Waveguide Quantum Electrodynamics with Superconducting Qubits

by

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Abstract

Experiments in quantum optics have long been implemented with atoms in 3D free space or with atoms interacting with cavities. Over the past decade, the field of microwave quantum optics using superconducting circuits has gained a tremendous amount of attention. In particular, the confinement of photonic modes to 1D enables a new parameter regime of strong interactions between qubits and open waveguides. In these setups, known as waveguide quantum electrodynamics (WQED), superconducting qubits interact with a continuum of propagating photonic modes. In this thesis, we will explore the physics of WQED devices that consist of multiple qubits and their potential application to quantum information and simulation.

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Chapter 1

Introduction

The field of experimental quantum optics has flourished over the past several decades. The ability to investigate light-matter interactions became possible due to the advancements in the measurement and control of individual atoms with light. This has primarily been investigated in systems involving 3D space [1–5]. These types of experimental demonstrations eventually led to the 2012 Nobel prize that was awarded to Serge Haroche and David Wineland. In Haroche’s experiments, an atom was placed between two mirrors that form a cavity. The atom then interacts with the quantized modes of the cavity. This setup became known as cavity quantum electrodynamics (CQED). The first demonstrations of light-matter interactions in superconducting systems were accomplished by coupling superconducting qubits to the electric field of superconducting transmission line resonators [6, 7]. This field of study is now known as circuit quantum electrodynamics (cQED). The 1D nature of the photonic modes in the resonator enables the qubit-cavity coupling strength to be much stronger than all dissipative processes. This is known as the strong coupling regime. Instead of coupling qubits to cavities, we will investigate systems where qubits are coupled to the guided continuum of light in a waveguide, otherwise known as waveguide quantum electrodynamics (WQED). In superconducting systems, a coplanar transmission line is used at the waveguide. Reaching the strong coupling regime in WQED is much harder than in CQED. This is because photons in a cavity can interact with the atom for many “cycles” before leaking out whereas photons in WQED are propa-
gating. Superconducting circuits are a good platform to study the physics of WQED in the strong coupling regime because the light in the waveguide is confined to one dimension. The first demonstration of superconducting WQED came about in 2010 [8, 9] where resonance fluorescence of a superconducting flux qubit was observed when coupled to a superconducting transmission line. Since then, a wide variety of applications of this technology have emerged. For example, superconducting quantum networks will likely involve a WQED architecture for the control, generation, and transfer of quantum information [10–13]. These systems have also shown promise as an interesting platform for quantum simulations [14–16].

This thesis is structured as follows. In the remainder of Chapter 1, we will briefly review superconductivity, introduce the Josephson junction as a circuit element, and discuss its relevant properties. Chapter 2 will focus on the construction and analysis of quantum circuits. We will derive the Hamiltonian for the transmission line, the transmon qubit, and how a WQED setup can be constructed with these two objects. Chapter 3 will then use these results to discuss the theory of WQED systems. For a single qubit, we will show how light in the waveguide can act as a probe of the qubit through coherent scattering. We will then show that the waveguide can act as an interface for qubit-qubit interactions and as a medium for correlated decay when more than one qubit is coupled to it. In Chapter 4, we will present the experimental results for the theory presented in Chapter 3. Finally, we will summarize our work and comment on our future plans in Chapter 5.

1.1 Superconductivity

1.1.1 A Brief Overview

Superconductivity was first observed in 1911 by Heike Kamerlingh Onnes. It was shown that the electrical resistance of mercury sharply falls to 0 when cooled below a certain critical temperature, $T_c = 4.2K$ [17]. Nearly half a century later, a microscopic quantum theory for this phenomenon was discovered [18]. This theory is now known
as the BCS theory, named after its founders Bardeen, Cooper, and Shrieffer. It was shown that electrons can be effectively attracted to each other to form bound states between pairs of electrons, known as Cooper pairs. To see how this comes about one must include the effects of electron-phonon interactions. Consider the many-body Hamiltonian [19]

$$\hat{H} = \sum_k \hbar \epsilon_k \hat{c}_k^\dagger \hat{c}_k + \sum_q \hbar \omega_q \hat{a}_q^\dagger \hat{a}_q + \sum_{k,q} V_q \hat{c}_{k+q}^\dagger \hat{c}_k (\hat{a}_{-q} + \hat{a}_q),$$

(1.1)

where $\hat{c}_k$ are electron (fermionic) annihilation operators with momentum $k$ and $\hat{a}_q$ are phonon (bosonic) annihilation operators with momentum $q$. The electron band structure and phonon dispersion are accounted for in $\epsilon_k$ and $\omega_q$, respectively. The final term in equation 1.1 is the interaction between electrons and phonons, where an electron scatters off of a phonon with strength $V_q$. Taking this interaction to be a small perturbation, a Schrieffer-Wolf transformation can be applied to eliminate the phonon operators to first order in the electron-phonon coupling:

$$\hat{H}_{\text{eff}} \approx \sum_k \hbar \epsilon_k \hat{c}_k^\dagger \hat{c}_k + \sum_{k,k',q} \frac{\hbar V_q \omega_q}{(\epsilon_k - \epsilon_{k+q})^2 - \omega_q^2} \hat{c}_{k+q}^\dagger \hat{c}_{k'}^\dagger \hat{c}_k \hat{c}_{k+q}.$$  

(1.2)

This Hamiltonian now contains an interaction between electrons that is mediated through a virtual phonon. Note that when $|\epsilon_k - \epsilon_{k+q}| < \omega_q$, the interaction will be attractive. The problem can be simplified by imposing a set of restrictions. First, the only electrons that will partake in this interaction will be those near the Fermi surface of the material. This is because electrons within the Fermi surface will not have any nearby unoccupied states to scatter to. Next, energy conservation will restrict the set of electrons that can interact such that $V_q = 0$ if $|\epsilon_k - \epsilon_{k'}| \geq \omega_D$, where $\omega_D$ is the Debye cutoff frequency. Finally, the center-of-mass momentum between any two electrons will need to be conserved. As a result, the interaction is most likely when $k = -k'$. The formation of Cooper pairs will be favored when the energy of two interacting electrons is less than twice the Fermi Energy $E_F$. Under these conditions, pairs of the electrons with opposite momentum and spin (for s-wave superconductors)
will bond to form Cooper pairs. This effect is described by the BCS Hamiltonian

\[ \hat{H}_{\text{BCS}} = \sum_k (\epsilon_k - E_F) (c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}) - V_{\text{eff}} \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{k\uparrow} c_{-k\downarrow}, \]  

where \( V_{\text{eff}} \) is the effective strength of the interaction. The Cooper-pair bonds are very weak and will only form at low temperatures. The energy required to break up the Cooper pairs is given by the energy gap \( 2\Delta \), which also determines the critical temperature of the superconducting phase transition.

Since Cooper pairs are composed of two electrons, they obey bosonic statistics. As a result, the Cooper pairs will condense into a collective ground state that can be described by a coherent state in terms of the number of Cooper pairs \( n(r,t) \) and a phase \( \phi(r,t) \)

\[ \Psi(r,t) = \sqrt{n(r,t)} e^{i\phi(r,t)}. \]  

The macroscopic nature of this wavefunction is one of the key aspects of superconductivity in the context of quantum circuits. The following sections and chapters of this thesis will show how this wavefunction can be used to describe the physics of quantum circuits.

### 1.1.2 Flux Quantization

In quantum mechanics, a probability current can be defined as a flow of probability per unit time and is given by

\[ J_\psi = \text{Re} \left( \Psi^* \dot{\psi} \Psi \right), \]  

where \( \Psi \) and \( \dot{\psi} \) are the wavefunction and velocity operator, respectively. The macroscopic wavefunction of a superconductor can be thought of as a wavefunction that describes the location of an ensemble of particles. In this sense, the probability current for a superconductor can be thought of as an actual flow of particles, i.e. a physical current. It will be important to consider the effect of an externally applied
magnetic field. The canonical momentum for a particle in an electromagnetic field is
\[ \hat{P} = m\hat{v} + q\hat{A}, \]  
(1.6)
where \( m \) is the mass of the particle, \( q \) is the charge of the particle, and \( \hat{A} \) is the magnetic vector potential. Substituting this into equation 1.5 we can arrive at the supercurrent equation
\[ J_s = Re\left[\Psi^*(\frac{\hat{P}}{m} - \frac{q}{m}\hat{A})\Psi\right] = Re\left[\Psi^*(\frac{\hbar}{m}\nabla - \frac{q}{m}\hat{A})\Psi\right]. \]  
(1.7)
For Cooper pairs we have \( q = 2e \) and \( m = 2m_e \). Next, we substitute in the wavefunction of the superconductor and rearrange terms to arrive at the phase gradient along a path of superconductor
\[ \phi(r_2, t) - \phi(r_1, t) = \frac{m_e}{\epsilon n_0\hbar} \int_{r_1}^{r_2} J_s(r)dr + \frac{q}{\hbar} \int_{r_1}^{r_2} A(r)dr. \]  
(1.8)
We have assumed the Cooper pair density \( n_0 \) to be constant throughout the superconductor. The first term describes a current-induced phase difference from the points \( r_1 \) to \( r_2 \) whereas the second term describes the field-induced phase difference. We can now consider the special case where the path from \( r_1 \) to \( r_2 \), forms a loop. The superconducting phase must be single valued. Therefore, the phase difference around any superconducting loop must be an integer multiple of \( 2\pi \).
\[ 2\pi n = \frac{m}{\epsilon n_0\hbar} \oint_C J_s(r)dr + \frac{q}{\hbar} \oint_C A(r)dr. \]  
(1.9)
Using Stokes’ theorem, the line integral over the vector potential can be converted into a surface integral over the magnetic field
\[ \oint_C A \cdot dr = \iint_S B \cdot dS = \Phi_{ext}, \]  
(1.10)
where $\Phi_{\text{ext}}$ is the external magnetic flux going through the loop. The flux quantization condition can be summarized as follows

$$\Phi_0 n = \frac{m}{2e^2 n_0} \oint_C J_s(r) dr + \Phi_{\text{ext}},$$  \hspace{1cm} (1.11)$$

where $\Phi_0 = h/2e \approx 2.067 \times 10^{-15}$ Wb is known as the superconducting magnetic flux quantum. In general, the "flux" between any two points on a superconductor can be related to the phase difference through this constant $\Phi = \Phi_0 \Delta \phi/2\pi$. This is true for both circuits with and without loops. The physical interpretation of this "flux" will be discussed in the following chapter.

### 1.2 The Josephson Junction

The Josephson effect was first proposed in 1962 [20] to describe the flow of Cooper pairs between two independent superconductors. This became known as a Josephson junction. The most common form of a Josephson junction used in quantum computing consists of two superconductors that are separated by an insulated barrier, as shown in Figure 1-1. Cooper pairs can tunnel across the insulating barrier and couple the two superconductors. In this section, we will derive the equations that describe the physics of a Josephson junction.
1.2.1 The Josephson Equations

The derivation of the Josephson equation begins by defining the wavefunction of each superconductor. As discussed in a previous section, each superconductor can be described by a macroscopic wavefunction in terms of the number of Cooper pairs \( n_i \) and a phase \( \phi_i \)

\[
\Psi_1 = \sqrt{n_1} e^{i\phi_1}
\]

\[
\Psi_2 = \sqrt{n_2} e^{i\phi_2}.
\]  

(1.12)

The capacitive energy per Cooper pair of each superconductor due to a voltage \( V_i \) is given by \( 2eV_i \). Assuming the insulator forms a barrier of constant energy \( E_J \), the physics of the Josephson junction can be described by a Hamiltonian that couples the two pieces of superconductor via Cooper pair tunneling

\[
\hat{H} = \begin{bmatrix} 2eV_1 & -E_J \\ -E_J & 2eV_2 \end{bmatrix}.
\]  

(1.13)

From this Hamiltonian, we can arrive at the Schröedinger equations describing the time evolution of each superconductor

\[
i\hbar \frac{\partial \Psi_1}{\partial t} = 2eV_1 \Psi_1 - \frac{E_J}{n_\Sigma} \Psi_2
\]

\[
i\hbar \frac{\partial \Psi_2}{\partial t} = 2eV_2 \Psi_2 - \frac{E_J}{n_\Sigma} \Psi_1.
\]  

(1.14) \hspace{1cm} (1.15)

We define \( n_\Sigma = n_1 + n_2 \). \( E_J \) is known as the Josephson energy and is determined by a variety of factors, such as the superconducting material and the insulator thickness. Substituting the wavefunctions from 1.12 into the Schröedinger equations and multiplying by \( \Psi_1^* \) and \( \Psi_2^* \) respectively, we find

\[
i\hbar \frac{\dot{n}_1}{2} - \hbar n_1 \dot{\phi}_1 = 2eV_1 n_1 - \frac{E_J}{n_\Sigma} \sqrt{n_1 n_2} e^{i\phi}
\]

\[
i\hbar \frac{\dot{n}_2}{2} - \hbar n_1 \dot{\phi}_2 = 2eV_2 n_2 - \frac{E_J}{n_\Sigma} \sqrt{n_1 n_2} e^{-i\phi},
\]  

(1.16) \hspace{1cm} (1.17)
where $\phi = \phi_1 - \phi_2$ is the phase difference between the superconductors. Separating these equations into their real and imaginary parts and taking differences, we arrive at the equations for $n = n_2 - n_1$ and $\phi$

$$\dot{\phi} = \frac{2e(V_2 - V_1)}{\hbar} - \frac{E_1}{\hbar n_\Sigma} \left( \sqrt{\frac{n_2}{n_1}} - \sqrt{\frac{n_1}{n_2}} \right) \cos(\phi)$$

(1.18)

$$\dot{n} = \frac{2E_1 \sqrt{n_1 n_2}}{\hbar n_\Sigma} \sin(\phi).$$

(1.19)

To arrive at the Josephson equations we will assume that the two superconductors are the same material such that $n_1 \approx n_2$. Next, we can define the current through a junction as $I = 2e\dot{n}$. This simplifies equations 1.18 and 1.19 to

$$\dot{\phi} = \frac{2eV}{\hbar}$$

(1.20)

$$I = I_c \sin(\phi),$$

(1.21)

where $I_c = 2eE_1/\hbar$ is known as the critical current of the junction and $V = V_2 - V_1$ is the voltage difference across the junction. These equations show that a current can flow across the junction in the absence of an applied voltage. Instead, a phase difference between the superconductors is all that is required. This is known as the DC Josephson effect. Equation 1.20 states that a DC voltage across the junction will evolve the phase difference linearly in time $\phi = 2eVt/\hbar$. Therefore, the current through the junction will oscillate with frequency $\omega = 2eV/\hbar$

$$I = I_c \sin \left( \frac{2eV}{\hbar} t \right).$$

(1.22)

This is known as the AC Josephson effect. Next, we can calculate the inductance of the Josephson junction

$$L = V \left( \frac{dI}{dt} \right)^{-1} = \frac{\Phi_0}{2\pi I_c \cos(\phi)} = \frac{\Phi_0}{2\pi I_c \sqrt{1 - \left( \frac{I}{I_c} \right)^2}} = \frac{L_{J0}}{\sqrt{1 - \left( \frac{I}{I_c} \right)^2}},$$

(1.23)
where \( L_{J0} = \Phi_0 / 2\pi I_c \) is known as the Josephson inductance. The inductance of the junction is highly nonlinear with respect to the phase difference and current. This property is critical for the construction of superconducting qubits. Finally, the energy of a Josephson junction can be found by integrating the power \( IV \)

\[
E(\phi) = \int IV dt = -E_J \cos(\phi). \tag{1.24}
\]

### 1.2.2 DC SQUID

Another important circuit element is the DC SQUID (superconducting quantum interference device), which consists of two Josephson junctions in parallel. An external magnetic flux \( \Phi_{\text{ext}} \) may then be threaded through the loop formed by the junctions. This can be used to tune the properties of the circuit. The current passing through the SQUID is given by the sum of the currents through each junction

\[
I = I_{c,L} \sin(\phi_L) + I_{c,R} \sin(\phi_R), \tag{1.25}
\]

where \( I_{c,L(R)} \) and \( \phi_{L(R)} \) are the critical current and phase drop across the left (right) junction. Assuming the SQUID is symmetric \( I_{c,L} = I_{c,R} = I_c \), equation 1.25 can be re-written as

\[
I = 2I_c \cos(\phi_m) \sin(\phi_p), \tag{1.26}
\]

where we have defined \( \phi_p = (\phi_R + \phi_L) / 2 \) and \( \phi_m = (\phi_R - \phi_L) / 2 \). We must now impose flux quantization around the loop formed by junctions. This condition can be expressed as

\[
\phi_R - \phi_L - \frac{2\pi \Phi_{\text{ext}}}{\Phi_0} = 2\pi n, \tag{1.27}
\]

where \( n \) is an integer. Choosing \( n = 0 \), we can solve for \( \phi_R \) and substitute back into equation 1.26 to obtain

\[
I = 2I_c \left| \cos \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_0} \right) \right| \sin(\phi_p). \tag{1.28}
\]
Figure 1-2: A) A circuit diagram of a SQUID, with the crosses representing Josephson junctions. The two junctions form a loop through which an external magnetic flux $\Phi_{\text{ext}}$ can be applied. B) The Josephson energy of a SQUID as a function of external flux through the loop.

This is very similar to the current-phase relation for a single Josephson junction. In fact, the SQUID can be treated as a single Josephson junction with a tunable critical current. Similarly, the Josephson energy $E_J$ of the SQUID becomes tunable

$$E_J(\Phi_{\text{ext}}) = 2E_J \left| \cos \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_0} \right) \right|. \quad (1.29)$$

$E_J$ is plotted in Figure 1-2B. The Josephson energy is periodic with respect to the external flux and reaches its maximum value every $\Phi_0$. It will be shown in the next chapter that this tunability enables the construction of qubits that are tunable in frequency.
Chapter 2

Circuit Quantum Electrodynamics

Superconductivity and the Josephson junction account for much of the interesting physics associated with superconducting quantum mechanical circuits. The former enables dynamics with very low loss by operating the device in question under the transition temperature $T_c$ of the superconducting material. Our circuits are made from aluminum with a $T_c \approx 1.2$ K. However, we operate at a temperature much lower than this to suppress thermal excitations. The quantum states of interest typically have a frequency of $\approx 5$ GHz, which corresponds to a temperature of $T_{th} \approx 250$ mK. Therefore in order to cool our circuits down to the ground state, we must satisfy the condition $T \ll T_{th}$. We accomplish this though the use of a dilution refrigerator where temperatures below 30 mK are achievable. Given this capability, a controllable quantum circuit can be constructed through combinations of linear inductors, capacitors, and Josephson junctions. In the experiments presented in this thesis, Josephson junctions are primarily used to introduce a nonlinearity, and thus an anharmonicity, that is required to form a qubit. In this chapter, we will present the methods of circuit quantization in the context of transmission lines and the transmon qubit.
2.1 Circuit Quantization of a Transmission Line

In this section, we will quantize the guided modes along a one-dimensional superconducting transmission line, i.e. a waveguide. We will follow the standard procedure of introducing the classical Lagrangian and Hamiltonian, describing the system as a set of linear harmonic oscillators, and performing canonical quantization of these modes.

2.1.1 Classical Treatment

The circuit for a transmission line can be approximated as a distributed element model, as shown in Figure 2.1, and can be treated as a 1D EM field theory. Here, we consider infinitesimal unit cells of length $dx$ with series inductance $Ldx$ and capacitance to ground $Cdx$. We can define a generalized flux variable $\Phi(x,t)$ in terms of the voltage along the transmission line [21]

$$\Phi(x,t) = \int_{-\infty}^{t} V(x,\tau) \, d\tau \rightarrow V(x,t) = \Phi(x,t). \quad (2.1)$$

The constitutive equations that describe this system can be found by analyzing a single unit cell of the chain. The voltage and current drop between the positions $x$...
and $x + dx$ are given as follows,

\begin{align*}
V(x + dx, t) &= V(x, t) - L dx \frac{\partial I}{\partial t}, \\
I(x + dx, t) &= I(x, t) - C dx \frac{\partial V}{\partial t}.
\end{align*}

Taking the limit $dx \to 0$ and rearranging the terms will yield the constitutive equations

\begin{align*}
\frac{\partial V(x, t)}{\partial x} &= -L \frac{\partial I(x, t)}{\partial t}, \\
\frac{\partial I(x, t)}{\partial x} &= -C \frac{\partial V(x, t)}{\partial t}.
\end{align*}

These are known as the “telegraph equations” and describe the voltage and current flow through a transmission line. We can then define the flux at each node as $\Phi_n = \Phi(ndx, t)$ and write down the capacitive and inductive energy densities

\begin{align*}
U_C &= \frac{C}{2} V(x, t)^2 = \frac{C}{2} \dot{\Phi}(x, t)^2, \\
U_I &= \frac{L}{2} I(x, t)^2 = \frac{L}{2} \left( \int \frac{dt}{L} \frac{\partial V}{\partial x} \right)^2 = \frac{1}{2L} \left( \frac{\partial \Phi}{\partial x} \right)^2.
\end{align*}

In classical mechanics, the Lagrangian is typically given by the difference between kinetic energy $T$ and potential energy $V$ such that $\mathcal{L} = T - V$. Choosing $\Phi$ to act as the “position” and $\dot{\Phi}$ to act as the “velocity”, the Lagrangian density of the circuit can be found to be

\begin{equation}
\mathcal{L} = U_C - U_I = C \dot{\Phi}(x, t)^2 - \frac{1}{2L} \left( \frac{\partial \Phi}{\partial x} \right)^2.
\end{equation}

We now obtain that the conjugate momentum of the node flux $\Phi(x, t)$ is the charge density $q(x, t)$

\begin{equation}
q(x, t) = \frac{\partial \mathcal{L}}{\partial \Phi} = C \dot{\Phi}(x, t) = CV(x, t).
\end{equation}

This shows that in the quantum treatment, the node charges and fluxes will become conjugate observables that do not commute. The Euler-Lagrange equations yield the
equations of motion for the flux

\[ C\dddot{\Phi}(x, t) - \frac{1}{L} \frac{\partial^2 \Phi(x, t)}{\partial x^2} = 0. \quad (2.8) \]

We can then expand the solution in terms of plane waves and normalization factor \( N \)

\[ \Phi(x, t) = \frac{1}{\sqrt{N}} \sum_k \Phi_k(t) e^{ikx}. \quad (2.9) \]

We will now impose periodic boundary conditions for some length \( l \) such that \( k = 2\pi n/l \) and \( n = 0, \pm 1, \pm 2, \ldots \). This may seem counter-intuitive since the waveguide does not have any boundary conditions along the direction of propagation and should therefore support a continuum of modes. However, this trick simply defines the density of modes in the waveguide, and we will take \( l \to \infty \) to produce a continuum.

The differential equation 2.8 in terms of the \( k^{th} \) mode becomes

\[ k^2 \Phi_k(t) + \frac{k^2}{LC} \Phi_k(t) = 0. \quad (2.10) \]

with linear dispersion \( \omega_k = k/\sqrt{LC} = kv \) and \( v \) being the speed of light in the transmission line. The flux is a real quantity, and thus its Fourier components satisfy \( \Phi_k(t) = \Phi_k^*(t) \). Using this, we can write down a general solution to the differential equation 2.10

\[ \Phi_k(t) = c_k e^{-i\omega_k t} + c_k^* e^{i\omega_k t}. \quad (2.11) \]

Defining a time dependent mode amplitude \( A_k(t) = A_k(0)e^{-i\omega_k t} \) with \( A_k(0) = \sqrt{2\omega_k c_k} \) and substituting into equation 2.9 reveals the flux in position space

\[ \Phi(x, t) = \frac{1}{N\sqrt{2}} \sum_k \frac{1}{\omega_k} (A_k(0)e^{i(kx - \omega_k t)} + A_k^*(0)e^{i(kx + \omega_k t)}) \]

\[ = \frac{1}{N\sqrt{2}} \sum_k \frac{1}{\omega_k} (A_k(0)e^{i(kx - \omega_k t)} + A_k^*(0)e^{-i(kx - \omega_k t)}). \quad (2.12) \]
The second equality can be made since the sum over $k$ runs over $\pm \infty$. The charge
density can be computed by substituting equation 2.12 into equation 2.7

$$q(x, t) = i \frac{C}{N\sqrt{2}} \sum_k A_k(0)e^{i(kx - \omega_k t)} - A_k^*(0)e^{-i(kx - \omega_k t)}$$  \hspace{1cm} (2.13)

Finally, we can perform the Legendre transformation and integrate over the trans-
mission line while taking $l \to \infty$ to obtain the Hamiltonian of the system.

$$H = \int_l \frac{q(x, t)^2}{2C} + \frac{1}{2L} \left( \frac{\partial \Phi(x, t)}{\partial x} \right)^2 \, dx$$  \hspace{1cm} (2.14)

Substituting equations 2.12 and 2.13 into the Hamiltonian and ignoring fast rotating
terms yields

$$H = \frac{Cl}{2N^2} \sum_k A_k(t)A_k(t)^* + A_k^*(t)A_k(t) = 2 \sum_k \omega_k^2 |c_k|^2,$$  \hspace{1cm} (2.15)

where we have defined the normalization constant to be $N = \sqrt{Cl}$. Finally, we can
compute the voltage and current as these are an experimental quantities of interest

$$V(x, t) = \frac{i}{\sqrt{2Cl}} \sum_k A_k(0)e^{i(kx - \omega_k t)} - A_k^*(0)e^{-i(kx - \omega_k t)}.$$  \hspace{1cm} (2.16)

$$I(x, t) = \frac{1}{L\sqrt{2Cl}} \sum_k \frac{1}{\omega_k} (A_k(0)e^{i(kx - \omega_k t)} + A_k^*(0)e^{-i(kx - \omega_k t)}).$$  \hspace{1cm} (2.17)

2.1.2 Quantum Treatment

Equation 2.7 showed that the charge density $q(x, t)$ and flux $\Phi(x, t)$ are canonically
conjugate variables. Using the standard quantization procedure for harmonic oscilla-
tors, the charge and flux can be promoted to non-commuting operators that satisfy

$$\left[ \hat{\Phi}(x, t), \hat{q}(x', t) \right] = i\hbar \delta(x - x').$$  \hspace{1cm} (2.18)
Therefore, the amplitude of each mode $k$ will obey

$$\left[ \hat{A}_k, \hat{A}^\dagger_{k'} \right] = \hbar \omega_k \delta_{k,k'}.$$  \hfill (2.19)

Using this commutation relation, the conventional creation and annihilation operators for the mode $k$ can be identified from the mode amplitude

$$\hat{a}_k = \frac{1}{\sqrt{\hbar \omega_k}} \hat{A}_k.$$  \hfill (2.20)

Substituting this into the central expression in equation 2.14, we arrive at the standard quantum harmonic oscillator Hamiltonian for each mode $k$

$$\hat{H} = \sum_k \hbar \omega_k \left( \hat{a}^\dagger_k \hat{a}_k + \frac{1}{2} \right).$$  \hfill (2.21)

The voltage and flux operators can be found to be

$$\hat{V}(x,t) = \frac{i}{\sqrt{2C_1}} \sum_k \sqrt{\hbar \omega_k} \left( \hat{a}_k e^{i(kx - \omega_k t)} - \hat{a}^\dagger_k e^{-i(kx - \omega_k t)} \right),$$  \hfill (2.22)

$$\hat{\Phi}(x,t) = \frac{1}{\sqrt{2C_1}} \sum_k \sqrt{\hbar / \omega_k} \left( \hat{a}_k e^{i(kx - \omega_k t)} + \hat{a}^\dagger_k e^{-i(kx - \omega_k t)} \right).$$  \hfill (2.23)

Finally, the continuum limit of equations 2.22 and 2.23 can be taken to arrive at the continuum form of the voltage and flux

$$\hat{V}(x,t) = i \int \frac{d\omega}{\sqrt{2\pi}} \sqrt{\hbar \omega Z} \left( \hat{a}(\omega) e^{i(kx - \omega t)} - \hat{a}^\dagger(\omega) e^{-i(kx - \omega t)} \right)$$  \hfill (2.24)

$$\hat{\Phi}(x,t) = \int \frac{d\omega}{\sqrt{2\pi}} \sqrt{\hbar Z / 2\omega} \left( \hat{a}(\omega) e^{i(kx - \omega t)} + \hat{a}^\dagger(\omega) e^{-i(kx - \omega t)} \right).$$  \hfill (2.25)

where the continuum creation and annihilation operators $\hat{a}^\dagger(\omega)$ and $\hat{a}(\omega)$ are given by

$$\hat{a}(\omega) = \sqrt{\frac{2\pi n}{l}} \sum_k \hat{a}_k \delta(\omega - \omega_k)$$  \hfill (2.26)
and we have defined the characteristic line impedance as $Z = \sqrt{L/C}$.

### 2.2 The Transmon Qubit

In this section, we will discuss the construction of a charge qubit [22] and its properties. We will then show how a charge qubit with large capacitance can significantly improve qubit coherence. In this regime, the qubit is known as a transmon. The transmon qubit is the workhorse of modern superconducting quantum information due to its relative simplicity and high coherence. For these reasons, the experiments presented in this thesis will use transmon qubits as well.

#### 2.2.1 The Charge Qubit

The charge qubit, also known as a Cooper pair box, can be modeled as a capacitor in parallel with one or two Josephson junctions, as shown in Figure 2-2. We choose to model a charge qubit with two Josephson junctions in parallel, forming a SQUID. As discussed in Section 1.2.2, a SQUID can be modeled as a single Josephson junction with a tunable $E_J(\Phi_{\text{ext}})$. We absorb the capacitances of the Josephson junctions into the capacitance $C_Q$. This capacitor can be biased by a voltage $V_g$ and an external magnetic flux $\Phi_{\text{ext}}$ can be threaded in the loop formed by the two junctions. We will show that the purpose of these terms is to tune the frequency of the qubit. We begin by writing down the capacitive and inductive energies of the circuit

$$ U_C = \frac{1}{2} C_Q \left( \frac{h \dot{\phi}}{2e} \right)^2 + \frac{1}{2} C_S \left( \frac{h \phi}{2e} - V_g \right)^2 $$

$$ U_1 = -E_J(\Phi_{\text{ext}}) \cos(\phi) $$

where both junctions are symmetric such that $E_J(\Phi_{\text{ext}}) = E_J(\pi\Phi_{\text{ext}}/\Phi_0)$ and the phase degree of freedom is defined by the flux difference across the junction $\phi = \Phi/\Phi_0$. The Lagrangian can be written (by completing the square on the capacitive energy

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Figure 2-2: A circuit diagram of a Cooper pair box. It consists of a SQUID in parallel with a shunt and gate capacitance. The frequency of the qubit can be tuned by an externally applied magnetic flux through the SQUID or a gate voltage $V_g$.

and omitting constant terms) along with the conjugate momentum of the circuit

$$\mathcal{L} = \frac{C_\Sigma}{2} \left( \frac{\hbar \dot{\phi}}{2e} - \frac{C_g V_g}{C_\Sigma} \right)^2 - E_J(\Phi_{\text{ext}})(1 - \cos(\phi)) \quad (2.29)$$

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\hbar C_\Sigma}{2e} \left( \frac{\hbar \dot{\phi}}{2e} - \frac{C_g V_g}{C_\Sigma} \right) \quad (2.30)$$

where $C_\Sigma = C_q + C_g$. Finally, we can promote $\phi \to \hat{\phi}$ and $Q \to \hat{Q}$ and obtain the Hamiltonian of the Cooper pair box

$$\hat{\mathcal{H}}_{\text{CPB}} = 4E_C(\hat{n} - n_g)^2 - E_J(\Phi_{\text{ext}})\cos(\hat{\phi}) \quad (2.31)$$

where $E_C = e^2/2C_\Sigma$, $\hat{n} = \hat{Q}/2e$ represents the number of cooper pairs on the island and $n_g = -C_g V_g/2e$ is an offset charge that is induced by the gate voltage $V_g$. Typically, the energy scales $E_C$ and $E_J$ are quoted in units of frequency. $\hat{\phi}$ can be re-written such that the Hamiltonian is fully expressed in the number basis

$$\hat{\mathcal{H}}_{\text{CPB}} = 4E_C \sum_n (n - n_g)^2 |n\rangle \langle n| - \frac{E_J(\Phi_{\text{ext}})}{2} \sum_n |n\rangle \langle n+1| + |n+1\rangle \langle n|. \quad (2.32)$$

The charge qubit operates in the regime where $E_C \gg E_J$ such that the eigenstates are mostly number-like $|n\rangle$. The eigenspectrum as a function of $n_g$ is plotted in figure 2-3A. As expected, the eigenenergies scale with $n_g^2$. It is clear from this spectrum that gating the island does indeed tune the frequency of the qubit (i.e. the ground to first
Figure 2-3: A) The energy spectrum of a Cooper Pair Box qubit with $E_J/E_C = 0.1$ as a function of the gate charge $n_g$. B) Magnification of the spectrum near $n_g = 0.5$. An avoided level crossing appears at this point. The spectrum is first order insensitive to charge fluctuations at $n_g = 0.5$. Operating the qubit at this point will minimize the effects of charge noise.

excited state energy difference). To operate the Cooper pair box as a qubit, $n_g$ must not be an integer, since the first and second excited states would be nearly degenerate. Instead, it is best to operate the qubit at odd integer multiples of $n_g = 0.5$. At these points, states with $|n\rangle$ and $|n+1\rangle$ cooper pair will hybridize to form an avoided level crossing, shown in Figure 2-3B. The ground and first excited states become equal superpositions of $|n\rangle$ and $|n+1\rangle$ and can be assigned as the computational $|0\rangle$ and $|1\rangle$ states. Note that the qubit has the largest anharmonicity at this point, which helps prevent leakage into the higher non-computational states. Another advantage is that the qubit frequency is first-order insensitive to noise on the gate charge. Therefore, the qubit will also be insensitive to small charge fluctuations that cause dephasing. For this reason, this operation point is known as a sweet spot.

2.2.2 The Transmon Regime

Even though the charge qubit is first-order insensitive to charge noise at the sweet spot, the coherence time of charge qubits typically do not exceed $\sim 100$ ns. However, one can overcome the issue of charge sensitivity by increasing the capacitance of the
Figure 2-4: The spectrum of a Cooper Pair Box qubit for several values of $E_j/E_C$. As this ratio increases, the bands begin to flatten and the transmon regime is entered. The flatter bands reduce sensitivity to charge noise and significantly improve the coherence of the qubit.

superconducting island [23]. That is, to operate the qubit in a regime where $E_C \ll E_J$. The eigenspectrum of $\mathcal{H}_{\text{CPB}}$ is plotted as a function of $n_g$ for several values of $E_j/E_C$ in Figure 2-4. As the capacitance of the island is increased, the bands representing the eigenenergies flatten. Thus, the sensitivity to charge noise on the island decreases. The eigenstates for qubits with higher $E_j/E_C$ become more “phase-like” and overlap with many charge states. When operating the charge qubit in this regime, it is known as a transmon qubit.

Modern transmon qubits can reach coherence times of $T_1, T_2 \approx 100 \mu s$. However, improving the qubit coherence by increasing the capacitance does not come for free.
As can be seen in Figure 2-4, the anharmonicity of the qubit in the transmon regime is much lower. Since the cosine potential is the dominant term, the lowest-lying bound states will be nearly harmonic. To explicitly calculate the anharmonicity, we can expand the Hamiltonian to fourth order and obtain an analytical form of the eigenenergies.

\[ \hat{H}_{\text{Transmon}} \approx 4E_C \hat{n}^2 - E_J(\Phi_{\text{ext}}) \left( 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24} \right) \]  

(2.33)

Since the number and phase operators are conjugate variables, they can be written in terms of the creation and annihilation operators \( \hat{n} = i(\frac{E_J(\Phi_{\text{ext}})}{32E_C})^{1/4}(\hat{a}^\dagger - \hat{a}) \) and \( \hat{\phi} = (\frac{2E_C}{E_J(\Phi_{\text{ext}})})^{1/4}(\hat{a}^\dagger + \hat{a}) \). We can obtain the frequency of the harmonic component of the Hamiltonian to be \( \omega \approx \sqrt{8E_J(\Phi_{\text{ext}})E_C} \) by ignoring the fourth order term in the potential and treating the circuit as a linear LC oscillator. Note that the applied external flux can now be used to tune the qubit frequency. We find the leading order corrections \( E_m \) to the excited state energies to be

\[ E_m = -\frac{E_J(\Phi_{\text{ext}})}{24} \langle m | \hat{\phi}^4 | m \rangle = -\frac{E_C}{12} \left( 6m^2 + 6m + 3 \right) \]  

(2.34)

where the state \( |m\rangle \) corresponds to the \( m^{th} \) eigenstate of the transmon. With the anharmonic correction, it can be shown that \( \omega_{21} - \omega_{10} = E_C \). The parameters of a transmon qubit are typically \( E_C \approx 200 - 300 \text{ MHz} \), \( E_J \approx 10 - 20 \text{ GHz} \), and \( \omega/2\pi \approx 3 - 6 \text{ GHz} \). Because the anharmonicity of the qubit, \( E_C \), is only \( 200 - 300 \text{ MHz} \), qubit operations of length \( < 5 \text{ ns} \) can cause leakage into higher non-computational levels. However, control techniques have been developed to shape the microwave pulse such that this leakage is reduced [24].

### 2.3 Coupling Transmon Qubits to Transmission Lines

In this section, we will derive the Hamiltonian of the circuit shown in Figure 2-5. Here, a flux-tunable transmon is capacitively coupled to the transmission line at a
Figure 2-5: A circuit diagram of a transmon qubit that is capacitively coupled to a microwave transmission line.

single node. The discretized Lagrangian of the circuit is given by

\[
\mathcal{L} = \sum_n C_{dx} \dot{\Phi}_n(t)^2 - \frac{(\Phi_{n+1}(t) - \Phi_n(t))^2}{2L_{dx}} + \frac{C_q}{2} \dot{\Phi}_q(t)^2 + E_3(\Phi_{ext}) \cos\left(\frac{2e\Phi_q}{\hbar}\right) \\
+ \frac{C_c}{2}(\dot{\Phi}_0(t) - \dot{\Phi}_q(t))^2,
\]

where \( \Phi_n(t) = \Phi(ndx, t) \), \( \Phi_q \) is the node flux of the transmon island, and the qubit is coupled to the transmission line at \( x = 0 \). This discretized model of the interaction is a valid approximation when the physical length of the coupling capacitor \( d \sim 100 \mu m \) is much less than the wavelength associated with the qubits frequency \( \lambda \sim 10 \text{ mm} \).

We have modeled the flux-tunable transmon as a single junction transmon with a flux-tunable \( E_3(\Phi_{ext}) \). The conjugate momentum of the flux on the transmission line was shown to be the charge density \( q(x, t) \) in equation 2.7. The conjugate momentum for the transmon degree of freedom is

\[
Q_q(t) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_q} = C_q \dot{\Phi}_q(t) + C_c(\dot{\Phi}_q(t) - \dot{\Phi}_0(t)).
\]

To simplify the calculation, we will assume \( Cd \gg C_q, C_c \). Under this approximation, the qubit capacitances act as a small perturbation on the transmission line. We can
solve for the Hamiltonian of the circuit

\[ \hat{H} \approx \int_{-\infty}^{\infty} \frac{\dot{\hat{q}}(x,t)^2}{2C} + \frac{1}{2L} \left( \frac{\partial \hat{\Phi}(x,t)}{\partial x} \right)^2 \, dx + \frac{\dot{\hat{Q}}_q(t)^2}{2(C_c + C_q)} - E_j(\Phi_{\text{ext}}) \cos \left( \frac{2e\hat{\Phi}_q}{\hbar} \right) + \frac{C_c\dot{\hat{q}}(0,t)\dot{\hat{Q}}_q(t)}{C'} , \]

where \( C' = C_qC + C_cC \). Finally, we may impose canonical quantization \( \hat{q}(x,t) \rightarrow \hat{\Phi}(x,t), \hat{Q}_q(t) \rightarrow \hat{\Phi}_q(t), \Phi(x,t) \rightarrow \hat{\Phi}(x,t), \) and \( \Phi_q(t) \rightarrow \hat{\Phi}_q(t) \) such that the Hamiltonian becomes

\[ \hat{H} = \int_{-\infty}^{\infty} \frac{\dot{\hat{q}}(x,t)^2}{2C} + \frac{1}{2L} \left( \frac{\partial \hat{\Phi}(x,t)}{\partial x} \right)^2 \, dx + \frac{\dot{\hat{Q}}_q(t)^2}{2(C_c + C_q)} - E_j(\Phi_{\text{ext}}) \cos \left( \frac{2e\hat{\Phi}_q}{\hbar} \right) + \frac{C_c\dot{\hat{q}}(0,t)\dot{\hat{Q}}_q(t)}{C'} . \]

The capacitive coupling of a transmon qubit to a transmission line is shown to be a “charge-like” transverse interaction. The continuum form of the quantized charge density \( \hat{q}(x,t) = C\hat{\Phi}(x,t) \) and flux \( \hat{\Phi}(x,t) \) of the transmission line can be found from equations 2.24 and 2.25

\[ \hat{q}(x,t) = iC \int \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{\hbar\omega Z}{2}} \left( \hat{a}(\omega)e^{i(kx - \omega t)} - \hat{a}^\dagger(\omega)e^{-i(kx - \omega t)} \right) \]

\[ \hat{\Phi}(x,t) = \int \frac{d\omega}{\sqrt{2\pi}} \sqrt{\frac{\hbar Z}{2\omega}} \left( \hat{a}(\omega)e^{i(kx - \omega t)} + \hat{a}^\dagger(\omega)e^{-i(kx - \omega t)} \right) . \]
Chapter 3

Theory for WQED with Superconducting Qubits

In this chapter, we will present a master equation to describe the state evolution of the qubits. An input-output theory will also be presented to keep track of the photons in the waveguide. We will then use these results to derive the single- and multi-qubit effects that will be demonstrated experimentally.

3.1 Master Equation Formalism

Having derived the interaction of a single transmon qubit with a transmission line, we can now generalize this treatment to include multiple qubits. We begin by re-writing the Hamiltonian in equation 2.38 in second quantization by substituting in equations 2.39 and 2.40. We invoke the two-level approximation on the qubits such that $\hat{Q}_{q,j} \sim \hat{\sigma}_x^j$ for qubit $j$, where $\hat{\sigma}_x$ is the Pauli X operator. In this form, the Hamiltonian describing the electromagnetic fields in the transmission line is

$$\hat{H}_{EM} = \int d\omega \ h \omega \ (\hat{a}_r^\dagger(\omega)\hat{a}_r(\omega) + \hat{a}_l^\dagger(\omega)\hat{a}_l(\omega)).$$

(3.1)

Here, $\hat{a}_{r(l)}(\omega)$ represent the right (left) propagating modes in the transmission line for the modes with positive (negative) wavevector $k = \pm 2\pi/\lambda$. The Hamiltonian for the
The Hamiltonian for qubits is given by

\[ \hat{H}_q = \hbar \sum_{j=1}^{N} \omega_j \hat{\sigma}_j^+ \hat{\sigma}_j^- \] (3.2)

where \( \omega_j \) is the frequency of the \( j^{th} \) qubit for \( N \) number of qubits and \( \hat{\sigma}_j^\pm \) are the Pauli raising/lowering operators. The interaction Hamiltonian is then

\[ \hat{H}_I = \hbar \sum_{j=1}^{N} g_j \hat{\sigma}_j^\tau \int d\omega \sqrt{\omega} \left( (\hat{a}_j^\dagger(\omega)e^{ikx_j} - \hat{a}_j(\omega)e^{-ikx_j}) + (\hat{a}_j^\dagger(\omega)e^{-ikx_j} - \hat{a}_j(\omega)e^{ikx_j}) \right) \] (3.3)

where \( x_j \) is the position of the \( j^{th} \) qubit. The coupling strength \( g_j \) between the \( j^{th} \) qubit and the transmission line is

\[ g_j = \sqrt{\frac{e^2C}{2\hbar \pi v(C_j/C_{c,j})^2}} \left( \frac{E_j}{E_C} \right)^\frac{1}{4}, \] (3.4)

where the definitions of these variables are the same as in the previous chapter.

### 3.1.1 The Master Equation

Even though this Hamiltonian fully describes the system of interest (to within the stated approximations), it is typically difficult to perform calculations with due to the continuum in the transmission line. Fortunately, we can switch to a master equation formalism, which treats the degrees of freedom of the transmission line as a dissipative bath that is coupled to the qubits. That is, it only keeps track of the dynamics of the qubits and their mutual interaction via the transmission line; the state evolution of the transmission line is not explicitly accounted for. The derivation of the master equation from the Hamiltonians in equations 3.1 - 3.3 is shown in Appendix A and follows similar calculations done in references [25, 26]. Our model will assume that the transmission line begins in a coherent state and is coupled to \( N \) qubits that are equally spaced apart. The master equation for the qubits is given by

\[ \dot{\hat{\rho}} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_{j,k=1}^{N} \Gamma_{j,k}(\hat{\sigma}_j^- \hat{\rho} \hat{\sigma}_k^+ - \frac{1}{2} \{ \hat{\sigma}_k^+ \hat{\sigma}_j^-, \hat{\rho} \}). \] (3.5)
The effective Hamiltonian $\hat{H}$ describes the Hermitian evolution of the qubits and is given by

$$\hat{H} = \hbar \sum_{j=1}^{N} \omega_j \hat{\sigma}_j^+ \hat{\sigma}_j^- + \hbar \sum_{j=1}^{N} \alpha_j(t) \hat{\sigma}_j^z + \hbar \sum_{j \neq k}^{N} J_{j,k} \hat{\sigma}_j^- \hat{\sigma}_k^+.$$ (3.6)

The first term in the Hamiltonian describes the energy of each qubit. The second term describes the coherent drive on the qubits from the photons in the transmission line. The drive amplitude $\alpha_j(t)$ can be shown to be

$$\alpha_j(t) = \frac{\Omega_{d,j}}{2} \sin(\omega_{d,l}(t + t_j + \theta_l)) + \frac{\Omega_{d,j}}{2} \sin(\omega_{d,r}(t - t_j + \theta_r))$$ (3.7)

where $\Omega_{d,(r),j}$ is the amplitude of the drive on the $j^{th}$ qubit, $\omega_{d,l(r)}$ is the frequency of the drive, and $\theta_{l(r)}$ is the phase of the drive applied from the left (right). The position of each qubit is defined in $t_j = \frac{z_j}{c}$. The correlated decay rate $\Gamma_{j,k}$ and two qubit exchange coupling $J_{j,k}$ terms are given by

$$\Gamma_{j,k} = 2\pi g_j g_k (\omega_j e^{i\omega_j t_{j,k}} + \omega_k e^{-i\omega_k t_{j,k}})$$ (3.8)

$$J_{j,k} = -i\pi g_j g_k (\omega_j e^{i\omega_j t_{j,k}} - \omega_k e^{-i\omega_k t_{j,k}})$$ (3.9)

where $t_{j,k} = |t_j - t_k|$ is the time delay for a photon to travel between qubits $j$ and $k$.

For the specific case where qubits $j$ and $k$ are resonant, these terms simplify to

$$\Gamma_{j,k} = 4\pi g_j g_k \omega_j \cos(\omega_j t_{j,k})$$ (3.10)

$$J_{j,k} = 2\pi g_j g_k \omega_j \sin(\omega_j t_{j,k}).$$ (3.11)

This implies that when qubits $j$ and $k$ are spaced apart by integer multiples of $\lambda/2$, the correlated decay rate $\Gamma_{j,k}$ is maximized and the coherent exchange rate $J_{j,k} = 0$. Conversely, when qubits $j$ and $k$ are spaced apart by odd integer multiples of $\lambda/4$, $\Gamma_{j,k} = 0$ and $J_{j,k}$ is maximized. Note that $J_{j,k} \leq \frac{\Gamma_{d,j}}{2}, \frac{\Gamma_{d,k}}{2}$ when $g_j = g_k$, where $\Gamma_{d,j}$ is the self-decoherence rate for qubit $j$ due to the transmission line. We will discuss the multi-qubit properties of the system in more detail in Section 3.3.
To simplify the problem even further, we can assume that there is only a single coherent drive propagating from left to right. In this case, we may move into the rotating frame of the drive and apply the rotating wave approximation (RWA) to obtain a simple time-independent effective Hamiltonian

$$\hat{H}_{rf} = \hbar \sum_{j=1}^{N} \Delta_j \hat{\sigma}_j^+ \hat{\sigma}_j^- + \hbar \sum_{j=1}^{N} \left( \frac{\Omega_{ij}}{2} \hat{\sigma}_j^+ + h.c. \right) + \hbar \sum_{j \neq k}^{N} J_{j,k} \hat{\sigma}_j^- \hat{\sigma}_k^+, \quad (3.12)$$

where $\Delta_j = \omega_j - \omega_d$ is the qubit-drive detuning and $\Omega_{ij} = -i \sqrt{2 \Gamma_{j,i} \omega_d / \omega_j} \langle a_{1,1n} \rangle e^{-i \omega_d t_j}$ is the input drive amplitude. In this rotating frame, it is clear that a coherent drive in the transmission line will drive Rabi oscillations on the qubits. So far in this model, we have not included spontaneous emission rate $\Gamma_{nr}$ to modes outside the transmission line, nor have we considered the qubit’s pure dephasing rate $\Gamma_{\phi}$. The rate $\Gamma_{nr}$ is colloquially known as the “nonradiative” and is borrowed from atomic systems. It is important to note that physical decay channels that this term accounts for are radiative, but do not radiate into the mode of interest. We can account for $\Gamma_{nr,j}$ and $\Gamma_{\phi,j}$ by modifying the qubit decoherence rate $\Gamma'_{j,k} = \Gamma_{j,k} + \delta_{j,k} \Gamma_{nr,j}$ and adding an additional dissipator $\Gamma_{\phi,j} (\sigma_j^x \rho \sigma_j^x - \rho) / 2$ to the master equation. While we do include these contributions in our simulations, our devices will operate in the regime where $\Gamma_{j,j} \gg \Gamma_{nr}, \Gamma_{\phi}$. This is known as the strong coupling regime. The master equation with these contributions is

$$\dot{\rho} = -\frac{i}{\hbar} [\hat{H}_{rf}, \rho] + \sum_{j,k} \Gamma'_{j,k} (\hat{\sigma}_j^- \hat{\rho} \hat{\sigma}_k^+ - \frac{1}{2} \{\hat{\sigma}_j^+ \hat{\sigma}_j^-, \hat{\rho}\}) + \sum_{j} \frac{\Gamma_{\phi,j}}{2} (\sigma_j^x \hat{\rho} \sigma_j^x - \hat{\rho}). \quad (3.13)$$

3.1.2 Input-Output Theory

Even though the master equation 3.13 does not explicitly keep track of the modes in the transmission line, it is still possible to gain some of this information through the use of input-output theory, where the transmission line field modes can be described in terms of the input field modes and the qubit states. We begin with the Heisenberg
equation of motion for the right propagating fields

\[ \hat{\dot{a}}_r(\omega) = -i\hat{a}_r(\omega) + \sum_{j=1}^{N} g_j \sqrt{\omega} \hat{\sigma}_j^x e^{-ikx_j}. \]  

(3.14)

A similar expression for \( a_{l,k} \) can be found with the replacement of \( x_j \rightarrow -x_j \). Performing formal integration from an initial time \( t_0 = 0 \) yields

\[ \hat{\dot{a}}_r(\omega, t) = \hat{a}_r(\omega, 0)e^{-i\omega t} + \sum_{j=1}^{N} g_j \sqrt{\omega} \int_{0}^{t} e^{-i\omega(t-\tau+t_j)} \hat{\sigma}_j^x(\tau)d\tau \]  

(3.15)

By writing the master equation in Lindblad form, we have already made an assumption that the dynamics of the system are not fast on the time-scales that we care about. That is, we can only compute the dynamics for \( \omega_k t \rightarrow \infty \). Since WQED with superconducting qubits typically deals with photons in the GHz frequency range, the dynamics of interest must occur on a time scale \( t \gg 0.1 \) ns, which is easily satisfied. When considering more than one qubit, this solution will only be valid for \( t \gg t_{jk} \sim 0.1 \) ns. We can also compute \( a_r(\omega, t) \) by integrating up to a time \( t_f > t \) after the interaction

\[ \hat{\dot{a}}_r(\omega, t) = \hat{a}_r(\omega, t_f)e^{-i\omega t} - \sum_{j=1}^{N} g_j \sqrt{\omega} \int_{t}^{t_f} e^{-i\omega(t-\tau+t_j)} \hat{\sigma}_j^x(\tau)d\tau. \]  

(3.16)

Subtracting equation 3.15 from 3.16 and integrating over \( \omega \) gives an input-output boundary condition which must be satisfied

\[ \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \hat{\dot{a}}_r(\omega, t_f)e^{-i\omega t} = \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \hat{a}_r(\omega, 0)e^{-i\omega t} \]

+ \[ \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \sum_{j=1}^{N} g_j \sqrt{\omega} \int_{0}^{t_f} d\tau e^{-i\omega(t-\tau+t_j)} \hat{\sigma}_j^x(\tau). \]  

(3.17)

The input and output fields can now be defined to be as follows

\[ \hat{\dot{a}}_r^{\text{out}}(t) = \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \hat{a}_r(\omega, t_f)e^{-i\omega t} \]  

(3.18)
The final challenge that remains is to simplify the second expression on the right hand side of equation 3.17 into a more useful form. The complexity in this expression is the dependence of $\hat{\sigma}^x$ on the integration variable $\tau$. To simplify the expression, we begin by rewriting

$$\hat{\sigma}_{j}^{x}(\tau) = e^{i\hat{H}_{T}(\tau-t)/\hbar} \hat{\sigma}_{j}^{-}(t)e^{-i\hat{H}_{T}(\tau-t)/\hbar} + e^{i\hat{H}_{T}(\tau-t)/\hbar} \hat{\sigma}_{j}^{+}(t)e^{-i\hat{H}_{T}(\tau-t)/\hbar}, \quad (3.20)$$

where $\hat{H}_{T} = \hat{H}_{EM} + H_{q} + H_{1}$. To simplify this expression even further, the interaction Hamiltonian $H_{1}$ can be neglected when compared to the free Hamiltonian $\hat{H}_{q}$ of the qubits. Doing so will result in $\hat{\sigma}_{j}^{x}(\tau) \approx \hat{\sigma}_{j}^{-}(t)e^{-i\omega_{j}(\tau-t)}$. Additionally, we will assume that we are only interested in the long time limit $\omega_{j}(t_{f} - t) \rightarrow \infty$. Applying these approximations to equation 3.17 and evaluating the integrals will yield a simpler form of the input-output relation

$$\hat{a}_{r}^{\text{in}}(t) = \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \hat{a}_{r}(\omega, 0)e^{-i\omega t} \quad (3.19)$$

The boundary condition and input-ouput relation for the left propagating modes can be computed in a similar fashion and is given by

$$\hat{a}_{l}^{\text{out}}(t) = \hat{a}_{l}^{\text{in}}(t) + \sum_{j=1}^{N} e^{-i\omega_{j}t} \sqrt{\frac{\gamma_{j}^{l}}{2}} \hat{\sigma}_{j}^{-}(t). \quad (3.21)$$

Using these relations, information about the state of the qubits can be extracted by measuring the moments of the output field. This will be discussed in more detail in Chapter 4.

### 3.2 Scattering of a Qubit in a Transmission Line

To characterize the single qubit properties of the devices, we typically measure the scattering properties of coherent light incident on the qubits. This consists of mea-
suring the reflection and/or transmission of a probe signal. In this section, we will derive the equations that describe the scattering behavior of a single qubit coupled to a transmission line. This method of characterization will be extended to devices with more than one qubit per transmission line.

3.2.1 Semi-classical Treatment

Before considering the full quantum case, it is much simpler and instructive to perform a semi-classical calculation. The input fields are assumed to be classical (coherent) waves that are incident on a qubit. We return to the telegraph equations

\[ \frac{\partial V(x,t)}{\partial x} = -L \frac{\partial I(x,t)}{\partial t} \]  
(3.23)

\[ \frac{\partial I(x,t)}{\partial x} = -C \frac{\partial V(x,t)}{\partial t}, \]  
(3.24)

except now there is a boundary condition at the location of the qubit \( x = 0 \)

\[ I(0^+,t) = I(0^-,t) - C_c \frac{\partial \langle \hat{V}_q \rangle}{\partial t} \]  
(3.25)

\[ V(0^+,t) = V(0^-,t). \]  
(3.26)

This boundary condition defines the current draw of the transmon and \( \hat{V}_q = \hat{Q}_q/(C_c + C_q) \) is the voltage on the qubit island. Since this calculation treats the transmission line classically, the electromagnetic fields in the transmission line are taken to be in a coherent state. Thus the voltage and current will take the form of sinusoidal waves. We can separate the classical field into components that are left \((x < 0)\) and right \((x > 0)\) of the qubit

\[ V(x < 0, t) = V_0(e^{i(kx-\omega t)} - r e^{-i(kx+\omega t)}) \]  
(3.27)

\[ V(x > 0, t) = tV_0 e^{i(kx-\omega t)} \]  
(3.28)
Here, we defined the complex reflection and transmission amplitudes $r$ and $t$, respectively. The voltage to the left of the qubit $V(x < 0, t)$ consists of a weak forward propagating input probe (first term) and the reflected signal (second term). The voltage to the left of the qubit $V(x > 0, t)$ is solely given by the transmitted signal. The boundary condition in equation 3.26 gives $t = 1 - r$. Substituting the voltage into 3.24 and integrating over $x$ gives the current

$$I(x < 0, t) = \frac{CV_0\omega}{k} (e^{i(kx - \omega t)} + r e^{-i(kx + \omega t)}) \tag{3.29}$$

$$I(x > 0, t) = \frac{tCV_0\omega}{k} e^{i(kx - \omega t)} \tag{3.30}$$

Finally, applying the boundary condition 3.25 will give the reflection and transmission amplitude

$$r = \frac{C_c}{2I_0(C_c + C_q)} \frac{\partial \langle \hat{Q}_q \rangle}{\partial t} e^{i\omega t} \tag{3.31}$$

$$t = 1 - \frac{C_c}{2I_0(C_c + C_q)} \frac{\partial \langle \hat{Q}_q \rangle}{\partial t} e^{i\omega t}. \tag{3.32}$$

### 3.2.2 Master Equation Solution

Now that we have found the form of the scattering amplitudes, we will need to solve for $\langle \hat{Q}_q \rangle$. For a transmon qubit this is [23]

$$\langle \hat{Q}_q \rangle = 2e^\frac{i}{\sqrt{2}} \left( \frac{E_J(\Phi_{\text{ext}})}{8E_c} \right) \frac{1}{\langle \hat{\sigma}^x \rangle} \tag{3.33}$$

The expectation value $\langle \hat{\sigma}^x \rangle$ is given by $\text{Tr}(\hat{\rho}\hat{\sigma}^x) = \rho_{01} + \rho_{10} = 2\text{Re}(\rho_{10})$. We have now reduced the problem to computing the state evolution $\dot{\rho}(t)$, which can found by using the master equation 3.13 for a single qubit. Dephasing will be the main decoherence channel that is not to the waveguide, and thus we will include it in this computation.

We will ignore $\Gamma_{nr}$ at first, but will reintroduce its contribution at the end. We are interested in the steady state solution of the master equation $\dot{\rho} = 0$. Solving for $\rho_{10}$
Figure 3-1: A schematic diagram of the interference effect between a weak coherent tone incident upon a qubit. The emission from the qubit destructively interferes with the forward propagating field and will therefore reflect the signal.

in the lab frame will yield

\[
\rho_{10} = -\frac{\Omega}{2\Gamma_2} \left( i + \frac{\Delta}{\Gamma_2} \right) e^{i\omega t},
\]

(3.34)

where \( \Gamma_2 = \Gamma_{1,1} + \Gamma_\phi \) is the total decoherence rate of the qubit, \( \Gamma_1 = 2\Gamma_{1,1} \) is the spontaneous emission rate of the qubit into the transmission line, and \( \Delta = \omega_1 - \omega_p \) is the qubit-probe detuning. We can then obtain \( \langle \hat{\sigma}^x \rangle \)

\[
\langle \hat{\sigma}^x \rangle = -\frac{\frac{\Omega}{\Gamma_2}}{1 + \left( \Delta \Gamma_2 \right)^2 + \frac{\Omega^2}{\Gamma_1 \Gamma_2}} \left( \frac{\Delta}{\Gamma_2} \cos(\omega t) - \sin(\omega t) \right).
\]

(3.35)

In our model we assumed the drive is propagating to the right. Therefore when calculating the reflection, we should only consider negative frequency modes that propagate to the left.

\[
\langle \hat{\sigma}^z_L \rangle = -\frac{\frac{\Omega}{\Gamma_2}}{1 + \left( \Delta \Gamma_2 \right)^2 + \frac{\Omega^2}{\Gamma_1 \Gamma_2}} \left( 1 - i \frac{\Delta}{\Gamma_2} \right) e^{-i\omega t}.
\]

(3.36)
Figure 3-2: The reflectivity and transmittivity of a weak coherent tone as a function of the detuning of the tone from the qubit. The scattering will only be perfectly coherent \((R + T = 1)\) for infinitesimally weak drives and zero dephasing and nonradiative decay of the qubit.

Before obtaining the scattering coefficients, we need to specify \(\Gamma_{1,1}\) in terms of the physical parameters of the device:

\[
\Gamma_{1,1} = \frac{|\langle 0 | \hat{Q}_q | 1 \rangle |^2}{2\hbar} \left( \frac{C_c}{C_c + C_q} \right)^2 \omega_q \sqrt{\frac{L}{C}}. \quad (3.37)
\]

We insert equation 3.36 into equation 3.31 to obtain the reflection coefficient

\[
\frac{r}{2} = \frac{\Gamma_1}{2\Gamma_2} \frac{1 - \frac{i\Delta}{\Gamma_2}}{1 + \left( \frac{\Delta}{\Gamma_2} \right)^2 + \frac{\Omega^2}{\Gamma_1 \Gamma_2}} \quad (3.38)
\]

We plot the reflectance \(R = |r|^2\) and transmittance \(T = |t|^2\) as a function of \(\Delta/\Gamma_1\) assuming \(\Gamma^o = 0\) and low power \(\Omega \ll \Gamma_1\) in Figure 3-2. The lack of transmission when \(\Delta = 0\) can be understood from interference. The forward propagating radiation of the qubit into the transmission line acquires a \(\pi\) phase shift. This causes the input tone to destructively interfere with the qubit radiation. Similarly, the reflected signal constructively interferes with the emission from the qubit. Higher drive powers
will saturate the qubit and reduce the degree of interference. Real qubits will have some finite amount of decay to other modes. We can reintroduce the effect of the non-radiative decay by modifying the reflection amplitude in equation 3.39

\[ r = \frac{\eta \Gamma_1}{2 \Gamma_2} \frac{1 - \frac{i \Delta}{\Gamma_2}}{1 + \left( \frac{\Delta}{\Gamma_2} \right)^2 + \frac{\Omega^2}{\Gamma_1 \Gamma_2}}. \]  

(3.39)

\( \eta \) is given by the ratio of emission into the waveguide vs. other modes. These processes will cause incoherent scattering such that \( R + T < 1 \).

Assuming a probe from the left, the reflection and transmission can also be calculated from input-out theory

\[ T = \frac{|\langle \hat{a}^\text{out}_1 \rangle|^2}{|\langle \hat{a}^\text{in}_1 \rangle|^2}, \]  

(3.40)

\[ R = \frac{|\langle \hat{a}^\text{out}_r \rangle|^2}{|\langle \hat{a}^\text{in}_1 \rangle|^2}. \]  

(3.41)

For devices with more than one qubit, we use the python package QuTiP (Quantum Toolkit in Python) to numerically calculate the reflection and transmission from input-output theory.

### 3.3 Two Qubits in a Waveguide

Thus far, we have only considered the physics of a single qubit coupled to a transmission line. We will now discuss the interesting effects that emerge when two resonant qubits are coupled to a common waveguide. In particular, we will see how the distance \( d \) between the qubits plays a key role.

#### 3.3.1 Correlated Dissipation

Consider the case of two resonant qubits \( \omega_1 = \omega_2 = \omega_q \) that are spatially separated by one wavelength \( d = \lambda \) with respect to \( \omega_q \). Furthermore, we will assume that there is no drive and, for the time being, we will ignore all decoherence channels that are unrelated to waveguide. Under these conditions, the master equation will only
depend on the dissipators. Equations 3.10 and 3.11 state that the correlated decay rate between the qubits will be maximal, and that there will be no exchange coupling.

To deal with the correlated decay, we can perform a change of basis that diagonalizes the dissipators. The master equation can then be written as

$$\dot{\rho} = \sum_{k=B,D} \Gamma_k D[\sigma_k^-] \rho,$$

(3.42)

where $D[\sigma_k^-] \rho = \sigma_k^- \rho \sigma_k^+ - \{\sigma_k^+ \sigma_k^- \}, \rho \}$. The indices B, D correspond to the bright and dark states that form the new basis. The bright state will have an enhanced decay rate, whereas the decay of the dark state will be suppressed. This is similar to the formation of bright and dark states between qubits in a cavity [27]. The diagonalized decay rates and lowering operators can be shown to be [25]

$$\Gamma_{B,D} = \frac{\Gamma_{1,1} + \Gamma_{2,2}}{2} \pm \sqrt{\left(\frac{\Gamma_{1,1} - \Gamma_{2,2}}{2}\right)^2 - \Gamma_{1,2}^2}$$

(3.43)

$$\hat{\sigma}_{B,D} = \frac{(\Gamma_{B,D} - \Gamma_{2,2}) \hat{\sigma}_{1}^- + \Gamma_{1,2} \hat{\sigma}_{1}^-}{\sqrt{(\Gamma_{B,D} - \Gamma_{2,2})^2 + \Gamma_{1,2}^2}}.$$

(3.44)

The bright and dark states can then be defined as the zero eigenvalue eigenstates of their respective lowering operator. All qubits in our experiments will be coupled to the waveguide equally $\Gamma_{1,1} = \Gamma_{2,2} = \Gamma$. In this case, the bright state will be the triplet state $|B\rangle = |eg\rangle + |ge\rangle$ and the dark state will be the singlet state $|D\rangle = |eg\rangle - |ge\rangle$.

Additionally, the symmetric coupling implies that the decay rate for these states will

Figure 3-3: Two resonant qubits that are coupled to a common 1D waveguide and are spaced apart by one wavelength corresponding to their frequency.
Figure 3-4: A) The level diagram for two qubits at a λ spacing. The dark state is decoupled from the other states and can only be populated by nonradiative decay from |ee⟩ or dephasing from |B⟩. The states |ee⟩ and |B⟩ decay with an enhanced emission rate due to superradiance. B) The transmission of a weak coherent tone when two resonant qubits are present at a λ spacing. The spectrum is broadened by superradiance. The transmission increases at zero detuning due to the formation of a dark state.

be \( \Gamma_B = 2\Gamma, \Gamma_D = 0 \). That is, the dark state is completely decoupled from the waveguide. One can intuitively understand this by looking at the symmetries of the setup. Figure 3-3 shows that the two qubits see the same phase of light in the waveguide at the qubit frequency. Therefore, the waveguide can only induce transitions to and from the symmetric state |eg⟩ + |ge⟩. Additionally, the decay of the qubits in the symmetric state will constructively interfere to enhance the emission rate. Decay from the anti-symmetric state |eg⟩ − |ge⟩, however, will destructively interfere and suppress the emission rate. This phenomenon of collectively enhanced and suppressed emission rates is known as super-radiance and sub-radiance, respectively [28].

Super- and sub-radiance will occur for qubits that are spaced apart by any integer multiple of \( \lambda/2 \). For even integer multiples, the effect is the same as what was described for \( d = \lambda \). For odd integer multiples, however, the roles of symmetric and anti-symmetric states are reversed. This is because the qubits will now see an opposite phase of light in the waveguide with frequency \( \omega_q \). Finally, we can reintroduce the effects of decoherence into other channels. By definition, the non-radiative decay \( \Gamma_{nr} \) will occur for all states. In particular, the dark state will now spontaneously emit
with a rate $\Gamma_D = \Gamma_{nr}$. The dephasing rate of each qubit will cause transitions between the dark and bright state, creating another decay pathway for the dark state. While these rates are small in comparison to the qubits coupling to the waveguide, they will impose limitations on the length of any operation that involves the dark state. Figure 3-4A shows the level diagram of the two qubits and the decay channels.

The bright and dark states can be probed by a transmission measurement through the waveguide. Using input-output theory and equation 3.41, the transmission for two resonant qubits at $\lambda$ spacing is calculated and plotted in Figure 3-4B. A dip in the transmission with twice the linewidth $2\Gamma$ can be seen and is due to the enhanced emission of the bright state. The data showing this effect will be presented in the following chapter. A sharp peak at zero detuning where the transmission approaches unity is also present. This is due to the formation of a dark state. Since the dark state is decoupled from the waveguide, it will not participate in the interference effects that cause the suppression in transmission. Instead, light in the waveguide will, ideally, transmit perfectly. It is difficult to observe this feature in practice due to its very narrow linewidth.

### 3.3.2 Two-Qubit Exchange Coupling

We will now consider the same setup in the previous section with a different qubit spacing $d = 3\lambda/4$. Equations 3.10 and 3.11 now state that the correlated decay rate is 0 and the qubit exchange coupling is maximized. The Hamiltonian of the two qubits in the rotating frame is then given by

$$\hat{H} = \hbar \sum_{j \neq k}^N J_{j,k} \hat{\sigma}_j^- \hat{\sigma}_k^+.$$  \hspace{1cm} (3.45)

Therefore, the two qubits will hybridize and the eigenstates will split in energy by $2J_{j,k}$. The lack of correlated decay is expected because each qubit is at a node with respect to emission from the other. Note that the qubits still decay independently into the waveguide (i.e. $\Gamma_{jj} \neq 0$). The maximized exchange coupling can be understood by considering the nature of the interaction. Each qubit will emit and absorb
virtual photons (ω ≠ ω_q) from each other. The coupling due to each mode k can be approximated as J_k = g_{1,k}g_{2,k}/δ_k, where δ_k = ω_k - ω_q. Although this form of J_k may not be correct for modes near the qubit frequency, the parity with respect to g and δ will remain the same. Figure 3-5 shows an illustration of modes above and below the qubit frequency. It can be seen that g_{2,k} will have an opposite sign for modes with δ_k > 0 and δ_k < 0. However, the change in sign of δ_k makes the overall sign of J_k the same for all modes. A finite exchange interaction can be achieved when summing up the contributions from each mode. The same reasoning can be applied to explain why there was no exchange interaction for d = λ. In this case, J_k will have an opposite sign for modes above and below the qubit frequency. The contributions then cancel to 0 when summing them all up. The physics of d = 3λ/4 will remain the same for d = (2n + 1)λ/4, where n is an integer. The overall sign of J_{1,2} will be positive for even n and negative for odd n.
Chapter 4

Measurements

We will present our experimental results in this chapter. First, we will describe the cryogenic setup inside our fridge. Next, we will present our techniques for single qubit characterization. This will include both VNA scans of transmission as well as pulsed time-domain measurements. We will then present our data for two-qubit experiments. Finally, we will propose a method to dissipatively entangle two qubits using when they are spaced apart by $\lambda$ along the waveguide.

4.1 Cryogenic Setup

The cryogenic setup for measuring our devices is shown in Figure 4-1. The sample consists of three qubits that are capacitively coupled to a 50 $\Omega$ coplanar waveguide. The sample sits at the mixing chamber of the fridge which is cooled down to $< 12$ mK. Cryogenic circulators are installed at both ends of the sample to allow us to apply signals and measure the output from either end. Attenuation is added to the input lines at the different temperature stages to minimize the thermal noise at the sample. 20 dB of attenuation is added at the 4K stage, 10 dB at the 1K stage, and 40 dB at the mixing chamber. Cryogenic isolators, a 4 GHz high-pass filter, and a 12 GHz low-pass filter are added at the mixing chamber (filters not shown in Figure 4-1) to suppress thermal noise from the readout lines. The readout chain consists of three amplifiers: a traveling wave parametric amplifier (TWPA) at the mixing chamber, a
Figure 4-1: Schematic diagram of cryogenic setup inside the dilution fridge. The three qubit sample is placed at the mixing chamber and inside mu-metal and superconducting shields. Circulators are used for dual-sided input and readout. Attenuation is added to the suppress thermal noise on the input lines. Isolators and filters are added to suppress thermal noise on the output lines. The three amplifiers used for readout are a TWPA at the mixing chamber, a Low Noise Factory (LNF) HEMT at the 4K stage, and a MITEQ HEMT at room-temperature. Superconducting NbTi coaxial cables are used from the mixing chamber to the still on the readout lines.
Local Flux Bias

Figure 4-2: The transmon design with local charge and flux control. X/Y rotations are applied using a microwave drive $\Omega$ on the charge line. Local flux control is achieved by applying a current near the SQUID loop of the qubit. This current then generates a magnetic flux through the loop. A global flux can also be applied in the loop through the use of the bobbin coil on the sample package.

Low Noise Factory (LNF) high electron mobility transistor (HEMT) amplifier at the 4K stage, and a MITEQ HEMT amplifier at room temperature. The TWPA acts as a broadband linear amplifier with noise performance that is nearly quantum limited [29]. Figure 4-3 shows the gain spectrum of one of the TWPAAs used. Here, we see a gain of 20 dB - 30 dB for frequencies between 5.5 GHz and 7 GHz. The LNF and MITEQ amplifiers provide an additional $\sim$40 dB and $\sim$30 dB of gain, respectively.

The sample is placed within a gold plated package with a DC coil, known as a bobbin, for global flux control and six microwave inputs/outputs. Each qubit in the sample also has local microwave and flux control, as shown in Figure 4-2. However, only two qubits have their local controls connected to the package due to constraints on the number of connectors available. Using the two local flux control lines and the bobbin, a crosstalk matrix can be measured and diagonalized for independent flux control. The local microwave drive lines are weakly coupled to the qubit to minimize the relaxation into them. VLFX-300 MHz low-pass filters are added to the local flux lines. The DC bobbin lines are filtered with a RC filter at the 4K stage ($R = 50 \, \Omega$, $C = 100 \, nF$) followed by an LC and stainless steel powder filter at the mixing chamber.
Figure 4-3: A plot of the broadband gain spectrum of one of the TWPA used in the experiments. The TWPA dispersion feature is visible and begins at 7.7 GHz.

4.2 Single Qubit Characterization

4.2.1 Scattering Measurements

We begin the characterization of our devices through coherent scattering measurements. As shown in Chapter 3, the qubits will reflect weak coherent tones in the transmission line that are incident upon them. Each qubit can be treated independently as long as they are detuned far from one another $|\omega_i - \omega_j| \gg \Gamma_{i,j}, \Gamma_{j,j}$. We perform low-power transmission measurements through the sample using a Keysight vector network analyzer (VNA). This is shown in the simplified setup diagram in Figure 4-4A. The VNA will measure the transmission of coherent signals $\langle V_{\text{out}} \rangle / \langle V_{\text{in}} \rangle$. A normalized transmission measurement of one of the qubits is shown in Figures 4-4B and 4-4C. From equation 3.39, the transmission coefficient can be calculated to be

$$t = 1 - \frac{\eta \Gamma_1}{2\Gamma_2} \frac{1 - \frac{i\Delta}{\Gamma_2} + \frac{\Omega^2}{\Gamma_1 \Gamma_2}}{1 + \left( \frac{\Delta}{\Gamma_2} \right)^2}$$  \hspace{1cm} (4.1)
Figure 4-4: A) A schematic diagram of the transmission measurements. 75dB of attenuation is placed at the input and a 100dB of amplification is present at the output. B) Real part of the transmission experiment with fitted theory. C) Imaginary part of the transmission experiment with fitted theory.

We fit the data to this equation and are able to extract the qubit frequency to be \( \sim 5.845 \) GHz. The coupling strength to the waveguide can be extracted from the half-width-half-max \( \Gamma_{1,1}/2\pi = 0.6 \) MHz. Next, we perform the same VNA measurement while also sweeping the probe power, as shown in Figure 4-5. The qubit will absorb energy from and emit into the waveguide. The probe is mostly reflected at low powers due to coherent interference between the probe and emission from the qubit. As the probe power increases, the qubit will saturate. As a result, the degree of interference between the probe and emission from the qubit is reduced until the resonant transmission is unity. The power scan can be used to find a variety of parameters, such as the total attenuation between the source of the coherent tone and the qubit. To find the attenuation, we apply a coherent tone of known power at room temperature. A fit to the transmission of this tone using Equation 4.1 will
give the strength of the input tone at the qubit from the fitting parameter $\Omega$. The difference between the power at the qubit and the initial input power gives us the total attenuation between the source of the coherent tone at room temperature and the qubit. From this measurement, we find the attenuation to be $\sim 75$ dB. This measurement is also used to find the intrinsic dephasing rate of the qubit. This is once again found by fitting the transmission vs. power scan to Equation 4.1. From the fitting parameters $\Gamma_1$ and $\Gamma_2$, the dephasing can be calculated to be $\Gamma_\phi = \Gamma_2 - \Gamma_1/2$. From the data presented in Figure 4-5, we find the dephasing rate to be $\Gamma_\phi/2\pi \approx 10$ kHz.

Figure 4-6 shows the resonant transmittance $|t|^2$ as a function of probe power. We find the reflection amplitude $r_0 = \eta \Gamma_1/2 \Gamma_2$ is greater than 99%. Thus, we will ignore the effects of nonradiative decay in our analysis unless otherwise stated. Finally, our qubits are frequency tunable transmons with local and global control. A 1 k$\Omega$ resistor is placed at the output of a voltage source to construct a current source. We plot in Figure 4-7 the transmittance as a function of voltage that is applied on the resistor and bobbin. All three qubits on the chip can now be seen clearly. The peak frequency offset from zero voltage (zero current) is due to spurious magnetic fields inside the fridge. The local control lines are used identify each qubit with its physical location on the chip. All of the data presented thus far was taken at the flux insensitive frequency of one of the qubits. The dephasing rate will increase for measurements taken away from the sweet spots.

### 4.2.2 A Three-Level Atom

In this section, we will take a brief departure from the two-level system approximation in order to characterize the properties of the higher levels of the transmons. As shown in section 2.2, the transmon is a weakly nonlinear oscillator. One parameter of interest is the charging energy $E_C$. We will need this value to find the Josephson energy $E_J$ of the qubits

$$E_J = \frac{(\omega_{01} + E_C)^2}{8E_C},$$  

(4.2)
Figure 4-5: A transmission scan of a qubit as a function probe power. The real and imaginary part of the experimental data is plotted on the left whereas fitted theory is plotted on the right.

Figure 4-6: The resonant transmitivity as a function of the probe power. The experimental data theoretical fit are plotted together.
Figure 4-7: A scan of the transmission through the three qubit sample as a function of external bobbin voltage. The bobbin threads a flux through all qubits on the sample. Three distinct lobes are clearly visible and correspond to a unique qubit on the sample.

where $\omega_{01} = 2\pi f_{01}$ is the frequency of the 0-1 transition (the qubit frequency). With $E_J$ and $E_C$ in hand, we can calibrate our Josephson junction critical currents in fabrication to achieve a device at a desired frequency. We can obtain $E_c = |f_{12} - f_{01}|$ by performing a transmission measurement while saturating the 0-1 transition. We plot in Figure 4-8 the $S_{21}$ measurements near the $f_{01}$ and $f_{12}$ transition frequencies. First, we perform the same transmission measurement as before and see the resonance associated with the 0-1 transition. Next, we apply a strong coherent tone at $f_{01}$ and measure transmission using the weak tone from the VNA as the probe ($f_p$). The resonance at $f_{01}$ is now suppressed, indicating that this transition is indeed saturated. We can clearly see a new resonance appear when $E_C = f_p - f_{01} \approx 335$ MHz. This resonance if from the 1-2 transition of the transmon.

We use the third level of the transmon to investigate the effects of electromagnetically induced transparency (EIT) using two-tone spectroscopy. Consider a probe field at frequency $\omega_p$ and a pump field at frequency $\omega_p$ that are used to drive the 0-1 and 1-2 transition, respectively. Under the rotating wave approximation, the Hamiltonian
Figure 4-8: Unnormalized transmission measurements for finding the $E_c$ of a qubit. The probe frequency is $f_p = \omega_p/2\pi$. A) A transmission measurement near the 0-1 transition frequency with and without a strong pump at the 0-1 transition frequency. The strong pump saturates the qubit and we do not see a dip in the transmission. B) A transmission measurement near the 1-2 transition frequency with and without a strong pump at the 0-1 transition frequency. The strong pump saturates the 0-1 transition and allows us to observe a dip in the transmission at the 1-2 transition frequency.

will be

$$\dot{H} = \hbar \delta_p |1\rangle\langle 1| + \hbar (\delta_{p'} + \delta_p) |2\rangle\langle 2| + \frac{1}{2} (\Omega_p |0\rangle\langle 1| + \Omega_{p'} |1\rangle\langle 2| + \text{h.c.}),$$

(4.3)

where $\delta_{p'} = \omega_{12} - \omega_{p'}$, $\delta_p = \omega_{01} - \omega_p$, $\Omega_{p'}$ is the pump field drive strength, and $\Omega_p$ is the probe field drive strength. To obtain the correct master equation, dissipators for the 0-1, 1-2 and 0-2 transition must be added to the Lindbladian with decay rates $\Gamma_{01}$, $\Gamma_{12}$, and $\Gamma_{02}$. We have dropped the indexes for multiple qubits here since we are only considering a single qubit. Using input output theory, we can compute the transmission of the probe field near the 0-1 transition frequency

$$\langle \hat{a}^\text{out}_t \rangle = \langle \hat{a}^\text{in}_t \rangle + \sqrt{\frac{\Gamma_{01}}{2}} \langle \rho_{01} \rangle,$$

(4.4)

where $\langle \rho_{01} \rangle$ is the steady state expectation value of the 0-1 matrix element of the qubit density matrix $\rho$. Solving the master equation and substituting into equation
Figure 4-9: The Autler-Townes splitting for a three level atom in a waveguide. The experimental data is plotted on the left and the fitted theory is plotted on the right.

4.4, we can compute the transmission $\frac{\langle \hat{a}^{\text{out}} \rangle}{\langle \hat{a}^{\text{in}} \rangle}$

$$
\begin{align*}
    t &= 1 - \frac{2\Gamma_{01}}{2(\Gamma_{01} - i\delta_p) + \frac{\Omega_p^2}{2\Gamma_{02} - 2i(\delta_p + \delta_{p'})}}.
\end{align*}
$$

To obtain this solution, we assumed a very weak probe field $\Omega_p \ll \Gamma_{01}$ and $\Omega_p \ll \Omega_{p'}$. we apply a strong and resonant pump field such that $\delta_{p'} = 0$. Figure 4-9 shows our data alongside a fit to equation 4.5. As we sweep the power of the pump, a splitting of the resonance at the 0-1 transition appears. This due to the pump field dressing the transmon. This effect is also known as the Autler-Townes splitting (ATS) [30]. In general, the ATS is a type of AC Stark effect where the absorption/emission spectrum of an atomic transition is altered due to an applied oscillating electric field. The splitting between the dips is given by the pump strength $\Omega_{p'}$. The transmission
at $\delta_p = 0$ approaches unity as the strength of the pump field increases. Therefore, the ATS is one way of obtaining EIT.

4.2.3 Time Domain Measurements

Room-Temperature Setup

Thus far, we have only presented S-parameter and continuous wave measurements. While many of the qubit properties can be extracted through these kinds of measurements, time-domain and pulsed measurements are necessary for measuring the qubit state evolution. Our room-temperature setup for time-domain measurements is shown in Figure 4-10. We use Keysight 8267D vector signal generators (PSG) for our qubit drives and local oscillator (LO). The drives are pulse shaped by IQ mixing the high frequency tone from the PSG with a waveform from a Keysight PXI M3202A arbitrary waveform generator (AWG). The pulse-shaped drive passes through a high frequency gate to suppress mixer leakage. A microwave switch is used to switch between an input from the left or right side of the sample. Signals from either side of the sample are first amplified and then down-converted to an intermediary frequency of 40 MHz ($f_{IF} = f_{\text{qubit}} - f_{\text{LO}} = 40$ MHz). The I and Q quadratures of both outputs
from the fridge are then digitized in a Keysight PXI M3102A digitizer. To cancel any temporal phase drifts, a reference is captured on a second digitizer. With this setup, we are able to input signals into and simultaneously measure the emission from either end of the sample.

**Temporal Mode Matching**

The field emitted by the qubit into the time dependent output mode $a_{t}^{\text{out}}(t)$ is given by the input-output relation in equation 3.21 (3.22) for right (left) propagating modes. For pulsed measurements, we will define an initial condition for the state of the qubit at $t = 0$ and ignore any input fields that were used for state preparation during $t < 0$. To obtain a single time independent mode, we demodulate the down-converted signal while applying a time-weighted filter $f(t)$

$$
\langle \hat{a}_{t,1} \rangle = \int_{0}^{t} f(t) \langle \hat{a}_{t,1}^{\text{out}}(t) \rangle e^{-i\omega_D t} dt,
$$

where the averaging $\langle \cdot \rangle$ is over many repetitions of the experiment, $t_f$ is the time for a single measurement, and $\omega_D$ is the demodulation frequency. A demodulation frequency of $\omega_D = 2\pi f_{IF}$ corresponds to demodulating the signal to obtain the output mode at the qubit frequency. We include the weighting function $f(t)$ to maximize the detection efficiency via temporal mode-matching. That is, we would like to have $\langle a \rangle = \langle \sigma_j^{-}(0) \rangle / \sqrt{2}$ for the $j^{th}$ qubit. The factor of $\sqrt{2}$ comes from the fact that the qubits will emit into both right and left propagating channels. Therefore, the detection efficiency (in power) is limited to 0.5 for each channel. To preserve the canonical commutation $[\hat{a}, \hat{a}^\dagger] = 1$, the filter function must be normalized $\int |f(t)|^2 dt = 1$. Assuming there is no input on the qubit during the emission process, we have $\langle a_{t}^{\text{out}}(t) \rangle = \sqrt{\Gamma_{j,j}/2} \langle \sigma_{j}^{-}(t) \rangle$. Using the master equation, the dynamics of qubit in the lab frame can be found to be given by $\langle \sigma_j^{-}(t) \rangle = e^{-\Gamma_{j,j}t/2} e^{i\omega_{J}t} \langle \sigma_j^{-}(0) \rangle$. Therefore, we choose $f(t) = \sqrt{\Gamma_{j,j}} e^{-\Gamma_{j,j}t/2}$ as our filter and apply it digitally during the demodulation process. It is important to note that a different choice for $f(t)$ would simply decrease our signal-to-noise ratio (SNR) and would not change the statistics.
of our measurements. In general, we will have $\langle \hat{a} \rangle = \sqrt{\eta/2} \langle \sigma^- \rangle$, where $\eta \leq 1$ is the effective detection efficiency of the measurement chain. The exact value of $\eta$ will be determined by a variety of factors, including the choice of filter function $f(t)$ and any losses in the measurement chain. Finally, the measurement time must be long enough to capture most of the emission from the qubit $t_f \gg \Gamma_{jj}$.

**Amplification**

The signal from the sample will go through several amplifiers before being digitized at room temperature. To capture the correct statistics, we must account for noise added by these amplifiers. The first amplifier in the measurement chain is a TWPA that is operated as a phase-insensitive amplifier. The phase-insensitive linear amplifier can be modeled by [31-33]

$$\hat{S}_{t,1} = \sqrt{G} \hat{a}_{t,1} + \sqrt{G-1} \hat{h}^\dagger,$$

where $\hat{a}$ is the mode being amplified, $\hat{h}$ is the noise channel, $\hat{S}$ is the output mode of the amplifier, and $G$ is the power gain of the amplifier. All aspects of detection inefficiency, such as imperfect mode matching, cable losses, and amplifier noise, are lumped into the effective noise channel $\hat{h}$. We assume that all three of our amplifiers
act as a single amplifier with gain $G$. This is a valid approximation when the gain of the first amplifier is much larger than 1, ensuring the noise added at the first amplifier is the dominant noise source. To find the statistics of $\hat{\alpha}$, we perform a calibration measurement where $\hat{a}$ is in a vacuum state. The gain of the amplification chain is found by sending a weak coherent tone into the fridge. The measurements shown in figure 4-6 are used to determine the power of this tone at the qubit. Finally, this tone is amplified and digitized at room temperature. The gain is calculated by comparing the power at the digitizer and the power at the qubit. For large $G$ we can make the approximation $\sqrt{G-1} \approx \sqrt{G}$. Putting all of this together, we can find the first moment of the qubit $\hat{\sigma}^-$ to be

$$\langle \hat{\sigma}^- \rangle = \sqrt{2} \langle \hat{a} \rangle = \sqrt{2} \left( \frac{\langle \hat{S} \rangle}{\sqrt{G}} - \langle \hat{h} \rangle \right). \quad (4.8)$$

We will use this equation to convert measured signals into information about the state of the qubit. Typically, the noise channel is taken to be in a thermal state. The density matrix describing the state of the noise channel for each mode $k$ is then given by

$$\hat{\rho}_n = \sum_{n=0} e^{\frac{\hbar k_n}{k_B T}} \frac{1}{e^{\frac{\hbar k_{n+1}}{k_B T}}} |n\rangle \langle n|, \quad (4.9)$$

where $T$ is the effective temperature of the noise channel. Since the density matrix is diagonal in the Fock basis, the first moment $\langle \hat{h} \rangle = \text{Tr}(\hat{\rho}_n \hat{h})$ can easily be shown to be zero. When this the case, the measured signal $\langle \hat{S} \rangle$ can simply be scaled by $\sqrt{G/2}$ to obtain $\langle \hat{\sigma}^- \rangle$

$$\langle \hat{\sigma}^- \rangle = \frac{\langle \hat{S} \rangle}{\sqrt{G/2}}. \quad (4.10)$$

**Data Acquisition**

To initialize our qubits, a waveform is generated by the AWG and mixed with the coherent tone from the drive PSG. The width of the pulse is fixed at 20 ns. The degree to which we excite the qubit is then controlled by the amplitude of the pulse. When the qubit is initialized into the excited state $|1\rangle$, no signal is seen at the digitizer.
Figure 4-12: A) Emission amplitude from the qubit as a function of the pulse amplitude and demodulation frequency. Rabi oscillations can be seen when measuring the signal that is at the qubit frequency. B) A horizontal line-cut of (A) for signal at the qubit frequency. The anti-nodes correspond to preparing the qubit on the equator whereas the nodes correspond to preparing the qubit at the poles.

This is because a Fock state has no well defined phase and the signal will average to zero over many repetitions. The ensemble average of the phase of the emitted field will be maximized when the state of the qubit is on the equator of the Bloch sphere. Figure 4-11A shows the averaged time traces of the digitizer for three different qubit state initializations: $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $|1\rangle$. This signal is then digitally demodulated, filtered, and processed to obtain $\langle \hat{\sigma}^- \rangle$. By sweeping the demodulation frequency, we can capture the full emission spectrum of the qubit, as shown in Figure 4-11B. Demodulating at $\omega_D = \omega_{IF}$ will provide the signal that is resonant with the qubit. Finally, $\langle \hat{\sigma}^- \rangle$ is rotated in phase such that it only has a real component.

We produce coherent quantum oscillations, shown in Figure 4-12, by sweeping the pulse amplitude. These are known as Rabi oscillations between the ground and excited state of the qubit. In Figure 4-12A, a 2D scan of amplitude and emission frequency vs. $\text{Real}[\langle \hat{\sigma}^- \rangle]$ is shown. Figure 4-12B shows a line-cut of this 2D scan for the emission at the qubit frequency. This is used to calibrate our $\pi$ and $\pi/2$ pulses for state preparation. Contrary to resonator based readout in circuit QED, the anti-nodes of the Rabi oscillation correspond to states on the equator and the nodes correspond to states at the poles. This is simply due to the difference in observables that are measured. Note that the maximum value of the Rabi oscillation does not reach $0.5$,
as it should for a state on the equator. A simply back of the hand calculation can show that most of this error is due to decay during the pulse. Our qubits have a $T_1$ of $\approx 135$ ns. Given a $\pi$-pulse length of 20 ns, the probability of decay in this window can be estimated to be $1 - e^{-20/135} \approx 0.86$. Note that the pulse in the presented data was not optimized. Higher fidelities can be obtained by shortening the length of this pulse as well as implementing pulse-shaping techniques that minimize excited state leakage [24].

Using these calibrated pulses, we can measure the $T_2$ and $T_1$ of the qubit. To measure $T_2$, we initialize the qubit to the equator with a $\pi/2$ pulse. The time between this initialization and measurement is then swept. To measure $T_1$, we initialize the qubit to the excited state with a $\pi$ pulse. A $\pi/2$ pulse is then applied after some time to project the readout from $|0\rangle$ and $|1\rangle$ to $|0\rangle + |1\rangle / \sqrt{2}$. By sweeping the time between the initialization and projection pulses, the decay from excited to ground state can be measured. Figure 4-13 shows the data for $T_2$ and $T_1$ measurements along with fitted exponentials curves. The measured $T_2 \approx 269$ ns is in line with the coupling strength that was extracted from the VNA measurements $\Gamma_{1,1} = \frac{1}{2\pi T_2} \approx 0.6$ MHz. Furthermore, $T_2 \approx 2T_1$ implies that the dephasing rate is much smaller than the qubits coupling to the waveguide. However, this is only the case when the qubit is at its flux insensitive point.

4.3 Two-Qubit Experiments

Multi-qubit effects become prominent when two or more qubits are brought close to resonance. We begin by describing the methods that will determine the frequencies at which the qubits are spaced apart by $\lambda/2$ and $3\lambda/4$. Measuring the phase delay between the qubits is not a trivial task. While the physical distance between the qubits is set in the design, it is difficult to precisely predict the speed of light in the transmission line. The measurements will involve two qubit scattering measurements (VNA) as well as time-domain measurements that probe the left-right emission symmetry. Finally, we will present our protocol for the dissipative preparation of entangled dark
Figure 4-13: A) A time-domain $T_2$ measurement of a qubit. The pulse sequence involves applying a $\pi$-half pulse and waiting some time before measuring the emission amplitude. B) A time-domain $T_1$ measurement of a qubit. The pulse sequence begins with a $\pi$ pulse, then waiting for some time, followed by a $\pi$-half pulse for readout.

4.3.1 Spacing Calibration

We begin by performing VNA scans while sweeping the frequencies of the two qubits through each other using flux control. Figure 4-14 shows the real and imaginary part of this scan. The multi-qubit input-output boundary condition from equation 3.21 is used to compute the theoretical transmission coefficients. This is simulated in QuTiP and fit to the experimental data. From this fit, we extract the qubit spacing to be $d = 1.02\lambda$ at a frequency of 6.58 GHz. This implies that the spacing will be $\lambda$ at a frequency of 6.71 GHz. We were unable to probe any higher than 6.58 GHz as this was the peak frequency of the qubits in this sample. The imperfect spacing is also the reason for the slight asymmetry in the VNA scan. Even though this scan is slightly off the ideal frequency, we still observe the increase in linewidth of the transmission scan when the two qubits are resonant. This is a signature of a superradiant (bright) state. Samples with a high enough frequency have recently been fabricated (spectrum shown later in Figure 4-17) and will be used
Figure 4-14: The transmission spectrum of two qubits that are spaced apart \( \sim \lambda \) as a function of the qubit-qubit detuning and the qubit-probe detuning. The experimental data is plotted on the left and the fitted theory is plotted on the right. The broadened linewidth (half width half max) is labeled and is a signature of the formation of a superradiant (Bright) state for the entanglement protocol that will be described in the following section.

The spacing between two qubits can also be probed by comparing the emission into left and right propagating modes. The directional emission will be symmetric \( \hat{a}_l = \hat{a}_r \sim \sigma_1^- + \sigma_2^- \) for a spacing of \( \lambda \). We observe this by preparing the first qubit on the equator and measuring the coherent emission \( \langle \hat{a}_{1,r} \rangle \) from both ends (the 1st qubit is defined to be leftmost qubit in this discussion). The data and simulation is shown in Figure 4-15A. The amplitude of the coherent emission is plotted as a function of the qubit-qubit detuning and the qubit-emission detuning. As expected, the left and right modes are nearly indistinguishable. Next, the same experiment can be done at a spacing of \( \frac{3\lambda}{4} \). The left-right symmetry is now broken.
Figure 4-15: The left and right propagating emission spectrum for two resonant qubits that are A) a distance of $L = \lambda$ and B) a distance of $L = 3\lambda/4$ apart. The left qubit is prepared on the equator prior to measurement. Simulated theory is plotted alongside each experiment. The emission is directionally symmetric for $L = \lambda$ and asymmetric for $L = 3\lambda/4$. The distance between the qubits $L$ is varied by tuning the qubit frequencies since $\hat{a}_1 \sim \sigma_1^- - i\sigma_2^-$ and $\hat{a}_r \sim \sigma_1^- + i\sigma_2^-$. We find the spacing of $3\lambda/4$ to be at $f_1, f_2 \approx 5$ GHz as it is the frequency that showed the maximal asymmetry between the left and right emission. This is in good agreement with the distance obtained from the two-qubit scattering measurements. The data and simulation is shown in Figure 4-15. The coherence of the emission is enhanced in left propagating modes and suppressed in right propagating modes. The exchange interaction (see the Hamiltonian in Equation 3.12) plays an important role since the asymmetry is determined by the emission from the second qubit. If the second qubit remained in the ground state for all times, the directional emission would be symmetric. With the exchange interaction, the second qubit becomes partially excited.
4.3.2 Dissipative Entanglement

High-fidelity entanglement of superconducting qubits in waveguide QED systems is a challenging task. Single-qubit dissipation into the waveguide disrupts the generation of entanglement through conventional techniques that involve coherent control. In this section, we will describe an alternative approach to qubit entanglement that takes advantage of the correlated dissipation from the qubits into the waveguide.

Entanglement Protocol

A bright state and a dark state are formed when two resonant qubits are spaced apart by one wavelength $\lambda$, as discussed in Section 3.3.1. The bright state will decay with an enhanced emission rate whereas the dark state will be decoupled from the waveguide. If the coupling strength $\Gamma$ of each qubit to the waveguide is the same, the dark state will be the maximally entangled singlet state. However, by definition, the dark state cannot be prepared through the waveguide alone. Instead, we will introduce a local drive on the first qubit with strength $Q_1$. The act of the local drive on the four-level system formed by the two qubits is shown in Figure 4-16C. In the basis formed by the states $\{|gg\rangle, |eg\rangle, |ge\rangle, |ee\rangle\}$, the act of the local drive is to induce transitions between $|gg\rangle \leftrightarrow |eg\rangle$ and $|ge\rangle \leftrightarrow |ee\rangle$. At first, it does not seem that anything interesting will result. Describing the same process in the singlet-triplet basis, shown in Figure 4-16D, sheds some light on the interference effects. In this basis, the local drive will drive the $|gg\rangle \leftrightarrow \frac{1}{\sqrt{2}}(|eg\rangle \pm |ge\rangle)$ and $\frac{1}{\sqrt{2}}(|ge\rangle \pm |eg\rangle) \leftrightarrow |ee\rangle$ transitions equally. However, the triplet states will decay with rate $2\Gamma$. Therefore, any population in these states will quickly relax to the ground state. Population in the singlet state (dark state) will not decay into the waveguide. If the strength of the local drive is weak $Q_1 \ll \Gamma$, the system of two qubits become an effective two level system that oscillates between the ground and dark state.

Figure 4-16B shows a simulation of this protocol. We plot the overlap of state of the two qubits with the dark state as a function of time. The state will oscillate between the ground and dark state while progressively dampening. Under ideal con-
Figure 4-16: A) A schematic setup of two qubits coupling to a common waveguide that are spaced apart by $\lambda$. The first qubit is locally driven with strength $\Omega_1$ drive the qubits into the dark state. B) Simulated evolution of the two qubit system shown in (A). The overlap of the state of the two qubits with the dark state is plotted as a function of time for two different local drive powers. As the drive power increases, the maximum achievable fidelity reduces. C) Level diagram of the driven-dissipative system in the qubit basis. D) Level diagram of the driven-dissipative system in the singlet-triplet (bright-dark) basis. E) Effective two level system when the local drive is very weak.

ditions, the maximum achievable fidelity will approach unity as the drive strength approaches zero. This is because stronger drives will involve the states that we chose to ignore and our approximation will begin to break down. A lower limit will be placed on the drive strength due to nonradiative decay and dephasing. That is, the dark state must be prepared faster than the rates that will cause it to decohere. An optimal drive strength $\Omega_1$ can be found for a given $\Gamma$, $\Gamma_{nr}$, and $\Gamma_\phi$. For example, the maximal dark state overlap given the parameters of $\Gamma/2\pi = 0.6$ MHz, $\Gamma_{nr}/2\pi = 5$ kHz, and $\Gamma_\phi/2\pi = 50$ kHz will only be 63%.
Figure 4-17: A scan of the transmission through the sample with three asymmetric transmons as a function of external bobbin voltage. As expected, a minimum and maximum frequency are present. Theory curves for the frequencies of the qubits are drawn on top of the transmission spectrum to clearly distinguish the three qubits on the sample.

The fidelity can be improved by making three adjustments: increasing the coupling strength of the qubit to the waveguide, decreasing the dephasing due to flux noise, and introducing a second local drive. We have designed new samples with a much stronger coupling strength $\Gamma/2\pi = 5$ MHz. The dephasing rate could be reduced if the qubit was fixed in frequency or the frequency corresponding to a spacing of $\lambda$ was at the flux insensitive point of a qubit. However, the qubit frequencies with current fabrication techniques can vary by $\sim 3\%$ on the chip. This corresponds to a variance of 100’s of MHz. Since this variance is much larger than the qubit linewidth, it is impractical to use fixed frequency transmons for our experiments. Instead, we have made our transmons with asymmetric SQUID loops. The Josephson energy for an asymmetric SQUID is given by

$$E_J(\Phi_{\text{ext}}) = \left| E_J \cos \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_0} \right) \right| \sqrt{1 + d^2 \tan^2 \left( \frac{\pi \Phi_{\text{ext}}}{\Phi_0} \right)},$$  \hspace{1cm} (4.11)
where $E_{J\Sigma} = E_{J1} + E_{J2}$ is the total Josephson energy of the two junctions and $d = (E_{J1} - E_{J2})/(E_{J1} + E_{J2})$ is the degree of asymmetry between the junctions. A transmon with an asymmetric SQUID will have two flux insensitive points and a reduced sensitivity to flux noise in between them. A plot of transmission vs. bobbin voltage for the first sample with asymmetric transmons is shown in Figure 4-17. Finally, the rate at which the dark state is achieved can be doubled by locally driving the second qubit $\Omega_2 = -\Omega_1$. With all of these improvements, an entanglement fidelity of over 90% can be achieved.

**Entanglement Verification**

In order to verify that the qubits have been entangled, we must expand upon the readout techniques presented thus far; we will need to measure beyond the first moment of the emitted field. From equation 4.7, the normally ordered moments of the signal from the fridge can be calculated to be

$$
\langle \hat{S}^n \hat{S}^m \rangle = G^{n+m} \sum_{i,j=0}^{n,m} \binom{n}{i} \binom{m}{j} \langle \hat{a}^i \hat{a}^j \rangle \langle \hat{h}^{n-i} \hat{h}^{m-j} \rangle.
$$

(4.12)

We have assumed that the noise channel $\hat{h}$ and $\hat{a}$ are uncorrelated. The anti-normally ordered moments of $\hat{h}$ are found by measuring the signal when the qubit is left in the ground state $\langle \hat{h}^n \hat{h}^{tm} \rangle = G^{-n-m} \langle \hat{S}^n \hat{S}^m \rangle$. To obtain the moments of $\hat{S}$, the I and Q quadratures of each measurement record is stored. The Husimi Q-function $Q(S)$ of the signal $\hat{S}$ can be constructed from a normalized histogram of the collected I-Q data points. The anti-normally ordered moments of $\hat{S}$ are then found using $Q(S)$

$$
\langle \hat{S}^n \hat{S}^m \rangle = \int S^n S^m Q(S).
$$

(4.13)

Using the bosonic commutation relation $[\hat{S}, \hat{S}^\dagger] = 1$, a generating function for the normally ordered moments of $S$ can be found to be

$$
\hat{S}^n \hat{S}^m = \partial^n \partial^m \left( e^{-\sigma r} e^{+\sigma \hat{S}} e^{\sigma \hat{S}^\dagger} \right) \bigg|_{\sigma = \tau = 0}.
$$

(4.14)
That is, the normally ordered moments of $S$ can now be found from a linear combination of the anti-normally ordered moments of $S$. For example, with $n=1$ and $m=3$ we have

$$\langle \hat{S}^4 \rangle = \langle \partial_\sigma \partial_\tau \left( e^{-\sigma \tau} e^{\tau \sigma} \hat{S} e^{\sigma \tau} \right) \rangle \bigg|_{\sigma=\tau=0} = \langle \hat{S}^3 \rangle - 3\langle \hat{S}^2 \rangle. \quad (4.15)$$

Equation 4.12 can then be inverted to extract the normally ordered moments of $\hat{a}$.

Solely measuring the moments of the emitted field will not be sufficient to fully reconstruct the state of the qubits. First, the dark state, by definition, will not emit into the waveguide. Additionally, the moments of the emitted field constitute a joint measurement of the two qubits. That is, we will have $\hat{a}_+ \sim \hat{\sigma}_1^- + \hat{\sigma}_2^-$ from input-output theory. To solve these issues, we must make use of single qubit gates. Applying a local $\pi$-pulse about the $z$ axis of the second qubit will give us $\hat{a}_- \sim \hat{\sigma}_1^- - \hat{\sigma}_2^-$. The moments of $\hat{\sigma}_{1,2}$ and correlations between qubits $\hat{\sigma}_1 \hat{\sigma}_2$ can then be obtained from linear combinations of the moments of $\hat{a}_\pm$. The rotation about the $z$ axis will be implemented with an $x$ and $y$ rotation. The pulses for $x$ and $y$ rotations will simply differ in the phase of the input drive. From this, the two qubit density matrix can be constructed.
Chapter 5

Conclusions and Outlook

Superconducting circuits are a promising platform to study and implement the physics of WQED. In part, this is because they can easily achieve strong coupling regimes, can be designed to match a specific set of parameters, and can be scaled lithographically. These properties can be applied to many WQED applications, such as networking quantum information or quantum simulation of many-body models that involve interactions with a bosonic continuum [15]. We have presented the theory of superconducting WQED systems and demonstrated our ability to control and measure such devices. We observed an extinction of transmission of over 99% through coherent scattering. Next, we observed the nonlinearity of the transmon in two-tone spectroscopy and produced the Autler-Townes splitting. Pulsed time-domain experiments were then demonstrated as a means of qubit state initialization and readout. We also presented our preliminary measurements that involve two qubits. Here, we observed a broadened linewidth in scattering measurements due to the formation of a super-radiant state. Finally, we outlined a novel protocol to achieve entanglement through correlated dissipation.

Outlook

Our immediate goal is to demonstrate the dissipative entanglement protocol that was described in this thesis. This will be done with new samples with a much higher qubit-
waveguide coupling strength. The moment based readout technique will also require the demodulation of the signal to be done on an FPGA (field programmable gate array). The number of averages required will increase rapidly for higher moments. Performing the demodulation on a personal computer is impractical, because it would require large amounts of data transfer between the digitizer and the computer through a slow PCIe interface. The amount of data transferred, and thus the measurement time, is reduced dramatically by performing the demodulation with the on-board FPGA of the Keysight digitizer. This has recently been implemented and tested.

It will also be interesting to demonstrate dark state preparation with more than two qubits. To achieve this, we will operate at the frequency where the three qubits on the presented sample are resonant and consecutively spaced apart by λ/2. Samples with up to nine qubits have also been designed. Finally, we plan to investigate a novel WQED configuration where individual qubits are coupled to multiple locations on the waveguide [34]. We will use the experimental techniques presented in this thesis to investigate the interesting interference effects that arise from such a setup.
Appendix A

Master Equation Derivation

In this appendix, we will derive the master equation presented in Chapter 3. This calculation is similar to those presented in references [25, 26]. We begin with the full Hamiltonian of the system

\[ \hat{H} = \int d\omega \hbar \omega \left( \hat{a}_r^\dagger(\omega)\hat{a}_r(\omega) + \hat{a}_l^\dagger(\omega)\hat{a}_l(\omega) \right) + \hbar \sum_j \omega_j \hat{\sigma}_j^+ \hat{\sigma}_j^- + \]

\[ i\hbar \sum_j g_j \hat{\sigma}_j^Z \int d\omega \sqrt{\omega} \left( (\hat{a}_r^\dagger(\omega)e^{i\omega t_j} - \hat{a}_r(\omega)e^{-i\omega t_j}) + (\hat{a}_l^\dagger(\omega)e^{-i\omega t_j} - \hat{a}_l(\omega)e^{i\omega t_j}) \right). \]  

(A.1)

We define the left and right propagating electric field at the \( j \)th qubit as

\[ \hat{E}_r^l(t) = i \int d\omega \sqrt{\omega} \left( \hat{a}_r^\dagger(\omega)e^{i\omega t_j} - \hat{a}_r(\omega)e^{-i\omega t_j} \right) \]

(A.2)

\[ \hat{E}_l^r(t) = i \int d\omega \sqrt{\omega} \left( \hat{a}_l^\dagger(\omega)e^{-i\omega t_j} - \hat{a}_l(\omega)e^{i\omega t_j} \right) \]

(A.3)

The Heisenberg equations of motion for the EM field modes can be solved for to obtain

\[ \hat{a}_r(\omega, t) = \hat{a}_r(\omega, 0)e^{-i\omega t} + \sum_j g_j \sqrt{\omega} \int_0^t e^{-i\omega(t-\tau+t_j)} \hat{\sigma}_j^Z(\tau)d\tau. \]  

(A.4)

\[ \hat{a}_l(\omega, t) = \hat{a}_l(\omega, 0)e^{-i\omega t} + \sum_j g_j \sqrt{\omega} \int_0^t e^{-i\omega(t-\tau-t_j)} \hat{\sigma}_j^Z(\tau)d\tau. \]  

(A.5)
Substituting these into the electric field yields

\[ \hat{E}_i^j = i \int d\omega \sqrt{\omega} \left( \hat{a}_i^\dagger(\omega, 0)e^{i\omega(t+t_j)} + \sum_k g_k^i \sqrt{\omega} \int_0^t d\tau e^{i\omega(t-\tau-t_k)} \hat{d}_k^\pm(\tau) + h.c. \right) \quad (A.6) \]

\[ \hat{E}_r^j = i \int d\omega \sqrt{\omega} \left( \hat{a}_r^\dagger(\omega, 0)e^{i\omega(t-t_j)} + \sum_k g_k^r \sqrt{\omega} \int_0^t d\tau e^{i\omega(t-\tau+t_k)} \hat{d}_k^\pm(\tau) + h.c. \right), \quad (A.7) \]

where \( t_{kj} = t_k - t_j \) is the transit time for light between qubits \( k \) and \( j \). We define the initial electric fields to be

\[ \hat{E}_{i,\text{in}}^j = i \int d\omega \sqrt{\omega} \left( \hat{a}_i^\dagger(\omega, 0)e^{i\omega(t+t_j)} + h.c. \right) \quad (A.8) \]

\[ \hat{E}_{r,\text{in}}^j = i \int d\omega \sqrt{\omega} \left( \hat{a}_r^\dagger(\omega, 0)e^{i\omega(t-t_j)} + h.c. \right) \quad (A.9) \]

As explained in Chapter 3 of the main text, we invoke an approximation such that \( \hat{d}_j^\dagger(\tau) \approx \hat{d}_k^\dagger(\tau)e^{-i\omega(t-\tau)} \). Defining the total electric field to be \( \hat{E}_{\Sigma}^j = \hat{E}_i^j + \hat{E}_r^j \), we can integrate equations A.7 and A.6 to obtain

\[ \hat{E}_{\Sigma}^j = \hat{E}_{i,\text{in}}^j + \hat{E}_{r,\text{in}}^j - \frac{1}{g_j} \sum_k \left( \Lambda_{k,j}^+ \hat{d}_k^+ + \left( \Lambda_{k,j}^- + \frac{i\gamma_{k,j}}{2} \right) \hat{d}_k^- \right) + h.c. \quad (A.10) \]

where we define

\[ \Lambda_{k,j}^\pm = 2g_kg_j P \int_0^\infty d\omega \frac{\omega \cos(\omega t_{kj})}{\omega \pm \omega_k} \quad (A.11) \]

\[ \gamma_{k,j} = 4\pi g_kg_j \omega_k \cos(\omega_k t_{kj}) \quad (A.12) \]

and \( P \) is the Cauchy principle value. Following references [25, 26], a master equation for the qubit degrees of freedom can be found by considering an operator \( \hat{O}^j \) that only acts on the qubit degrees of freedom. Given this condition, we will have \([\hat{O}, \hat{E}_{i,r}^j] = 0\).
The Heisenberg equation of motion for the operator $\hat{O}$ is then given by

$$\dot{\hat{O}} = i \left[ \sum_{j} \omega_j \hat{\sigma}_j^+ \hat{\sigma}_j^- + \sum_{j} \kappa_{0j} \hat{E}_{2j}^2 \hat{\sigma}_j^2, \hat{O} \right] + \sum_{j} \sum_{k} \left( -i \Lambda_{kj}^+(\hat{\sigma}_j^+ \hat{\sigma}_k^+ - \hat{\sigma}_j^+ \hat{\sigma}_k^+ - \text{h.c.}) - i \Lambda_{kj}^-(\hat{\sigma}_j^- \hat{\sigma}_k^- - \hat{\sigma}_j^- \hat{\sigma}_k^- - \text{h.c.}) \right) + \frac{\gamma_{kj}}{2} \left( \hat{\sigma}_j^+ \hat{\sigma}_k^- - \hat{\sigma}_j^- \hat{\sigma}_k^+ + \text{h.c.} \right). \quad (A.13)$$

If we define the density matrix of the total system to be $\hat{\rho}_\Sigma$, then the density matrix of the qubit degrees of freedom can be obtained by tracing out the EM degrees of freedom $\hat{\rho} = \text{Tr}_{EM}(\hat{\rho}_\Sigma)$. To do this, we must define the state of the modes in the waveguide. We will take this to be a coherent state at the frequency $\omega_{d}$ such that

$$\hat{a}_{l(r)}(\omega, 0)|\alpha\rangle = \sqrt{\frac{P_{l(r)}}{\omega_{d,l(r)}}} e^{-i\omega_{d,l(r)}\theta_{l(r)}} \delta(\omega - \omega_{d,l(r)})|\alpha\rangle, \quad (A.14)$$

where $P_{l(r)}$ and $\theta_{l(r)}$ are the power and phase of the left (right) propagating coherent tones. Therefore, the expected value of the total electric field can be found to be given by

$$\langle \hat{E}_\Sigma \rangle = \alpha(t) = \frac{\Omega_{d,l}}{2} \sin(\omega_{d,l}(t + t_l + \theta_l)) + \frac{\Omega_{d,r}}{2} \sin(\omega_{d,r}(t - t_r + \theta_r)), \quad (A.15)$$

where the drive strength is $\Omega_{l(r),j} = -\sqrt{4\pi \sigma_j^2 P_{l(r)}} / \omega_{d,l(r)}$. Using the rotating wave approximation and $\text{Tr}(\hat{O} \hat{\rho}_\Sigma) = \text{Tr}_q(\hat{O} \hat{\rho})$, where $\text{Tr}_q(\cdot)$ represents a trace over the qubit degrees of freedom, the master equation for $\hat{\rho}$ can be found to be

$$\dot{\hat{\rho}} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + \sum_{j,k} \Gamma_{j,k}(\hat{\sigma}_j^- \hat{\rho} \hat{\sigma}_k^+ + \frac{1}{2} \{\hat{\sigma}_k^+ \hat{\sigma}_j^-, \hat{\rho}\}) \quad (A.16)$$

$$\hat{H} = \sum_{j} \hbar \omega_j \hat{\sigma}_j^+ \hat{\sigma}_j^- + \sum_{j} \hbar \alpha_j(t) \sigma_j^2 + \sum_{j \neq k} \hbar J_{j,k} \sigma_j^- \sigma_k^+. \quad (A.17)$$
The correlated decay rate $\Gamma_{j,k}$ and qubit exchange coupling $J_{j,k}$ are

$$\Gamma_{j,k} = 2\pi g_j g_k (\omega_j e^{i\omega_j t_{j,k}} + \omega_k e^{-i\omega_k t_{j,k}})$$ \hspace{1cm} (A.18)$$

$$J_{j,k} = -i\pi g_j g_k (\omega_j e^{i\omega_j t_{j,k}} - \omega_k e^{-i\omega_k t_{j,k}}).$$ \hspace{1cm} (A.19)$$

We absorbed the terms due to the Lamb shift into the qubit frequencies, dropped small non-positive terms, and made a rotating-wave approximation when going from equation A.13 to A.16. For more details on this calculation, see reference [25].
Bibliography


