Institutional Theory of Naive Money

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Abstract

In the first chapter, I propose a theoretical framework to elucidate how capital from unsophisticated investors (naive money) is associated with fund performance dynamics. In the framework, when naive money invested in a fund exceeds the ideal amount for the manager's skill, it leads funds to underperform persistently. In contrast, the model predicts that, when the amount of invested naive money is smaller than the ideal size of a fund reflecting the manager's skill, the fund performs the same as the market on a risk-adjusted basis. Empirical results using mutual fund data support this prediction.

In the second chapter, I develop a model that characterizes how naive money influences the decisions of active mutual fund managers: in particular, managerial effort, fees, marketing expenses, private benefit-seeking, and risk-taking. My model predicts that managers who receive a surplus of naive money are inclined to reduce their managerial effort, charge higher fees, allocate more resources towards marketing, and pursue their private benefit by sacrificing returns to investors. In addition, it also predicts that a manager is most likely to increase idiosyncratic risk when the amount of invested naive money gets closer to a certain size of the fund that reflects the manager's skill.

In the third chapter, I build a model to study how naive money affects funds' survivorship and entry decisions. Sufficient capital provision from unsophisticated investors elongates the survival of unskilled managers. Competition among funds determines the industry equilibrium, and the equilibrium is affected by several key market conditions: the aggregate investment opportunities, the aggregate capital inflows from unsophisticated investors, and the supply of skilled managers. When AM markets are heterogeneous in investor sophistication, the model shows, AM markets with more sophisticated investors (say, hedge fund markets) differentiate from those with less sophisticated investors (say, mutual fund markets). Skilled managers generate more value in hedge fund markets, and choose to enter those markets.

Thesis Supervisor: Leonid Kogan
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Dedication

To Steve Ross

Acknowledgments

Firstly, I would like to thank God for his endless grace, mercy and love.

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Introduction

Why do active asset management (henceforth, AM) markets exist and persist at all? This question has puzzled many financial economists, since neoclassical finance and neoclassical economics, despite their kindred standpoints on most issues in financial economics, give starkly different answers to the question. The answer from the conventional perspective of neoclassical finance is that market efficiency prevents fund managers from beating the market even before fees but, nonetheless, actively managed funds exist since all the investors in those funds are simply irrational. In contrast, traditional neoclassical economists take the view that investors are rational, and that the active AM industry persists because active fund managers have skill. These two different intellectual traditions offer sharply contrasting views on the rationality of investors and the existence of active managers' skill.

Empirical findings on the performance of active mutual funds are in line with the view of neoclassical finance. Active mutual funds underperform the market net of fees on average (e.g., Fama and French (2010)), and persistently outperforming funds are hard to find (e.g., Carhart (1997)). The former evidence supports a view that (at least a significant portion of) investors in active mutual funds are irrational. The latter evidence appears to be consistent with a view that active mutual fund managers rarely have skill. These two high-profile empirical results seem unfavorable to neoclassical economists' view, where investors are rational and active managers have skill.

In order to overcome inconsistency with the empirical evidence, the seminal work of Berk and Green (2004) refines the view of neoclassical economics. From the perspective of traditional neoclassical economists, active AM markets are markets for managers' skill, i.e., rational investors look for skilled managers and invest in those managers' funds. On the other hand, the perspective of asset market efficiency tells us that the average risk-adjusted excess return on active funds must be zero after fees, as long as investors are rational and they consider active funds as one type of asset class. Reconciling these two perspectives leads to a seemingly peculiar idea: active managers are skilled, but in equilibrium they perform the same as the market on a risk-adjusted basis. This idea is consistent with the non-persistence of good performance, but fails to account for the average underperformance in active mutual fund markets.

Since neoclassical economists' view on the active AM industry is not consistent with (some) empirical findings on performance, can't we simply take the view of neoclassical finance and move on? In fact, this view, where active managers do not have skill and there are only irrational investors in active funds, has difficulties in explaining the cross-sectional fund performance. There are many empirical studies (e.g., Gruber (1996)) showing that some active funds perform better than others do, and may outperform the market in the short run. The traditional view of neoclassical finance cannot very well accommodate such empirical findings.

In this thesis, I attempt to fill this lacuna by developing a theoretical framework for understanding the AM industry that reconciles the perspectives of neoclassical finance and neoclassical economics. This thesis adopts the view of neoclassical finance that there is a substantial number of irrational investors in AM markets. On the other hand, this thesis adopts a view that skilled managers exist in AM markets, partially taking the perspective of neoclassical economics, and this is crucially different from that of neoclassical finance. This new framework provides a natural explanation for the two major empirical findings in active mutual fund markets: the non-persistence of good performance and the average underperformance.

Although Berk and Green (2004) do not take irrational investors into account, it is worth highlighting their insight into the efficiency of AM markets. When skill is subject to decreasing returns to scale, fierce competition among investors in capital provision drives the performance of active funds down to that of the passive benchmarks. Since investors in active funds break even, the entire value from skill goes to the managers in the form of fee profits. Therefore, from the investor rationality perspective, the competitive nature of capital markets translates into the efficiency of AM markets as markets for skill: rational investors correctly evaluate and compensate managers' skill.

By introducing irrational investors, this thesis opens up the possibility of inefficiency in AM markets: overpricing of managers' skill. Decreasing returns to scale at the fund level imply that there is a fund size that correctly prices the manager's skill, given the fee schedule. If there is only a small amount of capital from irrational investors (henceforth, "naive money" or "naive capital") invested in a fund, rational investors flow into the fund until the fund reaches its correct size and, consequently, the manager's skill is fairly priced. In contrast, if the amount of naive money invested in a fund exceeds that size, the skill of the manager is overpriced, i.e., the manager receives fee revenues greater than the surplus (or value, interchangeably) that she is expected to generate. However, rational investors cannot correct for such overpricing, since they are not able to short-sell funds.

This thesis studies how the overpricing of managers' skill shapes AM markets. One of the most prominent implications of the new theoretical framework is the performance dynamic of active funds. In equilibrium, overpriced funds underperform, and the rest perform the same as the market on a risk-adjusted basis. Moreover, the underperformance of overpriced funds persists. Such performance dynamics of active funds, particularly that of active mutual funds, is studied in chapter 1.

Overpricing of skill distorts managers' incentives, and changes their behavior. Smart investors induce managers to make their best efforts to create value for investors. However, the possibility of skill overpricing may lead to deviation of managers' behavior from the optimal behavior from the perspective of smart investors. Existing managers may choose to adjust their choices (intensive margin) such as fees, effort, etc.. Managers may even want to change their entry and exit decisions (extensive margin). These industrial organization (IO) aspects of the AM industry are studied in chapter 2 and 3.

In chapter 1, I build a model that associates naive money with fund performance dynamics. If funds receive excessive capital from naive investors exceeding the fair size, at which the managers' skill is fairly priced, those funds underperform persistently. In contrast, when funds only receive small amounts of capital from naive investors, those funds perform the same as the market on a risk-adjusted basis. Empirical tests using mutual fund data are consistent with these predictions.

In chapter 2, I propose a model of active mutual fund managers' decisions, particularly effort, fees, idiosyncratic risk, marketing and private benefit-seeking. More invested naive money is associated with less effort, higher fees, more marketing and more private benefit. In addition, managers choose to take the maximum amount of idiosyncratic risk when fee profits from naive investors are at the same magnitude of those from smart investors.

In chapter 3, I develop a model to examine how naive money influences managers' entry and exit decisions, and how those decisions shape the AM industry. In particular, when there is heterogeneity of investor sophistication across markets, AM markets with smarter investors (say, hedge fund markets) differentiate from those with naive investors (say, mutual fund markets). Hedge funds attract relatively skilled managers compared to mutual funds do. In addition, hedge fund managers create more value than they would generate by managing mutual funds.

Lastly, I would like to emphasize that the conclusions of this thesis do not rely on a set of possibly questionable premises regarding the behavioral patterns of irrational investors. Theoretical studies that involve irrational investors are often criticized for being sensitive to how their behavior is modeled. Since we lack a satisfactory understanding of investor choice in AM markets, such sensitivity is an undesirable feature for theories. Here, the only crucial assumption in chapter 1 is that naive money is persistent. In chapter 2, in addition to the assumption of chapter 1, I make an assumption that unsophisticated investors are less sensitive to fees than sophisticated investors are. In chapter 3, I make an additional assumption regarding naive money: competition among funds deteriorates the average naive money flow that a fund can attract. The modesty of these assumptions should assuage the concern about theories involving irrational investors.

The remainder of this thesis is as follows. In the remainder of the introduction, related studies on active AM markets, especially those on active mutual fund markets, are briefly discussed. Chapter 1 discusses the implication of the theoretical framework for the performance dynamics of active funds, and presents the analysis of empirical tests on those implications. Chapter 2 theoretically investigates how naive money affects the decisions of active mutual fund managers, particularly fees, effort, idiosyncratic risk, marketing and seeking private benefit. Chapter 3 theoretically examines how naive money influences entry and exit decisions of managers in the AM industry, and how changes in those decisions shape the AM industry, particularly when AM markets are heterogeneous in investor sophistication.

Related literature

Empirical studies on the average underperformance of active mutual funds go back to Jensen (1968). More recent studies (Gruber (1996), French (2008), Fama and French (2010)) confirm the underperformance. Carhart (1997) is a representative study that empirically shows the non-persistence of good performance. I would like to emphasize Fama and French (2010), especially their contribution showing that the aggregate portfolio of actively managed US equity mutual funds is close to the market (with 99% R^2), but underperforms the market after fees roughly at the magnitude of the average fee. This result makes it difficult to argue that the average underperformance of active mutual funds is simply due to benchmark misspecification or missing risk factors.

Many empirical studies on active mutual funds associate the characteristics of funds with their future performance in the short run (usually the following quarter). Gruber (1996) finds that the past risk-adjusted performance (alpha) predicts the future performance. Similarly, the "return gap" (Kacperczyk, Sialm and Zheng (2008)) and the "active share" (Cremers and Petajisto (2009)) predict performance. Cohen, Coval and Pastor (2005) find that managers whose portfolios consist of stocks that other successful managers hold outperform. These studies suggest that, to a certain extent, a group of managers have the ability to make portfolio choices that outperform the market in the short run.

There are theoretical studies on the active AM industry based on the standard notion of investor rationality. Berk and Green (2004) is the first study that theoretically addresses the non-persistence of the performance of active mutual funds, by assuming decreasing returns to scale at the fund level. Pastor and Stambaugh (2012) assumes decreasing returns to scale at the industry level, and justifies the substantial size of the active AM industry despite the average poor track record. Glode (2011) associates the underperformance of active mutual funds with the counter-cyclical component of their performance, which provides insurance against market downturns. One common issue with these approaches is that it is hard to see how active funds can underperform on a risk-adjusted basis without assuming the existence of irrational investors.

Among many empirical studies that document the irrationality of investor decisions (e.g., Benartzi and Thaler (2001)), a set of empirical studies that investigate the role of brokers and/or financial advisors in the AM industry are particularly interesting. Bergstresser, Chalmers and Tufano (2009) show that brokers do not benefit investors in most tangible dimensions. Del Guercio and Reuter (2014) provide evidence that funds sold through brokers underperform index funds. These studies show not only that unsophisticated retail investors exist, but also that the AM industry possesses effective means of directing those investors towards underperforming funds.

Some theoretical studies on the active AM industry are based on irrational or "distorted" behavior of investors. A study by Gennaioli, Shleifer and Vishny (2015) proposes a trust-based model in which trust distorts the investors' perception of riskiness, and allows the manager to charge high fees despite the underperformance. Carlin (2009) presents a model of retail financial markets in which firms make their price structure complex in order to attract less knowledgeable consumers. While there are other pertinent theoretical studies (e.g., Gabaix and Laibson (2006)) that are not explicitly written in the context of the AM industry, their implications for the structure of that industry fall outside the scope of this thesis.

Chapter 1

Theory and Evidence: Mutual Fund Performance Dynamics

1.1 Introduction

Are investors in active AM markets rational or not? It is quite well documented that active mutual funds on average underperform the market (e.g., Jensen (1968)). Hence, aggregate investors in active mutual fund markets do not seem to make optimal capital allocation decisions. This supports the view that a non-negligible portion of investors, especially those in active mutual fund markets, are not fully rational. The term 'naive money' well captures the nature of such investors, although there is an ambiguity in the precise meaning of this term.

The rationality of investors is closely linked to the question whether active fund managers have skills or not. If managers do not have any skill, provided that investors pay fees, all the investors in active management markets must be 'naive'. On the other hand, if all the investors are not able to evaluate managers' skill correctly, managers do not have any incentive to realize their skill. Therefore, the existence of rational investors in active AM markets implies the existence of skilled managers.

Surprisingly, when the average risk-adjusted return (alpha) is used to measure active mutual fund managers' skill, the evidence is not favorable to the view that managers have skills. Compared to the market, active mutual funds underperform on average (e.g., Jensen (1968)). Besides, the performance of active mutual funds does not persist (e.g., Cahart (1997)), with an exception that poor performance persists. In addition, in order to examine whether alphas of active funds come from luck or skill, Fama and French (2010) conducted a thorough cross-sectional study of mutual funds, and found little evidence for skill.

However, if alpha does not correctly measure active managers' skill, there may well be skilled managers. A rational framework introduced by Berk and Green (2004) theoretically justifies that view. According to their model, rational investors identify active funds that exhibit skills and flow into those funds. But there is a limit to how much a fund can scale up its skill, because of diseconomies of scale (decreasing returns to scale). In equilibrium, a skilled active fund grows up to a size such that the net alpha becomes zero, and, as a result, the fund performs exactly the same as the market. In Berk and Green (2004)'s framework, active managers have skills, but the managers do not show any superior alpha to the market. Their framework suggests that alpha may not be a good measure for managers' skill.

Berk and Green (2004) is the model of 'smart money' as opposed to 'naive money': all the investors are rational in active AM markets. Yet, even empirical studies that do not involve alpha are not quite supportive of this view. Previous studies (e.g., Frazzini and Lamont (2008)) find that aggregate investors in mutual fund markets lose wealth in the long run from their capital allocation decisions. Moreover, there is more direct evidence on the irrationality of investors in studies of investors' choice of funds within 401(k) plans (e.g., Agnew and Balduzzi (2012); Madrian and Shea (2001)).

In this chapter, I propose a theoretical framework of AM markets, where active managers have skill, and both smart and naive investors participate in the AM markets. This framework can be viewed as an extension of Berk and Green (2004) in a sense that funds are subject to decreasing returns to scale. Due to diseconomies of scale, there exists a fund size that correctly compensates the manager's skill, and I call the size the 'fair size'. If the amount of invested naive money exceeds the fair size, smart investors withdraw all of their capital from that fund, and the fund is dominated by naive money. In this case, the skill of the manager is overpriced. On the other hand, if the amount of invested naive money is smaller than the fair size, smart investors provide capital up to the fair size. In this case, the marginal investors are smart investors, and the skill of the manager is fairly priced.

In the equilibrium of the model, overpriced funds underperform, and the rest perform the same as the market on a risk-adjusted basis. Moreover, the underperformance of overpriced funds persists. When a manager's skill is overpriced, unsophisticated investors subsidize the overpriced amount in the form of negative average excess returns (alpha). Since sophisticated investors do not invest in funds with negative alpha, only unsophisticated investors remain invested in those funds. As long as naive capital invested in overpriced funds persists, those funds continue being overpriced and, hence, keep underperforming. On the other hand, when a manager's skill is fairly priced, both the sophisticated and unsophisticated investors are invested in the fund, and those investors receive zero risk-adjusted excess returns on average. Since funds, at best, perform the same as the market, good performance does not persist. In addition, because some funds underperform while the others do not outperform, on average active funds underperform.

Using mutual fund data, I test a prediction that funds dominated by naive money underperform. The main challenge for this test is identifying those overpriced funds, since both skill and the amount of naive money are not directly observable. In particular, since size and skill are correlated, large funds are not necessarily dominated by naive money. In order to overcome the challenge, I identify funds with the worst track records but the largest capital flows as most likely to be overpriced. In contrast, funds with the best track records but the smallest capital flows are identified as least likely to be overpriced. Results are consistent with the prediction that overpriced funds underperform, and fairly priced funds neither underperform nor outperform. Additional empirical tests support the mechanism of the model as well.

In the remainder of the chapter, section 2 presents the model. Section 3 discusses the implications of the model for the performance dynamics of active funds. Section 4 presents the analysis of empirical tests on those implications. Section 5 discusses the limitation of the model and future directions for research, and summarizes the conclusions.

1.2 The Model

In this model, funds charge a flat proportional fee $f dt$ between t and $t + dt$. I assume that f is constant and uniform across funds[1]. Figure 1 illustrates types of investors and funds in the model:

[See figure 1]

1.2.1 Heterogeneity in skill across managers

There are two types of active managers: skilled (high or H-type) and unskilled (low or L-type). Skilled managers generate surplus while unskilled managers cannot create any value. A skilled manager faces decreasing returns to scale (DRS) as her assets under management (AUM) increase. I assume an extreme form of DRS at fund level: a skilled manager can generate a fixed amount of value (per time) regardless of her AUM. Qualitative results of the chapter do not depend on the specific form of DRS. A skilled manager generates

$$A\left(dt + \frac{1}{s}dZ_t\right)$$

over the passive benchmark between t and $t + dt$, where A is the average dollar value per time that the manager creates, Z_t is a standard Brownian motion, and s is signal-to-noise ratio of the fund's excess return. Hence, a skilled manager creates surplus on average, but there is uncertainty associated with the surplus. The uncertainty is idiosyncratic across managers, i.e., for manager i and j

$$dZ_{i,t} \cdot dZ_{j,s} = 0 \quad , \quad \forall t, s \ .$$

An unskilled manager generates no value on average, but the volatility of the

[1]Endogenous choice of fees is addressed in chapter 2.

excess return over the passive benchmark is the same as that of a skilled manager. This guarantees that the skill of managers is not immediately revealed[2].

1.2.2 Information and learning

There is no informational asymmetry on managers' skill between investors and managers, i.e., all the agents in the model share the same information about the managers' skill. When a new manager enters an AM market, the manager and investors have the same prior on the manager's skill. New information about the manager's skill comes solely from the manager's performance over time. From the manager's track record, both investors and the manager learn about her skill. Since the performance of managers is public information, all the agents solve the same inference problem. Therefore, all the agents share the same posterior on managers' skill at all times. The signal-to-noise ratio s determines the speed of learning: small s implies that the learning process is slow, and large s implies the opposite.

The probability distribution of a manager's skill follows the Bernoulli distribution, since the manager is either skilled (H-type) or unskilled (L-type). At time t, the probability $p_{i,t}$ of being H-type is a sufficient statistic for the probability distribution of skill for manager (fund) i, given the prior and track record of the manager's performance. Unless the type of manager i is known with certainty, agents cannot directly observe the physical Brownian motion $Z_{i,t}$, which governs the uncertainty of the value created by manager i. Instead, define

$$d\tilde{Z}_{i,t} = s(\mathbb{1}_{i,H} - p_{i,t})dt + dZ_{i,t} \ , \tag{1.1}$$

where $\mathbb{1}_{i,H}$ is 1 if manager i is H-type and 0 otherwise. $\tilde{Z}_{i,t}$ is a standard Brownian motion under the available information set at time t. Observing the track record of

[2]Since the model is in continuous time, the volatility of skill can be accurately measured in an infinitesimally short period of time. If the volatility of the excess return is different between skilled managers and unskilled managers, agents can instantaneously infer whether a manager is skilled or not.

the manager between t and $t + dt$, the Bayes' rule for $p_{i,t}$ is

$$dp_{i,t} = sp_{i,t}(1 - p_{i,t})d\tilde{Z}_{i,t} \ ,$$

using results on filtering from Liptser and Shiryayev (1977). Since $d\tilde{Z}_{i,t}$ is orthogonal to $d\tilde{Z}_{j,t}$ for all $j \neq i$, the learning process for one manager's skill is independent of that for another manager's skill.

1.2.3 Naive investors

Regarding fund performance dynamics, what matters is the amount of invested capital from naive investors. Hence, I do not explicitly model capital allocation decisions of naive investors. Rather, I model the amount of invested naive money in a reduced form. In order to obtain the fund performance results of this chapter, I only need one assumption: the amount $\tilde{q}_t$ of invested naive money is persistent. In continuous-time setup, this assumption is automatically guaranteed as long as there are no jumps in $\tilde{q}_t$. Any process of $\tilde{q}_t$ satisfying this assumption (e.g., geometric Brownian motion, Cox-Ingersoll-Ross process, etc.) results in fund performance results that follow in the next section. Note that due to the short-sale constraint, $\tilde{q}_t$ cannot be negative.

1.3 Theoretical Results

In this section, I first consider a case without naive investors as a benchmark, and then consider general cases with naive investors.

1.3.1 Benchmark: no naive investors

Surplus that a manager generates is distributed to the investors, in proportion to the amount of investment that each investor made in the fund. Therefore, all the investors in a certain fund receive the same (gross and net) excess return to the passive benchmark. The gross excess return on a fund between t and $t + dt$ is given by

$$dR_t^{ex} = \frac{A\left(\mathbb{1}_H dt + \frac{1}{s} dZ_t\right)}{q_t} ,$$

where $\mathbb{1}_H$ is 1 if the manager of the fund is H-type and 0 otherwise, and q_t is the AUM of the fund. Investors pay a proportional fee $f dt$ between t and $t + dt$. Hence, the net excess return on a fund between t and $t + dt$ is

$$dr_t^{ex} = dR_t^{ex} - f dt = \frac{A\left(\mathbb{1}_H dt + \frac{1}{s} dZ_t\right)}{q_t} - f dt .$$

Investors can diversify away idiosyncratic risks by themselves. Therefore, the stochastic discount factor does not depend on idiosyncratic risks. If the net expected excess return (net alpha) on a fund is strictly positive, sophisticated investors are strictly better off increasing their investment in the fund slightly. As a result, the net alpha must be non-positive in equilibrium.

On the other hand, if the net alpha is strictly negative, sophisticated investors do not invest any dollars in the fund. Therefore, strictly negative net alpha cannot constitute an equilibrium. Therefore, the net alpha must be zero in equilibrium. Mathematically, this statement translates into

$$\mathbb{E}_t[dr_t^{ex}] = 0 \quad \Longleftrightarrow \quad q_t = \frac{p_t A}{q_t} \equiv q^*(p_t) ,$$

where $q^*(p)$ is the fair size of the fund. The skill of the fund manager is correctly priced and compensated: between t and $t + dt$ the manager receives fee revenues of

$$f q_t dt = p_t A dt = \mathbb{E}_t\left[A\left(\mathbb{1}_H dt + \frac{1}{s} dZ_t\right)\right] ,$$

which is the expected surplus generated by the manager. Therefore, when there are no naive investors, funds perform the same as the market on a risk-adjusted basis.

1.3.2 With naive investors

If the net alpha on a fund is strictly positive, sophisticated investors are strictly better off increasing their investment in the fund slightly. As a result, the net alpha must be non-positive in equilibrium. On the other hand, if the net alpha on a fund is strictly negative, sophisticated investors are strictly better off decreasing their investment in the fund. However, since investors cannot short-sell funds, there is no way that sophisticated investors can benefit beyond withdrawing the entire investment from the fund. Therefore, the net alpha on a fund must be either zero, if any sophisticated investor is remaining in the fund, or negative, if only unsophisticated investors remain invested in the fund.

Mathematically, these statements translate into

$$\mathbb{E}_t[dr_t^{ex}] = \left(\frac{p_t A}{q_t} - f\right) dt = \begin{cases} 0 & , \quad \tilde{q}_t < q_t \\ \leq 0 & , \quad \tilde{q}_t = q_t \end{cases} ,$$

where q_t is the AUM of the fund at t, and $\tilde{q}_t$ is the amount of naive capital invested in the fund at t. When there are sophisticated investors invested in the fund,

$$q_t = q^*(p_t) = \frac{p_t A}{f} ,$$

where $q^*(p)$ is the fair size of the fund. As long as $\tilde{q}_t < q^*(p_t)$, i.e., naive capital does not exceed the fair size of the fund, the net alpha of the fund is zero, and sophisticated investors remain invested in the fund. The amount of invested capital from sophisticated investors in the fund is $q^*(p_t) - \tilde{q}_t$. In this case, the skill of the fund manager is correctly priced and compensated: between t and $t + dt$ the manager receives fee revenues of

$$f q_t dt = p_t A dt = \mathbb{E}_t \left[A \left(\mathbb{1}_H dt + \frac{1}{s} dZ_t \right) \right] ,$$

which is the expected surplus generated by the manager.

If $\tilde{q}_t \geq q^*(p_t)$, i.e., the amount of naive capital exceeds the fair size of the fund,

$$\mathbb{E}_t[dr_t^{ex}] = \left(\frac{p_t A}{q_t} - f\right) dt \leq \left(\frac{p_t A}{\tilde{q}_t} - f\right) dt \leq \left(\frac{p_t A}{q^*(p_t)} - f\right) dt = 0 \; ,$$

where the first inequality comes from the fact that investors cannot short-sell funds. Hence, the net alpha on a fund is negative when the amount of naive capital goes beyond the fair size of the fund. Since the net alpha is negative, sophisticated investors do not invest in the fund, and only unsophisticated investors remain in the fund, i.e., $q_t = \tilde{q}_t$. In this case, the skill of the fund manager is overpriced: between t and $t + dt$ the manager receives fee revenues of

$$f q_t dt = f \tilde{q}_t dt \geq p_t A dt = \mathbb{E}_t \left[A \left(\mathbb{1}_H dt + \frac{1}{s} dZ_t \right) \right] \; .$$

The fund manager is compensated more than the surplus that she is expected to generate, if there is excessive naive capital invested in the fund. Therefore, the manager does not have an incentive to block capital from unsophisticated investors. The following proposition summarizes the analysis.

Proposition 1.1 *The AUM of a fund is*

$$q_t = \max\left\{ \frac{p_t A}{f}, \tilde{q}_t \right\} \; ,$$

i.e., the AUM is the fair size $\frac{p_t A}{f}$ *if the amount of invested naive capital is smaller than the fair size, and the AUM is* $\tilde{q}_t$ *if the amount of invested naive capital exceeds the fair size.*

The net expected excess return on the fund is

$$\mathbb{E}_t[dr_t^{ex}] = \left(\frac{p_t A}{q_t} - f\right) dt = \begin{cases} 0 & , \quad \tilde{q}_t \leq \frac{p_t A}{f} \\ < 0 & , \quad \tilde{q}_t > \frac{p_t A}{f} \end{cases} \; ,$$

i.e., the fund underperforms if the amount of naive capital exceeds the fair size.

The investors are a mix of sophisticated and unsophisticated investors if the net *alpha is zero, and only unsophisticated investors remain invested if the net alpha is negative.*

Proof. Proof provided in the above analysis. ■

The mechanism of skill overpricing is graphically illustrated in figure 2:

[See figure 2]

The following corollary gives some intuitions regarding flows to funds:

Corollary 1.1 *Given the skill of a manager at t and $t+T$ (p_t and p_{t+T}), flows to the fund between t and $t+T$ are*

$$\Delta q_{t,t+T} = q_{t+T} - q_t = \max\left\{\frac{p_{t+T}A}{f}, \tilde{q}_{t+T}\right\} - \max\left\{\frac{p_t A}{f}, \tilde{q}_t\right\} .$$

The flows are the largest for $\tilde{q}_{t+T} = q_{t+T}$ and $\tilde{q}_t < q_t$, and the smallest for $\tilde{q}_{t+T} < q_{t+T}$ and $\tilde{q}_t = q_t$, i.e., given p_t and p_{t+T}, flows are the largest if fairly priced funds become overpriced, and the smallest if overpriced funds become fairly priced.

Proof. If $\tilde{q}_{t+T} = q_{t+T}$ and $\tilde{q}_t < q_t$, then

$$\Delta q_{t,t+T} = \tilde{q}_{t+T} - \frac{p_t A}{f} \geq \frac{p_{t+T}A}{f} - \frac{p_t A}{f} ,$$

i.e., flows to overpriced funds that were previously fairly priced are always greater than flows to continuously fairly priced funds, and

$$\Delta q_{t,t+T} = \tilde{q}_{t+T} - \frac{p_t A}{f} \geq \tilde{q}_{t+T} - \max\left\{\frac{p_t A}{f}, \tilde{q}'_t\right\} , \quad \forall \tilde{q}'_t ,$$

i.e., flows to overpriced funds that were previously fairly priced are always greater than flows to continuously overpriced funds.

Similarly, if $\tilde{q}_{t+T} < q_{t+T}$ and $\tilde{q}_t = q_t$,

$$\Delta q_{t,t+T} = \frac{p_{t+T}A}{f} - \tilde{q}_t \leq \frac{p_{t+T}A}{f} - \frac{p_t A}{f} ,$$

i.e., flows to fairly priced funds that were previously overpriced are always smaller than flows to continuously fairly priced funds, and

$$\Delta q_{t,t+T} = \frac{p_{t+T}A}{f} - \tilde{q}_t \leq \max\left\{\frac{p_{t+T}A}{f}, \tilde{q}'_{t+T}\right\} - \tilde{q}_t \ , \quad \forall \tilde{q}'_{t+T} \ ,$$

i.e., flows to fairly priced funds that were previously overpriced are always smaller than flows to continuously overpriced funds.

Combining these two sets of results, given p_t and p_{t+T}, flows are the largest to overpriced funds that were previously fairly priced, and the smallest to fairly priced funds that were previously overpriced. ■

Lastly, the following corollary provides additional implications for the magnitude of underperformance:

Corollary 1.2 *When a fund is overpriced, i.e., the amount of invested naive money exceeds the fair size of the fund, the magnitude of underperformance increases in the amount $\tilde{q}_t$ of naive capital, and decreases in the expected skill $p_t A$.*

Proof. When a fund is overpriced, the magnitude of underperformance is given by

$$|\mathbb{E}_t[dr_t^{ex}]| = f - \frac{p_t A}{q_t} = f - \frac{p_t A}{\tilde{q}_t} \ .$$

The magnitude is increasing in $\tilde{q}_t$ and decreasing in $p_t A$. ■

1.4 Empirical Results

1.4.1 Empirical challenges

Empirical tests of this chapter involve active mutual funds. The model suggests that a mutual fund underperforms when only naive capital is invested in the fund (when the manager's skill is overpriced). However, identifying funds that are dominated by naive money is quite a challenging task, since we cannot directly observe both the skill of managers and the amount of naive money invested in funds. In particular, the size of a fund may not be a good indicator for whether the fund is overpriced or not, because skill and size are positively correlated.

In order to overcome this challenge, I employ capital flows instead of size. This by itself does not fully address the issue, since capital flows may be from sophisticated investors and/or unsophisticated investors. Corollary 1.1 is helpful since, controlling for the perceived skill, the largest capital flows can be associated with the funds becoming overpriced. Similarly, the smallest capital flows can be associated with the funds becoming fairly priced, controlling for changes in perceived skill. Thus, extreme flows can be exploited in order to distinguish overpriced funds and fairly priced funds.

In addition to the main challenge of identifying funds dominated by naive money, the performance evaluation of mutual funds is subject to several types of bias. Survivorship bias (Brown et al. (1992)) and incubation bias (Evans (2010)) are upward biases, and are widely recognized in mutual fund studies. Reverse-survivorship bias (Linnainmaa (2013)) is a more recent concern, and is a downward bias. Since the empirical tests of this chapter examine underperformance, downward bias is a particular concern. The concern is that, if reverse-survivorship bias contributes to the estimate of underperformance, the true performance of (those identified as) overpriced funds may not be significantly different from the market performance, on a risk-adjusted basis.

In order to address the concern about reverse-survivorship bias, I take a portfolio approach: I form portfolios and estimate the performance of those portfolios, instead of measuring the performance of individual funds. Reverse-survivorship bias comes from the positive correlation between idiosyncratic shocks to fund performance and the survival time. Simplistically, suppose that the risk-adjusted excess return on fund i is

$$r_{i,t}^{ex} = \alpha_i + \epsilon_{i,t} \ ,$$

where $\epsilon_{i,t}$ is the mean-zero idiosyncratic component of fund returns that is orthogonal to systematic risks that the fund bears. The conventional estimate of alpha of an individual fund can be written as

$$\hat{\alpha}_i = \frac{1}{T_i} \sum_{t=1}^{T_i} r_{i,t}^{ex} \ ,$$

where T_i is the survival time of fund i. The estimate of alpha is downward biased:

$$\mathbb{E}\left[\hat{\alpha}_i\right] = \mathbb{E}\left[\frac{1}{T_i} \sum_{t=1}^{T_i} r_{i,t}^{ex}\right] = \alpha_i + \mathbb{E}\left[\frac{1}{T_i} \sum_{t=1}^{T_i} \epsilon_{i,t}\right] < \alpha_i \ ,$$

as long as the survival time T_i and idiosyncratic shocks $\sum_{t=1}^{T} \epsilon_{i,t}$ are positively correlated. On the other hand, by forming a portfolio of funds, the conventional estimate of alpha of the portfolio can be written as

$$\hat{\alpha}_p = \frac{1}{T_E} \sum_{t=1}^{T_E} \sum_{i=1}^{N_t} w_{i,t} r_{i,t}^{ex} \ ,$$

where T_E is the length of the estimation period that is exogenously set by the researcher, N_t is the number of funds existing in period t, and $w_{i,t}$ is the portfolio weight on fund i in period t. Suppose that all the funds in the portfolio have the same $\alpha_i = \alpha_p$. Then, the estimate of the portfolio alpha is

$$\mathbb{E}\left[\hat{\alpha}_p\right] = \mathbb{E}\left[\frac{1}{T_E} \sum_{t=1}^{T_E} \sum_{i=1}^{N_t} w_{i,t} r_{i,t}^{ex}\right] = \mathbb{E}\left[\frac{1}{T_E} \sum_{t=1}^{T_E} \sum_{i=1}^{N_t} w_{i,t} \alpha_i\right] + \mathbb{E}\left[\frac{1}{T_E} \sum_{t=1}^{T_E} \sum_{i=1}^{N_t} w_{i,t} \epsilon_{i,t}\right] .$$

Since $w_{i,t}$ is determined before $\epsilon_{i,t}$ is realized, the estimate of the portfolio alpha is unbiased:

$$\mathbb{E}\left[\frac{1}{T_E} \sum_{t=1}^{T_E} \sum_{i=1}^{N_t} w_{i,t} \epsilon_{i,t}\right] = 0 \quad \Longleftrightarrow \quad \mathbb{E}\left[\hat{\alpha}_p\right] = \mathbb{E}\left[\frac{1}{T_E} \sum_{t=1}^{T_E} \sum_{i=1}^{N_t} w_{i,t} \alpha_i\right] = \alpha_p \ .$$

Therefore, by taking a portfolio approach, reverse-survivorship bias is no longer a concern.

However, if funds in a portfolio have heterogeneous α, there is another concern. Since low α funds are more likely to die out than high α funds, portfolio weights on high α funds are likely to increase in time within the estimation period for any sensible weighting scheme. For instance, consider equal weighting. If the attrition rate is independent of α,

$$\mathbb{E}\left[\hat{\alpha}_p\right] = \mathbb{E}\left[\frac{1}{T_E}\sum_{t=1}^{T_E}\frac{1}{N_t}\sum_{i=1}^{N_t}\alpha_i\right] = \frac{1}{N}\sum_{i=1}^{N}\alpha_i \, ,$$

where N is the number of funds that exist at the beginning of the estimation period. In this case, the expected value of the estimated portfolio alpha is exactly the average alpha of all the funds included in the portfolio. Yet, if the attrition rate is higher for low α funds,

$$\mathbb{E}\left[\hat{\alpha}_p\right] = \mathbb{E}\left[\frac{1}{T_E}\sum_{t=1}^{T_E}\frac{1}{N_t}\sum_{i=1}^{N_t}\alpha_i\right] > \frac{1}{N}\sum_{i=1}^{N}\alpha_i \, .$$

In this case, the expected value of the estimated portfolio alpha is above the average alpha of all the funds included in the portfolio: the estimate is upward-biased. I refer to this upward bias as "general survivorship bias," as opposed to the narrow interpretation of survivorship bias (Brown et al. (1992)) due to missing data of dead funds. While the general survivorship bias is a valid concern, underperformance cannot be overestimated due to such upward bias. Therefore, the bias is less of a concern of this chapter.

1.4.2 Empirical tests

I employ an identification strategy that compares two groups of mutual funds: one group of funds that are most likely to be dominated by naive money, and another group of funds that are least likely to be dominated by naive money. If the former group mostly consists of overpriced funds, and the latter group mostly consists of fairly priced funds, then the model predicts that the former group underperforms, and the latter group performs the same as the market on a risk-adjusted basis.

In the main test, I identify funds with the worst past performance, but the largest capital flows, as most likely to be overpriced; likewise, funds with the best past performance, but the smallest capital flows, are least likely to be overpriced. In the model, past performance is the most directly related to changes in the perceived skill of funds. By controlling for past performance, from corollary 1.1, funds having received the largest flows are likely to be overpriced, and those having received the smallest flows are likely to be fairly priced. Among the former, funds with the worst track records and largest flows are the most likely to be overpriced, because past underperformance implies that a significant portion of those funds are overpriced. Similarly, funds with the best track records and smallest flows are the least likely to be overpriced.

In the second test, I identify funds with the smallest return gap, but the largest capital flows, as most likely to be overpriced; likewise, funds with the largest return gap but smallest capital flows are least likely to be overpriced. The return gap, proposed by Kacperczyk, Sialm and Zheng (2008), is defined by the difference between the gross return of a fund and the hypothetical return on the most recently disclosed portfolio of the fund. The return gap measures whether managers improve performance by changing their holdings, and hence the return gap is an indirect measure of manager skill. Funds are more likely to be overpriced when the skill of managers is low, given the same amount of invested naive money. Therefore, from corollary 1.1, funds with the smallest return gap and largest flows are the most likely to be overpriced, and funds with the largest return gap and smallest flows are the least likely to be overpriced.

In the third test, I take a different approach that provides less direct support for the model's mechanism than the other tests do. Instead of looking into the entire set of active (domestic equity) funds, I consider a subset of funds that are involved in mergers. Mutual fund mergers are quite common in the industry: 4-9% of mutual funds exit each year, and roughly half of them are merged into other funds. However, our understanding of why some funds choose to acquire other funds is quite limited. In particular, the investor rationality view does not provide a good explanation.

This chapter takes the view that mergers benefit acquiring funds by transferring naive money from target funds to acquiring funds. In order for acquiring funds to benefit, they have to be overpriced after mergers. Under this view, from corollary 1.2, the performance of acquiring funds worsens after mergers since the amount of naive money invested in those funds increases as a consequence of mergers. In particular, corollary 1.2 implies that the magnitude of underperformance after mergers increases with the size of the target funds: funds that acquire larger target funds obtain more naive capital and, hence, underperform more after mergers compared with their performance before mergers. I test this prediction, which is a joint test of corollary 1.2 and the hypothesis that mergers are transfers of naive money.

1.4.3 Data and methodology

The main data source of this chapter is the CRSP Survivor-Bias-Free US Mutual Fund Database. The database provides the characteristics of mutual funds and their net returns. As previously discussed, I take a portfolio approach by double-sorting funds into 5×5 bins. In order to have sufficient numbers of funds in each bin, I focus on data from January 1991 through December 2016[3]. Since the CRSP database assigns separate identifiers for different share classes of a single fund, I construct returns of funds by value-weighting returns of their share classes. I only include US domestic equity funds based on the CRSP style code, and exclude index funds and ETF/ETN in order to include active funds only.

I take further steps in order to address known biases - in particular, incubation bias. The concern of (the narrow interpretation of) survivorship bias is mostly addressed by using the CRSP Survivor-Bias-Free US Mutual Fund Database. However, incubation bias is still a valid concern. In order to assuage the concern, I only include returns of a fund after the point that the fund reaches $15 million AUM (in end of 2016 dollars) for the first time. For computing the real AUM of funds, I use the GDP implicit price deflator, which is obtained from the FRED website[4].

In the main test, I double-sort funds on their performance and flow. I use the Fama-French-Carhart four-factor (FFC4F) model for evaluating the performance of

[3] In the CRSP database, after data cleaning, there are 656 domestic equity funds in January 1991, and 2,776 domestic equity funds in December 2016. In 1990, there are less than 100 domestic equity funds in the CRSP database.

[4] `https://fred.stlouisfed.org`

portfolios of funds. The four factors are obtained from Kenneth French's website[5]. Funds are sorted on their past 12-month performance (FFC4F alpha), and then are sorted on the flow in the past 12 months, which is defined by

$$Flow_t = \frac{AUM_t - (1 + r_t)AUM_{t-1}}{AUM_{t-1}} ,$$

where r_t is the net rate of return on the fund between $t - 1$ and t, and the time interval is one year (12 months). I double-sort funds at the beginning of each year, starting from the 1st of January 1992 through the 1st of January 2016, and, hence, annually rebalance portfolios in each bin. This procedure gives me returns of 5×5 mutual fund portfolios from January 1992 through December 2016. I measure the performance of these 25 portfolios.

In the second test, I double-sort funds on their return gap and flow. In order to compute the return gap, I obtain the holdings of mutual funds from the Thomson-Reuters Mutual Fund Holdings Database. In order to compute the hypothetical return, I obtain return data of individual stocks from the CRSP US Stock Databases. The gross return of a fund is computed by adding the expense ratio to the net return. When linking data from the Thomson-Reuters Database to data from the CRSP database, I employ the Mutual Fund Links (MFLINKS) database. I rebalance 5×5 portfolios annually based on the past year's return gap and flow, similar to the main test. These procedures moderately reduce the number of samples compared with the main test.

In the third test, I sort mergers by the size of target funds. Since I am interested in the total amount of naive money acquired through mergers, I aggregate the AUM of all the target funds if mergers involve multiple target funds at the same time. Mergers of one share class into another share class of the same fund are not included. Since mergers occur at different points of time, I cannot take a portfolio approach here: I need to estimate the performance of individual funds. This brings up the concern of reverse-survivorship bias. In order to address reverse-survivorship bias,

[5] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

I only include acquiring funds that have at least 12-month returns before and after mergers (24-month returns at minimum). Since I only consider acquiring funds that survive for longer than 12 months after mergers, there is no reverse-survivorship bias. However, there can be survivorship bias for after-merger performance[6]. Since the bias is upward, the magnitude of the underperformance estimate is never overestimated by the bias. For the performance of target funds, there is a concern about selection bias, since negative idiosyncratic shocks are likely associated with their attrition. Thus, the performance of target funds is likely downward-biased.

1.4.4 Results

Performance-Fund flow double-sort

[See table 1]

The past year's underperformance predicts this year's underperformance[7]. In Appendix Table 1, when funds are sorted on the past year's FFC4F, poor-performing funds tend to underperform in the next year. However, the difference in the next year's performance between the best-performing funds and the worst-performing funds is only marginally significant (at the 10% level), when the difference in performance is measured by the FFC4F alpha. In addition, when that difference is measured by the CAPM alpha, the difference is not statistically significant.

Similarly, the past year's fund flow predicts this year's underperformance. In Appendix Table 2, when funds are sorted on the past year's fund flow, funds that receive large flows tend to underperform in the next year. However, the difference in the next year's performance between the funds receiving the largest flow and those receiving the smallest flow (or the largest outflow) is only marginally significant (at the 10% level), when the difference in performance is measured by the FFC4F alpha. Yet, when the difference of performance is measured by the CAPM alpha, the difference becomes quite significant (at the 1% level).

[6]I do not have survivorship bias for before-merger performance, because acquiring funds *must* survive until the point of mergers in order to acquire target funds.

[7]This is a well known empirical regularity since Carhart (1997).

When funds are double-sorted on performance and fund flow, the next year's net performance measured by the FFC4F alpha is perceptibly aligned: the (risk-adjusted) performance of funds that previously outperformed and received small flows is not distinguishable from zero, and funds that previously underperformed and received large flows keep significantly underperforming. The performance difference between best-performance-smallest-flow funds and worst-performance-largest-flow funds is about 4% annually (both for equal-weighted and value-weighted portfolios), and statistically significant at the 1% level.

When the next year's performance is measured by the CAPM alpha, the result is qualitatively quite similar, and can be seen in Appendix Table 4. Quantitatively, the annual performance difference between best-performance-smallest-flow funds and worst-performance-largest-flow funds is 4.35% for equal-weighted portfolios, and 5.18% for value-weighted portfolios. Those differences are statistically significant at the 1% level.

Return gap-Fund flow double-sort

[See table 2]

The past year's small return gap predicts this year's underperformance. In Appendix Table 3, when funds are sorted on the past year's return gap, funds with small return gaps tend to underperform during the next year. In Kacperczyk, Sialm and Zheng (2008), the return gap predicts the short-run outperformance (in the next quarter) of funds. Yet, in this chapter, the return gap does not predict outperformance in the long run (in the next year).

Small return gaps do not predict underperformance as well as other variables (underperformance and large flows) do. In Appendix Table 3, only the equal-weighting scheme combined with the FFC4F alpha as the performance measure generates a statistically significant difference in performance between funds with the largest return gap in the past year and those with the smallest return gap.

When funds are double-sorted on return gap and fund flow, the next year's net performance measured by the FFC4F alpha is aligned: the (risk-adjusted) performance of funds that previously had large return gaps and received small flows is not distinguishable from zero, and funds that previously had small return gaps and received large flows keep significantly underperforming. The annual performance difference between largest-return gap-smallest-flow funds and smallest-return gap-largest-flow funds is 2.42% for equal-weighted portfolios, and 3.14% for value-weighted portfolios. These differences are statistically significant at the 5% level.

When the next year's performance is measured by the CAPM alpha, the performance pattern is less evident, as can be seen in Appendix Table 5. This is partly due to the finding (in Appendix Table 3) that the return gap does not predict the next year's CAPM alpha very well. However, the annual performance difference between largest-return gap-smallest-flow funds and smallest-return gap-largest-flow funds is still statistically significant at the 5% level: the difference is 2.36% for equal-weighted portfolios, and 3.75% for value-weighted portfolios.

Performance of funds before/after mergers

[See table 3]

As predicted, the difference between the performance of acquiring funds after mergers and that before mergers increases with the size of target funds. In particular, the difference is pronounced for funds that acquire target funds the size of which is in the biggest quintile. The after-merger performance of those acquiring funds is 1.57% lower than the before-merger performance of the same funds, and statistically significant at the 1% level, when the FFC4F alpha is used as the performance measure. When performance is measured by the CAPM alpha, funds that acquired the largest target funds perform 1.64% lower after mergers than they did before mergers, and the difference is statistically significant at the 5% level.

There are other interesting aspects that this chapter does not address. While target funds significantly underperform acquiring funds before mergers, the magnitude of underperformance tends to decrease with the size of target funds. After mergers, the combined funds perform better than target funds, except for those which acquired the largest target funds. Funds that acquired the smallest target funds seem to improve their performance after mergers, but the improvement is only statistically significant when the performance is measured by the CAPM alpha.

1.5 Discussions and Conclusions

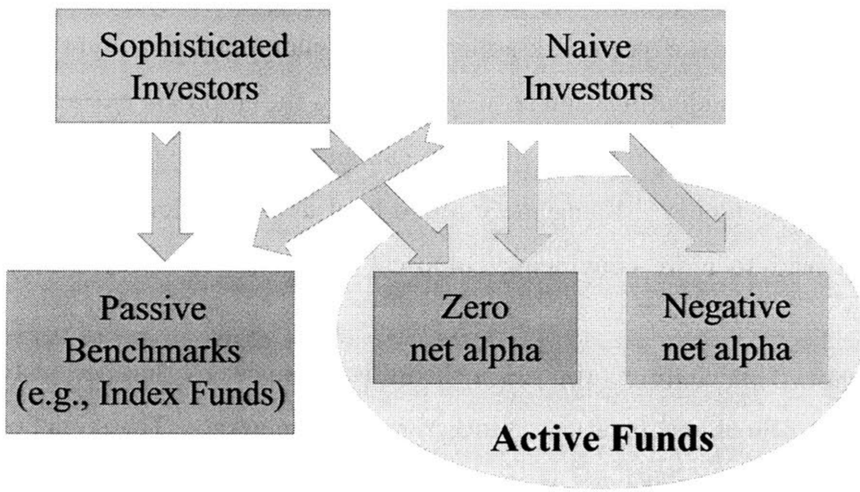
The main empirical challenge for the predictions of the model is that neither the skill of managers nor the amount of invested naive money is observable. In order to address this challenge, I employ mismatches between changes of the perceived skill of managers, measured by the recent risk-adjusted performance of funds, and the changes of fund size. While this identification strategy results in the empirical pattern of future fund performance as predicted by the model, it cannot still perfectly distinguish 'naive money' from 'smart money'. Using more micro-level data in order to identify naive money would lead to more convincing empirical results for the predictions of the model.

In conclusion, this chapter proposes a theoretical framework that seriously takes into account the role of unsophisticated investors in AM markets. The model predicts that the skill of active managers is "overpriced" once the amount of capital from unsophisticated investors exceeds the size that fairly prices the skill. Overpriced funds underperform the passive benchmark, while fairly priced funds perform the same as the benchmark. The mechanism of the model delivers several predictions on the fund performance dynamics, which are empirically tested and supported.

Figures

Figure 1: Types of investors and types of funds

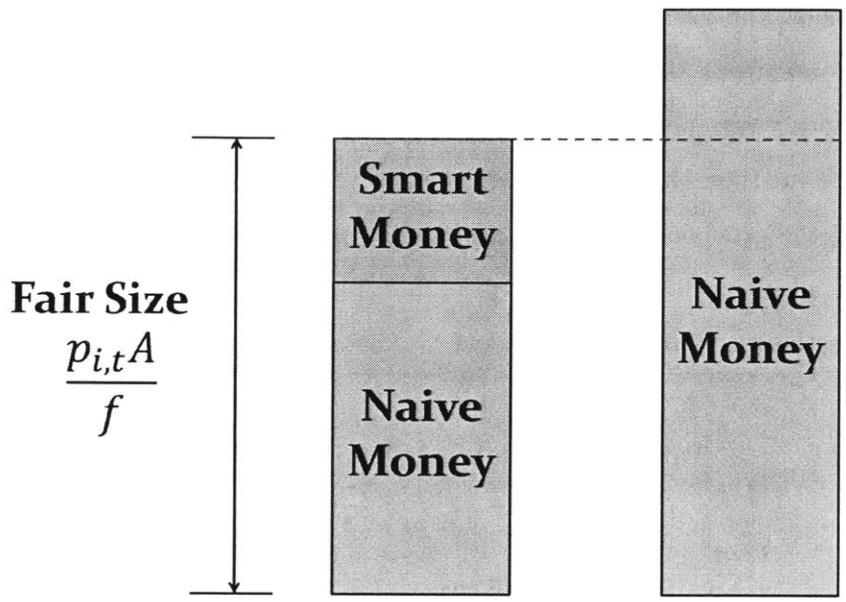
The following figure illustrates two types of investors, sophisticated (smart) investors and unsophisticated (naive) investors, and two types of funds, active funds and passive benchmarks. Among active funds, smart investors never choose to invest in active funds with negative net alpha, while naive investors do invest in those with negative net alpha.



Types of investors and types of funds

Figure 2: Mechanism of skill overpricing

The following figure illustrates the mechanism that excessive inflows of naive money overprice the skill of managers.



Mechanism of skill overpricing by naive investors

Tables

Table 1: Performance-Fund flow double sort

The tables show the Fama-French-Carhart 4-factor model (FFC4F) monthly alpha for 25 portfolios from 1992 to 2016. The 25 (5×5) portfolios are double-sorted on the previous year's performance and the previous year's fund flow (firstly on the performance and then the flow). The previous year's performance is measured by FFC4F, and the previous year's fund flow is the percentage growth of the previous year's AUM adjusting for the net return. The portfolios are rebalanced at annual frequency on the 1st of January each year.

Table 1a: Equal-weighted portfolios

		Flow				
	$12 \times \alpha$	Low	2	3	4	High
	Low	-1.83**	-1.67*	-2.87***	-2.71***	-3.62***
	2	-1.18**	-1.41***	-1.05**	-1.63***	-1.75***
FFC4F α	3	-1.03**	-0.85*	-1.14***	-1.41***	-1.36***
	4	-0.78	-0.47	-0.74	-1.18**	-1.23***
	High	0.39	-0.21	-0.53	-0.78	-2.16**

(High α & Low flow) minus (Low α & High flow): 4.02*** (std err: 1.22)

Standard errors

		Flow				
	$12 \times \alpha$	Low	2	3	4	High
	Low	0.81	0.91	1.03	0.86	0.86
	2	0.54	0.49	0.49	0.54	0.50
FFC4F α	3	0.49	0.48	0.41	0.40	0.41
	4	0.57	0.65	0.58	0.50	0.46
	High	0.89	0.82	0.83	0.80	0.98

Table 1b: Value-weighted portfolios

	$12 \times \alpha$	Flow: Low	2	3	4	High
	Low	-2.17**	-2.68***	-2.69***	-2.63***	-4.41***
	2	0.07	-0.68	-0.61	-1.27*	-2.14***
FFC4F α	3	-1.54**	-1.22**	-1.27***	-1.12*	-1.61***
	4	-1.64**	-0.09	0.05	-1.03*	-1.89***
	High	-0.27	-0.07	-0.20	-1.89*	-2.97**

(High α & Low flow) minus (Low α & High flow): 4.14*** (std err: 1.41)

Standard errors

	$12 \times \alpha$	Flow: Low	2	3	4	High
	Low	0.94	0.74	0.77	0.77	0.99
	2	0.90	0.58	0.52	0.70	0.56
FFC4F α	3	0.63	0.56	0.46	0.65	0.62
	4	0.76	0.72	0.60	0.59	0.55
	High	0.88	0.86	0.87	1.01	1.16

Table 2: Return gap-Fund flow double sort

The tables show the Fama-French-Carhart 4-factor model (FFC4F) monthly alpha for 25 portfolios from 1992 to 2016. The 25 (5×5) portfolios are double-sorted on the previous year's return gap and the previous year's fund flow (firstly on the return gap and then the flow). The previous year's return gap is measured as in Kacperczyk, Sialm and Zheng (2008), and the previous year's fund flow is the percentage growth of the previous year's AUM adjusting for the net return. The portfolios are rebalanced at annual frequency on the 1st of January each year.

Table 2a: Equal-weighted portfolios

		Flow				
	$12 \times \alpha$	Low	2	3	4	High
	Low	-1.88***	-2.00**	-2.03***	-1.99***	-2.66***
	2	-0.87	-1.27**	-0.73	-1.67***	-1.31***
Return Gap	3	-1.14*	-1.19*	-0.64	-0.96	-1.17**
	4	0.19	-0.70	-0.63	-1.20*	-1.87***
	High	-0.24	-0.63	-0.51	-0.44	-1.52*

(High return gap & Low flow) minus (Low return gap & High flow): 2.42**

(std err: 1.17)

Standard errors

		Flow				
	$12 \times \alpha$	Low	2	3	4	High
	Low	0.71	0.85	0.67	0.73	0.76
	2	0.58	0.54	0.46	0.46	0.48
Return Gap	3	0.69	0.63	0.50	0.65	0.49
	4	0.80	0.75	0.66	0.67	0.52
	High	0.89	0.93	1.02	0.89	0.81

Table 2b: Value-weighted portfolios

	$12 \times \alpha$	Flow: Low	2	3	4	High
Return Gap	Low	-2.23**	-1.34	-1.71*	-1.33	-2.58**
	2	-0.18	-2.62***	0.02	-0.46	-1.46**
	3	-1.42*	-0.70	-1.27**	-0.54	-0.67
	4	-0.62	-0.49	-0.03	-1.88***	-2.38***
	High	0.56	-0.23	0.22	-1.59*	-2.99***

(High return gap & Low flow) minus (Low return gap & High flow):
3.14**
(std err: 1.49)

Standard errors

	$12 \times \alpha$	Flow: Low	2	3	4	High
Return Gap	Low	1.10	1.01	0.96	1.00	1.14
	2	0.67	0.72	0.69	0.60	0.72
	3	0.73	0.77	0.51	0.74	0.66
	4	0.78	0.88	0.76	0.67	0.62
	High	1.01	1.04	1.05	0.92	1.03

Table 3: Performance of funds before/after mergers

The tables show the performance of target and acquiring funds before mergers, and the performance of acquiring funds after mergers, for mergers in period 1991-2016. When a single fund acquires multiple target funds at the same time, those target funds are aggregated, and their returns are value-weighted. The significance level is based on the p-value of the null that the difference in the average performance before mergers and that after mergers is zero.

Table 3a: FFC4F alpha (annualized)

	FFC4F α					
	Target	Before	After	Before-Target	After-Target	After-Before
Total	-2.95	-0.86	-1.02	2.09***	1.93***	-0.16
Small	-3.47	-1.32	-0.41	2.15***	3.06***	0.91
2	-2.83	-0.45	-0.84	2.38***	1.99***	-0.39
3	-3.42	-0.89	-1.32	2.53***	2.10***	-0.43
4	-3.17	-1.30	-0.63	1.88***	2.55***	0.67
Big	-1.86	-0.35	-1.92	1.51**	-0.06	-1.57***

***p*-values**

	FFC4F α		
	Before-Target	After-Target	After-Before
Total	0.00	0.00	0.55
Small	0.00	0.00	0.13
2	0.00	0.00	0.52
3	0.00	0.00	0.48
4	0.00	0.00	0.26
Big	0.01	0.93	0.01

Table 3b: CAPM alpha (annualized)

	CAPM α					
	Target	Before	After	Before-Target	After-Target	After-Before
Total	-2.96	-0.29	-0.15	2.66***	2.80***	0.14
Small	-3.91	-1.50	1.82	2.41***	5.73***	3.32***
2	-2.68	-0.20	-0.42	2.48***	2.26***	-0.22
3	-2.34	0.56	-0.10	2.90***	2.23***	-0.67
4	-3.60	-0.26	-0.35	3.35***	3.26***	-0.09
Big	-2.25	-0.07	-1.70	2.18***	0.55	-1.64**

***p*-values**

	FFC4F α		
	Before-Target	After-Target	After-Before
Total	0.00	0.00	0.65
Small	0.00	0.00	0.00
2	0.00	0.00	0.74
3	0.00	0.00	0.33
4	0.00	0.00	0.90
Big	0.00	0.42	0.02

Appendix Tables

Appendix Table 1: Performance single sort

The tables show the Fama-French-Carhart 4-factor model (FFC4F) monthly alpha and the CAPM monthly alpha for 10 portfolios from 1992 to 2016. The 10 portfolios are sorted on the previous year's performance. The previous year's performance is measured by FFC4F. The portfolios are rebalanced at annual frequency on the 1st of January each year. The standard errors are reported in parentheses.

Appendix Table 1a: FFC4F alpha (annualized)

	FFC4F α			
	Equal-weighted		Value-weighted	
Low	-2.90**	(1.15)	-3.75***	(0.97)
2	-2.21***	(0.56)	-2.62***	(0.56)
3	-1.47***	(0.46)	-0.72	(0.55)
4	-1.34***	(0.44)	-1.08**	(0.47)
5	-1.42***	(0.38)	-1.24***	(0.48)
6	-0.90**	(0.40)	-1.29***	(0.43)
7	-1.13**	(0.47)	-0.80**	(0.38)
8	-0.64	(0.51)	-0.47	(0.54)
9	-0.85	(0.60)	-1.00*	(0.59)
High	-0.47	(0.99)	-1.31	(1.04)
High-Low	2.43*	(1.43)	2.44*	(1.38)

Appendix Table 1b: CAPM alpha (annualized)

	CAPM α			
	Equal-weighted		Value-weighted	
Low	-2.62**	(1.33)	-3.68***	(1.07)
2	-1.73**	(0.71)	-2.52***	(0.62)
3	-1.10**	(0.56)	-0.66	(0.55)
4	-0.98*	(0.50)	-0.96**	(0.46)
5	-0.99**	(0.46)	-1.04**	(0.49)
6	-0.46	(0.46)	-0.98**	(0.48)
7	-0.87*	(0.52)	-0.63	(0.39)
8	-0.35	(0.58)	-0.62	(0.56)
9	-0.81	(0.73)	-1.13	(0.74)
High	-0.35	(1.41)	-1.70	(1.50)
High-Low	2.27	(1.48)	1.98	(1.50)

Appendix Table 2: Fund flow single sort

The tables show the Fama-French-Carhart 4-factor model (FFC4F) monthly alpha and the CAPM monthly alpha for 10 portfolios from 1992 to 2016. The 10 portfolios are sorted on the previous year's fund flow. The previous year's fund flow is measured by the percentage growth of the previous year's AUM adjusting for the net return. The portfolios are rebalanced at annual frequency on the 1st of January each year. The standard errors are reported in parentheses.

Appendix Table 2a: FFC4F alpha (annualized)

	FFC4F α			
	Equal-weighted		Value-weighted	
Low	-0.85	(0.55)	-1.28*	(0.70)
2	-1.12*	(0.61)	-0.83	(0.68)
3	-1.23**	(0.60)	-1.34**	(0.62)
4	-1.09**	(0.53)	-0.78	(0.50)
5	-1.13**	(0.52)	-0.68	(0.45)
6	-1.38***	(0.50)	-0.72*	(0.41)
7	-1.24**	(0.49)	-0.66	(0.50)
8	-1.56***	(0.51)	-1.57***	(0.57)
9	-1.55***	(0.51)	-1.68***	(0.53)
High	-2.09***	(0.59)	-2.94***	(0.78)
High-Low	-1.24*	(0.65)	-1.66*	(0.87)

Appendix Table 2b: CAPM alpha (annualized)

	CAPM α			
	Equal-weighted		Value-weighted	
Low	0.05	(0.82)	-0.22	(1.02)
2	-0.28	(0.73)	0.24	(0.76)
3	-0.63	(0.68)	-0.57	(0.64)
4	-0.81	(0.59)	-0.36	(0.51)
5	-1.06*	(0.59)	-0.72	(0.45)
6	-1.29**	(0.56)	-0.95**	(0.40)
7	-1.18*	(0.61)	-1.17**	(0.53)
8	-1.46**	(0.68)	-1.96***	(0.67)
9	-1.43**	(0.73)	-1.88***	(0.67)
High	-2.02**	(0.90)	-3.15***	(1.14)
High-Low	-2.07***	(0.69)	-2.93***	(0.94)

Appendix Table 3: Return gap single sort

The tables show the Fama-French-Carhart 4-factor model (FFC4F) monthly alpha and the CAPM monthly alpha for 10 portfolios from 1992 to 2016. The 10 portfolios are sorted on the previous year's return gap. The previous year's return gap is measured as in Kacperczyk, Sialm and Zheng (2008) The portfolios are rebalanced at annual frequency on the 1st of January each year. The standard errors are reported in parentheses.

Appendix Table 3a: FFC4F alpha (annualized)

	FFC4F α			
	Equal-weighted		Value-weighted	
Low	-2.17***	(0.75)	-1.78*	(1.01)
2	-2.03***	(0.56)	-2.21**	(0.88)
3	-1.32***	(0.43)	-1.47**	(0.63)
4	-1.03***	(0.39)	-0.25	(0.53)
5	-0.94*	(0.51)	-0.88*	(0.52)
6	-1.10**	(0.49)	-1.01*	(0.53)
7	-0.96*	(0.54)	-0.77	(0.48)
8	-0.75	(0.67)	-2.06***	(0.64)
9	-0.69	(0.82)	-1.16	(0.72)
High	-0.65	(0.84)	-1.32	(0.85)
High-Low	1.52**	(0.74)	0.47	(0.88)

Appendix Table 3b: CAPM alpha (annualized)

	CAPM α			
	Equal-weighted		Value-weighted	
Low	-1.82	(1.20)	-1.99	(1.40)
2	-1.28	(0.92)	-2.22*	(1.27)
3	-0.84	(0.64)	-1.44*	(0.77)
4	-0.49	(0.57)	-0.33	(0.60)
5	-0.48	(0.57)	-1.06*	(0.55)
6	-0.78	(0.57)	-1.05**	(0.54)
7	-0.64	(0.61)	-0.95*	(0.49)
8	-0.54	(0.74)	-2.52***	(0.69)
9	-0.67	(0.94)	-1.22	(0.97)
High	-0.66	(1.37)	-1.50	(1.45)
High-Low	1.16	(0.75)	0.49	(0.88)

Appendix Table 4: Performance-Fund flow double sort - CAPM α

The table shows the CAPM monthly alpha for 25 portfolios from 1992 to 2016. The 25 (5×5) portfolios are double-sorted on the previous year's performance and the previous year's fund flow (firstly on the performance and then the flow). The previous year's performance is measured by FFC4F, and the previous year's fund flow is the percentage growth of the previous year's AUM adjusting for the net return. The portfolios are rebalanced at annual frequency on the 1st of January each year.

Appendix Table 4a: Equal-weighted portfolios

		Flow				
	$12 \times \alpha$	Low	2	3	4	High
	Low	-0.70	-0.92	-2.97**	-2.66***	-3.50***
	2	-0.32	-0.93*	-0.89	-1.52***	-1.53**
FFC4F α	3	-0.16	-0.35	-0.90**	-1.07**	-1.09**
	4	0.02	-0.06	-0.71	-1.09*	-1.12*
	High	0.85	-0.01	-0.90	-0.80	-2.00

(High α & Low flow) minus (Low α & High flow): 4.35*** (std err: 1.19)

Standard errors

	$12 \times \alpha$	Flow Low	2	3	4	High
	Low	1.08	1.02	1.18	0.99	1.03
	2	0.67	0.54	0.55	0.58	0.62
FFC4F α	3	0.60	0.52	0.43	0.49	0.53
	4	0.67	0.70	0.63	0.56	0.60
	High	1.08	0.90	1.00	1.16	1.49

Appendix Table 4b: Value-weighted portfolios

	$12 \times \alpha$	Flow Low	2	3	4	High
	Low	-1.08	-2.09***	-2.49***	-2.96***	-4.73***
	2	0.91	0.01	-0.53	-1.52**	-2.53***
FFC4F α	3	-0.48	-0.40	-0.85*	-1.63**	-1.98***
	4	-0.58	0.23	-0.41	-1.25**	-1.94***
	High	0.46	-0.12	-0.90	-2.06	-3.07*

(High α & Low flow) minus (Low α & High flow): 5.18*** (std err: 1.40)

Standard errors

	$12 \times \alpha$	Flow Low	2	3	4	High
	Low	1.22	0.79	0.81	0.84	1.10
	2	0.94	0.62	0.51	0.69	0.65
FFC4F α	3	0.72	0.59	0.47	0.66	0.74
	4	0.92	0.81	0.65	0.61	0.66
	High	1.08	0.84	0.99	1.38	1.69

Appendix Table 5: Return gap-Fund flow double sort - CAPM α

The table shows the CAPM monthly alpha for 25 portfolios from 1992 to 2016. The 25 (5×5) portfolios are double-sorted on the previous year's return gap and the previous year's fund flow (firstly on the return gap and then the flow). The previous year's return gap is measured as in Kacperczyk, Sialm and Zheng (2008), and the previous year's fund flow is the percentage growth of the previous year's AUM adjusting for the net return. The portfolios are rebalanced at annual frequency on the 1st of January each year.

Appendix Table 5a: Equal-weighted portfolios

		Flow				
	$12 \times \alpha$	Low	2	3	4	High
	Low	-1.06	-1.07	-1.81*	-1.58	-2.28*
	2	0.36	-0.70	-0.71	-1.32**	-0.88
Return Gap	3	-0.18	-0.68	-0.42	-0.98	-0.89
	4	0.95	-0.28	-0.57	-1.28*	-1.68***
	High	0.08	-0.61	-0.76	-0.72	-1.30

(High return gap & Low flow) minus (Low return gap & High flow): 2.36**

(std err: 1.16)

Standard errors

		Flow				
	$12 \times \alpha$	Low	2	3	4	High
	Low	1.14	1.15	0.98	1.15	1.18
	2	0.87	0.64	0.56	0.58	0.71
Return Gap	3	0.79	0.68	0.54	0.69	0.60
	4	0.87	0.81	0.69	0.73	0.65
	High	1.12	1.08	1.16	1.24	1.34

Appendix Table 5b: Value-weighted portfolios

	$12 \times \alpha$	Flow Low	2	3	4	High
	Low	-0.96	-0.57	-1.97	-1.67	-2.76*
	2	1.22	-2.06**	-0.61	-0.67	-1.66*
Return Gap	3	-0.74	-0.27	-1.45***	-0.99	-0.92
	4	0.14	-0.20	-0.11	-2.58***	-2.83***
	High	0.98	-0.20	-0.37	-2.16	-2.89*

(High return gap & Low flow) minus (Low return gap & High flow): 3.75**

(std err: 1.47)

Standard errors

	$12 \times \alpha$	Flow Low	2	3	4	High
	Low	1.59	1.25	1.26	1.41	1.49
	2	0.95	0.80	0.75	0.72	0.88
Return Gap	3	0.79	0.77	0.55	0.75	0.71
	4	0.90	0.88	0.74	0.71	0.70
	High	1.35	1.15	1.18	1.43	1.52

Chapter 2

IO of Active Mutual Funds

2.1 Introduction

This chapter theoretically addresses the decisions of active mutual fund managers under an assumption that a significant portion of investors in mutual fund markets are unsophisticated. In particular, this chapter focuses on the managerial choice of fees, effort, idiosyncratic risk, marketing and the pursuit of private benefit. The key variables that govern the choices of managers are the skill of managers and the amount of invested naive money. To be more precise, the relative magnitude of fee profits that managers can earn from naive investors compared to that from smart investors is the most crucial variable that determines the managerial choices.

If funds attract more naive money, managers raise fees and reduce their effort to create value for investors. Since naive investors are less sensitive to fees than smart investors are, increasing fees leads to higher fee profits from naive investors and lower fee profits from smart investors. If the amount of invested naive money increases, marginal fee profits from naive investors increase while those from smart investors decrease. As a result, managers would want to charge higher fees when more naive money flows into their funds. Because managerial effort is compensated by smart investors, as marginal fee profits from smart investors decrease, managers choose to reduce their effort.

Managers choose to bear the maximum idiosyncratic risk when the magnitude of expected profits from naive investors is the same as that from smart investors because fee profits follow the same payoff structure as a call option. When there is only a small amount of invested naive money, fee profits are determined by smart investors, and the magnitude is fixed at a certain level that reflects managers' skill. Hence, fee profits from smart investors can be thought of as the strike price. In contrast, when excessive naive money flows into funds, fee profits are determined by naive investors, and are proportional to the amount of invested naive money. Since the call option vega, which is the sensitivity of option price to changes in the volatility of the stock, is maximized when stock price is the same as the strike price, the marginal benefit of increasing idiosyncratic risk is maximized when the magnitude of expected profits from naive investors is the same as that from smart investors.

As funds receive more naive money, managers choose more marketing and pursue more private benefit. Since the marginal benefit of marketing is proportional to the probability that naive money dominates funds, the marginal benefit increases as more naive money flows into funds. Hence, managers would want to increase the level of marketing if they receive more capital from naive investors. Similarly, because the marginal cost of private benefit obtained by sacrificing returns to investors is proportional to the probability that smart investors are the marginal investors, the marginal cost decreases as funds attract more naive money. As a result, managers choose to pursue more private benefit as more naive money is invested in their funds.

In the remainder of the chapter, section 2 presents the baseline model. Section 3 discusses the equilibrium of the model and its implication for managerial decisions of fees and effort. Section 4 discusses the extensions of the baseline model with the endogenous choice of i) idiosyncratic risk, ii) marketing, and iii) private benefit. Section 5 discusses the limitations of the model and future directions for research, and summarizes the conclusions.

2.2 Baseline Model

Managers have skill, and the skill is captured by a parameter a. Managers choose an effort level e_t in each period. The AUM of funds in time t is denoted by q_t. Managers generate value subject to decreasing returns to scale.

In time t, a manager generates

$$A_t = (ae_t)^{1-\alpha} q_t^{\alpha} \tag{2.1}$$

dollar amount of value in expectation, where $0 < \alpha < 1$. As a result, the gross excess return on the fund in time t is

$$R_t^{ex} = \frac{A_t}{q_t} + \epsilon_t \quad , \quad \epsilon_t \sim N(0, \sigma_\epsilon^2) \ ,$$

where ϵ_t is an idiosyncratic component of the gross excess return, and the distribution of ϵ_t follows a normal distribution with zero mean and the standard deviation of σ_ϵ.

The fund charges fees f_t that is proportional to the AUM of the fund in time t. Fees f_t in time t is determined in the previous period $t-1$. Therefore, the net excess return on the fund in time t is

$$r_t^{ex} = \frac{A_t}{q_t} - f_t + \epsilon_t \ .$$

In each period, the fund bears costs that are proportional to the AUM, and costs of fixed amount:

$$\phi q_t + F \ .$$

In addition, the fund also bears an effort cost $\Psi_e(e_t)$ that is convex in the level of a managerial effort e_t. For the sake of analytic convenience, I choose the following quadratic form of $\Psi_e(e_t)$:

$$\Psi_e(e_t) = \frac{1}{2} a (e_t)^2$$

As a result, the net profit to the fund is given by

$$(f_t - \phi)q_t - F - \Psi_e(e_t) \ ,$$

and the objective of the manager in time t is to maximize the discounted sum of expected profits:

$$\max_{\{f_{s+1}, e_s\}_{s=t,t+1,\cdots}} \sum_{s=t}^{\infty} \frac{1}{(1+r_f)^{s-t}} \mathbb{E}_t \left[(f_s - \phi)q_s - F - \Psi_e(e_s) \right] \ ,$$

where r_f is the constant risk-free rate. Fee profits are discounted at the risk-free rate since all the shocks in the model are idiosyncratic. Funds cannot choose to exit.

There are two types of investors: smart investors (sophisticated investors) and naive investors (unsophisticated investors). Smart investors invest in a fund as long as the net expected excess return (net alpha) of the fund is greater than or equal to zero[1]. In contrast, naive investors invest in a fund partly based on the fund's recent performance and partly based on unmodeled reasons. In addition, the amount of capital invested by naive investors (naive money) responds to fees: higher fees lead to lower amounts of invested naive money in the fund. The amount of invested naive money in a fund is modeled as

$$q_t^u = \max\{0, \tilde{q}_t^u\} \quad , \quad \tilde{q}_t^u = g(f_t)\hat{\mu}_t^q + \gamma_u(f_t)u_{t-1} + \gamma_\epsilon(f_t)\epsilon_{t-1} \quad , \quad u_t \sim N(0,1) \ , \ (2.2)$$

where $g(f)$ is a decreasing function in f, and captures the responsiveness of naive investors to fees. $\hat{\mu}_t^q$ captures the expected value of the invested naive money in time t measured in time $t-1$, and is modeled as

$$\hat{\mu_t}^q = \mu_q + \rho(\hat{\mu}_{t-1}^q - \mu_q) + \eta_u u_{t-2} + \eta_\epsilon \frac{\epsilon_{t-2}}{\sigma_\epsilon} \ ,$$

where ρ captures how sticky naive money is.

[1]One crucial assumption is that smart investors can diversify idiosyncratic risk by themselves.

I make the following assumption:

$$(f_t - \phi)\gamma_u(f_t) \equiv \hat{\gamma}_u = const \quad , \quad (f_t - \phi)\gamma_\epsilon(f_t) \equiv \hat{\gamma}_\epsilon = const \; . \tag{2.3}$$

This assumption assures that a fee choice f_t only affects the mean of the distribution of $(f_t - \phi)\tilde{q}_t^u$, which determines fee profits from naive investors. Therefore, under the assumption (2.3), changes in fee choices f_t do not affect the volatility of fee profits from naive investors. In an extended version of the model, the optimal choice of volatility (of fee profits from naive investors) will be examined[2].

For the sake of analytic convenience, I choose the following form of $g(f_t)$:

$$g(f_t) = 1 - \frac{f_t}{\kappa} \; .$$

I make an additional assumption on the value of κ:

$$\kappa > \left(\frac{2}{\alpha} - 1\right)\phi \; . \tag{2.4}$$

Since κ is the inverse sensitivity of naive money to fees, large κ means that naive investors are insensitive to fees. Assumption (2.4) implies that, roughly speaking, naive investors are less sensitive to fees than smart investors are.

2.3 Equilibrium

As benchmark cases, I consider two cases: one where there are no naive investors and the other where there are no smart investors. Then, I solve for the equilibrium of general cases.

[2]This analysis provides the marginal benefit of increasing the volatility of fee profits from naive investors. Suppose that changes of fees affect both the mean and the volatility of fee profits from naive investors. Adding the marginal benefit of increasing the volatility to the marginal benefit of increasing the mean, one can easily figure out the marginal benefit of increasing fees in this case. The marginal cost stays the same for the same fees.

2.3.1 First benchmark - no naive investors

Investors' capital allocations

Smart investors competitively provide capital to a fund as long as the net alpha of the fund is positive. In contrast, smart investors do not provide capital to the fund if the net alpha is negative. Therefore, in equilibrium, the net alpha of the fund is zero , i.e.,

$$\mathbb{E}_t\left[r_t^{ex}\right] = \frac{A_t}{q_t} - f_t = 0 \iff q_t = \frac{A_t}{f_t} \equiv q_t^* ,$$

where q_t^* is the fair size of the fund. From (2.1) the AUM of the fund reads

$$q_t = \frac{(ae_t)^{1-\alpha} q_t^\alpha}{f_t} \iff q_t = ae_t(f_t)^{-\frac{1}{1-\alpha}}$$

Managers' decisions

Fee profits in time t read

$$(f_t - \phi)q_t - F - \Psi_e(e_t) = (f_t - \phi)ae_t(f_t)^{-\frac{1}{1-\alpha}} - F - \frac{1}{2}a(e_t)^2 .$$

Since the effort choice e_t in time t only affects fee profits in time t, given f_t, the optimal choice of e_t solves the following first-order condition:

$$a(f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}} - ae_t = 0 \iff e_t = (f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}} .$$

As a result, fee profits in time t can be written as

$$\frac{1}{2}a\left[(f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}}\right]^2 - F .$$

The proportional fee f_t is determined in the previous period $t-1$. The optimal fee choice maximizes expected fee profits in time t:

$$f_t = \arg\max_{f_t} \frac{1}{2}a\left[(f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}}\right]^2 - F ,$$

which leads to the following first-order condition:

$$\frac{d}{df_t}\left[(f_t-\phi)(f_t)^{-\frac{1}{1-\alpha}}\right]=(f_t)^{-\frac{1}{1-\alpha}-1}\left(f_t-\frac{1}{1-\alpha}(f_t-\phi)\right)=0\,.$$

Therefore, the optimal choice of f_t is

$$f_t=\frac{1}{\alpha}\phi\equiv f^s>\phi\,.$$

2.3.2 Second benchmark - no smart investors

Since there are no sophisticated investors, investors' capital allocation decision is given by

$$q_t=q_t^u=\max\{0,\tilde{q}_t^u\}\;,$$

where $\tilde{q}_t^u$ is given by (2.2). Fee profits in time t read

$$(f_t-\phi)q_t^u-F-\frac{1}{2}a(e_t)^2$$

The optimal choice of e_t is zero, since fee revenues are unaffected by the choice of e_t but there are costs associated with positive e_t. Define

$$Q_t^u(f_t;\hat{\mu}_t^q)\equiv(f_t-\phi)g(f_t)\hat{\mu}_t^q+\hat{\gamma}_u u_{t-1}+\hat{\gamma}_\epsilon\epsilon_{t-1}\,.$$

Note that the distribution of $Q_t^u(f_t)$ in time $t-1$ is a normal distribution with the mean of

$$(f_t-\phi)g(f_t)\hat{\mu}_t^q,$$

and the variance of

$$\hat{\gamma}_u^2+\hat{\gamma}_\epsilon^2\sigma_\epsilon^2\,.$$

Fee profits now read

$$(f_t-\phi)q_t^u-F=\max\{0,Q_t^u(f_t)\}-F\,.$$

The optimal choice of fee f_t maximizes the expected profits in time t:

$$f_t = \arg\max_{f_t} \mathbb{E}_{t-1}\left[\max\{0, Q_t^u(f_t)\} - F\right] .$$

Since only the mean of $Q_t^u(f_t)$ is affected by changes in f_t, the optimal choice of f_t maximizes the mean of $Q_t^u(f_t)$. The optimal f_t satisfies the following first-order condition:

$$\frac{d}{df_t}\left[(f_t - \phi)\left(1 - \frac{f_t}{\kappa}\right)\right] = \frac{1}{\kappa}(\kappa + \phi - 2f_t) = 0 \quad \Longleftrightarrow \quad f_t = \frac{\phi + \kappa}{2} \equiv f^u .$$

Therefore, when there are no smart investors, the optimal choices are $e_t = 0$ and $f_t = \frac{\phi+\kappa}{2}$. Note that assumption (2.4) guarantees that f^u is greater than f^s.

2.3.3 General cases

Investors' capital allocations

Smart investors competitively provide capital to a fund as long as the net alpha of the fund is positive. Therefore, in equilibrium, the net alpha of the fund is nonpositive , i.e.,

$$\mathbb{E}_t\left[r_t^{ex}\right] = \frac{A_t}{q_t} - f_t \le 0 \quad \Longleftrightarrow \quad q_t \ge \frac{A_t}{f_t} \equiv q_t^* ,$$

where the the net alpha of the fund is zero at $q_t = q_t^*$ (fair size). The AUM of a fund is the sum of capital invested by smart investors (smart money) and capital invested by naive investors (naive money):

$$q_t = q_t^s + q_t^u .$$

If the amount of invested naive money in a fund is smaller than q_t^*, smart money flows into the fund until the fund reaches the size q_t^*, at which the net alpha of the fund becomes zero:

$$q_t = q_t^* \quad , \quad q_t^s = q_t^* - q_t^u .$$

If the amount of invested naive money is greater than q_t^*, smart investors withdraw their capital invested in the fund since the net alpha of the fund is negative. However, because investors cannot short-sell funds, the amount of capital q_t^s invested by smart investors cannot be negative. As a result, smart investors choose to invest zero amount of capital in those funds in which the amount of invested naive money is greater than q_t^*:

$$q_t = q_t^u \quad , \quad q_t^s = 0 \, .$$

Therefore, the equilibrium AUM of a fund is either the fair size q_t^*, if the amount of invested naive money is smaller than the fair size, or the amount q_t^u of invested naive money, if the amount q_t^u is greater than the fair size:

$$q_t = \max\{q_t^*, q_t^u\} = \max\left\{\frac{A_t}{f_t}, q_t^u\right\} \, . \tag{2.5}$$

Combining (2.1) and (2.5) leads to

$$q_t = \max\left\{\frac{(ae_t)^{1-\alpha} q_t^\alpha}{f_t}, q_t^u\right\} \, .$$

When q_t^u is smaller than the fair size

$$q_t = \frac{(ae_t)^{1-\alpha} q_t^\alpha}{f_t} \quad \Longleftrightarrow \quad q_t = ae_t(f_t)^{-\frac{1}{1-\alpha}} \, .$$

As a result, q_t can be rewritten as

$$q_t = \max\left\{ae_t(f_t)^{-\frac{1}{1-\alpha}}, q_t^u\right\} \, .$$

Managers' decisions

Fee profits in time t read

$$(f_t - \phi)q_t - F - \Psi_e(e_t) = \max\left\{(f_t - \phi)ae_t(f_t)^{-\frac{1}{1-\alpha}}, (f_t - \phi)q_t^u\right\} - F - \frac{1}{2}a(e_t)^2 \, .$$

Since the effort choice e_t in time t only affects fee profits in time t, given f_t, the optimal choice of e_t solves

$$e_t = \arg\max_{e_t} \left\{ (f_t - \phi) a e_t (f_t)^{-\frac{1}{1-\alpha}}, (f_t - \phi) q_t^u \right\} - F - \frac{1}{2} a (e_t)^2 .$$

Suppose that the amount q_t^u of invested naive money is smaller than the fair size of the fund. In this case, the optimal choice of e_t solves the following first-order condition:

$$a(f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}} - a e_t = 0 \quad \Longleftrightarrow \quad e_t = (f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}} . \tag{2.6}$$

In contrast, suppose that the amount q_t^u of invested naive money is larger than the fair size of the fund. In this case, the marginal cost of effort e_t is always negative for $e_t > 0$:

$$\frac{d}{de_t} \left[(f_t - \phi) q_t^u - F - \frac{1}{2} a (e_t)^2 \right] = -a e_t < 0 .$$

As a result, in this case, the optimal choice is $e_t = 0$.

In the former case where q_t^u is smaller than the fair size, fee profits net of costs in time t can be written as

$$\frac{1}{2} a \left[(f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}} \right]^2 - F .$$

In the latter case where q_t^u is greater than the fair size, fee profits net of costs in time t read

$$(f_t - \phi) q_t^u - F .$$

Therefore, the threshold $\bar{q}_t^u$ above which naive money dominates is determined by

$$(f_t - \phi) \bar{q}_t^u - F = \frac{1}{2} a \left[(f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}} \right]^2 - F \quad \Longleftrightarrow \quad \bar{q}_t^u = \frac{1}{2} a (f_t - \phi)(f_t)^{-\frac{2}{1-\alpha}} .$$

To summarize, if the amount of invested naive money is smaller than the threshold $\bar{q}_t^u$, the manager chooses e_t as determined in (2.6). In contrast, if the amount of invested naive money is larger than $\bar{q}_t^u$, the manager chooses $e_t = 0$. As a result, fee profits in time t can be written as

$$\max\left\{\frac{1}{2}a\left[(f_t-\phi)(f_t)^{-\frac{1}{1-\alpha}}\right]^2, (f_t-\phi)q_t^u\right\} - F .$$

Define

$$Q^s(f_t) \equiv \frac{1}{2}a\left[(f_t-\phi)(f_t)^{-\frac{1}{1-\alpha}}\right]^2 , \tag{2.7}$$

where the value of $Q^s(f_t)$ is maximized at $f_t = f^s = \frac{1}{\alpha}\phi$. Then, fee profits in time t can be rewritten as

$$\max\{Q^s(f_t), (f_t-\phi)\max\{0,\tilde{q}_t^u\}\} - F = \max\{Q^s(f_t), Q_t^u(f_t)\} - F ,$$

where the equality comes from the fact $Q^s(f_t) > 0$. The optimal choice of f_t maximizes the expected fee profits in time t:

$$f_t = \arg\max_{f_t} \mathbb{E}_{t-1}\left[\max\{Q^s(f_t), Q_t^u(f_t)\}\right] - F .$$

The following proposition characterizes the range of the optimal f_t:

Proposition 2.1 *The optimal f_t is between f^s and f^u, i.e., $f^s < f_t < f^u$.*

Proof. The optimal f_t solves the following first-order condition:

$$\frac{d}{df_t}\mathbb{E}_{t-1}\left[\max\{Q^s(f_t), Q_t^u(f_t)\}\right] = 0 . \tag{2.8}$$

Define

$$\mu_t^Q(f_t;\hat{\mu}_t^q) = (f_t-\phi)\left(1-\frac{f_t}{\kappa}\right)\hat{\mu}_t^q \quad , \quad \sigma_Q^2 = \hat{\gamma}_u^2 + \hat{\gamma}_\epsilon^2\sigma_\epsilon^2 ,$$

where $\mu_t^Q(f_t)$ is the mean and σ_Q^2 is the variance of $Q_t^u(f_t)$, respectively. Note that $\mu_t^Q(f_t)$ is maximized at $f_t = f^u = \frac{\phi+\kappa}{2} > f^s$. Then, the FOC (2.8) can be written as

$$\frac{d}{df_t}\left[\int_{-\infty}^{Q^s(f_t)} Q^s(f_t)\varphi\left(\frac{x-\mu_t^Q(f_t)}{\sigma_Q}\right)\frac{dx}{\sigma_Q} + \int_{Q^s(f_t)}^{\infty} x\varphi\left(\frac{x-\mu_t^Q(f_t)}{\sigma_Q}\right)\frac{dx}{\sigma_Q}\right] = 0 ,$$

where $\varphi(x)$ is the probability distribution function (PDF) of the standard normal distribution function. Substituting $y = \frac{x-\mu_t^Q(f_t)}{\sigma_Q}$ leads to

$$\frac{d}{df_t}\left[\mu_t^Q(f_t) + \int_{-\infty}^{\frac{Q^s(f_t)-\mu_t^Q(f_t)}{\sigma_Q}} \left(Q^s(f_t) - \mu_t^Q(f_t)\right) \varphi(y)\, dy \right.$$
$$\left. +\sigma_Q \int_{\frac{Q^s(f_t)-\mu_t^Q(f_t)}{\sigma_Q}}^{\infty} y\varphi(y)\, dy\right] = 0\,,$$

which reads

$$\frac{d}{df_t}\mu_t^Q(f_t) + \Phi\left(\frac{Q^s(f_t) - \mu_t^Q(f_t)}{\sigma_Q}\right)\frac{d}{df_t}\left(Q^s(f_t) - \mu_t^Q(f_t)\right) = 0\,,$$

where $\Phi(x)$ is the cumulative distribution function (CDF) of the standard normal distribution function. The left-hand side can be written as

$$(1-\alpha(f_t))\frac{d}{df_t}\mu_t^Q(f_t) + \alpha(f_t)\frac{d}{df_t}Q^s(f_t)\,,$$

which is the marginal fee profits to the manager as f_t increases, and $\alpha(f_t)$ is defined by

$$\alpha(f_t) \equiv \Phi\left(\frac{Q^s(f_t) - \mu_t^Q(f_t)}{\sigma_Q}\right)\,,$$

where $0 < \alpha(f_t) < 1$. $\mu_t^Q(f_t)$ and $Q^s(f_t)$ are both smooth functions and have only one maximum, respectively. Since $\mu_t^Q(f_t)$ is maximized at $f_t = f^u$ and $Q^s(f_t)$ is maximized at $f^s < f^u$,

$$\frac{d}{df_t}\mu_t^Q(f_t) = \begin{cases} >0 \ , & \text{if } f_t < f^u \\ =0 \ , & \text{if } f_t = f^u \\ <0 \ , & \text{if } f_t > f^u \end{cases} \quad , \quad \frac{d}{df_t}Q^s(f_t) = \begin{cases} >0 \ , & \text{if } f_t < f^s \\ =0 \ , & \text{if } f_t = f^s \\ <0 \ , & \text{if } f_t > f^s \end{cases}\ .$$

Therefore at $f_t = f^s$,

$$(1-\alpha(f^s))\frac{d}{df_t}\mu_t^Q(f^s) + \alpha(f^s)\frac{d}{df_t}Q^s(f^s) > 0\,.$$

As a result, the manager can increase fee profits to the fund by increasing fees at $f_t = f^s$, which implies the optimal f_t is strictly greater than f^s. On the other hand, at $f_t = f^u$,

$$(1 - \alpha(f^u))\frac{d}{df_t}\mu_t^Q(f^u) + \alpha(f^u)\frac{d}{df_t}Q^s(f^u) < 0 \, .$$

As a result, the manager can increase fee profits to the fund by decreasing fees at $f_t = f^u$, which implies the optimal f_t is strictly smaller than f^u. In sum, the optimal fee choice f_t satisfies $f^s < f_t < f^u$. ■

The intuition of this result is as follows: due to the existence of naive investors, who are relatively less sensitive to fees than smart investors are, managers want to choose fees that are higher than f^s. However, since there is a possibility that they only receive small amounts of naive money, managers would not want to increase fees up to f^u.

2.3.4 Comparative statics

The skill of a manager (captured by a) and the amount of invested naive money in the fund (captured by $\hat{\mu}_t^q$) affect the optimal choices of e_t and f_t. In the benchmark cases where there are either no naive investors or no smart investors, optimal e_t and f_t are affected neither by the skill of the manager nor by the amount of invested naive money. However, in general cases where investors are a mix of both smart and naive investors, both skill and the amount of invested naive money affect the optimal choice of e_t and f_t.

The following proposition characterizes how the optimal choices are affected by the skill of a manager:

Proposition 2.2 *Suppose that there exists a unique f_t that satisfies the first-order condition (2.8). As the skill a of a manager increases, the optimal fee f_t decreases. The optimal effort e_t becomes first-order stochastically dominant as a increases.*

Proof. By the definition (2.7) of $Q^s(f_t)$,

$$Q^s(f_t; a) \equiv \frac{1}{2}a\left[(f_t - \phi)(f_t)^{-\frac{1}{1-\alpha}}\right]^2 ,$$

$Q^s(f_t)$ is proportional to a. In contrast, $Q_t^u(f_t)$ is not affected by changes of a. Denote the optimal fee choice before an increase of a by f_t^0. At f_t^0,

$$(1-\alpha(f_t^0;a))\frac{d}{df_t}\mu_t^Q(f_t^0)+\alpha(f_t^0;a)\frac{d}{df_t}Q^s(f_t^0;a)=0\ ,$$

which implies that the marginal fee profit is zero at $f=f_t^0$. Now consider an infinitesimal increase of a by δa. At f_t^0,

$$(1-\alpha(f_t^0;a+\delta a))\frac{d}{df_t}\mu_t^Q(f_t^0)+\alpha(f_t^0;a+\delta a)\frac{d}{df_t}Q^s(f_t^0;a+\delta a)<0\ ,$$

since

$$\alpha(f_t^0;a+\delta a)>\alpha(f_t^0;a)\quad ,\quad \frac{d}{df_t}Q^s(f_t^0;a+\delta a)<\frac{d}{df_t}Q^s(f_t^0;a)<0\ .$$

Denote the optimal fee choice after the increase of a by f_t^1. From the above analysis, $f_t^1<f_t^0$. Since $f^s<f_t^1<f_t^0<f^u$, the optimal choice of e_t when naive money does not dominate satisfies

$$e_t(f_t^1)=(f_t^1-\phi)(f_t^1)^{-\frac{1}{1-\alpha}}>(f_t^0-\phi)(f_t^0)^{-\frac{1}{1-\alpha}}=e_t(f_t^0)\ ,$$

because

$$\frac{d}{df_t}e_t(f_t)=-\frac{\alpha}{1-\alpha}(f_t)^{-\frac{1}{1-\alpha}-1}\left(f_t-\frac{1}{\alpha}\phi\right)<0\ ,\ \forall f_t>f^s=\frac{1}{\alpha}\phi\ .$$

In addition,

$$\mu_t^Q(f_t^0)>\mu_t^Q(f_t^1)\ ,$$

since

$$\frac{d}{df_t}\mu_t^Q(f_t)=\frac{2}{\kappa}\hat{\mu}_t^q(f^u-f_t)>0\ ,\ \forall f_t<f^u\ .$$

Since the variance of $Q_t^u(f_t)$ is unaffected by the fee choice f_t, $Q_t^u(f_t^0)$ first-order stochastically dominates $Q_t^u(f_t^1)$. On the other hand, $Q^s(f_t^0;a)$ is smaller than $Q^s(f_t^1;a+$

δa):

$$Q^s(f_t^0;a)=\frac{1}{2}a\left[(f_t^0-\phi)(f_t^0)^{-\frac{1}{1-\alpha}}\right]^2<\frac{1}{2}(a+\delta a)\left[(f_t^1-\phi)(f_t^1)^{-\frac{1}{1-\alpha}}\right]^2=Q^s(f_t^1;a+\delta a)\,.$$

The optimal choice of e_t is given by

$$\begin{aligned}
e_t(f_t^0;a) &= \begin{cases} (f_t^0-\phi)(f_t^0)^{-\frac{1}{1-\alpha}} &, \ Q_t^u(f_t^0)\le Q^s(f_t^0;a) \\ 0 &, \ Q_t^u(f_t^0)> Q^s(f_t^0;a) \end{cases}, \\
e_t(f_t^1;a+\delta a) &= \begin{cases} (f_t^1-\phi)(f_t^1)^{-\frac{1}{1-\alpha}} &, \ Q_t^u(f_t^1)\le Q^s(f_t^1;a+\delta a) \\ 0 &, \ Q_t^u(f_t^1)> Q^s(f_t^1;a+\delta a) \end{cases}.
\end{aligned}$$

Therefore, $e_t(f_t^1;a+\delta a)$ first-order stochastically dominates $e_t(f_t^0;a)$. ■

The intuition of the result is as follows: as the skill of managers increase, because smart investors provide more capital to managers with higher skill, it becomes less likely that naive money dominates those funds. Hence, marginal fee profits from naive investors decreases as managers' skill increases. On the other hand, marginal fee profits from smart investors increases as managers' skill increases. Therefore, managers face less incentive to increase fees when their skill is high. Managers also find it optimal to increase their effort because their effort is more likely to be compensated by smart investors when their skill is high.

The following proposition characterizes how the optimal choices are affected by the amount of invested naive money:

Proposition 2.3 *Suppose that there exists a unique f_t that satisfies the first-order condition (2.8). As $\hat{\mu}_t^q$ (capturing the amount of invested naive money) increases, the optimal fee f_t increases. The optimal effort e_t becomes first-order stochastically dominant as $\hat{\mu}_t^q$ decreases.*

Proof. First note that $Q^s(f_t)$ is unaffected by changes of $\hat{\mu}_t^q$. On the other hand,

$$\mu_t^Q(f_t;\hat{\mu}_t^q)=(f_t-\phi)\left(1-\frac{f_t}{\kappa}\right)\hat{\mu}_t^q$$

is proportional to $\hat{\mu}_t^q$. Denote the optimal fee choice before an increase of $\hat{\mu}_t^q$ by f_t^0. At f_t^0,

$$(1-\alpha(f_t^0;\hat{\mu}_t^q))\frac{d}{df_t}\mu_t^Q(f_t^0;\hat{\mu}_t^q)+\alpha(f_t^0;\hat{\mu}_t^q)\frac{d}{df_t}Q^s(f_t^0)=0\ ,$$

which implies that the marginal fee profit is zero at $f = f_t^0$. Now consider an infinitesimal increase of $\hat{\mu}_t^q$ by $\delta\hat{\mu}^q$. At f_t^0,

$$(1-\alpha(f_t^0;\hat{\mu}_t^q+\delta\hat{\mu}^q))\frac{d}{df_t}\mu_t^Q(f_t^0;\hat{\mu}_t^q+\delta\hat{\mu}^q)+\alpha(f_t^0;\hat{\mu}_t^q+\delta\hat{\mu}^q)\frac{d}{df_t}Q^s(f_t^0)>0\ ,$$

since

$$\alpha(f_t^0;\hat{\mu}_t^q+\delta\hat{\mu}^q)<\alpha(f_t^0;\hat{\mu}_t^q)\quad,\quad\frac{d}{df_t}\mu_t^Q(f_t^0;\hat{\mu}_t^q+\delta\hat{\mu}^q)>\frac{d}{df_t}\mu_t^Q(f_t^0;\hat{\mu}_t^q)>0\ .$$

Denote the optimal fee choice after the increase of a by f_t^1. From the above analysis, $f_t^1 > f_t^0$. Since $f^s < f_t^0 < f_t^1 < f^u$, the optimal choice of e_t when naive money does not dominate satisfies

$$e_t(f_t^1)=(f_t^1-\phi)(f_t^1)^{-\frac{1}{1-\alpha}}<(f_t^0-\phi)(f_t^0)^{-\frac{1}{1-\alpha}}=e_t(f_t^0)\ ,$$

because

$$\frac{d}{df_t}e_t(f_t)=-\frac{\alpha}{1-\alpha}(f_t)^{-\frac{1}{1-\alpha}-1}\left(f_t-\frac{1}{\alpha}\phi\right)<0\ ,\ \forall f_t>f^s=\frac{1}{\alpha}\phi\ .$$

As a result,

$$Q^s(f_t^0)=\frac{1}{2}a\left[(f_t^0-\phi)(f_t^0)^{-\frac{1}{1-\alpha}}\right]^2>\frac{1}{2}a\left[(f_t^1-\phi)(f_t^1)^{-\frac{1}{1-\alpha}}\right]^2=Q^s(f_t^1)\ .$$

In addition,

$$\mu_t^Q(f_t^0;\hat{\mu}_t^q)<\mu_t^Q(f_t^1;\hat{\mu}_t^q+\delta\hat{\mu}^q)\ ,$$

since

$$\begin{aligned} \mu_t^Q(f_t^0;\hat{\mu}_t^q) &= (f_t^0-\phi)\left(1-\frac{f_t^0}{\kappa}\right)\hat{\mu}_t^q < (f_t^1-\phi)\left(1-\frac{f_t^1}{\kappa}\right)\hat{\mu}_t^q \\ &< (f_t^1-\phi)\left(1-\frac{f_t^1}{\kappa}\right)(\hat{\mu}_t^q+\delta\hat{\mu}^q) = \mu_t^Q(f_t^1;\hat{\mu}_t^q+\delta\hat{\mu}^q)\ . \end{aligned}$$

Because the variance of $Q_t^u(f_t;\hat{\mu}_t^q)$ is unaffected by the fee choice f_t, $Q_t^u(f_t^1;\hat{\mu}_t^q+\delta\hat{\mu}^q)$ first-order stochastically dominates $Q_t^u(f_t^0;\hat{\mu}_t^q)$. The optimal choice of e_t is given by

$$\begin{aligned} e_t(f_t^0;\hat{\mu}_t^q) &= \begin{cases} (f_t^0-\phi)(f_t^0)^{-\frac{1}{1-\alpha}} &, \quad Q_t^u(f_t^0;\hat{\mu}_t^q) \le Q^s(f_t^0) \\ 0 &, \quad Q_t^u(f_t^0;\hat{\mu}_t^q) > Q^s(f_t^0) \end{cases}, \\ e_t(f_t^1;\hat{\mu}_t^q+\delta\hat{\mu}^q) &= \begin{cases} (f_t^1-\phi)(f_t^1)^{-\frac{1}{1-\alpha}} &, \quad Q_t^u(f_t^1;\hat{\mu}_t^q+\delta\hat{\mu}^q) \le Q^s(f_t^1) \\ 0 &, \quad Q_t^u(f_t^1;\hat{\mu}_t^q+\delta\hat{\mu}^q) > Q^s(f_t^1) \end{cases}. \end{aligned}$$

Therefore, $e_t(f_t^0;\hat{\mu}_t^q)$ first-order stochastically dominates $e_t(f_t^1;\hat{\mu}_t^q+\delta\hat{\mu}^q)$. ■

The intuition of this result is quite similar to the intuition of Proposition 2.2. As the amount of invested naive money increases, it is more likely that naive money dominates those funds. Marginal fee profits from naive investors increases as the amount of invested naive money increases. On the other hand, marginal fee profits from smart investors decreases as the amount of invested naive money increases. Therefore, managers have more incentive to increase fees when they expect more naive money invested in their funds. Managers also find it optimal to decrease their effort because their effort is less likely to be compensated by smart investors due to the dominance of naive money.

2.3.5 Numerical examples

For the baseline numerical computation, I make the following parameter choices:

$$\begin{aligned} &\alpha = 0.5\ ,\ \phi = 0.003\ ,\ \kappa = 0.027\ ,\ r_f = 0.03\ ,\ \sigma_\epsilon = 0.1\ , \\ &\hat{\gamma}_u = 1.2\ ,\ \hat{\gamma}_\epsilon = 16\ ,\ a = 0.00015\ ,\ \hat{\mu}_t^q = 100\ ,\ F = 0\ . \end{aligned} \tag{2.9}$$

Figure 1 plots variables of interest in the baseline model:

[See figure 1]

Comparative statics with respect to changes of a is plotted in Figure 2:

[See figure 2]

Similarly, Comparative statics with respect to changes of $\hat{\mu}_t^q$ is plotted in Figure 3:

[See figure 3]

2.4 Extensions

I consider three extended versions of the baseline model by incorporating choices of i) idiosyncratic risk, ii) marketing and iii) private benefit.

2.4.1 Idiosyncratic risk

In this version of the model, managers are allowed to choose the variance σ_ϵ^2 of the idiosyncratic risk ϵ_t of the fund return in each period. Denote the variance in time t by $\sigma_{\epsilon,t}^2$. There are costs associated with the choice of idiosyncratic risk:

$$\Psi_\sigma(\sigma_{\epsilon,t}^2) = \frac{1}{2}\xi_\sigma \left(\sigma_{\epsilon,t}^2\right)^2 .$$

The objective of a manager is to maximize

$$\max_{\left\{f_{s+1}, e_s, \sigma_{\epsilon,s}^2\right\}_{s=t,t+1,\cdots}} \sum_{s=t}^{\infty} \frac{1}{(1+r_f)^{s-t}} \mathbb{E}_t \left[(f_s - \phi)q_s - F - \Psi_e(e_s) - \Psi_\sigma(\sigma_{\epsilon,t}^2)\right] .$$

Changes in $\sigma_{\epsilon,t}^2$ do not affect capital allocation decisions of smart investors since smart investors can diversify idiosyncratic risk by themselves. Therefore, the optimal choice of e_t is the same as in the baseline model. On the other hand, changes in $\sigma_{\epsilon,t}^2$ affect the distribution of invested naive money in time $t+1$. Therefore, a manager chooses

$\sigma^2_{\epsilon,t}$ that maximizes

$$-\Psi_\sigma(\sigma^2_{\epsilon,t}) + \frac{1}{(1+r_f)}\mathbb{E}_t\left[(f_{t+1}-\phi)q_{t+1} - F - \Psi_e(e_{t+1})\right] ,$$

which leads to the following first-order condition:

$$-\xi_\sigma\sigma^2_{\epsilon,t} + \frac{1}{(1+r_f)}\frac{\partial}{\partial\left(\sigma^2_{\epsilon,t}\right)}\mathbb{E}_t\left[\max\left\{Q^s(f_{t+1}), Q^u_{t+1}(f_{t+1};\sigma_{\epsilon,t})\right\}\right] = 0 , \qquad (2.10)$$

where the first term is the marginal cost of increasing $\sigma^2_{\epsilon,t}$, and the second term is the marginal benefit.

The following proposition characterizes the optimal choice of $\sigma_{\epsilon,t}$:

Proposition 2.4 *Suppose that there exists a unique solution to the first-order condition (2.10). The optimal choice of $\sigma_{\epsilon,t}$ is the largest when the expected profits from naive investors are of the same magnitude of profits from smart investors, i.e.,*

$$Q^s(f_{t+1}) = \mu^Q_{t+1}(f_{t+1}) .$$

As $Q^s(f_{t+1})$ deviates from $\mu^Q_{t+1}(f_{t+1})$, the optimal $\sigma_{\epsilon,t}$ decreases.

Proof. Given f_{t+1}, the marginal benefit of increasing $\sigma^2_{\epsilon,t}$ can be written as

$$\frac{1}{(1+r_f)}\frac{d}{d\sigma^2_{\epsilon,t}}\left[\int_{-\infty}^{Q^s(f_{t+1})} Q^s(f_{t+1})\varphi\left(\frac{x-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}\right)\frac{dx}{\sigma_{Q,t}} + \int_{Q^s(f_{t+1})}^{\infty} x\varphi\left(\frac{x-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}\right)\frac{dx}{\sigma_{Q,t}}\right] ,$$

where

$$\sigma^2_{Q,t} = \hat{\gamma}^2_u + \hat{\gamma}^2_\epsilon\sigma^2_{\epsilon,t} .$$

Substituting $y = \frac{x-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}$ leads to

$$\frac{1}{(1+r_f)}\frac{d}{d\sigma^2_{\epsilon,t}}\left[\mu^Q_{t+1}(f_{t+1}) + \int_{-\infty}^{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}}\left(Q^s(f_{t+1}) - \mu^Q_{t+1}(f_{t+1})\right)\right.$$

$$\left.\times\varphi(y)\,dy + \sigma_{Q,t}\int_{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}}^{\infty} y\varphi(y)\,dy\right].$$

Therefore, the marginal benefit of increasing $\sigma^2_{\epsilon,t}$ is

$$\frac{1}{(1+r_f)}\frac{1}{2\sigma_{\epsilon,t}}\frac{d\sigma_{Q,t}}{d\sigma_{\epsilon,t}}\int_{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}}^{\infty} y\varphi(y)\,dy\;,$$

where

$$\frac{d\sigma_{Q,t}}{d\sigma_{\epsilon,t}} = \frac{\hat{\gamma}^2_\epsilon\sigma_{\epsilon,t}}{\sqrt{\hat{\gamma}^2_u + \hat{\gamma}^2_\epsilon\sigma^2_{\epsilon,t}}} = \hat{\gamma}^2_\epsilon\frac{\sigma_{\epsilon,t}}{\sigma_{Q,t}}\;.$$

Then, the FOC (2.10) reads

$$-\xi_\sigma\sigma^2_{\epsilon,t} + \frac{1}{(1+r_f)}\frac{\hat{\gamma}^2_\epsilon}{2\sigma_{Q,t}}\int_{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}}^{\infty} y\varphi(y)\,dy = 0\;.$$

Note that there exists a solution to this FOC. At $\sigma_{\epsilon,t} = 0$, the marginal benefit of increasing $\sigma^2_{\epsilon,t}$ net of cost is

$$\frac{1}{(1+r_f)}\frac{\hat{\gamma}^2_\epsilon}{2\sigma_{Q,t}}\int_{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}}^{\infty} y\varphi(y)\,dy > 0\;.$$

On the other hand, as $\sigma_{\epsilon,t} \to \infty$, the marginal benefit of increasing $\sigma^2_{\epsilon,t}$ net of cost becomes

$$\lim_{\sigma_{\epsilon,t}\to\infty}\left[-\xi_\sigma\sigma^2_{\epsilon,t} + \frac{1}{(1+r_f)}\frac{\hat{\gamma}^2_\epsilon}{2\sigma_{Q,t}}\int_{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1})}{\sigma_{Q,t}}}^{\infty} y\varphi(y)\,dy\right] \to -\infty\;.$$

Therefore, there exists $\sigma_{\epsilon,t}$ that satisfies the FOC (2.10).

Define $\Delta_{t+1}^{s-u} \equiv Q^s(f_{t+1}) - \mu_{t+1}^Q(f_{t+1})$. Also define

$$h\left(\frac{\Delta_{t+1}^{s-u}}{\sigma_{Q,t}}\right) \equiv \int_{\frac{Q^s(f_{t+1})-\mu_{t+1}^Q(f_{t+1})}{\sigma_{Q,t}}}^{\infty} y\varphi\left(y\right) dy \ .$$

First note that $h(z)$ is maximized at $z = 0$:

$$h'(z) = \frac{d}{dz}\int_z^{\infty} y\varphi(y)dy = -z\varphi(z) = \begin{cases} >0 \ , & z<0 \\ =0 \ , & z=0 \\ <0 \ , & z>0 \end{cases} ,$$

and $h(z)$ decreases as $|z|$ increases. Denote the optimal $\sigma_{\epsilon,t}$ at Δ_{t+1}^{s-u} by $\sigma_{\epsilon,t}(\Delta_{t+1}^{s-u})$. For $\Delta_{t+1}^{s-u} \neq 0$, consider a transformation $\Delta_{t+1}^{s-u} \to 0$, the marginal benefit of increasing $\sigma_{\epsilon,t}^2$ net of cost at $\sigma_{\epsilon,t}(\Delta_{t+1}^{s-u})$ is

$$-\xi_\sigma \sigma_{\epsilon,t}^2(\Delta_{t+1}^{s-u}) + \frac{1}{(1+r_f)}\frac{\hat{\gamma}_\epsilon^2}{2\sigma_{Q,t}(\sigma_{\epsilon,t}(\Delta_{t+1}^{s-u}))}h\left(0\right) > 0 \ ,$$

for an arbitrary $\Delta_{t+1}^{s-u} \neq 0$. Therefore,

$$\sigma_{\epsilon,t}(0) > \sigma_{\epsilon,t}(\Delta_{t+1}^{s-u}) \ , \ \forall \Delta_{t+1}^{s-u} \neq 0 \ .$$

Now consider the following transformation for $\Delta_{t+1}^{s-u} \neq 0$:

$$\Delta_{t+1}^{s-u} \longrightarrow \Delta_{t+1}^{s-u}(1+\delta) \ ,$$

which is an infinitesimal expansion of Δ_{t+1}^{s-u} in the direction of Δ_{t+1}^{s-u}. Then, the marginal benefit of increasing $\sigma_{\epsilon,t}^2$ net of cost at $\sigma_{\epsilon,t}(\Delta_{t+1}^{s-u})$ is

$$-\xi_\sigma \sigma_{\epsilon,t}^2(\Delta_{t+1}^{s-u}) + \frac{1}{(1+r_f)}\frac{\hat{\gamma}_\epsilon^2}{2\sigma_{Q,t}(\sigma_{\epsilon,t}(\Delta_{t+1}^{s-u}))}h\left(\frac{\Delta_{t+1}^{s-u}(1+\delta)}{\sigma_{Q,t}(\sigma_{\epsilon,t}(\Delta_{t+1}^{s-u}))}\right) < 0 \ .$$

Therefore, as Δ_{t+1}^{s-u} deviates further from zero, the optimal choice of $\sigma_{\epsilon,t}$ decreases more. ■

The intuition of this result is as follows: the expected fee profits

$$\frac{1}{(1+r_f)}\mathbb{E}_t\left[\max\left\{Q^s(f_{t+1}), Q^u_{t+1}(f_{t+1};\sigma_{\epsilon,t})\right\}\right]$$

has a call option-like payoff structure as a function of the realization of $Q^u_{t+1}(f_{t+1};\sigma_{\epsilon,t})$, which can be interpreted as fee profits from naive investors. When the realized value of $Q^u_{t+1}(f_{t+1};\sigma_{\epsilon,t})$ is lower than $Q^s(f_{t+1})$, i.e., when naive money does not dominate, fee profits are determined by smart investors at $Q^s(f_{t+1})$. On the other hand, when the realized value of $Q^u_{t+1}(f_{t+1};\sigma_{\epsilon,t})$ is higher than $Q^s(f_{t+1})$, i.e., when naive money dominates, fee profits are determined by naive investors at $Q^u_{t+1}(f_{t+1};\sigma_{\epsilon,t})$. Therefore, $Q^s(f_{t+1})$, which can be interpreted as fee profits from smart investors, serves as the strike price of the call option. Since a call option vega (the sensitivity of an option's price to changes in the volatility of its underlying) is maximized at the strike price, the marginal benefit of increasing $\sigma_{\epsilon,t}$ is maximized at

$$Q^s(f_{t+1}) = \mu^Q_{t+1}(f_{t+1}) \ .$$

The marginal benefit decreases as $Q^s(f_{t+1})$ deviates from $\mu^Q_{t+1}(f_{t+1})$. Since the marginal cost of increasing $\sigma_{\epsilon,t}$ is increasing in $\sigma_{\epsilon,t}$, the optimal choice of $\sigma_{\epsilon,t}$ is maximized when the marginal benefit is maximized.

Denote the optimal choice of $\sigma_{\epsilon,t}$ given fee f_{t+1} by $\sigma_{\epsilon,t}(f_{t+1})$. The optimal fee f_{t+1} solves

$$\frac{\partial}{\partial f_{t+1}}\mathbb{E}_t\left[\max\left\{Q^s(f_{t+1}), Q^u_{t+1}(f_{t+1};\sigma_{\epsilon,t}(f_{t+1}))\right\}\right] = 0 \ .$$

The range of the optimal f_{t+1} is between f^s and f^u as in the baseline model:

Corollary 2.1 *The optimal f_{t+1} satisfies $f^s < f_{t+1} < f^u$.*

Proof. Since Proposition 2.1 holds regardless of the value of σ_ϵ, the optimal choice of f_{t+1} lies between f^s and f^u regardless of the choice $\sigma_{\epsilon,t}(f_{t+1})$. ■

The following proposition characterizes how the optimal choice of $\sigma_{\epsilon,t}$ is affected by changes of a and $\hat{\mu}^q_{t+1}$:

Proposition 2.5 *Suppose that there exists a unique solution to the first-order condition (2.10). As a and $\hat{\mu}^q_{t+1}$ changes, the optimal choice of $\sigma_{\epsilon,t}$ is the largest when the expected profits from naive investors are of the same magnitude of profits from smart investors, i.e.,*

$$Q^s(f_{t+1}(a, \hat{\mu}^q_{t+1}); a) = \mu^Q_{t+1}(f_{t+1}(a, \hat{\mu}^q_{t+1}); \hat{\mu}^q_{t+1}) \ .$$

As $Q^s(f_{t+1}(a, \hat{\mu}^q_{t+1}); a)$ deviates from $\mu^Q_{t+1}(f_{t+1}(a, \hat{\mu}^q_{t+1}); \hat{\mu}^q_{t+1})$, the optimal $\sigma_{\epsilon,t}$ decreases.

Proof. The marginal benefit of increasing $\sigma^2_{\epsilon,t}$ is

$$\frac{1}{(1+r_f)} \frac{\hat{\gamma}^2_\epsilon}{2\sigma_{Q,t}} \int^{\infty}_{\frac{Q^s(f_{t+1}(a,\hat{\mu}^q_{t+1});a)-\mu^Q_{t+1}(f_{t+1}(a,\hat{\mu}^q_{t+1});\hat{\mu}^q_{t+1})}{\sigma_{Q,t}}} y\varphi(y)\, dy \ ,$$

which is maximized at, given $\sigma_{Q,t}$,

$$Q^s(f_{t+1}(a, \hat{\mu}^q_{t+1}); a) = \mu^Q_{t+1}(f_{t+1}(a, \hat{\mu}^q_{t+1}); \hat{\mu}^q_{t+1}) \ , \tag{2.11}$$

and decreases as $Q^s(f_{t+1}(a, \hat{\mu}^q_{t+1}); a)$ deviates from $\mu^Q_{t+1}(f_{t+1}(a, \hat{\mu}^q_{t+1}); \hat{\mu}^q_{t+1})$. Therefore, following the same logic as in the proof of Proposition 2.4, the optimal choice of $\sigma_{\epsilon,t}$ is maximized when (2.11) holds. In addition, the optimal choice of $\sigma_{\epsilon,t}$ decreases as $Q^s(f_{t+1}(a, \hat{\mu}^q_{t+1}); a)$ deviates from $\mu^Q_{t+1}(f_{t+1}(a, \hat{\mu}^q_{t+1}); \hat{\mu}^q_{t+1})$. ■

The intuition of this result is the same as that of Proposition 2.4.

2.4.2 Marketing

In this version of the model, managers can choose the level of marketing m_t. In the context of the model, I define marketing as activities that attract capital from naive investors by spending costs[3]. By choosing m_t, a manager can increase the mean of $\hat{\mu}^q_{t+1}$ by m_t:

$$\hat{\mu}^q_{t+1} \to \hat{\mu}^q_{t+1} + m_t \ ,$$

[3]In this context, marketing may be interpreted as deceiving and alluring unsophisticated investors. Another interpretation is that marketing reduces search costs for unsophisticated investors. Since sophisticated investors in this model do not face any frictions (including search frictions), such reduction of search costs only affects capital allocation decisions of naive investors.

by incurring marketing costs

$$\Psi_m(m_t) = \frac{1}{2}\xi_m m_t^2 \, .$$

The objective of a manager is to maximize

$$\max_{\{f_{s+1}, e_s, m_s\}_{s=t,t+1,\cdots}} \sum_{s=t}^{\infty} \frac{1}{(1+r_f)^{s-t}} \mathbb{E}_t \left[(f_s - \phi) q_s - F - \Psi_e(e_s) - \Psi_m(m_t) \right] \, .$$

Changes in m_t do not affect capital allocation decisions of smart investors. Therefore, the optimal choice of e_t is the same as in the baseline model. On the other hand, changes in m_t shift the mean of invested naive money in time $t+1$. A manager chooses m_t that maximizes

$$-\Psi_m(m_t) + \frac{1}{(1+r_f)} \mathbb{E}_t \left[(f_{t+1} - \phi) q_{t+1} - F - \Psi_e(e_{t+1}) \right] \, ,$$

which leads to the following first-order condition:

$$-\xi_m m_t + \frac{1}{(1+r_f)} \frac{\partial}{\partial m_t} \mathbb{E}_t \left[\max \left\{ Q^s(f_{t+1}), Q^u_{t+1}(f_{t+1}; m_t) \right\} \right] = 0 \, , \tag{2.12}$$

where the first term is the marginal cost of increasing m_t, and the second term is the marginal benefit.

The following proposition characterizes the optimal choice of m_t:

Proposition 2.6 *Suppose that there exists a unique solution to the first-order condition (2.12). Given f_{t+1}, the optimal choice of marketing m_t decreases in a and increases in $\hat{\mu}^q_t$.*

Proof. First note that for m_t

$$\mu^Q_{t+1}(f_{t+1}; m_t) = (f_{t+1} - \phi) \left(1 - \frac{f_{t+1}}{\kappa} \right) \left(\hat{\mu}^q_{t+1} + m_t \right) \, .$$

Given f_{t+1}, the marginal benefit of increasing m_t can be written as

$$\frac{1}{(1+r_f)}\frac{d}{dm_t}\left[\int_{-\infty}^{Q^s(f_{t+1})} Q^s(f_{t+1})\varphi\left(\frac{x-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}\right)\frac{dx}{\sigma_Q}\right.$$
$$\left.+\int_{Q^s(f_{t+1})}^{\infty} x\varphi\left(\frac{x-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}\right)\frac{dx}{\sigma_Q}\right],$$

Substituting $y=\frac{x-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}$ leads to

$$\frac{1}{(1+r_f)}\frac{d}{dm_t}\left[\mu^Q_{t+1}(f_{t+1};m_t)+\int_{-\infty}^{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}}\left(Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1};m_t)\right)\right.$$
$$\left.\times\varphi\left(y\right)dy+\sigma_Q\int_{\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}}^{\infty} y\varphi\left(y\right)dy\right],$$

which can be simplified as

$$\frac{1}{(1+r_f)}\left(1-\Phi\left(\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}\right)\right)\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};m_t).$$

Hence, the FOC (2.12) reads

$$-\xi_m m_t+\frac{1}{(1+r_f)}\left(1-\Phi\left(\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}\right)\right)\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};m_t)=0.$$

Note that there exists a solution to this FOC. Since the mean $\mu^Q_{t+1}(f_{t+1};m_t)$ of the profits $Q^u_{t+1}(f_{t+1};m_t)$ from naive investors is increasing in m_t, at $m_t=0$ the marginal benefit of increasing m_t net of cost is

$$\frac{1}{(1+r_f)}\left(1-\Phi\left(\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1};m_t=0)}{\sigma_Q}\right)\right)\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};m_t=0)>0.$$

On the other hand, as $m_t\to\infty$, the marginal benefit of increasing m_t net of cost becomes

$$\begin{aligned}
&\lim_{m_t\to\infty}\Bigg[-\xi_m m_t+\frac{1}{(1+r_f)}\left(1-\Phi\left(\frac{Q^s(f_{t+1})-\mu^Q_{t+1}(f_{t+1};m_t)}{\sigma_Q}\right)\right)\\
&\times\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};m_t)\Bigg]\\
=\;&\lim_{m_t\to\infty}\left[-\xi_m m_t+\frac{1}{(1+r_f)}(f_{t+1}-\phi)\left(1-\frac{f_{t+1}}{\kappa}\right)\right]\to-\infty\,.
\end{aligned}$$

Therefore, there exists m_t that satisfies the FOC (2.12).

Given f_{t+1}, the optimal choice of m_t decreases in a. Denote the optimal choice of m_t at a by $m_t(a)$. Since

$$Q^s(f_{t+1};a)\equiv\frac{1}{2}a\left[(f_{t+1}-\phi)(f_{t+1})^{-\frac{1}{1-\alpha}}\right]^2\,,$$

changes in a affect $Q^s(f_{t+1};a)$, but does not affect $Q^u_{t+1}(f_{t+1})$. Consider an increase in a: $a\to a+\delta a$.

$$\begin{aligned}
&-\xi_m m_t(a)+\frac{1}{(1+r_f)}\left(1-\Phi\left(\frac{Q^s(f_{t+1};a+\delta a)-\mu^Q_{t+1}(f_{t+1};m_t(a))}{\sigma_Q}\right)\right)\\
&\times\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};m_t(a))<0\,.
\end{aligned}$$

Therefore, $m_t(a+\delta a)<m_t(a)$, i.e., m_t decreases in a given f_{t+1}.

Given f_{t+1}, the optimal choice of m_t increases in $\hat{\mu}^q_{t+1}$. Denote the optimal choice of m_t at $\hat{\mu}^q_{t+1}$ by $m_t(\hat{\mu}^q_{t+1})$. Changes in $\hat{\mu}^q_{t+1}$ do not affect $Q^s(f_{t+1})$ but $Q^u_{t+1}(f_{t+1};\hat{\mu}^q_{t+1})$ by increasing the mean $\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1})$. Consider an increase in $\hat{\mu}^q_{t+1}$: $\hat{\mu}^q_{t+1}\to\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q$. At $m_t(\hat{\mu}^q_{t+1})$,

$$\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1},m_t(\hat{\mu}^q_{t+1}))<\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q,m_t(\hat{\mu}^q_{t+1}))\,,$$

and

$$\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1},m_t(\hat{\mu}^q_{t+1}))=\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q,m_t(\hat{\mu}^q_{t+1}))\,.$$

As a result,

$$-\xi_m m_t(\hat{\mu}^q_{t+1}) + \frac{1}{(1+r_f)}\left(1 - \Phi\left(\frac{Q^s(f_{t+1}) - \mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1} + \delta\hat{\mu}^q; m_t(\hat{\mu}^q_{t+1}))}{\sigma_Q}\right)\right)$$
$$\times\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1} + \delta\hat{\mu}^q, m_t(\hat{\mu}^q_{t+1})) > 0\ .$$

Therefore, $m_t(\hat{\mu}^q_{t+1} + \delta\hat{\mu}^q) > m_t(\hat{\mu}^q_{t+1})$, i.e., m_t increases in $\hat{\mu}^q_{t+1}$ given f_{t+1}. ■

The intuition of this result is as follows: the marginal benefit of increasing marketing m_t is proportional to the probability that naive money dominates. This is because m_t simply shifts the expected amount $\hat{\mu}^q_{t+1}$ of naive money. Given fee f_{t+1}, an increase in a leads to a lower probability that naive money dominates. On the other hand, an increase in $\hat{\mu}^q_{t+1}$ raises the probability that naive money dominates funds. Therefore, the marginal benefit of increasing m_t becomes lower as a increases, and becomes higher as $\hat{\mu}^q_{t+1}$ increases. Since the marginal cost of marketing is an increasing function of m_t, the optimal m_t, given f_{t+1}, is decreasing in a and increasing in $\hat{\mu}^q_{t+1}$.

Denote the optimal choice of m_t given fee f_{t+1} by $m_t(f_{t+1})$. The optimal fee f_{t+1} solves

$$\frac{\partial}{\partial f_{t+1}}\mathbb{E}_t\left[\max\left\{Q^s(f_{t+1}; m_t(f_{t+1})), Q^u_{t+1}(f_{t+1})\right\}\right] = 0\ . \tag{2.13}$$

The range of the optimal f_{t+1} is between f^s and f^u as in the baseline model:

Corollary 2.2 *The optimal f_{t+1} satisfies $f^s < f_{t+1} < f^u$.*

Proof. Choosing m_t is equivalent to shifting $\hat{\mu}^q_{t+1}$:

$$\hat{\mu}^q_{t+1} \to \hat{\mu}^q_{t+1} + m_t\ .$$

Since Proposition 2.1 holds regardless of the value of $\hat{\mu}^q_{t+1}$, the optimal choice of f_{t+1} lies between f^s and f^u. ■

The following proposition characterizes the optimal choice of f_t:

Proposition 2.7 *Suppose that there exists a unique solution to the first-order condition (2.13). The optimal f_{t+1} decreases in a and increases in $\hat{\mu}^q_{t+1}$.*

Proof. The FOC (2.13) can be written as

$$\frac{d}{df_{t+1}}\mu^Q_{t+1}(f_{t+1}; m_t(f_{t+1})) + \Phi\left(\frac{Q^s(f_{t+1}) - \mu^Q_{t+1}(f_{t+1}; m_t(f_{t+1}))}{\sigma_Q}\right)$$
$$\times \frac{d}{df_{t+1}}\left(Q^s(f_{t+1}) - \mu^Q_{t+1}(f_{t+1}; m_t(f_{t+1}))\right) = 0 \,.$$

Consider an increase in a: $a \to a+\delta a$. Denote the optimal f_{t+1} at a by $f_{t+1}(a)$. From Proposition 2.6, $m_t(f_{t+1}; a)$ decreases in a. Note that given f_{t+1}

$$Q^s(f_{t+1}; a+\delta a) > Q^s(f_{t+1}; a) \,,\ \hat{\mu}^q_{t+1}(f_{t+1}) + m_t(f_{t+1}; a+\delta a) < \hat{\mu}^q_{t+1}(f_{t+1}) + m_t(f_{t+1}; a) \,,$$

and

$$\frac{d}{df_{t+1}}Q^s(f_{t+1}; a+\delta a) < \frac{d}{df_{t+1}}Q^s(f_{t+1}; a) < 0 \,,$$
$$0 < \frac{d}{df_{t+1}}\mu^Q_{t+1}(f_{t+1}; m_t(f_{t+1}; a+\delta a)) < \frac{d}{df_{t+1}}\mu^Q_{t+1}(f_{t+1}; m_t(f_{t+1}; a)) \,.$$

Then,

$$\frac{d}{df_{t+1}}\mu^Q_{t+1}(f_{t+1}(a); m_t(f_{t+1}(a); a+\delta a))$$
$$+\Phi\left(\frac{Q^s(f_{t+1}(a); a+\delta a) - \mu^Q_{t+1}(f_{t+1}(a); m_t(f_{t+1}(a); a+\delta a))}{\sigma_Q}\right)$$
$$\times \frac{d}{df_t}\left(Q^s(f_{t+1}(a); a+\delta a) - \mu^Q_{t+1}(f_{t+1}(a); m_t(f_{t+1}(a); a+\delta a))\right) < 0 \,.$$

Therefore, the optimal $f_{t+1}(a)$ is decreasing in a.

Similarly, consider an increase in $\hat{\mu}^q_{t+1}$: $\hat{\mu}^q_{t+1} \to \hat{\mu}^q_{t+1} + \delta\hat{\mu}^q$. Denote the optimal f_{t+1} at $\hat{\mu}^q_{t+1}$ by $f_{t+1}(\hat{\mu}^q_{t+1})$. From Proposition 2.6, $m_t(f_{t+1}; \hat{\mu}^q_{t+1})$ increases in $\hat{\mu}^q_{t+1}$. Note that given f_{t+1}

$$\mu^Q_{t+1}(f_{t+1}; \hat{\mu}^q_{t+1} + \delta\hat{\mu}^q, m_t(f_{t+1}; \hat{\mu}^q_{t+1} + \delta\hat{\mu}^q)) > \mu^Q_{t+1}(f_{t+1}; \hat{\mu}^q_{t+1}, m_t(f_{t+1}; \hat{\mu}^q_{t+1})) \,,$$

and

$$
\begin{aligned}
&\frac{d}{df_{t+1}}\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q, m_t(f_{t+1};\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q)) \\
> \; &\frac{d}{df_{t+1}}\mu^Q_{t+1}(f_{t+1};\hat{\mu}^q_{t+1}, m_t(f_{t+1};\hat{\mu}^q_{t+1})) > 0 \,.
\end{aligned}
$$

In contrast, given f_{t+1}, $Q^s(f_{t+1})$ is unaffected by changes in $\hat{\mu}^q_{t+1}$. Then,

$$
\begin{aligned}
&\frac{d}{df_{t+1}}\mu^Q_{t+1}(f_{t+1}(\hat{\mu}^q_{t+1});\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q, m_t(f_{t+1}(\hat{\mu}^q_{t+1});\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q)) \\
&+\Phi\left(\frac{Q^s(f_{t+1}(\hat{\mu}^q_{t+1}))-\mu^Q_{t+1}(f_{t+1}(\hat{\mu}^q_{t+1});\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q, m_t(f_{t+1}(\hat{\mu}^q_{t+1});\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q))}{\sigma_Q}\right) \\
&\times\frac{d}{df_{t+1}}\Big(Q^s(f_{t+1}(\hat{\mu}^q_{t+1})) \\
&-\mu^Q_{t+1}(f_{t+1}(\hat{\mu}^q_{t+1});\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q, m_t(f_{t+1}(\hat{\mu}^q_{t+1});\hat{\mu}^q_{t+1}+\delta\hat{\mu}^q))\Big) > 0 \,.
\end{aligned}
$$

Therefore, the optimal $f_{t+1}(\hat{\mu}^q_{t+1})$ is increasing in $\hat{\mu}^q_{t+1}$. ■

The intuition of this result is as follows: for fixed f_{t+1}, marketing m_t decreases as a increases, and increases as $\hat{\mu}^q_{t+1}$ increases as shown in Proposition 2.6. This implies that the choice of m_t reinforces the effect of changes in a and $\hat{\mu}^q_{t+1}$. As a increases, naive money becomes less likely to dominate funds, and the corresponding decrease of m_t makes naive money even less likely to dominate. A similar argument applies to changes in $\hat{\mu}^q_{t+1}$.

Therefore, as a increases, marginal fee profits from naive investors decrease, and marginal fee profits from smart investors increase. Hence, managers have incentive to lower fees as a increases. In contrast, as $\hat{\mu}^q_{t+1}$ increases, marginal fee profits from naive investors increase, and marginal fee profits from smart investors decrease. Hence, managers would want to raise fees as $\hat{\mu}^q_{t+1}$ increases.

The following corollary confirms that the result of Proposition 2.6 holds without fixing fees f_{t+1}, i.e., f_{t+1} optimally responds to changes in a and $\hat{\mu}^q_{t+1}$:

Corollary 2.3 *Suppose that there exist unique solutions to the first-order conditions (2.12) and (2.13). The optimal choice of marketing m_t decreases in a and increases* *in* $\hat{\mu}^q_{t+1}$.

Proof. Following the logic of Proposition 2.6, and using the result of Proposition 2.7,

$$-\xi_m m_t(a) + \frac{1}{(1+r_f)}$$
$$\times \left(1 - \Phi\left(\frac{Q^s(f_{t+1}(a+\delta a); a+\delta a) - \mu^Q_{t+1}(f_{t+1}(a+\delta a); m_t(a))}{\sigma_Q}\right)\right)$$
$$\times \frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1}(a+\delta a); m_t(a)) < 0 \; .$$

Therefore, $m_t(a)$ is decreasing in a. Similarly,

$$-\xi_m m_t(\hat{\mu}^q_{t+1}) + \frac{1}{(1+r_f)}\frac{d}{dm_t}\mu^Q_{t+1}(f_{t+1}(\hat{\mu}^q_{t+1} + \delta\hat{\mu}^q); \hat{\mu}^q_{t+1} + \delta\hat{\mu}^q, m_t(\hat{\mu}^q_{t+1})) \times \Bigg(1$$
$$-\Phi\left(\frac{Q^s(f_{t+1}(\hat{\mu}^q_{t+1} + \delta\hat{\mu}^q)) - \mu^Q_{t+1}(f_{t+1}(\hat{\mu}^q_{t+1} + \delta\hat{\mu}^q); \hat{\mu}^q_{t+1} + \delta\hat{\mu}^q; m_t(\hat{\mu}^q_{t+1}))}{\sigma_Q}\right)\Bigg)$$
$$> 0 \; .$$

Therefore, $m_t(\hat{\mu}^q_{t+1})$ is increasing in $\hat{\mu}^q_{t+1}$. ∎

2.4.3 Private benefit

In this version of the model, managers can choose to gain private benefit by sacrificing dollar value that they generate for investors[4]. To be more specific, in time t, managers gain

$$\Theta(b_t) \; ,$$

where b_t captures the amount of value that managers sacrifice. Choosing $b_t \in [0,1]$ decreases the value of a as follows:

$$a \to a(1-b_t) \; .$$

[4]Soft dollars may be thought of as an example. Conrad, Johnson, and Wahal (2002) documents that soft-dollar trades incur higher costs.

b_t is determined in the previous period $t-1$. The marginal gain $\Theta'(b_t)$ of private benefit decreases in b_t, i.e., $\Theta(b_t)$ is convex in b_t. For analytic simplicity, I choose the following form of $\Theta(b_t)$:

$$\Theta(b_t) = \xi_b\left(b_t - \frac{1}{2}b_t^2\right) .$$

The object of a manager is to maximize

$$\max_{\{f_{s+1},e_s,b_{s+1}\}_{s=t,t+1,\cdots}} \sum_{s=t}^{\infty} \frac{1}{(1+r_f)^{s-t}} \mathbb{E}_t\left[(f_s-\phi)q_s - F - \Psi_e(e_s) + \Theta_b(b_t)\right] .$$

Since the optimal choice of e_t does not explicitly depend on a, the optimal e_t is the same as in the baseline model. Hence, $Q^s(f_t;b_t)$ can be written as

$$Q^s(f_t;b_t) \equiv \frac{1}{2}a(1-b_t)\left[(f_t-\phi)(f_t)^{-\frac{1}{1-\alpha}}\right]^2 .$$

On the other hand, $Q_t^u(f_t)$ is unaffected by changes in b_t. A manager chooses b_t that maximizes

$$\Theta(b_t) + \mathbb{E}_{t-1}\left[(f_t-\phi)q_t - F - \Psi_e(e_t)\right] ,$$

which leads to the following first-order condition:

$$\xi_b(1-b_t) + \frac{\partial}{\partial b_t}\mathbb{E}_{t-1}\left[\max\left\{Q^s(f_t;b_t), Q_t^u(f_t)\right\}\right] = 0 , \tag{2.14}$$

where the first term is the marginal private benefit of increasing b_t, and the second term is the marginal cost. Note that the existence of a solution to the FOC (2.14) is guaranteed if ξ_b is sufficiently large:

$$\xi_b > \frac{1}{2}a(1-\alpha)^2\left(\frac{\phi}{\alpha}\right)^{-\frac{2\alpha}{1-\alpha}} , \tag{2.15}$$

because

$$\xi_b > \frac{\partial}{\partial b_t}Q^s(f_t;b_t=0) \geq \frac{\partial}{\partial b_t}\mathbb{E}_{t-1}\left[\max\left\{Q^s(f_t;b_t), Q_t^u(f_t)\right\}\right]\Big|_{b_t=0} ,$$

and

$$0 < \frac{\partial}{\partial b_t}\mathbb{E}_{t-1}\left[\max\left\{Q^s(f_t;b_t), Q^u_t(f_t)\right\}\right]\Big|_{b_t=1} .$$

The following proposition characterizes the optimal choice of b_t:

Proposition 2.8 *Suppose that there exists a unique solution to the first-order condition (2.14). Given f_t, the optimal choice of b_t decreases in a and increases in $\hat{\mu}^q_t$.*

Proof. Given f_t, the marginal cost of increasing b_t can be written as

$$\frac{d}{db_t}\left[\int_{-\infty}^{Q^s(f_t;b_t)} Q^s(f_t;b_t)\varphi\left(\frac{x-\mu^Q_t(f_t)}{\sigma_Q}\right)\frac{dx}{\sigma_Q} + \int_{Q^s(f_t;b_t)}^{\infty} x\varphi\left(\frac{x-\mu^Q_t(f_t)}{\sigma_Q}\right)\frac{dx}{\sigma_Q}\right] .$$

Substituting $y = \frac{x-\mu^Q_t(f_t)}{\sigma_Q}$ leads to

$$\frac{d}{db_t}\left[\mu^Q_t(f_t) + \int_{-\infty}^{\frac{Q^s(f_t;b_t)-\mu^Q_t(f_t)}{\sigma_Q}} \left(Q^s(f_t;b_t) - \mu^Q_t(f_t)\right)\varphi(y)\,dy\right.$$
$$\left. +\sigma_Q \int_{\frac{Q^s(f_t;b_t)-\mu^Q_t(f_t)}{\sigma_Q}}^{\infty} y\varphi(y)\,dy\right] ,$$

which can be simplified as

$$\Phi\left(\frac{Q^s(f_t;b_t)-\mu^Q_t(f_t)}{\sigma_Q}\right)\frac{d}{db_t}Q^s(f_t;b_t) = -\Phi\left(\frac{Q^s(f_t;b_t)-\mu^Q_t(f_t)}{\sigma_Q}\right)Q^s(f_t;b_t) .$$

Hence, the FOC (2.14) reads

$$\xi_b(1-b_t) - \Phi\left(\frac{Q^s(f_t;b_t)-\mu^Q_t(f_t)}{\sigma_Q}\right)Q^s(f_t;b_t) = 0 .$$

Given f_t, the optimal choice of b_t decreases in a. Denote the optimal choice of b_t at a by $b_t(a)$. Changes in a affect $Q^s(f_t;a,b_t)$, but does not affect $Q^u_t(f_t)$. Consider an increase in a: $a \to a + \delta a$.

$$\xi_b(1-b_t(a)) - \Phi\left(\frac{Q^s(f_t;a+\delta a,b_t(a))-\mu^Q_t(f_t)}{\sigma_Q}\right)Q^s(f_t;a+\delta a,b_t(a)) < 0 .$$

Therefore, $b_t(a + \delta a) < b_t(a)$, i.e., b_t decreases in a given f_t.

Given f_t, the optimal choice of b_t increases in μ_t^q. Denote the optimal choice of b_t at $\hat{\mu}_t^q$ by $b_t(\hat{\mu}_t^q)$. Changes in $\hat{\mu}_t^q$ do not affect $Q^s(f_t; b_t)$ but $Q_t^u(f_t; \hat{\mu}_t^q)$ by increasing the mean $\mu_t^Q(f_t; \hat{\mu}_t^q)$. Consider an increase in $\hat{\mu}_t^q$: $\hat{\mu}_t^q \to \hat{\mu}_t^q + \delta\hat{\mu}^q$. Since

$$\mu_t^Q(f_t; \hat{\mu}_t^q) < \mu_t^Q(f_t; \hat{\mu}_t^q + \delta\hat{\mu}^q) ,$$

at $b_t(\hat{\mu}_t^q)$,

$$\xi_b(1 - b_t(\hat{\mu}_t^q)) - \Phi\left(\frac{Q^s(f_t; b_t(\hat{\mu}_t^q)) - \mu_t^Q(f_t; \hat{\mu}_t^q + \delta\hat{\mu}^q)}{\sigma_Q}\right) Q^s(f_t; b_t(\hat{\mu}_t^q)) > 0 .$$

Therefore, $b_t(\hat{\mu}_t^q + \delta\hat{\mu}^q) > b_t(\hat{\mu}_t^q)$, i.e., b_t increases in $\hat{\mu}_t^q$ given f_t. ■

The intuition of this result is as follows: the marginal cost of increasing b_t is proportional to the probability that naive money does not dominate. This is because b_t simply shifts the skill a of managers. Given fee f_{t+1}, an increase in a leads to a lower probability that naive money dominates. On the other hand, an increase in $\hat{\mu}_{t+1}^q$ raises the probability that naive money dominates funds. Therefore, the marginal cost of increasing b_t becomes higher as a increases, and becomes lower as $\hat{\mu}_{t+1}^q$ increases. Since the marginal private benefit is a decreasing function of b_t, the optimal b_t, given f_{t+1}, is decreasing in a and increasing in $\hat{\mu}_{t+1}^q$.

Denote the optimal choice of b_t given fee f_t by $b_t(f_t)$. The optimal fee f_t solves

$$\frac{\partial}{\partial f_t}\mathbb{E}_{t-1}\left[\max\left\{Q^s(f_t; b_t(f_t)), Q_t^u(f_t)\right\}\right] = 0 . \tag{2.16}$$

The range of the optimal f_t is between f^s and f^u as in the baseline model:

Corollary 2.4 *The optimal f_t satisfies $f^s < f_t < f^u$.*

Proof. Choosing b_t is equivalent to shifting a:

$$a \to a(1 - b_t) .$$

Since Proposition 2.1 holds regardless of the value of a, the optimal choice of f_t lies between f^s and f^u. ■

The following proposition characterizes the optimal choice of f_t:

Proposition 2.9 *Suppose that there exists a unique solution to the first-order condition (2.16). The optimal f_t decreases in a and increases in $\hat{\mu}_t^q$.*

Proof. The FOC (2.16) can be written as

$$\frac{d}{df_t}\mu_t^Q(f_t) + \Phi\left(\frac{Q^s(f_t; b_t(f_t)) - \mu_t^Q(f_t)}{\sigma_Q}\right)\frac{d}{df_t}\left(Q^s(f_t; b_t(f_t)) - \mu_t^Q(f_t)\right) = 0 \, .$$

Consider an increase in a: $a \to a + \delta a$. Denote the optimal f_t at a by $f_t(a)$. From Proposition 2.8, $b_t(f_t; a)$ decreases in a. Note that given f_t

$$Q^s(f_t; a + \delta a, b_t(f_t; a + \delta a)) > Q^s(f_t; a, b_t(f_t; a)) \, ,$$

and

$$\frac{d}{df_t}Q^s(f_t; a + \delta a, b_t(f_t; a + \delta a)) < \frac{d}{df_t}Q^s(f_t; a, b_t(f_t; a)) < 0 \, .$$

Then, at $f_t(a)$

$$\frac{d}{df_t}\mu_t^Q(f_t(a)) + \Phi\left(\frac{Q^s(f_t(a); a + \delta a, b_t(f_t; a + \delta a)) - \mu_t^Q(f_t(a))}{\sigma_Q}\right)$$
$$\times\frac{d}{df_t}\left(Q^s(f_t(a); a + \delta a, b_t(f_t; a + \delta a)) - \mu_t^Q(f_t(a))\right) < 0 \, .$$

Therefore, the optimal $f_t(a)$ is decreasing in a.

Similarly, consider an increase in $\hat{\mu}_t^q$: $\hat{\mu}_t^q \to \hat{\mu}_t^q + \delta\hat{\mu}^q$. Denote the optimal f_t at $\hat{\mu}_t^q$ by $f_t(\hat{\mu}_t^q)$. From Proposition 2.8, $b_t(f_t; \hat{\mu}_t^q)$ increases in $\hat{\mu}_t^q$. Note that given f_t

$$Q^s(f_t; b_t(f_t; \hat{\mu}_t^q + \delta\hat{\mu}^q)) < Q^s(f_t; b_t(f_t; \hat{\mu}_t^q)) \, , \; \mu_t^Q(f_t; \hat{\mu}_t^q + \delta\hat{\mu}^q) > \mu_t^Q(f_t; \hat{\mu}_t^q) \, ,$$

and

$$0 > \frac{d}{df_t} Q^s(f_t; b_t(f_t; \hat{\mu}_t^q + \delta\hat{\mu}^q)) > \frac{d}{df_t} Q^s(f_t; b_t(f_t; \hat{\mu}_t^q)) ,$$
$$\frac{d}{df_t} \mu_t^Q(f_t; \hat{\mu}_t^q + \delta\hat{\mu}^q) > \frac{d}{df_t} \mu_t^Q(f_t; \hat{\mu}_t^q) > 0 .$$

Then, at $f_t(\hat{\mu}_t^q)$

$$\begin{aligned}
&\frac{d}{df_t} \mu_t^Q(f_t(\hat{\mu}_t^q); \hat{\mu}_t^q + \delta\hat{\mu}^q) \\
&+\Phi\left(\frac{Q^s(f_t(\hat{\mu}_t^q); b_t(f_t(\hat{\mu}_t^q); \hat{\mu}_t^q + \delta\hat{\mu}^q)) - \mu_t^Q(f_t(\hat{\mu}_t^q); \hat{\mu}_t^q + \delta\hat{\mu}^q)}{\sigma_Q}\right) \\
&\times \frac{d}{df_t}\left(Q^s(f_t(\hat{\mu}_t^q); b_t(f_t(\hat{\mu}_t^q); \hat{\mu}_t^q + \delta\hat{\mu}^q)) - \mu_t^Q(f_t(\hat{\mu}_t^q); \hat{\mu}_t^q + \delta\hat{\mu}^q)\right) > 0 .
\end{aligned}$$

Therefore, the optimal $f_t(\hat{\mu}_t^q)$ is increasing in $\hat{\mu}_t^q$. ■

The intuition of this result is as follows: for fixed f_{t+1}, b_t decreases as a increases, and increases as $\hat{\mu}_{t+1}^q$ increases as shown in Proposition 2.8. This implies that the choice of b_t reinforces the effect of changes in a and $\hat{\mu}_{t+1}^q$. As a increases, naive money becomes less likely to dominate funds, and the corresponding decrease of b_t makes naive money even less likely to dominate. A similar argument applies to changes in $\hat{\mu}_{t+1}^q$.

Therefore, as a increases, marginal fee profits from naive investors decrease, and marginal fee profits from smart investors increase. Hence, managers have incentive to lower fees as a increases. In contrast, as $\hat{\mu}_{t+1}^q$ increases, marginal fee profits from naive investors increase, and marginal fee profits from smart investors decrease. Hence, managers would want to raise fees as $\hat{\mu}_{t+1}^q$ increases.

The following corollary confirms that the result of Proposition 2.8 holds without fixing fees f_t, i.e., f_t optimally responds to changes in a and $\hat{\mu}_t^q$:

Corollary 2.5 *Suppose that there exist unique solutions to the first-order conditions (2.12) and (2.13). The optimal choice of marketing m_t decreases in a and increases in $\hat{\mu}_t^q$.*

Proof. Following the logic of Proposition 2.8, and using the result of Proposition 2.9,

$$\xi_b(1 - b_t(a)) - \Phi\left(\frac{Q^s(f_t(a+\delta a); a+\delta a, b_t(a)) - \mu_t^Q(f_t(a+\delta a))}{\sigma_Q}\right)$$
$$\times Q^s(f_t(a+\delta a); a+\delta a, b_t(a)) < 0 \ .$$

Therefore, $b_t(a)$ is decreasing in a. Similarly,

$$\xi_b(1 - b_t(\hat{\mu}_t^q)) - \Phi\left(\frac{Q^s(f_t(\hat{\mu}_t^q + \delta\hat{\mu}^q); b_t(\hat{\mu}_t^q)) - \mu_t^Q(f_t(\hat{\mu}_t^q + \delta\hat{\mu}^q); \hat{\mu}_t^q + \delta\hat{\mu}^q)}{\sigma_Q}\right)$$
$$\times Q^s(f_t(\hat{\mu}_t^q + \delta\hat{\mu}^q); b_t(\hat{\mu}_t^q)) > 0 \ .$$

Therefore, $b_t(\hat{\mu}_{t+1}^q)$ is increasing in $\hat{\mu}_{t+1}^q$. ■

2.4.4 Numerical examples

In addition to the parameter choices (2.9) of the baseline model, I make the following parameter choices:

$$\xi_\sigma = 0.2 \ , \ \xi_m = 0.0002 \ , \ \xi_b = 0.5 \ . \tag{2.17}$$

An extension of the baseline model with the endogenous choice of $\sigma_{\epsilon,t-1}$ is plotted in Figure 4, 5 and 6. Figure 4 plots the optimal σ_ϵ as a function of f_t:

[See figure 4]

Figure 5 plots the optimal σ_ϵ as a function of a:

[See figure 5]

Figure 6 plots the optimal σ_ϵ as a function of $\hat{\mu}_t^q$:

[See figure 6]

An extension of the baseline model with the endogenous choice of marketing m_{t-1} is plotted in Figure 7 and 8. Figure 7 plots comparative statics with respect to changes of a:

[See figure 7]

Figure 8 plots comparative statics with respect to changes of $\hat{\mu}_t^q$:

[See figure 8]

An extension of the baseline model with the endogenous choice of b_t, which reflects private benefit that managers gain by sacrificing returns to investors, is plotted in Figure 9 and 10. Figure 9 plots comparative statics with respect to changes of a:

[See figure 9]

Figure 10 plots comparative statics with respect to changes of $\hat{\mu}_t^q$:

[See figure 10]

2.5 Discussions and Conclusions

For analytic simplicity, this chapter makes several assumptions. However, relaxing some of them may provide interesting perspectives on how unsophisticated investors affect decisions of mutual fund managers. One crucial assumption is that there is no information asymmetry between managers and investors. If managers know more about their skill than investors, managers may want to make decisions, particularly those which are easily observable by investors (e.g., fees), in order to signal their skill. The signaling channel is not investigated in this chapter, and remains to be explored.

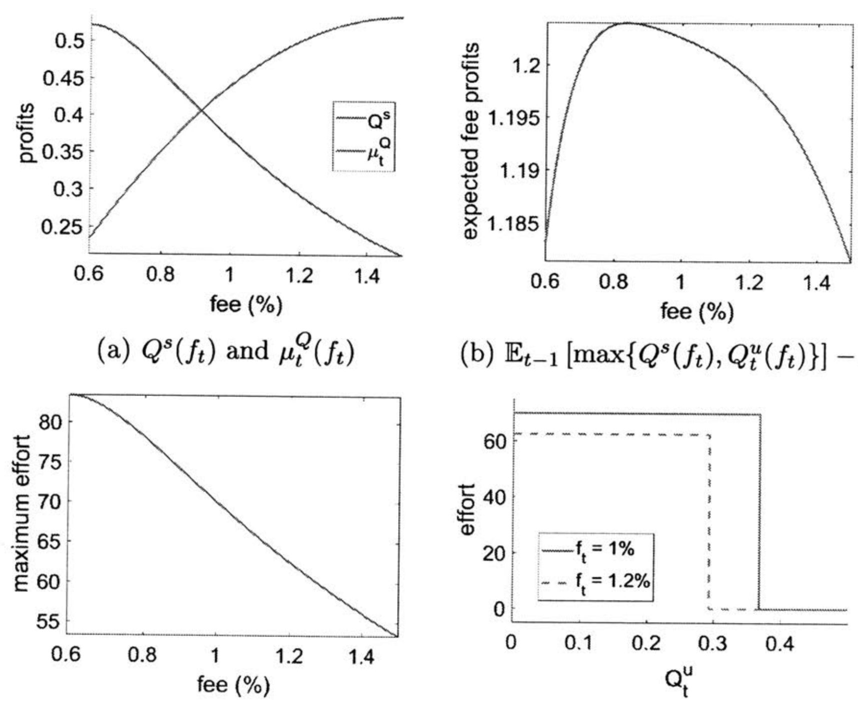
In addition, this chapter concerns only the short-term effect of managerial choices. The infinite-horizon model in this chapter can be reduced to a two-period model since managerial choices do not affect the future dynamics of invested naive money. If managers' decisions influence the accumulation of naive money invested in funds, the managers have incentives to adjust their decisions in order to receive more naive money in the future. How the long-term effect influences managerial decisions remains a topic for future research.

In conclusion, this chapter builds a model on how naive money affects the decisions of active mutual fund managers, particularly fees, effort, idiosyncratic risk, marketing and private benefit. The model shows that more naive money is associated with higher fees, lower managerial effort, more marketing and seeking greater private benefit. In addition, the model proves that managers choose to take higher idiosyncratic risk when expected fee profits from naive investors are close to those from smart investors.

Figures

Figure 1: Baseline model

The following figures plot (a) the value of $Q^s(f_t)$ and $\mu_t^Q(f_t)$ as functions of fee f_t, (b) the expected fee profits $\mathbb{E}_{t-1}\left[\max\{Q^s(f_t), Q_t^u(f_t)\}\right] - F$ net of costs as a function of fee f_t, (c) the maximum effort e_t as a function of fee f_t, and (d) the optimal choice of effort e_t as a function of the realization of Q_t^u for two different fees $f_t = 1\%$ and $f_t = 1.2\%$. Parameter choices are given in (2.9).



(a) $Q^s(f_t)$ and $\mu_t^Q(f_t)$

(b) $\mathbb{E}_{t-1}\left[\max\{Q^s(f_t), Q_t^u(f_t)\}\right] - F$

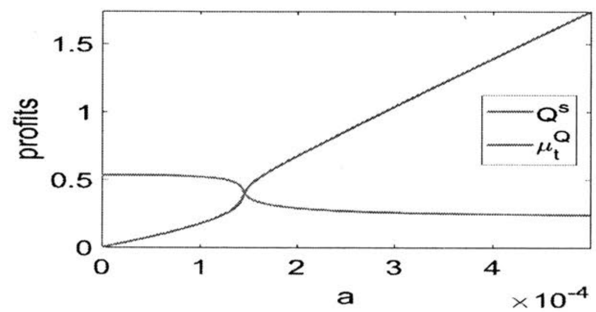
(c) Maximum effort e_t

(d) Optimal choice of effort e_t

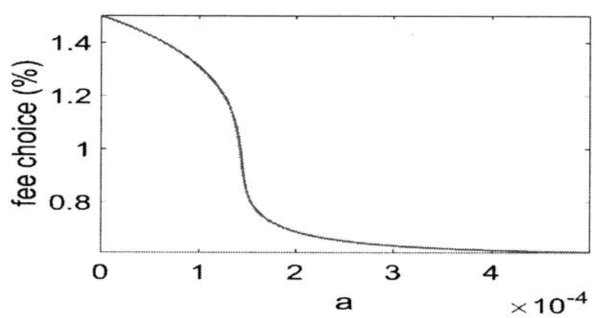
Baseline model

Figure 2: Baseline model - changes of a

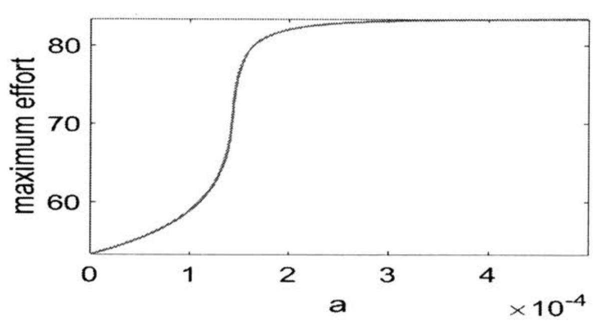
The following figures plot (a) the value of $Q^s(f_t)$ and $\mu_t^Q(f_t)$, (b) the optimal fee choice f_t, and (c) the maximum effort e_t as functions of a. Parameter choices are given in (2.9).



(a) $Q^s(f_t)$ and $\mu_t^Q(f_t)$



(b) Optimal fee f_t

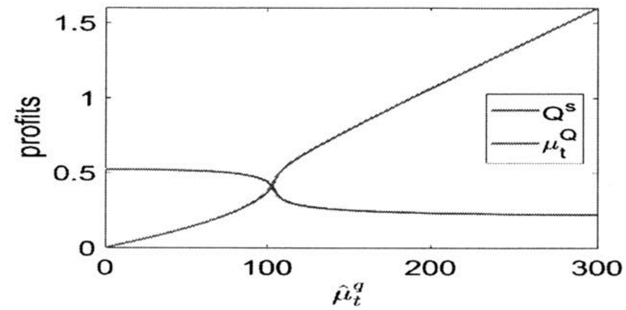


(c) Maximum effort e_t

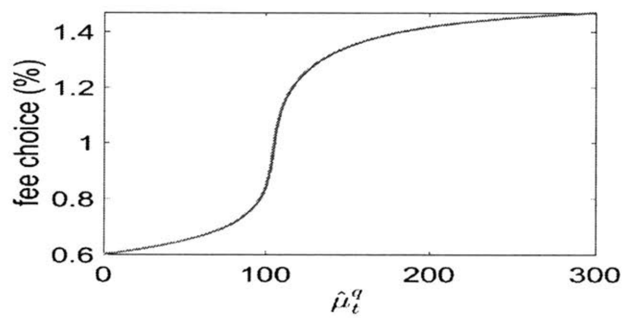
Comparative statics with respect to changes of a

Figure 3: Baseline model - changes of $\hat{\mu}_t^q$

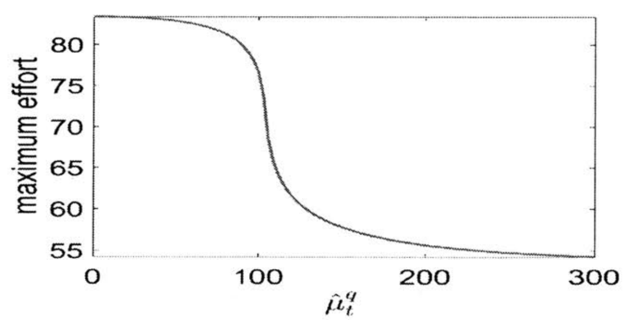
The following figures plot (a) the value of $Q^s(f_t)$ and $\mu_t^Q(f_t)$, (b) the optimal fee choice f_t, and (c) the maximum effort e_t as functions of $\hat{\mu}_t^q$. Parameter choices are given in (2.9).



(a) $Q^s(f_t)$ and $\mu_t^Q(f_t)$



(b) Optimal fee f_t

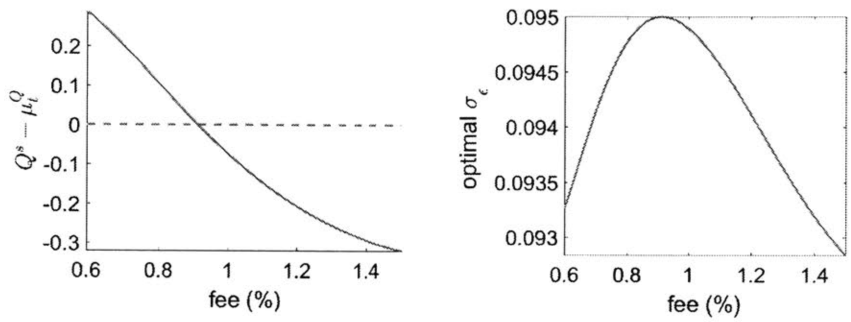


(c) Maximum effort e_t

Comparative statics with respect to changes of $\hat{\mu}_t^q$

Figure 4: Extensions - Endogenous choice of σ_ϵ

The following figures plot (a) the value of $Q^s(f_t) - \mu_t^Q(f_t)$ and (b) the optimal choice of $\sigma_{\epsilon,t-1}$ as functions of fee f_t. Parameter choices are given in (2.9) and (2.17).

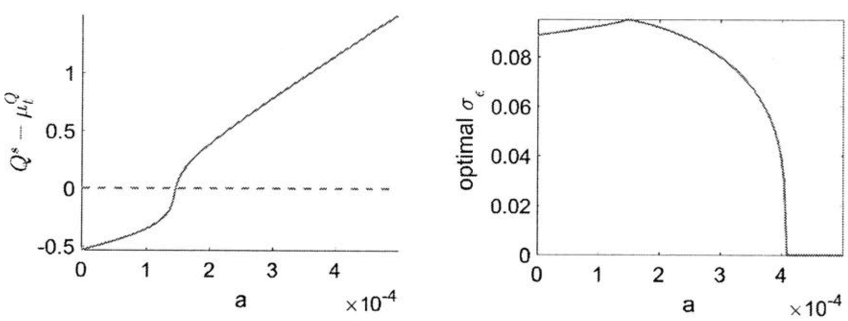


(a) $Q^s(f_t) - \mu_t^Q(f_t)$ (b) Optimal idiosyncratic risk $\sigma_{\epsilon,t-1}$

Optimal choice of idiosyncratic risk $\sigma_{\epsilon,t-1}$ with respect to f_t

Figure 5: Endogenous choice of σ_ϵ - changes of a

The following figures plot (a) the value of $Q^s(f_t(a); a) - \mu_t^Q(f_t(a))$ and (b) the optimal choice of $\sigma_{\epsilon,t-1}$ as functions of a. Parameter choices are given in (2.9) and (2.17).

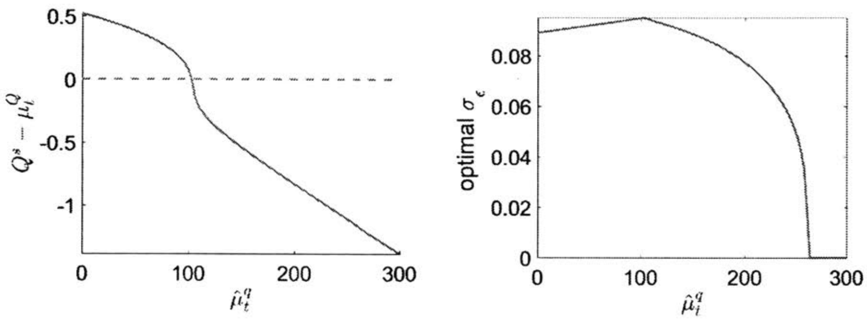


(a) $Q^s(f_t(a); a) - \mu_t^Q(f_t(a))$ (b) Optimal idiosyncratic risk $\sigma_{\epsilon,t-1}$

Optimal choice of idiosyncratic risk $\sigma_{\epsilon,t-1}$ with respect to a

Figure 6: Endogenous choice of σ_ϵ - changes of $\hat{\mu}_t^q$

The following figures plot (a) the value of $Q^s(f_t(\hat{\mu}_t^q)) - \mu_t^Q(f_t(\hat{\mu}_t^q); \hat{\mu}_t^q)$ and (b) the optimal choice of $\sigma_{\epsilon,t-1}$ as functions of $\hat{\mu}_t^q$. Parameter choices are given in (2.9) and (2.17).

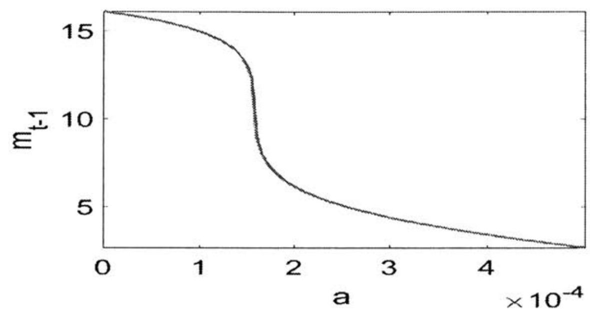


(a) $Q^s(f_t(\hat{\mu}_t^q)) - \mu_t^Q(f_t(\hat{\mu}_t^q); \hat{\mu}_t^q)$ (b) Optimal idiosyncratic risk $\sigma_{\epsilon,t-1}$

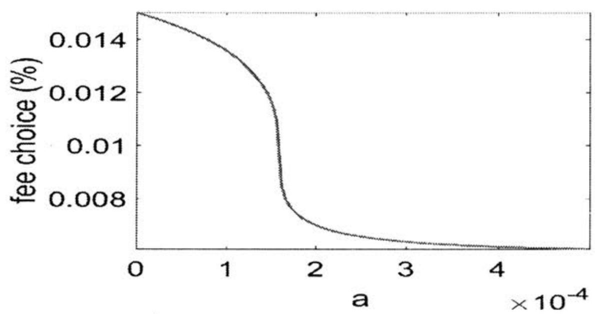
Optimal choice of idiosyncratic risk $\sigma_{\epsilon,t-1}$ with respect to $\hat{\mu}_t^q$

Figure 7: Endogenous choice of m_{t-1} - changes of a

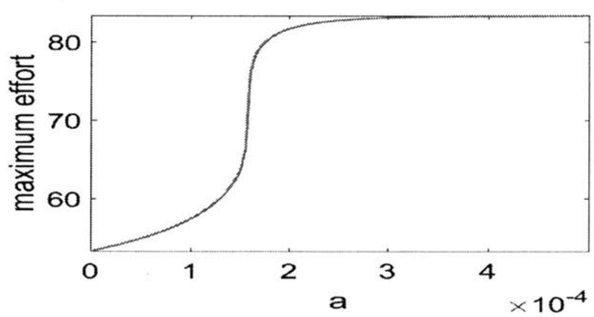
The following figures plot (a) the optimal choice of marketing m_{t-1}, (b) the optimal fee choice f_t, and (c) the maximum effort e_t as functions of a. Parameter choices are given in (2.9) and (2.17).



(a) Optimal marketing m_{t-1}



(b) Optimal fee f_t

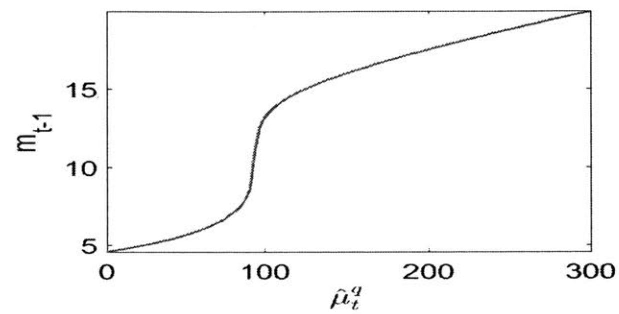


(c) Maximum effort e_t

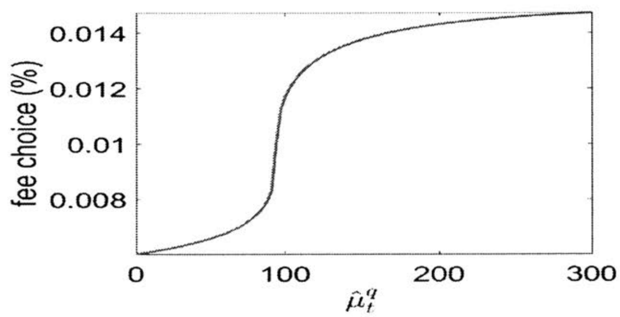
Comparative statics with respect to changes of a

Figure 8: Endogenous choice of m_{t-1} - changes of $\hat{\mu}_t^q$

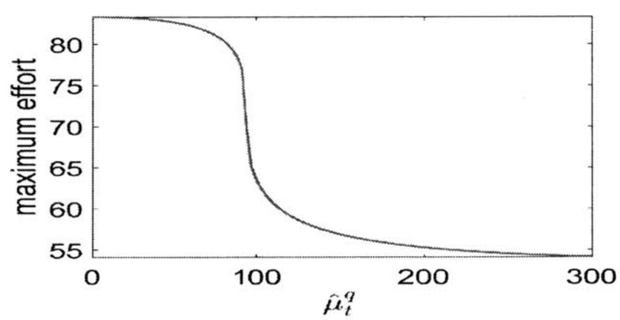
The following figures plot (a) the optimal choice of marketing m_{t-1}, (b) the optimal fee choice f_t, and (c) the maximum effort e_t as functions of $\hat{\mu}_t^q$. Parameter choices are given in (2.9) and (2.17).



(a) Optimal marketing m_{t-1}



(b) Optimal fee f_t

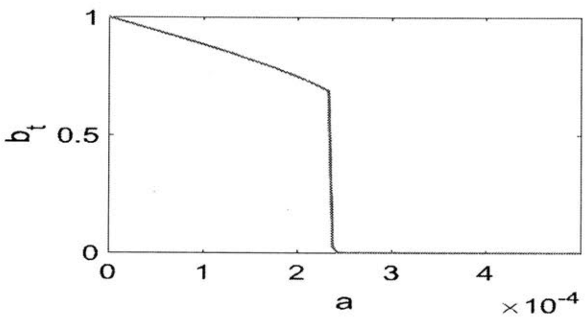


(c) Maximum effort e_t

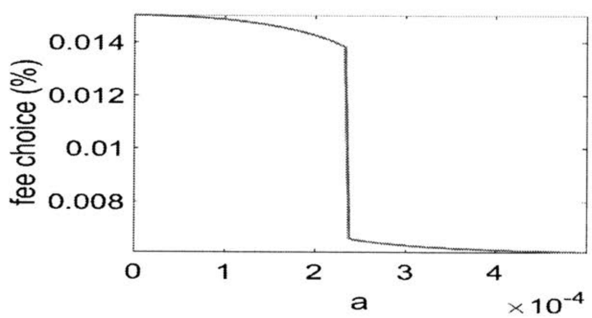
Comparative statics with respect to changes of $\hat{\mu}_t^q$

Figure 9: Endogenous choice of b_t - changes of a

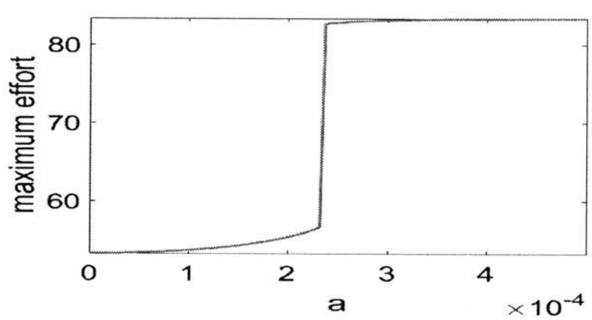
The following figures plot (a) the optimal choice of b_t, (b) the optimal fee choice f_t, and (c) the maximum effort e_t as functions of a. Parameter choices are given in (2.9) and (2.17).



(a) Optimal choice of b_t



(b) Optimal fee f_t

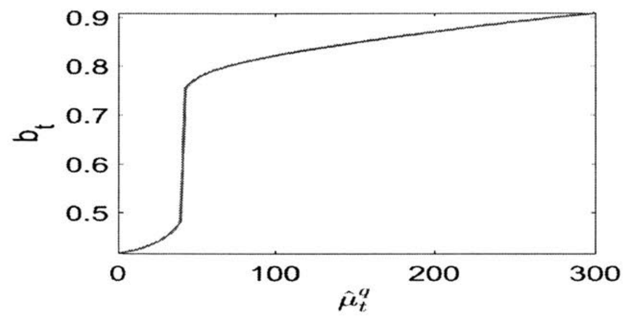


(c) Maximum effort e_t

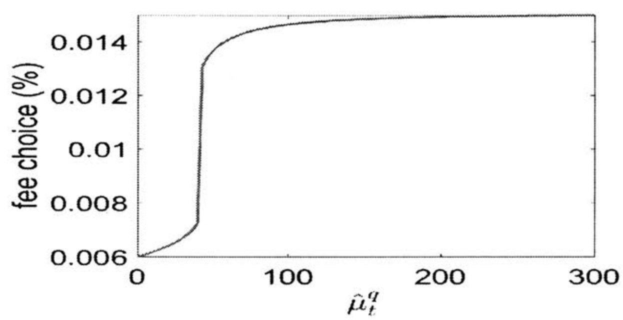
Comparative statics with respect to changes of a

Figure 10: Endogenous choice of b_t - changes of $\hat{\mu}_t^q$

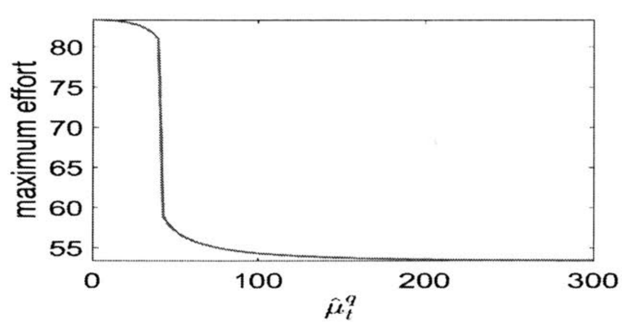
The following figures plot (a) the optimal choice of b_t, (b) the optimal fee choice f_t, and (c) the maximum effort e_t as functions of $\hat{\mu}_t^q$. Parameter choices are given in (2.9) and (2.17).



(a) Optimal choice of b_t



(b) Optimal fee f_t



(c) Maximum effort e_t

Comparative statics with respect to changes of $\hat{\mu}_t^q$

Appendix

On the FOC approach

Most of the proofs (including comparative statics) in this chapter use the FOC approach. Since the objective function (expected fee profits) is smooth, the FOC approach is legitimate as long as there is a unique local and global maximum. However, that condition is not guaranteed for all possible range of parameter values. In this section, I show that all the proofs of this chapter still hold when the uniqueness of the local maximum is not satisfied.

Since the logic is the same across all the proofs, I choose Proposition 2.2 as a representative example. In this example, there may exist multiple solutions to the FOC (2.8), i.e., there are multiple extrema of

$$h(f_t; a) = \mathbb{E}_{t-1}\left[\max\left\{Q^s(f_t; a), Q_t^u(f_t)\right\}\right] .$$

Denote f_t at the global maximum of $h(f_t; a)$ by f_t^*. Even if there exist multiple extrema, Proposition 2.2 holds locally. This means that as long as other local maxima are smaller than the current global maximum, the global maximum shifts towards smaller $f_t < f_t^*$ as a increases. In order for Proposition 2.2 to be violated, two conditions must be satisfied: i) another local maximum becomes larger than the current global maximum as a increases, ii) the new global maximum is located at $f_t > f_t^*$. I show that these two conditions cannot hold at the same time.

Suppose that at a certain a, there are two local maxima at the same value, i.e., there exist f_1^* and f_2^*, where $f_1^* < f_2^*$, satisfying

$$h(f_1^*; a) = h(f_2^*; a) = \max_{f \in [f^s, f^u]} h(f; a) .$$

Suppose that

$$h(f_1^*; a - \delta a) = \max_{f \in [f^s, f^u]} h(f; a - \delta a) > h(f_2^*; a - \delta a) ,$$

and

$$h(f_1^*; a+\delta a) < h(f_2^*; a+\delta a) = \max_{f\in[f^s,f^u]} h(f; a+\delta a) \ .$$

The slope of $h(f;a)$ can be written as

$$\begin{aligned}\frac{dh(f;a)}{df} &= \left(1-\Phi\left(\frac{Q^s(f;a)-\mu_t^Q(f)}{\sigma_Q}\right)\right)\frac{d}{df}\mu_t^Q(f) \\ &\quad +\Phi\left(\frac{Q^s(f;a)-\mu_t^Q(f)}{\sigma_Q}\right)\frac{d}{df}Q^s(f;a) \ .\end{aligned}$$

Note that

$$\frac{dh(f;a+\delta)}{df} < \frac{dh(f;a)}{df} \ , \ \forall f \in [f^s, f^u] \ .$$

Define

$$\frac{d\Delta h(f;a)}{df} \equiv \frac{dh(f;a+\delta)}{df} - \frac{dh(f;a)}{df} < 0 \ , \ \forall f \in [f^s, f^u] \ .$$

Then,

$$\begin{aligned}h(f_2^*; a+\delta a) - h(f_1^*; a+\delta a) &= h(f_2^*; a) - h(f_1^*; a) + \int_{f_1^*}^{f_2^*} \frac{d\Delta h(f;a)}{df} df \\ &< h(f_2^*; a) - h(f_1^*; a) = 0 \ ,\end{aligned}$$

which is a contradiction. Although this proof considers two maxima at the same value, it can be easily generalized to cases with more than two maxima at the same value.

From this proof, if there is a transition of the global maximum from one local maximum to another, the transition must be towards lower f_t. Therefore, Proposition 2.2 holds even if there are multiple solutions to the FOC (2.8).

Chapter 3

IO of the Active AM Industry: Entries and Exits

3.1 Introduction

This chapter addresses the structure of the active AM industry: in particular, entries and survivorship. Given that funds have access to the same technology for attracting naive money, only those whose perceived skill is higher than a certain threshold enter AM markets. Existing managers choose to exit if their track records are poor and/or the amount of invested naive money is small. While fund entry and exit decisions affect the degree of competition in the AM industry, competition also, in turn, influences the entry and exit decisions of managers. Therefore, competition among managers and their entry and exit decisions jointly characterize an equilibrium of the model, which I refer to as an "industry equilibrium." In the long run, the economy converges to the stationary equilibrium.

Entries of and the long-run survivorship of unskilled managers are of particular interest, and the model characterizes what components are associated with them, and how. Among newly entering managers, the portion of unskilled managers is higher if there are 1) more aggregate investment opportunities, 2) more aggregate naive capital flows, 3) less supply of skilled managers to the AM industry, and 4) lower entry costs. These changes induce likely unskilled managers to enter AM markets who would not have, absent the changes. Among those changes, only more aggregate naive capital flows and less supply of skilled managers are associated with a higher probability of the long-run survivorship of unskilled managers. Only these changes increase the average flow of naive capital to an individual fund.

When AM markets are heterogeneous in investor sophistication, AM markets with more sophisticated investors (say, hedge fund markets) naturally differentiate from AM markets with less sophisticated investors (say, mutual fund markets). In equilibrium, skilled managers generate more value in hedge fund markets than they do in mutual fund markets. As a result, relatively high-skilled new managers tend to enter hedge fund markets. Since the supply of skilled managers to mutual fund markets decreases, the probability of the long-term survivorship of unskilled managers increases in mutual fund markets. Therefore, roughly speaking, mutual fund markets are characterized as markets for naive money with a relatively high portion of unskilled managers, and hedge fund markets are characterized as markets for smart money with a relatively high portion of skilled managers.

A certain type of regulations for the AM industry may be detrimental to the wealth of unsophisticated investors. In the model, unsophisticated investors lose wealth by investing in underperforming active funds compared with investing in a passive benchmark (e.g., index funds) with similar risk characteristics. Regulations that restrict value-generating activities of active funds reduce entries of unskilled managers, but raise the probability of the long-term survivorship of unskilled managers. Since funds run by unskilled managers underperform the most, regulations of these types may harm the wealth of unsophisticated investors. On the other hand, regulations that discourage activities that induce retail investors to invest in underperforming funds decrease both entries and the long-run survivorship of unskilled managers. Therefore, regulations of these types are beneficial to the wealth of unsophisticated investors. Furthermore, regulations that apply to one type of AM markets may have unintended long-term effects on the other types. One crucial caveat is that this chapter does not provide a rationale for retail investor protection, and, therefore, the regulatory implications of this chapter must not be over-interpreted.

In the remainder of the paper, section 2 presents the model. Section 3 discusses the industry equilibrium implications of the model for the structure of the active AM industry. In particular, section 4 investigates the structure of the AM industry when AM markets are heterogeneous in investor sophistication. Section 5 examines the implications of the model for retail investor protection. Section 6 discusses the limitation of the model and future directions for research, and summarizes the conclusions.

3.2 The Model

There are two types of managers who are different in skill as described in subsection 1.2.1. The information structure and the learning process are the same as in chapter 1, and are described in subsection 1.2.2.

3.2.1 Naive capital

This chapter views unsophisticated investors as irrational: they invest in underperforming active funds although those investors have all the information that is available to sophisticated investors, including the track record of funds. Those unsophisticated investors can improve returns by investing in passive benchmarks (e.g., index funds) with similar risk characteristics, but they do not choose to invest in the passive benchmarks. This is different from the concept of uninformed investors in Grossman and Stiglitz (1976, 1980): uninformed investors do not have private information, but may trade for reasons such as hedging or liquidity needs. Here, unsophisticated investors have all the information, and there are liquid passive alternatives that provide similar systematic risk as active funds.

I model capital from unsophisticated investors (naive money) in a reduced form, and abstract from detailed capital allocation decisions of unsophisticated investors. naive capital flows to fund (manager) i are as follows:

$$d\tilde{q}_{i,t} = (b - \eta\tilde{q}_{i,t})dt + \sqrt{\tilde{q}_{i,t}}\left(\sigma dW_{i,t} + \sigma_z d\tilde{Z}_{i,t}\right) \ , \tag{3.1}$$

where $\tilde{q}_{i,t}$ is the amount of naive capital invested in the fund, b is the average naive capital inflow, η is the rate of average naive capital outflow, $W_{i,t}$ is a standard Brownian motion that captures the component of naive capital flows irrelevant to the fund performance, σ is a parameter that determines the volatility of naive capital flows that are irrelevant to the fund performance, and σ_z is a parameter that determines the volatility of naive capital flows that respond to the fund performance. Note that σ_z may be either positive or negative[1], and

$$dW_{i,t} \cdot d\tilde{Z}_{i,t} = dW_{i,t} \cdot d\tilde{Z}_{j,t} = dW_{i,t} \cdot dW_{j,t} = 0 \;, \quad \forall j \neq i \;,$$

i.e., $W_{i,t}$ captures the idiosyncratic component of naive money flows, and is orthogonal to the fund performance.

The process (3.1) is the Cox-Ingersoll-Ross (CIR) process (Cox, Ingersoll and Ross (1985)), since the process can be rewritten as

$$d\tilde{q}_{i,t} = (b - \eta\tilde{q}_{i,t})dt + \sigma_T\sqrt{\tilde{q}_{i,t}}d\tilde{W}_{i,t} \;,$$

where

$$\sigma_T = \sqrt{\sigma^2 + \sigma_z^2} \quad , \quad d\tilde{W}_{i,t} = \frac{\sigma}{\sigma_T}dW_{i,t} + \frac{\sigma_z}{\sigma_T}d\tilde{Z}_{i,t} \;,$$

and $\tilde{W}_{i,t}$ is a standard Brownian motion as well. The CIR process as the model of naive capital flows has several attractive properties: the amount of naive capital is always non-negative, and possesses a stationary distribution.

I would like to emphasize that insights from the model do not significantly depend on the process of naive money. The only essential property of naive money that the model needs is that naive capital invested in a fund is persistent, i.e., unsophisticated investors reallocate their capital slowly. In fact, in a continuous-time setup without any jump process, this property is automatically assured. As long as the property is satisfied, the detail of the process of naive capital does not affect the main qualitative results of the chapter.

[1]Empirical studies (e.g., Gruber (1996)) suggest that fund flows positively respond to good performance.

3.2.2 Fees and operating costs

A fund manager charges a fixed percentage fee f per time, i.e., between time t and $t + dt$, investors in the fund pay

$$f dt$$

per dollar invested in the fund at time t. For modeling simplicity, I assume that f is constant and uniform across funds. Each fund pays a fixed operating cost ϕdt between t and $t + dt$ as long as the fund exists.

3.2.3 Discount rate and utility

The risk-free rate r is constant and does not change over time. The only relevant discount rate in the model is the risk-free rate because all the shocks are idiosyncratic. I assume that managers are risk-neutral. Therefore, managers' objective is to maximize the discounted expected fee profits.

The risk neutrality can be justified if the market is sufficiently complete. In this case, managers can hedge risk associated with fee profits, and the form of their utility functions does not matter. Market completeness allows managers to perfectly smooth out their consumption over time, and the value of the consumption stream needs to equal the entry value of fee profits net of the entry cost.

3.2.4 Entries and exits

At each point of time, prospective managers are born. The expected skill of a newly born manager is summarized by the probability p of the manager being H-type. The prior cumulative distribution of the expected skill of managers born between t and $t + dt$ is given by

$$F(p)dt \; .$$

One crucial assumption is that the distribution does not change over time. This assumption implies that the supply of skilled managers is inelastic. Define

$$G(1-p) \equiv \int_p^1 F'(p')dp' \ , \tag{3.2}$$

which is the cumulative distribution measured from p to 1, i.e., $G(1-p)$ is the reverse-order cumulative distribution of the prior skill. Another assumption that I make is that

$$\lim_{p \to 0^+} G(1-p) \to \infty \ , \tag{3.3}$$

i.e., skilled managers are scarce, but surely unskilled managers are abundant. This assumption is needed for the existence of an equilibrium, but is not necessary as long as $\lim_{p\to 0^+} G(1-p)$ is sufficiently large.

A new fund needs to pay a lump-sum entry cost Φ at entry, and the cost becomes sunk once the fund starts its operation. A prospective manager enters an AM market if the value of fee profits net of the entry cost is positive. When a new manager enters an AM market, she starts with zero amount of naive capital at entry.

Existing managers may exit AM markets. There are two types of exits in the model: exogenous exits and endogenous exits. I model the exogenous exits as a Poisson process: between t and $t + dt$, existing managers independently exit with probability λdt. On the other hand, at each point of time, existing managers may endogenously choose to exit if the value of fee profits is not positive. There are no costs associated with exits.

3.2.5 Competition

There are two types of competition among managers: one is competition for positive NPV investment opportunities, and the other is competition for naive capital inflows. These types of competition lead to decreasing returns to scale at the industry level. One crucial assumption is that high competition drives down both A, the average value per time that a skilled manager generates, and b, the average naive money inflows to a fund. Mathematically, this assumption translates into

$$A = \bar{A}h_A(N) \ , \quad b = \bar{b}h_b(N) \ , \tag{3.4}$$

where N is the total number of existing managers in the AM industry, $\bar{A}$ and $\bar{b}$ are constants, and $h_A(N)$ and $h_b(N)$ are smooth monotone (strictly) decreasing functions in N. Here, I take the total number of managers as the equilibrium variable that captures the degree of competition in the industry. Taking another sensible variable (e.g., aggregate capital in the industry) instead as the measure for competition does not change qualitative results.

3.3 Structure of the AM industry: Entries and Survivorship

The full characterization of the equilibrium of the model involves the determination of equilibrium variables A and b. These variables are determined by the number (measure) of managers by (3.4), which I repeat:

$$A = \bar{A}h_A(N) \ , \quad b = \bar{b}h_b(N) \ ,$$

where $\bar{A}$ and $\bar{b}$ are constants, and $h_A(N)$ and $h_b(N)$ are arbitrary decreasing functions. In order to guarantee an interior solution, additional assumptions are made:

$$\lim_{N \to 0} h_A(N) \to \infty \ , \quad \lim_{N \to 0} h_b(N) \to \infty \ , \quad \lim_{N \to \infty} h_A(N) \to 0 \ , \quad \lim_{N \to \infty} h_b(N) \to 0 \ , \tag{3.5}$$

i.e., active funds become extremely profitable if there are no competitors, and become extremely unprofitable if there are too many competitors.

Since there is no aggregate shock, the model focuses on the stationary equilibrium. In the stationary equilibrium, the equilibrium variables A and b are constants over time.

3.3.1 Fee revenues

A and b are endogenously determined in equilibrium through competition among funds. In this subsection, I focus on characterizing fee revenues of active funds by taking the value of A and b as given. For notational simplicity, I omit indices for individual managers (funds) unless necessary.

From the analysis in subsection 1.3.2, the net excess return on a fund between t and $t + dt$ is

$$dr_t^{ex} = dR_t^{ex} - f\,dt = \frac{A\left(\mathbb{1}_H dt + \frac{1}{s} dZ_t\right)}{q_t} - f\,dt \; .$$

In equilibrium, the net expected excess return (net alpha) is either zero or negative:

$$\mathbb{E}_t[dr_t^{ex}] = \left(\frac{p_t A}{q_t} - f\right) dt = \begin{cases} 0 & , \quad \tilde{q}_t < q_t \\ \leq 0 & , \quad \tilde{q}_t = q_t \end{cases} ,$$

where q_t is the AUM of the fund at t, and $\tilde{q}_t$ is the amount of naive capital invested in the fund at t. From Proposition 1.1, the AUM of the fund is

$$q_t = \max\left\{\frac{p_t A}{f}, \tilde{q}_t\right\} ,$$

and fee revenues between t and $t + dt$ are given by

$$f q_t dt = \max\left\{p_t A, f\tilde{q}_t\right\} dt \; .$$

3.3.2 Stationary equilibrium

The state variables for an individual fund manager are p and $\tilde{q}$. The aggregate state of the economy is the distribution of agents across states. In order to define the aggregate state variables formally, define the state space for an individual manager. The state space of a manager is the Cartesian product $S : [0, 1] \times [0, \infty)$ with Borel σ algebra $\mathcal{B}$. For any set $\mathcal{S} \in \mathcal{B}$, $\rho(\mathcal{S})$ is the measure of managers in the set $\mathcal{S}$. A stationary equilibrium is defined as follows:

Definition 1 *A stationary equilibrium is the value function $V : S \to \mathbb{R}_{\geq 0}$; the surplus* *rate A that a skilled manager generates per time and the average naive capital inflow rate b; and the stationary measure ρ^* such that*

- *Given A and b, $V(p, \tilde{q})$ is the value function of fee profits.*
- *Given ρ^*, $N = \int\int \rho^*(p, \tilde{q}) dp d\tilde{q}$ is consistent with the values of A and b by (3.4).*
- *Prospective managers arrive with prior skill distribution $F(p)$, and chooses to enter the AM industry if their $V(p, 0)$ is greater than the entry cost Φ.*
- *Managers choose to exit if $V(p, \tilde{q}) = 0$. In addition, managers exogenously exit under the Poisson process with probability λdt between t and $t + dt$.*
- *ρ^* is invariant under entries, exits and the transition of states of existing managers, given by*

$$dp = sp(1-p)d\tilde{Z} \ , \quad d\tilde{q} = (b - \eta\tilde{q})dt + \sqrt{\tilde{q}}\left(\sigma dW + \sigma_z d\tilde{Z}\right) \ .$$

3.3.3 Value of fee profits, exits and entries

The following analysis assumes a stationary equilibrium, and takes A and b as given. Recall that an existing fund bears the operating cost of ϕdt between t and $t + dt$.

Value of fee profits

Fee profits between t and $t + dt$ are

$$\left(\max\{p_t A, f\tilde{q}_t\} - \phi\right) dt \ ,$$

by Proposition 1.1. The value of fee profits at t can be written as

$$V_t = \mathbb{E}_t\left[\int_t^{T^D} e^{-r(u-t)} \left(\max\{p_u A, f\tilde{q}_u\} - \phi\right) du\right] \ ,$$

where T^D is the point of time when the manager exits. Under the given information set at t, the joint distribution of p_u and $\tilde{q}_u$ for $u \geq t$ solely depends on p_t and $\tilde{q}_t$. As a result, the state variables for the value of fee profits at t are p_t and $\tilde{q}_t$. Hence, the value of fee profits can be rewritten as $V_t = V(p_t, \tilde{q}_t)$.

T^D is determined either by an exogenous exit, with probability λdt between t and $t + dt$ independently, or by an endogenous exit, where the manager chooses to exit. A manager chooses to exit if the value of staying is less than (or equal to) the value of exiting, which is zero. These statements can be translated into dynamic programming language as follows:

$$V(p_t, \tilde{q}_t) = \max\left\{(\max\{p_t A, f\tilde{q}_t\} - \phi)\, dt + (1 - r dt - \lambda dt)\mathbb{E}_t\left[V(p_{t+dt}, \tilde{q}_{t+dt})\right], 0\right\} ,$$

where

$$\begin{aligned} dp_t &= sp_t(1-p_t)d\tilde{Z}_t , \\ d\tilde{q}_t &= (b - \eta\tilde{q}_t)dt + \sqrt{\tilde{q}_t}\left(\sigma dW_t + \sigma_z d\tilde{Z}_t\right) . \end{aligned}$$

When the value is greater than zero, the value function $V(p, \tilde{q})$ is smooth, and solves the following partial differential equation (PDE):

$$\begin{aligned} (r+\lambda)V(p,\tilde{q}) &= (\max\{pA, f\tilde{q}\} - \phi) + \frac{1}{2}\frac{\partial^2 V(p,\tilde{q})}{\partial p^2}s^2p^2(1-p)^2 + \frac{\partial V(p,\tilde{q})}{\partial \tilde{q}}(b - \eta\tilde{q}) \\ &\quad + \frac{1}{2}\frac{\partial^2 V(p,\tilde{q})}{\partial \tilde{q}^2}(\sigma^2 + \sigma_z^2)\tilde{q} + \frac{\partial^2 V(p,\tilde{q})}{\partial p \partial \tilde{q}} sp(1-p)\sigma_z\sqrt{\tilde{q}} . \end{aligned}$$

There is no analytic solution to the PDE. The following lemmas characterize several properties of $V(p, \tilde{q})$ that can be derived without directly solving for $V(p, \tilde{q})$.

Lemma 3.1 *$V(p, \tilde{q})$ is increasing in p and $\tilde{q}$.*

Proof. See Appendix. ■

Lemma 3.2 *$V(p, \tilde{q})$ is convex in the direction of the diffusion $d\tilde{Z}$ in p and $\tilde{q}$, which is proportional to the vector $(sp(1-p), \sigma_z\sqrt{\tilde{q}})$, i.e.,*

$$\begin{bmatrix} sp(1-p) & \sigma_z\sqrt{\tilde{q}} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 V(p,\tilde{q})}{\partial p^2} & \frac{\partial^2 V(p,\tilde{q})}{\partial p \partial \tilde{q}} \\ \frac{\partial^2 V(p,\tilde{q})}{\partial p \partial \tilde{q}} & \frac{\partial^2 V(p,\tilde{q})}{\partial \tilde{q}^2} \end{bmatrix} \begin{bmatrix} sp(1-p) \\ \sigma_z\sqrt{\tilde{q}} \end{bmatrix} \geq 0 .$$

Proof. See Appendix. ■

Exits

Since the value of fee profits changes over time only through p and $\tilde{q}$, endogenous exit decisions by managers solely depend on p and $\tilde{q}$ as well. If $V(p,\tilde{q}) > 0$ for all $(p,\tilde{q})$, the manager never chooses to exit. If $V(p,\tilde{q}) = 0$ for some $(p,\tilde{q})$, the manager exits. The following theorem characterizes the set of $(p,\tilde{q})$ where a fund manager chooses to exit, i.e.,

$$E \equiv \{(p,\tilde{q})|V(p,\tilde{q}) = 0\}. \tag{3.6}$$

Proposition 3.1 *Suppose that E is not empty. E is characterized by a downward sloping curve $\tilde{q} = h(p)$ that crosses the $\tilde{q}$-axis (p=0) and the p-axis ($\tilde{q} = 0$):*

$$E = \{(p,\tilde{q})|\tilde{q} \leq h(p)\} .$$

Proof. See Appendix. ■

The set E defined by (3.6) may or may not be empty. If E is nonempty, managers choose to exit immediately after they reach the curve $\tilde{q} = h(p)$ in continuous time. Hence, the curve $\tilde{q} = h(p)$ is the exit threshold. The downward-sloping exit threshold shows that there are two reasons why managers may choose to exit: managers are perceived as low-skilled and/or there is insufficient naive capital invested in their funds.

Entries

The prior cumulative distribution of the skill of managers newly born between t and $t + dt$ is given by $F(p)dt$. A prospective manager with probability p of being H-type chooses to enter the AM industry if the participation constraint

$$V(p,0) - \Phi \geq 0 \tag{3.7}$$

holds, since a prospective manager enters with zero naive capital. Therefore, in order to study entry decisions of prospective managers, it is useful to characterize properties of $V(p, 0)$ further. Define

$$p_{ex} \equiv \sup\{p|V(p,0) = 0\} . \tag{3.8}$$

By Proposition 3.1, p_{ex} exists if E defined by (3.6) is nonempty, but does not exist if E is empty. The following corollary elucidates further properties of $V(p, 0)$:

Corollary 3.1 *$V(p, 0)$ is convex in p. $V(p, 0)$ is strictly increasing if p_{ex} does not exist, and strictly increasing for $p \geq p_{ex}$ if p_{ex} exists.*

Proof. See Appendix. ∎

Now define p^* that satisfies

$$V(p^*, 0) = \Phi . \tag{3.9}$$

p^* is greater than p^{ex} (if p^{ex} exists). Based on the participation constraint (3.7), the following proposition characterizes the set of prospective managers who enter the AM industry in a stationary equilibrium:

Proposition 3.2 *In a stationary equilibrium, $p^* \in (0, 1)$ exists and is unique. Prospective managers enter the AM industry if their prior skill (the prior probability of being H-type) is higher than p^*, i.e., prospective managers with $p \in [p^*, 1]$ choose to enter.*

Proof. See Appendix. ∎

The intuition of Proposition 3.2 is as follows: the prior skill of a prospective manager matters for her decision to start an active fund, given that funds are homogeneous in their ability to attract naive money. While the supply of skilled prospective managers is limited, there are plenty of low-skilled managers[2], and competition among funds does not allow surely unskilled managers to enter the AM industry. Therefore, only prospective managers whose prior skill is above a certain threshold (p^*) choose to enter AM markets.

[2]This is captured by assumption (3.3).

3.3.4 Long-run survivorship and entries of unskilled managers

While this chapter accounts for exit and entry decisions of both skilled and unskilled managers, those of unskilled managers are of particular interest. Once a manager survives for a sufficiently long time, her true skill is eventually revealed[3]. The following corollary shows that old funds run by skilled managers never choose to exit in a stationary equilibrium.

Corollary 3.2 *Denote the survival time of fund i by t_i, and the probability of the manager being H-type by $p_{i,t}$. If the manager of fund i is H-type, as $t_i \to \infty$, the fund never chooses to exit.*

Proof. See Appendix. ■

Therefore, the long-term survivorship of skilled managers is relatively trivial compared with that of unskilled managers.

Long-term survivorship of unskilled managers

As the survival time becomes sufficiently long, the true (lack of) skill of unskilled managers is revealed, i.e., p_t converges to 0 almost surely as $t \to \infty$. Since the fair size of funds run by those managers is zero, these funds are always overpriced. The value of fee profits to unskilled managers (with their type being revealed) is

$$V_t = V(0, \tilde{q}_t) = \mathbb{E}_t \left[\int_t^{T^D} e^{-r(u-t)} \left(f\tilde{q}_u - \phi \right) du \right] ,$$

where

$$d\tilde{q}_t = (b - \eta\tilde{q}_t)dt + \sqrt{\tilde{q}_t} \left(\sigma dW_t + \sigma_z d\tilde{Z}_t \right) .$$

Therefore, the value to surely unskilled managers is determined by how much naive money they currently have, and how much naive money they can attract in the future.

The following lemma is helpful for the rest of the analysis:

[3]Formally, as $t \to \infty$, p_t for a skilled manager converges to 1 almost surely, and p_t for an unskilled manager converges to 0 almost surely.

Lemma 3.3 *The set E defined by (3.6) diminishes as A and b increase. Precisely,*

$$E(A', b') \subseteq E(A, b) , \quad \forall A' \geq A , \; b' \geq b .$$

Proof. See Appendix. ∎

Before making formal statements, it is useful to define the precise meaning of the long-run survivorship. Here, I define the long-run survivorship of unskilled managers as the expected survival time once the type of those managers is revealed, given the current naive money $\tilde{q}_t$ fixed. A more general definition is the probability of an unskilled manager's survival for sufficiently long time after entry. These two definitions are closely related, and the relation is examined in the following analysis.

Proposition 3.3 *The long-run survivorship of unskilled managers increases in b, i.e., the expected survival time of surely unskilled managers increases in b, given $\tilde{q}_t$ fixed.*

Proof. See Appendix. ∎

Given the amount $\tilde{q}_t$ of currently invested naive capital fixed, higher b allows surely unskilled funds to attract more naive capital in the future. Hence, those managers are expected to survive longer than they are for lower b. The result of Proposition 3.3 is valid regardless of the value of A, since fee profits of surely unskilled managers do not depend on the value of A.

On the other hand, the following corollary requires a certain condition on A.

Corollary 3.3 *Suppose that $A_1 \geq A_2$ and $b_1 \geq b_2$. The probability of an unskilled manager's survival is higher for (A_1, b_1) than it is for (A_2, b_2) at any point of time, given the prior skill p_0 at entry fixed.*

Proof. See Appendix. ∎

Therefore, as long as A does not decrease, an increase in b raises the probability of an unskilled manager's survival at any point of time, given her prior skill fixed. This is equivalent to the statement that the distribution of unskilled managers' survival time becomes (first-order stochastically) dominant as b increases, which implies an increase in the expected survival time of those managers. Therefore, in this case, for unskilled managers, the probability of survival and the expected survival time move in the same direction as b changes.

However, if A decreases, the result of Corollary 3.3 does not hold. For $A_1 < A_2$ and $b_1 \geq b_2$, the probability of survival for (A_1, b_1) does not necessarily dominate that for (A_2, b_2). Hence, the expected survival time for (A_1, b_1) may not be longer than that for (A_2, b_2). Therefore, the short-run survivorship of unskilled managers depends both on the value of A and b, and does not monotonically increase in the value of b.

In the long run, the true (lack of) skill of unskilled managers is revealed regardless of the prior. Therefore, after sufficiently long time, the survivorship of unskilled managers is solely determined by their current AUM (from unsophisticated investors), and the value of b, which represents their ability to attract naive capital in the future. The long-run survivorship of unskilled managers, defined by their expected survival time for fixed $\tilde{q}_t$ once their true skill is revealed, increases in b by Proposition 3.3. In the limit of sufficiently long time after entry, the survival probability of unskilled managers increases in b as well[4]. The following proposition proves that the long-term survival probability is greater for higher b regardless of A:

Proposition 3.4 *Suppose that $b_1 > b_2$. There exists T such that for all $s > T$, the survival probability of unskilled managers for b_1 is greater than that for b_2.*

Proof. See Appendix. ∎

Therefore, in the remainder of the chapter, the long-run survivorship of unskilled managers indicates both the expected survival time for surely unskilled managers and the survival probability of unskilled managers in the long run.

In an extreme case that b is sufficiently large compared with ϕ, unskilled managers never choose to exit and, hence, no managers choose to exit.

[4]Simply put, the long-run survival probability of unskilled managers is dominantly determined by the value of b. A simple analogy is as follows. Compare two processes where one decays with rate λ_1 until T_1 and with rate λ_1' afterwards, and the other decays with rate λ_2 until T_2 and with rate λ_2' afterwards. Suppose $\lambda_1 > \lambda_2$ and $\lambda_1' < \lambda_2'$. In the limit of infinite time, the survival probability of the former process is always greater than that of the latter.

Corollary 3.4 *If*

$$b > \frac{(r+\lambda+\eta)\phi}{f}, \tag{3.10}$$

no managers choose to exit.

Proof. See Appendix ■

Entries of unskilled managers

Skilled managers and unskilled managers are not distinguishable at entry. New managers enter the AM industry with a certain prior about their skill, and this prior is correct: among a group of managers with the same prior probability p of being H-type, p portion of managers in the group are skilled, and $(1-p)$ portion of managers in the group are unskilled. Therefore, given the prior skill distribution fixed, the entry threshold p^* defined by (3.9) determines the number (measure) of newly entering unskilled managers.

When p^* is lower, more unskilled managers enter the AM industry, while more skilled managers enter as well. The following proposition characterizes when p^* is lower, and the implications of lower p^*.

Proposition 3.5 *Suppose that $A_1 \geq A_2$ and $b_1 \geq b_2$. p^* is lower for (A_1, b_1) than that for (A_2, b_2). Given the prior skill distribution fixed, the number (measure) of entries of both skilled and unskilled managers is greater for (A_1, b_1) than those for (A_2, b_2). Also, the portion of unskilled managers among newly entering managers is higher for (A_1, b_1) than it is for (A_2, b_2).*

Proof. See Appendix. ■

The entry threshold p^* decreases in A and b. When p^* is lower, there are more entries of skilled and unskilled managers, but disproportionately more entries of unskilled managers.

3.3.5 Determinants of the stationary equilibrium

The previous analysis shows that the long-term survivorship of unskilled managers crucially depends on b, and the entry threshold p^* determines the entries of unskilled managers as well as those of skilled managers. Based on these analyses, I investigate how exogenous components of the model influences the equilibrium variables and, consequently, the long-run survivorship and entries of unskilled managers, by changing the stationary equilibrium.

In this subsection, I assume that a stationary equilibrium exists and it is unique. The existence and the uniqueness are shown in the next subsection.

Aggregate invest opportunities ($\bar{A}$) and aggregate naive capital ($\bar{b}$)

The following proposition characterizes how an increase in $\bar{A}$, the parameter that governs the aggregate invest opportunities, affects the stationary equilibrium.

Proposition 3.6 *Ceteris paribus, an increase in $\bar{A}$ increases A, decreases b, and increases the number N of active managers in the AM industry.*

Proof. See Appendix. ∎

In general cases, an increase in $\bar{A}$ induces more entries of prospective managers. In particular, when $h_A(N)$ and $h_b(N)$ are not steep, p^* decreases in $\bar{A}$, i.e., there are more entries of prospective managers when there are more investment opportunities. This is proved in the following proposition:

Proposition 3.7 *Suppose that in the stationary equilibrium the exit set E defined by (3.6) is nonempty. Denote the equilibrium measure of active funds by N^*. There exist $c_A > 0$ and $c_b > 0$ such that if $|h'_A(N^*)| \leq c_A$ and $|h'_b(N^*)| \leq c_b$, a small increase in $\bar{A}$ leads to a decrease in p^*.*

Proof. See Appendix. ∎

As long as $h_A(N)$ and $h_b(N)$ are sufficiently flat, i.e., returns to scale at the industry level are not steeply decreasing, there are more entries of prospective managers to the AM industry as there are more aggregate investment opportunities available.

When the exit set E is empty, it is straightforward to show that the entry threshold p^* decreases in $\bar{A}$:

Corollary 3.5 *Suppose that in the stationary equilibrium the exit set defined by (3.6) is empty, a small increase in $\bar{A}$ leads to a decrease in p^*.*

Proof. See Appendix. ■

The following proposition characterizes how an increase in $\bar{b}$, the parameter that governs the aggregate amount of naive money, affects the stationary equilibrium.

Proposition 3.8 *Ceteris paribus, an increase in $\bar{b}$ decreases A, increases b, and increases the number N of active managers in the AM industry.*

Proof. The proof is quite similar to that of Proposition 3.6. See Appendix. ■

In general cases, an increase in $\bar{b}$ induces more entries of prospective managers. In particular, when $h_A(N)$ and $h_b(N)$ are not steep, p^* decreases in $\bar{b}$, i.e., there are more entries of prospective managers when there are more capital inflows from unsophisticated investors. This is proved in the following proposition:

Proposition 3.9 *Suppose that in the stationary equilibrium the exit set E defined by (3.6) is nonempty. Denote the equilibrium measure of active funds by N^*. There exist $c_A > 0$ and $c_b > 0$ such that if $|h'_A(N^*)| \leq c_A$ and $|h'_b(N^*)| \leq c_b$, a small increase in $\bar{b}$ leads to a decrease in p^*.*

Proof. The proof is quite similar to that of Proposition 3.7. See Appendix. ■

As long as $h_A(N)$ and $h_b(N)$ are sufficiently flat, i.e., returns to scale at the industry level are not steeply decreasing, there are more entries of prospective managers to the AM industry as there are more aggregate naive money available.

When the exit set E is empty, it is straightforward to show that the entry threshold p^* decreases in $\bar{b}$:

Corollary 3.6 *Suppose that in the stationary equilibrium the exit set defined by (3.6) is empty, a small increase in $\bar{b}$ leads to a decrease in p^*.*

Proof. The proof is quite similar to that of Corollary 3.5. See Appendix. ■

Entry costs (Φ) and distribution ($G(1-p)$) of the prior skill

So far, I have investigated exogenous parameters that directly affect the value of A, the average value creation per time by a skilled manager, and b, the average naive money inflow rate. Here, I consider two other exogenous components of the model: the entry cost and the distribution of the prior skill of prospective managers.

The following proposition shows how an increase in the entry cost Φ influences the stationary equilibrium:

Proposition 3.10 *Ceteris paribus, an increase in Φ increases A, increases b, increases p^*, and decreases the number N of active managers in the AM industry.*

Proof. See Appendix. ■

Lastly, I study how the distribution of the prior skill of prospective managers affects the stationary equilibrium. I consider $G(1-p) \equiv \int_p^1 F'(p')dp'$ instead of $F(p)$ because of the condition (3.3) that is imposed in order to guarantee the existence of a stationary equilibrium[5]. I define that $G_1(x)$ dominates $G_2(x)$ if

$$G_2(x) < G_1(x) \ , \quad \forall 0 \leq x \leq 1 \ .$$

$G_1(1-p)$ dominating $G_2(1-p)$ implies that the former distribution has more skilled prospective managers than the latter does, at any point p.

Proposition 3.11 *Suppose that $G_1(1-p)$ dominates $G_2(1-p)$. Ceteris paribus, $A_1 < A_2$, $b_1 < b_2$, $p_1^* > p_2^*$, and $N_1 > N_2$.*

Proof. See Appendix. ■

3.3.6 Existence and uniqueness of a stationary equilibrium

The following proposition proves the existence and uniqueness of a stationary equilibrium.

[5]If $F(p)$ is used as the cumulative distribution, the condition (3.3) implies $\lim_{p\to 0^+} F(p) = \infty$.

Proposition 3.12 *A stationary equilibrium defined by Definition 1 exists and is unique.*

Proof. See Appendix. ■

3.3.7 Numerical Examples

For the baseline numerical computation, I make the following parameter choices:

$$\begin{gathered} \bar{A} = 1 \ , \quad \bar{b} = 3.5 \ , \quad \phi = 1.5 \ , \quad \Phi = 3 \ , \quad f = 0.015 \ , \\ s = 0.3 \ , \quad \lambda = 0.05 \ , \quad \eta = 0.05 \ , \quad r = 0 \ , \quad \sigma = 0 \ , \quad \sigma_z = 2 \ , \end{gathered} \tag{3.11}$$

and decreasing returns to scale

$$h_A(N) = h_b(N) = N^{-1} \ ,$$

and the distribution density

$$F'(p) = 0.4(1-p)^3 \ .$$

The value of fee profits is plotted as follows:

[See figure 1]

Entries among all prospective managers are characterized as

[See figure 2]

The exit threshold is plotted in the following:

[See figure 3]

Lastly, the stationary equilibrium distribution of managers in $(p, \tilde{q})$ is

[See figure 4]

3.4 Heterogeneity in Investor Sophistication

In this section, I consider two different types of AM markets that are heterogeneous in investor sophistication. I refer to markets with more sophisticated investors as hedge fund markets, and markets with less sophisticated investors as mutual fund markets. In order to contrast these two types of markets, I model the heterogeneity in investor sophistication in an extreme form: unsophisticated investors only exist in mutual fund markets.

Since no unsophisticated investors exist in hedge fund markets, further analysis on those markets is needed in order to characterize the stationary equilibrium. On the other hand, the analyses of the previous section, especially the partial equilibrium results, apply to mutual fund markets.

3.4.1 The model of hedge funds

When a skilled (H-type) manager runs a hedge fund, she can generate

$$A_h\left(dt + \frac{1}{s}dZ_t\right)$$

dollars between t and $t + dt$. Similar to the baseline setup, Z_t is a physical Brownian motion and idiosyncratic. The hedge fund sector is also subject to the decreasing returns to scale at the industry level, which is modeled in a reduced-form as follows:

$$A_h = \bar{A}_h h_h(N_h) \ , \tag{3.12}$$

where N_h is the number (measure) of active hedge fund managers, and $h_h(\cdot)$ is a smooth monotone (strictly) decreasing function in N_h. A condition similar to (3.5) is imposed on $h_h(N_h)$:

$$\lim_{N_h \to 0} h_h(N_h) \to \infty \ , \quad \lim_{N_h \to \infty} h_h(N_h) \to 0 \ . \tag{3.13}$$

A hedge fund manager can offer any short-term per-invested-dollar fee contracts between time t and $t+dt$ to the investors, i.e., at time t managers can offer a contract that pays off

$$f_{t,t+dt}$$

to the manager at $t+dt$ as a function of verifiable variables at time $t+dt$, including excess returns between t and $t+dt$, per invested dollar at time t.

When an unskilled (L-type) manager runs a hedge fund, she generates no value on average, but the volatility of the value is the same as that of a skilled manager. Hedge fund managers are risk-neutral as mutual fund managers are. Hedge fund managers bear the same operating costs ϕdt between t and $t+dt$ as mutual fund managers do. At entry, hedge fund managers pay the same entry cost Φ as mutual fund managers do.

The following analysis assumes a stationary equilibrium, and takes A_h as given.

3.4.2 Fee revenues

The gross excess return on a hedge fund between t and $t+dt$ is

$$dR^{ex}_{h,t} = \frac{A_h \left(\mathbb{1}_H dt + \frac{1}{s} dZ_t \right)}{q_t} ,$$

where $\mathbb{1}_H$ is 1 if the manager of the fund is H-type and 0 otherwise. Given that the short-term compensation contract offers $f_{t,t+dt}$ per invested dollar, the net excess return on the fund between t and $t+dt$ is

$$dr^{ex}_{h,t} = dR^{ex}_{h,t} - f_{t,t+dt} = \frac{A_h \left(\mathbb{1}_H dt + \frac{1}{s} dZ_t \right)}{q_t} - f_{t,t+dt} .$$

Since investors can diversify away idiosyncratic risks by themselves, they accept the compensation contract only if the net expected excess return (net alpha) is non-negative. On the other hand, if the net alpha on a hedge fund is strictly negative, sophisticated investors do not invest any dollars in the fund. Since there are no unsophisticated investors in hedge fund markets (by assumption), strictly negative net alpha cannot constitute an equilibrium. Therefore, the net alpha must be zero in equilibrium. This condition reads

$$\mathbb{E}_t[dr^{ex}_{h,t}] = \frac{p_t A_h dt}{q_t} - \mathbb{E}_t[f_{t,t+dt}] = 0 \quad \Longleftrightarrow \quad \mathbb{E}_t[f_{t,t+dt}] = \frac{p_t A_h dt}{q_t} .$$

The intuition for this result that the net alpha is zero in equilibrium is the same as that of Berk and Green (2004). Fee revenues at $t + dt$ expected at t are

$$\mathbb{E}_t[q_t f_{t,t+dt}] = p_t A_h dt = \mathbb{E}_t\left[A_h\left(\mathbb{1}_H dt + \frac{1}{s} dZ_t\right)\right] .$$

Thus, regardless of the form of the short-term compensation contract $f_{t,t+dt}$, the skill of the hedge fund manager is correctly priced and compensated. The following proposition summarizes the analysis.

Proposition 3.13 *If all the investors are rational, and can freely move capital, the skill of active managers is fairly compensated, i.e., the expected fee revenue is equivalent to the amount of dollar value that the manager is expected to generate. The fee structure does not matter for fee revenues and, consequently, fee profits.*

Proof. Proof provided in the above analysis. ■

When there are no unsophisticated investors in AM markets, AM markets are indeed markets for active managers' skill. Since skill is scarce, managers take all the rents from surplus that they are expected to generate no matter what compensation contracts are.

3.4.3 The value of fee profits and exits of hedge funds

Expected fee profits that a hedge fund receives between t and $t + dt$ are

$$(p_t A_h - \phi) dt$$

The value of fee profits at t can be written as

$$V_{h,t} = \mathbb{E}_t\left[\int_t^{T_h^D} e^{-r(u-t)} \left(p_u A_h - \phi\right) du\right] ,$$

where T_h^D is the point in time when the hedge fund manager exits. T_h^D is determined by either an exogenous exit or an endogenous exit, similar to the case of mutual funds. Since the conditional distribution p_u for $u \geq t$ at time t depends entirely on p_t, the state variable for the value of fee profits at t is p_t. Hence, the value of fee profits can be rewritten as $V_{h,t} = V_h(p_t)$.

In dynamic programming language, the value of fee profits reads

$$V_h(p_t) = \max\left\{(p_t A_h - \phi)\, dt + (1 - rdt - \lambda dt)\mathbb{E}_t\left[V_h(p_{t+dt})\right], 0\right\} ,$$

where

$$dp_t = sp_t(1-p_t)d\tilde{Z}_t .$$

When the value is greater than zero, the value function $V_h(p)$ is smooth, and solves the following ordinary differential equation (ODE):

$$(r+\lambda)V_h(p) = (pA_h - \phi) + \frac{1}{2}V_h''(p)s^2p^2(1-p)^2 .$$

There exist analytic solutions to the ODE. One can decompose $V_h(p)$ into

$$V_h(p) = V_h^b(p) + V_h^o(p) ,$$

where $V_h^b(p)$ is the particular solution, and $V_h^o(p)$ is the homogeneous solution. The particular solution is

$$V_h^b(p) = \frac{1}{r+\lambda}(pA_h - \phi) ,$$

and the homogeneous solution takes the following form:

$$V_h^o(p) = c_1\sqrt{p}^{1-\mu}\sqrt{1-p}^{1+\mu} + c_2\sqrt{p}^{1+\mu}\sqrt{1-p}^{1-\mu} ,$$

where

$$\mu = \sqrt{1 + \frac{8(r+\lambda)}{s^2}} > 1 , \tag{3.14}$$

and the coefficients c_1 and c_2 are determined by boundary conditions.

The following lemma characterizes some properties of $V_h(p)$:

Lemma 3.4 *$V_h(p)$ is increasing and convex.*

Proof. See Appendix. ■

Since the state variable for a hedge fund is p, endogenous exit decisions by hedge fund managers depend only on p. Define the set of p where hedge fund managers choose to exit:

$$E_h \equiv \{p | V_h(p) = 0\} .$$

Since $V_h(0) = 0$, the exit set E_h of hedge funds is always nonempty, unlike the exit set E_m of mutual funds which may (or not) be empty. The set E_h is fully characterized by p_{ex}^h as follows:

$$E_h \equiv \left\{p | p \leq p_{ex}^h\right\} , \tag{3.15}$$

since $V_h(p)$ is increasing in p. In continuous time, hedge fund managers choose to exit immediately after p reaches p_{ex}^h and, hence, p_{ex}^h is the exit threshold for hedge fund managers. Therefore, hedge fund managers choose to exit if their skill is perceived as low.

At $p = 0$ and $p = 1$, the manager's type is fully revealed, and there is no additional learning about her skill. As a result,

$$\begin{aligned} V_h(0) &= \max\{-\phi dt + (1 - rdt - \lambda dt)V_h(0), 0\} = 0 , \\ V_h(1) &= \max\{(A_h - \phi)dt + (1 - rdt - \lambda dt)V_h(1), 0\} = \max\left\{\frac{A_h - \phi}{r + \lambda}, 0\right\} . \end{aligned}$$

If $V_h(1) = 0$, $p_{ex}^h = 1$ and $V_h(p)$ is uniformly zero. If $V_h(1) > 0$, i.e., $A_h > \phi$, $p_{ex}^h < 1$ and $c_2 = 0$. The smooth pasting condition at $p = p_{ex}^h$ further pins down c_1 and p_{ex}^h as follows:

$$c_1 = \frac{2\phi\sqrt{\mu - 1}^{\mu-1}}{(r + \lambda)\sqrt{\mu + 1}^{\mu+1}} \sqrt{\frac{\frac{\phi}{A_h}}{1 - \frac{\phi}{A_h}}}^{\mu-1} , \tag{3.16}$$

and

$$p_{ex}^h = \frac{(\mu - 1)\frac{\phi}{A_h}}{\mu + 1 - \frac{2\phi}{A_h}} , \tag{3.17}$$

where μ is defined in (3.14). p^h_{ex} is increasing in $\frac{\phi}{A_h}$ and μ. Therefore, the exit set E_h of hedge funds diminishes as A_h increases:

Proposition 3.14 *For $A_{h,1} > A_{h,2}$, $E_h(A_{h,1}) \subset E_h(A_{h,2})$.*

Proof. It is straightforward from (3.15) and (3.17). ∎

3.4.4 Stationary equilibrium

Since there are two types of AM markets, the a stationary equilibrium must be redefined. The state variables for an individual mutual fund manager are p and $\tilde{q}$, and the state variable for an individual hedge fund manager is p. The aggregate state of the economy is the distribution of agents across states in mutual fund markets and hedge fund markets.

In order to define the aggregate state variables formally, define the state space for an individual manager. The state space of a mutual fund manager is the Cartesian product $S_m : [0,1] \times [0,\infty)$ with Borel σ algebra $\mathcal{B}_m$, and that of a hedge fund is $S_h : [0,1]$ with Borel σ algebra $\mathcal{B}_h$. For any set $\mathcal{S}_m \in \mathcal{B}_m$, $\rho_m(\mathcal{S}_m)$ is the measure of mutual fund managers in the set $\mathcal{S}_m$. Similarly, for any set $\mathcal{S}_h \in \mathcal{B}_h$, $\rho_h(\mathcal{S}_h)$ is the measure of hedge fund managers in the set $\mathcal{S}_h$.

Definition 2 *A stationary equilibrium is value functions $V_m : S_m \to \mathbb{R}_{\geq 0}$ and $V_h : S_h \to \mathbb{R}_{\geq 0}$; the base surplus A, the additional surplus for hedge funds ξ and the average naive capital inflow rate b; and stationary measures ρ^*_m and ρ^*_h such that*

- *Given (A_m, A_h, b), $V_m(p,\tilde{q})$ and $V_h(p)$ are value functions of mutual fund managers and hedge fund managers, respectively.*

- *Given ρ^*_m and ρ^*_h, $N_m = \int\int \rho^*_m(p,\tilde{q})dpd\tilde{q}$ and $N_h = \int \rho^*_h(p)dp$ are consistent with the values of (A_m, A_h, b) by (3.4) and (3.12).*

- *Prospective managers arrive with the prior skill distribution $F(p)$, and chooses to enter mutual fund markets if $V_m(p,0) > V_h(p)$ and $V_m(p,0) \geq \Phi$, and enter hedge fund markets if $V_m(p,0) < V_h(p)$ and $V_h(p) \geq \Phi$.*

- *Mutual fund managers choose to exit if $V_m(p,\tilde{q}) = 0$, and hedge fund managers choose to exit if $V_h(p) = 0$. In addition, existing managers exogenously exit under the Poisson process with probability λdt between t and $t+dt$.*

- *ρ_m^* and ρ_h^* are invariant under entries, exits and the transition of states of existing managers, given by*

$$dp = sp(1-p)d\tilde{Z} \ , \quad d\tilde{q} = (b - \eta\tilde{q})dt + \sqrt{\tilde{q}}\left(\sigma dW + \sigma_z d\tilde{Z}\right) \ .$$

3.4.5 Entries

A prospective manager with probability p of being H-type compares $V_m(p,0)$ and $V_h(p)$, and chooses to enter an AM market with a higher value of fee profits, as long as the participation constraint

$$\max\{V_m(p,0), V_h(p)\} - \Phi \geq 0 \tag{3.18}$$

holds. Define p_m^* and p_h^* such that

$$V_m(p_m^*,0) = V_h(p_h^*) = \Phi \ . \tag{3.19}$$

A stationary equilibrium imposes the following conditions:

Lemma 3.5 *In a stationary equilibrium, both p_m^* and p_h^* exist, and are unique, respectively. $V_m(p,0)$ or $V_h(p)$ cannot dominate the other, i.e.,*

$$V_m(p,0) > V_h(p) \ , \quad \forall p > p_m^* \ ,$$

or

$$V_m(p,0) < V_h(p) \ , \quad \forall p > p_h^* \ ,$$

is inconsistent with a stationary equilibrium.

Proof. See Appendix. ■

From this lemma, one can prove the following proposition straightforwardly:

Proposition 3.15 *In a stationary equilibrium, $A_h > A_m$.*

Proof. See Appendix. ■

This proposition implies that in a stationary equilibrium, hedge fund managers generate more value than mutual fund managers do. Mutual fund managers enjoy additional fee profits when they succeed attracting sufficient capital from unsophisticated investors, but hedge fund managers are always fairly compensated. Therefore, for prospective managers to be motivated to enter hedge fund markets, they must be able to generate greater value in hedge fund markets than they are in mutual fund markets.

This result implies that the sensitivity of the entry value to the perceived skill of the manager is generally lower for mutual funds than for hedge funds. The following lemma justifies this statement.

Lemma 3.6 *The sensitivity of the hedge fund entry value to the perceived skill of the manager satisfies*

$$V_h'(p) \leq \frac{A_h}{r+\lambda},$$

where the equality holds for $p = 1$. On the other hand, the sensitivity of the mutual fund entry value to the perceived skill of the manager satisfies

$$\frac{\partial V_m(p,0)}{\partial p} < \frac{A_m}{r+\lambda} < \frac{A_h}{r+\lambda}$$

for all p.

Proof. See Appendix. ■

This lemma shows that the maximum sensitivity of the entry value to the perceived skill is strictly higher for hedge funds than for mutual funds. However, $V_h'(p)$ cannot be strictly higher than $\frac{\partial V_m(p,0)}{\partial p}$ for all $p > p_{ex}^h$, because otherwise $V_h(p)$ strictly dominates $V_m(p,0)$. Still, the sensitivity of hedge fund entry value to skill (p) is strictly higher than that of mutual fund entry value for high p.

A high-skilled (high p) manager generates more value by managing a hedge fund, and the value of her fee profits is more sensitive to her skill when she manages a hedge fund, compared to the case where she runs a mutual fund. This implies that better-skilled (with higher prior p of being skilled) prospective managers tend to enter hedge fund markets. Although this sounds intuitive, it is hard to formally prove that in general cases, mainly because $V(p, \tilde{q})$ does not have an analytic form.

In order to formally prove that high-skilled prospective managers choose to enter hedge fund markets, I make an additional assumption: the signal-to-noise ratio s of managers' skill is sufficiently small. This is a reasonable assumption considering how noisy the actual track records of asset managers are. The following Lemma defines what sufficiently small s means.

Lemma 3.7 *There exists signal-to-noise ratio $s = \bar{s}$ such that for all $s < \bar{s}$*

$$V_h'(p_h^*; s) > \frac{A_m}{r + \lambda},$$

where p_h^* is defined in (3.19).

Proof. See Appendix. ■

Given that the signal-to-noise ratio s being sufficiently low, prospective managers sort themselves based on the investor sophistication: high-skilled managers enter hedge fund markets, and low-skilled mangers enter mutual fund markets. This is proved in the following proposition:

Proposition 3.16 *Suppose that s is smaller than $\bar{s}$ as defined in Lemma 3.7. There exists $0 < \hat{p} < 1$ such that prospective managers with $p \in (p_m^*, \hat{p})$ enter mutual fund markets, and prospective managers with $p \in (\hat{p}, 1]$ enter hedge fund markets.*

Proof. See Appendix. ■

The intuition is, when s is low, the value of the exit option for hedge fund managers becomes small. This implies that the sensitivity of the value of hedge fund fee profits to skill (p) is close to the maximum slope $\frac{A_h}{r+\lambda}$ as long as the perceived skill p is sufficiently far from the exit threshold p_{ex}^h. Therefore, the slope of the hedge fund fee value with respect to p is strictly greater than the slope of the mutual fund fee value for most values of p. Since hedge fund markets compensate managers' skill better than mutual fund markets do, high-skilled managers enter hedge fund markets.

When the mutual fund exit set E_m is empty, it is also possible to prove that prospective managers strictly sort themselves based on the investor sophistication, regardless of the value of s. This is proved in the following proposition:

Proposition 3.17 *Suppose that the mutual fund exit set E_m is empty. There exists $0 < \hat{p} < 1$ such that prospective managers with $p \in (p_m^*, \hat{p})$ enter mutual fund markets, and prospective managers with $p \in (\hat{p}, 1]$ enter hedge fund markets.*

Proof. See Appendix. ■

Since the value of hedge fund fee profits tends to be more sensitive to skill compared with the value of mutual fund fee profits, prospective managers with higher skill tend to enter hedge fund markets.

While unskilled hedge fund managers eventually choose to exit as they reveal their (lack of) skill, unskilled mutual fund managers may survive as long as they attract sufficient naive capital. In the limit of zero operating costs, the attrition rate of hedge fund managers is strictly higher than that of mutual fund managers.

Corollary 3.7 *In the limit of $\phi \to 0$, the hedge fund attrition rate is strictly greater than the mutual fund attrition rate.*

Proof. See Appendix. ■

In summary, hedge fund markets are characterized by relatively high-skilled managers attracting smart money. On the other hand, mutual fund markets are characterized by relatively low-skilled managers attracting naive money.

3.4.6 Existence and uniqueness of a stationary equilibrium

When there are two types of markets, formally proving the existence and/or the uniqueness of a stationary equilibrium is quite difficult. The primary reason is that prospective managers choose between those two types when they enter. The binary choice does not guarantee continuous changes of entry decisions as equilibrium parameters change infinitesimally. The proof of the existence and/or the uniqueness of a stationary equilibrium relies on such continuity (see the proof of Proposition 3.12).

For example, consider an extreme case where $V_m(p,0) = V_h(p) - \delta$ where δ is infinitesimal. In this case, all entries of prospective managers are into mutual fund markets. Now suppose that A_h increases slightly. In this case, prospective managers choose to enter hedge fund markets only. This example highlights why small changes in equilibrium parameters may not translate into small changes in entries. While that example is quite extreme, without knowing properties of $V_m(p,0)$ sufficiently, it is hard to exclude discontinuities in entry decisions.

Therefore, when there are two types of markets heterogeneous in investor sophistication, I provide a limited set of results regarding the existence and uniqueness of a stationary equilibrium. For example, Proposition 3.16 shows that, for s lower than $\bar{s}$ defined in Lemma 3.7, there exists a cutoff of the prior skill such that prospective managers with skill higher than the cutoff enter hedge fund markets, and prospective managers with skill lower than the cutoff enter mutual fund markets. Such sorting of prospective managers is helpful for proving the existence and uniqueness of a stationary equilibrium, because the structure (sorting) guarantees the continuity of entry decisions. However, since $\bar{s}$ depends on the values of A_m and A_h, it is possible that there is no finite $\bar{s}$ that guarantees the sorting of prospective managers for the entire space of A_m and A_h. In an extreme case where $s \to 0$, the result of Proposition 3.16 always holds. Therefore, I prove the existence and uniqueness of a stationary equilibrium in this extreme case:

Proposition 3.18 *In the limit of $s \to 0$, there exists a stationary equilibrium and it is unique.*

Proof. See Appendix. ■

Similar to Proposition 3.18, I consider an extreme case where the assumption of Proposition 3.17 always holds. The mutual fund exit set E_m may be either empty or nonempty depending on the value of b. The only case where the mutual fund exit set is always nonempty regardless of b is the limit case of zero operating cost: $\phi \to 0$. I prove the existence and uniqueness of a stationary equilibrium in this extreme case:

Proposition 3.19 *In the limit of $\phi \to 0$, there exists a stationary equilibrium and it is unique.*

Proof. See Appendix ■

Note that these results are in limited cases, as opposed to the general proof of the existence and uniqueness of a stationary equilibrium in Proposition 3.12 when markets are homogeneous in investor sophistication.

3.4.7 Numerical Examples

I use the baseline parameter choices given in (3.11) for the mutual fund industry. In addition, I make the following parameter choices for the hedge fund industry:

$$\bar{A}_h = 0.6\ , \quad h_h(N_h) = N_h^{-1}\ . \tag{3.20}$$

$V_m(p, 0)$ and $V_h(p)$ are plotted as follows:

[See figure 5]

Entries to mutual fund markets and hedge fund markets among all prospective managers are characterized as

[See figure 6]

The stationary equilibrium distribution of mutual fund managers in $(p, \tilde{q})$ is

[See figure 7]

The stationary equilibrium distribution of hedge fund managers in p is

[See figure 8]

3.5 Regulatory Implications for Retail Investor Protection

This discussion has implications (although limited) for regulations regarding retail investor protection. One crucial caveat is that this chapter does not provide a rationale for retail investor protection, since financial transactions are merely the redistribution of wealth and do not necessarily improve the aggregate welfare. In addition, this chapter does not address how the structure of AM markets affects the efficiency (or price discovery) of asset markets. Therefore, no regulatory implications are offered regarding welfare, and the regulatory focus is limited to identifying and elucidating effective measures for retail investor protection. Such are the limitations of the regulatory implications of the chapter.

3.5.1 Protection of unsophisticated investors

In this chapter, sophisticated investors always break even since they only invest in fairly priced active funds or passive benchmarks. In contrast, unsophisticated investors lose wealth when they invest their capital (naive capital) in underperforming (overpriced) funds. While those investors can improve the outcome by investing in passive benchmarks with similar risk characteristics, why they do not switch is not the focus of this chapter. On the other hand, unsophisticated investors can break even by investing in fairly priced funds. Therefore, the aggregate wealth transfer from unsophisticated investors to active funds depends on how much naive money is allocated to overpriced funds.

Whether a fund is overpriced, or not, depends on the skill of the manager and the amount of invested naive money. By Proposition 1.1, a fund is less likely to be overpriced when the (perceived) skill of the manager is high, and when the amount of invested naive money is small. Translating this statement into the language of the model, a fund is less likely to be overpriced when A, the amount of value per time that a skilled manager generates, is high, and b, the average naive capital inflow rate to a fund, is low. In addition, low b reduces naive money allocations to the fund. At the individual fund level, the protection of unsophisticated investors seems straightforward: encourage active funds to generate more value, and prevent them from attracting unsophisticated investors.

However, regulations that affect the incentive structure of active funds may also affect those funds' entry and exit decisions. Specifically, entries of unskilled managers and their survivorship are of particular concern. While unskilled managers enter AM markets with a certain expectation of having skill, their true skill is revealed as their track records accumulate. Therefore, funds run by unskilled managers are more likely to become overpriced, and eventually *are* overpriced when their (lack of) skill is fully revealed. If those managers survive in the long run, they negatively influence the wealth of their (unsophisticated) investors for long periods.

Therefore, regulations that aim to protect retail investors must take into account both the intensive margin (individual funds being overpriced and receiving more naive capital) and the extensive margin (more entries of unskilled managers and their long-term survivorship). In particular, high A decreases the probability of individual funds being overpriced, but increases entries of unskilled managers. In contrast, low b decreases the probability of individual funds being overpriced, discourages entries of unskilled managers, and reduces the long-run surviviorship of unskilled managers. Regulations that intend to influence one of them (e.g., raise or lower A) may also affect the equilibrium value of the other (the value of b) through competition among funds.

Regulations affect the industry-wide parameters (e.g., $\bar{A}$, a parameter governing the aggregate amount of value creation, and $\bar{b}$, a parameter influencing the aggregate naive money inflow), and may change the industry equilibrium. For example, regulations that restrict the value-generating activities of active funds (e.g., strict disclosure rules) can be thought of as decreasing $\bar{A}$, and may discourage entries of unskilled managers by raising the entry threshold p^*. However, such regulations can increase the long-term survivorship of unskilled managers by increasing b. This type of regulation may be detrimental to the wealth of unsophisticated investors. On the other hand, regulations that discourage active funds from attracting capital from unsophisticated investors (e.g., fiduciary rules for brokers and advisors) can be regarded as lowering $\bar{b}$. Such regulations may reduce entries of unskilled managers, and decrease the long-run survivorship of unskilled managers (lower p^* and lower b). This type of regulation is likely to be beneficial to the wealth of unsophisticated investors.

Another regulatory implication of this chapter is that regulations for one type of AM market (e.g., the hedge fund industry) may affect other types of AM markets (e.g., the mutual fund industry) in the long run. For example, imposing strict regulations on the hedge fund industry may decrease the profitability of individual hedge funds, and, as a result, induce better skilled prospective managers to enter mutual fund markets. In the long run, this may increase the competition in the mutual fund industry, and, hence, lower the long-term survivorship of unskilled managers. Regulations for the AM industry must take into account the long-run interactions among different types of AM markets.

3.5.2 Fee structure

The model does not address how funds make fee choices, mainly because endogenizing fee choices involves modeling how unsophisticated investors respond to fees. Such modeling requires additional specific assumptions about the behavior of unsophisticated investors, and these assumptions are not easy to justify, particularly because this chapter studies industry equilibria. Different industry equilibria correspond to different degrees of competition, and it is hard to imagine that the way in which unsophisticated investors react to fees stays unchanged when competition in the AM industry becomes more (or less) fierce.

While there is a clear reason why this chapter does not model fee choices, its framework can address a certain aspect of mutual fund fee choices. The Investment Company Amendments Act of 1970 in the US prohibited mutual funds from charging asymmetrical performance fees. As a consequence, in the US, mutual funds face a restricted set of fee choices: flat fees, which are overwhelmingly used in the mutual fund industry, or fulcrum (symmetrical) fees, which are employed by only a handful of mutual funds. Since only a tiny portion of US mutual funds adopt fulcrum fees, researchers find it difficult to conduct meaningful empirical investigations on the fund choice of fee structure, not merely the level of fees.

Under an additional assumption, this chapter can show that mutual funds prefer flat fees to fulcrum fees. Between t and $t + dt$, a fund may charge a proportional fee of

$$f_{t,t+dt} = f dt + \psi(dr_t^{ex}) ,$$

where dr_t^{ex} is the net return of the fund between t and $t + dt$ excess the benchmark. $\psi(x)$ is an arbitrary increasing function[6] satisfying $\psi(x) = -\psi(-x)$, since mutual funds in the US are restricted from charging asymmetrical performance fees. One crucial assumption that I make is that unsophisticated investors only pay attention to f, i.e., the fixed component, and their capital flows to a fund only depend on f. Note that this is a fairly strong assumption.

Suppose that a fund may choose its fee schedule once at entry, and cannot change it afterwards[7]. The following proposition proves that the fund (strictly) prefers flat fees to fulcrum fees.

Proposition 3.20 *Suppose that flows of naive capital only depend on f (the fixed component of fees). At entry, managers choose flat fees over symmetrical fees, i.e.,*

$$\psi(dr_t^{ex}) = 0$$

is the optimal fee choice for managers.

Proof. See Appendix. ■

The intuition of Proposition 3.20 is as follows: symmetrical fees hurt funds when the funds are expected to underperform, while fee structure does not matter (for fee revenues) when the funds are expected to perform the same as the market. Therefore,

[6] I only consider fees that monotonically increase in performance. Fees that do not belong to this class (e.g., fees that locally decrease in performance) may cause serious moral hazard problems (e.g., managers intentionally lowering their performance in order to receive higher fees).

[7] This assumption is not needed, but adopted for illustrative purpose. The conclusion does not change even if funds may change the fee schedule continuously.

symmetrical fees reduce the value of fee profits, by lowering fee profits when the managerial skill is overpriced. When funds are restricted to charging symmetrical fees, those funds find it optimal to choose flat fee structures, since that maximizes the amount of fees that those funds can extract from unsophisticated investors. Although the result of Proposition 3.20 relies on the strong assumption that naive capital only depends on f, the result still holds if the behavior of unsophisticated investors is sufficiently insensitive to the choice of $\psi(\cdot)$[8].

This discussion points to the possibility that active funds choose their fee structure in order to exploit unsophisticated investors. Therefore, regulations for the AM industry, particularly those restricting fee choices, must take this aspect into account.

Other possible explanations for fee structure choice

There are other valid explanations, from the perspective of investor rationality, for why US mutual funds overwhelmingly choose flat fees over fulcrum fees. If investors are rational and can freely move capital, as discussed in Proposition 3.13, fee structure does not matter for fee profits. Yet, if managers are risk-averse and are not able to hedge risk associated with fee profits, they may prefer flat fees to fulcrum fees in order to receive less volatile streams of fee profits. Another explanation comes from a view that fee contracts are means of aligning the incentives of the principal (investors) and the agent (managers). If symmetrical performance fees discourage managers from taking actions that are beneficial to the investors, and/or encourage managers to take actions that are detrimental to the investors, the optimal fee contract can take the form of flat fees.

There is a study by Drago, Lazzari and Navone (2010) that challenges these explanations. The authors examine the fee structure choice of funds in Italian mutual fund markets, where no (significantly) restrictive regulations for fee structure existed until 2006. Their study documents that, in the Italian equity mutual fund industry before 2006, the "bonus plan", i.e., fixed fees plus rewards for outperforming (but no

[8] Under a cheap assumption that unsophisticated investors "hate" nontrivial $\psi(\cdot)$, the result holds as well.

penalties for underperforming), is the standard fee structure and no funds employ fulcrum fees. Those explanations based on investor rationality need to account for both why funds (strictly) prefer flat fees to symmetrical fees, and prefer asymmetrical fees (the bonus plan) to flat fees.

There may be other explanations for such fee structure choices. One possibility is that those fee choices are path-dependent: once the industry standard is set up, new funds cannot easily adopt other fee structures. Another explanation is that the difference in fee choices comes from, for example, heterogeneity in investor characteristics between the US and Italian mutual fund industry. While these types of explanations are valid, those hypotheses are not easily verifiable (or falsifiable), and violate the expected universality of economic perspectives.

3.6 Discussions and Conclusions

This chapter concerns only the extensive margin, i.e., entries and exits of funds, but not the intensive margin, i.e., actions other than entries and exits that individual funds may take. In contrast, chapter 2 deals only with the intensive margin. It would be interesting to study the interactions between the extensive margin and the intensive margin in the context of the industry equilibrium, which remain a topic for future research.

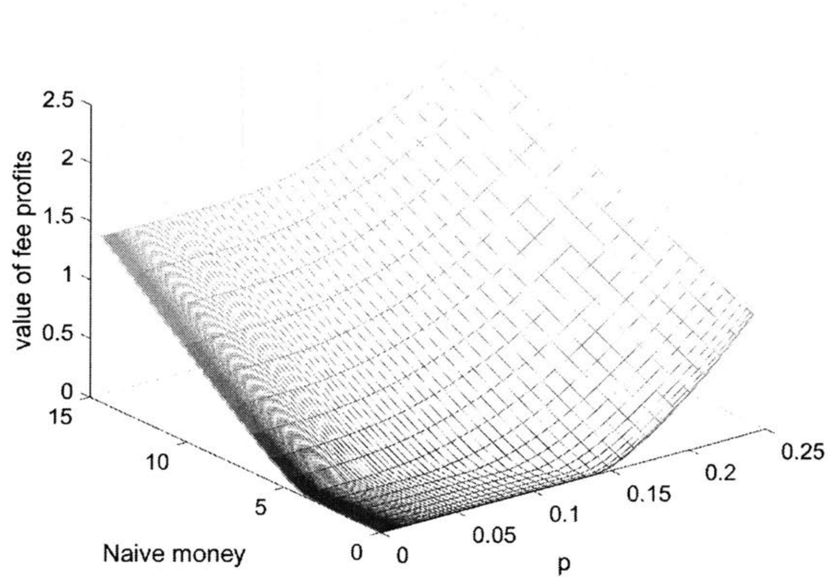
I would like to note that labor market dynamics are not dealt with, since one of the underlying assumptions of the model entails the fixed distribution of the prior skill of prospective managers regardless of how competitive AM markets are. However, in reality, it is hard to imagine that the supply of prospective managers is unaltered by the degree of competition in the AM industry. For instance, when skilled managers are relatively highly compensated due to comparatively low competition, those who are sufficiently competent outside AM markets are prone to enter the labor market, expanding the supply of skilled prospective managers. Numerous questions on the interactions between the AM markets and the labor market for prospective managers remain to be explored, both theoretically and empirically.

In conclusion, this chapter proposes a model that associates naive money with entry and exit decisions of active funds. Under a small set of modest assumptions, this chapter offers insights into the structure of the AM industry by examining the industry equilibrium. These insights may be useful for evaluating the impact of regulations that are designed to protect retail investors.

Figures

Figure 1: Value of fee profits

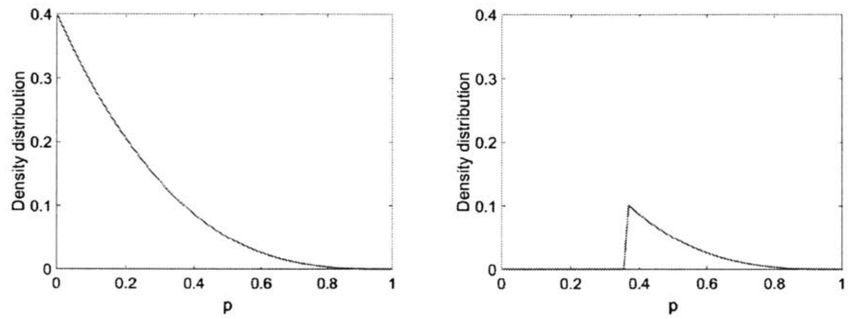
The following figure plots the value of fee profits as a function of the probability p of being H-type and the amount of invested naive money $\tilde{q}$. Parameter choices are given in (3.11).



Value of fee profits

Figure 2: Entry threshold

The following figures plot the density distribution of the prior skill of prospective managers. The left figure plots the density distribution for all prospective managers, and the right figure plots the density distribution for managers who choose to enter the AM industry. Parameter choices are given in (3.11).

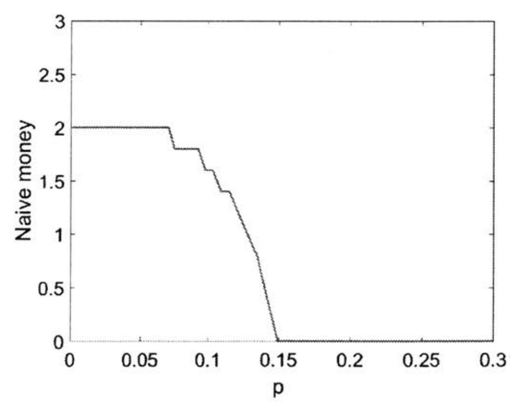


(a) All prospective managers (b) Those entering the AM industry

Density distribution of the prior skill of prospective managers

Figure 3: Exit threshold

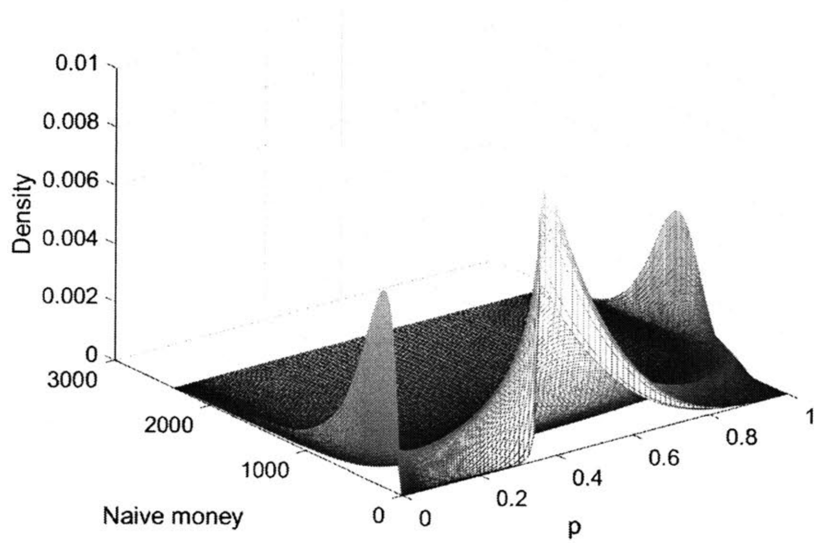
The following figures plot the exit threshold as a function of the probability p of being H-type. Parameter choices are given in (3.11).



Exit threshold

Figure 4: Distribution of managers

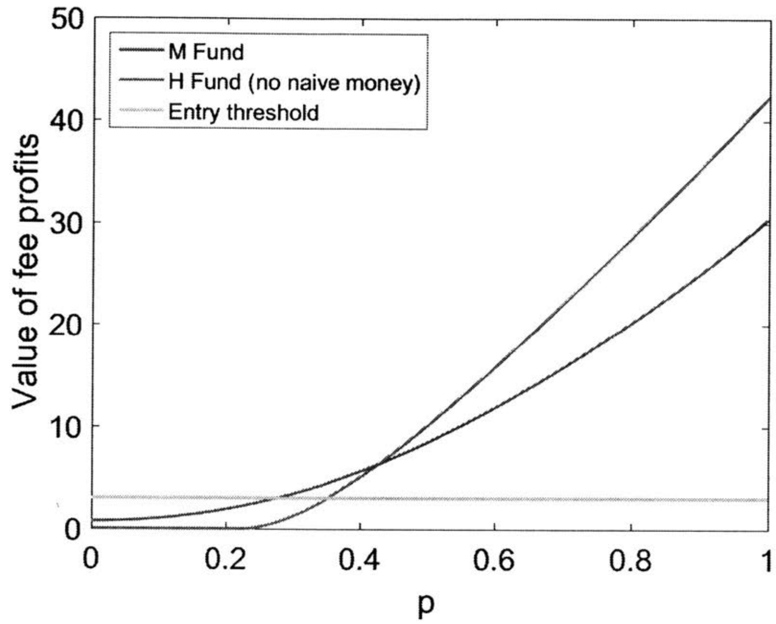
The following figure plots the stationary equilibrium density distribution of managers in p and $\tilde{q}$. Parameter choices are given in (3.11).



Distribution of managers

Figure 5: Heterogeneous AM markets - Value of fee profits

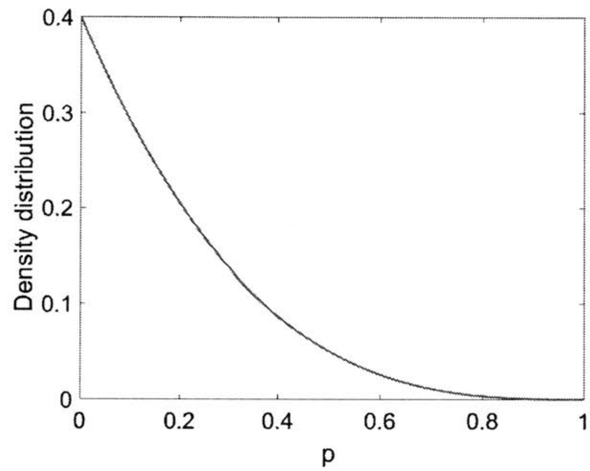
The following figure plots $V_m(p, \tilde{q})$, $V_h(p)$ and the entry threshold. Parameter choices are given in (3.11) and (3.20).



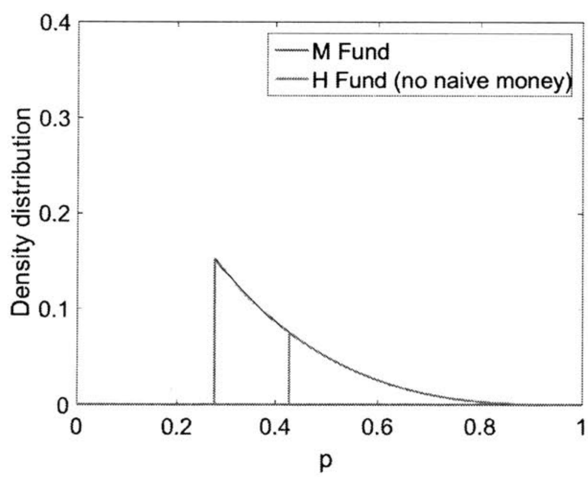
Value of fee profits - mutual funds and hedge funds

Figure 6: Heterogeneous AM markets - Entry decisions

The following figures plot the density distribution of the prior skill of prospective managers. The left figure plots the density distribution for all prospective managers, and the right figure plots the density distribution for managers who choose to enter the mutual fund industry and hedge fund industry. Parameter choices are given in (3.11) and (3.20).



(a) All prospective managers

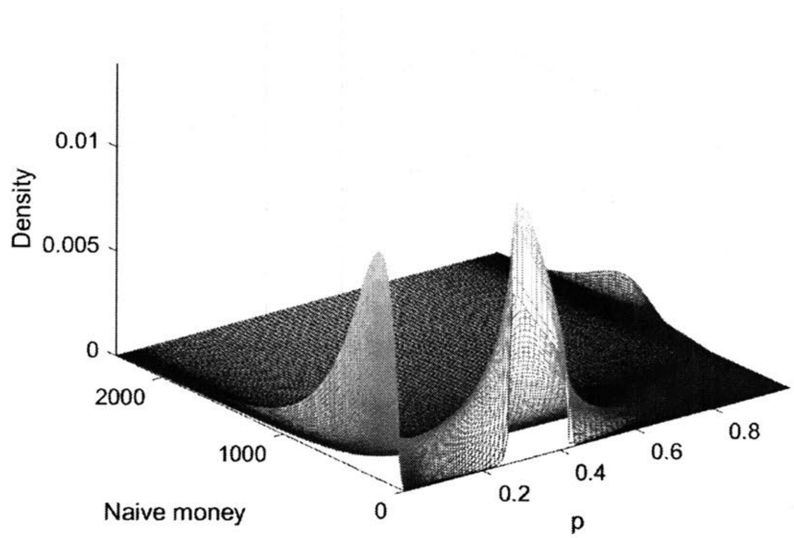


(b) Managers entering the AM industry - mutual funds and hedge funds

Density distribution of the prior skill of prospective managers

Figure 7: Distribution of mutual fund managers - Heterogenous AM markets

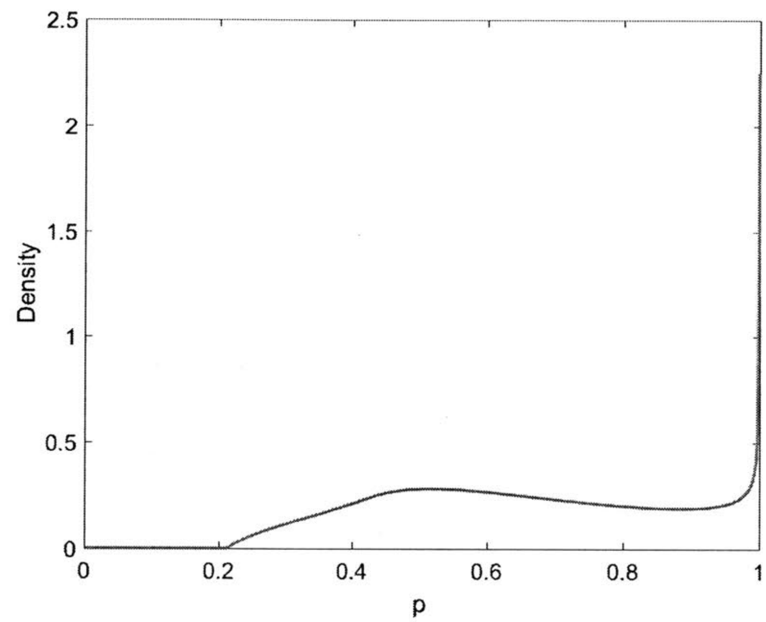
The following figure plots the stationary equilibrium density distribution of mutual fund managers in p and $\tilde{q}$. Parameter choices are given in (3.11) and (3.20).



Distribution of mutual fund managers

Figure 8: Distribution of hedge fund managers - Heterogenous AM markets

The following figure plots the stationary equilibrium density distribution of hedge fund managers in p. Parameter choices are given in (3.11) and (3.20).



Distribution of hedge fund managers

Appendix

Proofs

Proof of Lemma 3.1

$V(p, \tilde{q})$ is the fixed point of the following map:

$$\mathbb{T}f(p,\tilde{q}) = \max\left\{(\max\{pA, f\tilde{q}\} - \phi)\,dt + (1 - rdt - \lambda dt)\mathbb{E}\left[f(p',\tilde{q}')|p,\tilde{q}\right], 0\right\}\ , \quad (3.21)$$

where

$$\begin{aligned} p'|p &= p + sp(1-p)dZ\ , \\ \tilde{q}'|\tilde{q} &= \tilde{q} + (b - \eta\tilde{q})dt + \sqrt{\tilde{q}}\left(\sigma dW + \sigma_z d\tilde{Z}\right)\ , \end{aligned}$$

where Z is a standard Brownian motion. The contraction mapping theorem guarantees that

$$V(p,\tilde{q}) = \lim_{k\to\infty} \mathbb{T}^k f(p,\tilde{q})$$

for an arbitrary function $f(p, \tilde{q})$.

Suppose that $f(p, \tilde{q})$ is an (weakly) increasing function in p and $\tilde{q}$. For $p_1 < p_2$,

$$\mathbb{E}\left[f(p',\tilde{q}')|p_1,\tilde{q}\right] \leq \mathbb{E}\left[f(p',\tilde{q}')|p_2,\tilde{q}\right]\ ,$$

since $p'|p_2$ first-order stochastically dominates $p'|p_1$ asymptotically as $dt \to 0$, and the distribution of $\tilde{q}'$ is unaffected. As a result,

$$\begin{aligned} &\mathbb{T}f(p_1,\tilde{q}) = \max\left\{(\max\{p_1 A, f\tilde{q}\} - \phi)\,dt + (1 - rdt - \lambda dt)\mathbb{E}\left[f(p',\tilde{q}')|p_1\right], 0\right\} \\ \leq\ &\mathbb{T}f(p_2,\tilde{q}) = \max\left\{(\max\{p_2 A, f\tilde{q}\} - \phi)\,dt + (1 - rdt - \lambda dt)\mathbb{E}\left[f(p',\tilde{q}')|p_2\right], 0\right\}\ , \end{aligned}$$

since $\max\{p_1 A, f\tilde{q}\} \leq \max\{p_2 A, f\tilde{q}\}$. Hence, $\mathbb{T}$ maps an increasing function in p to an increasing function in p. Similarly, for $\tilde{q}_1 < \tilde{q}_2$,

$$\mathbb{E}\left[f(p', \tilde{q}')|p, \tilde{q}_1\right] \leq \mathbb{E}\left[f(p', \tilde{q}')|p, \tilde{q}_2\right] ,$$

since $\tilde{q}'|\tilde{q}_2$ first-order stochastically dominates $\tilde{q}'|\tilde{q}_1$ asymptotically as $dt \to 0$, and the distribution of p' is unaffected. As a result, together with $\max\{pA, f\tilde{q}_1\} \leq \max\{pA, f\tilde{q}_2\}$,

$$\mathbb{T}f(p, \tilde{q}_1) \leq \mathbb{T}f(p, \tilde{q}_2) .$$

Hence, $\mathbb{T}$ maps an increasing function in $\tilde{q}$ to an increasing function in $\tilde{q}$. Since $f(p, \tilde{q})$ is increasing in p and $\tilde{q}$,

$$V(p, \tilde{q}) = \lim_{k \to \infty} \mathbb{T}^k f(p, \tilde{q})$$

is increasing in p and $\tilde{q}$ as well.

Proof of Lemma 3.2

Given the same map $\mathbb{T}$ as in (3.21), suppose that $f(p, \tilde{q})$ is (weakly) convex in the direction of $(sp(1-p), \sigma\sqrt{\tilde{q}})$ at each $(p, \tilde{q})$. For an arbitrary point $(p, \tilde{q})$, define

$$(p_1, \tilde{q}_1) = (p, \tilde{q}) - (\delta p, \delta \tilde{q}) , \quad (p_2, \tilde{q}_2) = (p, \tilde{q}) + (\delta p, \delta \tilde{q}) ,$$

where δp and $\delta \tilde{q}$ are infinitesimal and

$$(\delta p, \delta \tilde{q}) \propto (sp(1-p), \sigma_z \sqrt{\tilde{q}}) .$$

The order of magnitude of δp and $\delta \tilde{q}$ is set to be the same as or smaller than that of dt. This assumption on the magnitude of δp and $\delta \tilde{q}$ guarantees that first- and second-order derivatives of $V(p, \tilde{q})$ are constants in an infinitesimal region within δp and $\delta \tilde{q}$.

I first show that the following holds:

$$\mathbb{E}\left[f(p', \tilde{q}')|p, \tilde{q}\right] \leq \frac{1}{2}\mathbb{E}\left[f(p', \tilde{q}')|p_1, \tilde{q}_1\right] + \frac{1}{2}\mathbb{E}\left[f(p', \tilde{q}')|p_2, \tilde{q}_2\right] ,$$

i.e., $\mathbb{E}[f(p',\tilde{q}')|p,\tilde{q}]$ is locally convex in the direction of $(sp(1-p),\sigma\sqrt{\tilde{q}})$. Expanding the LHS yields

$$\begin{aligned}
\mathbb{E}[f(p',\tilde{q}')|p,\tilde{q}] &= \mathbb{E}\left[f\left(p+sp(1-p)dZ,\tilde{q}+(b-\eta\tilde{q})dt+\sqrt{\tilde{q}}\left(\sigma dW+\sigma_z d\tilde{Z}\right)\right)\right] \\
&= f(p,\tilde{q})+\left[\frac{\partial f(p,\tilde{q})}{\partial\tilde{q}}(b-\eta\tilde{q})+\frac{1}{2}\frac{\partial^2 f(p,\tilde{q})}{\partial p^2}s^2p^2(1-p)^2\right. \\
&\left.+\frac{1}{2}\frac{\partial^2 f(p,\tilde{q})}{\partial\tilde{q}^2}(\sigma^2+\sigma_z^2)\tilde{q}+\frac{\partial^2 f(p,\tilde{q})}{\partial p\partial\tilde{q}}sp(1-p)\sigma_z\sqrt{\tilde{q}}\right]dt \\
&+\mathcal{O}(\sqrt{dt}^3)\;.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}[f(p',\tilde{q}')|p_1,\tilde{q}_1]+\frac{1}{2}\mathbb{E}[f(p',\tilde{q}')|p_2,\tilde{q}_2] \\
=\;&\frac{1}{2}\left(f(p_1,\tilde{q}_1)+f(p_2,\tilde{q}_2)\right)+\left[\frac{\partial f(p,\tilde{q})}{\partial\tilde{q}}(b-\eta\tilde{q})+\frac{1}{2}\frac{\partial^2 f(p,\tilde{q})}{\partial p^2}s^2p^2(1-p)^2\right. \\
&\left.+\frac{1}{2}\frac{\partial^2 f(p,\tilde{q})}{\partial\tilde{q}^2}(\sigma^2+\sigma_z^2)\tilde{q}+\frac{\partial^2 f(p,\tilde{q})}{\partial p\partial\tilde{q}}sp(1-p)\sigma_z\sqrt{\tilde{q}}\right]dt+\mathcal{O}(\sqrt{dt}^3)\}\;.
\end{aligned}$$

If $f(p,\tilde{q})$ is strictly convex at $(p,\tilde{q})$ in the direction of diffusion $d\tilde{Z}$ in p and $\tilde{q}$, i.e.,

$$\begin{bmatrix} sp(1-p) & \sigma_z\sqrt{\tilde{q}} \end{bmatrix}\begin{bmatrix} \frac{\partial^2 f(p,\tilde{q})}{\partial p^2} & \frac{\partial^2 f(p,\tilde{q})}{\partial p\partial\tilde{q}} \\ \frac{\partial^2 f(p,\tilde{q})}{\partial p\partial\tilde{q}} & \frac{\partial^2 f(p,\tilde{q})}{\partial\tilde{q}^2} \end{bmatrix}\begin{bmatrix} sp(1-p) \\ \sigma_z\sqrt{\tilde{q}} \end{bmatrix}>0\,,$$

the following is (strictly) positive:

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}[f(p',\tilde{q}')|p_1,\tilde{q}_1]+\frac{1}{2}\mathbb{E}[f(p',\tilde{q}')|p_2,\tilde{q}_2]-\mathbb{E}[f(p',\tilde{q}')|p,\tilde{q}] \\
=\;&\frac{1}{2}\left[\frac{\partial^2 f(p,\tilde{q})}{\partial p^2}\delta p^2+2\frac{\partial^2 f(p,\tilde{q})}{\partial p\partial\tilde{q}}\delta p\delta\tilde{q}+\frac{\partial^2 f(p,\tilde{q})}{\partial\tilde{q}^2}\delta\tilde{q}^2\right] \\
&+\max\{\mathcal{O}(\delta p^2 dt),\mathcal{O}(\delta\tilde{q}^2 dt)\}\;.
\end{aligned}$$

If $f(p,\tilde{q})$ is flat at $(p,\tilde{q})$ in the direction of the diffusion $d\tilde{Z}$ in p and $\tilde{q}$, i.e.,

$$\begin{bmatrix} sp(1-p) & \sigma_z\sqrt{\tilde{q}} \end{bmatrix}\begin{bmatrix} \frac{\partial^2 f(p,\tilde{q})}{\partial p^2} & \frac{\partial^2 f(p,\tilde{q})}{\partial p\partial\tilde{q}} \\ \frac{\partial^2 f(p,\tilde{q})}{\partial p\partial\tilde{q}} & \frac{\partial^2 f(p,\tilde{q})}{\partial\tilde{q}^2} \end{bmatrix}\begin{bmatrix} sp(1-p) \\ \sigma_z\sqrt{\tilde{q}} \end{bmatrix}=0\,,$$

then

$$
\begin{aligned}
& \frac{1}{2}\mathbb{E}\left[f(p',\tilde{q}')|p_1,\tilde{q}_1\right] + \frac{1}{2}\mathbb{E}\left[f(p',\tilde{q}')|p_2,\tilde{q}_2\right] - \mathbb{E}\left[f(p',\tilde{q}')|p,\tilde{q}\right] \\
= \; & \frac{1}{2}\frac{\partial f(p,\tilde{q})}{\partial \tilde{q}}\left((b-\eta\tilde{q}_1)+(b-\eta\tilde{q}_2)-2(b-\eta\tilde{q})\right)dt \\
& +\frac{1}{4}\frac{\partial^2 f(p,\tilde{q})}{\partial \tilde{q}^2}\sigma^2\left(\tilde{q}_1+\tilde{q}_2-2\tilde{q}\right)dt + \max\{\mathcal{O}(\delta p^2\sqrt{dt}^3),\mathcal{O}(\delta\tilde{q}^2\sqrt{dt}^3)\} \\
= \; & \max\{\mathcal{O}(\delta p^2\sqrt{dt}^3),\mathcal{O}(\delta\tilde{q}^2\sqrt{dt}^3)\}\;.
\end{aligned}
$$

Hence, the second-order derivative of $\mathbb{E}\left[f(p',\tilde{q}')|p,\tilde{q}\right]$ in the direction of $(sp(1-p),\sigma_z\sqrt{\tilde{q}})$ is at the order of $\sqrt{dt}^3$ or smaller. In the continuous-time limit where terms of the order of dt dominate, the second-order derivative is zero. Therefore, $\mathbb{E}\left[f(p',\tilde{q}')|p,\tilde{q}\right]$ is convex in the direction of $(sp(1-p),\sigma_z\sqrt{\tilde{q}})$.

Note that this logic cannot be applied to $(\delta p,\delta\tilde{q})$ that is orthogonal to $(sp(1-p),\sigma_z\sqrt{\tilde{q}})$. If $f(p,\tilde{q})$ is strictly convex in the direction of $(sp(1-p),\sigma_z\sqrt{\tilde{q}})$, but flat in the orthogonal direction, the second-order derivative of $\mathbb{E}\left[f(p',\tilde{q}')|p,\tilde{q}\right]$ in the orthogonal direction may be negative at the order of dt. Although one starts from strictly convex $f(p,\tilde{q})$ (in all directions) in order to avoid such a situation, the limit $\mathbb{T}^k f(p,\tilde{q})$ as $k\to\infty$ may be weakly convex, since the set of strictly convex functions is not closed. Hence, $\mathbb{E}\left[\mathbb{T}^k f(p',\tilde{q}')|p,\tilde{q}\right]$ may be locally concave in the direction that is orthogonal to $(sp(1-p),\sigma_z\sqrt{\tilde{q}})$.

Having proven that $\mathbb{E}\left[f(p',\tilde{q}')|p,\tilde{q}\right]$ is convex in the direction of $(sp(1-p),\sigma_z\sqrt{\tilde{q}})$ if $f(p,\tilde{q})$ is convex in that direction, I next prove that $\mathbb{T}f(p,\tilde{q})$ is convex in the same direction. Since

$$\max\{pA, f\tilde{q}\}$$

is a convex function,

$$\left(\max\{pA,f\tilde{q}\}-\phi\right)dt + (1-rdt-\lambda dt)\mathbb{E}\left[f(p',\tilde{q}')\right]$$

is convex in the direction of $(sp(1-p),\sigma_z\sqrt{\tilde{q}})$. Since taking the pointwise maximum of two functions preserves convexity,

$$\mathbb{T}f(p,\tilde{q}) = \max\left\{(\max\{pA, f\tilde{q}\} - \phi)\, dt + (1 - rdt - \lambda dt)\mathbb{E}\left[f(p',\tilde{q}')|p,\tilde{q}\right], 0\right\}$$

is convex in the direction of $(sp(1-p), \sigma_z\sqrt{\tilde{q}})$ in the continuous-time limit.

Since the map $\mathbb{T}$ preserves the convexity in the direction of the diffusion $d\tilde{Z}$ in p and $\tilde{q}$,

$$V(p,\tilde{q}) = \lim_{k\to\infty} \mathbb{T}^k f(p,\tilde{q})$$

is convex in the direction of $(sp(1-p), \sigma_z\sqrt{\tilde{q}})$.

Proof of Proposition 3.1

Since $V(p,\tilde{q})$ is nonnegative and increasing in p and $\tilde{q}$, if $V(p',\tilde{q}') = 0$, $V(p'',\tilde{q}'') = 0$ for all $p'' \leq p'$ and $\tilde{q}'' \leq \tilde{q}'$. Suppose that $(p_1, \tilde{q}_1)$ is on the curve $\tilde{q} = h(p)$, i.e., $V(p_1, \tilde{q}') > 0$ for all $\tilde{q}' > \tilde{q}_1$ and $V(p_1,\tilde{q}') = 0$ for all $\tilde{q}' \leq \tilde{q}_1$. In addition, suppose that the curve is (strictly) upward sloping at $(p_1,\tilde{q}_1)$, i.e., there exists $(p_2,\tilde{q}_2)$ on the curve $\tilde{q} = h(p)$, where $p_2 > p_1$ and $\tilde{q}_2 > \tilde{q}_1$. However, this implies $V(p_1,\tilde{q}_2) = 0$, which leads to a contradiction. Therefore, $\tilde{q} = h(p)$ is downward-sloping.

Since $\lim_{\tilde{q}\to\infty} V(p,\tilde{q}) \to \infty$, the continuity of $V(p,\tilde{q})$ implies that there exists $\tilde{q}'$ such that $V(0,\tilde{q}') > 0$. Therefore, $\tilde{q} = h(p)$ crosses the $\tilde{q}$-axis at some $\tilde{q} < \tilde{q}'$. If $\tilde{q} = h(p)$ does not cross the p-axis, $V(p, 0)$ is uniformly zero. This implies that there is no entry, which contradicts a stationary equilibrium. Therefore, $\tilde{q} = h(p)$ crosses the p-axis at some $0 < p < 1$.

Proof of Corollary 3.1

At $\tilde{q} = 0$, the direction of diffusion $d\tilde{Z}$ is in the direction of p-axis. Therefore, Lemma 3.2 implies that $V(p, 0)$ is convex in p. p_{ex} does not exist if and only if the set E is empty by Proposition 3.1.

If p_{ex} does not exist, I first show that $V(\delta p, 0) > V(0,0)$ for an arbitrary small δp.

In order to show this, note that

$$V(0,0) = \mathbb{E}\left[\int_0^{T^D} e^{-ru}\left(f\tilde{q}_u - \phi\right) du \middle| \tilde{q}_0 = 0\right] ,$$

where the right-hand side is strictly greater than zero from the definition of E. Since E is empty, T^D is solely determined by the Poisson (exogenous) exit process. Similarly,

$$V(\delta p, 0) = \mathbb{E}\left[\int_0^{T^D} e^{-ru}\left(\max\{p_u A, f\tilde{q}_u\} - \phi\right) du \middle| p_0 = \delta p, \tilde{q}_0 = 0\right] .$$

Since E is empty, the process that determines T^D is the same as that of $V(0,0)$. Since the process of $\tilde{q}_u$ is independent of p_0, and the distribution of p_u given $p_0 = \delta p$ has a positive probability for $p_u > 0$,

$$V(\delta p, 0) > V(0,0) ,$$

for an arbitrary small δp. Therefore, $V(p,0)$ is strictly increasing in p at $p = 0$, if E is empty. Then, the convexity of $V(p,0)$ yields

$$\frac{\partial V(p,0)}{\partial p} > 0 , \quad \forall p > 0 ,$$

This proves that $V(p,0)$ is strictly increasing if p_{ex} does not exist.

If p_{ex} exists, by the definition of p_{ex}, for $p > p_{ex}$,

$$V(p,0) > V(p_{ex}, 0) = 0 .$$

This implies

$$\frac{\partial V(p,0)}{\partial p} > 0 , \quad \forall p > p_{ex} ,$$

due to the convexity of $V(p,0)$. Therefore, $V(p,0)$ is strictly increasing for $p \geq p_{ex}$.

Proof of Proposition 3.2

Suppose p^* does not exist, i.e., there is no p satisfying

$$V(p, 0) = \Phi \, .$$

Since $V(p, 0)$ is (weakly) increasing, this implies that either $V(p, 0) < \Phi$ for all $p \in [0, 1]$, or $V(p, 0) > \Phi$ for all $p \in [0, 1]$.

If $V(p, 0) < \Phi$ for all p, the participation constraint (3.7) does not hold for any prospective managers. In this case, there is no entry and, as a result, no managers remain in the AM industry since existing managers eventually exit. This is inconsistent with a stationary equilibrium because of assumption (3.5).

If $V(p, 0) > \Phi$ for all p, the participation constraint (3.7) holds for all prospective managers. In this case, an infinite number (measure) of prospective managers enter AM markets because of assumption (3.3). Since $V(p, \tilde{q}) > \Phi > 0$, funds do not choose to exit and, as a result, the number (measure) of existing managers becomes infinite. This is inconsistent with a stationary equilibrium because of assumption (3.5).

Given the existence of p^*, the uniqueness naturally follows from Corollary 3.1. Since $V(p, 0)$ is strictly increasing for $V(p, 0) > 0$, if p^* exists, i.e., $V(p^*, 0) = \Phi > 0$, it is unique. Since $V(p, 0)$ is increasing, any $p \in [p^*, 1]$ satisfies the participation constraint (3.7).

Proof of Corollary 3.2

As $t_i \to \infty$, $p_{i,t}$ converges to 1 almost surely. If p_{ex} exists, since $p_{ex} < p^* < 1$ by Proposition 3.2 and $V(1, \tilde{q}) \geq V(p, \tilde{q}) \geq V(p, 0)$ by Lemma 3.1, $V(1, \tilde{q}) > 0$ by Corollary 3.1 and the fund never chooses to exit. If p_{ex} does not exist, the set E defined by (3.6) is empty by Proposition 3.1, and any fund never chooses to exit.

Proof of Lemma 3.3

Consider $A_1 > A_2$. Since the process of p_t and $\tilde{q}_t$ do not depend on the value of A, fee profits for A_1 at any $s > t$

$$\max\{p_s A_1, f\tilde{q}_s\} - \phi$$

first-order stochastically dominate those for A_2. Hence, for any $(p_t, \tilde{q}_t)$, the value of fee profits for A_1 is greater than the value of fee profits for A_2, i.e.,

$$V(p_t, \tilde{q}_t; A_1, b) \geq V(p_t, \tilde{q}_t; A_2, b) \ . \tag{3.22}$$

Therefore, the set $E(A_1, b)$ defined by (3.6) is a subset of $E(A_2, b)$. Suppose this is not the case, i.e., there exists $(p', \tilde{q}')$ such that

$$V(p', \tilde{q}'; A_1, b) > 0 \ , \quad V(p', \tilde{q}'; A_2, b) = 0 \ ,$$

which violates (3.22), and is a contradiction.

Similarly, consider $b_1 > b_2$. Given $\tilde{q}_t$,

$$\tilde{q}_s(b) = \tilde{q}_t + \int_t^s (b - \eta\tilde{q}_u)du + \int_t^s \sqrt{\tilde{q}_u}\left(\sigma dW_u + \sigma_z d\tilde{Z}_u\right) \ , \quad \forall s > t \ .$$

$\tilde{q}_s(b_1)$ first-order stochastically dominates $\tilde{q}_s(b_2)$ for all $s > t$. As a result, fee profits for b_1 at any $s > t$

$$\max\{p_s A, f\tilde{q}_s(b_1)\} - \phi$$

first-order stochastically dominates those for b_2. Hence, for any $\tilde{q}_t$, the value of fee profits for b_1 is greater than the value of fee profits for b_2, i.e.,

$$V(p_t, \tilde{q}_t; A, b_1) \geq V(p_t, \tilde{q}_t; A, b_2) \ . \tag{3.23}$$

Therefore, the set $E(A, b_1)$ is a subset of $E(A, b_2)$. Suppose this is not the case, i.e., there exists $(p', \tilde{q}')$ such that

$$V(p', \tilde{q}'; A, b_1) > 0 \ , \quad V(p', \tilde{q}'; A, b_2) = 0 \ ,$$

which violates (3.23), and is a contradiction. Consequently, for all $A' \geq A$ and $b' \geq b$,

$$E(A', b') \subseteq E(A', b) \subseteq E(A, b) \ .$$

Proof of Proposition 3.3

Given $\tilde{q}_t$[9],

$$\tilde{q}_s(b) = \tilde{q}_t + \int_t^s (b - \eta\tilde{q}_u)du + \int_t^s \sqrt{\tilde{q}_u}\,(\sigma dW_u + \sigma_z dZ_u) \ , \quad \forall s > t \ .$$

For $b_1 > b_2$, $\tilde{q}_s(b_1)$ first-order stochastically dominates $\tilde{q}_s(b_2)$ for all $s > t$. In addition, by Lemma 3.3, the exit threshold at $p = 0$ is lower for b_1 than for b_2. Therefore, since $\tilde{q}_s(b_1)$ first-order stochastically dominates $\tilde{q}_s(b_2)$, and the exit threshold for $\tilde{q}_s(b_1)$ is lower than that for $\tilde{q}_s(b_2)$, the probability of an unskilled manager's survival is higher for b_1 than that is for b_2, at any $s > t$.

Denote the cumulative probability distribution function of survival time, conditional on the survival at t, by $F(u|t)$, where the support is (t, ∞). The probability of an unskilled manager's survival at $s > t$ (conditional on the survival at t) is closely related to the distribution of the survival time in the following way:

$$P(\text{survival between } t \text{ and } s) = 1 - F(s|t) \ .$$

Therefore, the statement that the probability of an unskilled manager's survival is higher for b_1 than that is for b_2 at any $s > t$ is equivalent to

$$F(s|t; A_1, b_1) \leq F(s|t; A_2, b_2) \ , \quad \forall s > t \ ,$$

[9]In this equation, the physical Brownian motion Z_t is used instead of the "perceived" Brownian motion $\tilde{Z}_t$, because the physical distribution is determined by Z_t, not $\tilde{Z}_t$. At $p_t = 0$, $dZ_t = d\tilde{Z}_t$ for unskilled managers by (1.1).

i.e., the survival time of surely unskilled managers for b_1 first-order stochastically dominates that for b_2, given $\tilde{q}_t$.

Proof of Corollary 3.3

Given the prior skill p_0 fixed, the process of p_t, conditional on survival, is independent of A and b. The process of $\tilde{q}_t$ does not depend on A, but does depend on b. Given $\tilde{q}_0 = 0$,

$$\tilde{q}_t(b) = \int_0^t (b - \eta \tilde{q}_u) du + \int_0^t \sqrt{\tilde{q}_u} \left(\sigma dW_u + \sigma_z(-sp_u du + dZ_u) \right) \ , \quad \forall t > 0 \ ,$$

by (1.1). Hence, $\tilde{q}_t(b_1)$ first-order stochastically dominates $\tilde{q}_t(b_2)$ for all $t > 0$. In addition, by Lemma 3.3, $E(A_1, b_1)$ is a subset of $E(A_2, b_2)$. Therefore, the probability of an unskilled manager's survival for (A_1, b_1) is higher than that for (A_2, b_2) at any point of time.

Proof of Proposition 3.4

Since $b_1 > b_2$, regardless of the values of A_1 and A_2, there exists p_ϵ such that $h_1(p) < h_2(p)$ for all $0 \leq p < p_\epsilon$, where $h(p)$ is defined by Proposition 3.1. Consider an arbitrary small p_ϵ. The perceived skill p (the probability of being H-type) of unskilled managers converges to 0 almost surely as their survival time grows to infinity. This implies that there exists T_ϵ such that for all $s > T_\epsilon$ after entry, unskilled managers' p_s lies between $p = 0$ and $p = p_\epsilon$, conditional on survival, with probability $1 - \epsilon$ for an arbitrary small ϵ, both for economy 1 and economy 2.

Now consider the following hypothetical exit rule: from T_ϵ, unskilled managers exit if their $\tilde{q}_s$ is smaller than $h(p)$ for $s = T_\epsilon + k\Delta T$ for $k = 1, 2, \cdots$. The hypothetical exit rate is strictly lower than the true exit rate, because the true exit rule imposes that unskilled managers exit if their $\tilde{q}_s$ is smaller than $h(p)$ for all $s > T_\epsilon$. Since the CIR process is an ergodic process, for an arbitrarily small δ, there exists sufficiently large ΔT such that, under the hypothetical rule, for all k,

$$\begin{aligned}&\tilde{P}(\text{survival between } T_\epsilon + (k-1)\Delta T \text{ and } T_\epsilon + k\Delta T)\\ = \; &(1-\epsilon)\left(1 - \frac{1}{\Gamma(\frac{2b}{\sigma^2})}\gamma\left(\frac{2b}{\sigma^2}, \frac{2\eta}{\sigma^2}h(0)\right)\right) + \mathcal{O}(\epsilon) + \mathcal{O}(\delta) + \mathcal{O}(p_\epsilon) < 1\,,\end{aligned}$$

where γ is the lower incomplete gamma function, and $\tilde{P}$ indicates probability under the hypothetical exit rule. Therefore, under the hypothetical rule, for sufficiently small ϵ and p_ϵ, as T grows to infinity

$$\tilde{P}(\text{survival between } T_\epsilon \text{ and } T) \sim \left(1 - \frac{1}{\Gamma(\frac{2b}{\sigma^2})}\gamma\left(\frac{2b}{\sigma^2}, \frac{2\eta}{\sigma^2}h(0)\right) + \mathcal{O}(\delta)\right)^{\frac{T-T_\epsilon}{\Delta T}}\,.$$

Therefore, under the hypothetical exit rule, the survival probability decreases exponentially in time, where the exponent is

$$\frac{T-T_\epsilon}{\Delta T}\log\left(1 - \frac{1}{\Gamma(\frac{2b}{\sigma^2})}\gamma\left(\frac{2b}{\sigma^2}, \frac{2\eta}{\sigma^2}h(0)\right) + \mathcal{O}(\delta)\right)\,.$$

The actual survival rate is strictly lower than the hypothetical survival rate. Hence, the actual survival probability asymptotically decreases exponentially or faster than exponential functions as T becomes large. Since $b_1 > b_2$, $h_1(0) < h_2(0)$, and the asymptotic attrition rate (for large T) is strictly lower for b_1 than that for b_2. Therefore, regardless of the survival probability at T_ϵ, there exists T such that the survival probability of unskilled managers for b_1 is greater than that for b_2 for all $s > T$.

Proof of Corollary 3.4

$V(0,\tilde{q})$ solves the following ODE:

$$(r+\lambda)V(0,\tilde{q}) = (f\tilde{q} - \phi) + \frac{\partial V(0,\tilde{q})}{\partial \tilde{q}}(b - \eta\tilde{q}) + \frac{1}{2}\frac{\partial^2 V(0,\tilde{q})}{\partial \tilde{q}^2}(\sigma^2 + \sigma_z^2)\tilde{q}\,.$$

The particular solution is

$$V^b(0,\tilde{q}) = \frac{1}{r+\lambda+\eta}\left(f\tilde{q} - \frac{fb}{\eta}\right) + \frac{1}{r+\lambda}\left(\frac{fb}{\eta} - \phi\right) ,$$

and the homogeneous solution is

$$V^o(0,\tilde{q}) = c_1 M\left(\frac{r+\lambda}{\eta}, \frac{2b}{\sigma^2+\sigma_z^2}, \frac{2\eta}{\sigma^2+\sigma_z^2}\tilde{q}\right) + c_2 U\left(\frac{r+\lambda}{\eta}, \frac{2b}{\sigma^2+\sigma_z^2}, \frac{2\eta}{\sigma^2+\sigma_z^2}\tilde{q}\right) ,$$

where $M(a_1,a_2,x)$ is the Kummer's function, and $U(a_1,a_2,x)$ is the Tricomi confluent hypergeometric function. Note that c_1 and c_2 are nonnegative because the particular solution $V^b(0,\tilde{q})$ is the hypothetical value of fee profits if the manager does not take any action. The manager can always choose to exit, and $V^o(0,\tilde{q})$ represents the value of the exit option.

For large $\tilde{q}$

$$V(0,\tilde{q}) \sim \frac{1}{r+\lambda+\eta}f\tilde{q} ,$$

and this implies that $c_1 = 0$. For $b > \frac{(r+\lambda+\eta)\phi}{f}$,

$$V^b(0,\tilde{q}) > 0 , \quad \forall\tilde{q} \geq 0 .$$

This implies that $c_2 = 0$, since $c_2 > 0$ suggests that $\lim_{\tilde{q}\to 0} V(0,\tilde{q}) \to \infty$. Therefore, for $b > \frac{(r+\lambda+\eta)\phi}{f}$,

$$V(0,\tilde{q}) = V^b(0,\tilde{q}) = \frac{1}{r+\lambda+\eta}\left(f\tilde{q} - \frac{fb}{\eta}\right) + \frac{1}{r+\lambda}\left(\frac{fb}{\eta} - \phi\right) > 0 , \quad \forall\tilde{q} \geq 0 .$$

This implies, since $V(p,\tilde{q})$ is increasing in p and $\tilde{q}$,

$$V(p,\tilde{q}) > 0 , \quad \forall p \geq 0 , \ \forall\tilde{q} \geq 0 .$$

Therefore, the exit set E is empty, and no managers choose to exit.

Proof of Proposition 3.5

In the proof of Lemma 3.3,

$$V(p, \tilde{q}; A_1, b_1) \geq V(p, \tilde{q}; A_2, b_2) \tag{3.24}$$

is shown. By the definition (3.9) of p^*, the following holds:

$$p^*(A_1, b_1) \leq p^*(A_2, b_2) \ .$$

Suppose this is not the case, i.e., $p_1 \equiv p^*(A_1, b_1) > p_2 \equiv p^*(A_2, b_2)$. Then

$$V(p_1, 0; A_1, b_1) = \Phi = V(p_2, 0; A_2, b_2) < V(p_1, 0; A_2, b_2) \ ,$$

where the inequality comes from Corollary 3.1. This contradicts (3.24).

Therefore, p^* is lower for (A_1, b_1) than for (A_2, b_2). By Proposition 3.2, prospective managers with prior skill $p \in [p^*, 1]$ choose to enter the AM industry. The measure of entries is given by $G(1 - p^*)$, where $G(1 - p) \equiv \int_p^1 F'(p')dp'$ and $F(p)$ is the cumulative distribution of the prior skill of prospective managers. Define $g(x) = G'(x) = F'(1 - x)$. Then the measure of entries of skilled managers and that of unskilled managers are, respectively,

$$n_H = \int_{p^*}^1 pg(1 - p)dp \ , \quad n_L = \int_{p^*}^1 (1 - p)g(1 - p)dp \ .$$

Since $g(x) \geq 0$ for all $x \in [0, 1]$, n_H and n_L are both decreasing in p^*. $p^*(A_1, b_1) \leq p^*(A_2, b_2)$ implies that

$$n_H(A_1, b_1) \geq n_H(A_2, b_2) \ , \quad n_L(A_1, b_1) \geq n_L(A_2, b_2) \ .$$

Therefore, the measure of entries, that of skilled managers and that of unskilled managers all increase in A and b.

Given p^*, the portion of unskilled managers is lower than $1-p^*$:

$$\frac{n_L}{n} = \frac{n_L}{n_L + n_H} = \frac{\int_{p^*}^1 (1-p)g(1-p)dp}{\int_{p^*}^1 g(1-p)dp} \le 1 - p^* \ .$$

When p^* is lowered by an infinitesimal amount δp^*, the portion of unskilled managers among the marginally entering managers is $1-p^*$. Therefore, lower p^* leads to a higher portion of unskilled managers among newly entering managers.

Proof of Proposition 3.6

Consider an increase in $\bar{A}$: $\bar{A} \to \bar{A}'$. Suppose that the number N' of active managers does not change in the new stationary equilibrium. This implies that

$$A' = \bar{A}' h_A(N) > A \ , \quad b' = \bar{b} h_b(N) = b \ ,$$

i.e., A increases and b is unchanged. By Proposition 3.5, the entry threshold p^* decreases, and by Lemma 3.3, the exit set E diminishes. Since changes in A does not change the process of p_t and $\tilde{q}_t$, there are more entries and less exits compared with the stationary equilibrium for $\bar{A}$. This contradicts the assumption that the number of active managers does not change.

Now suppose that the number of managers increases such that A' is the same as A. Denote this number by N''. This implies

$$A' = \bar{A}' h_A(N'') = A \ , \quad b' = \bar{b} h_b(N'') < b \ ,$$

i.e., A is unchanged and b decreases. The entry threshold p^* increases, and the exit set E expands. The process of p_t does not change, but the process of $\tilde{q}_t(b')$ is first-order stochastically dominated by $\tilde{q}_t(b)$. Hence, there are less entries and more exits compared with the stationary equilibrium for $\bar{A}$. This contradicts the assumption that N increases.

Therefore, in the new stationary equilibrium, $N < N' < N''$ must be satisfied.

Consequently, $A' > A$ and $b' < b$ in the stationary equilibrium.

Proof of Proposition 3.7

Consider an infinitesimal increase in $\bar{A}$: $\bar{A}' = \bar{A} + \delta\bar{A}$ where $\delta\bar{A} > 0$. Let p^* be the initial entry threshold and $p' = p^* + \delta p$ be the entry threshold after the increase in $\bar{A}$, where the sign of δp is not determined. Now assume the following hypothetical entry rule: prospective managers whose perceived skill is above p^* choose to enter. This hypothetical entry rule is different from the actual entry rule under $\bar{A}'$ where prospective managers whose perceived skill is above p' choose to enter. The parameter values A and b at N^* is

$$A = \bar{A}' h_A(N^*) > \bar{A} h_A(N^*)\ , \quad b = \bar{b} h_b(N^*)\ . \tag{3.25}$$

Assume the hypothetical exit rule where the exit set $E(A, b)$ defined by (3.6) is characterized by above A and b. Under the hypothetical entry and exit rules, the stationary measure $\tilde{N}$ of active managers is greater than N^*:

$$\tilde{N} = N^* + \delta\tilde{N}\ , \quad \delta\tilde{N} > 0\ ,$$

because the hypothetical entry rule is the same as that for $\bar{A}$, and the hypothetical exit set E is strictly smaller than that for $\bar{A}$. The parameter values A and b at $\tilde{N}$ are

$$\begin{aligned} A &= \bar{A}'(h_A(N^*) + h_A'(N^*)\delta\tilde{N}) = \bar{A} h_A(N^*) + \delta\bar{A} h_A(N^*) + \bar{A} h_A'(N^*)\delta\tilde{N}\ , \\ b &= \bar{b}(h_b(N^*) + h_b'(N^*)\delta\tilde{N}) = \bar{b} h_b(N^*) + \bar{b} h_b'(N^*)\delta\tilde{N}\ . \end{aligned}$$

Since the entry threshold $p^*(A, b)$ is strictly decreasing in both A and b, there exist $c_A > 0$ and $c_b > 0$ such that

$$p^* \equiv p^*(\bar{A} h_A(N^*), \bar{b} h_b(N^*)) = p^*(\bar{A} h_A(N^*) + \delta\bar{A} h_A(N^*) - \bar{A} c_A \delta\tilde{N}, \bar{b} h_b(N^*) - \bar{b} c_b \delta\tilde{N})\ .$$

Note that there are infinite numbers of (c_A, c_b) that satisfies this condition. Given $|h'_A(N^*)| \leq c_A$ and $|h'_b(N^*)| \leq c_b$, $p^*(A, b)$, where A and b are determined under the hypothetical entry and exit rule, is weakly lower than $p^*(\bar{A}h_A(N^*), \bar{b}h_b(N^*))$.

Now, consider another hypothetical entry and exit rule: prospective managers whose perceived skill is above p^* choose to enter, and the exit rule is the same as the actual exit rule for $\bar{A}'$. Since the actual exit set E for $\bar{A}'$ is strictly greater than the hypothetical exit set $E(A, b)$ determined by (3.25), the new hypothetical measure $\tilde{N}'$ of active managers is strictly less than $\tilde{N}$. This implies

$$p^*(\bar{A}h_A(N^*), \bar{b}h_b(N^*)) \geq p^*(\bar{A}'h_A(\tilde{N}), \bar{b}h_b(\tilde{N})) > p^*(\bar{A}'h_A(\tilde{N}'), \bar{b}h_b(\tilde{N}')) \ .$$

This implies that the actual p' is strictly lower than p^*, i.e., $\delta p < 0$.

Proof of Corollary 3.5

When the exit set E is empty, no managers choose to exit. Therefore, in this case, the number N of active mangers is determined by

$$N = \frac{G(1 - p^*)}{\eta} \ ,$$

where p^* is the entry threshold, and $G(\cdot)$ is defined by (3.2). Since the number N of active managers in the AM industry increases as $\bar{A}$ increases by Proposition 3.6, the number of entries must increase as well. Therefore, p^* decreases as $\bar{A}$ increases.

Proof of Proposition 3.8

Consider an increase in $\bar{b}$: $\bar{b} \to \bar{b}'$. Suppose that the number N' of active managers does not change in the new stationary equilibrium. This implies that

$$A' = \bar{A}h_A(N) = A \ , \quad b' = \bar{b}'h_b(N) > b \ ,$$

i.e., A is unchanged and b increases. By Proposition 3.5, the entry threshold p^* decreases, and by Lemma 3.3, the exit set E diminishes. Since an increase in b does not change the process of p_t, but makes the process of $\tilde{q}_t$ first-order stochastically dominant, there are more entries and less exits compared with the stationary equilibrium for $\bar{b}$. This contradicts the assumption that N does not change.

Now suppose that the number of managers increases such that b' is the same as b. Denote this number by N''. This implies

$$A' = \bar{A}h_A(N'') < A \ , \quad b' = \bar{b}'h_b(N'') = b \ ,$$

i.e., A decreases and b is unchanged. The entry threshold p^* increases, and the exit set E expands. The process of p_t and $\tilde{q}_t$ do not change. Hence, there are less entries and more exits compared with the stationary equilibrium for $\bar{b}$. This contradicts the assumption that N increases.

Therefore, in the new stationary equilibrium, $N < N' < N''$ must be satisfied. Consequently, $A' < A$ and $b' > b$ in the stationary equilibrium.

Proof of Proposition 3.9

Consider an infinitesimal increase in $\bar{b}$: $\bar{b}' = \bar{b} + \delta\bar{b}$ where $\delta\bar{b} > 0$. Let p^* be the initial entry threshold and $p' = p^* + \delta p$ be the entry threshold after the increase in $\bar{b}$, where the sign of δp is not determined. Now assume the following hypothetical entry rule: prospective managers whose perceived skill is above p^* choose to enter. This hypothetical entry rule is different from the actual entry rule under $\bar{b}'$ where prospective managers whose perceived skill is above p' choose to enter. The parameter values A and b at N^* is

$$A = \bar{A}h_A(N^*) \ , \quad b = \bar{b}'h_b(N^*) > \bar{b}h_b(N^*) \ . \tag{3.26}$$

Assume the hypothetical exit rule where the exit set $E(A, b)$ defined by (3.6) is characterized by above A and b. Under the hypothetical entry and exit rules, the stationary measure $\tilde{N}$ of active managers is greater than N^*:

$$\tilde{N} = N^* + \delta\tilde{N} \ , \quad \delta\tilde{N} > 0 \ ,$$

because the hypothetical entry rule is the same as that for $\bar{b}$, and the hypothetical exit set E is strictly smaller than that for $\bar{b}$. The parameter values A and b at $\tilde{N}$ are

$$\begin{aligned} A &= \bar{A}(h_A(N^*) + h'_A(N^*)\delta\tilde{N}) = \bar{A}h_A(N^*) + \bar{A}h'_A(N^*)\delta\tilde{N} , \\ b &= \bar{b}'(h_b(N^*) + h'_b(N^*)\delta\tilde{N}) = \bar{b}h_b(N^*) + \delta\bar{b}h_b(N^*) + \bar{b}h'_b(N^*)\delta\tilde{N} . \end{aligned}$$

Since the entry threshold $p^*(A, b)$ is strictly decreasing in both A and b, there exist $c_A > 0$ and $c_b > 0$ such that

$$p^* \equiv p^*(\bar{A}h_A(N^*), \bar{b}h_b(N^*)) = p^*(\bar{A}h_A(N^*) - \bar{A}c_A\delta\tilde{N}, \bar{b}h_b(N^*) + \delta\bar{b}h_b(N^*) - \bar{b}c_b\delta\tilde{N}) .$$

Note that there are infinite numbers of (c_A, c_b) that satisfies this condition. Given $|h'_A(N^*)| \le c_A$ and $|h'_b(N^*)| \le c_b$, $p^*(A, b)$, where A and b are determined under the hypothetical entry and exit rule, is weakly lower than $p^*(\bar{A}h_A(N^*), \bar{b}h_b(N^*))$.

Now, consider another hypothetical entry and exit rule: prospective managers whose perceived skill is above p^* choose to enter, and the exit rule is the same as the actual exit rule for $\bar{b}'$. Since the actual exit set E for $\bar{b}'$ is strictly greater than the hypothetical exit set $E(A, b)$ determined by (3.26), the new hypothetical measure $\tilde{N}'$ of active managers is strictly less than $\tilde{N}$. This implies

$$p^*(\bar{A}h_A(N^*), \bar{b}h_b(N^*)) \ge p^*(\bar{A}h_A(\tilde{N}), \bar{b}'h_b(\tilde{N})) > p^*(\bar{A}h_A(\tilde{N}'), \bar{b}'h_b(\tilde{N}')) .$$

This implies that the actual p' is strictly lower than p^*, i.e., $\delta p < 0$.

Proof of Corollary 3.6

When the exit set E is empty, no managers choose to exit. Therefore, in this case, the number N of active mangers is determined by

$$N = \frac{G(1 - p^*)}{\eta} ,$$

where p^* is the entry threshold, and $G(\cdot)$ is defined by (3.2). Since the number N of active managers in the AM industry increases as $\bar{b}$ increases by Proposition 3.8, the number of entries must increase as well. Therefore, p^* decreases as $\bar{b}$ increases.

Proof of Proposition 3.10

By the definition (3.9) of entry threshold p^*, $p^*(\Phi)$ is a strictly increasing function of Φ from Corollary 3.1, given A and b fixed. Consider an increase in Φ: $\Phi \to \Phi'$. Suppose that the number N' of active managers does not change in the new stationary equilibrium. This implies that both A and b are unchanged as the entry cost increases. However, if this is the case, $p^*(\Phi') > p^*(\Phi)$: the number of entries is smaller than that of the stationary equilibrium before the change of Φ. On the other hand, the exit threshold is unchanged, since A and b are not changed. This contradicts the assumption that the number of active managers does not change.

Now suppose that the number N' of active managers decreases such that the entry threshold does not change. Denote this number of managers by N'', which is smaller than N. This implies

$$p^*(\Phi'; A', b') = p^*(\Phi; A, b) \ , \quad A' = \bar{A} h_A(N'') > A \ , \ b' = \bar{b} h_b(N'') > b \ .$$

Hence, the number of entries is the same as that of the stationary equilibrium for Φ, but the number of exits is smaller: the exit set E defined by (3.6) diminishes by Lemma 3.3. This contradicts the assumption that N'' is smaller than N.

Therefore, the number N' of active managers in the industry satisfies $N'' < N' < N$. As a result, A and b increases, and p^* increases.

Proof of Proposition 3.11

Suppose that $N_1 = N_2$. This implies that $A_1 = A_2$ and $b_1 = b_2$, which also implies that $p_1^* = p_2^*$. By proposition 3.2, the measure $G(1 - p^*)$ of new managers enter the AM industry per time. Since $G_2(1 - p_2^*) < G_1(1 - p_1^*)$, there are more entries to the first stationary equilibrium than there are to the second stationary equilibrium. Since the exit set E is the same for both equilibria, the exit rate is higher for the second stationary equilibrium. This contradicts the assumption that $N_1 = N_2$.

Therefore, $N_1 > N_2$, and

$$A_1 = \bar{A}h_A(N_1) < \bar{A}h_A(N_2) = A_2 \ , \quad b_1 = \bar{b}h_b(N_1) < \bar{b}h_b(N_2) = b_2 \ .$$

By proposition 3.5, $p_1^* > p_2^*$.

Proof of Proposition 3.12

Consider the following map $\hat{N}(N) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$: take N as the number of active managers that determines A and b, and taken those $A(N)$ and $b(N)$ as given, compute the stationary number $\hat{N}$ of managers. A fixed point N^* of the map $N = \hat{N}(N)$ determines a stationary equilibrium.

To be more explicit, for N, $A(N)$ and $b(N)$ are determined by

$$A(N) = \bar{A}h_A(N) \ , \quad b(N) = \bar{b}h_b(N) \ .$$

Entry decisions of prospective managers for $A(N)$ and $b(N)$ are given by Proposition 3.2. Existing managers exit exogenously with rate λ, or endogenously choose to exit once they reach the exit threshold $\tilde{q} = h(p)$ defined by Proposition 3.1, given $A(N)$ and $b(N)$. The entry and exit decisions, taken $A(N)$ and $b(N)$ as given, pin down the stationary number (measure) $\hat{N}$ of managers. If $N = \hat{N}$, N is a stationary equilibrium number of active managers in the AM industry.

$\hat{N}(N)$ is a decreasing function in N, since $A(N)$ and $b(N)$ are decreasing in N. Lemma 3.3 suggests that the exit set E diminishes in A and b, and Proposition 3.5 implies that p^* decreases in A and b. Therefore, as N increases, there are more exits and less entries, and the stationary number $\hat{N}$ decreases. Note that $\hat{N}$ is continuous in N, since an infinitesimal change of N leads to infinitesimal decreases of A and b and, consequently, $\hat{N}$ decreases infinitesimally as well. The condition (3.5) implies that

$$\lim_{N \to 0} \hat{N}(N) \to \infty \ , \quad \lim_{N \to \infty} \hat{N}(N) = 0 \ .$$

The continuity of $\hat{N}(N)$ guarantees the existence of a fixed point: $N^* \equiv N = \hat{N}(N)$. Therefore, there exists a stationary equilibrium.

The uniqueness of a stationary equilibrium can be shown straightforwardly from the strict monotonicity of $\hat{N}(N)$. In order to show that $\hat{N}(N)$ is strictly decreasing in N, it suffices to show that p^* is strictly increasing in N[10]. For $N_1 > N_2$, $A_1 = A(N_1) < A_2 = A(N_2)$ and $b_1 = b(N_1) < b_2 = b(N_2)$. For all $p > p_{ex}(A_1, b_1)$, where p_{ex} is given by (3.8),

$$\begin{aligned} & V(p, 0; A_2, b_2) = \mathbb{E}\left[\int_0^{T^D} e^{-ru}\left(\max\{p_u A_2, f\tilde{q}_u(b_2)\} - \phi\right) du \middle| p_0 = p, \tilde{q}_0 = 0\right] \\ > \ & V(p, 0; A_1, b_1) = \mathbb{E}\left[\int_0^{T^D} e^{-ru}\left(\max\{p_u A_1, f\tilde{q}_u(b_1)\} - \phi\right) du \middle| p_0 = p, \tilde{q}_0 = 0\right] \ , \end{aligned}$$

since $\tilde{q}_t(b_2)$ first-order stochastically dominates $\tilde{q}_t(b_1)$. Since p^* is determined by (3.9)

$$p^*(A_2, b_2) < p^*(A_1, b_1) \ .$$

Hence, p^* is strictly increasing in N. The exit set E expands in N, and the process of $\tilde{q}_t$ becomes first-order stochastically dominated as N increases. Therefore, $\hat{N}(N)$ is strictly decreasing in N, which proves the uniqueness of the stationary equilibrium.

Proof of Lemma 3.4

The proof is similar to that of Lemma 3.1 and Lemma 3.2. $V_h(p)$ is the fixed point of the following map:

$$\mathbb{T}f(p) = \max\left\{(pA_h - \phi)\, dt + (1 - rdt - \lambda dt)\mathbb{E}\left[f(p')|p\right], 0\right\} \ ,$$

[10]The exit set E cannot be strictly expanding for the entire range of N, since the set becomes empty for a sufficiently small N.

where

$$p'|p = p + sp(1-p)dZ \ ,$$

where Z is a standard Brownian motion. The contraction mapping theorem guarantees that

$$V_h(p) = \lim_{k\to\infty} \mathbb{T}^k f(p)$$

for an arbitrary function $f(p)$. Suppose that $f(p)$ is an (weakly) increasing function in p. For $p_1 < p_2$, $\mathbb{E}[f(p')|p_1] \leq \mathbb{E}[f(p')|p_2]$, since $p'|p_2$ first-order stochastically dominates $p'|p_1$ asymptotically as $dt \to 0$. As a result, $\mathbb{T}f(p_1) \leq \mathbb{T}f(p_2)$. Hence, $\mathbb{T}$ maps an increasing function to an increasing function. Since $f(p)$ is increasing, $V_h(p)$ is increasing as well.

Now suppose that $f(p)$ is (weakly) convex. For an arbitrary point p, define

$$p_1 = p - \delta p \ , \quad p_2 = p + \delta p \ ,$$

where the order of magnitude of δp is set to be the same as or smaller than that of dt. If $f(p)$ is (locally) strictly convex at p, i.e., $\frac{\partial^2 f(p)}{\partial p^2} > 0$,

$$\frac{1}{2}\mathbb{E}\left[f(p')|p_1\right] + \frac{1}{2}\mathbb{E}\left[f(p')|p_2\right] - \mathbb{E}\left[f(p')|p\right] = \frac{1}{2}\frac{\partial^2 f(p)}{\partial p^2}\delta p^2 + \mathcal{O}(\delta p^2 dt) > 0 \ .$$

and if $f(p)$ is (locally) flat at p, i.e., $\frac{\partial^2 f(p)}{\partial p^2} = 0$,

$$\frac{1}{2}\mathbb{E}\left[f(p')|p_1\right] + \frac{1}{2}\mathbb{E}\left[f(p')|p_2\right] - \mathbb{E}\left[f(p')|p\right] = \mathcal{O}(\delta p^2 \sqrt{dt}^3) \ .$$

In the continuous-time limit, terms of the order dt (or lower) dominate. Hence, $\mathbb{E}\left[f(p')|p\right]$ is convex. As a result,

$$(pA_h - \phi)\, dt + (1 - rdt - \lambda dt)\mathbb{E}\left[f(p')|p\right]$$

is convex and, consequently, $\mathbb{T}f(p)$ is convex since taking the pointwise maximum of two functions preserves convexity. Since the map $\mathbb{T}$ preserves convexity, $V_h(p) =$

$\lim_{k\to\infty} \mathbb{T}^k f(p)$ is convex.

Proof of Lemma 3.5

If p_m^* does not exist in a stationary equilibrium, this implies

$$V_m(p,0) < \Phi\ , \quad \forall p\ ,$$

in which case there are no entries to mutual fund markets, or

$$V_m(p,0) > \Phi\ , \quad \forall p\ ,$$

in which case there are infinite entries to mutual fund markets by (3.5). These contradict a stationary equilibrium.

If p_h^* does not exist in a stationary equilibrium, this implies

$$V_h(p) < \Phi\ , \quad \forall p\ ,$$

which contradict a stationary equilibrium.

If $V_m(p,0)$ dominates $V_h(p)$, there are no entries to hedge fund markets. If $V_h(p)$ dominates $V_m(p)$, there are no entries to mutual fund markets. Both of these contradict a stationary equilibrium.

Proof of Proposition 3.15

Suppose that $A_h \leq A_m$. Fee profit rates for mutual funds are

$$\max\{p_t A_m, f\tilde{q}_t\} - \phi\ ,$$

and fee profit rates for hedge funds are

$$p_t A_h - \phi\ .$$

Since the process of p_t is the same for hedge funds and mutual funds, given the same $p_0 = p$, fee profits for mutual funds strictly dominate those for hedge funds. Therefore, $V_m(p,0)$ dominates $V_h(p)$, which contradicts a stationary equilibrium by Lemma 3.5.

Proof of Lemma 3.6

The sensitivity of the hedge fund entry value to p can be easily computed from the closed form of the value of hedge fund fee profits:

$$\begin{aligned} V_h(p) &= \frac{1}{r+\lambda}(pA_h - \phi) + c_1 \sqrt{p}^{1-\mu}\sqrt{1-p}^{1+\mu} , \\ V_h'(p) &= \frac{A_h}{r+\lambda} - c_1\left(p + \frac{\mu-1}{2}\right)\sqrt{p}^{-1-\mu}\sqrt{1-p}^{-1+\mu} \le \frac{A_h}{r+\lambda} , \end{aligned}$$

and

$$V_h'(1) = \frac{A_h}{r+\lambda} .$$

Since $V_h(p)$ is convex, $V_h'(p)$ increases in p.

The sensitivity of the mutual fund entry value to p is not easy to compute since $V_m(p,0)$ does not take a closed form. For $p > p_{ex}^m$, $V_m(p,0)$ can be decomposed as follows:

$$V_m(p,0) = \frac{1}{r+\lambda}(pA_m - \phi) + V_m^o(p,0) ,$$

where $V_m^o(p,0)$ satisfies

$$(r+\lambda)V_m^o(p,0) = \frac{1}{2}\frac{\partial^2 V_m^o(p,0)}{\partial p^2}s^2p^2(1-p)^2 + \frac{\partial V_m^o(p,0)}{\partial \tilde{q}}b .$$

Therefore,

$$\frac{\partial V_m(1,0)}{\partial p} = \frac{A_m}{r+\lambda} + \frac{\partial^2 V_m^o(1,0)}{\partial p \partial \tilde{q}}b .$$

Now I show that $\partial_p \partial_{\tilde{q}} V_m^o(1,0) < 0$. First note that

$$\partial_p \partial_{\tilde{q}} V_m(p,0) = \partial_p \partial_{\tilde{q}} V_m^o(p,0) .$$

Since $\tilde{q}_s$ for $s \geq t$ is independent of p_t,

$$\begin{aligned} \partial_p V_m(1,0) &= \frac{\partial}{\partial p_t} \mathbb{E}\left[\int_t^{T^D} e^{-rs} \left(\max\{p_s A_m, f\tilde{q}_s\} - \phi \right) ds \middle| p_t = 1, \tilde{q}_t = 0 \right] \\ &= \mathbb{E}\left[\int_t^{\infty} e^{-(r+\lambda)s} A_m \mathbb{1}_{\{A_m \geq f\tilde{q}_s\}} ds \right] , \end{aligned}$$

since $p_s = 1$ for $s > t$ given $p_t = 1$. As a result,

$$\begin{aligned} \partial_{\tilde{q}} \partial_p V_m(1,0) &= \frac{\partial}{\partial q_t} \mathbb{E}\left[\int_t^{\infty} e^{-(r+\lambda)s} A_m \mathbb{1}_{\{A_m \geq f\tilde{q}_s\}} ds \right] \\ &= -A_m \mathbb{E}\left[\int_t^{\infty} e^{-(r+\lambda)s} \frac{\partial \tilde{q}_s}{\partial \tilde{q}_t} \delta\left(\tilde{q}_s - \frac{A_m}{f} \right) ds \right] < 0 , \end{aligned}$$

because

$$\frac{\partial \tilde{q}_s}{\partial \tilde{q}_t} \geq 0 , \quad \forall s > t ,$$

and strictly positive for finite s. Note that $\delta(\cdot)$ is the Dirac delta function. As a result,

$$\frac{\partial V_m(1,0)}{\partial p} < \frac{A_m}{r+\lambda} .$$

Since $V_m(p,0)$ is convex by Corollary 3.1,

$$\frac{\partial V_m(p,0)}{\partial p} \leq \frac{\partial V_m(1,0)}{\partial p} < \frac{A_m}{r+\lambda} < \frac{A_h}{r+\lambda} .$$

Therefore, the sensitivity of the mutual fund entry value to p is strictly lower than $\frac{A_m}{r+\lambda}$.

Proof of Lemma 3.7

Since μ defined by (3.14) is a monotone decreasing function of s, $s < \bar{s}$ is equivalent to $\mu > \bar{\mu}$ where

$$\bar{\mu} = \sqrt{1 + \frac{8(r+\lambda)}{\bar{s}^2}}$$

p_h^* is determined by the following condition:

$$\Phi = \frac{p_h^* A_h - \phi}{r+\lambda} + c_1 \sqrt{p_h^*}^{1-\mu} \sqrt{1-p_h^*}^{1+\mu} ,$$

where c_1 is defined by (3.16). In the limit $\mu \to \infty$, i.e., $s \to 0$,

$$p_h^* \to p_h^{*,\infty} \equiv \frac{(r+\lambda)\Phi + \phi}{A_h} .$$

p_h^* can be decomposed as follows:

$$p_h^* = p_h^{*,\infty} + \delta p_h^*(\mu) .$$

Then, for sufficiently large μ, $\delta p_h^*(\mu)$ is determined by

$$\begin{aligned} 0 \;\approx\; & \frac{A_h}{r+\lambda}\delta p_h^*(\mu) + \frac{2\phi}{r+\lambda}\frac{1}{\mu+1}\sqrt{\left(\frac{\mu-1}{\mu+1}\right)}^{\mu-1} \left(1 - \frac{(r+\lambda)\Phi+\phi}{A_h} - p_h^*(\mu)\right) \\ & \times \left(\frac{\phi}{(r+\lambda)\Phi+\phi}\right)^{\mu-1} \left(\frac{A_h - (r+\lambda)\Phi - \phi}{A_h - \phi}\right)^{\mu-1} \\ & \times \left(1 + \frac{1-\mu}{\frac{(r+\lambda)\Phi+\phi}{A_h}\left(1 - \frac{(r+\lambda)\Phi+\phi}{A_h}\right)}\delta p_h^*(\mu)\right) , \end{aligned}$$

which can be approximated as

$$\begin{aligned} \delta p_h^*(\mu) \approx & -\frac{1}{\mu}\frac{2\phi}{A_h}\left(1 - \frac{(r+\lambda)\Phi+\phi}{A_h}\right)\left(\frac{\phi}{(r+\lambda)\Phi+\phi}\right)^{\mu-1} \\ & \times \left(\frac{A_h - (r+\lambda)\Phi - \phi}{A_h - \phi}\right)^{\mu-1} . \end{aligned}$$

For large μ, $V_h'(p_h^*)$ reads

$$\begin{aligned}
V_h'(p_h^*) &= \frac{A_h}{r+\lambda} - c_1 \frac{p_h^* + \frac{\mu-1}{2}}{p_h^*} \sqrt{\frac{p_h^*}{1-p_h^*}}^{1-\mu} \\
&\approx \frac{A_h}{r+\lambda} - \frac{2\phi}{r+\lambda} \frac{p_h^{*,\infty} + \frac{\mu-1}{2}}{(\mu+1)p_h^{*,\infty}} \sqrt{\left(\frac{\mu-1}{\mu+1}\right)}^{\mu-1} \left(\frac{\phi/A_h}{p_h^{*,\infty}}\right)^{\mu-1} \left(\frac{1-p_h^{*,\infty}}{1-\phi/A_h}\right)^{\mu-1} \\
&\approx \frac{A_h}{r+\lambda} - \frac{\phi}{(r+\lambda)p_h^{*,\infty}} \left(\frac{\phi}{(r+\lambda)\Phi+\phi}\right)^{\mu-1} \left(\frac{A_h-(r+\lambda)\Phi-\phi}{A_h-\phi}\right)^{\mu-1} .
\end{aligned}$$

Hence, $V_h'(p_h^*)$ is strictly decreasing in μ for large μ, and

$$\lim_{\mu\to\infty} V_h'(p_h^*;\mu) = \frac{A_h}{r+\lambda} > \frac{A_m}{r+\lambda} ,$$

by Proposition 3.15. Therefore, there exists $\bar{\mu}$ such that for $\mu > \bar{\mu}$

$$V_h'(p_h^*;\mu) > \frac{A_m}{r+\lambda} .$$

This implies that there exists $\bar{s}$ such that, for all $s < \bar{s}$,

$$V_h'(p_h^*;s) > \frac{A_m}{r+\lambda} .$$

Proof of Proposition 3.16

I first show that $p_m^* < p_h^*$. Suppose that this is not the case, i.e., $p_m^* \geq p_h^*$. This implies

$$\Phi = V_h(p_h^*) \geq V_m(p_h^*, 0) .$$

Since $V_h(\cdot)$ is convex, for $p \geq p_h^*$

$$V_h'(p) > \frac{A_m}{r+\lambda} > \frac{\partial V_m(p,0)}{\partial p} ,$$

where the second inequality comes from Lemma 3.6. Combining the two relations, for $p > p_h^*$

$$V_h(p) > V_m(p,0) ,$$

which contradicts the stationary equilibrium by Lemma 3.5. Therefore, $p_m^* < p_h^*$, and this implies

$$\Phi = V_h(p_h^*) < V_m(p_h^*, 0) .$$

Now, I show that there exists $p_h^* < \hat{p} < 1$ such that

$$V_h(\hat{p}) = V_m(\hat{p}, 0) .$$

Suppose there does not exist such $\hat{p}$. Since $\Phi = V_h(p_h^*) < V_m(p_h^*, 0)$, this implies that, for all $p > p_m^*$,

$$V_h(p) < V_m(p, 0) ,$$

which contradicts the stationary equilibrium by Lemma 3.5. Therefore, there exists $p_h^* < \hat{p} < 1$ where

$$V_h(\hat{p}) = V_m(\hat{p}, 0) .$$

Next I show that $\hat{p}$ is unique and

$$V_m(p,0) - V_h(p) = \begin{cases} > 0 \ , & \text{if } p < \hat{p} \\ < 0 \ , & \text{if } p > \hat{p} \end{cases} .$$

The uniqueness of $\hat{p}$ is straightforward from

$$V_h'(p) > \frac{A_m}{r+\lambda} > \frac{\partial V_m(p,0)}{\partial p} ,$$

for $p \geq p_h^*$. Since the slope of $V_h(p)$ is strictly greater than the slope of $V_m(p,0)$ for $p \geq p_h^*$, once those two functions cross, they never cross again. The continuity of $V_h(p)$ and $V_m(p,0)$ guarantees that $V_h(p) < V_m(p,0)$ for $p < \hat{p}$, and $V_h(p) > V_m(p,0)$ for $p > \hat{p}$. Consequently, prospective managers with $p \in (p_m^*, \hat{p})$ enter mutual fund markets, and prospective managers with $p \in (\hat{p}, 1]$ enter hedge fund markets.

Proof of Proposition 3.17

Given that E_m is empty, no mutual fund managers choose to exit. On the other hand, the hedge fund exit set E_h defined in (3.15) is always nonempty. These together imply that there exists $0 < \hat{p} < 1$ such that

$$V_m(p, 0) > V_h(p) , \quad \forall p \in [0, \hat{p}) ,$$

and for an arbitrary small $\delta p > 0$

$$V_m(\hat{p} + \delta p, 0) < V_h(\hat{p} + \delta p) ,$$

by Lemma 3.5. Next I show that

$$V_m(p, 0) < V_h(p) , \quad \forall p \in (\hat{p}, 1] .$$

Since $V_m(p, 0)$ and $V_h(p)$ are smooth,

$$V_h'(\hat{p}) > \partial_p V_m(\hat{p}, 0) .$$

In order to proceed, I first need to show that

$$\partial_{\tilde{q}} \partial_p V_m(p, \tilde{q}) \leq 0 . \tag{3.27}$$

Since $\tilde{q}_s$ for $s \geq t$ is independent of p_t,

$$\begin{aligned} \partial_p V_m(p, \tilde{q}) &= \frac{\partial}{\partial p_t} \mathbb{E}\left[\int_t^{T^D} e^{-rs} \left(\max\{p_s A_m, f\tilde{q}_s\} - \phi\right) ds\right] \\ &= \mathbb{E}\left[\int_t^{T^D} e^{-rs} \frac{\partial p_s}{\partial p_t} A_m \mathbb{1}_{\{p_s A_m \geq f\tilde{q}_s\}} ds\right] . \end{aligned}$$

Note that T^D is independent of the change in p_t[11], since the mutual fund exit set E_m

[11]This property is crucial to prove that $\partial_{\tilde{q}} \partial_p V_m(p, \tilde{q})$ is nonpositive.

is empty. As a result,

$$\begin{aligned}
\partial_{\tilde{q}}\partial_p V_m(p,\tilde{q}) &= \frac{\partial}{\partial \tilde{q}_t}\mathbb{E}\left[\int_t^{T^D} e^{-rs}\frac{\partial p_s}{\partial p_t} A_m \mathbb{1}_{\{p_s A_m \geq f\tilde{q}_s\}} ds\right] \\
&= -A_m \mathbb{E}\left[\int_t^{T^D} e^{-rs}\frac{\partial p_s}{\partial p_t}\frac{\partial \tilde{q}_s}{\partial \tilde{q}_t}\delta\left(\tilde{q}_s - \frac{p_s A_m}{f}\right) ds\right] \leq 0 ,
\end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function. The sign of $\partial_{\tilde{q}}\partial_p V_m(p,\tilde{q})$ comes from

$$\frac{\partial p_s}{\partial p_t} \geq 0 , \quad \frac{\partial \tilde{q}_s}{\partial \tilde{q}_t} \geq 0 , \quad \forall s \geq t .$$

Now decompose $V_m(p,0)$ as follows:

$$V_m(p,0) = V_m^b(p,0) + V_m^o(p,0) ,$$

where

$$V_m^b(p,0) = \frac{pA_m - \phi}{r+\lambda} ,$$

and $V_m^o(p,0)$ solves

$$(r+\lambda)V_m^o(p,0) = \frac{1}{2}\frac{\partial^2 V_m^o(p,0)}{\partial p^2}s^2 p^2(1-p)^2 + \frac{\partial V_m^o(p,0)}{\partial \tilde{q}}b .$$

Note that (3.27) implies

$$\partial_p \partial_{\tilde{q}} V_m(p,0) = \partial_p \partial_{\tilde{q}} V_m^o(p,0) \leq 0 ,$$

i.e., $\frac{\partial V_m^o(p,0)}{\partial \tilde{q}}$ is decreasing in p.

Define

$$\tilde{V}_m(p,0) = \frac{pA_m - \phi}{r+\lambda} + c_{m,0} + c_{m,1}\sqrt{p}^{1-\mu}\sqrt{1-p}^{1+\mu} ,$$

where $c_{m,0}$ and $c_{m,1}$ are chosen such that

$$\tilde{V}_m(\hat{p},0) = V_m(\hat{p},0) , \quad \partial_p \tilde{V}_m(\hat{p},0) = \partial_p V_m(\hat{p},0) . \tag{3.28}$$

I first need to show that

$$c_{m,0} \leq \frac{b}{r+\lambda}\frac{\partial V_m^o(\hat{p},0)}{\partial \tilde{q}} .$$

Suppose that this is not the case, i.e., $c_{m,0} > \frac{b}{r+\lambda}\frac{\partial V_m^o(\hat{p},0)}{\partial \tilde{q}}$. For $p' = \hat{p} + \delta p$ with an arbitrary small δp,

$$\begin{aligned}
&\partial_p V_m(p',0) - \partial_p V_m(\hat{p},0) = \int_{\hat{p}}^{p'} \partial_p^2 V_m(p,0) dp \\
= &\int_{\hat{p}}^{p'} \frac{2}{s^2p^2(1-p)^2}\left[V_m(p,0) - \frac{pA_m - \phi}{r+\lambda} - \frac{b}{r+\lambda}\frac{\partial V_m^o(p,0)}{\partial \tilde{q}}\right] dp \\
> &\int_{\hat{p}}^{p'} \frac{2}{s^2p^2(1-p)^2}\left[\tilde{V}_m(p,0) - \frac{pA_m - \phi}{r+\lambda} - c_{m,0}\right] dp = \partial_p \tilde{V}_m(p',0) - \partial_p \tilde{V}_m(\hat{p},0) ,
\end{aligned}$$

because of (3.28). Since the first-order derivative of $V_m(p',0)$ dominates that of $\tilde{V}_m(p',0)$,

$$V_m(p',0) > \tilde{V}_m(p',0) .$$

A similar argument shows that

$$V_m(1,0) = \frac{A_m - \phi}{r+\lambda} + \frac{b}{r+\lambda}\frac{\partial V_m^o(1,0)}{\partial \tilde{q}} > \tilde{V}_m(1,0) = \frac{A_m - \phi}{r+\lambda} + c_{m,0} ,$$

which is a contradiction since $\frac{b}{r+\lambda}\frac{\partial V_m^o(1,0)}{\partial \tilde{q}} \leq \frac{b}{r+\lambda}\frac{\partial V_m^o(\hat{p},0)}{\partial \tilde{q}} < c_{m,0}$. Therefore, $c_{m,0} \leq \frac{b}{r+\lambda}\frac{\partial V_m^o(\hat{p},0)}{\partial \tilde{q}}$.

Now I can prove that for all $p > \hat{p}$,

$$\tilde{V}_m(p,0) \geq V_m(p,0) , \quad \partial_p \tilde{V}_m(p,0) \geq \partial_p V_m(p,0) . \tag{3.29}$$

For $p' = \hat{p} + \delta p$ with an arbitrary small δp,

$$\begin{aligned}
&\partial_p V_m(p',0) - \partial_p V_m(\hat{p},0) = \int_{\hat{p}}^{p'} \partial_p^2 V_m(p,0) dp \\
= &\int_{\hat{p}}^{p'} \frac{2}{s^2p^2(1-p)^2}\left[V_m(p,0) - \frac{pA_m - \phi}{r+\lambda} - \frac{b}{r+\lambda}\frac{\partial V_m^o(p,0)}{\partial \tilde{q}}\right] dp \\
\leq &\int_{\hat{p}}^{p'} \frac{2}{s^2p^2(1-p)^2}\left[\tilde{V}_m(p,0) - \frac{pA_m - \phi}{r+\lambda} - c_{m,0}\right] dp = \partial_p \tilde{V}_m(p',0) - \partial_p \tilde{V}_m(\hat{p},0) ,
\end{aligned}$$

where the equality holds if $\frac{\partial V^o_m(p,0)}{\partial \bar{q}}$ is constant over $p \in [\hat{p}, p')$. Otherwise, the strict inequality holds. This implies that $V_m(p', 0) \leq \tilde{V}_m(p', 0)$. Suppose that (3.29) is violated. Then, from the continuity of $V_m(p, 0)$ and $\tilde{V}_m(p, 0)$, there must exist $p'' \in (\hat{p}, 1)$ such that

$$V_m(p'', 0) \leq \tilde{V}_m(p'', 0) \ , \quad \partial_p V_m(p'', 0) = \partial_p \tilde{V}_m(p'', 0) \ ,$$

and for an arbitrary small $\delta p > 0$

$$\partial_p V_m(p'' + \delta p, 0) > \partial_p \tilde{V}_m(p'' + \delta p, 0) \ .$$

Suppose that there exists such p''. Then one can construct

$$\tilde{V}'_m(p, 0) = \frac{pA_m - \phi}{r + \lambda} + c'_{m,0} + c'_{m,1} \sqrt{p}^{1-\mu} \sqrt{1-p}^{1+\mu} \ ,$$

such that

$$V_m(p'', 0) = \tilde{V}'_m(p'', 0) \ , \quad \partial_p V_m(p'', 0) = \partial_p \tilde{V}_m(p'', 0) = \partial_p \tilde{V}'_m(p'', 0) \ .$$

This implies that $c'_{m,0} \leq c_{m,0}$ and $c'_{m,1} = c_{m,1}$. A similar argument to that of $\tilde{V}_m(p, 0)$ shows that

$$\partial_p V_m(p'' + \delta p, 0) \leq \partial_p \tilde{V}'_m(p'' + \delta p, 0) = \partial_p \tilde{V}_m(p'' + \delta p, 0) \ ,$$

which contradicts the existence of p''. Therefore, for all $p > \hat{p}$,

$$\tilde{V}_m(p, 0) \geq V_m(p, 0) \ , \quad \partial_p \tilde{V}_m(p, 0) \geq \partial_p V_m(p, 0) \ .$$

Finally, I show that

$$\tilde{V}_m(p, 0) < V_h(p) \ , \quad \forall p \in (\hat{p}, 1] \ .$$

Denote

$$g(p) = \frac{d}{dp}\left(\sqrt{p}^{1-\mu}\sqrt{1-p}^{1+\mu}\right) < 0 \, .$$

Note that $g'(p) > 0$. Suppose that $c_1 \geq c_{m,1}$. Since

$$V_h'(\hat{p}) = \frac{A_h}{r+\lambda} + c_1 g(\hat{p}) > \partial_p V_m(\hat{p}, 0) = \partial_p \tilde{V}_m(\hat{p}, 0) = \frac{A_m}{r+\lambda} + c_{m,1} g(\hat{p}) \, ,$$

the following holds for all $p > \hat{p}$:

$$V_h'(p) = \frac{A_h}{r+\lambda} + c_1 g(p) > \frac{A_m}{r+\lambda} + c_{m,1} g(p) = \partial_p \tilde{V}_m(p, 0) \, .$$

Now suppose that $c_1 < c_{m,1}$. It is straightforward that $V_h'(p)$ is (strictly) greater than $\partial_p \tilde{V}_m(p, 0)$ for all $p > \hat{p}$. Therefore, since $V_h(\hat{p}) = \tilde{V}_m(\hat{p}, 0)$,

$$V_h(p) > \tilde{V}_m(p, 0) \geq V_m(p, 0) \, ,$$

for all $p > \hat{p}$.

Proof of Corollary 3.7

The mutual fund exit set E_m is empty in the limit of $\phi \to 0$, but the hedge fund exit set E_h is nonempty. Since $p = 0$ is always included in E_h, unskilled hedge fund managers choose to exit once they reach p^h_{ex}, which converges to zero in the limit of $\phi \to 0$. Therefore, hedge funds exit at a higher rate than mutual funds do.

Proof of Proposition 3.18

Consider a map $\hat{N}(N_m, N_h) = (\hat{N}_m, \hat{N}_h) : \mathbb{R}^2_{>0} \to \mathbb{R}^2_{>0}$. The map takes the numbers N_m and N_h for the values of $A_m(N_m)$, $b(N_m)$ and $A_h(N_h)$, and computes the stationary numbers $\hat{N}_m$ and $\hat{N}_h$ of mutual fund managers and hedge fund managers, respectively. A fixed point $N^* = (N^*_m, N^*_h)$ of the map $(N_m, N_h) = \hat{N}(N_m, N_h)$ determines a stationary equilibrium.

Compared with the proof of Proposition 3.12, it is much more tricky to show the existence and the uniqueness of a fixed point of a two-dimensional map than that of a single-dimensional map. In order to address this issue, the proof consists of two steps: first, I define a map that reduces the dimensionality of the map $\hat{N}(N_m, N_h)$, and next, I show the existence and the uniqueness of a fixed point of the reduced map. A fixed point of the reduced map solves for a fixed point of $\hat{N}(N_m, N_h)$, and vice versa.

Define a map $\tilde{N}_m(N_m; N_h) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. The map regards N_h as an exogenous parameter, and uses the value for computing $A_h(N_h)$ and, consequently, $V_h(p)$. For each N_m, the map takes the value of N_m to compute $A_m(N_m)$ and $b(N_m)$. Then, the map computes the stationary number $\tilde{N}_m$ of mutual fund managers as a function of N_m. I first show that a fixed point of this map $\tilde{N}_m(N_m; N_h)$ exists and it is unique, given N_h. Proposition 3.16 proves that, if there exists $\hat{p}$ such that $V_m(\hat{p}, 0) = V_h(\hat{p})$, $V_m(p, 0) > V_h(p)$ for $p < \hat{p}$ and $V_m(p, 0) < V_h(p)$ for $p > \hat{p}$. Since $\hat{p}$ changes continuously as N_m changes, entry decisions are continuous in N_m. Since $V_m(0, 0) > V_h(0)$, $\hat{p}$ exists as long as $V_m(1, 0) \leq V_h(1)$. If N_m is sufficiently large, $V_m(\hat{p}, 0) < \Phi$ and there are no entries to mutual fund markets, i.e., $\tilde{N}_m$ is zero. If N_m is sufficiently small, $V_m(p, 0) > V_h(p, 0)$ for all p, and all entries are into mutual fund markets. As N_m goes to zero, $\tilde{N}_m$ diverges to infinity from (3.3). Therefore, from the continuity of $\tilde{N}_m(N_m; N_h)$, there exists a fixed point $N_m^{**}(N_h)$ satisfying $N_m^{**}(N_h) = \tilde{N}_m(N_m^{**}(N_h); N_h)$. The uniqueness of the fixed point comes from the strict monotonicity of $\tilde{N}_m(N_m; N_h)$ for $\tilde{N}_m > 0$.

Now define a map $\tilde{N}_h(N_h) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. The map is defined by

$$\tilde{N}_h(N_h) \equiv \hat{N}_h(N_m^{**}(N_h), N_h) ,$$

where $N_m^{**}(N_h)$ is the fixed point of $N_m = \tilde{N}_m(N_m; N_h)$. In other words, $\tilde{N}_h(N_h)$ computes the stationary number of hedge fund managers taking N_h and $N_m^{**}(N_h)$ as given. Note that $N_m^{**}(N_h)$ is continuous in N_h, which can be proven by the continuity of entry decisions in N_m. If N_h is sufficiently large, there are no entries to hedge fund markets and $\tilde{N}_h$ is zero. In contrast, if N_h goes to zero, $\tilde{N}_h$ diverges to infinity.

The continuity of $N_m^{**}(N_h)$ implies the continuity of $\tilde{N}_h(N_h)$ and, hence, there exists a fixed point $N_h^{**} = \tilde{N}_h(N_h^{**})$. Then, $N_m^{**}(N_h^{**})$ and N_h^{**} jointly solve

$$(N_m^{**}(N_h^{**}), N_h^{**}) = \hat{N}(N_m^{**}(N_h^{**}), N_h^{**}) .$$

Therefore, there exists a stationary equilibrium.

The proof of the uniqueness of the stationary equilibrium requires several steps. Denote the stationary measure of mutual fund managers and hedge funds managers in a stationary equilibrium by $N^* = (N_m^*, N_h^*)$. Suppose that there exists another stationary equilibrium, and denote the stationary measure of mutual fund managers and hedge fund managers by $N^\dagger = (N_m^\dagger, N_h^\dagger)$. There are four possible cases:

$$\begin{array}{lll} 1.\ N_m^* \geq N_m^\dagger\ ,\ N_h^* \geq N_h^\dagger & , & 2.\ N_m^* \leq N_m^\dagger\ ,\ N_h^* \leq N_h^\dagger\ , \\ 3.\ N_m^* \geq N_m^\dagger\ ,\ N_h^* \leq N_h^\dagger & , & 4.\ N_m^* \leq N_m^\dagger\ ,\ N_h^* \geq N_h^\dagger\ , \end{array}$$

where $N^* \neq N^\dagger$.

In case 1,

$$A_m(N_m^*) \leq A_m(N_m^\dagger)\ , \quad b(N_m^*) \leq b(N_m^\dagger)\ , \quad A_h(N_h^*) \leq A_h(N_h^\dagger)\ ,$$

with at least one strict inequality. This implies that there must be more entries to one type of the AM markets for the stationary equilibrium with $N^\dagger$ than entries to the same type for the stationary equilibrium with N^*. This implication contradicts $N_m^* \geq N_m^\dagger$ or $N_h^* \geq N_h^\dagger$. A similar argument excludes case 2.

In case 3,

$$A_m(N_m^*) \leq A_m(N_m^\dagger)\ , \quad b(N_m^*) \leq b(N_m^\dagger)\ , \quad A_h(N_h^*) \geq A_h(N_h^\dagger)\ ,$$

with at least one strict inequality. This implies that there must be more entries to mutual fund markets for the stationary equilibrium with $N^\dagger$ than entries to mutual fund markets for the stationary equilibrium with N^*. This implication contradicts

$N_m^* \geq N_m^\dagger$. A similar argument excludes case 4.

Therefore, since $N^\dagger \neq N^*$ is not consistent with a stationary equilibrium, the stationary equilibrium for N^* is unique.

Proof of Proposition 3.19

Consider a map $\hat{N}(N_m, N_h) = (\hat{N}_m, \hat{N}_h) : \mathbb{R}^2_{>0} \to \mathbb{R}^2_{>0}$. The map takes the numbers N_m and N_h for the values of $A_m(N_m)$, $b(N_m)$ and $A_h(N_h)$, and computes the stationary numbers $\hat{N}_m$ and $\hat{N}_h$ of mutual fund managers and hedge fund managers, respectively. A fixed point $N^* = (N_m^*, N_h^*)$ of the map $(N_m, N_h) = \hat{N}(N_m, N_h)$ determines a stationary equilibrium.

Define a map $\tilde{N}_m(N_m; N_h) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. The map regards N_h as an exogenous parameter, and uses the value for computing $A_h(N_h)$ and, consequently, $V_h(p)$. For each N_m, the map takes the value of N_m to compute $A_m(N_m)$ and $b(N_m)$. Then, the map computes the stationary number $\tilde{N}_m$ of mutual fund managers as a function of N_m. I first show that a fixed point of this map $\tilde{N}_m(N_m; N_h)$ exists and it is unique, given N_h. Proposition 3.17 proves that, if there exists $\hat{p}$ such that $V_m(\hat{p}, 0) = V_h(\hat{p})$, $V_m(p, 0) > V_h(p)$ for $p < \hat{p}$ and $V_m(p, 0) < V_h(p)$ for $p > \hat{p}$. Since $\hat{p}$ changes continuously as N_m changes, entry decisions are continuous in N_m. Since $V_m(0, 0) > V_h(0)$, $\hat{p}$ exists as long as $V_m(1, 0) \leq V_h(1)$. If N_m is sufficiently large, $V_m(\hat{p}, 0) < \Phi$ and there are no entries to mutual fund markets, i.e., $\tilde{N}_m$ is zero. If N_m is sufficiently small, $V_m(p, 0) > V_h(p, 0)$ for all p, and all entries are into mutual fund markets. As N_m goes to zero, $\tilde{N}_m$ diverges to infinity from (3.3). Therefore, from the continuity of $\tilde{N}_m(N_m; N_h)$, there exists a fixed point $N_m^{**}(N_h)$ satisfying $N_m^{**}(N_h) = \tilde{N}_m(N_m^{**}(N_h); N_h)$. The uniqueness of the fixed point comes from the strict monotonicity of $\tilde{N}_m(N_m; N_h)$ for $\tilde{N}_m > 0$.

Now define a map $\tilde{N}_h(N_h) : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. The map is defined by

$$\tilde{N}_h(N_h) \equiv \hat{N}_h(N_m^{**}(N_h), N_h) ,$$

where $N_m^{**}(N_h)$ is the fixed point of $N_m = \tilde{N}_m(N_m; N_h)$. In other words, $\tilde{N}_h(N_h)$

computes the stationary number of hedge fund managers taking N_h and $N_m^{**}(N_h)$ as given. Note that $N_m^{**}(N_h)$ is continuous in N_h, which can be proven by the continuity of entry decisions in N_m. If N_h is sufficiently large, there are no entries to hedge fund markets and $\tilde{N}_h$ is zero. In contrast, if N_h goes to zero, $\tilde{N}_h$ diverges to infinity. The continuity of $N_m^{**}(N_h)$ implies the continuity of $\tilde{N}_h(N_h)$ and, hence, there exists a fixed point $N_h^{**} = \tilde{N}_h(N_h^{**})$. Then, $N_m^{**}(N_h^{**})$ and N_h^{**} jointly solve

$$(N_m^{**}(N_h^{**}), N_h^{**}) = \hat{N}(N_m^{**}(N_h^{**}), N_h^{**}) .$$

Therefore, there exists a stationary equilibrium.

The proof of the uniqueness of the stationary equilibrium requires several steps. Denote the stationary measure of mutual fund managers and hedge funds managers in a stationary equilibrium by $N^* = (N_m^*, N_h^*)$. Suppose that there exists another stationary equilibrium, and denote the stationary measure of mutual fund managers and hedge fund managers by $N^\dagger = (N_m^\dagger, N_h^\dagger)$. There are four possible cases:

$$\begin{array}{lcl} 1.\ N_m^* \geq N_m^\dagger\ ,\ N_h^* \geq N_h^\dagger & , & 2.\ N_m^* \leq N_m^\dagger\ ,\ N_h^* \leq N_h^\dagger\ , \\ 3.\ N_m^* \geq N_m^\dagger\ ,\ N_h^* \leq N_h^\dagger & , & 4.\ N_m^* \leq N_m^\dagger\ ,\ N_h^* \geq N_h^\dagger\ , \end{array}$$

where $N^* \neq N^\dagger$.

In case 1,

$$A_m(N_m^*) \leq A_m(N_m^\dagger)\ , \quad b(N_m^*) \leq b(N_m^\dagger)\ , \quad A_h(N_h^*) \leq A_h(N_h^\dagger)\ ,$$

with at least one strict inequality. This implies that there must be more entries to one type of the AM markets for the stationary equilibrium with $N^\dagger$ than entries to the same type for the stationary equilibrium with N^*. This implication contradicts $N_m^* \geq N_m^\dagger$ or $N_h^* \geq N_h^\dagger$. A similar argument excludes case 2.

In case 3,

$$A_m(N_m^*) \leq A_m(N_m^\dagger)\ , \quad b(N_m^*) \leq b(N_m^\dagger)\ , \quad A_h(N_h^*) \geq A_h(N_h^\dagger)\ ,$$

with at least one strict inequality. This implies that there must be more entries to mutual fund markets for the stationary equilibrium with $N^\dagger$ than entries to mutual fund markets for the stationary equilibrium with N^*. This implication contradicts $N^*_m \geq N^\dagger_m$. A similar argument excludes case 4.

Therefore, since $N^\dagger \neq N^*$ is not consistent with a stationary equilibrium, the stationary equilibrium for N^* is unique.

Proof of Proposition 3.20

Given f and nontrivial $\psi(\cdot)$, i.e., $\psi(\cdot)$ is not uniformly zero, the net alpha is

$$\mathbb{E}_t\left[dr^{ex}_t\right] = \frac{p_t A dt}{q_t} - f dt - \mathbb{E}_t\left[\psi(dr^{ex}_t)\right]$$

Rational investors provide capital until the net alpha becomes zero. If the net alpha becomes negative, rational investors withdraw their capital and do not invest in the fund. If rational investors invest in the fund,

$$\mathbb{E}_t\left[dr^{ex}_t\right] = \frac{p_t A dt}{q_t} - f dt - \mathbb{E}_t\left[\psi(dr^{ex}_t)\right] = 0 \quad \Longleftrightarrow \quad q_t = \frac{p_t A}{f} ,$$

because dr^{ex}_t is symmetric and mean-zero and, as a result, $\mathbb{E}_t\left[\psi(dr^{ex}_t)\right] = 0$. On the other hand, if the amount $\tilde{q}_t$ of naive money exceeds the size $\frac{p_t A}{f}$, the net alpha becomes negative, which implies

$$\mathbb{E}_t\left[dr^{ex}_t\right] < 0 \quad \Longleftrightarrow \quad \mathbb{E}_t\left[\psi(dr^{ex}_t)\right] < 0 ,$$

since the conditional distribution of dr^{ex}_t has a negative mean and is symmetric about the mean.

Then, the expected fee revenues between t and $t + dt$ at time t is

$$\mathbb{E}_t\left[\max\left\{\frac{p_t A}{f}, \tilde{q}_t\right\}\left(f dt + \psi(dr^{ex}_t)\right)\right] = \begin{cases} p_t A dt & , \quad \tilde{q}_t \leq \frac{p_t A}{f} \\ \tilde{q}_t\left(f dt + \mathbb{E}_t\left[\psi(dr^{ex}_t)\right]\right) < \tilde{q}_t f dt & , \quad \tilde{q}_t > \frac{p_t A}{f} \end{cases}$$

Therefore, the conditional expectation of fee revenues at time t is smaller for any nontrivial $\psi(\cdot)$ than for uniformly zero $\psi(\cdot)$. At entry, a manager make choices in order to maximize the discounted fee profits

$$\begin{aligned} V &= \max_{f,\psi(\cdot),T^D} \mathbb{E}\left[\int_0^{T^D} e^{-rt}\left(\max\left\{\frac{p_t A}{f}, \tilde{q}_t\right\}(f dt + \psi(dr_t^{ex})) - \phi dt\right)\right] \\ &= \max_{f,\psi(\cdot),T^D} \mathbb{E}\left[\int_0^{T^D} e^{-rt}\left(\mathbb{E}_t\left[\max\left\{\frac{p_t A}{f}, \tilde{q}_t\right\}(f dt + \psi(dr_t^{ex}))\right] - \phi dt\right)\right], \end{aligned}$$

where the second inequality comes from the law of iterated expectations. For any choice of f and T^D, because the dynamics of $\tilde{q}_t$ does not depend on the choice of $\psi(\cdot)$, the manager is better off by choosing $\psi(\cdot) = 0$, i.e., flat fees.

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