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# Fast convergence in evolutionary models: A Lyapunov approach\*

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## Abstract

Evolutionary models in which  $N$  players are repeatedly matched to play a game have “fast convergence” to a set  $A$  if the models both reach  $A$  quickly and leave  $A$  slowly, where “quickly” and “slowly” refer to whether the expected hitting and exit times remain bounded when  $N$  tends to infinity. We provide simple and general Lyapunov criteria which are sufficient for reaching quickly and leaving slowly. We use these criteria to determine aspects of learning models that promote fast convergence.

Keywords: hitting time, learning model, local interaction, Lyapunov function, Markov chain, recency

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# 1 Introduction

The study of stochastic stability in evolutionary models focuses on the long-run outcomes of various sorts of adjustment processes that combine best response or learning dynamics with mutations, errors, or other sorts of random fluctuations. Because the stochastic terms make these systems ergodic, they have a unique invariant distribution which corresponds to their long-run outcome, and since these outcomes typically single out a single equilibrium they provide a way to do equilibrium selection, as in Kandori, Mailath, and Rob (1993) and Young (1993). However, the long-run outcome is only relevant if it is reached in a reasonable amount of time in populations of the relevant size, and this is not the case if agents intend to play a best response to the current state and the stochastic term arises from a constant probability of error, as here the expected time for the population to shift to the risk-dominant equilibrium is exponentially large in the population (Ellison, 1993).

For this reason, there has been considerable interest in understanding when the long run outcome is reached “quickly” in the sense that the expected number of time periods to reach a neighborhood of the selected outcome is bounded above, independent of the population size. Past work in economics has used one of two methods for showing that this occurs: either an argument using coupled Markov processes as in Ellison (1993), or a two-step approach of first showing that the associated deterministic, continuous time, mean field has a global attractor, and then showing that the discrete-time stochastic system behaves approximately the same way when  $N$  is large, as in Kreindler and Young (2013, 2014.) We say more about these papers below.<sup>1</sup>

This paper provides a simple Lyapunov condition for quickness that covers this past work. We also provide a complementary Lyapunov condition under which the process leaves the target set “slowly,” in the sense that the probability of getting more than  $\epsilon$  away from the set in any fixed time goes to zero as the population size increases. As this latter property seems necessary for convergence to the target set to be interesting, we only say that there is “fast convergence” when both conditions hold. By providing a unified approach to proving fast convergence, we highlight the connection between them; this may provide intuition about other settings where the same result will apply. Our conditions are also relatively tractable and portable, which lets us prove that there is fast convergence in a number of new cases that are more complicated than those already in the literature. As one example of this, Section 2 presents a model of local interaction with a small-world element,

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<sup>1</sup>In mathematical biology hitting times for evolutionary processes have been characterized by diffusion approximations as in Ethier and Kurtz (1986) and Ewens (2004) and have been explicitly calculated for birth-death processes as in Nakajima and Masuda (2015).

where players interact both with their neighbors and with randomly drawn members of the whole population; here neither of the past techniques can be applied.

Section 3 presents the general model, which is based on a collection of time-homogeneous Markov chains  $S^N = \{S^N(t) : t = 0, \frac{1}{N}, \frac{2}{N}, \dots\}$  with finite state spaces  $\Omega_N$ , where  $N$  indexes the number of players in the population. These Markov chains may track for example the play of each player at each location of a network. We then define functions  $\phi_N$  on  $\Omega_N$  that map to a space  $\mathcal{X}$  of fixed dimension, and consider the processes given by  $X^N(t) = \phi_N(S^N(t))$ . For example, these processes can describe the share of agents using each of a finite number of pure strategies in a game.

Section 4 presents a pair of general results that use a Lyapunov function  $V$  to provide sufficient conditions for “reaching quickly” and “leaving slowly.” Proposition 1 says roughly that the system reaches a subset  $A$  of  $\mathcal{X}$  quickly if the expectation of  $V(X^N(t))$  decreases by at least a fixed rate  $c > 0$  whenever the state is outside of  $A$ , that is,

$$E\left[V\left(X^N\left(t + \frac{1}{N}\right)\right) - V\left(X^N(t)\right) \middle| S^N(t)\right] \leq -\frac{c}{N} \text{ when } X^N(t) \notin A.$$

Intuitively, if  $X^N(t)$  is deterministic and  $V$  is nonnegative, this condition implies that  $X^N(t)$  will reach  $A$  from initial state  $x$  in  $V(x)/c$  periods; Proposition 1 extends this to a probabilistic statement. Proposition 2 provides a closely related condition for leaving  $A$  slowly: If the maximum rate at which the state can move is bounded and  $V(X^N(t))$  is decreasing in expectation at rate  $c$  whenever  $X^N(t)$  is outside  $A$ , then the system will leave  $A$  slowly. Intuitively, this is because getting more than  $\epsilon$  away from  $A$  would take large number of steps against the drift. Propositions 3 and 4 extend the analysis to the case where the expected decrease of the Lyapunov function decreases as the target set is approached, and to the case where a set is reached quickly through a multi-step process that need not monotonically approach the set.

Sections 5 and 6 use our general results to examine whether convergence is fast in various evolutionary game dynamics. In Section 5 we study models where the population shares using each strategy provide an adequate state space, as when the noisy-sampling model of Sandholm (2001) is extended to include random errors, or when agents play a stochastic best response to the current state, with less costly errors occurring less often, as in Kreindler and Young (2013, 2014).<sup>2</sup> Our examples further explore each of these mechanisms. Here, there will be fast convergence to a neighborhood of the state in which all

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<sup>2</sup>Sandholm analyzed a continuum population model, but the approximation results of Benaim and Weibull (2003) can be used to show that there is fast convergence in finite populations if errors are added to Sandholm’s model. (If the only source of noise is sampling, then the population can stay at the dominated equilibrium.)

agents use one particular strategy whenever the popularity of that strategy is increasing in expectation when the state is outside the neighborhood, and conversely there will not be fast convergence to such a neighborhood if the popularity is decreasing in expectation whenever the popularity is sufficiently low. Section 5.2 examines models with stochastic choice, and Section 5.3 considers models where agents receive noisy signals of the current state. In most of the examples we show that if there is enough noise, there is fast convergence to the set where the majority of agents use the risk-dominant action in a coordination game, because the share of agents playing the risk-dominant action increases in expectation whenever it is less than one-half. Intuitively, pure noise will increase the share of any strategy when it is near zero, and at intermediate states risk-dominance pushes in one direction; the last part of the argument is that in both sorts of examples the dynamics have a convexity that generates a reinforcing interaction between noise and risk dominance.

Section 6 analyzes two examples that require larger and more complex state spaces: a two-neighbor local interaction model like Ellison (1993), where the state must track the actions taken at each location in the network; and a model of learning with recency, where the state must track past observations. In these more complex models, it is very helpful that our Lyapunov results can be applied directly, without first establishing that the model converges in an appropriate sense to some deterministic limit dynamic as  $N \rightarrow \infty$ , because the relevant approximation theorems assume that the dimension of the state space is fixed and finite independent of  $N$ .

To help clarify the difference between our approach and past work, we will review simplified versions of the Ellison (1993) and Kreindler and Young (2013) results in a bit more detail to explain their proof techniques and how our Lyapunov-function approach applies to them. Suppose then that a single population of agents plays a symmetric  $2 \times 2$  coordination game with two strict Nash equilibria, one of which is both risk-dominant and Pareto-dominant, and one agent adjusts their play per time step.

In Kreindler and Young (2013), agents observe the population fractions that are currently using each action, and choose a logit best response to it. Here the state of the system is simply these fractions, and their result shows that the state moves quickly to a neighborhood of the risk dominant equilibrium when there is enough noise, i.e. that the agent puts sufficient probability on the suboptimal choice. To prove this, they first showed that when there is “enough noise” (as parameterized in their paper by the logit parameter) the logit best reply curves have a unique intersection,<sup>3</sup> so the associated deterministic, contin-

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<sup>3</sup>This intersection is what Fudenberg and Kreps (1993) called a Nash distribution and McKelvey and Palfrey (1995) subsequently called a quantal response equilibrium. The uniqueness of the intersection relies on the convexity alluded to earlier, as shown in example 2.

uous time mean field has a unique steady state; because the system is one-dimensional this means that the share of agents playing the risk-dominant action increases monotonically whenever it is below the steady-state level. They then used the results of Benaïm and Weibull (2003) to show that the discrete-time stochastic system behaves approximately the same way when  $N$  is large, and specifically that the limit of the expected waiting time to the neighborhood of the attractor is the same.

In the two-neighbor local interaction model of Ellison (1993), agents are arranged on a circle. With probability  $1 - \epsilon$  the adjusting agent chooses a best reply to the play of his neighbors, while with probability  $\epsilon$  the agent chooses the other strategy. Here the intuitive reason for fast convergence is that a small pocket of agents playing the risk-dominant action is very robust to noise or mutations, while a cluster of agents using the other action is not. Ellison’s formal argument uses a coupling of Markov processes. It starts by showing that there is a  $T$  such that even in the worst case where no players outside a small fixed neighborhood ever use the risk-dominant action, the neighborhood will with high probability have all players using the risk dominant action after  $T$  periods. The original model can be seen as a coupling together of a collection of these neighborhood-specific processes and the linkages can only further speed convergence to the risk dominant action.<sup>4</sup>

Our approach to these and other examples is to use the intuition for fast convergence to identify a suitable Lyapunov function. In the case where agents play a stochastic best response to the aggregate population fractions, as in Kreindler and Young (2013), we use the share of agents playing the risk dominant action. Here the properties of the response function imply that the expected value of this share is strictly increasing whenever it is no greater than  $\frac{1}{2}$ , which implies fast convergence to that set. As we will show in section 5, the argument for more general payoffs and for games with more than 2 actions is similar: it requires that there be “enough noise” that the risk dominated equilibria disappear- which depends on the extent of risk dominance- and it also requires that the noise be sufficiently small that the intersection of the response curves gives probability more than  $\frac{1}{2}$  to the risk-dominant action. Unlike the deterministic approximation results, our result applies directly to the discrete-time stochastic process and does not require the technical conditions needed for the approximation theorem to apply.

In the case of local interaction, the state of the system must encode what each agent is doing, so it corresponds to a vector in  $\{A, B\}^N$ . For this reason it would be difficult to prove fast convergence here using approximation by a deterministic mean field, as the dimension

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<sup>4</sup>Young (2011) uses a similar argument.

of the state of the system changes with  $N$ . To use our approach, we must identify some quantity that is increasing in expectation until most agents play  $A$ . The most obvious function to try, the share of agents playing  $A$ , is increasing in expectation when fewer than half of the players play  $A$ , but decreasing (albeit at less than an  $\epsilon$  rate) in states like  $(A, A, A, \dots, A, B, B, \dots, B)$  where more than half of the players are already using strategy  $A$ . In these problem states, the expected number of agents who have  $A$  as a best response is strictly increasing. That function also does not work as a Lyapunov function because it has an expected change of zero at some other states like  $(B, A, B, B, A, B, \dots, B, A, B)$ . However, adding the two functions with appropriate weights does provide a valid Lyapunov function, and so implies that there is fast convergence to a set where most players have  $A$  as their best response. One of our general results also lets us show further that the system quickly reaches the set where most agents play  $A$ .

## 2 Local Interaction in a Small World

In this section we present an example that readers can keep in mind as they read our more abstract model and results.

Consider the following local interaction model with a “small-world” element:  $N$  players are arranged around circle and are randomly matched to play a  $2 \times 2$  coordination game. One player chosen at random considers updating his strategy in each period. With probability  $1 - \epsilon > 1/2$  the updating player plays a best response to the average of the play in the previous period of four players: his two immediate neighbors and two players selected uniformly at random;<sup>5</sup> with probability  $\epsilon$  the updating player chooses the opposite strategy. This combination of local and global interaction seems a reasonable description of many social games. We focus on the case where strategy 1 is risk-dominant, but not  $\frac{1}{4}$ -dominant: an updating player has strategy 1 as the best response if at least two of the four players in his sample are using it, but strategy 2 is the best response if at least 3 players in the sample are using strategy 2. An example is the familiar coordination game where players get 2 if they coordinate on strategy 1, 1 if they coordinate on strategy 2, and 0 if they miscoordinate.

In the noiseless ( $\epsilon = 0$ ) version of this model, strategy 1 will not spread contagiously as it does in the two-neighbor local interaction model. Instead, in states with just a few isolated clusters of players using strategy 1, the clusters will tend to shrink from the edges and

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<sup>5</sup>To simplify the algebra we assume that the random samples are independent draws from the full population (including the player himself and the immediate neighbors).

disappear. However, the model does have both some local interaction and some sampling-based beliefs, both of which can promote fast evolution, so it is plausible that evolution may be fast.

Neither of the two standard proof techniques can be readily applied: The dimension of the state space increases with the number of players, which rules out using the Benaïm and Weibull (2003) analysis of deterministic approximations of large-population models, and the model does not have the sort of purely local stable “clusters” that underlie Ellison’s coupling argument.

An application of our Lyapunov condition shows that the model has fast convergence to the risk dominant equilibrium if the noise level is above a critical threshold. Let  $x_1$  denote the fraction of the agents using strategy 1. Intuitively, when  $x_1$  is very small the state tends to increase because of the  $\epsilon$  noise. When  $x_1$  is larger there are additional adoptions from players who have neighbors playing strategy 1 and/or see it in their random sample. This force dominates until most players are playing the risk dominant action.

**Example 1.** *Consider the model above. Suppose that strategy 1 is risk dominant. Then, for  $\epsilon > 0.065$  the model has fast convergence to  $\{x|x_1 > 1 - 1.5\epsilon\}$ .*

The proof is in the Appendix. In outline, we use as Lyapunov function the share of  $x_1$  of agents using strategy 1. Then at any state  $s$  in which a fraction  $x_1$  of the players play strategy 1, the expected change in the share of players using strategy 1 will be  $\frac{1}{N}(y(1 - \epsilon) + (1 - y)\epsilon - x_1)$ , where  $y$  is the probability that the updating player has strategy 1 as a best response in state  $s$ . We show that this is bounded below by a positive constant whenever  $x_1 \in [0, 1 - 1.5\epsilon]$ . The probability  $y$  is not a function of  $x_1$ , but we can put a lower bound on  $y$  for each  $x_1$  by partitioning  $[0, 1 - 1.5\epsilon]$  into three subintervals. When  $x_1$  is small the worst case is to have the players using strategy 1 be completely isolated. In such states players adopt strategy 1 both due to  $\epsilon$  mutations (which by themselves make strategy 1 grow in popularity whenever  $x_1 < \epsilon$ ) and when they randomly get two observations of strategy 1 in their sample. When  $x_1$  is large the worst case becomes having all of the players using strategy 1 in a single cluster. Here, the  $\epsilon$  mutations tend to decrease  $x_1$ , but this is outweighed by adoptions from players who randomly sample multiple players from the cluster unless almost all players are already in the cluster.



### 3 Model and definitions

Suppose that for each integer  $N = 1, 2, \dots$  we are given a discrete time homogeneous Markov chain  $S^N = \{S^N(t) : t = 0, \frac{1}{N}, \frac{2}{N}, \dots\}$  with finite state space  $\Omega_N$ .<sup>6</sup> Let  $\mathcal{X}$  be a compact subset of  $\mathbb{R}^m$  for some  $m$ . Given some function  $\phi_N : \Omega_N \rightarrow \mathcal{X}$ , let  $X^N$  be the stochastic process on  $\mathcal{X}$  determined by  $X^N(t) = \phi_N(S^N(t))$ . In most of our applications to models of populations playing a game with  $m$  actions, we take  $\mathcal{X}$  to be the  $(m - 1)$ -dimensional unit simplex  $\Delta = \{x \in [0, 1]^m : x_1 + \dots + x_m = 1\}$  and a point  $x \in \Delta$  gives the fraction of players using each of the  $m$  strategies. We denote the conditional probability and the conditional expectation given  $S^N(0) = s$  by  $P_s$  and  $E_s$ . For every  $A \subset \mathcal{X}$  let  $\bar{A}$  denote its closure.

To discuss the speed with which the process  $X^N$  reaches the set  $A \subset \mathcal{X}$  we write

$$\tau_A^N = \inf\{t \geq 0 : X^N(t) \in A\}$$

for the random variable giving the time at which  $A$  is first reached and define

$$W^N(A, s) = E_s(\tau_A^N)$$

to be the expected wait to reach  $A$  conditional on the process  $S^N$  starting at state  $s$ . Our first main definition is

**Definition 1.** The family  $\{X^N\}$  reaches  $A$  quickly if

$$\limsup_{N \rightarrow \infty} \sup_{s \in \Omega_N} W^N(A, s) < \infty. \quad (1)$$

Note that as in Kreindler and Young (2013) this is an asymptotic property meaning that the expected waiting time remains bounded (uniformly over all starting points) in the  $N \rightarrow \infty$  limit. We think that this is a natural definition of “quickly” for many applications. For example, our model could capture a situation in which a large number of players are asynchronously randomly matched to play some game with each player being matched on average once per unit of calendar time. Here, the  $\frac{1}{N}$  period length would correspond to the interval between encounters, and quickly would mean that the calendar time required for some behavior to arise was bounded independently of the population size.<sup>7</sup> In other

<sup>6</sup>We believe that the assumption that  $S^N$  has a finite state space is not critical; we use it as a convenient way to avoid technical difficulties.

<sup>7</sup>If for every  $N$ ,  $S^N$  is irreducible and  $\phi_N^{-1}(A) \neq \emptyset$ , or, more generally, if for every  $N$ ,  $\phi_N^{-1}(A)$  is accessible from its complement, then (1) is equivalent to the existence of a constant  $M < \infty$  such that  $W^N(A, s) < M$  for all  $s \in \Omega_N$  and all  $N$ . This stronger condition holds in all the examples in the present paper where we show that a set is reached quickly. We have made the definition less demanding to include the possibility that a set is reached quickly even if it is so small that, for finitely many  $N$ , it contains none of the states of  $X^N$ .

applications, one could want other definitions of fast convergence.<sup>8</sup>

Our second main definition is

**Definition 2.** The family  $\{X^N\}$  *leaves  $A$  slowly* if for any finite  $T$  and for every open set  $U$  containing  $\bar{A}$

$$\lim_{N \rightarrow \infty} \max_{s \in \phi_N^{-1}(A)} P_s(\tau_{\mathcal{X} \setminus U}^N \leq T) = 0. \quad (2)$$

Note that the requirement in the definition is stronger than requiring that

$$\lim_{N \rightarrow \infty} \min_{s \in \phi_N^{-1}(A)} W^N(\mathcal{X} \setminus U, s) = \infty.$$

We made the definition more demanding in this dimension so that we will not count a model as leaving a set slowly just because there is some probability of being trapped within  $A$  for a very long time. Instead, it must be the case that even for very large  $T$ , the probability of escaping within  $T$  periods vanishes in the  $N \rightarrow \infty$  limit.<sup>9</sup>

Finally, we define “fast convergence” as the combination of these two properties.

**Definition 3.** The family  $\{X^N\}$  *has fast convergence to  $A$*  if  $\{X^N\}$  reaches  $A$  quickly and leaves  $A$  slowly.

## 4 Lyapunov criteria

This section contains several sufficient conditions for “reaching quickly” and “leaving slowly.”

### 4.1 Main results

We first present results relating the two components of fast convergence to the existence of Lyapunov functions satisfying certain properties. Proposition 1 contains a Lyapunov condition for  $\{X^N\}$  to reach a given set quickly. Where applicable, it also provides an explicit upper bound for the expected hitting time.

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<sup>8</sup>For example, in a model of learning from others it might be reasonable to assume that each individual got more information per unit time in a larger population, e.g. each might get  $\alpha N$  pieces of information per unit time by observing a fraction  $\alpha$  of the others. If one modeled every instance of one player getting one piece of information as a period, then it would be natural to also regard evolution as fast if waiting times increased linearly in  $N$ .

<sup>9</sup>It would have been more accurate to say  $\{X^N\}$  “is likely to remain close to  $A$  for a long time” rather than “leaves  $A$  slowly,” but we prefer the simpler wording. For example, for  $m = 2$  a deterministic system which immediately transitions from  $(0, 1)$  to  $(\frac{1}{N}, 1 - \frac{1}{N})$ , but then immediately returns to  $(0, 1)$  is defined to leave  $\{(0, 1)\}$  slowly.

To provide some intuition for the result, suppose that the Markov processes are deterministic and there is a nonnegative function  $V$  for which  $V(X^N(t))$  decreases by at least  $c/N$  in the next  $1/N$ -length time interval whenever  $X^N(t)$  is outside  $A$ . Clearly, when such a process starts at  $x$  it must reach  $A$  within  $V(x)/c$  units of time. The proposition extends this simple argument to the case when the Markov process is not deterministic, but  $V(X^N(t))$  still decreases in expectation at rate  $c$ .

**Proposition 1.** *Let  $A \subset \mathcal{X}$ ,  $c \in (0, \infty)$ , and let  $V : \mathcal{X} \rightarrow [0, \infty)$  be a bounded function. If*

$$E_s [V(X^N(1/N)) - V(X^N(0))] \leq -\frac{c}{N} \text{ for all } s \in \phi_N^{-1}(\mathcal{X} \setminus A), \quad (3)$$

*then  $W^N(A, s) \leq V(\phi_N(s))/c$  for every  $s \in \Omega_N$ . In particular, if (3) holds for all  $N$  sufficiently large, then  $\{X^N\}$  reaches  $A$  quickly.*

**Remark.** Note that the implication in the sentence containing (3) holds for every single  $N$ . Thus, if one can show, for example, that there is a constant  $c_0 > 0$  so that for all  $N$ ,

$$E_s [V(X^N(1/N)) - V(X^N(0))] \leq -\frac{c_0}{N^2} \text{ for all } s \in \phi_N^{-1}(\mathcal{X} \setminus A),$$

one can conclude that  $\max_{s \in \Omega_N} W^N(A, s)$  grows at most linearly in  $N$ .

**Proof of Proposition 1.** Fix  $N \geq 1$ , define  $Y(t) = X^N(t \wedge \tau_A^N)$ , and let  $Z(t)$  be 0 or 1 according as  $Y(t) \in A$  or  $Y(t) \notin A$ . By (3), for  $t = 0, \frac{1}{N}, \frac{2}{N}, \dots$ , we have

$$E [V(Y(t+1/N)) - V(Y(t)) | Y(t) = y, S^N(t) = s_t] \leq -\frac{c}{N},$$

for any  $(y, s_t)$  with  $y \notin A$  that occurs with positive probability. The LHS of the above inequality is identically zero if  $y \in A$ . So we can combine these two observations and write

$$E [V(Y(t+1/N)) - V(Y(t)) | Y(t), S^N(t)] \leq -\frac{c}{N} Z(t).$$

The expected change in the value function conditional on the state  $s$  at  $t = 0$  can be computed by iterated expectations as

$$E_s [V(Y(t+1/N)) - V(Y(t))] = E_s [E [V(Y(t+1/N)) - V(Y(t)) | Y(t), S^N(t)]] .$$

This gives

$$E_s [V(Y(t+1/N)) - V(Y(t))] \leq -\frac{c}{N} E_s [Z(t)],$$

which is equivalent to

$$\frac{1}{N} E_s [Z(t)] \leq \frac{1}{c} (E_s [V(Y(t))] - E_s [V(Y(t+1/N))]).$$

The expected time to reach  $A$  is given by  $\frac{1}{N}E_s[\sum_{k=0}^{\infty} Z(\frac{k}{N})]$ . The partial sums are bounded above:

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{m-1} E_s Z\left(\frac{k}{N}\right) &\leq \frac{1}{c} \sum_{k=0}^{m-1} (E_s [V(Y(k/N))] - E_s [V(Y((k+1)/N))]) \\ &= \frac{1}{c} (E_s [V(Y(0))] - E_s [V(Y(m/N))]) \\ &\leq \frac{1}{c} E_s [V(Y(0))] = \frac{1}{c} V(\phi_N(s)). \end{aligned}$$

Hence, by monotone convergence,  $W^N(A, s) \leq V(\phi_N(s))/c$ .  $\square$

**Remark.** The proof resembles that of Foster's theorem (e.g. Brémaud (1999)). Like that proof, the one here has the flavor of martingale arguments although it does not appeal to martingale results.

Our second proposition provides a criterion for  $\{X^N\}$  to leave a set  $A$  slowly. It requires that the Lyapunov function decrease in expectation whenever  $X^N(t)$  is slightly outside  $A$ . This suffices because we also assume that there is an upper bound on the rate at which the process can move.<sup>10</sup> As a result, whenever the process does jump out of  $A$  it first reaches a point slightly outside  $A$ . The Lyapunov condition then ensures that for large  $N$  the system is unlikely to escape the neighborhood before being drawn back into  $A$ .

**Proposition 2.** *Suppose there is a constant  $K < \infty$  such that  $P_s(\|X^N(\frac{1}{N}) - X^N(0)\| \leq \frac{K}{N}) = 1$  for all  $N$  and all  $s \in \Omega_N$ , where  $\|\cdot\|$  is the Euclidean norm on  $\mathbb{R}^m$ . Let  $A \subset \mathcal{X}$  be a set with  $\phi_N^{-1}(A) \neq \emptyset$  for all  $N \geq N_0$  for some  $N_0$ . Let  $c \in (0, \infty)$  and let  $V : \mathcal{X} \rightarrow \mathbb{R}$  be a Lipschitz continuous function such that*

$$V(x) < V(y) \quad \text{for all } x \in \bar{A}, y \in \mathcal{X} \setminus \bar{A}. \quad (4)$$

*Suppose there is an open set  $U_0 \subset \mathcal{X}$  that contains  $\bar{A}$  and*

$$E_s [V(X^N(1/N)) - V(X^N(0))] \leq -\frac{c}{N} \quad (5)$$

*whenever  $s \in \phi_N^{-1}(U_0 \setminus \bar{A})$  and  $N \geq N_0$ . Then  $\{X^N\}$  leaves  $A$  slowly.*

**Remark.** In the Appendix we prove the stronger result that there is a constant  $\gamma$  such that for  $N$  large the exit time  $\tau_{\mathcal{X} \setminus U}^N$  first order stochastically dominates an exponential

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<sup>10</sup>For example, this would hold if at most  $K$  players change their strategies in each  $1/N$ -length time interval.

random variable with mean  $e^{\gamma N}$ . The argument writes the exit time as the sum over all instances that the process reaches  $\bar{A}$  of the transit times to either reach  $\mathcal{X} \setminus U$  or to return to  $\bar{A}$ . Each such transit takes at least one unit of time, so it suffices to show that the number of transits is large enough. The key step is to show that the probability of reaching  $\mathcal{X} \setminus U$  before returning to  $\bar{A}$  when starting in  $\bar{A}$  is small (and declines exponentially in  $N$ ). Intuitively, this is true because the bound on the speed with which the process can move implies that any initial step out of  $\bar{A}$  reaches a point that is bounded away from  $\mathcal{X} \setminus U$ , and because the drift is toward  $\bar{A}$  and many steps against the drift are needed to reach  $\mathcal{X} \setminus U$ , it becomes increasingly unlikely that a transit will lead to  $\mathcal{X} \setminus U$  rather than to  $\bar{A}$  when  $N$  is large.

The technical argument focuses on the random variable  $Y^N(t) \equiv e^{\delta_0 N V(X^N(t))}$ . Using the hypotheses that  $X^N(t)$  moves in bounded steps and  $V$  is Lipschitz continuous, we can approximate the exponential in the definition of  $Y^N$  using a Taylor expansion and show that for an appropriately small value of  $\delta_0$ , the random variable  $Y^N(t)$  is a positive supermartingale whenever  $X^N(t)$  is between  $\bar{A}$  and  $\mathcal{X} \setminus U$ . This allows us to apply general results on positive supermartingales to conclude that the probability of reaching  $\mathcal{X} \setminus U$  before returning to  $\bar{A}$  is at most  $e^{-\gamma N}$  for some constant  $\gamma$ .

The hypotheses of Propositions 1 and 2 are similar. One difference is that Proposition 2 is more demanding in that it has added a bound on the speed with which the process can move. Another is that Proposition 1 requires that the Lyapunov condition holds on a larger set (whenever  $X^N(t)$  is outside  $A$  versus just when  $X^N(t)$  is in some neighborhood of  $\bar{A}$ ). When the more restrictive version of each hypothesis holds the process will both reach  $A$  quickly and leave  $A$  slowly. Hence, we have fast convergence to  $A$ .

**Corollary 2.1.** *Suppose the hypotheses of Proposition 2 are satisfied and that condition (5) holds also for all  $s \in \phi_N^{-1}(\mathcal{X} \setminus A)$  when  $N \geq N_0$ . Then  $\{X^N\}$  has fast convergence to  $A$ .*

**Proof.** Any Lipschitz continuous function on  $\mathcal{X}$  is bounded, and replacing  $V(x)$  by  $V(x) - \min_{y \in \mathcal{X}} V(y)$  if necessary, one may assume that  $V$  is nonnegative. Hence, the hypotheses of Propositions 1 and 2 are satisfied.  $\square$

## 4.2 Extensions: Systems that slow at the limit and multistep arguments

Some models will not satisfy the hypotheses of the results above because the expected decrease in the Lyapunov function decreases to 0 as the state approaches the target set  $A$ .

In the case of Proposition 1, one will often be able to slightly weaken the desired conclusion and argue that for any open neighborhood  $U$  of  $A$ , the model reaches  $U$  quickly. This will follow if there is a positive lower bound on the rate at which the Lyapunov function decreases when  $x \in \mathcal{X} \setminus U$ . In the case of Proposition 2 we can do even better because a drift that vanishes at  $A$  is not a problem. To describe this formally, for  $A \subset \mathcal{X}$ , let  $U(A, \epsilon)$  denote the  $\epsilon$ -neighborhood of  $A$  in  $\mathcal{X}$ ,  $U(A, \epsilon) = \{x \in \mathcal{X} : \inf_{y \in A} \|x - y\| < \epsilon\}$ .

**Proposition 3.** *The conclusion of Proposition 2 that  $\{X^N\}$  leaves  $A$  slowly remains true if the Lyapunov hypothesis is replaced with “Suppose there is an open set  $U_0 \subset \mathcal{X}$  that contains  $\bar{A}$  and for every  $\epsilon > 0$  there are numbers  $c_\epsilon > 0$  and  $N_\epsilon \geq N_0$  such that*

$$E_s \left[ V \left( X^N \left( \frac{1}{N} \right) \right) - V \left( X^{(N)}(0) \right) \right] \leq -\frac{c_\epsilon}{N}$$

for all  $s \in \phi_N^{-1}(U_0 \setminus U(\bar{A}, \epsilon))$  and  $N \geq N_\epsilon$ .”

**Proof.** Let  $U$  be an open set containing  $\bar{A}$ . Let  $v_1 := \max_{x \in \bar{A}} V(x)$  and choose  $v_2 > v_1$  so small that  $\tilde{A} := \{x \in \mathcal{X} : V(x) \leq v_2\} \subset U \cap U_0$ . Then choose  $\epsilon > 0$  so that  $U(\bar{A}, \epsilon) \subset \tilde{A}$ . Now apply Proposition 2 with  $A, c, N_0$  replaced by  $\tilde{A}, c_\epsilon$  and  $N_\epsilon$ . This yields that  $\{X^N\}$  leaves  $\tilde{A}$  slowly. Thus, as  $A \subset \tilde{A} \subset U$ ,

$$\lim_{N \rightarrow \infty} \max_{s \in \phi_N^{-1}(A)} P_s(\tau_{\mathcal{X} \setminus U}^N \leq T) \leq \lim_{N \rightarrow \infty} \max_{s \in \phi_N^{-1}(\tilde{A})} P_s(\tau_{\mathcal{X} \setminus U}^N \leq T) = 0$$

for all  $T \in [0, \infty)$ . This proves that  $\{X^N\}$  leaves  $A$  slowly.  $\square$

In some cases models have fast convergence even though they do not always “drift” toward the selected set. For example, in a model where  $N$  agents are arranged on a circle to play a  $2 \times 2$  coordination game and play a best response to the average play of their  $2k$  closest neighbors unless a mutation occurs, the number of players playing the risk-dominant action is not monotonically increasing – it can decrease in expectation if there is not a sufficiently large cluster of players playing the risk-dominant action. Ellison (1993) nonetheless shows that play reaches a neighborhood of the state where everyone plays the risk-dominant equilibrium quickly. Intuitively, this occurs because evolution can proceed in a two-step manner: Each period there is a nonzero chance that a cluster of players using the risk-dominant action will form, and whenever such a cluster exists, the model drifts toward everyone playing the risk-dominant equilibrium.

Our results can be extended so that they apply to some models of this variety. Specifically, the proposition below shows that  $\{X^N\}$  reaches a set  $A$  quickly when three conditions hold: (1)  $\{X^N\}$  reaches a superset  $B$  of  $A$  quickly; (2)  $\{X^N\}$  does not stay too long in

$B \setminus A$ , that is,  $\{X^N\}$  reaches  $A \cup B^c$  quickly; and (3) the probability that  $\{X^N\}$  reaches  $A$  before  $B^c$  when starting anywhere in  $B \setminus A$  is bounded away from zero. The proof uses an argument related to Wald's equation for the expectation of a random sum of i.i.d. random variables. A transition from  $B^c$  to  $A$  consists of a random number of transitions from  $B^c$  to  $B \setminus A$  and back to  $B^c$  with a final transition to  $A$ . The assumptions ensure that the expected lengths of the individual transitions are bounded and that the number of these transitions has a finite expectation as well.

**Proposition 4.** *Let  $A \subset B \subset \mathcal{X}$ . Suppose  $\{X^N\}$  reaches  $B$  quickly,  $\{X^N\}$  reaches  $A \cup B^c$  quickly, and there exist  $c > 0$  and  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$ ,*

$$P_s(X^N(\tau_{A \cup B^c}^N) \in A) \geq c \quad \text{for all } s \in \phi_N^{-1}(B \setminus A). \quad (6)$$

*Then  $\{X^N\}$  reaches  $A$  quickly.*

The proof is contained in the Appendix. We will use this result in the proof of Proposition 7 in the following applications section.

Condition (6) can be hard to verify. The following variant of Proposition 4 does not contain this condition, but the assumption that  $\{X^N\}$  reaches  $B$  quickly is replaced by the stronger assumption that  $\{X^N\}$  has fast convergence to a suitable subset of  $\text{int}B$ , the interior of  $B$ .

**Proposition 5.** *Let  $A \subset B \subset \mathcal{X}$ . Suppose  $\{X^N\}$  has fast convergence to a set  $C$  with  $\bar{C} \subset \text{int}B$  and  $\{X^N\}$  reaches  $A \cup B^c$  quickly. Then  $\{X^N\}$  reaches  $A$  quickly.*

The proof is in the Appendix. We apply this result when we return to the two-neighbor local interaction model in Section 6. Another potential application for Propositions 4 and 5 that we do not pursue here is to develop results about the selection of iterated  $p$ -dominant equilibria. We believe that this would lead to results like those in Oyama, Sandholm, and Tercieux (2014).

## 5 Dynamics based on population shares

This section analyzes models where the evolution of the system depends only on the number of agents using each strategy, so that the state space  $\Omega_N$  can simply be these numbers. We connect fast convergence here with whether the popularity of some strategy is increasing in expectation, and derive several results about particular models as corollaries.

## 5.1 A general result

Suppose that  $N$  players are choosing strategies for a game with  $m$  pure strategies at  $t = 0, \frac{1}{N}, \frac{2}{N}, \dots$ . Throughout this section we assume that in each time period a single agent is chosen at random to revise his strategy. (With a time renormalization this could describe a model in which revision opportunities arrive in continuous time according to independent Poisson processes.) Let  $X_i^N(t)$  denote the share of agents that play strategy  $i$  at time  $t$ . If the current state of the population is  $x$ , suppose the revising agent chooses strategy  $i$  with probability  $f_i(x)$ , regardless of the agent's own current action.<sup>11</sup> For now, we let  $f_1, \dots, f_m : \Delta \rightarrow [0, 1]$  be arbitrary functions with  $f_1(x) + \dots + f_m(x) = 1$  for all  $x$ .

Part (a) of our general result shows that if the probability of choosing some strategy  $i$  exceeds its current share whenever that share is below some threshold  $a$ , then there is fast convergence to the states where the share of  $i$  exceeds  $a$ . Part (b) gives a partial converse.

**Proposition 6.** *Consider a dynamic  $\{X^N\}$  with choice rules  $f_1(x), \dots, f_m(x)$ . Let  $a, c \in (0, 1)$  and  $i \in \{1, \dots, m\}$ .*

(a) *If*

$$f_i(x) - x_i \geq c \text{ whenever } x_i \leq a,$$

*then  $\{X^N\}$  has fast convergence to  $\{x : x_i > a\}$  and  $W^N(\{x : x_i > a\}, X^N(0)) \leq \frac{1}{c}$ .*

(b) *If  $f_i$  is continuous and*

$$f_i(x) < a \text{ whenever } x_i = a,$$

*then  $\{X^N\}$  leaves  $\{x : x_i < a\}$  slowly.*

(c) *If  $f_i$  is continuous and*

$$f_i(x) > a \text{ whenever } x_i = a,$$

*then  $\{X^N\}$  leaves  $\{x : x_i > a\}$  slowly.*

**Remark.** We use (b) and (c) to show that fast convergence fails in Example 4. It can also be used to show that a sequential-move version of the KMR model will not have fast convergence in a  $2 \times 2$  coordination game unless there is so much noise (or large enough payoff differences) that one strategy is  $\epsilon$ -dominant.

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<sup>11</sup>The assumption that the  $f_i$  do not depend on the current action rules out models in which agents respond to the current play of all others not including themselves, but since the effect of any one agent's strategy on the overall state is of order  $\frac{1}{N}$ , we do not expect it to matter for large  $N$  except perhaps in knife-edge cases.



**Proof.** Here  $\phi_N(x) = x$ ,  $S^N = X^N$ , and  $\Omega_N = \{x \in \Delta : Nx \in \mathbb{N}_0^m\}$ .

(a) Let  $V(x) = 1 - x_i$ . Fast convergence to  $\{x : x_i > a\}$  then follows from Corollary 2.1 since for every  $x \in \Omega_N$  with  $x_i \leq a$ ,

$$\begin{aligned} E_x \left[ V \left( X^N \left( \frac{1}{N} \right) \right) - V \left( X^N(0) \right) \right] &= -E_x \left[ X_i^N \left( \frac{1}{N} \right) - X_i^N(0) \right] \\ &= -\frac{1}{N} (1 - x_i) f_i(x) + \frac{1}{N} x_i [1 - f_i(x)] = -\frac{1}{N} [f_i(x) - x_i] \leq -\frac{c}{N}. \end{aligned}$$

The uniform bound on the convergence time follows from Proposition 1.

(b) Set  $c' = -\frac{1}{2} \sup\{f_i(x) - x_i : x \in \Delta, x_i = a\}$ . The set of  $x$  with  $x_i = a$  is compact so  $c' > 0$ . The uniform continuity of  $f_i$  implies that we can choose an  $\epsilon > 0$  so that  $f_i(x) - x_i \leq -c'$  whenever  $x_i \in [a, a + \epsilon]$ . The assertion follows from Proposition 2 with  $V(x) = x_i$ .

(c) An argument similar to that in (b) shows that the present condition on  $f_i$  implies that there exist  $c' > 0$  and  $\epsilon > 0$  so that  $f_i(x) - x_i \geq c'$  whenever  $x_i \in [a - \epsilon, a]$ . The assertion follows again from Proposition 2 with  $V(x) = 1 - x_i$ .  $\square$

**Remark.** The continuity of  $f_i$  is used in part (b) only to show that  $f_i(x) - x_i$  is negative and bounded away from 0 when  $x_i$  is in an interval around  $a$ . It would suffice to instead assume that there exists a  $c > 0$  and an  $a' > a$  such that  $f_i(x) - x_i \leq -c$  whenever  $x_i \in (a, a')$ . In view of Proposition 3 it is also sufficient to assume that  $f_i$  is upper semicontinuous with  $f_i(x) - x_i < 0$  whenever  $x_i \in (a, a')$ . A similar remark applies to part (c).

The rest of this section presents several examples that use this result in coordination games under different dynamics. The first subsection looks at the case where fast convergence arises due to the noise caused by stochastic choice; the second subsection considers the case where players receive noisy signals of the current state. In each case we assume that strategy 1 is risk dominant.

## 5.2 Stochastic choice

Our first example here generalizes Kreindler and Young (2013)'s analysis of logit responses in  $2 \times 2$  games. We show that the key property of the logit responses they analyzed is that they can be generated by maximizing a perturbed utility function of the form  $U(p) = \sum_{i=1}^2 p_i \pi_i(x) - c(p_i)/\beta$ , where  $\pi_i(x)$  is the expected payoff of strategy  $i$  against distribution  $x$  and  $c : (0, 1] \rightarrow \mathbb{R}$  is continuously differentiable, strictly convex, and satisfies the Inada condition that  $\lim_{p \rightarrow 0} c'(p) = -\infty$ . Here  $1/\beta$  is a measure of the amount of

noise in the system: as  $\beta \rightarrow 0$  choice becomes uniform and as  $\beta \rightarrow \infty$  choice becomes almost deterministic.<sup>12</sup> Fudenberg, Iijima, and Strzalecki (2015) show that this form of stochastic choice corresponds to the behavior of an agent who is uncertain about his payoff function and so randomizes to guard against moves by a malevolent Nature; logit responses correspond to  $c(p) = p \ln p$ .

**Example 2.** *In the model of Section 5.1 suppose that  $f(x) = \operatorname{argmax}_p U(p)$  with  $c$  having the properties mentioned above, then there is fast convergence to the set  $\{x \in \Delta : x_1 > \frac{1}{2}\}$ , provided  $\beta$  is sufficiently small.*

Each of the results in this section result can be proven as an application of Proposition 6 with the share of a strategy as the Lyapunov function. We provide sketches here and leave the details to the Appendix.<sup>13</sup> In this example we can see that  $f_1(x) - x_1 > 0$  for all  $x$  with  $x_1 \in [0, \frac{1}{2}]$  using two cases. Let  $x^*$  be the strictly mixed Nash equilibrium of the unperturbed game. For  $x_1 \in (x_1^*, \frac{1}{2}]$ , the fact that strategy 1 is the unperturbed best response implies that  $f_1(x) > \frac{1}{2}$  which implies that  $f_1(x) - x_1 > 0$ . For  $x_1 \in [0, x_1^*]$ , note that the strict convexity of  $c$  implies that  $f_1(x)$  converges uniformly to  $\frac{1}{2}$  when  $\beta \rightarrow 0$ , so if we choose  $\beta$  small enough (enough noise), we will have  $f_1(x) - x_1 > 0$  for all  $x$  with  $x_1 \leq x_1^*$  as well.

Our next example generalizes Kreindler and Young in a different way, by considering games with more than 2 actions. Consider an  $m$ -action coordination game,  $m \geq 2$ , whose payoff matrix is a diagonal matrix with diagonal entries  $\alpha_1 > \alpha_2 \geq \dots \geq \alpha_m > 0$ . When an agent revises his action, he chooses action  $i$  with probability

$$f_i(x) = \frac{e^{\beta \alpha_i x_i}}{\sum_{j=1}^m e^{\beta \alpha_j x_j}},$$

where  $x = (x_1, \dots, x_m)$ ,  $x_j$  is the fraction of agents currently using action  $j$ , and  $1/\beta \in (0, \infty)$  is a measure of noise.

**Example 3.** *If*

$$\alpha_1 > \frac{2(m-2 + e^{\beta \alpha_2})}{e\beta}, \tag{7}$$

*then the logit dynamic  $\{X^N\}$  has fast convergence to  $\{x \in \Delta : x_1 > \frac{1}{2}\}$ .*

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<sup>12</sup>Blume (1993) studied logit responses in a model of local interaction and noted the role of the noise level there. Fudenberg and Kreps (1993) introduced stochastic choice rules in the study of fictitious play; Fudenberg and Levine (1995) show that stochastic choice generated by perturbed utility leads to “stochastic fictitious play,” and generates Hannan-consistent choice. Hofbauer and Sandholm (2002), Benaïm, Hofbauer, and Hopkins (2009), and Fudenberg and Takahashi (2011) use perturbed utility to construct Lyapunov functions for stochastic fictitious play, and study perturbed utility in an evolutionary model.

<sup>13</sup>The proof of Example 2 in the Appendix follows an alternate approach applying Proposition 8.

The proof of Example 3 is a simple application of the fast convergence criterion in Proposition 6(a). Because  $f$  is continuous, the hypothesis of Proposition 6(a) holds provided that  $f_1(x) > x_1$  when  $x_1 \leq \frac{1}{2}$ , and a short calculation based on the Schur-convexity of  $\sum_{i=2}^m e^{\beta\alpha_2 x_i}$  shows that the condition is implied by (7).

Condition (7) involves a relationship between  $\alpha_1$  and  $\beta$ . To interpret the condition denote the right hand side of (7) by  $\phi(\beta)$ , keeping  $\alpha_2 > 0$  fixed. The function  $\phi$  is strictly convex on  $(0, \infty)$  with  $\lim_{\beta \rightarrow 0} \phi(\beta) = \lim_{\beta \rightarrow \infty} \phi(\beta) = \infty$ , see Figure 1. Condition (7) can only hold if  $\alpha_1$  exceeds the minimum of  $\phi$ , that is, if the degree of risk dominance is sufficiently large.<sup>14</sup> If this is the case, then the set of  $\beta$  satisfying (7) is the nonempty level set  $\{\beta : \phi(\beta) < \alpha_1\}$ , which is a bounded interval with a positive distance from 0, and the interval increases to  $(0, \infty)$  as  $\alpha_1 \rightarrow \infty$ . Thus, Example 3 shows fast convergence if the degree of risk dominance is sufficiently large and there is enough but not too much noise.<sup>15</sup>

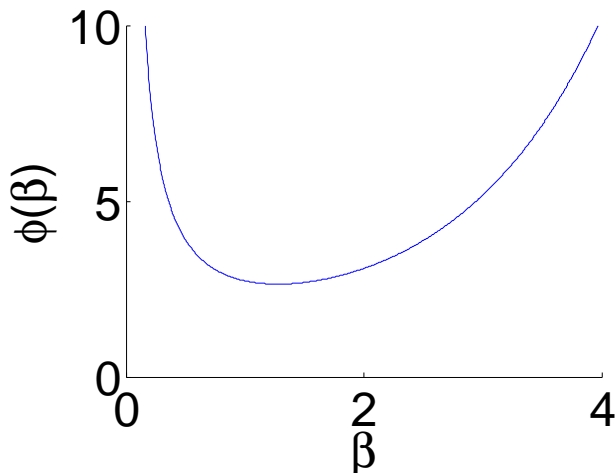


Figure 1: Graph of  $\phi$  when  $m = 3$  and  $\alpha_2 = 1$ . According to Example 3, the logit dynamic has fast convergence if the point  $(\beta, \alpha_1)$  lies above this graph.

The following example is a partial converse to Example 3. It gives three conditions under each of which  $\{X^N\}$  does not have fast convergence to  $\{x : x_1 > \frac{1}{2}\}$ : (a) If there is too little noise, fast convergence fails because play gets stuck at a dominated equilibrium. (b) Fast convergence fails if  $m \geq 3$  and there is too much noise. In this case choice is

<sup>14</sup>Note that  $e^{x-1} \geq x$  implies that condition (7) cannot be satisfied unless  $\alpha_1 > \frac{2(m-2)}{e\beta} + 2\alpha_2$ . Hence, something beyond  $\alpha_1 > 2\alpha_2$  will be necessary for the result to apply.

<sup>15</sup>In the  $m = 2$  case, even random choice will lead to half of the agents playing action 1, and as shown in Example 2 for more general dynamics, for every degree of risk dominance there will be fast convergence when there is enough noise. Kreindler and Young (2013) show that  $\alpha_1 > \alpha_2$  suffices whenever  $\beta \leq 2$ .

almost independent of payoffs, and the risk-dominant equilibrium will only be played by a  $1/m$  fraction of agents. (c) Fast convergence fails for every level of noise if the payoffs  $\alpha_i$  are too close and  $m \geq 3$ .

**Example 4.** (a) If

$$\frac{1}{\beta} \leq \frac{\alpha_2}{2 \log(m-1) + 2 \log\left(4 \frac{\alpha_1}{\alpha_2} - 1\right)}, \quad (8)$$

then the logit dynamic  $\{X^N\}$  leaves  $\{x : x_2 > 1 - \alpha_2/(4\alpha_1)\}$  slowly. In particular,  $\{X^N\}$  does not reach  $\{x : x_1 > \frac{1}{2}\}$  quickly.

(b) If

$$\frac{1}{\beta} \geq \left(1 + \frac{1}{m}\right) \left(\alpha_1 - \alpha_m + \frac{1}{m} \alpha_m\right), \quad (9)$$

then  $\{X^N\}$  leaves  $\{x : x_1 < \frac{1}{m} + \frac{1}{m^2}\}$  slowly. In particular, if  $m \geq 3$ , then (9) implies that  $\{X^N\}$  does not reach  $\{x : x_1 > \frac{1}{2}\}$  quickly.

(c) There is a constant  $\gamma > 1$  so that if  $m \geq 3$  and

$$\alpha_1 < \gamma \alpha_m, \quad (10)$$

then there is no  $\beta > 0$  so that  $\{X^N\}$  reaches  $\{x : x_1 > \frac{1}{2}\}$  quickly.

The proof is again an application of the Lyapunov criteria in Proposition 6 and consists mainly of showing that  $f_2(x) > x_2$  or  $f_1(x) < x_1$  on the boundaries of the specified sets.

### 5.3 Noisy signals

This subsection considers two examples where agents receive only a noisy signal of the state. In the first, players play an exact best response to a signal of the state which is noisy for two separate reasons: finite sampling and observation errors. At each time  $t = \frac{1}{N}, \frac{2}{N}, \dots$ , one randomly chosen agent  $i$  draws, with replacement, a sample of size  $r$  from the current population; with probability  $1 - \epsilon$  agent  $i$  correctly recognizes how  $j$  is playing, but with probability  $\epsilon$  agent  $i$  instead thinks  $j$  is playing another action, with all strategies other than the true one being equally likely, and observation errors are independent across all observations. (Note that if  $\epsilon = 1 - \frac{1}{m}$ , then each observation is equally likely to be each pure strategy irrespective of actual play.) The revising agent  $i$  then adopts the best response to this set of observations.

Here, we consider general  $m \times m$  games, but assume that strategy 1 is  $p$ -dominant for some  $p \leq \frac{1}{2}$ .<sup>16</sup> We find that there is fast convergence to the set of states where the

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<sup>16</sup>That is, strategy 1 is the unique best response to every mixed strategy that assigns at least probability  $p$  to strategy 1 (Morris, Rob, Shin, 1995).

popularity of strategy 1 exceeds a threshold provided that the sample size  $r$  is sufficiently small relative to the level of  $p$ -dominance (but at least two). Specifically, we require that  $r$  is small enough so having any observation of strategy 1 in his sample will lead agent  $i$  to choose strategy 1.

**Example 5.** *In the model described above, if  $r \in [2, \frac{1}{p}]$ , then  $\{X^N\}$  has fast convergence to  $\{x : x_1 > 1 - (1 - \frac{1}{m})^r\}$  for every  $\epsilon \in (0, 1 - \frac{1}{m})$ .*

We defer the calculations to the Appendix. The key once again is that the probability of choosing the risk-dominant strategy 1 is above its current share as long as the share is below the threshold. This follows from the fact that the above probability is bounded below by a concave function of that share, with the bound being above the share at  $x_1 = 0$  and equal to it at  $x_1 = 1 - (1 - \frac{1}{m})^r$ .

As a final application of Proposition 6 we suppose that beliefs derive from correctly observing a random sample of play (eliminating the second source of observation noise in the previous example) and return to stochastic choice in  $2 \times 2$  coordination games. Roughly, our results say that such noisy beliefs speed evolution in the sense that sampling enlarges the set of parameters for which evolution is fast. More precisely, we compare the Markov chain  $X^N$  describing the dynamics where the revising agent chooses strategy 1 with probability  $g(x)$  if the current state of the population is  $x$  with the chains  $X_r^N$  in which updating agents apply the same rule  $g$  to a random sample of size  $r$ .<sup>17</sup> In the random sampling model the probability  $f^{(r)}(x)$  that a revising agent chooses strategy 1 is

$$f^{(r)}(x) = \sum_{k=0}^r g\left(\frac{k}{r}\right) \binom{r}{k} x^k (1-x)^{r-k}, \quad x \in [0, 1].$$

We begin by considering when the implication

$$\{X^N\} \text{ reaches } (\frac{1}{2}, 1] \text{ quickly} \Rightarrow \{X_r^N\} \text{ reaches } (\frac{1}{2}, 1] \text{ quickly} \quad (11)$$

is true.<sup>18</sup> Proposition 7 provides a symmetry condition on  $g$  under which implication (11) holds for every  $r$ . This generalizes Kreindler and Young's (2013) result beyond logit best responses and strengthens the results to show there is fast convergence in the model with sampling beliefs whenever there is fast convergence with full information (as opposed to when a sufficient condition for reaching quickly holds). To understand the condition we

<sup>17</sup>Here, the state is the fraction of agents using strategy 1.

<sup>18</sup>In the analysis of the relation between  $\{X_r^N\}$  and  $\{X^N\}$  we take advantage of the fact that the functions  $f^{(r)}$  are the Bernstein polynomials of  $g$  to exploit known results about properties of these polynomials.

place on  $g$  and why it is a generalization, note that when  $\sup_{0 < x < \frac{1}{2}} g(x) < \frac{1}{2}$ ,  $\{X^N\}$  cannot reach  $(\frac{1}{2}, 1]$  quickly. So, ignoring a knife-edge case, we assume there exists  $x^* \in (0, \frac{1}{2})$  with  $g(x^*) \geq \frac{1}{2}$ . In two-action games with strategy 1 being risk dominant, the logit model satisfies a stronger version of the symmetry condition:  $g(x^* + x) + g(x^* - x) = 1$  for all  $x \in [0, x^*]$  where  $x^*$  is the mixed-strategy equilibrium. We loosen this to only require an inequality.<sup>19</sup>

**Proposition 7.** *In the above model of learning in a  $2 \times 2$  game let  $x^* \in (0, \frac{1}{2})$ . Suppose that  $g(x^* + x) + g(x^* - x) \geq 1$  for all  $x \in [0, x^*]$  and that  $g$  is strictly increasing. Suppose  $\{X^N\}$  reaches  $[x^*, 1]$  quickly. Then, there exists  $\xi > \frac{1}{2}$  such that  $\{X^N\}$  and  $\{X_r^N\}$  have fast convergence to  $(\xi, 1]$  for every  $r$ .*

Our next result sharpens the message of the previous one by showing that there is a range of parameter values for which there is fast convergence with random sampling beliefs, but not when players observe the full state. Consider as above a family of decision rules  $g(x, \beta)$ , where  $1/\beta$  is a measure of the level of noise.

**Proposition 8.** *Consider a 2-action game with affine payoff functions  $\pi_i$  and let  $x^* \in (0, \frac{1}{2})$  be such that  $\text{sign}(\pi_1(x) - \pi_2(x)) = \text{sign}(x - x^*)$ . Suppose that the choice rule  $g$  can be written in the form  $g(x, \beta) = P[\beta(\pi_1(x) - \pi_2(x)) \geq \epsilon]$ , where  $\epsilon$  is a random variable with support  $\mathbb{R}$ ,  $\epsilon$  and  $-\epsilon$  have the same distribution, and  $P[\epsilon = 0] < 1 - 2x^*$ . Then there exist  $0 < \beta^* < \beta_r^*$  such that*

(i) *if  $\beta \in (0, \beta^*)$  then both models have fast convergence to the set  $(\frac{1}{2}, 1]$ .*

(ii) *if  $\beta \in (\beta^*, \beta_r^*)$  then the system with random sampling has fast convergence to  $(\frac{1}{2}, 1]$  but the system with full information does not.*

Intuitively, fast convergence requires a sufficient amount of noise, and random sampling provides an additional stochastic element without breaking the monotonicity needed to appeal to Proposition 6. The proof of Example 2 shows that the condition of this proposition applies to choice rules generated by the perturbed utility functions considered there.<sup>20</sup>

<sup>19</sup>We also have strengthened (11) in that we relax the assumption that  $\{X^N\}$  reaches  $(\frac{1}{2}, 1]$  quickly to the assumption that  $\{X^N\}$  reaches  $[x^*, 1]$  quickly.

<sup>20</sup>Note also that the assumption that the support of  $\epsilon$  is  $\mathbb{R}$  implies that  $g(x, \beta)$  is always strictly between 0 and 1, i.e. choice is always stochastic. This is needed for the result: when  $c(p) = p^2$  as in Rosenthal (1989) and Voorneveld (2006) the choice rule is deterministic at some states and as we show in the Appendix the conclusion of the proposition is false. In this case, there is a  $\beta^* > 0$  such that both models have fast convergence to  $(\frac{1}{2}, 1]$  if  $0 < \beta < \beta^*$  and the state 0 is an absorbing state for both models if  $\beta \geq \beta^*$ .

## 6 More complex state spaces

In the examples of Section 5, the state space of the Markov process is naturally taken to be  $\Delta$ , as the dynamics only depend on the population shares using each strategy. Many interesting models require a more complex state space. For example, the probability that a player adopts some strategy may depend on the player's position in a network or with whom he has been matched previously in addition to the current population shares. An attractive feature of our Lyapunov approach is that it also applies to a variety of such models. In this section we discuss two applications: one involving learning from personal experience with recency weights, and a variant of Ellison's (1993) two-neighbor circle model.

First, consider a finite-memory fictitious-play style learning model in which  $N$  agents are matched to play an  $m$ -action two-player game  $G$  at  $t = 0, \frac{1}{N}, \frac{2}{N}, \dots$  and learn only from play in their own interactions.<sup>21</sup> Assume that strategy 1 is  $p$ -dominant for some  $p < \frac{1}{2}$  and that agents remember the outcomes of their  $k$  most recent matches. Within each period the players are randomly matched and play the game using strategies they selected at the end of the previous period. One player is then randomly selected to update his or her strategy. The updating player does one of two things. With probability  $1 - \epsilon$  he selects a strategy which is a best response to a weighted average  $w_1 a_{-i,t} + w_2 a_{-i,t-1} + \dots + w_k a_{-i,t-(k-1)}$  of the actions  $a_{-i,t}, \dots, a_{-i,t-(k-1)}$  that his opponents have used in the most recent  $k$  periods. With probability  $\epsilon$  he selects a strategy uniformly at random. The selected strategy will be used until the player is next chosen to update.

Informally, a motivation for using recency weights is that agents would want to place more weight on more recent observations if they believed that their opponents' play was changing over time. There is ample experimental evidence suggesting that beliefs are indeed more heavily influenced by recent observations both in decision problems and in games; see Cheung and Friedman (1997) for one of the first measures of recency bias in a game theory experiment, and Erev and Haruvy (2013) for a survey of evidence for recency effects in experimental decision problems. Benaïm, Hofbauer and Hopkins (2009) and Fudenberg and Levine (2014) provide theoretical analyses of recency, but neither consider the large-population limit that is our focus here.

We show that this model has fast convergence to a  $p$ -dominant equilibrium if the weights

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<sup>21</sup>Ellison (1997) studied the finite-memory variant as well as the more traditional infinite-memory fictitious play model with the addition of a single rational player, and showed that the rational player could manipulate the play of a large population of opponents when one action was strongly risk-dominant. Most other studies of fictitious play assume that all agents in the same player role have the same beliefs. One exception is Fudenberg and Takahashi (2011), who allow each of  $N$  agents to have different beliefs; they focus on the asymptotics for a fixed  $N$  rather than the large-population limit.

place enough emphasis on recent actions. Note that this result does not require that noise levels be above some threshold, and that play converges to an arbitrarily small neighborhood of the selected equilibrium when the level of noise is sufficiently low. To state the result formally, note that the model defines a Markov process  $S^N(t)$  on the state space  $\Omega_N = \{1, 2, \dots, m\}^{kN+N}$ : the first  $kN$  components of the state vector record what each player saw in the periods  $t - \frac{1}{N}, t - \frac{2}{N}, \dots, t - \frac{k}{N}$ ; and the last  $N$  components record the action that each player has selected for use in period  $t$ . Given any state  $S^N(t)$  we can define a random variable  $X^N(t) \in \Delta$  to be the fraction of players who have each pure strategy as their selected action in state  $S^N(t)$ .

**Example 6.** *Consider the model above with  $k > 1$ . Suppose that strategy 1 is  $p$ -dominant in  $G$  and the recency weights satisfy  $w_1 > p$  and  $w_2 + w_3 > p$ . Then, for any  $\epsilon > 0$  the model described above has fast convergence to  $\{x \in \Delta | x_1 > 1 - 1.2\epsilon\}$ .*

**Remarks.**

1. Even though the state space in this model grows with the number of players, we can use the share  $x_1$  of strategy 1 as the Lyapunov function here.
2. The result implies there can be fast convergence even with long memories and moderate levels of  $p$ -dominance provided that players place substantial weights on their most recent experiences. For example, if players place weight proportional to  $(\frac{2}{3})^{n-1}$  on their  $n$ th most recent observation, then  $w_1 > 1 / (1 + \frac{2}{3} + \frac{4}{9} + \dots) = \frac{1}{3}$  and  $w_2 + w_3 > \frac{2}{9} + \frac{4}{27} > \frac{1}{3}$ , so there is fast convergence to a neighborhood of strategy 1 if strategy 1 is  $\frac{1}{3}$ -dominant.
3. Note moreover that the noise from sampling is sufficient for fast convergence, the  $\epsilon$  error probability can be arbitrarily small provided it is bounded away from 0 uniformly in  $N$ : if there is any probability at all that the risk dominant action is played, the recency weighting will guarantee that its share grows.
4. Another form of recency weighting is to completely ignore all observations from more than  $k$  periods ago but weight all of the last  $k$  periods equally, so that  $w_n = \frac{1}{k}$  for  $n = 1, 2, \dots, k$ . In this case the result implies there is fast convergence to a neighborhood of the state where everyone plays strategy 1 if strategy 1 is  $\frac{1}{k}$ -dominant. This implies that we always have convergence to a risk-dominant equilibrium if memories are short enough (but longer than one period).



**Proof of Example 6.** Consider the two-period ahead dynamics of the model. By Propositions 1 and 2 it suffices to show that we can find  $c > 0$  for which

$$\inf_{s \in \phi_N^{-1}(x)} E[(X_1^N(2/N) - X_1^N(0)) | X^N(0) = x, S^N(t) = s] \geq \frac{c}{N}$$

for all  $x$  with  $x_1 \in [0, 1 - 1.2\epsilon]$ .

We can evaluate the change in the popularity of strategy 1 by counting every time a player playing strategy 1 is selected to update as a loss of 1 and every adoption by an updating player as a gain of 1. The expected losses are  $x_1$  from revisions at  $t = 0$  and at most  $(x_1 + \frac{1}{N})$  from revisions at  $t = \frac{1}{N}$ . There will be a gain without a mutation from the period  $t$  revision if the player selected to update at  $t$  is matched with a player who uses strategy 1 in period  $t$ , or if he saw strategy 1 in both periods  $t - \frac{2}{N}$  and  $t - \frac{1}{N}$ . At  $t = 0$  only the former is guaranteed to be possible for all  $s$  – the worst case state is that the matching was such that the sets of players who saw strategy 1 in periods  $t - \frac{1}{N}$  and  $t - \frac{2}{N}$  are disjoint – so all we can say is that the expected number of adoptions is at least  $x_1(1 - \epsilon)$ . But the  $t = \frac{1}{N}$  revision will produce an adoption of the latter type if the player who saw strategy 1 in period  $t = -\frac{1}{N}$  is randomly matched with a player playing strategy 1 at  $t = 0$  and is then randomly selected to update at  $t = \frac{1}{N}$ . So the expected number of non-mutation adoptions is at least  $((x_1 - \frac{1}{N}) + (1 - x_1 + \frac{1}{N})x_1(x_1 - \frac{1}{N})) (1 - \epsilon)$ . And there are  $\frac{\epsilon}{2}$  adoptions in expectation due to mutations in each of the two periods.

Adding all of these changes together and ignoring all of the  $\frac{1}{N}$  terms, it suffices by Corollary 2.1 to show that there is a  $c > 0$  for which

$$\phi(x_1) := -2x_1 + [2x_1 + (1 - x_1)x_1^2](1 - \epsilon) + \epsilon \geq c \text{ for all } x_1 \in [0, 1 - 1.2\epsilon].$$

If  $x_1 \in [0, \frac{1}{3}]$ , then  $\phi(x_1) \geq -2x_1 + 2x_1(1 - \epsilon) + \epsilon = (1 - 2x_1)\epsilon > 0$ . In the interval  $[\frac{1}{3}, 1]$ ,  $\phi$  is concave with  $\phi(\frac{1}{3}) > 0$ , and a numerical calculation shows that  $\phi(1 - 1.2\epsilon) = 0.2\epsilon - 1.68\epsilon^2 + 4.608\epsilon^3 - 1.728\epsilon^4 > 0$ . Thus  $\phi(x_1) > 0$  if  $x_1 \in [\frac{1}{3}, 1 - 1.2\epsilon]$ . Hence  $\phi$  is positive in  $[0, 1 - 1.2\epsilon]$  and the claim follows with  $c := \min\{\phi(x_1) : 0 \leq x_1 \leq 1 - 1.2\epsilon\}$ .  $\square$

**Remark.** The result could be strengthened to show that there is fast convergence to a somewhat smaller set by considering  $k$ -period ahead transitions instead of the two-period ahead transitions considered in the proof.

Now consider a variant of the two-neighbor local interaction model of Ellison (1993).<sup>22</sup>

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<sup>22</sup>The model differs from that of Ellison (1993) in that we consider a version where one randomly chosen player at a time considers revising, whereas Ellison (1993) assumes that all players revise simultaneously.

$N$  players arranged around a circle are choosing strategies for a  $2 \times 2$  coordination game. Assume that at each  $t = 0, \frac{1}{N}, \frac{2}{N}, \dots$  one player is chosen at random to consider revising his or her strategy. The selected player plays a best response to the average play of her two immediate neighbors with probability  $1 - \epsilon$  and plays the opposite strategy with probability  $\epsilon$ .

The description above defines a Markov process  $S^N(t)$  on the state space  $\Omega_N = \{1, 2\}^N$  which consists of vectors describing whether each player uses strategy 1 or strategy 2. Given any state  $S^N(t)$  we define  $X^N(t)$  to be the two-vector with the first component  $X_1^N(t)$  giving the fraction of players who have strategy 1 as a best response to their neighbors' actions in  $S^N(t)$  and with second component  $X_2^N(t)$  giving the fraction of players using strategy 1.

In this example, the share using strategy 1 can be used as a Lyapunov function to show that there is fast convergence to the set of states where at least a  $\frac{1}{2} - \delta$  share of the players use strategy 1. But it will not suffice for showing convergence to a smaller neighborhood of the risk dominant equilibrium, because in the state  $(1, 2, 1, 2, \dots, 1, 2)$  every player will switch to the opposite strategy with probability  $1 - \epsilon$ , so the expected share playing 1 is constant instead of increasing. The expected number of agents who have 1 as a best response is strictly increasing in that state. But that candidate Lyapunov function also does not suffice because it has zero expected change at some states including  $(2, 1, 2, 2, 1, 2, \dots, 2, 1, 2)$ . Adding these two functions with the appropriate weights does provide a valid Lyapunov function. Part (a) of the result below exploits this to say that there is fast convergence to a set of states where most players have strategy 1 as their best response. Part (b) provides an explicit bound on the convergence time that holds for any  $N$ .<sup>23</sup> Part (c) notes further that the system quickly reaches a state where most players are using strategy 1. The intuition for this is that the system cannot remain long in the set of states where most players have strategy 1 as a best response without having most players adopt strategy 1. The proof uses our result on multistep convergence, Proposition 5.

**Example 7.** *In the model above suppose that strategy 1 is risk dominant. Then,*

(a)  $\{X^N\}$  has fast convergence to  $\{x \in [0, 1]^2 : x_1 + 3\epsilon x_2 \geq 1 - 4\epsilon\}$ .

(b) For every  $N > 2$  and every initial state, the expected time until at least a  $1 - 4\epsilon$  fraction of the players have strategy 1 as a best response is at most  $2\epsilon^{-2}$ .

(c)  $\{X_2^N\}$  reaches  $(1 - 8\epsilon, 1]$  quickly.

**Proof of Example 7.** (a) Define  $V : [0, 1]^2 \rightarrow \mathbb{R}$  by  $V(x) = -(x_1 + 3\epsilon x_2)$ . Let  $A = \{x \in [0, 1]^2 : V(x) \leq 4\epsilon - 1\}$ . The Appendix contains a proof of the lemma below which says

<sup>23</sup>This result improves on the result in Ellison (1993) in providing an explicit bound on the convergence time and in providing a bound that applies for all  $\epsilon$ , not just sufficiently small  $\epsilon$ .

that this is a valid Lyapunov function:

**Lemma 1.** *If  $N > 2$  and  $s$  is any state in which the fraction of players who have strategy 2 as a best response is at least  $4\epsilon$  then*

$$E_s \left[ V \left( X^N \left( \frac{1}{N} \right) \right) - V \left( X^N(0) \right) \right] \leq -\frac{\epsilon^2}{N}.$$

Hence, Corollary 2.1 implies the system has fast convergence to  $A$ .

(b) The bound on the expected time until at least a  $1 - 4\epsilon$  fraction of the players have the risk-dominant action as a best response follows from Proposition 1 applied to the nonnegative Lyapunov function  $2 + V(x)$ .

(c) We apply Proposition 5 with  $\mathcal{X} = [0, 1]^2$ ,

$$\begin{aligned} B &= \{x \in [0, 1]^2 : x_1 + 3\epsilon x_2 > 1 - 4\epsilon - \epsilon^2\}, \\ A &= \{x \in B : x_2 > 1 - 8\epsilon\}, \\ C &= \{x \in [0, 1]^2 : x_1 + 3\epsilon x_2 \geq 1 - 4\epsilon\}. \end{aligned}$$

By part (a),  $\{X^N\}$  has fast convergence to  $C$ . We use Proposition 1 with  $V(x_1, x_2) = 1 - x_2$  to show that  $\{X^N\}$  reaches  $A \cup B^c$  quickly. The expectation of the increment  $X_2^N(\frac{1}{N}) - X_2^N(0)$  can be written as  $\frac{1}{N}$  times the probability that the selected player chooses action 1 minus the probability that the selected player had previously been using action 1. If  $s$  is a state with  $(x_1, x_2) \in A^c \cap B$ , then  $x_2 \leq 1 - 8\epsilon$  and  $x_1 \geq 1 - 7\epsilon$  and it follows that

$$E_s \left[ X_2^N \left( \frac{1}{N} \right) - X_2^N(0) \right] \geq \frac{1}{N} [(1 - \epsilon)x_1 - x_2] \geq \frac{1}{N} [(1 - \epsilon)(1 - 7\epsilon) - (1 - 8\epsilon)] = \frac{7\epsilon^2}{N}.$$

Thus,  $\{X^N\}$  reaches  $A \cup B^c$  quickly. By Proposition 5,  $\{X^N\}$  reaches  $A$  quickly. This implies that  $\{X_2^N\}$  reaches  $(1 - 8\epsilon, 1]$  quickly.  $\square$

In addition to these examples, we conjecture that our results could be fruitfully applied to the network models of Kreindler and Young (2014), where the dimension of the state space is the size of the population.

## 7 Related Literature

Ellison (1993) raises the issue of slow convergence; it shows that in the model of Kandori, Mailath, and Rob (1993) the rate of convergence slows as the noise level converges to zero,

and that the expected waiting time to reach the long-run equilibrium grows exponentially as the population size  $N$  increases. Blume (1993) provides conditions under which an infinite-population local interaction model is ergodic; evolution can be thought of as fast for large  $N$  if it still occurs when  $N = \infty$ . Möbius (2000) defines a model as “clustering” on a set of states  $A$  if the probability of  $A$  under the ergodic distribution converges to 1 as  $N \rightarrow \infty$ , and analyzes the limit of the worst case expected wait to reach  $A$ . Kreindler and Young (2013) defines a concept of “fast selection” which is roughly equivalent to what we call “reaches  $A$  quickly.” Relative to this the novelty in our concept consists of adding the “leaves  $A$  slowly” requirement.

There is also a sizable literature on factors that promote fast evolution, including local interaction (Ellison (1993), Blume (1993), Young (2011)), personal experience (Ellison (1997)), endogenous location choice (Ely (2002)), homophily (Golub and Jackson (2012)), and population structure (Hauert, Chen and Imhof (2014)). Samuelson (1994) and Ellison (2000) note that evolution is more rapid when it can proceed via a series of smaller steps. Kreindler and Young (2014) provides sufficient conditions on payoffs and noise for evolution to be fast on arbitrary networks.

The methodological parts of our paper are related to the literature on stochastic stability with constant step size, where the dimension of the Markov process is held fixed as the population grows.<sup>24</sup> Benaïm and Weibull (2003) considers models where all agents have the same beliefs (which is why the state space doesn’t depend on the population size) and a random player from an  $N$  agent population is chosen to revise his play at  $t = \frac{1}{N}, \frac{2}{N}, \frac{3}{N}, \dots$ . The paper shows that the maximum deviation between the finite  $N$  model and its continuous limit over a  $T$  period horizon goes to zero as  $N \rightarrow \infty$ , and also that the probability of exiting the basin of attraction of an attractor of the continuous model by time  $T$  goes to zero (exponentially fast) as the population size increases; this is related to our “leaving slowly” results. Roth and Sandholm (2013) develops more general results along these lines, and provides conditions that imply that play in the discrete-time model remains within  $\epsilon$  of the dynamics of the continuous-time model for at least  $T$  periods with probability that goes to one as  $N$  grows. The results can provide an alternate method for establishing some of our applied results in cases where the state space remains the same as the population grows.

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<sup>24</sup>Kaniovski and Young (1995) and Benaïm and Hirsch (1999) provide results connecting discrete-time and continuous-time limit dynamics in the context of fictitious-play style models, but the results themselves are not closely related because the approximation occurs in the  $t \rightarrow \infty$  limit with the population size held fixed.

## 8 Conclusion

We defined a notion of fast convergence for evolutionary models, which refines the previous literature by requiring that some set  $A$  is both reached quickly and left slowly. We then used Lyapunov functions to give sufficient conditions for fast convergence. One advantage of our approach is that it can apply to models with state spaces that do not have a finite-dimensional continuous time limit dynamic. At a conceptual level, our proofs provide a unified way of viewing examples that had previously been handled with a variety of specific techniques, and lead to proofs that directly highlight the reason for fast convergence. Our approach also separates the factors sufficient for fast convergence from technical conditions needed to assure a well-behaved approach to the mean field, and allow us to handle models with more complex state spaces where it is not clear how existing results on approximation by the mean field could be applied. We illustrated the use of our conditions in various examples without presenting quantitative results, but the simulations of e.g. Ellison (1993) and Kreindler and Young (2013) make us optimistic that when convergence is fast in our sense it will be fast enough to be of practical importance.

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## Appendix

**Proof of Example 1.** Let  $s$  be any state in which a fraction  $x_1$  of the players play strategy 1. The expected change in the fraction of players using strategy 1 will be

$$E(X_1(t + \frac{1}{N}) - x_1 | S^N(t) = s) = \frac{1}{N} [y(1 - \epsilon) + (1 - y)\epsilon - x_1],$$

where  $y$  is the expected fraction of players who have strategy 1 as a best response in state  $s$ . Note that the RHS can be reorganized as

$$\begin{aligned} y(1 - \epsilon) + (1 - y)\epsilon - x_1 &= y(1 - 2\epsilon) + \epsilon - x_1 \\ &= (y - x_1)(1 - 2\epsilon) + \epsilon(1 - 2x_1) \end{aligned}$$

Any state  $s$  with fraction  $x_1$  players playing strategy 1 will have fraction  $r$  players with two neighbors playing strategy 1, fraction  $2(x_1 - r)$  with one neighbor playing strategy 1, and fraction  $(1 - 2x_1 + r)$  with no neighbor playing strategy 1 for some  $r \in [0, x_1]$ . The value of  $y$  depends on the state  $s$  only through  $x_1$  and  $r$ . Because players with two neighbors playing strategy 1 will always have strategy 1 as their best response, those with one neighbor playing strategy 1 will have strategy 1 as their best response if at least one player they randomly sample uses strategy 1, and those with no neighbors playing strategy 1 must have both players in their sample using strategy 1, we have  $y = r + 2(x_1 - r)(2x_1 - x_1^2) + (1 - 2x_1 + r)x_1^2$ . Writing  $x_1 = r + (x_1 - r)$  and collecting terms gives

$$\begin{aligned} y - x_1 &= (x_1 - r)(-1 + 4x_1 - 2x_1^2) + ((1 - x_1) - (x_1 - r))x_1^2 \\ &= (x_1 - r)(-1 + 4x_1 - 3x_1^2) + (1 - x_1)x_1^2 \end{aligned}$$

Plugging back into the formula for the change in  $x_1$  gives

$$\begin{aligned} NE(X_1(t + \frac{1}{N}) - x_1 | S^N(t) = s) &= ((x_1 - r)(-1 + 4x_1 - 3x_1^2) + (1 - x_1)x_1^2) (1 - 2\epsilon) + \epsilon(1 - 2x_1), \end{aligned}$$

We show that the RHS can be bounded below by some positive constant  $c$  by considering three cases.

First, for  $x_1 \in [0, \frac{1}{3}]$  the quadratic  $(-1 + 4x_1 - 3x_1^2) = (3x_1 - 1)(1 - x_1)$  is negative. Hence, the RHS is minimized for  $r = 0$  in which case it is equal to

$$(-x_1 + 5x_1^2 - 4x_1^3)(1 - 2\epsilon) + (1 - 2x_1)\epsilon = (-x_1 + 5x_1^2 - 4x_1^3) + (1 - 10x_1^2 + 8x_1^3)\epsilon$$

The polynomial  $-x_1 + 5x_1^2 - 4x_1^3$  is only negative if  $x_1$  is additionally less than  $\frac{1}{4}$  and  $1 - 10x_1 + 8x_1^3$  is positive in this case, so the RHS will be bounded away from zero for  $x_1 \in [0, \frac{1}{3}]$  if  $\epsilon$  is chosen to be greater than

$$\sup_{x_1 \leq \frac{1}{4}} \frac{-x_1 + 5x_1^2 - 4x_1^3}{1 - 10x_1^2 + 8x_1^3}.$$

Evaluating this numerically shows that choosing  $\epsilon = 0.065$  suffices.

Second for  $x_1 \in [\frac{1}{3}, \frac{1}{2}]$  the expected change in  $x_1$  is minimized for  $r = x_1$  in which case it is simply  $(1 - x_1)x_1^2(1 - 2\epsilon) + \epsilon(1 - 2x_1)$ . This is obviously bounded away from zero for all  $\epsilon < \frac{1}{2}$ .

Finally, for  $x_1 \in [\frac{1}{2}, 1 - \ell\epsilon]$  the minimum again occurs for  $r = x_1$  and the value is again  $(1 - x_1)x_1^2(1 - 2\epsilon) + \epsilon(1 - 2x_1)$  which expands as  $x_1^2 - x_1^3 + (1 - 2x_1 - 2x_1^2 + 2x_1^3)\epsilon$ . The first term is positive and the second negative, so for each  $x_1$  the expression is minimized by choosing  $\epsilon$  as large as possible given  $x_1$ :  $\epsilon = (1 - x_1)/\ell$ . Factoring out the  $1 - x_1$  we find that this holds for all  $x_1$  in the range if  $\ell$  is chosen to be greater than  $\sup_{x_1 > 0.5} \frac{-2x_1^3 + 2x_1^2 + 2x_1 - 1}{x^2}$ . The maximum is about 1.42.  $\square$

**Proof of Proposition 2.** To show that condition (2) holds for every open set  $U$  containing  $\bar{A}$ , it suffices to show that this is true for every open  $U$  with  $\bar{A} \subset U \subset U_0$  and  $U \neq \mathcal{X}$ . Given this restriction, condition (5) always holds until the process has left  $U$ . Since  $V$  is Lipschitz continuous and the increments of  $X^N$  are bounded by  $\frac{K}{N}$ , there is a constant  $\kappa$  such that  $P_s(|V(X^N(\frac{1}{N})) - V(X^N(0))| \leq \frac{\kappa}{N}) = 1$  for all  $s \in \Omega_N$  and all  $N$ . Using this bound, Taylor's formula implies that

$$E_s e^{\delta N[V(X^N(\frac{1}{N})) - V(X^N(0))]} = 1 + \delta N E_s \left[ V \left( X^N \left( \frac{1}{N} \right) \right) - V(X^N(0)) \right] + R_{N,s},$$

where  $|R_{N,s}| \leq \frac{1}{2} \delta^2 \kappa^2 e^{\delta \kappa}$ . Pick  $\delta_0 > 0$  so that  $\delta_0 \kappa^2 e^{\delta_0 \kappa} < c$ . Let  $Y^N(t) = \exp(\delta_0 N V(X^N(t)))$  and  $Z^N(t) = Y^N(t \wedge \tau_{\bar{A}}^N \wedge \tau_{\mathcal{X} \setminus U}^N)$ . Suppose  $N \geq N_0$ . Then, by (5), for every  $s \in \phi_N^{-1}(U \setminus \bar{A})$ ,

$$\begin{aligned} E_s Y^N \left( \frac{1}{N} \right) &= e^{\delta_0 N V(\phi_N(s))} E_s e^{\delta_0 N [V(X^N(\frac{1}{N})) - V(X^N(0))]} \\ &\leq E_s Y^N(0) \left( 1 - \delta_0 c + \frac{1}{2} \delta_0^2 \kappa^2 e^{\delta_0 \kappa} \right) \leq E_s Y^N(0). \end{aligned}$$

Thus,  $\{Z^N(t)\}$  is a nonnegative supermartingale.

Let  $v_1 = \max_{x \in \bar{A}} V(x)$ ,  $v_3 = \min_{x \in \mathcal{X} \setminus U} V(x)$ . By (4),  $v_1 < v_3$ . Let  $v_2 \in (v_1, v_3)$  and  $\gamma = \frac{1}{2} \delta_0 (v_3 - v_2)$ . If  $s \in \Omega_N$  and  $V(\phi_N(s)) \leq v_2$ , then, by the maximal inequality (see e.g.

Shiryaev (1996), page 493),

$$\begin{aligned} P_s(\tau_{\mathcal{X} \setminus U}^N < \tau_A^N) &\leq P_s\left(\sup_{t \geq 0} Z^N(t) \geq e^{\delta_0 N v_3}\right) \leq e^{-\delta_0 N v_3} E_s Z^N(0) \\ &= e^{-\delta_0 N v_3} e^{\delta_0 N V(\phi_N(s))} \leq e^{\delta_0 N (v_2 - v_3)} = e^{-2\gamma N}. \end{aligned} \quad (12)$$

Let  $\sigma_{-1}^N = -1$ . Define stopping times  $\sigma_0^N \leq \sigma_1^N \leq \dots$  by

$$\sigma_k^N = \inf \left\{ t \in \frac{1}{N} \mathbb{N}_0 : t > \sigma_{k-1}^N, X^N(t) \in \bar{A} \cup (\mathcal{X} \setminus U) \right\}, \quad k \in \mathbb{N}_0.$$

In view of (5),  $P_s(V(X^N(\frac{1}{N})) < V(X^N(0))) > 0$  for all  $s \in \phi_N^{-1}(U \setminus \bar{A})$ . Thus,  $P_s(\tau_A^N < \infty) > 0$  for all  $s \in \phi_N^{-1}(U \setminus \bar{A})$ . This implies that  $P_s(\tau_A^N \wedge \tau_{\mathcal{X} \setminus U}^N < \infty) = 1$  for all  $s \in \Omega_N$ , see Durrett (1996), page 290. Hence  $P_s(\sigma_k^N < \infty \text{ for all } k) = 1$ .

Assume from now on that  $N \geq N_0$  is so large that  $v_1 + \frac{\kappa}{N} \leq v_2$ . Then for every  $s \in \phi_N^{-1}(\bar{A})$ ,  $P_s(V(X^N(\frac{1}{N})) \leq v_2) = 1$  and so, by (12)

$$\begin{aligned} P_s(X^N(\sigma_1^N) \in \bar{A}) &= \sum_{\xi \in \Omega_N: V(\phi_N(\xi)) \leq v_2} P_\xi(X^N(\sigma_0^N) \in \bar{A}) P_s\left(S^N\left(\frac{1}{N}\right) = \xi\right) \\ &\geq 1 - e^{-2\gamma N}. \end{aligned}$$

Hence, for  $s \in \phi_N^{-1}(\bar{A})$  and  $k = 0, 1, \dots$ ,

$$P_s(N\tau_{\mathcal{X} \setminus U}^N > k) \geq P_s(X^N(\sigma_j^N) \in \bar{A}, 0 \leq j \leq k) \geq (1 - e^{-2\gamma N})^k.$$

It follows that for all  $T \in [0, \infty)$ ,

$$P_s(\tau_{\mathcal{X} \setminus U}^N > T) = P_s(N\tau_{\mathcal{X} \setminus U}^N > \lfloor NT \rfloor) \geq (1 - e^{-2\gamma N})^{NT} \geq \exp(-Te^{-\gamma N}),$$

provided that  $N$  is also so large that  $2Ne^{-\gamma N} \leq 1$ . Here  $\lfloor NT \rfloor$  denotes the largest integer  $\leq NT$ . In the last step, it was used that  $1 - u \geq e^{-2u}$  for  $u \in [0, \frac{1}{2}]$ .  $\square$

The proofs of the results on multi-step evolution in Propositions 4 and 5 use the following lemma.

**Lemma 2.** *Let  $A, B, C \subset \mathcal{X}$  with  $A \subset B$ . Let  $N \in \mathbb{N}$ ,  $K \in (0, \infty)$ , and  $c \in (0, 1)$ . Suppose that*

$$E_s \tau_{A \cup B^c}^N \leq K \text{ and } E_s \tau_{A \cup C}^N \leq K \text{ for all } s \in \Omega_N \quad (13)$$

and

$$P_s(\tau_A^N < \tau_{B^c}^N) \geq c \text{ for all } s \in \phi_N^{-1}(C). \quad (14)$$

Then  $E_s \tau_A^N \leq 2K/c$  for all  $s \in \Omega_N$ .

**Proof.** Let  $\sigma_0 = 0$  and for  $j = 0, 1, \dots$ ,

$$\begin{aligned}\sigma_{2j+1} &= \inf\{t \geq \sigma_{2j} : X^N(t) \in A \cup C\}, \\ \sigma_{2j+2} &= \inf\{t \geq \sigma_{2j+1} : X^N(t) \in A \cup B^c\}.\end{aligned}$$

Condition (13) implies that  $P_s(\sigma_j < \infty \text{ for all } j) = 1$  for every  $s \in \Omega_N$ . Let  $J = \inf\{j \in \mathbb{N}_0 : X^N(\sigma_j) \in A\}$ . Then, by Fubini,

$$E_s \tau_A^N = E_s \sum_{j=1}^J (\sigma_j - \sigma_{j-1}) = \sum_{j=1}^{\infty} E_s [(\sigma_j - \sigma_{j-1}) 1_{\{J \geq j\}}].$$

Let  $\mathcal{F}_t$  denote the  $\sigma$ -algebra generated by  $S^N(0), \dots, S^N(t)$ , let  $\mathcal{F}$  denote the  $\sigma$ -algebra generated by  $\cup_t \mathcal{F}_t$ , and let  $\mathcal{F}_{\sigma_j}$  denote the  $\sigma$ -algebra up to time  $\sigma_j$ , that is,  $\mathcal{F}_{\sigma_j} = \{F \in \mathcal{F} : F \cap \{\sigma_j = t\} \in \mathcal{F}_t \text{ for all } t\}$ . For every  $j \geq 1$ ,

$$\begin{aligned}\{J \geq j\} &= \{X^N(\sigma_0) \notin A, \dots, X^N(\sigma_{j-1}) \notin A\} \\ &= \{S^N(\sigma_0) \notin \phi_N^{-1}(A), \dots, S^N(\sigma_{j-1}) \notin \phi_N^{-1}(A)\},\end{aligned}$$

so that  $\{J \geq j\} \in \mathcal{F}_{\sigma_{j-1}}$ . Hence,

$$\begin{aligned}E_s [(\sigma_j - \sigma_{j-1}) 1_{\{J \geq j\}}] &= E_s [E[(\sigma_j - \sigma_{j-1}) 1_{\{J \geq j\}} | \mathcal{F}_{\sigma_{j-1}}]] \\ &= E_s [1_{\{J \geq j\}} E[\sigma_j - \sigma_{j-1} | \mathcal{F}_{\sigma_{j-1}}]].\end{aligned}$$

By (13) and the strong Markov property,

$$E[\sigma_j - \sigma_{j-1} | \mathcal{F}_{\sigma_{j-1}}] = \begin{cases} E_{S^N(\sigma_{j-1})} \tau_{A \cup C}^N \leq K & \text{if } j \text{ is odd,} \\ E_{S^N(\sigma_{j-1})} \tau_{A \cup B^c}^N \leq K & \text{if } j \text{ is even.} \end{cases}$$

Thus,  $E_s \tau_A^N \leq K \sum_{j=1}^{\infty} P_s(J \geq j) \leq 2K \sum_{j=0}^{\infty} P_s(J \geq 2j + 1)$ . If  $P_s(J \geq 2j + 1) > 0$ , then

$$\begin{aligned}P_s(J \geq 2j + 1) &= P_s(X^N(\sigma_k) \notin A \text{ for } k = 0, \dots, 2j) \\ &= \prod_{k=1}^{2j} P_s(X^N(\sigma_k) \notin A | X^N(\sigma_{\kappa}) \notin A \text{ for } \kappa = 0, \dots, k-1).\end{aligned}$$

When  $k$  is even, the conditional probability is at most  $1 - c$  by (14). Hence  $E_s \tau_A^N \leq 2K \sum_{j=0}^{\infty} (1 - c)^j = 2K/c$ .  $\square$

**Proof of Proposition 4.** Since  $\{X^N\}$  reaches  $B$  and  $A \cup B^c$  quickly, there exist  $N_1 \in \mathbb{N}$  and  $K < \infty$  so that for all  $N \geq N_1$  and all  $s \in \Omega_N$ ,  $E_s \tau_B^N \leq K$  and  $E_s \tau_{A \cup B^c}^N \leq K$ . If

$N \geq N_0$ , then, by (6),  $P_s(\tau_A^N < \tau_{B^c}^N) \geq c$  for every  $s \in \phi_N^{-1}(B \setminus A)$ . Lemma 2 with  $C = B \setminus A$  shows that  $E_s \tau_A^N \leq 2K/c$  for all  $s \in \Omega_N$ , provided  $N \geq \max\{N_0, N_1\}$ .  $\square$

**Proof of Proposition 5.** Since  $\{X^N\}$  reaches  $C$  and  $A \cup B^c$  quickly, there exist  $N_1 \in \mathbb{N}$  and  $K < \infty$  so that for all  $N \geq N_1$  and all  $s \in \Omega_N$ ,  $E_s \tau_{A \cup C}^N \leq E_s \tau_C^N \leq K$  and  $E_s \tau_{A \cup B^c}^N \leq K$ . The last inequality implies by Markov's inequality that  $P_s(\tau_{A \cup B^c}^N \geq 2K) \leq \frac{1}{2}$ . Since  $\{X^N\}$  leaves  $C$  slowly and  $\bar{C} \subset \text{int} B$  there exists  $N_2 > N_1$  so that for all  $N \geq N_2$ ,

$$P_s(\tau_{B^c}^N \leq 2K) \leq \frac{1}{4} \text{ for all } s \in \phi_N^{-1}(C).$$

Thus, if  $N \geq N_2$ , then for every  $s \in \phi_N^{-1}(C)$ ,

$$P_s(\tau_A^N \geq \tau_{B^c}^N) = P_s(\tau_{B^c}^N = \tau_{A \cup B^c}^N) \leq P_s(\tau_{A \cup B^c}^N \geq 2K \text{ or } \tau_{B^c}^N \leq 2K) \leq \frac{3}{4}.$$

It now follows by Lemma 2 that  $E_s \tau_A^N \leq 8K$  for all  $s \in \Omega_N$  if  $N \geq N_2$ .  $\square$

**Proof of Example 2.** The result can again be shown by applying Proposition 6(a), but instead we apply Proposition 8 to also justify the claim at the end of Subsection 5.3. The probability  $f_1(x, \beta)$  is uniquely determined by  $\beta[\pi_1(x) - \pi_2(x)] = \psi(f_1(x, \beta))$ , where  $\psi(p) := c'(p) - c'(1-p)$ ,  $0 < p < 1$ . The function  $\psi$  is continuous and strictly increasing,  $\lim_{p \rightarrow 0^+} \psi(p) = -\infty$ , and  $\lim_{p \rightarrow 1^-} \psi(p) = \infty$ . The inverse function  $\psi^{-1}$  is therefore a continuous strictly increasing function on  $\mathbb{R}$  with  $\lim_{u \rightarrow -\infty} \psi^{-1}(u) = 0$  and  $\lim_{u \rightarrow \infty} \psi^{-1}(u) = 1$ . Let  $\epsilon$  be a random variable that has  $\psi^{-1}$  as its distribution function. Then the support of  $\epsilon$  is  $\mathbb{R}$ ,  $P(\epsilon = 0) = 0$ , and  $f_1(x, \beta) = \psi^{-1}(\beta[\pi_1(x) - \pi_2(x)]) = P(\beta[\pi_1(x) - \pi_2(x)] \geq \epsilon)$ . Since  $\psi(1-p) = -\psi(p)$  for all  $p$ ,  $1 - \psi^{-1}(u) = \psi^{-1}(-u)$  for all  $u$ , which implies that  $\epsilon$  and  $-\epsilon$  have the same distribution. The assertion now follows from Proposition 8.  $\square$

**Proof of Example 3.** For every  $x \in \Delta$ ,

$$\sum_{i=2}^m e^{\beta \alpha_i x_i} \leq \sum_{i=2}^m e^{\beta \alpha_2 x_i} \leq m - 2 + e^{\beta \alpha_2 (1-x_1)} \leq m - 2 + e^{\beta \alpha_2},$$

where the second inequality follows from the Schur-convexity of  $\sum_{i=2}^m e^{\beta \alpha_2 x_i}$ , see e.g. Marshall and Olkin (1979), page 64. Hence,  $f_1(x) \geq h(x_1)$ , where

$$h(x_1) = \frac{1}{1 + (m - 2 + e^{\beta \alpha_2}) e^{-\beta \alpha_1 x_1}}.$$

Since, by (7),  $\beta \alpha_1 > 2(m - 2 + e^{\beta \alpha_2})/e$  and  $e^u \geq eu$  for all  $u \geq 0$ ,  $e^{\beta \alpha_1 x_1} > 2(m - 2 + e^{\beta \alpha_2}) x_1$  for  $x_1 > 0$ . Consequently,

$$h(x_1) > \frac{1}{1 + \frac{1}{2x_1}} \geq x_1$$

for  $0 < x_1 \leq \frac{1}{2}$ . Since  $h$  is continuous and  $h(0) > 0$ , it follows that there exists  $c > 0$  so that  $f_1(x) - x_1 \geq h(x_1) - x_1 \geq c$  for all  $x \in \Delta$  with  $x_1 \leq \frac{1}{2}$ . The assertion follows from Proposition 6(a).  $\square$

**Proof of Example 4.** (a) For every  $x \in \Delta$ ,

$$\sum_{j \neq 2} e^{\beta \alpha_j x_j} \leq \sum_{j \neq 2} e^{\beta \alpha_1 x_j} \leq (m-2) + e^{\beta \alpha_1 (1-x_2)},$$

where the second inequality follows from the Schur-convexity of  $\sum_{j \neq 2} e^{\beta \alpha_1 x_j}$ . Thus,

$$\frac{1}{f_2(x)} = 1 + e^{-\beta \alpha_2 x_2} \sum_{j \neq 2} e^{\beta \alpha_j x_j} \leq 1 + (m-2)e^{-\beta \alpha_2 x_2} + e^{\beta[\alpha_1(1-x_2) - \alpha_2 x_2]}.$$

If  $x_2 = 1 - \alpha_2/(4\alpha_1)$ , then  $x_2 > \frac{3}{4}$  and

$$\alpha_1(1-x_2) - \alpha_2 x_2 < \frac{1}{4}\alpha_2 - \frac{3}{4}\alpha_2 = -\frac{1}{2}\alpha_2,$$

and it follows that

$$\frac{1}{f_2(x)} < 1 + (m-2)e^{-\frac{3}{4}\beta\alpha_2} + e^{-\frac{1}{2}\beta\alpha_2}. \quad (15)$$

Hence, if  $\beta$  satisfies (8), then for every  $x$  with  $x_2 = 1 - \alpha_2/(4\alpha_1)$ ,

$$f_2(x) > \frac{1}{1 + (m-1)e^{-\frac{1}{2}\beta\alpha_2}} \geq \frac{1}{1 + (4\alpha_1/\alpha_2 - 1)^{-1}} = 1 - \frac{\alpha_2}{4\alpha_1}.$$

This implies by Proposition 6(c) that  $\{X^N\}$  leaves  $\{x : x_2 > 1 - \alpha_2/(4\alpha_1)\}$  slowly. As  $1 - \alpha_2/(4\alpha_1) > \frac{3}{4}$ , it follows that  $\{X^N\}$  does not reach  $\{x : x_1 > \frac{1}{2}\}$  quickly.

(b) The inequality  $e^u \geq 1 + u$  implies that for all  $x \in \Delta$ ,

$$\frac{1}{f_1(x)} = 1 + \sum_{j \geq 2} e^{\beta(\alpha_j x_j - \alpha_1 x_1)} \geq m + \beta[\alpha_m(1-x_1) - (m-1)\alpha_1 x_1]. \quad (16)$$

Hence, if  $\beta$  satisfies (9), then for all  $x$  with  $x_1 = \frac{1}{m} + \frac{1}{m^2}$ ,

$$\begin{aligned} \frac{1}{f_1(x)} &\geq m + \beta \left[ \alpha_m \left( 1 - \frac{1}{m} - \frac{1}{m^2} \right) - (m-1) \left( \frac{1}{m} + \frac{1}{m^2} \right) \alpha_1 \right] \\ &= m - \beta \left[ (\alpha_1 - \alpha_m) \left( 1 - \frac{1}{m^2} \right) + \frac{\alpha_m}{m} \right] > m - \frac{m}{m+1} = \frac{1}{x_1}, \end{aligned}$$

so that by Proposition 6(b),  $\{X^N\}$  leaves  $\{x : x_1 < \frac{1}{m} + \frac{1}{m^2}\}$  slowly. If  $m \geq 3$ , then  $\frac{1}{m} + \frac{1}{m^2} < \frac{1}{2}$ , and it follows that  $\{X^N\}$  does not reach  $\{x : x_1 > \frac{1}{2}\}$  quickly.

(c) A numerical computation shows that  $e^{-9/4} + e^{-3/2} < \frac{1}{3}$ , and so there is a constant  $c \in (\frac{4}{5}, 1)$  such that  $e^{-9c/4} + e^{-3c/2} < \frac{1}{3}$ . Let

$$\phi(u) = 1 + (u - 2)e^{-3cu/4} + e^{-cu/2}.$$

We have for all  $u \geq 3$

$$\phi^{1-3cu/4} \left[ 1 - \frac{3}{4}c(u-2) - \frac{c}{2}e^{cu/4} \right] < e^{-3cu/4} \left[ 1 - \frac{3c}{4} - \frac{c}{2} \right] < 0,$$

so that  $1 < \phi(u) \leq \phi(3) < \frac{4}{3}$ . Set  $\gamma_1 = 1/[4(1 - 1/\phi(3))]$ . Then  $\gamma_1 > 1$ .

Suppose first that

$$\alpha_1 \leq \gamma_1 \alpha_2 \quad \text{and} \quad \beta \alpha_2 \geq cm. \quad (17)$$

Then for all  $x \in \Delta$  with  $x_2 = 1 - \alpha_2/(4\alpha_1)$ , by (15),

$$f_2(x) > \frac{1}{\phi(m)} \geq \frac{1}{\phi(3)} = 1 - \frac{1}{4\gamma_1} \geq 1 - \frac{\alpha_2}{4\alpha_1},$$

provided that  $m \geq 3$ . This implies by Proposition 6(c) that  $\{X^N\}$  does not reach  $\{x : x_1 > \frac{1}{2}\}$  quickly.

Suppose next that

$$\alpha_1 < \gamma_2 \alpha_m \quad \text{and} \quad \beta \alpha_2 < cm, \quad (18)$$

where  $\gamma_2 = 1 + (1 - c)^2/(4c) > 1$ . Then for every  $x \in \Delta$  with  $x_1 = 2/[m(1 + c)]$ , by (16),

$$\begin{aligned} \frac{1}{f_1(x)} &\geq m + \beta \alpha_m [1 - x_1 - (m - 1)\gamma_2 x_1] = m - \frac{\beta \alpha_m (1 - c)}{2c} + \frac{\beta \alpha_m (1 - c)^2}{2cm(1 + c)} \\ &> m - \frac{\beta \alpha_2 (1 - c)}{2c} > m - \frac{m(1 - c)}{2} = \frac{1}{x_1}. \end{aligned}$$

Thus, by Proposition 6(b),  $\{X^N\}$  leaves  $\{x : x_1 < 2/[m(1 + c)]\}$  slowly. If  $m \geq 3$ ,  $2/[m(1 + c)] < \frac{1}{2}$ , and it follows that  $\{X^N\}$  does not reach  $\{x : x_1 > \frac{1}{2}\}$  quickly.

To complete the proof set  $\gamma = \min(\gamma_1, \gamma_2)$  and observe that (10) implies that for every  $\beta > 0$  either (17) or (18) must hold.  $\square$

**Proof of Example 5.** For  $x \in \Delta$ ,  $0 \leq \epsilon \leq 1 - \frac{1}{m}$ , and  $0 \leq y \leq 1$  let

$$q(x, \epsilon) = x_1(1 - \epsilon) + (1 - x_1)\frac{\epsilon}{m - 1}, \quad H(y) = 1 - (1 - y)^r.$$

In state  $x$ , the probability that a randomly sampled agent is thought to have played strategy 1 is  $q(x, \epsilon)$ , so that in a sample of size  $r$ , the probability that a revising agent chooses strategy 1 is  $f_1(x, \epsilon) = H(q(x, \epsilon))$ .

Obviously,  $H$  is strictly increasing, and  $(\partial/\partial\epsilon)q(x, \epsilon) = (1 - mx_1)/(m - 1)$ . Hence, for every  $x \in \Delta$ ,  $f_1(x, \epsilon)$  is strictly increasing in  $\epsilon$  if  $x_1 < \frac{1}{m}$ , and  $f_1(x, \epsilon)$  is strictly decreasing in  $\epsilon$  if  $x_1 > \frac{1}{m}$ .

Let  $\xi = (\xi_1, \dots, \xi_m)$  be a point in  $\Delta$  with  $\xi_1 = 1 - (1 - \frac{1}{m})^r$ . Then  $f_1(\xi, 1 - \frac{1}{m}) = \xi_1$ . Since  $r \geq 2$ ,  $\xi_1 > \frac{1}{m}$ , so that  $f_1(\xi, \epsilon)$  is strictly decreasing in  $\epsilon$ . Thus,  $f_1(\xi, \epsilon) - \xi_1 > 0$  for every  $\epsilon \in (0, 1 - \frac{1}{m})$ . Also, if  $x \in \Delta$  and  $x_1 = 0$ , then  $f_1(x, \epsilon) > 0$  for  $\epsilon > 0$ . Since  $H$  is concave, it follows that for every  $\epsilon \in (0, 1 - \frac{1}{m})$ ,  $f_1(x, \epsilon) - x_1 > 0$  for all  $x \in \Delta$  with  $x_1 \leq \xi_1$ . In view of the continuity of  $f_1$  it now follows from Proposition 6(a) that  $\{X^N\}$  has fast convergence to  $\{x : x_1 > \xi_1\}$ .  $\square$

The proof of Proposition 7 uses two lemmas. The first notes that in a  $2 \times 2$  game Proposition 4 implies that we have fast convergence if we can show that  $\{X^N\}$  quickly reaches some threshold and the dynamics are monotone from that point. The second provides the desired monotonicity result for the dynamics with sampling. In the following the state  $x \in [0, 1]$  is the fraction of agents using strategy 1 and  $g(x)$  is the probability that a revising agent chooses strategy 1.

**Lemma 3.** *In the model described above let  $a$  and  $\xi$  be constants with  $0 < a < \xi < 1$  and suppose  $c > 0$ . Suppose that  $\{X^N\}$  reaches  $[a, 1]$  quickly and*

$$g(x) - x \geq c \text{ for all } x \in [a, \xi].$$

*Then  $\{X^N\}$  has fast convergence to  $(\xi, 1]$ .*

**Proof of Lemma 3.** Let  $A = (\xi, 1]$ ,  $B = [a, 1]$ . It follows from Proposition 1 with  $V(x) = 1 - x$  that  $\{X^N\}$  reaches  $A \cup B^c$  quickly. For  $x \in (B \setminus A) \cap \{0, \frac{1}{N}, \dots, \frac{N}{N}\}$ ,

$$\frac{P_x(X^N(\frac{1}{N}) = x + \frac{1}{N})}{P_x(X^N(\frac{1}{N}) = x - \frac{1}{N})} = \frac{(1-x)g(x)}{x(1-g(x))} \geq \frac{(1-x)(x+c)}{x(1-x-c)} \geq c_0,$$

where  $c_0 := (1+c)^2/(1-c)^2$ . Hence, by the formula for absorption probabilities of birth and death chains, see e.g. Karlin and Taylor (1975), page 113,

$$P_x(X^N(\tau_{A \cup B^c}^N) \in A) \geq \frac{1}{\sum_{k=0}^{\infty} c_0^{-k}} = 1 - c_0^{-1} > 0.$$

Thus, by Proposition 4,  $\{X^N\}$  reaches  $A$  quickly. That  $\{X^N\}$  leaves  $A$  slowly follows from Proposition 2 with  $V(x) = 1 - x$  as the Lyapunov function.  $\square$

**Lemma 4.** *Suppose  $g(0) > 0$ ,  $g(\frac{k}{r}) \geq \frac{k}{r}$  for all  $k < \frac{r}{2}$ ,  $g(\frac{k}{r}) + g(1 - \frac{k}{r}) \geq 1$  for all  $k = 0, \dots, r$ , and  $g(\frac{k}{r}) + g(1 - \frac{k}{r}) > 1$  for some  $k$ . Then  $\min\{f^{(r)}(x) - x : 0 \leq x \leq \frac{1}{2}\} > 0$ .*



**Proof of Lemma 4.** Set  $\tilde{g}(x) = g(x) - x$ . Then  $\tilde{g}(1 - \frac{k}{r}) \geq -\tilde{g}(\frac{k}{r})$  for all  $k$ . Hence for all  $x \in [0, 1]$ ,

$$\begin{aligned} f^{(r)}(x) - x &= \sum_{k \leq \frac{r}{2}} \tilde{g}\left(\frac{k}{r}\right) \binom{r}{k} x^k (1-x)^{r-k} + \sum_{k < \frac{r}{2}} \tilde{g}\left(1 - \frac{k}{r}\right) \binom{r}{k} x^{r-k} (1-x)^k \\ &\geq \sum_{k < \frac{r}{2}} \tilde{g}\left(\frac{k}{r}\right) \binom{r}{k} [x^k (1-x)^{r-k} - x^{r-k} (1-x)^k]. \end{aligned} \quad (19)$$

If  $k < \frac{r}{2}$ , then  $\tilde{g}(\frac{k}{r}) \geq 0$  and the term in square brackets is nonnegative for all  $0 \leq x \leq \frac{1}{2}$ . Thus for all  $0 \leq x < \frac{1}{2}$ ,

$$f^{(r)}(x) - x \geq g(0)[(1-x)^r - x^r] > 0.$$

Since  $g(\frac{k}{r}) + g(1 - \frac{k}{r}) > 1$  for some  $k$ , the inequality in (19) is strict for  $x = \frac{1}{2}$ , and so  $f^{(r)}(\frac{1}{2}) - \frac{1}{2} > 0$ .  $\square$

**Proof of Proposition 7.** Suppose  $\{X^N\}$  reaches  $[x^*, 1]$  quickly. To apply Lemma 4 note first that if there existed some  $x_0 \in (0, x^*)$  with  $g(x_0) < x_0$ , then, since  $g$  is nondecreasing, for all  $x \in (\frac{1}{2}(g(x_0) + x_0), x_0)$ ,

$$g(x) - x \leq g(x_0) - \frac{g(x_0) + x_0}{2} = \frac{g(x_0) - x_0}{2} < 0.$$

By the remark after Proposition 6, it would follow that  $\{X^N\}$  does not reach  $[x^*, 1]$  quickly. Thus  $g(x) \geq x$  for all  $0 \leq x \leq x^*$ . As  $g(x^*) \geq \frac{1}{2}$  and  $g$  is strictly increasing, there exists  $\delta > 0$  such that  $g(x) \geq x + \delta$  for all  $x \in [x^*, \frac{1}{2} + \delta]$ . For all  $x \in [0, \frac{1}{2}]$ ,

$$\frac{1}{2} + x > x^* + \min(x, x^*), \quad \frac{1}{2} - x \geq x^* - \min(x, x^*),$$

and so

$$g\left(\frac{1}{2} + x\right) + g\left(\frac{1}{2} - x\right) > g(x^* + \min(x, x^*)) + g(x^* - \min(x, x^*)) \geq 1.$$

In particular,  $g(0) > 0$ . It now follows from Lemma 4 that  $\min\{f^{(r)}(x) - x : 0 \leq x \leq \frac{1}{2}\} > 0$  for every  $r$ .

As  $g$  is nondecreasing, so is each  $f^{(r)}$ , and  $\lim_{r \rightarrow \infty} f^{(r)}(\frac{1}{2}) = \frac{1}{2}[g(\frac{1}{2}-) + g(\frac{1}{2}+)] > \frac{1}{2}$  (see Lorentz (1986), pages 23 and 27.) Since  $f^{(r)}(\frac{1}{2}) > \frac{1}{2}$  for every  $r$ , it follows that there exists  $\xi \in (\frac{1}{2}, \frac{1}{2} + \delta)$  so that  $f^{(r)}(\frac{1}{2}) > \xi$  for every  $r$ . Hence, if  $x \in [\frac{1}{2}, \xi]$ , then  $f^{(r)}(x) - x \geq f^{(r)}(\frac{1}{2}) - \xi > 0$ . Consequently,

$$\min_{0 \leq x \leq \xi} f^{(r)}(x) - x > 0 \quad \text{and} \quad \inf_{x^* \leq x \leq \xi} g(x) - x > 0.$$

Therefore, by Proposition 6(a),  $\{X_r^N\}$  has fast convergence to  $(\xi, 1]$  and by Lemma 3,  $\{X^N\}$  has fast convergence to the same set.  $\square$

**Proof of Proposition 8.** Write  $h(u)$  for  $P[u \geq \epsilon]$ . Note that the restrictions on the distribution of  $\epsilon$  imply that  $h$  is strictly increasing and satisfies

$$h(u) + h(-u) \geq 1 \text{ for all } u, \quad \lim_{u \rightarrow -\infty} h(u) < x^*, \quad \lim_{u \rightarrow 0^-} h(u) > x^*.$$

For  $\beta > 0$  let  $G(\beta) = \inf\{g(x, \beta) - x : 0 \leq x \leq x^*\}$ . Let  $\beta^* = \sup\{\beta > 0 : G(\beta) \geq 0\}$ . Since  $\lim_{u \rightarrow 0^-} h(u) > x^*$ ,  $G(\beta) > 0$  for some small  $\beta > 0$ , and so  $\beta^* > 0$ . If  $x \in (\lim_{u \rightarrow -\infty} h(u), x^*)$ , then  $\lim_{\beta \rightarrow \infty} g(x, \beta) - x < 0$ . Thus,  $\beta^* < \infty$ . To prove claims (i) and (ii) it will suffice to show that for some  $\beta_r^* > \beta^*$  three results hold:

- (a) if  $0 < \beta < \beta^*$ ,  $\{X^N\}$  has fast convergence to  $(\frac{1}{2}, 1]$ ;
- (b) if  $\beta > \beta^*$ ,  $\{X^N\}$  does not reach  $[x^*, 1]$  quickly;
- (c) if  $0 < \beta < \beta_r^*$ ,  $\{X_r^N\}$  has fast convergence to  $(\frac{1}{2}, 1]$ .

(a) Let  $0 < \beta < \beta^*$ . Then there exists  $\beta' \in (\beta, \beta^*]$  with  $G(\beta') \geq 0$ . As  $g(x^*, \beta) = h(0) \geq \frac{1}{2}$  and  $\lim_{u \rightarrow 0^-} h(u) > x^*$ , there exists  $\delta > 0$  such that  $g(x, \beta) - x \geq \delta$  for all  $x \in [x^* - \delta, x^*]$ . If  $x \in [0, x^* - \delta]$ , then  $x' := \frac{\beta}{\beta'}(x - x^*) + x^* \in [0, x^*]$  and  $\beta[\pi_1(x) - \pi_2(x)] = \beta'[\pi_1(x') - \pi_2(x')]$ , so that

$$g(x, \beta) - x = g(x', \beta') - x' + (x^* - x) \left(1 - \frac{\beta}{\beta'}\right) \geq G(\beta') + \delta \left(1 - \frac{\beta}{\beta'}\right).$$

Hence  $g(x, \beta) - x \geq \delta(1 - \beta/\beta')$  for all  $x \in [0, x^*]$ . Since  $g(x, \beta)$  is strictly increasing in  $x$  and  $g(x^*, \beta) \geq \frac{1}{2}$ , it follows that  $\inf_{0 \leq x \leq \frac{1}{2}} g(x, \beta) - x > 0$  and so, by Proposition 6(a),  $\{X^N\}$  has fast convergence to  $(\frac{1}{2}, 1]$ .

(b) Let  $\beta > \beta^*$ . Then  $G(\beta) < 0$ , so that for some  $x_0 \in (0, x^*]$ ,  $\delta := x_0 - g(x_0, \beta) > 0$ . Since  $g(x, \beta)$  is increasing in  $x$ ,  $g(x, \beta) - x \leq -\frac{\delta}{2}$  for all  $x \in [x_0 - \frac{\delta}{2}, x_0]$ . Hence, by the remark after Proposition 6,  $\{X^N\}$  does not reach  $[x^*, 1]$  quickly.

(c) To prove the assertion about  $\{X_r^N\}$  we first show that for  $r \geq 2$ ,

$$\inf_{\frac{1}{r} \leq x \leq \frac{1}{2}} g(x, \beta) - x \geq 1 - \frac{\beta}{\beta^*} \quad \text{if } \beta \geq \beta^* \text{ and } (r-2)\beta \leq r\beta^*. \quad (20)$$

Suppose  $\beta$  satisfies both conditions. If  $x^* \leq x \leq \frac{1}{2}$ , then  $g(x, \beta) - x \geq g(x^*, \beta) - \frac{1}{2} \geq 0 \geq 1 - \frac{\beta}{\beta^*}$ . Suppose now  $\frac{1}{r} \leq x \leq x^*$ . Let  $\{\beta_n\}$  be a sequence with  $G(\beta_n) \geq 0$  for all  $n$  and

$\beta_n \rightarrow \beta^*$ . Let  $x_n = \frac{\beta}{\beta_n}(x - x^*) + x^*$ . Then  $x_n \leq x^*$  for all  $n$  and  $\lim_{n \rightarrow \infty} x_n > 0$ . Thus, for  $n$  sufficiently large,  $x_n \in [0, x^*]$ , so that  $g(x_n, \beta_n) - x_n \geq G(\beta_n) \geq 0$  and

$$g(x, \beta) - x = g(x_n, \beta_n) - x_n + (x^* - x) \left(1 - \frac{\beta}{\beta_n}\right) \geq 1 - \frac{\beta}{\beta_n}.$$

Letting  $n \rightarrow \infty$  completes the proof of (20).

Since  $x^* < \frac{1}{2}$ ,  $\pi_1(1 - x) - \pi_2(1 - x) > \pi_2(x) - \pi_1(x)$ , and so,

$$g(x, \beta) + g(1 - x, \beta) > h(\beta[\pi_1(x) - \pi_2(x)]) + h(\beta[\pi_2(x) - \pi_1(x)]) \geq 1$$

for all  $x \in [0, 1]$  and  $\beta > 0$ . Consequently,

$$\begin{aligned} f^{(r)}(x, \beta) - x &= \sum_{k=0}^r \left( g\left(\frac{k}{r}, \beta\right) - \frac{k}{r} \right) p_{r,k}(x) \\ &\geq g(0, \beta)(1 - x)^r + [g(1, \beta) - 1]x^r \\ &\quad + \sum_{1 \leq k < \frac{r}{2}} \left( g\left(\frac{k}{r}, \beta\right) - \frac{k}{r} \right) [p_{r,k}(x) - p_{r,r-k}(x)], \end{aligned}$$

where  $p_{r,k}(x) = \binom{r}{k} x^k (1 - x)^{r-k}$ . Recall from (a) that  $g(\frac{k}{r}, \beta) \geq \frac{k}{r}$  if  $k \leq \frac{r}{2}$  and  $\beta < \beta^*$ . Moreover,  $0 \leq p_{r,k}(x) - p_{r,r-k}(x) \leq 1$  if  $k \leq \frac{r}{2}$  and  $x \in [0, \frac{1}{2}]$ . It now follows by (20) that for every  $\beta > 0$  with  $(r - 2)\beta \leq r\beta^*$ ,

$$\min_{x \in [0, \frac{1}{2}]} f^{(r)}(x, \beta) - x \geq [g(0, \beta) + g(1, \beta) - 1] \left(\frac{1}{2}\right)^r - \frac{r}{2} \left(1 - \frac{\beta}{\beta^*}\right)^-,$$

where  $u^- = -\min(u, 0)$ . Since  $\pi_1(1) - \pi_2(1) > \pi_2(0) - \pi_1(0)$ ,  $g(0, \beta) + g(1, \beta) > 1$  for every  $\beta > 0$ , and

$$\liminf_{\beta \rightarrow \beta^*} g(0, \beta) + g(1, \beta) \geq g(0, \beta^*+) + g(1, \beta^*-) > 1.$$

Therefore, for every  $r \in \mathbb{N}$  there exists  $\beta_r^* > \beta^*$  so that if  $\beta < \beta_r^*$ , then  $\inf_{x \in [0, \frac{1}{2}]} f^{(r)}(x, \beta) - x > 0$ . By Proposition 6(a), if  $\beta < \beta_r^*$ , then  $\{X_r^N\}$  has fast convergence to  $(\frac{1}{2}, 1]$ .  $\square$

**Proof of claim in footnote 20.** Assume the payoff functions  $\pi_i$  are as in Proposition 8 and the choice rule  $g(x, \beta)$  is generated by a perturbed utility function with cost function  $c(p) = p^2$ . To show that the conclusion of Proposition 8 does not hold we show that for every parameter  $\beta > 0$  either  $\{X^N\}$  and  $\{X_r^N\}$  have fast convergence to  $(\frac{1}{2}, 1]$  or neither system has.

For the present cost function  $c$ ,  $g(x, \beta) = h(\beta[\pi_1(x) - \pi_2(x)])$ , where

$$h(u) = \begin{cases} 0, & u \leq -2 \\ \frac{u+2}{4}, & -2 < u < 2 \\ 1, & u \geq 2. \end{cases}$$

Let  $\beta^* = 2/[\pi_2(0) - \pi_1(0)]$ . If  $\beta \geq \beta^*$ , then  $g(0, \beta) = 0$  and so 0 is an absorbing state of  $X^N$  and  $X_r^N$  for every  $N$ . In particular, neither  $\{X^N\}$  nor  $\{X_r^N\}$  has fast convergence to  $(\frac{1}{2}, 1]$ .

Suppose next that  $\beta < \beta^*$ . Then there exist  $\gamma_0 > 0$ ,  $\gamma_1 > 0$  so that  $g(x, \beta) = \min(\gamma_0 + \gamma_1 x, 1)$  for all  $x \in [0, 1]$ , and  $\gamma_0 + \gamma_1 x^* = \frac{1}{2}$ . If  $\gamma_0 + \gamma_1 \leq 1$ , then  $g(x, \beta) = \gamma_0 + \gamma_1 x$  for all  $x$ , and so  $f^{(r)}(x, \beta) = \gamma_0 + \gamma_1 x$  for all  $x$ . If  $\gamma_0 + \gamma_1 > 1$ , then  $g(x, \beta) \geq \gamma_0 + (1 - \gamma_0)x$  for all  $x$ , and so  $f^{(r)}(x) \geq \gamma_0 + (1 - \gamma_0)x$  for all  $x$ . In either case,  $g(x, \beta) > x$  and  $f^{(r)}(x, \beta) > x$  for every  $x \in [0, \frac{1}{2}]$ . Thus, by Proposition 6(a),  $\{X^N\}$  and  $\{X_r^N\}$  have fast convergence to  $(\frac{1}{2}, 1]$ .  $\square$

**Proof of Lemma 1.** For  $\epsilon > \frac{1}{4}$  the result is trivial. Otherwise, let  $y_1$  and  $y_2$  be the fraction of players who have 1 as a best response in  $s$  and the fraction who are using action 1 in  $s$ . We wish to show that

$$E_s \left[ V \left( Y^N \left( \frac{1}{N} \right) \right) - V(Y^N(0)) \right] \leq -\frac{\epsilon^2}{N}$$

whenever  $y_1 < 1 - 4\epsilon$ . We do this using two cases.

Case 1:  $s$  has two adjacent players using strategy 2.

Any such state can be written as a concatenation of one or more substrings by the following algorithm: initially pick a player using strategy 1 and view that player as belonging to the largest substring containing that player that does not have adjacent players using strategy 2; repeat adding additional substrings by starting with as-yet unassigned players using strategy 1 until there are no such players; if two of the resultant substrings are separated by two or more players using 2 regard the sequence of two or more 2's as another substring; otherwise if a single 2 separates two of the initially defined substrings treat the single 2 as belonging to the substring immediately to its right. This produces a division of  $s$  into some number  $M$  of substrings, each of which starts and ends with a 2. Further, each substring is of one of three possible types: (i) a sequence of two or more 2s, e.g. 22222; (ii) a string starting and ending with 2 containing at least one 1 and with no consecutive 2s, e.g. 21212 or 2121112112; or (iii) a string starting with 22, ending with 2 and containing at least one 1 and no consecutive 2s after the first pair, e.g. 2212 or 22121211112.

Let  $m = 1, 2, \dots, M$  index the substrings. Write  $N_m$  for the length of substring  $m$ ,  $k_m$  for the number of players within substring  $m$  who have 2 as their best response, and  $\ell_m$  for the number of players in substring  $m$  who are using strategy 1. Note that

$$E_s \left[ V \left( Y^N \left( \frac{1}{N} \right) \right) - V(Y^N(0)) \right] = \sum_{m=1}^M \frac{N_m}{N} \Delta V_m,$$

where  $\Delta V_m$  is the expected change in  $V$  conditional on the player chosen to update belonging to substring  $m$ . We will show below using three subcases that

$$\Delta V_m \leq -\frac{1}{N} \frac{1}{N_m} (k_m \epsilon - 3\ell_m \epsilon^2) \quad (21)$$

Using the formula above equation (21) will imply that

$$E_s \left[ V \left( Y^N \left( \frac{1}{N} \right) \right) - V (Y^N (0)) \right] \leq -\frac{1}{N^2} \sum_{m=1}^M k_m \epsilon - 3\ell_m \epsilon^2.$$

Note that  $\sum_{m=1}^M k_m = (1 - y_1)N$  and  $\sum_{m=1}^M \ell_m$  is simply the number of players using strategy 1. When  $y_1 < 1 - 4\epsilon$  the first of these is at least  $4\epsilon N$ . And the second is at most  $(1 - 4\epsilon)N$  because the number of players using strategy 1 is at most the number with 1 as a best response (because the player to the left of any player using 1 has 1 as a best response). Hence,

$$E_s \left[ V \left( Y^N \left( \frac{1}{N} \right) \right) - V (Y^N (0)) \right] \leq -\frac{1}{N^2} ((4N\epsilon)\epsilon - 3(1 - 4\epsilon)N\epsilon^2) \leq -\frac{1}{N}\epsilon^2.$$

We show that equation (21) is satisfied by considering three subcases:

Case 1(i) Substring  $m$  is of the form  $222 \dots 2$  with  $k_m \geq 1$  2s.

All players in substring  $m$  have 2 as a best response so  $k_m = N_m$ . Each player switches to 1 with probability  $\epsilon$ . A switch to 1 by the leftmost or rightmost player increases  $y_1$  by at least  $\frac{1}{N}$  because the player to his or her right/left now has 1 as a BR. And a switch to 1 by any player interior to substring  $m$  increases  $y_1$  by  $\frac{2}{N}$ . The expected change in  $y_2$  is positive. Hence,

$$\Delta V_m \leq -\frac{1}{N} \left( \frac{2}{N_m} \epsilon + \frac{N_m - 2}{N_m} 2\epsilon \right) = -\frac{1}{N} \frac{1}{N_m} (2k_m - 2)\epsilon.$$

For  $k_m \geq 2$  we have  $2k_m - 2 \geq k_m$  so  $\Delta V_m \leq -\frac{1}{N} \frac{1}{N_m} k_m \epsilon$  which is stronger than (21).

Case 1(ii) Substring  $m$  starts and ends with a single 2, contains at least one 1 and has no consecutive 2s.

Each player in such a substring is in one of four situations: playing 2; playing 1 and having both neighbors playing 2; playing 1 and being on the boundary of a cluster of two or more players using strategy 1; or playing 1 and being interior to a cluster of three or more players using 1. Write  $n_2$ ,  $n_{1i}$ ,  $n_{1b}$ , and  $n_{1m}$  for the number of players in each of these four situations. (The  $i$ ,  $b$ , and  $m$  can be thought of as referring to whether a 1 player is “isolated”, on the “boundary” of a cluster, or in the “middle” of a cluster. Note that only

players in the second situation (playing 1 and having both neighbors playing 2) have 2 as their best response. Hence,  $k_m = n_{1i}$ .

The number of players with 1 as a best response will increase if and only if a neighbor of one of the 1 players flanked by 2 players is selected and switches to 1. The number of players with 1 as a best response will decrease by one in two situations: if the 1-playing neighbor of a player who is using 1 and is on the boundary of a cluster of 1 players switches to 2; or if the 1-playing neighbor of the leftmost or rightmost player in the substring switches to 2. Hence the expected change in  $y_1$  conditional on a player from this cluster being chosen is  $\frac{1}{N} \left( \frac{2n_{1i}}{N_m}(1 - \epsilon) - \frac{n_{1b}}{N_m}\epsilon - \frac{1}{N_m}(2\epsilon + (1 - 2\epsilon)z) \right)$ , where  $z$  is the 0, 1, or 2 depending on whether the leftmost and rightmost players in the substring have a total of 0, 1, or 2 neighbors who are isolated 1-players.

All 2 players and isolated 1 players will switch strategies with probability  $1 - \epsilon$  if selected. Other players will switch from 1 to 2 with probability  $\epsilon$ . Hence, the expected change in  $y_2$  conditional on a player from this cluster being chosen is  $\frac{1}{N} \left( \frac{n_2}{N_m}(1 - \epsilon) - \frac{n_{1i}}{N_m}(1 - \epsilon) - \frac{n_{1b} + n_{1m}}{N_m}\epsilon \right)$ . Adding the expressions for the two components of the Lyapunov function gives

$$\begin{aligned} \Delta V_m &= -\frac{1}{N} \frac{1}{N_m} (n_{1i}(2 - 2\epsilon) - n_{1b}\epsilon - 2\epsilon - (1 - 2\epsilon)z + n_2 3\epsilon(1 - \epsilon) - n_{1i} 3\epsilon(1 - \epsilon) - (n_{1b} + n_{1m}) 3\epsilon^2) \\ &\leq -\frac{1}{N} \frac{1}{N_m} (n_{1i}(2 - 2\epsilon) - (1 - 2\epsilon)z + n_2 3\epsilon(1 - \epsilon) - (n_{1b}\epsilon + 2\epsilon + n_{1i} 3\epsilon(1 - \epsilon)) - (n_{1b} + n_{1m}) 3\epsilon^2) \end{aligned}$$

The number  $n_2$  of players playing 2 is  $\frac{n_{1b} + 2n_{1i}}{2} + 1$  because we can count them by counting the number of 2-playing neighbors that each 1-player has, dividing by 2 to account for the double counting, and adding 1 because the leftmost and rightmost 2 players were only counted once. Making this substitution for  $n_2$  in the middle of the above expression and cancelling (and using that  $\frac{3}{2}\epsilon(1 - \epsilon) > \epsilon$ ) gives

$$\Delta V_m \leq -\frac{1}{N} \frac{1}{N_m} (n_{1i}(2 - 2\epsilon) - (1 - 2\epsilon)z + \epsilon - 3\epsilon^2 - (n_{1b} + n_{1m}) 3\epsilon^2)$$

Comparing this expression to (21) we see that (21) will hold if

$$n_{1i}(2 - 2\epsilon) - (1 - 2\epsilon)z + \epsilon - 3\epsilon^2 \geq n_{1i}\epsilon,$$

which is equivalent to

$$n_{1i}(2 - 3\epsilon) + \epsilon - 3\epsilon^2 \geq (1 - 2\epsilon)z.$$

We must have  $n_{1i} \geq 1$  when  $z > 0$  and direct computations show that the equation is satisfied for all  $\epsilon < \frac{1}{3}$  when  $z = 0$ , when  $z = 1$  and  $n_{1i} \geq 1$ , and when  $z = 2$  and  $n_{1i} \geq 1$ .

Case 1(iii) Substring  $m$  starts with 22, ends with a single 2, contains at least one 1 and has no other consecutive 2s.

This case is almost identical to the previous one. The expected change in  $y_1$  is  $\frac{1}{N} \frac{1}{N_m} (1 - \epsilon)$  larger because the only difference is that  $y_1$  increases by an additional  $\frac{1}{N}$  with probability  $(1 - \epsilon)$  if the second player from the left is chosen because a switch by that player also makes the leftmost player have 1 as a best response. The expected change in  $y_2$  is  $\frac{1}{N} \frac{1}{N_m} \epsilon$  larger than what one gets from the formula for the previous case after plugging in  $(\frac{n_{1b} + 2n_{1i}}{2} + 1)$  for  $n_2$  because that counting of the number of 2-players misses the leftmost player who switches to 1 with probability  $\epsilon$  when selected. Hence,  $\Delta V_m$  is  $\frac{1}{N} \frac{1}{N_m} ((1 - \epsilon) + 3\epsilon^2(1 - \epsilon))$  larger in absolute value than the expression derived above for Case 1(ii). The number  $k_m$  of players with 2 as a best response is one larger than in Case (ii). So the fact that  $1 - \epsilon > \epsilon$  implies that inequality (21) continues to hold.

Case 2:  $s$  does not have two adjacent players using strategy 2.

In this case we view the state as a single string similar to that involved in Case 1(ii) above: it starts with a single 2, contains at least one 1, has no consecutive 2s, and ends with a 1 which is adjacent to the initial 2. Define  $n_2$ ,  $n_{1i}$ ,  $n_{1b}$ , and  $n_{1m}$  as in Case 1(ii). Again, the number of players with 2 as a best response is  $n_{1i}$ . By an argument similar to that above it will suffice to show that

$$E_s \left[ V \left( Y^N \left( \frac{1}{N} \right) \right) - V \left( Y^N (0) \right) \right] \leq -\frac{1}{N^2} (n_{1i} \epsilon - 3\ell \epsilon^2), \quad (22)$$

where  $\ell$  is the number of players using strategy 1. (This is sufficient because  $n_{1i} \geq 4\epsilon N$  and  $\ell_m \leq (1 - 4\epsilon)N$ .)

To show that equation (22) is satisfied we compute bounds on the expected changes in  $y_1$  and  $y_2$  similar to those used in Case 1(ii). The number of players with 1 as a best response will increase if and only if one of the neighbors of one of the 1 players flanked by 2 players is selected and switches to 1. The number of players with 1 as a best response will decrease if and only if the 1-playing neighbor of a player who is using 1 and is on the boundary of a cluster of 1 players switches to 2. Hence, the expected change in  $y_1$  is  $\frac{1}{N} \left( \frac{2n_{1i}}{N} (1 - \epsilon) - \frac{n_{1b}}{N} \epsilon \right)$ .<sup>25</sup>

All 2 players and isolated 1 players will switch strategies with probability  $1 - \epsilon$  if selected. Other players will switch from 1 to 2 with probability  $\epsilon$ . Hence, the expected change in  $y_2$  is  $\frac{1}{N} \left( \frac{n_2}{N} (1 - \epsilon) - \frac{n_{1i}}{N} (1 - \epsilon) - \frac{n_{1b} + n_{1m}}{N_m} \epsilon \right)$ . Adding the expressions for the two components of the Lyapunov function and writing  $\Delta V$  as shorthand for the expected change in the

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<sup>25</sup>This formula assumes that  $N \geq 2$  so that the neighbors of an isolated 1-player are distinct.

Lyapunov function gives

$$\begin{aligned}\Delta V &= -\frac{1}{N^2} (n_{1i}(2 - 2\epsilon) - n_{1b}\epsilon + n_2 3\epsilon(1 - \epsilon) - n_{1i} 3\epsilon(1 - \epsilon) - (n_{1b} + n_{1m}) 3\epsilon^2) \\ &\leq -\frac{1}{N^2} (n_{1i}(2 - 2\epsilon) + n_2 3\epsilon(1 - \epsilon) - (n_{1b}\epsilon + n_{1i} 3\epsilon(1 - \epsilon)) - (n_{1b} + n_{1m}) 3\epsilon^2)\end{aligned}$$

The number of players  $n_{1b}$  playing 2 is  $\frac{n_{1b} + 2n_{1i}}{2}$  because we can now double count them by counting the number of 2-players adjacent to each 1-player. Making this substitution gives

$$\begin{aligned}\Delta V &\leq -\frac{1}{N^2} \left( n_{1i}(2 - 2\epsilon) + \left( \frac{n_{1b}}{2} + n_{1i} \right) 3\epsilon(1 - \epsilon) - (n_{1b}\epsilon + n_{1i} 3\epsilon(1 - \epsilon)) - (n_{1b} + n_{1m}) 3\epsilon^2 \right) \\ &\leq -\frac{1}{N^2} (n_{1i}(2 - 2\epsilon) - (n_{1b} + n_{1m}) 3\epsilon^2) \\ &\leq -\frac{1}{N^2} (n_{1i}\epsilon - 3\ell\epsilon^2).\end{aligned}$$

This establishes equation (22) and completes the final case of the proof.  $\square$