$

$$\beta(x;y) = M\frac{y}{a}k_I\left[x + yq\right]$$

$$\alpha(y) = 2Mk_I a\left[q\left(\frac{y}{a}\right)^2 - \tan^{-1}q\right]$$

$$\alpha(0) = 0$$

$$q = \frac{1}{y}\left[a^2 - y^2\right]^{\frac{1}{2}}$$

Application of the "validity parameters" similar to those of equation (4.8) and a great deal of algebra not worth repeating, shows that the error associated with ψ_1 is still the order of $\frac{\Omega}{\omega}$ and M.

Hence iteration of ψ_0 to obtain the iterate ψ_1 is not justified; the process is not convergent. Inspection of the validity parameters however, reveals that in the far-field $(r \to \infty)$ all the parameters go to zero with the exception of those associated with the core region and the core surface. But this is just the expected situation, for the terms associated with the core must give rise to a scattered wave, a situation which has not been explicitly allowed for in this formulation. This difficulty may be removed however, by the following physical reasoning: Suppose the errors associated with the joining conditions and perturbation terms of the core are arranged to be arbitrarily small by the addition of a suitable wave to the WKBJ solution. Then in the far-field, this modified WKBJ process would lead to an arbitrarily small error in the iterate ψ_1. Physically, the wave which adjusts itself to produce an arbitrarily small error at the core is just the total wave (incident plus scattered) studied in connection with rotor motion in Chapter III*. Thus it is expected physically that if the incident wave

$$e^{ik_I x}$$

is replaced by the rotor wave in equation (4.9), ψ_1 is then a reasonable approximation to the total wave for ideal vortex scattering. Hence the total wave is just that due to scattering from the core (the rotor) and refraction in the external irrotational velocity field. Denote ψ_R

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*This statement is not quite correct, for although the net flow fields of the rotor and the core are identical, the joining conditions on ψ are slightly different. Hence the scattered amplitudes will differ by an additional surface integral term. Inspection of the solutions for ψ reveals however, that the scattered amplitude is mainly determined by the volume perturbation; the surface perturbations modify the result only secondarily.

to be the asymptotic rotor field, and let $r \to \infty$ in equations (4.9); the explicit result is:

$$\psi_1(r,\phi) \simeq \psi_R(r,\phi)\ e^{i\theta}$$

$$\psi_1^s(r,\phi) \simeq \psi_R(r,\phi)\ e^{i(\theta+\alpha)} \qquad (4.10)$$

where

$$\theta(r,\phi) = \theta(\phi) = Mk_I a \left[\tan^{-1}(\mathrm{ctn}\phi) \pm \frac{\pi}{2} \right]$$

(use + for ϕ +, - for ϕ -)

$$\theta(\pi) = 0$$

Equation (4.10) is to be regarded only as a semi-quantitative result from which several important qualitative physical interpretations may be drawn. First of all, the forward scattering amplitude is just that for rotor scattering; to this approximation the external field of the ideal vortex does not effect the total power scattered. Thus the assumptions made in Chapter III concerning the applicability of rotor theory to measurements on sound propagation over ground appear justifiable.

Secondly, the phase asymmetry caused by the ideal vortex is much more important than that caused by rotor scattering, for it effects the total wave and extends to infinity. For example, consider the points $(r,-\frac{\pi}{4})$ and $(r,\frac{\pi}{4})$. The phase difference is $(Mk_I a)(\frac{3\pi}{2})$, independent of r. For the range of important scattering ($Mk_I a > 1$), this phase difference is obviously important. However, even for small $Mk_I a$, the phase change may be measurable. Consider the case in which a vortex moves between a source and a receiver. Suppose the vortex has the properties considered in Chapter III: $M = 3 \times 10^{-2}$ and $2a = 10$ ft. Then at 1000 cps the total phase fluctuation at the receiver is greater than 240° as the measurement point passes from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$. At 100 cps, the phase fluctuation is greater than 24°, and represents a sizeable effect. Thus it can be concluded that ideal vortex motion may give rise to important phase fluctuation effects in sound propagated over the ground for almost the entire audio frequency range. Lindsay[26] was the first to recognize this effect. In his ray analysis, he was forced to neglect the scattering from the core, which is taken into account semi-quantitatively in the present analysis. Incidentally, the phase term Θ calculated by the WKBJ technique is identical to the result of Lindsay obtained by a ray technique.

C. Scattering Associated with the Concomitant

Inhomogeneity

All the scattering calculations thus far have been based upon approximate differential equations which neglected effects the order of M^2. Thus scattering due to density gradients has been disgarded because they are of that order. It is recalled that terms the order of M^2, $\frac{\Omega^2}{\omega^2}$, and $M\,\Omega/\omega$ were omitted in order to enable the vector potential $\underline{A}$ to be expressed simply, and thus reduce the coupled equations to a single equation in ψ. Subject to this approximation, the scattering cross-section for a rotor was determined, and was found to tend to a finite limit as the frequency tended to be very large. It was also shown that to a first approximation the scattering cross-section for an ideal vortex is identical to that of a rotor, indicating that the external irrotational flow field does not contribute to the scattered power. But it may be supposed that the medium inhomogeneity associated with the irrotational flow will give rise to non-negligible scattering because the inhomogeneity extends over an infinite domain.

Consequently it is appropriate to investigate the scattering cross-section associated with the concomitant inhomogeneity of the medium. In order to do this rigorously, the coupled equations in ψ and $\underline{A}$ must be solved, but this presents a hopelessly difficult task. Instead, in order to get a first approximation to the cross-section associated with the inhomogeneity, the following problem will be solved: Suppose that the density field of an ideal vortex is reproduced by the proper temperature distribution and by the proper addition of body forces, in such a way that the velocity may be identically zero everywhere.

Then the scattering from this stationary density field should provide an estimate of the scattering from the inhomogeneity of the vortex.

The equations appropriate for the stationary problem are (2.43) and (2.55). They may be written in the following form.

$$(\nabla^2 + k_I^2 + W^2)\psi(\underline{r}) = 0 \qquad (4.11)$$

where

$$k_I^2 + W^2 = \frac{\omega^2}{c^2} + \frac{1}{\gamma - 1}\nabla \ln c^2 . \nabla \ln \psi$$

and where

$$c^2 = \begin{cases} c_o^2\left[1 - \frac{\gamma-1}{2} M^2\left(2 - \frac{r^2}{a^2}\right)\right] & , r < a \\ c_o^2\left[1 - \frac{\gamma-1}{2} M^2 \frac{a^2}{r^2}\right] & , r > a \end{cases}$$

The joining conditions associated with equation (4.11) are that ψ and $\frac{\partial \psi}{\partial r}$ must be continuous across the surface of the core. Hence the integral equation form for the scattered amplitude may be readily found to be,

$$S(\underline{k}_1, \underline{k}_0) = \int W^2 \psi \, e^{-i\underline{k}_1 \cdot \underline{r}'} \, dV' , \qquad (4.11)$$

where the integral is to be taken over all space. The WKBJ trial function used in Chapter III will be used here also (see equation (3.29)). Thus the forward scattered amplitude is given by:

$$S(\underline{k}_0,\underline{k}_0) = \frac{2k_I}{i}\int_{-\infty}^{\infty}\left[e^{\frac{i}{2k_I}\int_{-\infty}^{x} W^2 dx'}\right]_{-\infty}^{\infty} dy. \qquad (4.12)$$

W^2 is defined differently in Region I and in Region II (corresponding to $r > a$, and $r < a$, respectively). Hence equation (4.12) must be expanded into the following form for calculation:

$$S(\underline{k}_0,\underline{k}_0) = \frac{2k_I}{i}\int_{-\infty}^{-a}\left[e^{\frac{i}{2k_I}\int_{-\infty}^{\infty} W_I^{\,2}\, dx'} -1\right] dy$$

$$+ \frac{2k_I}{i}\int_{a}^{\infty}\left[e^{\frac{i}{2k_I}\int_{-\infty}^{\infty} W_I^{\,2} dx'} -1\right] dy \qquad (4.13)$$

$$+ \frac{2k_I}{i}\int_{-a}^{a}\left[e^{\frac{i}{2k_I}\int_{-\infty}^{-b} W_I^{\,2} dx'}\, e^{\frac{i}{2k_I}\int_{-b}^{b} W_{II}^{\,2} dx'}\, e^{\frac{i}{2k_I}\int_{b}^{\infty} W_I^{\,2} dx'} -1\right] dy$$

where

$$b = \left[a^2 - y^2\right]^{\frac{1}{2}}$$

In order to proceed further, it will be convenient to evaluate the x' integrals of W^2.

From equation (4.10), W^2 may be reduced to

$$W^2 = Dk_I^{\,2} - \frac{1}{\gamma-1}\frac{\partial D}{\partial r}\frac{\partial \ln\psi}{\partial r} \qquad (4.14)$$

where

$$D = \begin{cases} \frac{\gamma-1}{2} M^2\left(2 - \frac{r^2}{a^2}\right), & r < a \\ \frac{\gamma-1}{2} M^2 \frac{a^2}{r^2}, & r > a \end{cases}$$

and where M^4 has been considered small compared to unity. In order to calculate W^2 as given in equation (4.14), ψ may be taken as an undistorted plane wave (this procedure is analogous to that followed in Chapter III). Then

$$W^2 = \begin{cases} W_{II}^2 = k_I^2 D + iM^2 k_I \frac{r}{a^2} \cos\phi & , r < a \\ W_I^2 = k_I^2 D + iM^2 k_I \frac{a^2}{r^3} \cos\phi & , r > a \end{cases} \qquad (4.15)$$

All the integrals of W_I^2 and W_{II}^2 appearing in equation (4.13) are easily evaluated in terms of elementary functions. After some amount of algebra, the imaginary part of $S(\underline{k}_0, \underline{k}_0)$ divided by k_I, i.e., the scattering cross-section σ, is:

$$\frac{\sigma}{2a} = \pi g \int_0^{\frac{\pi}{4} g} \frac{\sin^2 \lambda}{\lambda^2} \, d\lambda + 4 \int_0^1 \sin^2 \left[gF \right] \, d\beta \qquad (4.16)$$

where

$$g = \frac{\gamma - 1}{2} M^2 k_I a$$

$$F(\beta) = \frac{1}{2} \left[\frac{1}{\beta} \sin^{-1}\beta + \frac{2}{3} \sqrt{1-\beta^2} \, (5 - 2\beta^2) \right] .$$

σ has been plotted as a function of g, for g between 0 and 15; the range 0 to 5 is shown in Figure 11. The cross-section is proportional to g^2 in the limit of small g. As g approaches unity, σ becomes an oscillating function, the oscillations decreasing in importance for g greater than about 3. The average value about which the oscillations of σ occur may be expressed very closely by

$$\frac{\sigma}{2a} \simeq 5g \qquad (4.17)$$

for g greater than about 3.

Although σ increases without limit as g increases, for most cases of interest it is smaller than the cross-section computed for the rotor (or core) alone. For example, consider the case discussed in Chapter III: $M = 3 \times 10^{-2}$, and 2a = 10 ft. The core cross-section σ_c and the concomitant inhomogeneity cross-section σ_r are compared below for three frequencies:

frequency, cps	σ_c, ft	σ_r, ft
100	5×10^{-2}	5×10^{-5}
1000	4	5×10^{-3}
10,000	20	5×10^{-1}

Of course, the cross-section computed by equation (4.16) represents an artificial problem; nevertheless it seems reasonable that the stationary problem considered gives at least the order of magnitude of the effect of the inhomogeneity associated with the velocity field of vortex motion. Therefore it appears that the concomitant inhomogeneity of vortex motion may be neglected for most cases of practical interest.

Hence the results of this and the previous Section show that to a good approximation the total energy scattered by an ideal vortex is identical to that scattered by a rotor. In other words, the scattering is associated more nearly with the vortical region of the core than with the external irrotational region.

V Experimental Considerations

A. Preliminary Underwater Experiments

Preliminary experiments were attempted underwater in an effort to measure the effect of vortical flow on sound propagation, particularly for large $k_I a$. The equipment consisted of:

1. A 1 megacycle pulse modulated source driving an 1 cm diameter X - cut quartz transducer,
2. Another X - cut quartz transducer used as a receiver in conjunction with a calibrated attenuator, tuned amplifier, and display oscilloscope,
3. A thin plastic tube, 1 cm diameter, mounted in a large (compared to wavelength) water tank.

No effects were observed on sound pulses transmitted through regions of rotation. In the light of the theory developed in Chapters III and IV, this is not surprising. From the BK calculation, the scattered intensity for a 1 cm diameter rotor is at best the order of 30 to 50 decibels less than the incident wave, depending upon the Mach number and the distance of observation, and provided that the angle of observation is less than about 14° (the first zero of $\Lambda_2(Ka)$). However the first zero of the directivity pattern of the incident beam occurs at an angle of about 10°, as measured from its surface,* or perhaps 20° as measured from the center of the rotor. Thus the peak of the scattered wave is completely masked by the beam; the subsidiary peaks of the scattered wave

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*The directivity pattern of a plane piston radiator is $\Lambda_1(Ka)$.

will also be masked by the beam, for their directivity patterns are behaviorally similar.

This unfavorable situation is not helped materially by increasing frequency. Although it is true that the beam becomes sharper, the scattering peak negates this by moving to smaller angles. Furthermore, the beam cannot be made too directive, for in order to approximate a plane wave it is required that the incident amplitude and phase fronts are plane over at least the dimensions of the scatterer.

Thus it may be concluded that the measurement of vortical scattering by a single rotor or ideal vortex in water is highly unfavorable. Workers in other laboratories have come to the same conclusion.[28]

B. Atmospheric Field Measurements

The agreement cited between the BK and WKBJ solutions (equations(3.23)and (3.42)), and the available experimental data of propagation over the ground, can only be regarded as semi-quantitative; estimates of the maximum Mach number, the radius a of the core, and the number of vortices, were necessary in order to compare predicted and measured values. In future experiments designed to test the present theory with greater precision, it would be essential to measure the net flow field between the source and receiver. That is, a complete specification of the net flow field is required at essentially the same time at which the sound is propagated. This presents an almost overwhelming task (although it can be eased to some extent by a careful choice of terrain). Further more, careful attention must be given to other micrometeorological effects such as temperature gradients, molecular composition of the air, etc. In other words, the medium must be specified to an extent which is uncommonly detailed, even for micrometeorological research. Thus it appears that from a practical point of view it will be necessary to estimate some of the unknown parameters, and that semi-quantitative agreement between theory and measurement is all that can be hoped for.

There is the need for additional measurements of the type carried out by Ingard,[18] for varying terrains and for varying average micrometeorological conditions. With such measurements it would be possible to check the present theory further, at least semi-quantitatively.

VI Conclusions

The major findings of this thesis investigation may be summarized as follows:

In general the sound particle velocity in a moving inhomogeneous medium is composed of a longitudinal part and a transverse part. The longitudinal part is characteristic of its wave properties; the transverse part is associated with the properties of the net flow and the boundary conditions. If the net flow is zero, the transverse part must be zero (despite any inhomogeneities of the medium); if the net flow is irrotational, the transverse part is zero (if the boundary conditions specify a curl-less velocity); if the net flow is rotational, the transverse part is non-zero.

In the case of circular cylindrical vortex motion, the transverse part of the sound particle velocity has the following approximate properties: It is normal to both the vorticity of the flow and to the longitudinal part of the sound particle velocity. In addition, it is the order of Ω/ω compared to the longitudinal part, and 90° out of phase from it. (2Ω is the magnitude of the vorticity, and ω is the sound frequency).

The effects of net flow of the medium are greater than the effects of the concomitant inhomogeneity, as far as wave motion is concerned. To a good approximation, the effects of the latter may be neglected compared to the former.

According to a Born-Kirchoff approximation solution, the sound scattered by a rotor (essentially the core of an ideal vortex) is dipolar at low frequencies (the maxima occur at $\pm \pi/2$ from the forward direction), and is mainly forward at high frequencies. The scattered energy is symmetrical with respect to the forward direction; the scattered phase is asymmetrical.

According to a WKBJ integral equation solution, the total scattered energy depends upon Mk_Ia or $\frac{\Omega}{\omega} k_Ia$, where M is the maximum Mach number occuring in the flow and where k_Ia is the ratio of the rotor circumference to the wavelength. For Mk_Ia (or $\frac{\Omega}{\omega} k_Ia$) the order of unity, the scattered energy becomes important; the cross-section for scattering approaches twice the geometrical cross-section for $Mk_Ia > 1$.

Both the Born-Kirchoff and the WKBJ solutions show that scattering from a single rotor is too small to be detected. Scattering from a medium containing a large number of rotors, however, is measurable. This effect is important in atmospheric acoustics, and may account for the scattered energy appearing in shadow zones, and for long distance attenuation not accounted for by molecular or ground absorption. With the assumption of reasonable parameters for atmospheric velocities and scales, the Born-Kirchoff and WKBJ solutions agree with measurements.

For intermediate frequencies ($\pi < k_Ia < \frac{1}{M}$), the scattering cross-section predicted by the WKBJ integral equation solution agrees with a result obtained by Lighthill based on turbulence theory. For higher frequencies, however, the WKBJ calculated cross-section tends to a constant, whereas the Lighthill calculated cross-section becomes indefinitely large. This difference is attributed to the essential physical difference between ordered and statistical vorticity.

The scattering from an ideal vortex is shown to be semi-quantitatively determined by the scattering from its core or rotor alone; to a first approximation, the velocity field external to its core refracts the total wave, without changing the total scattered power. Similarly, the scattering from any vortical disturbance is expected to be described qualitatively by the scattering from a rotor of size properly adjusted to enclose the region of vorticity, provided that the external velocity field is short range.

Appendix I Cross-Section Theory

The scattered intensity can be written as

$$I_s = \frac{\omega\rho}{2} \, \mathrm{Im} \, (\psi_s^* \frac{\partial \psi_s}{\partial n}) \qquad (A1.1)$$

where ψ_s^* is the complex conjugate of ψ_s, and where $\underline{n}$ is the outward normal vector of the surface through which I_s is to be computed. The total scattered power passing through a closed surface, S, is

$$P = \frac{\omega\rho}{2} \, \mathrm{Im} \int_S \psi_s^* \frac{\partial \psi_s}{\partial n} \, dS.$$

The incident plane wave intensity is

$$I_{in} = \frac{1}{2} k_I \omega\rho .$$

Hence the ratio of the scattered power to the incident intensity, i.e., the scattering cross-section is

$$\sigma = \frac{1}{k_I} \, \mathrm{Im} \int_S \psi_s^* \frac{\partial \psi_s}{\partial n} \, dS. \qquad (A1.2)$$

With the use of the asymptotic form of ψ_s, σ may be written as

$$\sigma = \frac{1}{8\pi k_I} \int_0^{2\pi} \left| S(k_1, k_0) \right|^2 d\phi, \qquad (A1.3)$$

which proves equation (3.13).

To prove equation (3.14), the following method is used. Write

$$\psi_s = \psi_I - e^{i\underline{k}_0 \cdot \underline{r}}$$

then

$$\sigma = \frac{1}{k_I} \operatorname{Im}\left\{ \int_s \psi_I^* \frac{\partial \psi_I}{\partial n} dS - \int_s e^{-i\underline{k}_0 \cdot \underline{r}} \frac{\partial \psi_I}{\partial n} dS - \int_s \psi_I^* \frac{\partial}{\partial n} (e^{i\underline{k}_0 \cdot \underline{r}}) dS \right\}.$$

If there is no absorption in Region II, the power leaving the scatterer surface is equal to the power passing through s. Hence s may be taken as the rotor surface, and the continuity conditions of equations (3.3) and (3.4) applied. Thus

$$\sigma = \frac{1}{k_I} \operatorname{Im}\left\{ \int_s \psi_I^* \frac{\partial \psi_I}{\partial n_{II}} dS - \int_s e^{-i\underline{k}_0 \cdot \underline{r}} \frac{\partial \psi_{II}}{\partial n_{II}} dS \right. \tag{A1.4}$$

$$- i \frac{2}{a}\frac{\Omega}{\omega} \int_s e^{-i\underline{k}_0 \cdot \underline{r}} \frac{\partial \psi_{II}}{\partial \phi} dS - \int_s \psi_{II}^* \frac{\partial}{\partial n_{II}} (e^{i\underline{k}_0 \cdot \underline{r}}) dS$$

$$+ i \frac{\Omega}{\omega} \int_s \frac{\partial \psi_{II}^*}{\partial \phi} \frac{\partial}{\partial n_{II}} (e^{i\underline{k}_0 \cdot \underline{r}}) dS$$

The integral equation for ψ_{II} specialized for the measurement direction in the forward direction, and valid for r large compared to wavelength and the rotor radius, may be written as

$$\int_S \left[\frac{\partial \psi_{II}}{\partial n_{II}} \; e^{-i\underline{k}_0 \cdot \underline{r}} - \psi_{II} \frac{\partial}{\partial n_{II}} \left(e^{-i\underline{k}_0 \cdot \underline{r}} \right) \right] dS$$

$$+ \int W^2 \; \psi_{II} \; e^{-i\underline{k}_0 \cdot \underline{r}} \; dV = 0. \qquad (A1.5)$$

Then, by using the scattered amplitude $S(k_0, k_0)$ in the forward direction, and equation (A1.5), equation (A1.4) becomes

$$\sigma = \frac{1}{k_I} \; \text{Im} \left[\int_S \psi_I^* \; \frac{\partial \psi_I}{\partial n_{II}} \; dS + S(\underline{k}_0, \underline{k}_0) \right], \qquad (A1.6)$$

where use has been made of the fact that

$$\text{Im } F(z) = - \text{ Im } F^*(z).$$

The power lost to the scatterer by absorption is

$$P_a = - \frac{\omega\rho}{2} \; \text{Im} \int_S \psi_I^* \; \frac{\partial \psi_I}{\partial n_{II}} \; dS,$$

and the absorption cross-section is

$$\sigma_a = - \frac{1}{k_I} \quad \text{Im} \int_S \psi_I^* \; \frac{\partial \psi_I}{\partial n_{II}} \; dS.$$

Hence the total cross-section is

$$\sigma_t = \sigma + \sigma_a = \frac{1}{k_I} \; \text{Im } S(\underline{k}_0, \underline{k}_0) \qquad (A1.7)$$

Appendix II The Anger Function

The Anger function[22] may be defined by

$$\tilde{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\beta - z\sin\beta)\, d\beta.$$

Its expansion in ascending powers of z is

$$\tilde{J}_\nu(z) = \frac{\sin\nu\pi}{\pi}\, s_{0,\nu}(z) - \frac{\nu\sin\nu\pi}{\pi}\, s_{-1,\nu}(z)$$

where

$$s_{0,\nu}(z) = \frac{z}{1^2-\nu^2} - \frac{z^3}{(1^2-\nu^2)(3^2-\nu^2)} + \frac{z^5}{(1^2-\nu^2)(3^2-\nu^2)(5^2-\nu^2)} - \cdots$$

$$s_{-1,\nu}(z) = -\frac{1}{\nu^2} + \frac{z^2}{\nu^2(2^2-\nu^2)} - \frac{z^4}{\nu^2(2^2-\nu^2)(4^2-\nu^2)} + \cdots$$

are Lommel's functions of order 0 and -1. It can be shown that the Anger function may be connected with the Weber function,[22] and that for integer values of ν, the Anger function becomes the Bessel function.

The asymptotic expansion of $\tilde{J}_\nu$ is

$$\tilde{J}_\nu(z) = J_\nu(z) + \frac{\sin\nu\pi}{\pi z}\left[1 - \frac{1^2-\nu^2}{z^2} + \frac{(1^2-\nu^2)(3^2-\nu^2)}{z^4} - \cdots\right]$$

$$- \frac{\sin\nu\pi}{\pi z}\left[\frac{\nu}{z} - \frac{\nu(2^2-\nu^2)}{z^3} + \cdots\right]$$

where $J_\nu(z)$ is the Bessel function of order ν.

In the following table, values of $\frac{\pi}{2}$ times the sum and difference of the Anger function of order $\frac{1}{2}$ and $-\frac{1}{2}$

are given. Thus let

$$A(z) = \frac{\pi}{2}\left[\ \tilde{J}_{\frac{1}{2}}(z) + \tilde{J}_{-\frac{1}{2}}(z)\ \right]$$

$$B(z) = \frac{\pi}{2}\left[\ \tilde{J}_{\frac{1}{2}}(z) - \tilde{J}_{-\frac{1}{2}}(z).\ \right]$$

The values of A and B are given to the nearest hundreth, with an accuracy estimated at better than one hundredth.

Z	A	B	Z	A	B
0	2.00	0.00	3.5	- 0.89	0.66
0.2	1.98	0.27	4.0	- 0.91	0.18
0.4	1.92	0.52	4.5	- 0.73	- 0.24
0.6	1.80	0.77	5.0	- 0.40	- 0.50
0.8	1.66	0.99	5.5	- 0.02	- 0.58
1.0	1.50	1.19	6.0	0.33	- 0.47
1.2	1.30	1.35	6.5	0.58	- 0.22
1.4	1.06	1.48	7.0	0.66	0.10
1.6	0.84	1.57	7.5	0.58	0.40
1.8	0.60	1.62	8.0	0.37	0.63
2.0	0.34	1.63	8.5	0.08	0.72
2.2	0.10	1.60	9.0	- 0.21	0.66
2.4	- 0.16	1.43	9.5	- 0.44	0.48
3.0	- 0.64	1.14	10.0	- 0.55	0.22

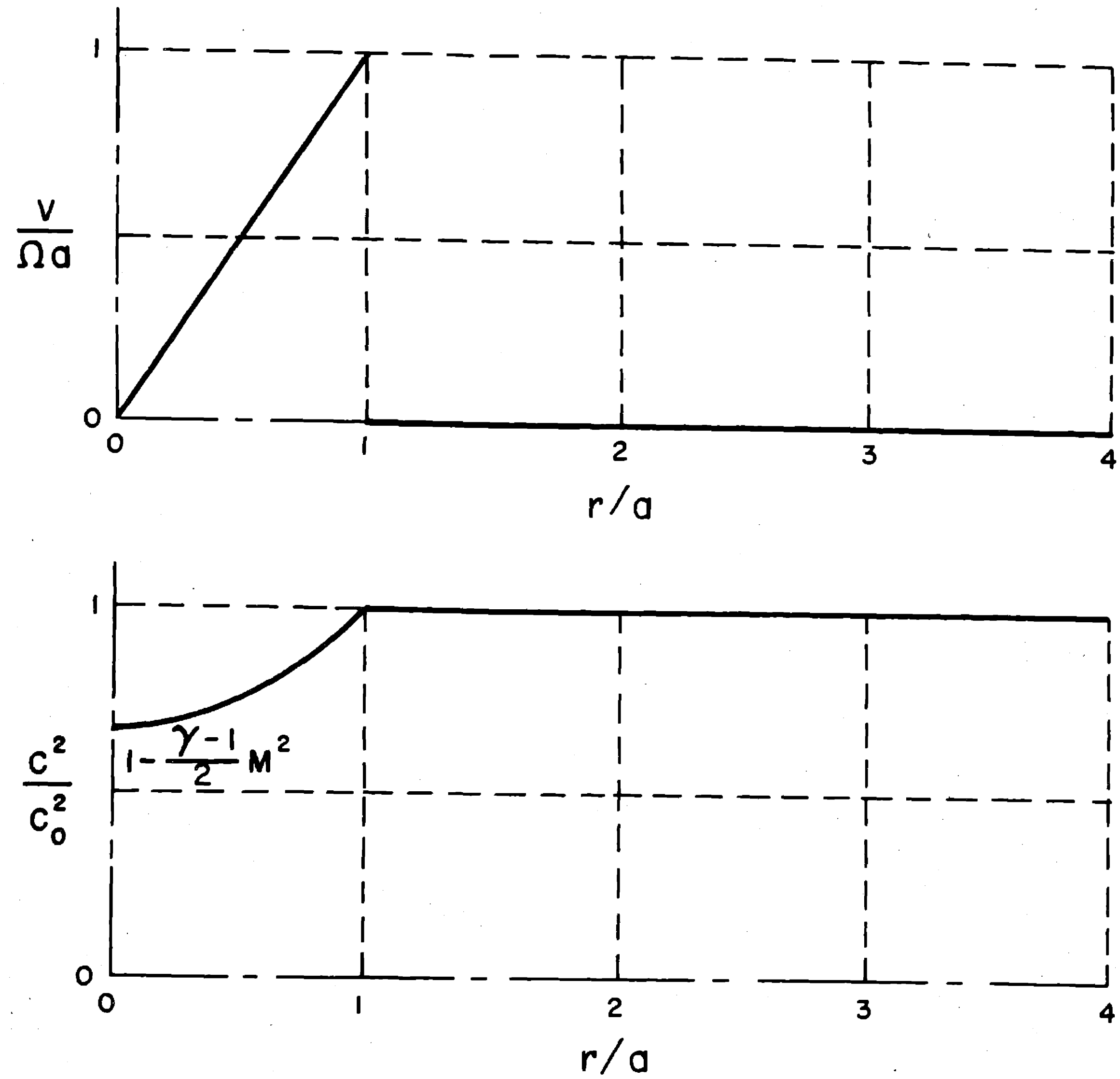

Figure 1

Net Flow and Concomitant Inhomogeneity
for Rotor Motion

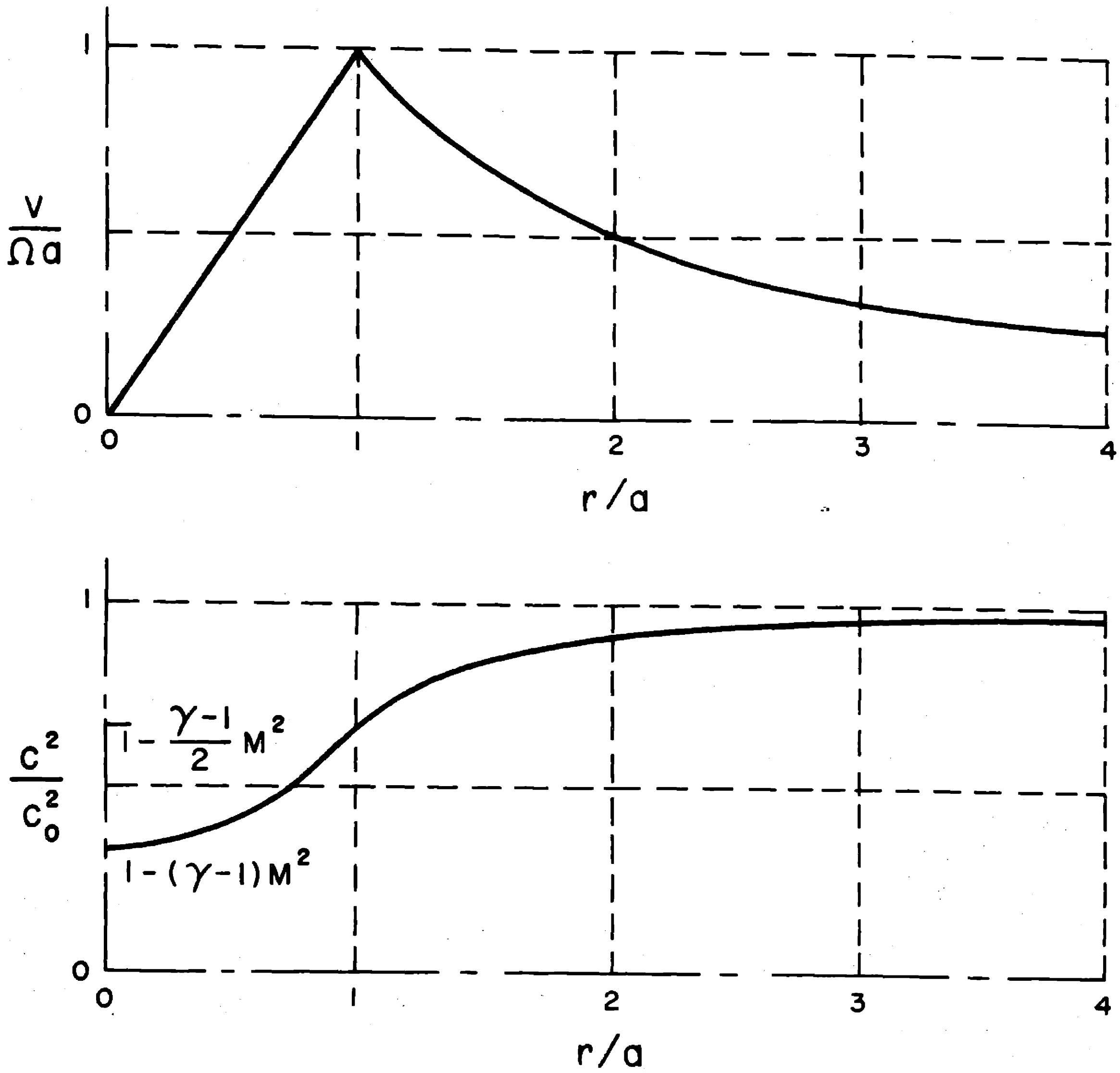

Figure 2

Net Flow and Concomitant Inhomogeneity for Ideal Vortex Motion

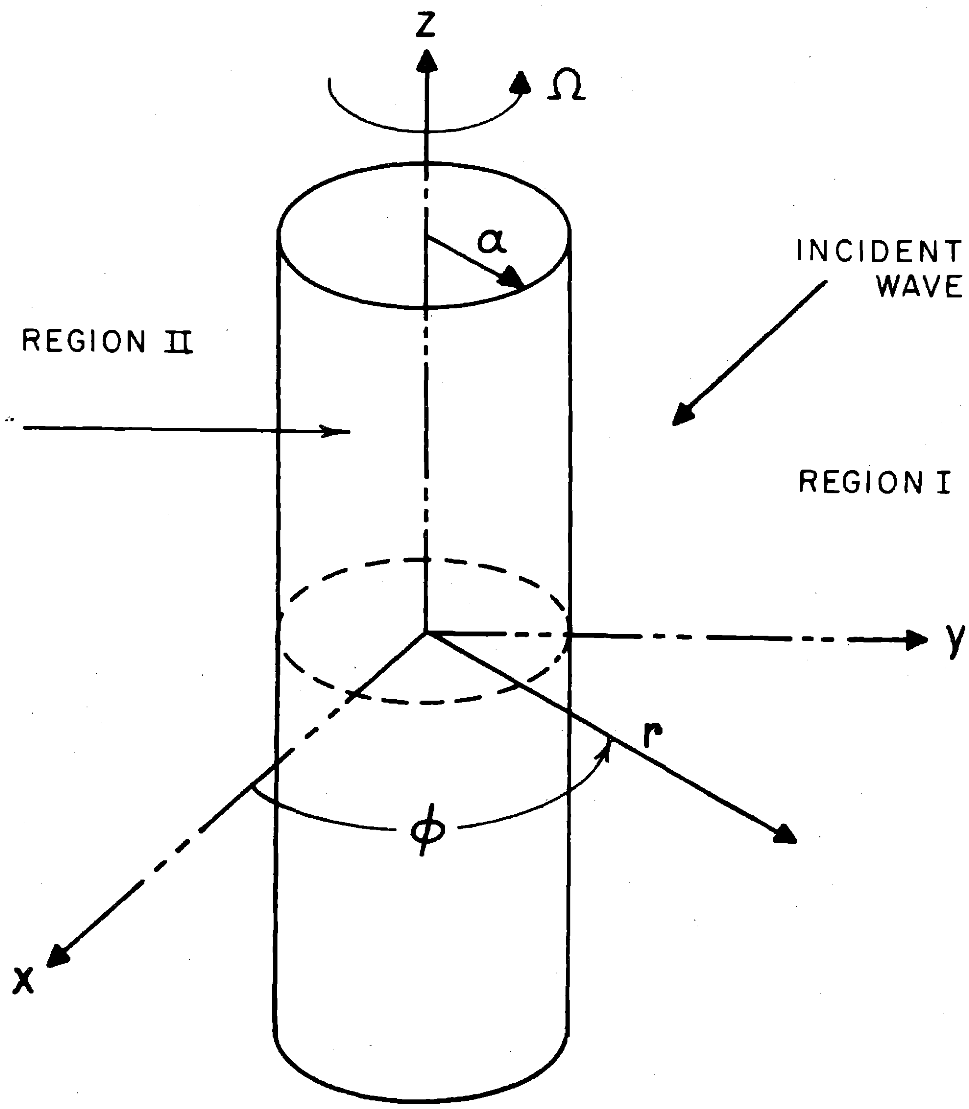

Figure 3

Coordinate System for Rotor Scattering

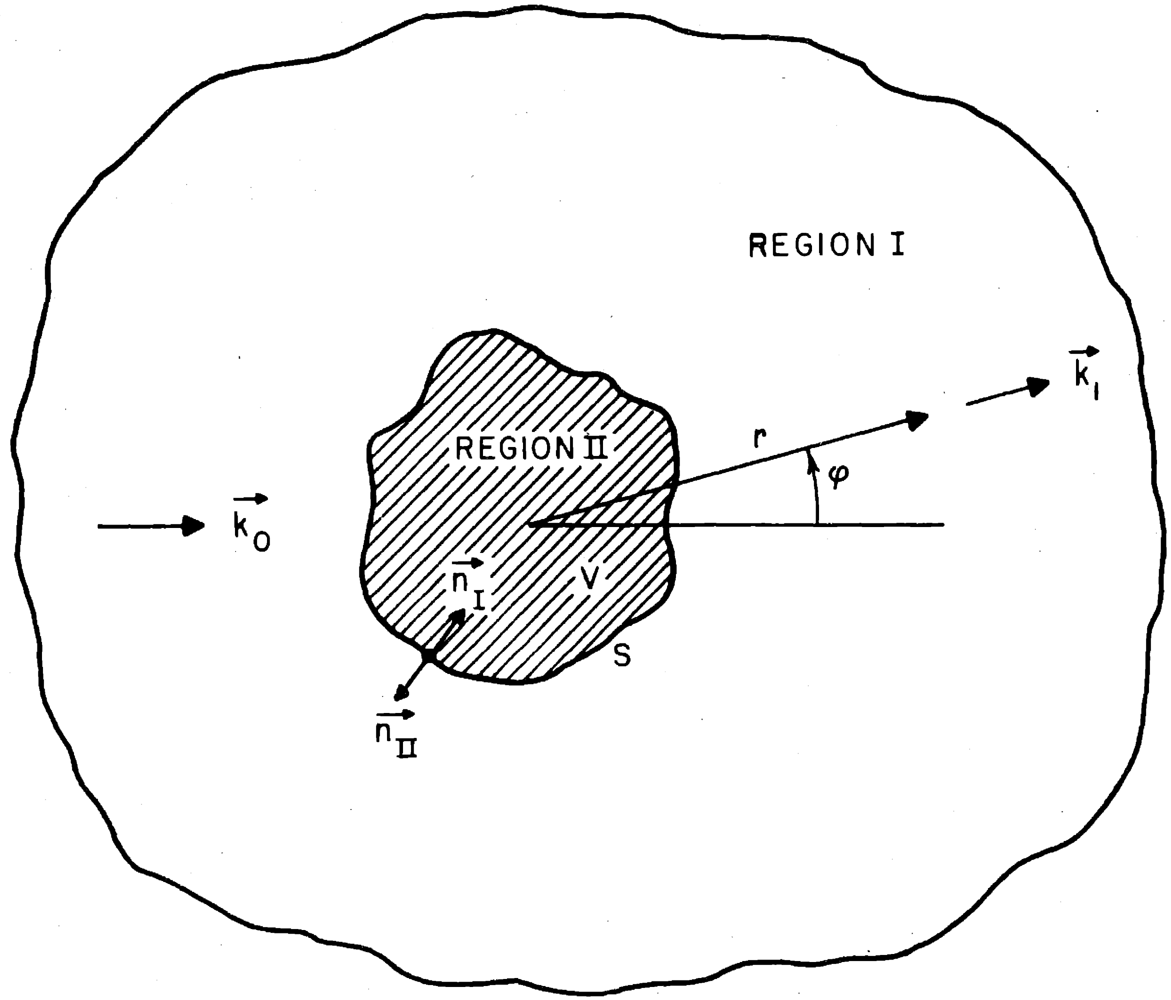

Figure 4

Coordinate System for the Integral Equations

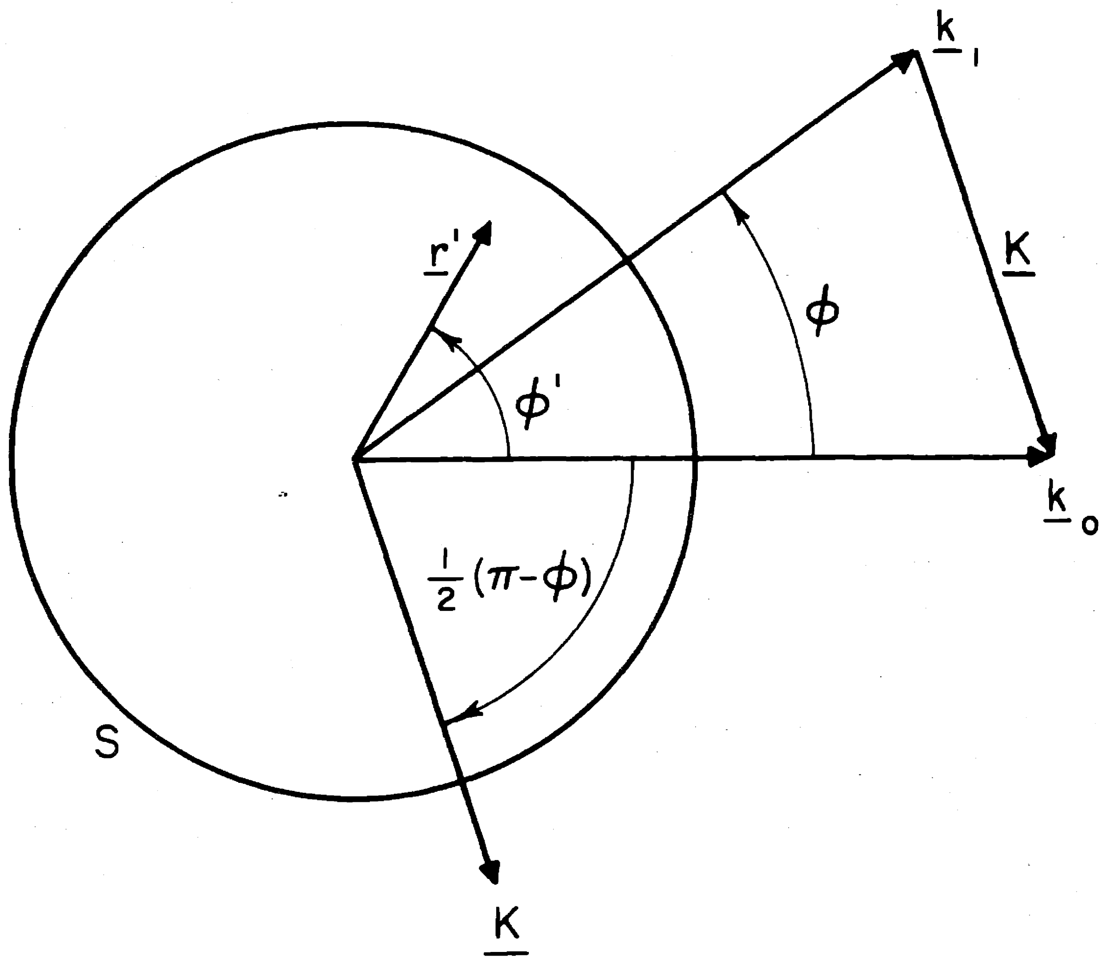

Figure 5

Coordinate Transformation to $\theta = \phi' + \frac{1}{2}(\pi - \phi)$

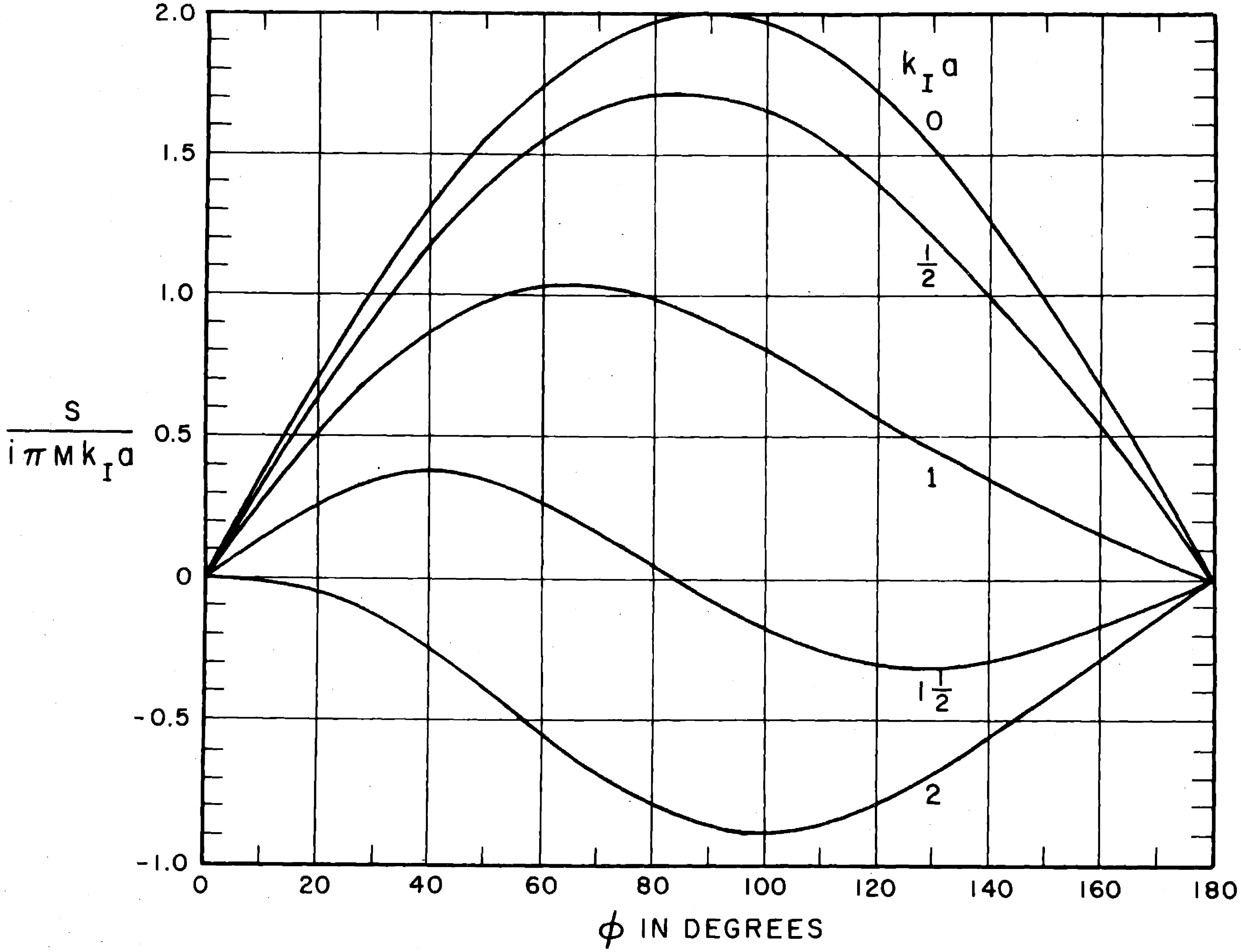

Figure 6 Scattered Amplitude for $k_I a$ 0 to 2

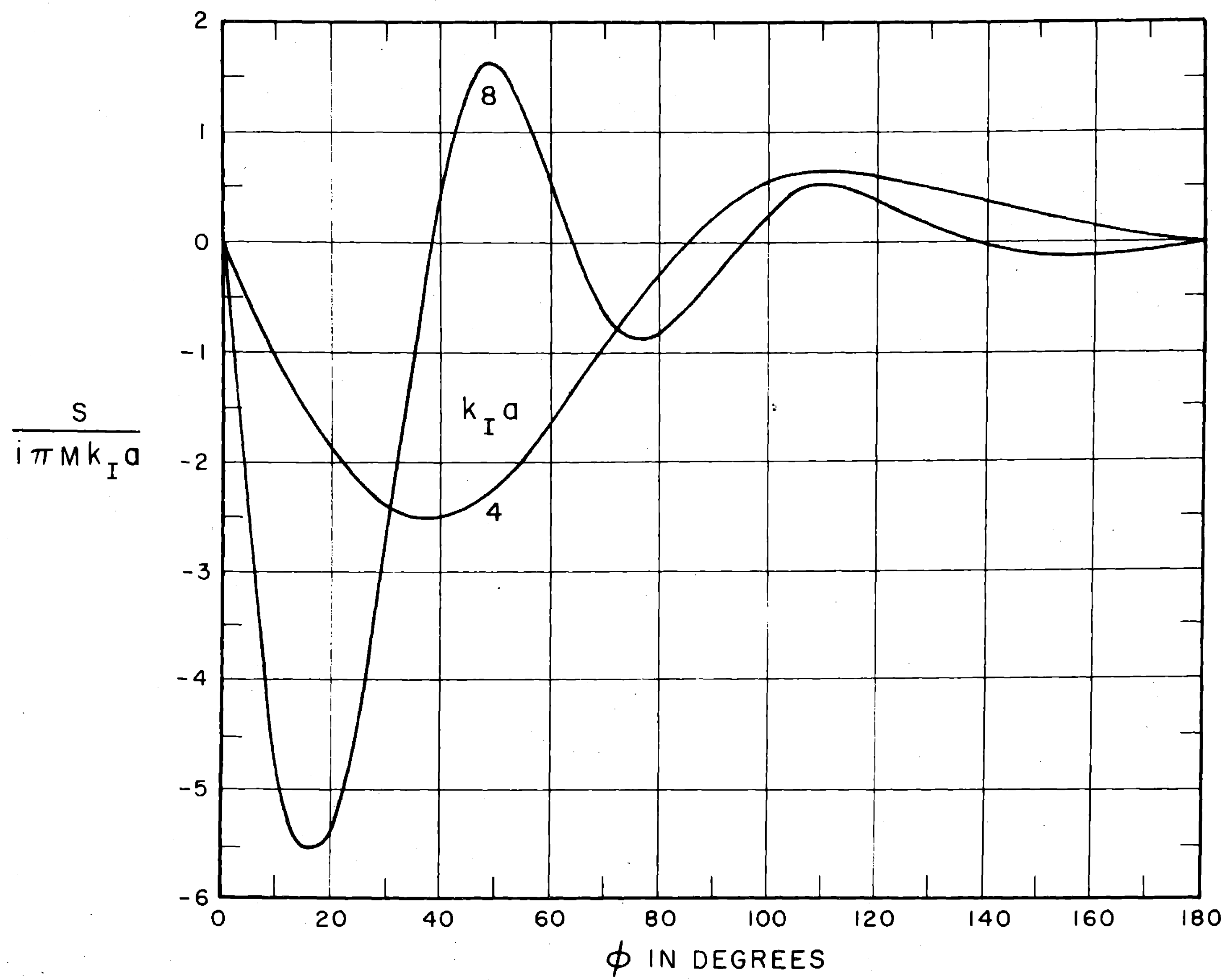

Figure 7 Scattered Amplitude for $k_I a$ 4 and 8

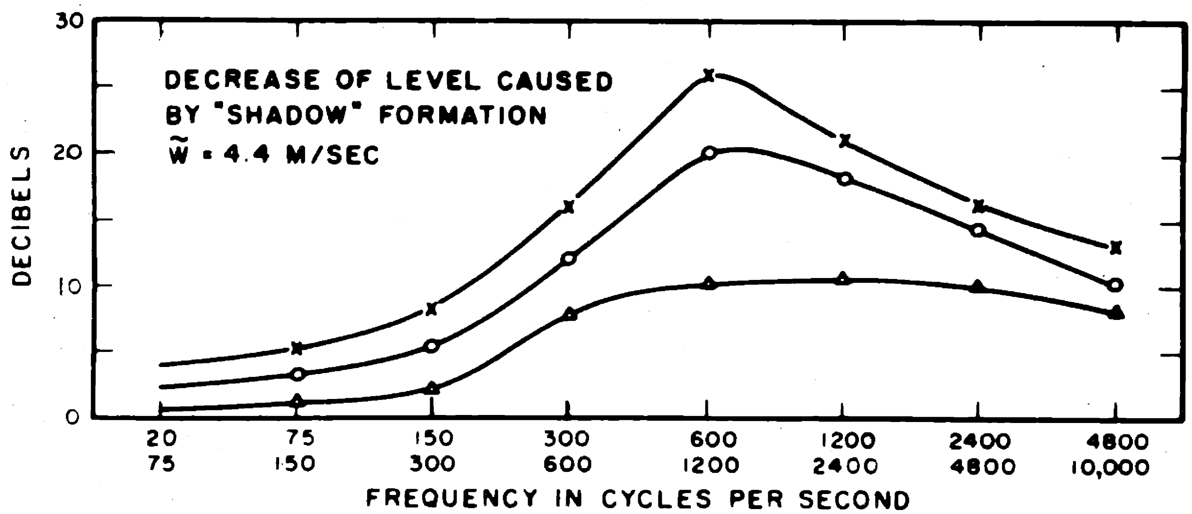

Figure 8

The decrease of sound pressure level in the shadow zone 1000 feet from the source. The source and receiver are both 10 feet above the ground. The average translational velocity is about 4.4 meters/sec or about 14 feet/sec. The upper curve refers to propagation against the wind, the center curve refers to propagation at 45 degrees against the wind, and the lower curve refers to propagation at right angles to the wind. (After Ingard (18)).

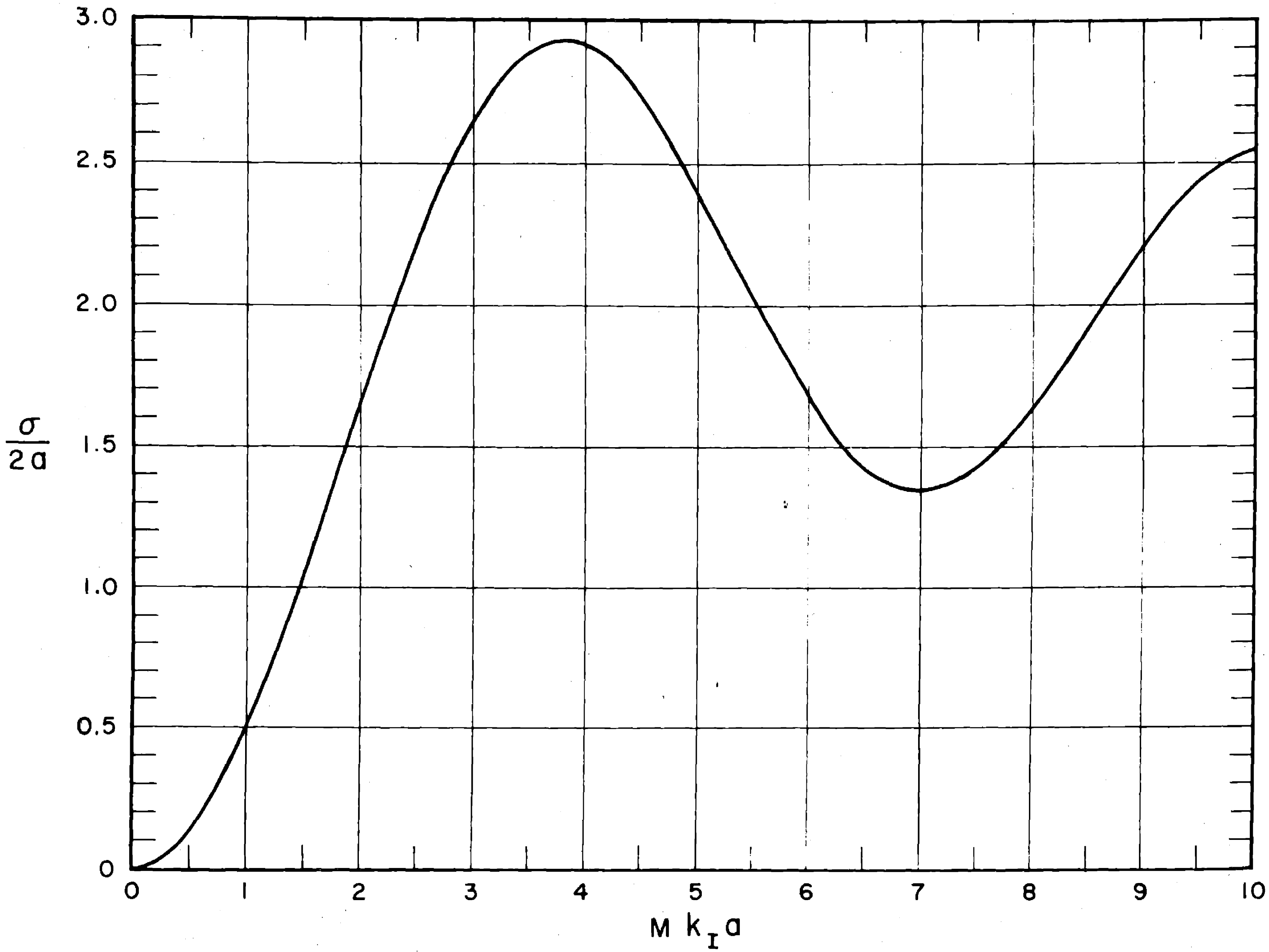

Figure 9 Scattering Cross-Section Based on a WKBJ Technique

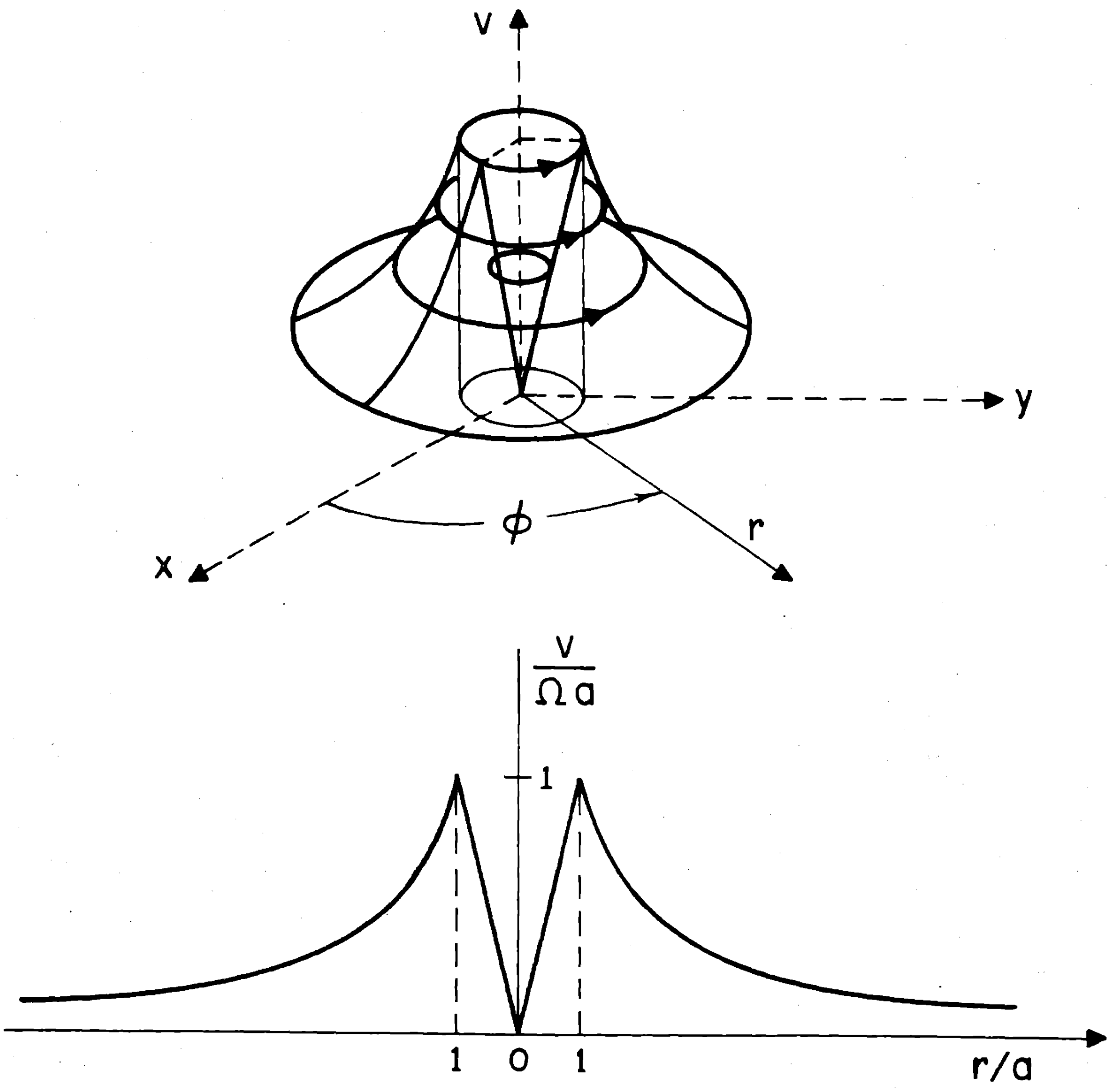

Figure 10

Coordinate System and Velocity Field for Ideal Vortex Scattering

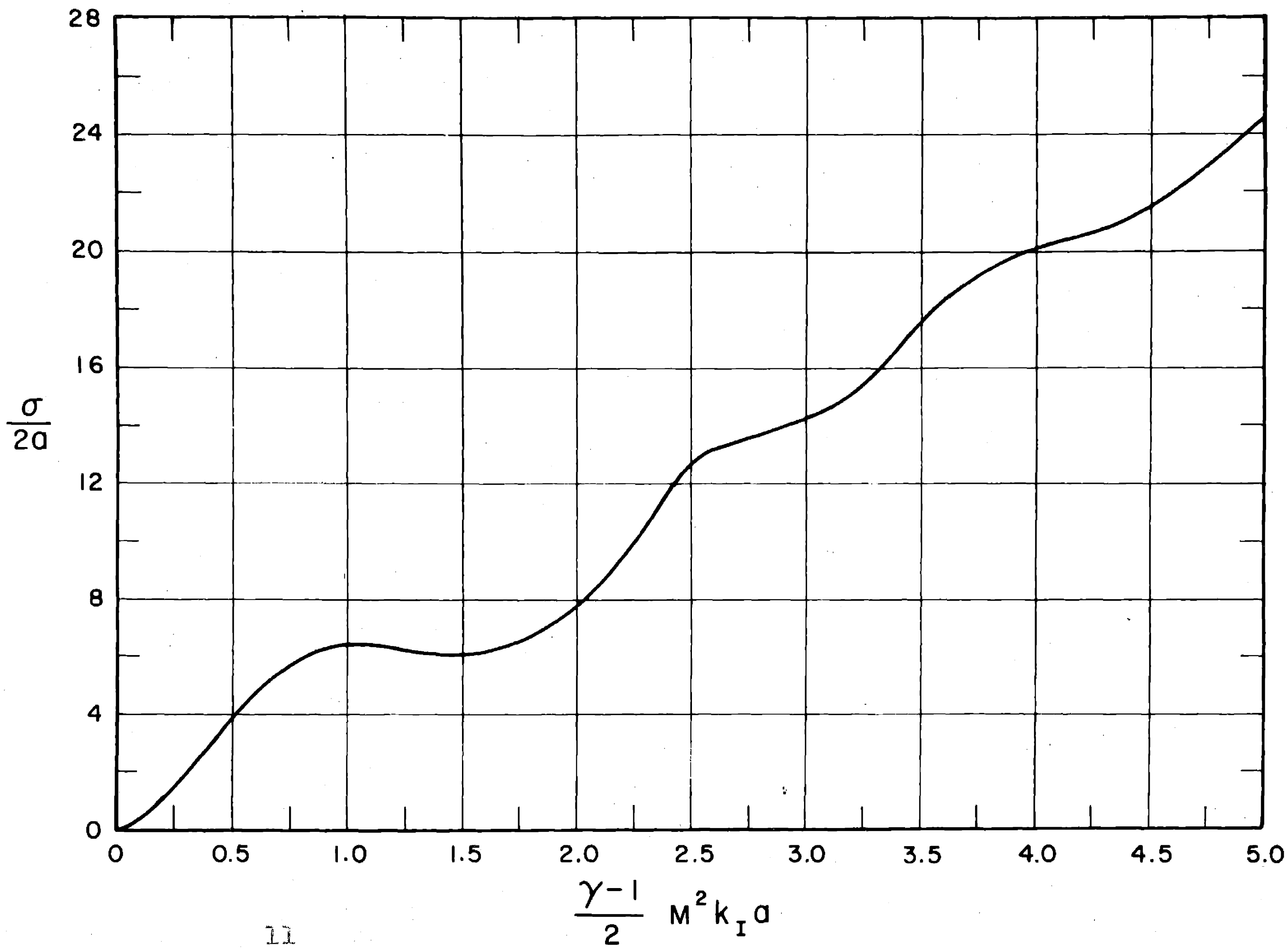

Figure ~~12~~ 11 Scattering Cross-Section Associated with the Concomitant Inhomogeneity of an Ideal Vortex

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Biographical Note

The author, Ira Dyer, was born in Brooklyn, New York, on June 14, 1925. He attended the Brooklyn Technical High School and was graduated in January 1944. In February 1944, he enlisted in the U. S. Army Air Corps as an Aviation Cadet and was in training until he was discharged at the termination of the War. He spent a brief period with the Federal Telecommunication Laboratories, and then entered M. I. T. in March, 1945 as a freshman. He was awarded three one-year part tuition scholarships while an undergraduate. In June 1949, he was awarded an S.B. degree in Physics, and entered the M. I. T. Graduate School with an appointment as a Research Assistant in Physics. He received an S.M. degree without specification in September 1951. He is a member of the Acoustical Society of America, Sigma Xi, and the Brooklyn Academy of Arts and Sciences.

He has published a paper entitled "On Noise of Aerodynamic Origin" in the Journal of the Acoustical Society, in conjunction with O. K. Mawardi, and has presented several papers before meetings of that Society. He has taught the course "Vibration and Sound" at M. I. T., and has assisted other instructors in that course and in freshman Physics courses.

In September 1949, he was married to Betty Ruth Schanberg. They have two children, Samuel Scott aged two and one-half years, and Debora Joan aged one-half year.