LIDS-P-2017

KALMAN FILTERING AND RICCATI EQUATIONS FOR DESCRIPTOR SYSTEMS

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January 1991

Abstract

In this paper we consider a general formulation of a discrete-time filtering problem for descriptor systems. It is shown that the nature of descriptor systems leads directly to the need to examine singular estimation problems. Using a "dual approach" to estimation we derive a so-called "3-block" form for the optimal filter and a corresponding 3-block Riccati equation for a general class of time-varying descriptor models which need not represent a well-posed system in that the dynamics may be either over- or under-constrained. Specializing to the time-invariant case we examine the asymptotic properties of the 3-block filter, and in particular analyze in detail the resulting 3-block algebraic Riccati equation, generalizing significantly the results in [23, 28, 33]. Finally, the noncausal nature of discrete-time descriptor dynamics implies that future dynamics may provide some information about the present state. We present a modified form for the descriptor Kalman filter that takes this information into account.

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1 Introduction

In this paper, we address the problem of recursive estimation for a general class of descriptor systems. Specifically the systems that we consider are of the form[1]:

$$\begin{aligned} E_{k+1}x(k+1) &= A_k x(k) + u(k), \quad k \geq 0 & (1.1)\\ y(k+1) &= C_{k+1}x(k+1) + r(k), \quad k \geq 0 & (1.2)\end{aligned}$$

where the matrix E_{k+1} is $l_k \times n_{k+1}$, A_k is $l_k \times n_k$, and C_{k+1} is $p_{k+1} \times n_{k+1}$. Here u and r are zero-mean white Gaussian noise sequences with

$$\mathcal{M}\left[\begin{pmatrix} u(k) \\ r(k) \end{pmatrix}\begin{pmatrix} u(j) \\ r(j) \end{pmatrix}^T\right] = \begin{pmatrix} Q_k & S_k \\ S_k^T & R_k \end{pmatrix}\delta(k-j) \tag{1.3}$$

where $\mathcal{M}(.)$ denotes the mean and $\delta(k) = 1$ if $k = 0$ and 0 otherwise. Also we assume that $x(0)$ is a Gaussian random variable independent of $(u^T(k), r^T(k))^T)$, with mean $\bar{x}_0$ and covariance P_0, independent of $(u^T(k), r^T(k))^T$. The problem that we consider in this paper is the recursive estimation of $x(k)$.

Several aspects of this model deserve comment. The study of descriptor systems, of course, has a rich and growing literature [9, 10, 18]. Some of the motivation for this activity comes from applications in which the natural descriptions of systems involve both dynamics and constraints among variables, leading to models of the form (1.1) with a possibly singular matrix E on the left-hand side. Furthermore in studies such as [35] it is argued that models of this type are a natural starting point for modeling when we are attempting to deduce <u>relations</u> among dynamically evolving quantities rather then imposing causative structure. Indeed in [19]–[21] as well as in our previous work [23]–[29], it has been emphasized that descriptor models such as (1.1) can be used to describe <u>noncausal</u> phenomena –e.g. where the variable "k" represents space rather than time– which typically involve such dynamics together with boundary conditions. In fact in a subsequent paper [26] the results that we develop here are used for constructing efficient smoothing algorithms for such boundary-value models.

A second point to note concerning the model (1.1) is that it allows the possiblity that the dimensions of the problem– the number l_k of dynamic constraints, the number p_k of measurements, and the dimension n_k of $x(k)$– may vary with k. As we shall see, this does not cause any difficulty in our analysis, but this is not our reason for including this level of generality. A better reason is that such a situation arises naturally in "recursive" descriptions of two-dimensional (2-D) phenomena. Specifically, as shown in [16], there are very natural directions of recursion for boundary-value models, namely in from or out towards the boundary. The inward propagation, for example, involves propagating boundary conditions into the domain of interest. In 1-D problems, where the boundary consists of two points, inward propagation leads to a new boundary which has also two points. In 2-D however, the boundary of a compact domain <u>changes size</u> as we shrink or expand the domain.

[1] The indexing choices in (1.1)-(1.2) have been made in part to simplify the subsequent development. For example, the use of the notation of $r(k)$ in (1.2) rather than $r(k+1)$ is consistent with and simplifies (1.3). Specifically with these choices, the noises $u(k)$ and $r(k)$ are possibly correlated, and both affect the information we acquire on $x(k+1)$.

Thus if we think of $x(k)$ as representing the values of a process along such a shrinking or expanding boundary, we see that we have no choice but to deal with changing dimensionality. While we do not explicitly focus on such problems in this paper, our analysis contains all the elements necessary to make it directly applicable to 2-D estimation problems.

Another reason for allowing the possibility of changing dimensionality is one that has other even more important implications. Specifically, rather than thinking of (1.1) as describing system dynamics, we may wish to think of it as providing a set of possibly noisy constraints on the behavior of x. From this perspective the information in (1.1) plays essentially the same role as that in (1.2), the only apparent difference being that each piece of information in (1.1) concerns x at two consecutive points in time rather than the single-point-in-time nature of the information (1.2). From this perspective allowing a change in dimensionality corresponds to allowing the possibility that at some points in time we might have more pieces of information than at others. Also, this perspective opens the question of the order in which these pieces of information are incorporated. For the most part in this paper we will use the obvious ordering, namely we use (1.1) through time k to estimate $x(k)$. However there are other possibilities. In particular we also consider in this paper the use of (1.1) over the entire time interval of interest, together with (1.2) through time k to estimate $x(k)$. As we will see, when $E_k = I$, there is no difference between these two cases, but there is a difference when one considers the case of E_k singular, again emphasizing the noncausality of such models. Furthermore, the possible singularity of E_k coupled with the interpretation of (1.1) as an additional source of "measurement" data, leads directly to the need to consider the possibility that some "measurements" are perfect. Thus, in our formulation we allow the possibility that R_k in (1.3) is singular.

Recursive estimation for descriptor systems has been the subject of several studies in recent years [8, 13, 23, 24, 28, 33]. In particular in [24] we addressed this problem in the context of optimal smoothing for well-posed, constant coefficient boundary-value descriptor systems, i.e. systems of the form (1.1)-(1.2) which are square and constant (i.e. $l_k = n_k \equiv n$, $p_k \equiv p$, and all matrices are constant), together with both the assumption that $\{E, A\}$ is a regular pencil and a set of boundary conditions which yield a unique solution to (1.1) for any input $u(k)$. In that paper we used the method in [1] to derive a $2n \times 2n$ Hamiltonian (boundary-value descriptor) system for the optimal smoother assuming also that the measurement noise covariance R was nonsingular. In addition, we introduced a new generalized algebraic Riccati equation and showed that if a solution existed to this equation, the Hamiltonian dynamics could be decoupled leading to parallel forward and backward recursions reminiscent of the Mayne-Fraser smoothing algorithm. In subsequent work in developing a system theory for such systems we obtained a set of necessary and sufficient conditions for the existence and uniqueness of positive definite solutions for this class of generalized Riccati equations [23, 28] and also provided a statistical interpretation for this solution. More recently Wang and Bernhard [33] have developed some closely-related results by dualizing their work on optimal control for descriptor systems [7]. Because of this perspective, less attention was paid to statistical interpretations of the results, and also their approach deals with estimating $Ex(k)$ rather than $x(k)$. On the other hand, Wang and Bernhard consider the more general case in which the pencil $\{E, A\}$ need not be regular and in fact may not even be square (so that $l \neq n$) and in this context develop analogous results on filter convergence and Riccati equations to those in [23, 28].

While the restriction to the estimation of $Ex(k)$ is not significant if E is invertible, it is substantive if E is singular. Furthermore, as we have hinted, the possible singularity of E an A, together with the objective of estimating all of $x(k)$, leads directly to the need to consider the possibility of perfect "measurements" either through the dynamics (1.1) or the observations (1.2). In this paper, we develop a procedure for optimal recursive estimation that is valid in the most general framework– with E and A not necessarily square nor invertible and with possibly singular measurement noise covariances. As we will see, considering such a problem leads to the introduction of what we refer to as "3-block" forms for Hamiltonians, filters, and Riccati equations. Such forms actually can be found in various contexts in a number of papers in estimation and control [3, 4, 5, 14, 22, 32, 34]. Our work builds most directly on the approach of Whittle [34], Chapter 11, and the machinery for singular estimation in Campbell and Meyer [11] to derive not only a new 3-block Hamiltonian form valid in our general context but also a new 3-block generalized Riccati equation. In addition in the constant dimension/constant matrix case we develop convergence and steady-state results for the algebraic version of this equation, thereby extending the earlier results in [23, 28, 33].

In the next section we present and review some of the basic concepts concerning maximum likelihood parameter estimation with particular emphasis on deriving a form that is valid when some of the measurements are perfect. These results allow us in Section 3 to address the filtering problem for the system (1.1)-(1.2) resulting in the 3-block form for the descriptor Kalman filter and a corresponding 3-block Riccati equation. In Sections 4 and 5 we then focus on the time-invariant case. In the first of these sections we generalize the results in [23, 28, 33] by studying in detail the asymptotic properties of the descriptor Kalman filter. In particular we provide conditions for filter stability and for the convergence of the solution to the Riccati equation. Conditions under which the resulting 3-block algebraic Riccati equation has a unique positive semi-definite solution are given, and in Section 5 we generalize the well-known eigenvector approach to solving standard Riccati equations [31, 32] to our 3-block form. Finally, in Section 6 we show how the estimation procedure we have developed can be modified to account for the information about $x(k)$ contained in the future dynamic constraints.

2 A Look at Maximum Likelihood Estimation

In this section we examine a few features of maximum likelihood (ML) linear estimation beginning with the simple problem of estimating an unknown n-dimensional vector x based on the p-dimensional measurement vector

$$y = Hx + v, \tag{2.1}$$

where v is a zero-mean Gaussian random vector with covariance R. While this is a well-studied problem, it is worth making a few comments about it. First, note that the study of this estimation problem actually includes least-squares Bayesian estimation for Gaussian vectors. Specifically, consider the problem of computing the least-squares estimate of a Gaussian random vector x with mean m and covariance P based on the measurement vector $z = Cx + n$ where n is zero-mean Gaussian, independent of x, with covariance N. It is straightforward to check that this problem yields the same estimate as the ML problem with

$$y = \begin{pmatrix} m \\ z \end{pmatrix}, \quad H = \begin{pmatrix} I \\ C \end{pmatrix}, \quad R = \begin{pmatrix} P & 0 \\ 0 & N \end{pmatrix}. \tag{2.2}$$

We focus here on the ML viewpoint, which in the next section will lead to our interpreting dynamic constraints as in (1.1) as additional pieces of information or measurements. A second point is that we focus here, for the most part, on the case in which x is estimable, i.e. in which (2.1) provides sufficient constraints so that we can in fact estimate all components of x. This is equivalent to assuming that H has rank $n = \dim(x)$, which, for example, is aways true in the Bayesian case (i.e. H in (2.2) obviously has rank n).

The third and most important point for us is that we wish to consider ML problems where R may not be of full rank. If H and R have full-rank, the solution to the ML problem is easy to write out explicitly:

$$\hat{x}_{ML} = (H^T R^{-1} H)^{-1} H^T R^{-1} y. \tag{2.3}$$

The error variance associated with this estimate is given by

$$P_{ML} = \mathcal{M}[(x - \hat{x}_{ML})(x - \hat{x}_{ML})^T] = (H^T R^{-1} H)^{-1}. \tag{2.4}$$

The fact that the calculations can be described in such explicit form is extremely important as it allows us to obtain an explicit recursive structure for sequential estimation problems. What we would like to do is to obtain an equally explicit form when R is singular. To do this, we begin by recasting the ML estimation problem as a quadratic minimization problem. This approach, described by Whittle [34] and by Campbell and Meyer [11], can also be traced to early optimal control-based derivations of the Kalman filter such as in [6].

Let V be a full-rank square root of R (so that $VV^T = R$). Then we can write $v = Vw$, where w is a zero-mean Gaussian random vector with covariance I. If we then view the measurement

$$y = Hx + Vw \tag{2.5}$$

as a linear constraint on x and w, the ML problem is simply one of finding a pair (x, w) satisfying (2.5) and maximizing the probability density of w or, equivalently, minimizing

$$J(w) = (1/2) w^T w. \tag{2.6}$$

This problem is readily solved using the Lagrange multipliers. Specifically, let

$$L(w, x, \lambda) = (1/2) w^T w + \lambda^T (y - Hx - Vw). \tag{2.7}$$

Setting the partials with respect to w, x, and λ to zero yields

$$w - V^T \lambda = 0 \tag{2.8a}$$

$$H^T \lambda = 0 \tag{2.8b}$$

$$y - Hx - Vw = 0. \tag{2.8c}$$

Using (2.8a) to eliminate w gives the 2-block $(p+n)$-dimensional set of equations

$$\begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ x \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}. \tag{2.9}$$

A first obvious question about this set of equations concerns the invertibility of the $(p+n)\times(p+n)$ matrix in (2.9). Note that one obvious necessary condition is that H must have full column rank, as otherwise the last n columns would not be linearly independent. A second immediate necessary condition is that the first p rows must be linearly independent. The following shows that this pair of conditions is also sufficient.

Lemma 2.1 *Let R be positive semi-definite and H a full column rank matrix. Then, if $[R \quad H]$ has full row rank, the matrix*

$$\begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix} \tag{2.10}$$

is invertible.

Proof: Suppose that

$$\begin{pmatrix} x^T & y^T \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix} = 0. \tag{2.11}$$

Then

$$x^T R + y^T H^T = 0 \tag{2.12}$$

and

$$x^T H = 0. \tag{2.13}$$

If we now take the transpose of (2.13) and multiply it on the left by y^T we get

$$y^T H^T x = 0, \tag{2.14}$$

which after substitution in (2.12) postmultiplied by x yields

$$x^T R x = 0, \tag{2.15}$$

which since R is positive semi-definite gives $x^T R = 0$. Together with (2.13), this yields $x^T[R \quad H] = 0$, which implies $x = 0$, since $[R \quad H]$ has full row rank. Then (2.12) implies that $y^T H^T = 0$, and since H has full rank we have $y = 0$, so that (2.10) is invertible. $\square$

Assuming that (2.10) is invertible, we have from (2.9) that

$$\hat{x}_{ML} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} y \tag{2.16}$$

and by direct calculation we find that $\hat{x}_{ML}$ is unbiased and has for error covariance

$$P_{ML} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{2.17}$$

Writing

$$\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R & 0 \\ H^T & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ H^T & 0 \end{pmatrix} \tag{2.18}$$

we obtain the following simpler expression for P_{ML}:

$$P_{ML} = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{2.19}$$

The condition that $[R \quad H]$ has full column rank has a simple physical interpretation, namely that there are no redundant perfect measurements (i.e. that no independent linear combinations of the observations yield noise free measurements of the same linear combination of components of x). While this would certainly seem to be a reasonable assumption and can in principle be enforced by identifying redundancies and eliminating them, it is convenient to have a result that applies even when (2.10) is not invertible[2]. In this case, as one might expect, it is necessary to use pseudo-inverses. As discussed in [11], there are various sets of properties one can impose in defining pseudo-inverses. For our purposes here it suffices to take the pseudo-inverse $Z^\dagger$ of a symmetric matrix Z to be any symmetric matrix for which

$$ZZ^\dagger Z = Z \tag{2.20}$$

(this is what is referred to in [11] as a (1)-inverse). Then we have the following:

Lemma 2.2 *Suppose that H has full column rank. The ML estimate of x based on the measurement vector (2.1) is given by*

$$\hat{x}_{ML} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} I \\ 0 \end{pmatrix} y. \tag{2.21}$$

This estimate is unbiased and has for error covariance

$$P_{ML} = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{2.22}$$

This result is proved in [11], although in our case we can say a bit more. Specifically, let

$$\begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} = \begin{pmatrix} W & U \\ U^T & T \end{pmatrix}. \tag{2.23}$$

Then in the case where H has full column rank, T is unique, while W and U will not be unless (2.10) is invertible. Note that from (2.21)

$$\hat{x}_{ML} = U^T y \tag{2.24}$$

so that the gains in the ML estimator may not be unique– reflecting the fact that there are nonunique ways in which to determine certain linear combinations of components of x exactly. On the other hand the resulting error covariance <u>should</u> be unambiguously defined, and from (2.22), (2.23) we see that it is, since $P_{ML} = -T$. We refer the reader to Appendix

[2]Indeed while it may be easy to keep track of and eliminate redundancies in a given set of measurements, it is more difficult to do this in an organized and easily expressible way when those redundancies may evolve dynamically and arise through the dynamic constraints as well as the measurements.

A for a summary of several of the calculations and some results from [11]. In particular, we prove the identity

$$\begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix} \begin{pmatrix} U \\ T \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}, \tag{2.25}$$

which is used below.

If H does not have full column rank, then, as we have indicated, x is not estimable so that the ML estimate of all of x is undefined. Nevertheless in such a situation various linear combinations of x may be estimable (e.g. obviously $(x_1 + x_2)$ is estimable from the observation $y = (x_1 + x_2)$). The precise definition of estimability given in [11] is that the linear combination of $c^T x$ is estimable if there exists a measurement linear combination $d^T y$ that is an unbiased estimate of $c^T x$. An essentially immediate necessary and sufficient condition for this is that c must be in the range of H^T. Let r denote the rank of H and let $H = H_1 H_2$ denote a full-rank factorization of H, i.e. H_1 is a $p \times r$ full column rank matrix and H_2 is an $r \times n$ full row rank matrix. Then it is precisely $z = H_2 x$ whose ML estimate can be computed from y. Furthermore from results in [11] we can deduce that

$$\hat{z}_{ML} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H_1 \\ H_1^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} I \\ 0 \end{pmatrix} y = \begin{pmatrix} 0 & H_2 \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} I \\ 0 \end{pmatrix} y \tag{2.26}$$

is an unbiased estimate of z with associated error covariance

$$P_{ML} = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H_1 \\ H_1^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ I \end{pmatrix} = -\begin{pmatrix} 0 & H_2 \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ H_2^T \end{pmatrix}. \tag{2.27}$$

Next, we prove several results that provide the justification for the recursive procedure that will be employed below for computing ML estimates. The first of these results states that for the purpose of estimating other variables, we can replace several measurements of a variable by its estimate based on these measurements.

Lemma 2.3 *Let x and z be unknown vectors and consider the observations*

$$a = Hx + v \tag{2.28}$$

$$b = Jx + Kz + w \tag{2.29}$$

where v and w are independent, zero-mean Gaussian vectors with covariance matrices V and W, respectively. Suppose that x is estimable based on (2.28) only, and that z is estimable based on both (2.28) and (2.29). Let $\hat{x}_1$ be the estimate of x based on (2.28), and let P_1 be the associated error covariance matrix. Then $\hat{x}$, the estimate of x based on both (2.28) and (2.29), and its associated estimation error covariance P, are identical to the estimate and estimation error covariance resulting from estimating x from (2.29) and the observation

$$\hat{x}_1 = x + u \tag{2.30}$$

where u is zero-mean and Gaussian, independent of w, with covariance P_1.

Furthermore, the estimate $\hat{z}$ and estimation error covariance of z based on (2.28) and (2.29) are the same as its estimate and error covariance based on (2.29) and (2.30).

Proof: Since x is estimable from (2.28), H must have full column rank. This, coupled with the assumption that z is estimable from both (2.28) and (2.29), implies that K has full column rank. Consider the joint estimation of x and z based on (2.28) and (2.29). If λ_a and λ_b denote the Lagrange multiplier vectors associated with observations (2.28) and (2.29), respectively, according to (2.9) the multiplier vectors and estimates $\hat{x}$ and $\hat{z}$ satisfy the system

$$\begin{pmatrix} V & 0 & H & 0 \\ 0 & W & J & K \\ H^T & J^T & 0 & 0 \\ 0 & K^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_b \\ \hat{x} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}. \tag{2.31}$$

We seek to compare this "batch" estimation method, where all the measurements are processed at the same time, with the "sequential" approach where we first estimate x based on (2.28) alone, and then combine the resulting estimate in the form of the summary measurement (2.30) with observation (2.29) in order to estimate both x and z. If λ_{1a} and $\hat{x}_1$ are the Lagrange multiplier vector and estimate of x based on (2.28) alone, and if U_1 and P_1 are the associated estimator matrix and error covariance, we see from (2.9) and from identity (2.25) that they satisfy

$$\begin{pmatrix} V & H \\ H^T & 0 \end{pmatrix} \begin{pmatrix} \lambda_{1a} & U_1 \\ \hat{x}_1 & -P_1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}. \tag{2.32}$$

Similarly, the estimates of x and z based on (2.29) and the summary measurement (2.30) are obtained by solving

$$\begin{pmatrix} P_1 & 0 & I & 0 \\ 0 & W & J & K \\ I & J^T & 0 & 0 \\ 0 & K^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_b \\ \hat{x} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \\ b \\ 0 \\ 0 \end{pmatrix} \tag{2.33}$$

where λ_1 denotes here the Lagrange multiplier vector associated to the measurement (2.30). What we need to show is that the solution of systems (2.32) and (2.33) combined is equivalent to solving (2.31). This requires eliminating λ_{1a}, $\hat{x}_1$ and λ_1 from (2.32) and (2.33) and finding an expression for the variable λ_a of (2.31) in terms of these quantities. Note that our choice of notation already indicates that the variables $\hat{x}$, $\hat{z}$ and λ_b which appear in both systems are the same.

First, we observe that the second and fourth block row of (2.31) already appear as the second and fourth block rows of (2.33), so that we need only to obtain the first and third block rows of (2.31) from (2.32) and (2.33).

First block row: We multiply the first block row of (2.33) on the left by H. This gives

$$HP_1\lambda_1 + H\hat{x} = H\hat{x}_1. \tag{2.34}$$

Now using the identities corresponding to the (1,1) and (1,2) blocks of (2.32), we find

$$VU_1\lambda_1 + H\hat{x} = a - V\lambda_{1a} \tag{2.35}$$

so that if we denote

$$\lambda_a \triangleq \lambda_{1a} + U_1\lambda_1, \tag{2.36}$$

we get the first block row of (2.31).

Third block row: Consider the third block row of (2.33):

$$\lambda_1 + J^T \lambda_b = 0. \tag{2.37}$$

Substituting the identities corresponding to the (2,1) and (2,2) blocks of (2.32) inside this identity, we obtain

$$H_x^T \lambda_a + J_x^T \lambda_b = 0 \tag{2.38}$$

where λ_a is defined as in (2.36). But (2.38) is just the third block row of (2.31).

Since the estimates of $\hat{x}$ and $\hat{z}$ obtained by the batch and sequential methods are identical functionals of a and b, and the statistical assumptions for a and b are the same under both methods, the error covariances are the same. □

Note that in the above proof, the assumption that x is estimable from (2.28) was needed to ensure that the second block column of (2.32) holds. On the other hand, the assumption that z is estimable from both (2.28) and (2.29), i.e. that K has full rank, was only required insofar as we wanted to discuss the properties of the ML estimate $\hat{z}$ and the associated error covariance. It is not necessary if we are only interested in estimable linear combinations of the entries of z, and the above result can easily be restated in a way that does not require z to be estimable.

Lemma 2.3 shows that previously processed measurements can be aggregated in the form of a summary measurement (2.30) for the variables that have been estimated. However the summary measurement (2.30) will include estimates of variables that do not appear in subsequent measurements and in which we are no longer interested. Conventional wisdom suggests that measurements associated to such exogenous variables can be discarded without affecting the estimation of the other variables. The following result, which is expressed in its most general form, provides a criterion for dropping unneeded measurements.

Lemma 2.4 *Consider the observations*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ H_2 & H_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.39}$$

where x_1 and x_2 are two unknown vectors, and $[v_1^T \ \ v_2^T]^T$ is a zero-mean Gaussian vector with covariance

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix}.$$

Suppose that H_3 has full row rank, and H_1 has full column rank. Then the ML estimate of x_1 based on both y_1 and y_2 is the same as that based on y_1 alone.

The assumption that H_1 has full column rank is introduced here to guarantee that x_1 is estimable from the y_1 measurement, but it can be removed if we only seek to estimate estimable linear combinations of the entries of x_1.

Proof: Let λ_1 and λ_2 be the Lagrange multiplier vectors associated to the y_1 and y_2 measurements, respectively, and let $\hat{x}_1$ and $\hat{x}_2$ be the estimates of x_1 and x_2 based on both

y_1 and y_2. According to (2.9), they satisfy the system

$$\begin{pmatrix} R_{11} & R_{12} & H_1 & 0 \\ R_{12}^T & R_{22} & H_2 & H_3 \\ H_1^T & H_2^T & 0 & 0 \\ 0 & H_3^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ 0 \\ 0 \end{pmatrix}. \tag{2.40}$$

Since H_3 has full row rank, the relation $H_3^T \lambda_2 = 0$ implies that $\lambda_2 = 0$, so that we can delete the second and fourth block rows and columns from (2.40). This gives

$$\begin{pmatrix} R_{11} & H_1 \\ H_1^T & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \hat{x}_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \tag{2.41}$$

which is precisely the system corresponding to the ML estimation of x_1 (or estimable linear combinations of its entries) from y_1 alone. □

The combination of Lemmas 2.3 and 2.4 provides a general mechanism for generating ML estimates recursively. Specifically, consider the two observations

$$a = Hx_1 + Gx_2 + v \tag{2.42}$$

$$b = Jx_1 + Kz + w \tag{2.43}$$

where we assume that x_1 and x_2 are jointly estimable from (2.42), i.e. $[H \quad G]$ has full column rank, and z is estimable from (2.42) and (2.43), i.e. K has full column rank. As in Lemma 2.3, we assume that v and w are zero-mean independent Gaussian vectors with covariance V and W, respectively. From Lemma 2.3, we see that the measurement (2.42) can be replaced by the summary measurements

$$\hat{x}_1 = x_1 + u_1 \tag{2.44a}$$

$$\hat{x}_2 = x_2 + u_2 \tag{2.44b}$$

where $\hat{x}_1$ and $\hat{x}_2$ denote the ML estimates of x_1 and x_2 based on (2.42) alone, and the covariance of $[u_1^T \quad u_2^T]^T$ is the corresponding estimation error covariance. Then, the ML estimates of x_1 and z based on both (2.43) and (2.44a)–(2.44b) are the same as those derived from (2.42) and (2.43). But x_2 does not appear in observation (2.43), and the system obtained by combining (2.43) and (2.44a)–(2.44b) satisfies the assumptions of Lemma 2.4, so that for the purpose of estimating x_1 and z we can drop the measurement (2.44b). This shows that the ML estimates of x_1 and z based on both (2.43) and the summary measurement (2.44a) are the same as those based on (2.42) and (2.43). This general procedure, whereby previously estimated variables are replaced by summary measurements, and irrelevant variables are discarded, constitutes the basis for the descriptor Kalman filtering method of Section 3.

Finally, we prove the intuitively obvious fact that for a given measurement set, if the noise covariance increases, the error covariance of the ML estimate also increases.

Lemma 2.5 *Consider the observations*

$$y_1 = Hx + v_1 \tag{2.45a}$$

$$y_2 = Hz + v_2 \tag{2.45b}$$

where v_1 and v_2 are zero-mean Gaussian vectors with covariances V_1 and V_2, and suppose that H has full column rank. Then, if $V_2 \geq V_1$, the estimation error variance associated with estimating x based on (2.45a) is less than or equal to the estimation error variance for estimating z based on (2.45b).

Proof: Let $\hat{x}_1$, P_1 and $\hat{z}_2$, P_2 denote the ML estimates and estimation error covariances associated with (2.45a) and (2.45b), respectively.

As shown in Appendix A

$$\hat{z}_2 = U_2^T y_2, \;\; P_2 = U_2^T V_2 U_2 \tag{2.46}$$

where U_2^T is a left-inverse of H satisfying some additional conditions. Consider then the following estimate of x:

$$\bar{x} = U_2^T y_1. \tag{2.47}$$

Since $U_2^T H = I$, this is an unbiased estimate; however it may be suboptimal. Thus

$$P_1 \leq \mathcal{M}[(x - \bar{x})(x - \bar{x})^T] = U_2^T V_1 U_2 \leq U_2^T V_2 U_2 = P_2. \tag{2.48}$$

□

3 The Descriptor Kalman Filter

We are now in a position to consider the recursive estimation problem for the system (1.1)-(1.3). As we have indicated, we adopt here an ML perspective, viewing the dynamics (1.1) and prior density on $x(0)$ as additional measurements. Specifically, we describe in this section the recursive computation of the filtered estimate $\hat{x}(j)$ which we define as the ML estimate of $x(j)$ based on the "measurements" in (1.1) and (1.2) for $k = 0, ..., j-1$ together with the "measurements" provided by the prior information about $x(0)$:

$$\bar{x}_0 = x(0) + \nu \tag{3.1}$$

where ν is zero-mean, Gaussian and independent of $u(k)$ and $r(k)$, and with covariance P_0. Specifically, (3.1) and (1.1), (1.2) for $k = 0, ..., j-1$ provide us with a set measurements of the unknown vector $[x^T(0), x^T(1), ..., x^T(j)]$. Examining this set of measurements we see that the only terms in these equations involving $x(j)$ are of the form $E_j x(j)$ and $C_j x(j)$. Thus a necessary condition for $x(j)$ to be estimable is that $\begin{pmatrix} E_j \\ C_j \end{pmatrix}$ have full column rank. By induction, using the recursive ML estimation procedure outlined in Section 2, and the fact that $x(0)$ is estimable from (3.1), we can show this is also a sufficient condition for estimability and, in fact, we can establish the following:

Lemma 3.1 *Let P_j denote the error covariance associated with the filtered estimate $\hat{x}(j)$, with $\hat{x}(0) = \bar{x}_0$ and P_0 given by the prior distribution for $x(0)$. Then $\hat{x}(j+1)$ and P_{j+1} are respectively equal to the ML estimate of $x(j+1)$ and its associated estimation error covariance based on the following observations*

$$y(j+1) = C_{j+1} x(j+1) + r(j) \tag{3.2}$$
$$A_j \hat{x}(j) = E_{j+1} x(j+1) + A_j \nu(j) - u(j) \tag{3.3}$$

where $\nu(j)$ is a Gaussian random vector, independent of $r(j)$, with zero mean and variance P_j.

Applying Lemma 2.1 to (3.2), (3.3) provides us with the 3-block form of the descriptor Kalman filter summarized in the following:

Theorem 3.1 *The filtered estimate $\hat{x}(j+1)$ and the corresponding error variance P_{j+1} can be obtained from the following recursions:*

$$\hat{x}(j+1) = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_jP_jA_j^T + Q_j & -S_j & E_{j+1} \\ -S_j^T & R_j & C_{j+1} \\ E_{j+1}^T & C_{j+1}^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} A_j\hat{x}(j) \\ y(j+1) \\ 0 \end{pmatrix}, \tag{3.4}$$

$$P_{j+1} = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_jP_jA_j^T + Q_j & -S_j & E_{j+1} \\ -S_j^T & R_j & C_{j+1} \\ E_{j+1}^T & C_{j+1}^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{3.5}$$

If past and present observations and dynamics do not supply redundant perfect informations, i.e. when

$$\begin{pmatrix} A_jP_jA_j^T + Q_j & S_j & E_{j+1} \\ S_j^T & R_j & C_{j+1} \end{pmatrix}$$

has full row rank, then the pseudo-inverse in (3.4), (3.5) is, in fact, an inverse.

Equation (3.4) is the 3-block form of the optimal filter, with the Hamiltonian matrix being the 3-block matrix whose pseudo-inverse is taken on the right-hand side. Van Dooren [32] introduced a similar Hamiltonian form in the case in which $E = I$ and R is invertible (so the Hamiltonian matrix is certainly invertible) and, by a limiting argument suggested that it was also valid for R singular. Whittle [34], Chapter 11 essentially shows this using a dual optimization approach as in (2.7)-(2.9) but applied to the dynamic problem directly. Arnold and Laub [3] also consider such a form for R invertible and $E \neq I$ but invertible. Perhaps closer to our work is that of Mehrmann [22] who considers a related Hamiltonian pencil (which we encounter in Section 5) for the case of E and R singular in the context of an optimal control problem for singular systems. However, no recursion of the type (3.4) is presented in [22] and, even more importantly, the 3-block descriptor Riccati equation (3.5) is not obtained in any of these references. For completeness, we note that Bender and Laub [4, 5] do introduce a Riccati equation for the case of E singular in the context of an optimal control problem. However theirs is a reduced-order Riccati equation of size equal to the rank of E, rather than the full-order Riccati equation (3.5)[3].

Finally, we remark that the previously derived results referred to in the introduction are indeed specializations of (3.4), (3.5). In particular, Wang and Bernhard [33] examine the case of time-invariant systems, where it is assumed not only that $\begin{pmatrix} E \\ C \end{pmatrix}$ has full column

[3]For this reason we conjecture that there is a connection between the estimation dual of Bender and Laub [4, 5] and the work in [33] in which the focus is on estimating $Ex(k)$. This is only of tangential interest here and thus is not pursued.

rank, but also that $\begin{pmatrix} E & Q \end{pmatrix}$ has full row rank, $S = 0$, and R is invertible. Their analysis can be interpreted as estimating $Ex(j) = Ax(j-1) + u(j-1)$ based on observations $y(k)$ for $0 \leq k \leq j-1$. In this case, they obtain the following Riccati equation for the error covariance Σ_j of this one-step predicted estimate:

$$\Sigma_{j+1} = \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} \Sigma_j & 0 \\ 0 & C^T R^{-1} C \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} + Q \tag{3.6}$$

with

$$M_j = \begin{pmatrix} \Sigma_j & E \\ E^T & -C^T R^{-1} C \end{pmatrix}. \tag{3.7}$$

To see how (3.6)-(3.7) can be obtained from (3.5), note that

$$\Sigma_j = AP_jA^T + Q \tag{3.8}$$

and thus using (3.5) and the fact that in Wang and Bernhard's case past and present observations and dynamics do not supply redundant perfect information, we obtain the following recursion for Σ_j:

$$\Sigma_{j+1} = -\begin{pmatrix} 0 & 0 & A \end{pmatrix} \begin{pmatrix} R & 0 & C \\ 0 & \Sigma_j & E \\ C^T & E^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ A^T \end{pmatrix} + Q. \tag{3.9}$$

Then the use of standard block-matrix inversion results allows us to express (3.9) as

$$\Sigma_{j+1} = -\begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} + Q. \tag{3.10}$$

where M_j is defined as in (3.7). It turns out that (3.10) is a simplified version of (3.6). To obtain (3.10) from (3.6), note that

$$\begin{aligned}
\Sigma_{j+1} &= \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} \Sigma_j & 0 \\ 0 & C^T R^{-1} C \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} + Q \\
&= \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \left[M_j - \begin{pmatrix} 0 & E \\ E^T & 2C^T R^{-1} C \end{pmatrix} \right] M_j - 1 \begin{pmatrix} 0 \\ A^T \end{pmatrix} + Q \\
&= \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} - \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 & E \\ 0 & C^T R^{-1} C \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} - \\
&\quad \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 & 0 \\ E^T & C^T R^{-1} C \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} + Q \\
&= \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} - \begin{pmatrix} 0 & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} - \\
&\quad \begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ A^T \end{pmatrix} + Q \\
&= -\begin{pmatrix} 0 & A \end{pmatrix} M_j^{-1} \begin{pmatrix} 0 \\ A^T \end{pmatrix} + Q.
\end{aligned} \tag{3.11}$$

If in addition, Σ_j is invertible, we can express the inverse of M_j in (3.10) in terms of its block entries, so that

$$\Sigma_{j+1} = A(E^T \Sigma_j^{-1} E + C^T R^{-1} C)^{-1} A^T + Q, \tag{3.12}$$

which is the Riccati equation that we had considered earlier in [28]. In the nondescriptor case ($E = I$), (3.12) reduces to the standard Riccati equation of Kalman filtering

$$\Sigma_{j+1} = A(\Sigma_j^{-1} + C^T R^{-1} C)^{-1} A^T + Q. \tag{3.13}$$

4 Stability and Convergence of the Descriptor Kalman Filter

In this section, we study the asymptotic properties of the descriptor Kalman filter in the time-invariant case:

$$\begin{aligned} Ex(k+1) &= Ax(k) + u(k), \quad k \geq 0 \qquad (4.1) \\ y(k+1) &= Cx(k+1) + r(k), \quad k \geq 0 \qquad (4.2) \end{aligned}$$

where matrices E and A are $l \times n$, C is $p \times n$, and u and r are zero-mean, white, Gaussian sequences with covariance

$$\mathcal{M}\left[\begin{pmatrix} u(k) \\ r(k) \end{pmatrix} \begin{pmatrix} u(j) \\ r(j) \end{pmatrix}^T\right] = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \delta(k-j). \tag{4.3}$$

In particular the results presented in this section generalize those in [23, 28, 33] for descriptor systems and the usual results for standard causal systems. Note that as in [33] in our development we do <u>not</u> require that $l = n$, so that (4.1) need not be square and even if it is, we do not require $\{E, A\}$ to define a regular pencil. In the context of viewing (4.1) as simply providing another source of measurements, we see that it is quite natural to remove this restriction. Also, as before, we allow the possibility that R is singular as well.

Definition 4.1 *The system (4.1)-(4.2) is called detectable if*

$$\begin{pmatrix} sE - tA \\ C \end{pmatrix}$$

has full column rank for all $(s,t) \neq (0,0)$ such that $|s| \geq |t|$.

It is called stabilizable if

$$\begin{pmatrix} sE - tA & Q & -S \\ sC & -S^T & R \end{pmatrix}$$

has full row rank for all $(s,t) \neq (0,0)$ such that $|s| \geq |t|$.

These definitions generalize other similar definitions in the literature [7, 22]. Note, for example, that these definitions reduce to the classical notions of detectability and stabilizability when $E = I$ and $R > 0$.[4]

The following generalizes the usual result that detectability implies the existence of a stable observer. An important point to note here, however, is that our observer (4.4) is an explicit causal system, even if (4.1), (4.2) is implicit, a direct consequence of estimability.

Theorem 4.1 *Let (4.1)-(4.2) be detectable, then there exists a stable filter*

$$x_s(k+1) = A_s x_s(k) + K_s y(k+1), \quad x_s(0) = \bar{x}_0 \tag{4.4}$$

i.e. such that A_s is a stable matrix (all eigenvalues have magnitude less than 1) and with

$$\lim_{k\to\infty} \mathcal{M}[(x(k) - x_s(k))(x(k) - x_s(k))^T] < \infty. \tag{4.5}$$

Proof of Theorem 4.1: We start the proof by showing the following lemma:

Lemma 4.1 *Let (4.1)-(4.2) be detectable. Then there exists a left inverse $(L_e \quad L_c)$ of $\begin{pmatrix} E \\ C \end{pmatrix}$, i.e.*

$$L_e E + L_c C = I, \tag{4.6}$$

such that $L_e A$ is stable.

Proof of Lemma 4.1: First note that the lemma is trivially true if C has full column rank, since we can simply take $L_e = 0$ and $L_c =$ a left-inverse of C. Assuming this is not the case but that $\begin{pmatrix} E \\ C \end{pmatrix}$ has full rank, we can find invertible $l \times l$ and $n \times n$ matrices U and V such that

$$UEV = \begin{pmatrix} I & 0 \\ 0 & E_{22} \end{pmatrix} \tag{4.7}$$

$$CV = \begin{pmatrix} 0 & C_2 \end{pmatrix} \tag{4.8}$$

where the partitions in (4.7), (4.8) are compatible, I denotes a square identity matrix, and C_2 has full column rank. If we partition UAV similarly as

$$UAV = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4.9}$$

the detectability of (C, E, A) implies that

$$\begin{pmatrix} (s/t)I - A_{11} \\ -A_{21} \end{pmatrix}$$

---

[4] In this case our detectability condition reduces to $\begin{pmatrix} sI - A \\ C \end{pmatrix}$ having full column rank for all $|s| > 1$. Also, with $R > 0$ and since (4.3) implies that $Q = BB^T$ and $S = BD^T$ for some matrices B and D, stabilizability in this case corresponds to $\begin{pmatrix} sI - A & B \end{pmatrix}$ having full row rank for $|s| > 1$.

has full column rank for $|s/t| \geq 1$, which means that $(A_{11}, -A_{21})$ is detectable in the usual sense. Thus, there exists a matrix D such that $A_{11} + DA_{21}$ is stable. Next, let F be any matrix satisfying

$$FC_2 = \begin{pmatrix} -DE_{22} \\ I \end{pmatrix}. \tag{4.10}$$

For example, we can take

$$F = \begin{pmatrix} -DE_{22}C_2^L \\ C_2^L \end{pmatrix} \tag{4.11}$$

where C_2^L is any left-inverse of C_2. Then

$$\begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & E_{22} \end{pmatrix} + F\begin{pmatrix} 0 & C_2 \end{pmatrix} = I \tag{4.12}$$

and

$$\begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} + DA_{21} & A_{12} + DA_{22} \\ 0 & 0 \end{pmatrix} \tag{4.13}$$

which is stable because $A_{11} + DA_{12}$ is stable. Thus by taking

$$L_e = V\begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix}U \tag{4.14a}$$

$$L_c = VF \tag{4.14b}$$

the lemma is proved. □

Continuing the proof of the theorem, note that using the above lemma, we can express $x(k+1)$ as

$$x(k+1) = L_eAx(k) + L_cy(k+1) + L_eu(k) - L_cr(k) \tag{4.15}$$

where L_eA is stable. If we now define

$$x_s(k+1) = L_eAx_s(k) + L_cy(k+1) \tag{4.16}$$

we can easily see that

$$\lim_{k\to\infty} \mathcal{M}[(x(k) - x_s(k))(x(k) - x_s(k))^T] = P_s \tag{4.17}$$

where P_s is the unique positive semi-definite solution of the Lyapunov equation

$$P_s - (L_eA)P_s(L_eA)^T = \begin{pmatrix} L_e & L_c \end{pmatrix}\begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix}\begin{pmatrix} L_e^T \\ L_c^T \end{pmatrix}. \tag{4.18}$$

The theorem is thus proved. □

Detectability alone, of course, does not guarantee that the descriptor Kalman filter converges to a stable filter. However, as would also be expected from what we know for causal systems, detectability does tell us something about the descriptor algebraic Riccati equation.

Theorem 4.2 *Let (4.1)-(4.2) be detectable. Then the algebraic descriptor Riccati equation*

$$P = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} APA^T + Q & -S & E \\ -S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \tag{4.19}$$

has a positive semi-definite solution.

Proof: We prove the existence of a positive semi-definite solution P to (4.19) by showing that the descriptor Riccati recursion

$$P_{j+1} = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} AP_jA^T + Q & -S & E \\ -S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \tag{4.20}$$

with $P_0 = 0$ is monotone increasing and bounded. This of course implies the convergence of P_k, which, from (4.20) implies that this limit, which must be positive semi-definite, satisfies (4.19).[5]

To see the boundedness of P_k, consider the stable filter (4.16) with $x_s(0) = \bar{x}_0$. It is then clear that the associated error variance matrices $P_s(k)$ converge asymptotically to P_s the unique solution of (4.18), and that thanks to the optimality of the descriptor Kalman filter, $P_k \leq P_s(k)$.

We show that P_k is monotone increasing by induction. Clearly

$$P_1 \geq P_0 = 0. \tag{4.21}$$

Now suppose that

$$P_j \geq P_{j-1}. \tag{4.22}$$

P_j is the estimation error covariance associated with estimating $x(j)$ based on

$$\begin{pmatrix} A\hat{x}(j-1) \\ y(j) \end{pmatrix} = \begin{pmatrix} E \\ C \end{pmatrix} x(j) + \begin{pmatrix} A\nu(j-1) - u(j-1) \\ r(j-1) \end{pmatrix} \tag{4.23}$$

where the covariance of $\begin{pmatrix} A\nu(j-1) - u(j-1) \\ r(j-1) \end{pmatrix}$ is $\begin{pmatrix} AP_{j-1}A^T + Q & -S \\ -S^T & R \end{pmatrix}$. Also P_{j+1} is the estimation error covariance associated with estimating $x(j+1)$ based on

$$\begin{pmatrix} A\hat{x}(j) \\ y(j+1) \end{pmatrix} = \begin{pmatrix} E \\ C \end{pmatrix} x(j+1) + \begin{pmatrix} A\nu(j) - u(j) \\ r(j) \end{pmatrix} \tag{4.24}$$

[5]The usual argument here involves a right-hand side which includes matrix inverses for which we can deduce convergence by the continuity of the inversion operation. While the full generalized inverse in (4.20) is not unique, the lower right-hand block is, due to the fact that $\begin{pmatrix} E \\ C \end{pmatrix}$ has full column rank. Indeed in Appendix A we give an explicit form for this block which involves true inverses (identify $\begin{pmatrix} E \\ C \end{pmatrix}$ with H in Appendix A).

where the covariance of $\begin{pmatrix} A\nu(j) - u(j) \\ r(j) \end{pmatrix}$ is $\begin{pmatrix} AP_jA^T + Q & -S \\ -S^T & R \end{pmatrix}$. But from (4.22) we see that

$$\begin{pmatrix} AP_jA^T + Q & -S \\ -S^T & R \end{pmatrix} \geq \begin{pmatrix} AP_{j-1}A^T + Q & -S \\ -S^T & R \end{pmatrix} \tag{4.25}$$

and thanks to Lemma 2.5, we conclude that $P_{j+1} \geq P_j$. □

In what follows it is useful to have available an alternate form for the Kalman filter equation (3.4), (3.5) in the time-invariant case. Obtaining this requires the following which is proved in Appendix A:

Lemma 4.2 *Let R be a positive semi-definite matrix and H a full column rank matrix. Then*

$$\begin{pmatrix} 0 & I \end{pmatrix} \left[\begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} + \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} \right] \begin{pmatrix} 0 \\ I \end{pmatrix} = 0 \tag{4.26}$$

and

$$\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} H \\ 0 \end{pmatrix} = I \tag{4.27}$$

Using this result, it is straightforward to verify that the Riccati recursion (4.20) can be rewritten as

$$P_{j+1} = (L_jA)P_j(L_jA)^T + \begin{pmatrix} L_j & K_j \end{pmatrix} \begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix} \begin{pmatrix} L_j^T \\ K_j^T \end{pmatrix} \tag{4.28}$$

with

$$\begin{pmatrix} L_j & K_j \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} AP_jA^T + Q & -S & E \\ -S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}. \tag{4.29}$$

Thus we have that the descriptor Kalman filter (3.4) takes the form

$$\hat{x}(k+1) = L_kA\hat{x}(k) + K_ky(k+1). \tag{4.30}$$

Via similar manipulations we can also rewrite the algebraic Riccati equation (4.19) as

$$P = (LA)P(LA)^T + \begin{pmatrix} L & K \end{pmatrix} \begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix} \begin{pmatrix} L^T \\ K^T \end{pmatrix} \tag{4.31}$$

where

$$\begin{pmatrix} L & K \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} APA^T + Q & -S & E \\ -S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}. \tag{4.32}$$

In the following we consider the behavior of the Kalman filter and Riccati equation when the system is both detectable and stabilizable, obtaining a generalization of well-known results for causal systems. Note that stabilizability implies that

$$\begin{pmatrix} E & Q & S \\ C & S^T & R \end{pmatrix}$$

has full row rank, which in turn implies that

$$\begin{pmatrix} E & AP_jA^T + Q & S \\ C & S^T & R \end{pmatrix}$$

has full row rank for all $P_j \geq 0$, so that in this case the pseudo-inverses in (4.29), (4.32) are in fact inverses.

Theorem 4.3 *Suppose that (4.1)-(4.2) is detectable and stabilizable. Then for any initial condition P_0, the solution P_k of the Riccati equation*

$$P_{k+1} = (L_kA)P_k(L_kA)^T + \begin{pmatrix} L_k & K_k \end{pmatrix} \begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix} \begin{pmatrix} L_k^T \\ K_k^T \end{pmatrix} \tag{4.33}$$

$$\begin{pmatrix} L_k & K_k \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} AP_kA^T + Q & -S & E \\ -S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \tag{4.34}$$

converges exponentially fast to the unique positive semi-definite sonlution of the algebraic descriptor Riccati equation

$$P = (LA)P(LA)^T + \begin{pmatrix} L & K \end{pmatrix} \begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix} \begin{pmatrix} L^T \\ K^T \end{pmatrix} \tag{4.35}$$

$$\begin{pmatrix} L & K \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} APA^T + Q & -S & E \\ -S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}. \tag{4.36}$$

Furthermore the steady-state Kalman filter

$$\hat{x}(k+1) = LA\hat{x}(k) + Ky(k+1) \tag{4.37}$$

is stable.

Proof ¿From Theorem 4.2 we know that there is at least one positive semi-definite solution to (4.35), (4.36). What we would like to show is that this positive semi-definite solution is unique, that P_j in (4.33,4.34) converges to P exponentially fast for any initial condition P_0, and that the resulting steady-state Kalman filter (4.37) is exponentially stable, i.e. that LA is a stable matrix. Note that by using (4.27), it is not difficult to show that

$$LE = I - KC. \tag{4.38}$$

Thus premultiplying (4.1) by L, using (4.2) and (4.37), and defining $\tilde{x}(k) = x(k) - \hat{x}(k)$, we see that

$$\tilde{x}(k+1) = LA\tilde{x}(k) + Lu(k) - Kr(k) \tag{4.39}$$

so that this will imply the stability of the error dynamics.

Let us first show that LA is stable when P is taken as any positive semi-definite solution of (4.31). Suppose LA is not stable. Then there exist a complex number λ and a complex row vector v such that $|\lambda| \geq 1$ and

$$vLA = \lambda v. \tag{4.40}$$

¿From (4.31) we get that

$$(1 - |\lambda|^2)vPv^H = v\begin{pmatrix} L & K \end{pmatrix}\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}\begin{pmatrix} L^T \\ K^T \end{pmatrix}v^H \tag{4.41}$$

where $(.)^H$ denotes the conjugate-transpose. Since the right hand side of (4.41) is non-negative and its left hand side, non-positive, we must have

$$v\begin{pmatrix} L & K \end{pmatrix}\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = 0. \tag{4.42}$$

But from (4.38) and (4.40) we have

$$\lambda vLE = vLA - \lambda vKC. \tag{4.43}$$

¿From (4.42) and (4.43) it follows that

$$v\begin{pmatrix} L & K \end{pmatrix}\begin{pmatrix} \lambda E - A & Q & S \\ \lambda C & S^T & R \end{pmatrix} = 0 \tag{4.44}$$

which since $v(L \quad K) \neq 0$ ((4.40) implies $vL \neq 0$) contradicts the stabilizability assumption. Thus LA is stable.

We can next show that there exists a unique positive semi-definite solution of (4.31). Specifically, suppose that P^1 and P^2 are two such solutions, and let $[L^1 \quad K^1]$, $[L^2 \quad K^2]$ denote the corresponding matrices in (4.32). Then L^1A and L^2A both are stable, and, as shown in Appendix B

$$P^1 - P^2 = (L^2A)(P^1 - P^2)(L^1A)^T, \tag{4.45}$$

so that iterating (4.45)

$$P^1 - P^2 = \lim_{k\to\infty}(L^2A)^k(P^1 - P^2)[(L^1A)^T]^k = 0. \tag{4.46}$$

Finally we can show that P_j converges to P exponentially fast for any initial condition P_0. First note that $P_j^0 \leq P_j$ where P_j^0 is the error covariance for the problem starting from $P_0 = 0$. We already know that $P_j^0 \to P$. Thus if we can find a sequence W_j so that $P_j \leq W_j$ and $W_j \to P$ exponentially fast, we will be finished. We accomplish this by letting W_j be the error covariance of the estimator defined by the steady-state filter (4.37) for <u>all</u>

k, starting with the same initial estimate as the optimal Kalman filter. Thus $W_0 = P_0$ and $W_j \geq P_j$ for all $j \geq 0$. Furthermore from (4.39)

$$W_{j+1} = (LA)W_j(LA)^T + (L \;\; K)\begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix}\begin{pmatrix} L \\ K \end{pmatrix} \tag{4.47}$$

which, thanks to the stability of LA, converges exponentially fast to the unique positive semi-definite solution of

$$W = (LA)W(LA)^T + (L \;\; K)\begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix}\begin{pmatrix} L \\ K \end{pmatrix} \tag{4.48}$$

and a comparison to (4.31) yields that $W = P$, completing the proof. □

Again we note that the results of this section represent a generalization of those in [28, 33]. Furthermore they represent what is, to our knowledge, a new derivation for the more frequently studied singular estimation problem ($E = I$, R singular).

5 Construction of the Steady State Filter

In this section, we show that the solution of the algebraic descriptor Riccati equation can be constructed by using the eigenvectors and generalized eigenvectors of the pencil:

$$\left\{ \begin{pmatrix} E & -Q & S \\ C & S^T & -R \\ 0 & A^T & 0 \end{pmatrix}, \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E^T & C^T \end{pmatrix} \right\}. \tag{5.1}$$

A similar pencil was also introduced in [22] for the study of the LQ control problem for descriptor systems, although 3-block Riccati equations are neither introduced nor studied.

We shall assume throughout this section that the system is detectable and stabilizable. The results we present here generalize the usual results [17, 31, 32] for standard causal ($E = I$) systems for the case of singular measurement noise, and our results of [23, 28] to the case where R may be singular and in addition $\{E, A\}$ need not be regular nor even square. Before beginning, let us introduce the following notation:

$$F = \begin{pmatrix} E \\ C \end{pmatrix}, \quad K = \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} Q & -S \\ -S^T & R \end{pmatrix}. \tag{5.2}$$

The pencil (5.1) can now be expressed as

$$\left\{ \begin{pmatrix} F & -G \\ 0 & K^T \end{pmatrix}, \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix} \right\} \tag{5.3}$$

and the descriptor Riccati equation as

$$P = -\begin{pmatrix} 0 & I \end{pmatrix}\begin{pmatrix} KPK^T + G & F \\ F^T & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{5.4}$$

We begin with the following

Lemma 5.1 *The pencil (5.1) is regular and has no eigenmode on the unit circle.*

Proof: All we need to show is that for all z on the unit circle,

$$\begin{pmatrix} F & -G \\ 0 & K^T \end{pmatrix} + z \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix} \tag{5.5}$$

is invertible. Note that thanks to the detectability assumption which can now be stated in terms of the new notation as: "$sF - tK$ has full column rank for $(s,t) \neq (0,0)$ and $|s| \geq |t|$", we can see that $F + zK$ has full column rank for all z on the unit circle. Now suppose that (5.5) is not invertible, which means that there exist u and v not simultaneously null such that

$$\begin{pmatrix} F + zK & -G \\ 0 & K^T + zF^T \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{5.6}$$

If we now let

$$\Gamma = zK + F, \tag{5.7}$$

from (5.6) it follows that

$$\begin{aligned} \Gamma u - Gv &= 0 & (5.8)\\ \Gamma^H v &= 0. & (5.9) \end{aligned}$$

If we now multiply (5.8) and (5.9) on the left by v^H and u^H respectively and take the transpose-conjugate of (5.9) and subtract from (5.8), we get

$$vGv^H = 0 \tag{5.10}$$

which since G is symmetric positive semi-definite implies that $Gv = 0$. Thus since Γ has full column rank, (5.8) implies that $u = 0$. But we also have that $v^H(\Gamma \quad G) = 0$ which thanks to the stabilizability assumption implies $v = 0$, contradicting the assumption that u and v are not simultaneously null. □

Lemma 5.2 *The pencil (5.1) has exactly n stable eigenmodes.*

Proof: Let

$$\begin{aligned} p(s,t) &= \det\left(s \begin{pmatrix} F & -G \\ 0 & K^T \end{pmatrix} + t \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix}\right) \\ &= \det\left(s \begin{pmatrix} 0 & K^T \\ F & -G \end{pmatrix} + t \begin{pmatrix} 0 & F^T \\ K & 0 \end{pmatrix}\right) = \det \begin{pmatrix} 0 & sK^T + tF^T \\ sF + tK & -sG \end{pmatrix}. \end{aligned} \tag{5.11}$$

Then

$$p(t,s) = \det\left(t \begin{pmatrix} 0 & K^T \\ F & -G \end{pmatrix} + s \begin{pmatrix} 0 & F^T \\ K & 0 \end{pmatrix}\right) = \det \begin{pmatrix} 0 & tK^T + sF^T \\ tF + sK & -tG \end{pmatrix}. \tag{5.12}$$

From

$$\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix} \begin{pmatrix} 0 & sK^T + tF^T \\ sF + tK & -sG \end{pmatrix} \begin{pmatrix} I/t & 0 \\ 0 & I \end{pmatrix} =$$

$$\begin{pmatrix} I & 0 \\ 0 & sI \end{pmatrix} \begin{pmatrix} 0 & tK^T + sF^T \\ tF + sK & -tG \end{pmatrix}^T \begin{pmatrix} I/s & 0 \\ 0 & I \end{pmatrix}, \tag{5.13}$$

we find

$$t^{l+p}p(s,t)t^{-n} = s^{l+p}p(t,s)s^{-n}, \tag{5.14}$$

so that

$$t^{l+p-n}p(s,t) = s^{l+p-n}p(t,s). \tag{5.15}$$

If we denote the number of zero eigenmodes by δ_0, stable but non-zero eigenmodes by δ_s, unstable eigenmodes by δ_u and infinite eigenmodes by δ_∞, from (5.15) and the fact that there are no eigenmodes on the unit circle, we conclude that

$$\delta_s = \delta_u \tag{5.16a}$$

$$\delta_\infty - \delta_0 = l + p - n. \tag{5.16b}$$

Finally noting that

$$\delta_0 + \delta_s + \delta_u + \delta_\infty = n + l + p \tag{5.17}$$

we get that the number of stable eigenmodes $\delta_0 + \delta_s = n$. □

Theorem 5.1 *Let the columns of*

$$\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix}$$

form a basis for the eigenspace of the pencil (5.1) associated with its n stable eigenmodes, i.e.

$$\begin{pmatrix} E & -Q & -S \\ C & -S^T & -R \\ 0 & A^T & 0 \end{pmatrix} \begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E^T & C^T \end{pmatrix} \begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} J \tag{5.18}$$

where J is stable. Then, P, the unique positive semi-definite solution of the algebraic Riccati equation (4.19) is given by

$$P = X(E^TY_1 + C^TY_2)^{-1}. \tag{5.19}$$

Proof: Using notation (5.2) and letting

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \tag{5.20}$$

we must show that

$$P = X(FY)^{-1}. \tag{5.21}$$

To construct a real basis

$$\begin{pmatrix} X \\ Y \end{pmatrix}$$

and a real stable matrix J satisfying (5.18), we need only to compute the generalized real Schur decomposition ([15], p. 396)

$$Q^T \begin{pmatrix} F & -G \\ 0 & K^T \end{pmatrix} Z = M \tag{5.22a}$$

$$Q^T \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix} Z = N \tag{5.22b}$$

of the pencil (5.1), where Q and Z are orthogonal matrices, M is quasi-upper-triangular, N is upper triangular, and where the $n \times n$ blocks M_s and N_s in the decomposition

$$M = \begin{pmatrix} M_s & M_{s\hat{s}} \\ 0 & M_{\hat{s}} \end{pmatrix}, \quad N = \begin{pmatrix} N_s & N_{s\hat{s}} \\ 0 & N_{\hat{s}} \end{pmatrix} \tag{5.23}$$

correspond to the stable eigenmodes of the pencil (5.1), i.e. $J = N_s^{-1} M_s$ is stable. Then, if Z_s is the matrix formed by the first n columns of Z, we have

$$Z_s = \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{5.24}$$

¿From (5.18) we have

$$FX - GY = KXJ \tag{5.25}$$

$$K^T Y = F^T Y J. \tag{5.26}$$

Premultiplying (5.25) by Y^T and taking into account the transpose of (5.26), we find that $Y^T F X$ satisfies the Lyapunov equation

$$Y^T F X = Y^T G Y + J^T Y^T F X J. \tag{5.27}$$

Let us show that $F^T Y$ is invertible. Suppose that $F^T Y$ is not invertible, so that there exists $w \neq 0$ such that $F^T Y w = 0$. Then, from (5.27) we see that

$$GYw = 0 \tag{5.28}$$

so that

$$w^T Y^T \begin{pmatrix} F & G \end{pmatrix} = 0. \tag{5.29}$$

But the stabilizability assumption implies that $\begin{pmatrix} F & G \end{pmatrix}$ has full row rank, so that

$$Yw = 0. \tag{5.30}$$

Multiplying (5.26) on the right by w and using (5.30) we see that

$$F^T Y J w = 0. \tag{5.31}$$

Thus, we have shown that the right null space of $F^T Y$ is J-invariant. This implies that there exists an eigenvector $w \neq 0$ of J in the right null space of $F^T Y$, i.e.

$$F^T Y w = 0, \quad Jw = \lambda w. \tag{5.32}$$

Multiplying (5.25) on the right by w and taking into account (5.32) gives

$$(F - \lambda K)Xw = 0. \tag{5.33}$$

Since J is stable, $|\lambda| < 1$. The detectability assumption then implies that $F - \lambda K$ has full column rank, so that

$$Xw = 0. \tag{5.34}$$

Combining (5.30) and (5.34) yields

$$\begin{pmatrix} X \\ Y \end{pmatrix} w = 0 \tag{5.35}$$

and since $\begin{pmatrix} X \\ Y \end{pmatrix}$ has full column rank, we must have $w = 0$. This is a contradiction, so that $F^T Y$ must be invertible.

Now, if we solve for J in (5.26) and substitute it in (5.25) we obtain

$$FX = [G + KX(F^TY)^{-1}K^T]Y \tag{5.36}$$

from which we get

$$\begin{pmatrix} G + KX(F^TY)^{-1}K^T & F \\ F^T & 0 \end{pmatrix} \begin{pmatrix} Y \\ -X \end{pmatrix} = \begin{pmatrix} 0 \\ F^TY \end{pmatrix}. \tag{5.37}$$

This implies

$$\begin{pmatrix} Y \\ -X \end{pmatrix} (F^TY)^{-1} = \begin{pmatrix} G + KX(F^TY)^{-1}K^T & F \\ F^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{5.38}$$

so that if $P = X(F^TY)^{-1}$, we have

$$P = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} G + KPK^T & F \\ F^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}, \tag{5.39}$$

i.e. P satisfies the algebraic descriptor Riccati equation (4.19). Since Y^TFX solves the Lyapunov equation (5.39), it is positive semi-definite, so that

$$P = (F^TY)^{-T}(Y^TFX)(F^TY)^{-1} \tag{5.40}$$

is also positive semi-definite. □

6 An Adjusted Estimate to Account for "Future" Dynamics

The estimation problem we have considered in the preceding sections involved the recursive computation of estimates of $x(k)$ based on dynamics and observations only in the past and present. As we have pointed out, descriptor dynamics allow the possibility of noncausal behavior and thus it also would seem reasonable to consider the recursive computation of estimates that incorporate future dynamics. Specifically suppose that we now define the estimate $\hat{x}(j)$ as the ML estimate of $x(j)$ based on the true measurements (1.2) for $k = 0, ..., j-1$, the "measurement" (3.1) provided by the prior information about $x(0)$, and the dynamics (1.1) for all k, as opposed to $0 \leq k \leq j-1$ as we did previously. In the usual causal case, i.e. $l_k = n_k = n$, $E = I$, the inclusion of these "future" dynamics provide no additional information about $x(j)$, as they provide no constraints on $x(j)$. That is, consider

$$0 = E_{j+1}x(j+1) - A_j x(j) - u(j). \tag{6.1}$$

If $E_{j+1} = I$ (or more generally, if it is surjective) then since $x(j+1)$ is completely unknown in the ML formulation, (6.1) provides no constraint on $x(j)$. However if E_{j+1} is singular, (6.1) does provide nontrivial information about $x(j)$ (e.g. consider the extreme case of $E_{j+1} = 0$). In general, of course, the situation is even more complex, since $x(j+1)$ may also be subject to constraints due to dynamics farther into the future. In the general time-varying case there is no bound on how far into the future one must look in order to capture all possible dynamics. In particular in such a case what we would need to do at each time is to filter backward the "measurements" corresponding to future dynamics in order to obtain the correct adjustment to the forward filtered estimate developed in the previous sections. However in the time-invariant case, the structure of the information provided by the future dynamics, i.e.

$$Ex(k+1) = Ax(k) + u(k) \qquad k \geq j \tag{6.2}$$

is independent of j. In this section we show that in this case we can replace the effect of future dynamics with just one observation:

$$Fx(j) = w(j) \tag{6.3}$$

where w is zero-mean, and F and the covariance V of w are time-invariant. Thus the problem is to find F and V in terms of E, A and Q (the covariance of u).

Rewriting (6.2) as a matrix equation we obtain

$$\begin{pmatrix} -A & E & & & & \\ & -A & E & & & \\ & & & -A & E & \\ & & & & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x(j) \\ x(j+1) \\ x(j+2) \\ \vdots \end{pmatrix} = \begin{pmatrix} u(j) \\ u(j+1) \\ u(j+2) \\ \vdots \end{pmatrix}. \tag{6.4}$$

The relation (6.4) provides some information not only about $x(j)$, but also about the vectors $x(k)$ for $k > j$, which are not directly of interest and can be viewed as exogenous variables. In order to isolate the information about $x(j)$ that is contained in (6.4), our first step will be to bring (6.4) to the form (2.39), so that Lemma 2.4 can be applied. This requires using block row manipulations to eliminate the vectors $x(k)$ with $k > j$ from as many equations as we can, thereby enabling us to drop the remaining measurements. Specifically, suppose that

$$\begin{pmatrix} T_0 & T_1 & T_2 & \dots \end{pmatrix} \begin{pmatrix} -A & E & & & & \\ & -A & E & & & \\ & & & -A & E & \\ & & & & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} -T_0 A & 0 & 0 & \dots \end{pmatrix}. \tag{6.5}$$

Then, from (6.4), we get

$$-T_0Ax(j)=\sum_i T_iu(j+i) \tag{6.6}$$

which is of the form (6.3). So the problem becomes one of finding the highest row rank matrix satisfying (6.5). We can rewrite (6.5) as

$$T(z)(zE-A)=-T_0A \tag{6.7}$$

where $T(z)=T_0+zT_1+z^2T_2+...$ and thus we need to find the polynomial matrix $T(z)$ of largest rank such that

$$T(z)(zE-A)=\text{constant matrix.} \tag{6.8}$$

Let us denote the unknown right-hand side of (6.8) by F, and let $U(z)$ and S be respectively unimodular and permutation matrices for which

$$U(z)(zE-A)S=\begin{pmatrix} N(z) & K(z) \\ 0 & 0 \end{pmatrix} \tag{6.9}$$

where $N(z)$ is square and invertible. Then, if we denote

$$FS = \begin{pmatrix} F_1 & F_2 \end{pmatrix} \tag{6.10a}$$

$$T(z)U^{-1}(z) = \begin{pmatrix} T_1(z) & T_2(z) \end{pmatrix} \tag{6.10b}$$

we must have

$$T_1(z)N(z) = F_1 \tag{6.11a}$$

$$T_1(z)K(z) = F_2 \tag{6.11b}$$

which implies that

$$K(z)N^{-1}(z)F_1=F_2 \tag{6.12}$$

or equivalently,

$$\begin{pmatrix} F_1 & F_2 \end{pmatrix}\begin{pmatrix} K(z)N^{-1}(z) \\ -I \end{pmatrix}=0. \tag{6.13}$$

Constant solutions $\begin{pmatrix} F_1 & F_2 \end{pmatrix}$ to (6.13) can be constructed by noting that if

$$\begin{pmatrix} K(z)N^{-1}(z) \\ -I \end{pmatrix}=\frac{1}{p(z)}\left(\sum_{i=1}^{m} L_iz^i\right) \tag{6.14}$$

where $p(z)$ is a scalar polynomial and L_i's are constant matrices, then (6.13) is equivalent to

$$\begin{pmatrix} F_1 & F_2 \end{pmatrix}\begin{pmatrix} L_0 & L_1 & ... & L_m \end{pmatrix}=0. \tag{6.15}$$

Let $\begin{pmatrix} \tilde{F}_1 & \tilde{F}_2 \end{pmatrix}$ be a highest rank solution to (6.15). Let W be the highest rank (full row rank) matrix for which

$$W\tilde{F}_1N^{-1}(z)=\text{polynomial.} \tag{6.16}$$

Then, let

$$F_1 = W\tilde{F}_1, \ \ F_2 = W\tilde{F}_2. \tag{6.17}$$

We get

$$F = \begin{pmatrix} F_1 & F_2 \end{pmatrix} S^{-1} \tag{6.18}$$

and

$$T(z) = \begin{pmatrix} F_1 N^{-1}(z) & T_2(z) \end{pmatrix} U(z) \tag{6.19}$$

where $T_2(z)$ is any arbitrary polynomial matrix. It turns out that, without loss of generality, we can pick $T_2(z) = 0$. This is due to an implicit assumption that was made throughout this paper, namely that the dynamic equations are consistent for all possible choices of inputs $u(k)$, i.e. no constraints on the inputs is imposed by the dynamic equations. It is straightforward to verify that this requires

$$\text{Left-ker } Q \supset \text{Left-ker } [zE - A], \tag{6.20}$$

which is called the *compatibility* assumption.

To see why the compatibility assumption implies that the choice of $T_2(z)$ does not matter, simply note that thanks to (6.20) we have

$$\begin{pmatrix} 0 & T_2(z) \end{pmatrix} U(z)Q = 0. \tag{6.21}$$

Finally, we get

$$Fx(j) = \sum_{i=1}^{p} T_i u(j+i) \tag{6.22}$$

where $\sum_{i=1}^{p} T_i z^i = T(z)$. Thus we obtain (6.3) with

$$V = \sum_{i=1}^{p} T_i Q T_i^T. \tag{6.23}$$

Using (6.3), we can construct the "true" or "adjusted" descriptor Kalman estimate by correcting the result of the Kalman filter to incorporate this additional observation. In particular using the methods developed in the previous sections, we construct the optimal estimate $\hat{x}(j)$ of $x(j)$ based on past dynamics and observations. This "information" is completely coded by the observation

$$\hat{x}(j) = x(j) + \nu(j), \tag{6.24}$$

where $\nu(j)$ is a zero-mean Gaussian vector with covariance P_j, where $P - j$ satisfies the Riccati equation described previously. If we now add future dynamics, we have to find the optimal estimate of $x(j)$ based on the observation

$$\begin{pmatrix} \hat{x}(j) \\ 0 \end{pmatrix} = \begin{pmatrix} I \\ F \end{pmatrix} x(j) + \begin{pmatrix} \nu(j) \\ w(j) \end{pmatrix}. \tag{6.25}$$

Since future dynamics are independent of past dynamics and observations, the new Kalman estimate $\check{x}(j)$ and the corresponding error covariance $\check{P}_j$ are given by

$$\check{x}(j) = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} P_j & 0 & I \\ 0 & V & F \\ I & F^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} \hat{x}(j) \\ 0 \\ 0 \end{pmatrix} \tag{6.26}$$

and

$$\check{P}_j = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} P_j & 0 & I \\ 0 & V & F \\ I & F^T & 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{6.27}$$

7 Conclusions

In this paper we have derived Kalman filtering recursions for a general class of discrete-time descriptor systems, where the noise covariances were allowed to be singular. By using a Hamiltonian (or dual) formulation of the ML estimation problem, the optimal filter and the associated Riccati equation for the error covariance were expressed in 3-block form. In the time-invariant case, the asymptotic behavior of the optimal filter was examined and characterized in terms of the corresponding 3-block algebraic Riccati equation. Finally, under standard detectability and stabilizability conditions, it was shown that the positive semi-definite solution of the algebraic Riccati equation could be obtained by constructing the generalized Schur form of a 3-block matrix pencil.

Although we have focused primarily on descriptor systems, it is worth noting that, because of the 3-block forms we have introduced, our results already present a number of advantages over existing Kalman filtering techniques for systems with standard dynamics ($E = I$) but with singular measurement noise. For example, in the absence of redundant perfect information, the 3-block filter and Riccati equations of Theorem 3.1 require only standard matrix inverses, whereas solutions proposed until now require the use of pseudo-inverses (see [12], section 7.4).

One obvious direction in which the results of our paper can be extended consists in dualizing our results by considering the descriptor LQ control problem. Preliminary results in this direction appear in section 6 of [30]. Other interesting results for the descriptor LQ control problem have been derived by Bernhard, Grimm and Wang [7], and Mehrmann [22]. Another possible extension would involve considering the continuous-time descriptor Kalman filtering problem. Unfortunately, the continuous-time version of the problem discussed here may not be completely meaningful. This is due to the fact that unlike the discrete-time case, where the singularity of the system dynamics gives rise to a noncausal impulse response, for continous-time systems the singularity manifests itself by the fact that the output contains derivatives of the system input. Since for the fitering problem the input is a white Gaussian noise, the output will contain white-noise derivatives, thereby necessitating a formulation of the filtering problem in terms of generalized stochastic processes.

Appendix A: Some Results on Block Pseudo-Inverses

Here we summarize and specialize several of the results in [11] concerning the generalized inverse (in the sense of (2.20)) of the matrix

$$Z = \begin{pmatrix} R & H \\ H^T & 0 \end{pmatrix} \tag{A.1}$$

when H has full column rank, i.e. when $(H^T H)$ is invertible. Let

$$Z^\dagger = \begin{pmatrix} W & U \\ U^T & T \end{pmatrix} \tag{A.2}$$

denote any symmetric matrix satisfying (2.20), which in this case reduces to

$$\begin{aligned} RWR + RUH^T + HU^TR + HTH^T &= R && \text{(A.3)} \\ RWH + HU^TH &= H && \text{(A.4)} \\ H^TWH &= 0. && \text{(A.5)} \end{aligned}$$

In [11] the following results are proved:

$$RWH = 0 \tag{A.6}$$

$$D = R - RWR \text{ is uniquely determined by } R \text{ and } H \tag{A.7}$$

$$\begin{aligned} HTH^T &= -D && \text{(A.8)} \\ RUH^T &= D. && \text{(A.9)} \end{aligned}$$

Note next that from (A.8) and the invertibility of $H^T H$, we have that T is also unique and given by

$$T = -(H^TH)^{-1}H^TDH(H^TH)^{-1}. \tag{A.10}$$

Also, from (A.4), (A.6) and the invertibility of $H^T H$

$$U^TH = (H^TH)^{-1}(H^TH) = I \tag{A.11}$$

proving (4.27) of Lemma 4.2, so that U^T is a left inverse[6] of H. Thus from (2.21)

$$\mathcal{M}(\hat{x}_{ML}) = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} W & U \\ U^T & T \end{pmatrix} \begin{pmatrix} H \\ 0 \end{pmatrix} x = x. \tag{A.12}$$

Also from (2.21)

$$P_{ML} = \begin{pmatrix} 0 & I \end{pmatrix} Z^\dagger \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} Z^\dagger \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{A.13}$$

[6]Note that U^T is not an arbitrary left-inverse of H as there is the additional constraint (A.9) that U must satisfy.

However from (A.8), (A.9) and (A.11) we see that

$$P_{ML} = U^T DH(H^T H)^{-1} = -U^T HTH^T H(H^T H)^{-1} = -T \tag{A.14}$$

proving (2.22) as well as (4.26) of Lemma 4.2.

Finally, to prove identity (2.25), we note that by summing (A.8) and (A.9) and post-multiplying by $H(H^T H)^{-1}$, we obtain

$$RU + HT = 0, \tag{A.15}$$

which when combined with (A.11) gives (2.25).

Appendix B: Derivation of (4.45)

Let

$$\Omega_i = \begin{pmatrix} AP_i A^T + Q & -S & E \\ -S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}, \quad i = 1, 2. \tag{B.1}$$

Then,

$$P^1 - P^2 = \begin{pmatrix} 0 & 0 & I \end{pmatrix} (\Omega_2^\dagger - \Omega_1^\dagger) \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{B.2}$$

¿From identity (2.25), there exists Δ such that

$$\begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} = \Omega_2 \Delta \tag{B.3}$$

so that

$$\Omega_2 \Omega_2^\dagger \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} = \Omega_2 \Omega_2^\dagger \Omega_2 \Delta \tag{B.4}$$

which using the property (2.20) of pseudo-inverses yields

$$\Omega_2 \Omega_2^\dagger \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{B.5}$$

Similarly we can show that

$$\begin{pmatrix} 0 & 0 & I \end{pmatrix} \Omega_1 \Omega_1^\dagger = \begin{pmatrix} 0 & 0 & I \end{pmatrix}. \tag{B.6}$$

Now, by using identities (B.5) and (B.6) in (B.2) we get

$$P^1 - P^2 = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \Omega_2^\dagger (\Omega_1 - \Omega_2) \Omega_1^\dagger \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{B.7}$$

But

$$\Omega_1 - \Omega_2 = \begin{pmatrix} A(P^1 - P^2)A^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} A(P^1 - P^2)A^T \begin{pmatrix} I & 0 & 0 \end{pmatrix}. \tag{B.8}$$

Thus,

$$\begin{aligned} P^1 - P^2 &= \left(\begin{pmatrix} 0 & 0 & I \end{pmatrix} \Omega_2^{\dagger} \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \right) A(P^1 - P^2)A^T \left(\begin{pmatrix} I & 0 & 0 \end{pmatrix} \Omega_1^{\dagger} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \right)^T \\ &= (L^2 A)(P^1 - P^2)(L^1 A^T). \end{aligned} \tag{B.9}$$

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