Star Measures and Dual Mixed Volumes

by

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S.B., Massachusetts Institute of Technology (1990)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1994

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Abstract

Hadwiger's characterization theorem classifies all convex-continuous measures on polyconvex sets that are invariant under rotations and translations as consisting of the linear span of the elementary Minkowski mixed volumes (Quermassintegrals). I present a characterization theorem for the *dual* elementary mixed volumes, giving the analogue to Hadwiger's result for the dual Brunn-Minkowski theory of star-shaped sets. Along the way I classify star-continuous measures satisfying such conditions as rotation-invariance, SL_n-invariance, and homogeneity with respect to dilation.

Thesis Supervisor: Gian-Carlo Rota
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Dedicated to my parents, Gerald and Barbara Klain.

Contents

Introduction

Since its creation by Brunn and Minkowski, what has become known as the Brunn-Minkowski theory has provided powerful machinery to solve a broad variety of inverse problems with stereological data. The machinery of the Brunn-Minkowski theory includes mixed volumes (of Minkowski), symmetrization techniques (such as those of Steiner and Blaschke), isoperimetric inequalities (such as the Brunn-Minkowski, Minkowski, and Aleksandrov-Fenchel inequalities), integral transforms (such as the Cosine transform), and important auxiliary bodies associated with these transforms (such as Minkowski's projection bodies). Schneider's recent book [37] on the Brunn-Minkowski theory is the best available introduction to the subject.

While the Brunn-Minkowski theory has proven to be of enormous value in answering inverse questions regarding projections of convex bodies onto subspaces, the theory has been of little value in answering inverse questions with data regarding *intersections* with subspaces.

Although the classical Brunn-Minkowski theory has developed rapidly over the last century, a dual Brunn-Minkowski theory has only begun to emerge during the last twenty years. The inspiration for such a theory can be traced to the program outlined by Busemann for the study of Minkowskian geometry. As Busemann has warned, significant results in Finsler geometry must be preceded by a deeper understanding of the local structure of Finsler spaces, which is Minkowskian.

In what was to become a highly influential paper in the development of a dual theory, Busemann and Petty [11] presented a list of problems from the field of Minkowskian geometry. Busemann and Petty translated these problems into questions regarding convex bodies in ordinary Euclidean space. Since these problems involved sections (rather than projections) of convex bodies, the available machinery at the time (the classical Brunn-Minkowski theory) was of little help in dealing with the questions posed by Busemann and Petty.

The first problem of the Busemann-Petty list became particularly notorious. It has in fact become known as *the* Busemann-Petty problem:

> Let K and L be convex bodies in R^n that are centrally symmetric about the origin, and suppose that for all $(n-1)$-dimensional subspaces ξ of R^n,
>
> $$V_{n-1}(K \cap \xi) \leq V_{n-1}(L \cap \xi).$$
>
> Does it then follow that $V_n(K) \leq V_n(L)$?

Here we denote by V_k the volume in R^k. Because this was a problem that could be easily explained to "the man in the street", the problem received wide exposure. Hadwiger, Bourgain, Larman, Klee, and Rogers published work on the problem, as did many others. (See the Bulletin Research Announcement by Gardner [15] for a fairly complete history of the problem.) The frustration over finding a solution to this easily stated question exposed the need for a theory that could deal with intersection questions.

The fact that such machinery would have to come from some sort of "dual" theory became evident when Shephard [39] posed the dual of the Busemann-Petty problem (with sections replaced by projections). The answer to Shephard's projection version of the Busemann-Petty problem was given shortly after Shephard posed the problem. Petty [33] and Schneider [36] independently used the machinery of the Brunn-Minkowski theory to show that, with strong symmetry assumptions, this dual of the Busemann-Petty problem had a positive answer. This later became known as the Petty-Schneider theorem. Petty and Schneider showed that, in general, the answer to the Shephard question was negative in all dimensions $n > 2$.

A first, but critical, step in the development of a dual theory was taken by Busemann himself [7] (See also Schneider [37]). In analogy to Minkowski's projection bodies, Busemann introduced new auxiliary bodies which, in today's terminology, are known as intersection bodies. With the aid of these bodies Busemann [7, 8] was able to present the first solution to the isoperimetric problem for Minkowski spaces. The surprise here was that the solutions turned out to be different from what would be expected based on Euclidean experience. Busemann [8] (see also [6]) presented a definition of Minkowskian surface area, and showed that, in a Minkowski space with unit ball M, the solutions to the isoperimetric problem are bodies that are homothetic to the polar of the intersection body of M. This is radically different from the situation in Euclidean spaces, where the solutions to the isoperimetric problem are homothetic to the ball. In fact, one of the major outstanding questions in Minkowskian geometry asks if Euclidean spaces are characterized (among Minkowski spaces) by the property that the solution of the isoperimetric problem is the ball of the space.

Unfortunately, two things were to delay the development of a dual theory for another quarter of a century. First, Busemann referred to his auxiliary bodies as *polar Z-bodies.* The name obscured the duality between Busemann's bodies and Minkowski's projection bodies. What was probably even more of a factor in delaying the development of a dual theory was the fact that Busemann restricted his attention to intersection bodies of centered (symmetric about the origin) convex bodies. Later, Croft [12] showed that the intersection bodies of (not necessarily centered) convex bodies are *not necessarily convex.* Croft's work was misinterpreted as indicating that, in order to obtain interesting results, Busemann's restriction to centered convex bodies was essential, as well as prophetic. Centered convex bodies now constituted the "proper" domain of the intersection operator.

As will be seen, the proper setting for a dual Brunn-Minkowski theory *must* be much larger than anything Busemann considered. Not only is the class of centered convex bodies insufficient, but even the class of all convex bodies is inadequate. To obtain results about centered convex bodies, one is forced to study the intersection bodies of objects more general than convex bodies. The correct dual analogue of Minkowski addition of bodies is impossible within the class of convex bodies. The critical importance of choosing the correct setting for a dual theory will be seen below. In fact, much of this dissertation is devoted to fixing a proper setting for a dual theory.

A quarter of a century after Busemann discovered (at least for centered convex bodies) the dual analogue of Minkowski's projection bodies, Lutwak [26] (see also Burago-Zalgaller [4] and Schneider [37]) discovered the dual analogue of Minkowski's mixed volumes. He also discovered the dual analogues of the classical mixed volume inequalities (such as the Aleksandrov-Fenchel and Minkowski mixed volume inequalities). Just as the classical mixed volumes of Minkowski (and the inequalities between them) proved to be powerful tools for the study of projection questions, the dual mixed volumes would play a critical role in later studies of intersection questions.

A good setting for the dual Brunn-Minkowski theory was presented by Lutwak [27]. All of the definitions are now given within the class of star bodies. In this paper a clear dual analogue is given for Minkowski combinations of bodies. Lutwak extends Busemann's definition of intersection bodies to the class of star bodies and obtains (rather effortlessly) the dual analogue of the Petty-Schneider Theorem.

By the early nineties, progress on the the Busemann-Petty problems appeared to be running out of steam. The work of Bourgain [2, 3] indicated that the techniques used to obtain negative answers to the Busemann-Petty problem in higher dimensions by Ball [1] and Giannopoulos [19] (and later Papadimitrakis [32] and Gardner [14]) would be dramatically less effective as the dimension of the space decreased. Bourgain showed that these techniques would ultimately fail to provide the answer to the Busemann-Petty problem in three dimensions. Bourgain's work clearly demonstrated the need for a new approach.

Gardner turned to the dual Brunn-Minkowski theory. Using the machinery of the dual theory, Gardner [14] was able to give a powerful extension of Lutwak's dual of the Petty-Schneider Theorem. He showed that the Busemann-Petty problem has an affirmative answer, in a given dimension, if and only if every strictly convex, sufficiently smooth, centered, convex body is an intersection body *of a star body.* This result enabled Gardner [14] to show that the answer to the Busemann-Petty problem was negative for all dimensions greater than four.

Gardner's Theorem clearly demonstrated the critical role that intersection bodies, as defined by Lutwak, must play in the ultimate answer to the Busemann-Petty question. With Gardner's Theorem in hand, Zhang began his investigation of intersection bodies. Zhang [45] quickly found dual mixed volume characterizations of intersection bodies that are the analogues of the mixed volume characterizations of projection bodies due to Weil [41] and Goodey [20]. Zhang [42, 43] then used his characterizations to give a negative answer to the Busemann-Petty question in four dimensions.

Finally, Gardner [16] used the new intersection bodies to obtain a surprising affirmative answer to the Busemann-Petty problem in three dimensions.

The extension of intersection bodies *to star-shaped sets* proved to be critical in the final solution to the Busemann-Petty problem *for convex bodies.* As Gardner [17, p. 306] states,

> ...the work [25] of Hadwiger already contained most of the ingredients for the solution: the necessary coordinate geometry for bodies of revolution, inversion of integral equations, and Hölder's inequality. What is missing is the essential use of star bodies, and the theory of dual mixed volumes; the point is that the theory of dual mixed volumes, via the dual Aleksandrov-Fenchel inequality, molds Hölder's inequality into precisely the right form.

Progress in the study of the dual theory is still rapid. Goodey, Fallert and Weil [13] have shown that just as the projections of projection bodies are projection bodies, central sections of intersection bodies are intersection bodies.

Unfortunately, there still remain fundamental and foundational problems with the dual Brunn-Minkowski theory. A rather obvious and glaring problem is its limitation to bodies that contain the origin. While this appears to present no difficulties in the study of centered convex bodies (as in the Busemann-Petty problem), from the viewpoint of Stereology this is far too restrictive. An extension of the dual theory to star bodies that do not contain the origin was recently presented by Gardner and Volčič [18].

However, there is an even more serious problem with the existing dual theory. One of the most beautiful and important results of twentieth century convexity is Hadwiger's characterization theorem for the elementary mixed volumes (also known as the quermass-integrals) [24, 34]. This result is of such fundamental importance that any candidate for a dual theory must possess a dual analogue of Hadwiger's characterization theorem. As will be seen below, the dual Brunn-Minkowski theory, as currently understood, is not sufficiently rich to be able to accommodate a dual of Hadwiger's theorem.

The purpose of this thesis is two-fold. First, it will be shown that the natural setting for a dual Brunn-Minkowski theory is larger than that envisioned by previous investigators. In the chapters that follow I present an extension of the dual Brunn-Minkowski theory to a broad new class of star-shaped sets previously inaccessible to the dual theory.

Second, I present a sequence of classification theorems for measures on star-shaped sets. Included among these results are classification theorems for continuous measures that satisfy the conditions of homogeneity, rotation invariance, SL_n-invariance, and monotonicity. These results lead in turn to a characterization theorem for the dual elementary mixed volumes (dual Quermassintegrals) that gives the dual analogue of Hadwiger's characterization of the elementary Minkowski mixed volumes.

Chapter 1 contains a summary of certain results from geometric convexity, the star analogues of which are developed in the subsequent chapters. Of particular importance is the new definition of the Hausdorff topology on the set of convex bodies, of which the dual analogue proves a crucial tool for understanding measures on the class of star-shaped sets.

Chapter 2 continues with a definition for "$\mathcal{L}^n$-stars", a new class of bodies with which we work throughout. This is followed by a new definition for the dual Hausdorff topology on $\mathcal{L}^n$-stars (corresponding to the $\mathcal{L}^n$ topology on radial functions). Extensions are given for the dual mixed volumes to this larger class of star-shaped sets.

Chapter 3 summarizes some important results and inequalities concerning the dual mixed volumes, which are extended to the broader context developed in Chapter 2. Definitions are given for measures (also called *star measures*) on the lattice of $\mathcal{L}^n$-stars. The elementary properties of this lattice are developed, and the measure structure of dual mixed volumes is worked out.

A key distinction to be noted in the dual theory is that the origin remains fixed throughout. Translations are no longer considered. As a result, one is led to study star measures in terms of their relation to the unit sphere, rather than to all of $\mathbf{R}^n$. The classification of star measures shall require the use of results concerning analysis on the sphere, especially those concerning the uniqueness of the Haar measure on the unit sphere and on the Grassmanians. Chapter 4 is devoted to the review of these results.

The three major results of this thesis are Theorems 5.9, 6.11, and 7.3. Chapter 5 is devoted to the classification of all continuous star measures that are homogeneous with respect to dilation (see Theorem 5.9). This classification leads in turn to a characterization theorem for dual mixed volumes of pairs of $\mathcal{L}^n$-stars. These results are then re-cast in a language similar (and, in some sense, dual) to that used by Goodey and Weil [21], and by McMullen [30] to classify homogeneous measures on *convex* sets. I also present a dual Hadwiger theorem for homogeneous measures that are rotation invariant.

Chapter 6 is concerned with the classification of rotation invariant measures. The collection of all continuous rotation invariant measures on the $\mathcal{L}^n$ stars turns out to be far larger than the collection of measures classified by Hadwiger in the convex case (see Theorem 6.11). While Hadwiger gave a finite basis for all convex-continuous rigid-motion measures, the vector space of all star-continuous rotation invariant measures turns out to have infinite dimension.

Chapter 7 concludes this investigation of star measures with Theorem 7.3, a classification of all continuous star measures that are invariant under the action of the group SL_n. This result is especially satisfying: the space of all continuous SL_n-invariant star measures is has only two dimensions, being spanned by the the Euler characteristic and the usual volume in $\mathbf{R}^n$.

Acknowledgements I would like to thank my advisor and mentor, Gian-Carlo Rota, for his help and encouragement in my study of geometric probability. I would also like to thank Barbara Klain and Glenn Tesler, for proofreading the entire draft, and Richard Dudley, Daniel Stroock, and David Jerison, for their advice concerning certain analytic questions. Thanks also to the members of my committee, Daniel Kleitman and Sergei Fomin. Finally, I would like to thank my parents, Gerald and Barbara Klain, for their invaluable support and encouragement over the past eight years.

Chapter 1

Convexity

In this chapter we summarize results from the classical theory of convexity. Since important modifications will be made in several of the classical definitions, all readers are encouraged to read this chapter.

We shall denote n-dimensional Euclidean space by $\mathbf{R}^n$. The symbol $\mathbf{S}^{n-1}$ shall denote the $(n-1)$-dimensional unit sphere, centered at the origin. The spherical Lebesgue measure on $\mathbf{S}^{n-1}$ shall be denoted by S. A function $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ shall be called *measurable* if f is a measurable function with respect to S. For $p \geq 1$, the $\mathcal{L}^p$ norm of a measurable function f on $\mathbf{S}^{n-1}$ is defined by the expression:

$$\|f\|_p = \left(\int_{\mathbf{S}^{n-1}} f^p \, dS\right)^{\frac{1}{p}}.$$

A measurable function f on $\mathbf{S}^{n-1}$ shall be called $\mathcal{L}^p$-integrable, or simply $\mathcal{L}^p$, if $\|f\|_p < \infty$.

Definition 1.1 *A set $A \subseteq \mathbf{R}^n$ is said to be* convex, *if, for any $x_0, x_1 \in A$ and any $t \in [0,1]$, the point $(1-t)x_0 + tx_1 \in A$. A* convex body *$K \subseteq \mathbf{R}^n$ is a convex set that is also compact. Let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbf{R}^n$.*

A convex body $K \in \mathcal{K}^n$ is determined uniquely by its *support function* $h_K : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, defined by

$$h_K(u) = \max_{x \in K}\{x \cdot u\},$$

where $\cdot$ denotes the standard inner product on $\mathbf{R}^n$. For all $K \in \mathcal{K}^n$, the support function h_K is a continuous function on the unit sphere [37, p. 37]. For all $K, L \in \mathcal{K}^n$, we have $K \subseteq L$ if and only if $h_K \leq h_L$.

Definition 1.2 *Given $K_1, K_2, \ldots, K_m \in \mathcal{K}^n$, and positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$, the* Minkowski linear combination *$K = \lambda_1 K_1 + \lambda_2 K_2 + \ldots + \lambda_m K_m$ is the convex body whose support function is given by*

$$h_K = \sum_{j=1}^{m} \lambda_j h_{K_j}.$$

It is not hard to show that K consists of all vector sums $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m$ of points $x_j \in K_j$.

Definition 1.3 *Let $K_1, K_2, K_3, \ldots \in \mathcal{K}^n$. The sequence $\{K_j\}_1^\infty$* converges *to the convex body K in the* Hausdorff topology *if $\|h_{K_j} - h_K\|_n \longrightarrow 0$ as $j \to \infty$.*

This definition differs from the usual description of the Hausdorff topology on $\mathcal{K}^n$, in which the support functions of a convergent sequence of bodies must be uniformly convergent rather than convergent in $\mathcal{L}^n$. However, Vitale [40] has shown that a sequence of support functions converges in the $\mathcal{L}^n$ topology if and only if the sequence converges uniformly [37, p. 59]. Hence the two definitions are equivalent. The following lemma gives yet another equivalent description of convergence in $\mathcal{K}^n$.

For all $\alpha > 0$, let αB denote the ball of radius α in $\mathbf{R}^n$, centered at the origin. Note that $h_{\alpha B}(u) = \alpha$ for all $u \in \mathbf{S}^{n-1}$.

Lemma 1.4 *Let $K, K_1, K_2, \ldots \in \mathcal{K}^n$. The sequence of support functions h_{K_i} converges uniformly to h_K, if and only if, for all $\epsilon > 0$, there exists $N > 0$ such that*

$$K \subseteq K_i + \epsilon B \quad \text{and} \quad K_i \subseteq K + \epsilon B$$

for all $i > N$.

Proof: The sequence of support functions h_{K_i} converges uniformly to h_K if and only if, for all $\epsilon > 0$, there exists $N > 0$ such that $|h_{K_i} - h_K| < \epsilon$ whenever $i > N$. But this holds if and only if

$$h_{K_i} < h_K + \epsilon = h_{K+\epsilon B} \quad \text{and} \quad h_K < h_{K_i} + \epsilon = h_{K_i+\epsilon B}$$

whenever $i > N$. Since $h_K(u) \leq h_L(u)$ for all $u \in \mathbf{S}^{n-1}$ if and only if $K \subseteq L$, the lemma follows. □

Let $\mathcal{K}_0^n$ denote the class of $K \in \mathcal{K}^n$ such that the origin is contained in the interior of K. Note that $K \in \mathcal{K}_0^n$ if and only if there exists $\delta > 0$ such that $h_K \geq \delta$.

If $0 \in K$, then K is also characterized by its *radial function* ρ_K. For all $u \in \mathbf{S}^{n-1}$, define

$$\rho_K(u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $K \in \mathcal{K}_0^n$, then there exists $\epsilon > 0$ such that $\rho_K \geq \epsilon$. In this case the function $1/\rho_K$ is the support function of another body $K^* \in \mathcal{K}_0^n$, called the *polar body* of K [37, p. 33]. The mapping $K \longrightarrow K^*$ is a bijective involution; that is, $1/\rho_{K^*} = h_K$, and so $K^{**} = K$.

Every convex body K has a volume, denoted by $V(K)$. For computing the volume of a Minkowski linear combination, we have the following theorem [37, p. 275]. Let $[m]$ denote the set of natural numbers $1, 2, \ldots, m$.

Theorem 1.5 *If $K_1, K_2, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$, then*

$$V(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m) = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n \in [m]} V(K_{i_1}, K_{i_2}, \ldots, K_{i_n}) \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n},$$

where each coefficient $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. □

Given $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$, the coefficient $V(K_1, K_2, \ldots, K_n)$ is called the *Minkowski mixed volume* of the convex bodies $K_1, K_2, \ldots, K_n$. It is well-known that the mixed volume $V(K_1, K_2, \ldots, K_n)$ is a non-negative function in n variables on the set $\mathcal{K}^n$, is continuous on $\mathcal{K}^n$, and is monotonic with respect to the subset partial ordering on $\mathcal{K}^n$

[37, p. 275].

For $0 \leq i \leq n$ and $K, L \in \mathcal{K}^n$, the following shorthand notation is standard:

$$V_i(K, L) = V(K, \ldots, K, L, \ldots, L),$$

where K appears $n - i$ times and L appears i times in the right-hand expression.

Important special cases of the Minkowski mixed volumes are the *quermassintegrals* or *elementary mixed volumes*, defined as follows. For all $K \in \mathcal{K}^n$, the i-th quermassintegral $W_i(K)$ is given by

$$W_i(K) = V_i(K, B).$$

The quermassintegrals are also known as the *mean projection measures.* Let v_i denote the i-dimensional volume measure on $\mathbf{R}^i$. We shall denote by $\mathrm{Gr}(n, i)$ the space of all i-dimensional vector subspaces of $\mathbf{R}^n$. The space $\mathrm{Gr}(n, i)$ is also known as the *Grassmannian* [31, p. 131]. For all $\xi \in \mathrm{Gr}(n, i)$ and all $K \in \mathcal{K}^n$, we denote by $K|\xi$ the orthogonal projection of the body K onto the vector subspace ξ.

For all $K \in \mathcal{K}^n$,

$$W_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\xi \in \mathrm{Gr}(n,i)} v_i(K|\xi)\, d\xi, \tag{1.1}$$

where κ_i denotes the i-dimensional volume of the unit ball in $\mathbf{R}^i$, and where the integration is taken with respect to the rotation invariant probability measure on the Grassmannian $\mathrm{Gr}(n, i)$ [31, p. 131]. In other words, $W_{n-i}(K)$ is a equal to the mean of the i-dimensional volumes of the projections of K onto i-dimensional vector subspaces ξ of $\mathbf{R}^n$ [37, p. 295].

Two convex bodies K and L are said to be *homothetic* if there exists a positive real number α such that L is a translate of αK. If $L = \alpha K$, then K and L are said to be *dilates.* It will be convenient to recall the following inequality for Minkowski mixed volumes [28]. This inequality follows from a successive application of the Alexandrov-Fenchel Inequality, followed by Minkowski's Inequality [37, p. 317].

Theorem 1.6 *Let $K_1, \ldots, K_n \in \mathcal{K}^n$. Then*

$$V(K_1, \ldots, K_n)^n \geq V(K_1) \cdots V(K_n),$$

with equality if and only if $K_1, \ldots, K_n$ are homothetic. □

Let $\mathcal{A}$ be a collection of subsets of $\mathbf{R}^n$. A real-valued function μ with domain $\mathcal{A}$ is called a *set function.* A set function $\mu : \mathcal{K}^n \longrightarrow \mathbf{R}$ is said to be a *measure* on $\mathcal{K}^n$ if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L) \tag{1.2}$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$ as well. Equation (1.2) is also known as the *inclusion-exclusion principle.* More generally, a set function μ that satisfies Equation (1.2) on its domain of definition will be called a measure on its domain.

A measure μ with domain $\mathcal{A}$ is said to be *countably additive* if, given any sequence $A_1, A_2, \ldots$ of disjoint sets in $\mathcal{A}$ such that $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$,

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i).$$

For our purposes, a given measure shall *not* be assumed to satisfy countable additivity unless this is postulated explicitly.

A measure μ on $\mathcal{K}^n$ is said to be *convex-continuous* if, for any convergent sequence $K_i \longrightarrow K$ in $\mathcal{K}^n$,

$$\lim_{i \to \infty} \mu(K_i) = \mu(K).$$

A well-known example of a convex-continuous measure on $\mathcal{K}^n$ is the volume V, or Lebesgue measure. It turns out that all of the quermassintegrals $W_0, W_1, \ldots, W_n$ are convex-continuous measures on $\mathcal{K}^n$ (see [37, p. 290]).

Let $\mathcal{A}$ be a subcollection of the σ-ring $\mathcal{M}^n$ of all Lebesgue measurable subsets of $\mathbf{R}^n$ such that $\mathcal{A}$ is closed under the action of the special orthogonal group $SO(n)$. A measure μ on $\mathcal{A}$ is said to be *rotation invariant* if $\mu(\phi A) = \mu(A)$ for all $\phi \in SO(n)$ and all $A \in \mathcal{A}$.

Similarly, a measure μ is said to be *translation invariant* if $\mu(\psi A) = \mu(A)$ for all translations ψ and all sets A in the domain of μ. A measure μ is *invariant under rigid motions* if μ is both translation and rotation invariant.

The following theorem of Hadwiger classifies all convex-continuous measures on $\mathcal{K}^n$ that are invariant under rigid motions. The proof is long and difficult (see [24, 34]).

Theorem 1.7 *Suppose that μ is a convex-continuous measure on $\mathcal{K}^n$, and that μ is invariant under rigid motions. Then there exist $c_0, c_1, \ldots, c_n \in \mathbf{R}$ such that, for all $K \in \mathcal{K}^n$,*

$$\mu(K) = \sum_{i=0}^{n} c_i W_i(K).$$

In other words, the convex-continuous measures that are invariant under rigid motions form a real vector space spanned by the quermassintegrals.

Let $i > 0$. A measure on $\mathcal{K}^n$ is *homogeneous of degree* i, if $\mu(cA) = c^i\mu(A)$ for all $c \geq 0$. In [30], McMullen proved the following theorem.

Theorem 1.8 *Suppose that μ is a convex-continuous translation invariant measure on $\mathcal{K}^n$ that is homogeneous of degree $n-1$. Then there exist sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$ in $\mathcal{K}^n$ such that*

$$\mu(K) = \lim_{j\to\infty}(V_1(K, L_j) - V_1(K, M_j))$$

for all $K \in \mathcal{K}^n$. □

In [21], Goodey and Weil give a similar classification for convex-continuous measures that are homogeneous of degree 1.

Theorem 1.9 *Suppose that μ is a convex-continuous translation invariant measure on $\mathcal{K}^n$. Then μ is homogeneous of degree 1 if and only if there exist sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$ in $\mathcal{K}^n$ such that, for all $\delta > 0$,*

$$\mu(K) = \lim_{j\to\infty}(V_1(L_j, K) - V_1(M_j, K))$$

uniformly for all convex bodies $K \subseteq \delta B$. □

In the chapters that follow we shall develop analogues to Hadwiger's theorem and to the McMullen-Goodey-Weil results in the context of star-shaped sets.

For a more detailed discussion of convex bodies, Minkowski mixed volumes, and quermassintegrals, see [37].

Chapter 2

$\mathcal{L}^n$-stars

We begin with a definition.

Definition 2.1 *A set* $A \subseteq \mathbf{R}^n$ *is said to be* star-shaped, *if the following conditions hold:*

- $0 \in A$.
- *For each line* ℓ *passing through the origin in* $\mathbf{R}^n$, *the set* $A \cap \ell$ *is a closed interval.*

Note that, for every $x \in A$, the line segment connecting x to 0 lies in A. A star-shaped set A is determined uniquely by its *radial function* $\rho_A : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$. For $u \in \mathbf{S}^{n-1}$, define

$$\rho_A(u) = \max\{\lambda \geq 0 : \lambda u \in A\}.$$

For example, $\rho_{\alpha B}(u) = \alpha$ for all $u \in \mathbf{S}^{n-1}$. If A and C are star-shaped sets, then $A \subseteq C$ if and only if $\rho_A \leq \rho_C$.

Definition 2.2 *Given star-shaped sets* $A_1, \ldots, A_m$, *and positive real numbers* $\lambda_1, \ldots, \lambda_m$, *the* radial linear combination $A = \lambda_1 A_1 \tilde{+} \lambda_2 A_2 \tilde{+} \ldots \tilde{+} \lambda_m A_m$ *is the star-shaped set whose radial function is given by*

$$\rho_A = \sum_{j=1}^{m} \lambda_j \rho_{A_j}.$$

Given a sequence of star-shaped sets $A_1, A_2, \ldots$, and an integer $m > 0$, the sets $\bigcup_{i=1}^{m} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are also star-shaped, having radial functions

$$\rho_{A_1 \cup \cdots \cup A_m}(u) = \max_{i \in [m]} \rho_{A_i}(u) \quad \text{and} \quad \rho_{A_1 \cap A_2 \cap \cdots}(u) = \inf_{i > 0} \rho_{A_i}(u). \tag{2.1}$$

Note that $\bigcup_{i=1}^{\infty} A_i$ is not necessarily a star-shaped set, for this set may intersect a line through the origin in an open interval. If, for all lines ℓ through the origin, the set $\ell \cap \bigcup_{i=1}^{\infty} A_i$ is closed, then the radial function of $\bigcup_{i=1}^{\infty} A_i$ is given by the equation:

$$\rho_{A_1 \cup A_2 \cup \cdots}(u) = \max_{i > 0} \rho_{A_i}(u).$$

Any non-negative function on $\mathbf{S}^{n-1}$ will determine a star-shaped set, but the set of all non-negative functions is far too large and contains too many pathologies to suit our purposes.

Definition 2.3 *Let $p > 0$. A set $K \subseteq \mathbf{R}^n$ is an* $\mathcal{L}^p$-star, *if the following conditions hold:*

- *K is star-shaped.*
- *The radial function ρ_K of K is an $\mathcal{L}^p$ function on $\mathbf{S}^{n-1}$.*

Two $\mathcal{L}^p$-stars K, L are defined to be equal whenever $\rho_K = \rho_L$ almost everywhere on $\mathbf{S}^{n-1}$. If ρ_K is a continuous function on $\mathbf{S}^{n-1}$, then K is called a star body.

Denote by $\mathcal{S}^n$ the set of all $\mathcal{L}^n$-stars in $\mathbf{R}^n$. Denote by $\mathcal{S}_c^n$ the set of all star bodies in $\mathbf{R}^n$. Both $\mathcal{S}^n$ and $\mathcal{S}_c^n$ are closed under finite unions, finite intersections, and radial combinations. It follows from Equation (2.1) that the collection $\mathcal{S}^n$ is also closed under countable intersections. A star body is an $\mathcal{L}^p$-star for all $p \geq 1$.

As is well-known, the set of all $\mathcal{L}^n$ functions on a measure space X is Cauchy complete. This motivates the following definition.

Definition 2.4 *Let $K_1, K_2, K_3, \ldots \in \mathcal{S}^n$. The sequence $\{K_j\}_1^\infty$* converges *to the $\mathcal{L}^n$-star K in the* dual Hausdorff topology, *also called the* star topology, *if $\|\rho_{K_j} - \rho_K\|_n \longrightarrow 0$ as $j \to \infty$. The convergence of the sequence $\{K_j\}_1^\infty$ to the $\mathcal{L}^n$-star K is denoted $K_j \longrightarrow K$.*

A set function $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ is called *star-continuous* if, for any convergent sequence $K_j \longrightarrow K$ in $\mathcal{S}^n$, we have $\lim_{j\to\infty} \mu(K_j) = \mu(K)$.

This definition disagrees with previous definitions of the topology of $\mathcal{S}^n$, in which uniform convergence of radial functions was required for a sequence of star bodies to converge [27]. While sufficient when dealing with star bodies, uniform convergence is too stringent a condition for convergence in the larger class $\mathcal{S}^n$.

The dual Hausdorff topology on $\mathcal{S}^n$ is the natural analogue of the Hausdorff topology on the class $\mathcal{K}^n$ of convex bodies in $\mathbf{R}^n$.

Definition 2.5 *A set function $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ is a* star measure *if*

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

for all $K, L \in \mathcal{S}^n$.

A star measure need not be countably additive. For $i > 0$, a star measure μ is *homogeneous of degree i*, if $\mu(\alpha A) = \alpha^i \mu(A)$ for all $\alpha \geq 0$.

We will use the terms *volume* and *Lebesgue measure* interchangeably in reference to the Lebesgue measure in $\mathbf{R}^n$. We will show that every $\mathcal{L}^n$-star has a volume. The volume of an $\mathcal{L}^n$-star K will be denoted $V(K)$. In the case where K is also a convex body, this volume will agree with the volume $V(K)$ mentioned in Chapter 1. Since the volume is a measure on the Lebesgue measurable subsets of $\mathbf{R}^n$ (see [35, p. 50]), its restriction to $\mathcal{S}^n$ is a star measure. Often it will be convenient to express $V(K)$ in terms of polar coordinates on $\mathbf{R}^n$. In order to make sense of this, it will be helpful to review the construction of the volume measure on the unit sphere $\mathbf{S}^{n-1}$. Some preliminary definitions are required.

Definition 2.6 *Let $A \subseteq \mathbf{R}^n$. The* star hull *of A is defined to be the set*

$$so(A) = \{\lambda x : x \in A,\ 0 \leq \lambda \leq 1\}.$$

In other words, $so(A)$ is formed by taking the union of all straight line segments from the origin to points of A.

Lemma 2.7 *Let $\alpha > 0$. For all $A_1, A_2, \ldots \subseteq \mathbf{R}^n$,*

$$so(A_1 \cup A_2 \cup \ldots) = \bigcup_{i=1}^{\infty} so(A_i), \quad \text{and} \quad so(A_1 \cap A_2 \cap \ldots) \subseteq \bigcap_{i=1}^{\infty} so(A_i).$$

Proof: For all $y \in \mathbf{R}^n$, we have $y \in so(\bigcup_{i=1}^{\infty} A_i)$ if and only if $y = \lambda x$, where $x \in \bigcup_{i=1}^{\infty} A_i$ and $\lambda \in [0,1]$. But this holds if and only if $y = \lambda x$, where $x \in A_i$ for some $i \in \{1, 2, \ldots\}$; i.e. if and only if $y \in so(A_i)$.

Let $y \in so(\bigcap_{i=1}^{\infty} A_i)$. Then $y = \lambda x$, where $x \in \bigcap_{i=1}^{\infty} A_i$ and $\lambda \in [0,1]$. In other words, $x \in A_i$ for all $i > 0$, and so $y \in so(A_i)$ for all $i > 0$. □

For all $\alpha > 0$, denote by $\alpha\mathbf{S}^{n-1}$ the sphere of radius α, centered at the origin. Similarly, denote by αB the n-dimensional ball of radius α, centered at the origin. The following is an important special case of the star hull.

Definition 2.8 *Let $\alpha > 0$, and let $A \subseteq \alpha\mathbf{S}^{n-1}$ be measurable with respect to the spherical Lebesgue measure. In this case the star hull $so(A)$ will be called a* spherical cone *with base A and* height *α. A collection of spherical cones $C_1, C_2, \ldots$ is said to be* disjoint *if, $C_i \cap C_j = \{0\}$ for each $i \neq j$.*

Note that, by definition, a spherical cone always has a *measurable* base.

The results of Lemma 2.7 may be sharpened in the case where the star hulls in question are spherical cones with bases in a common sphere $\alpha\mathbf{S}^{n-1}$.

Lemma 2.9 *Let $\alpha > 0$. For all $A_1, A_2, \ldots \subseteq \alpha\mathbf{S}^{n-1}$,*

$$so(A_1 \cup A_2 \cup \ldots) = \bigcup_{i=1}^{\infty} so(A_i), \quad \text{and} \quad so(A_1 \cap A_2 \cap \ldots) = \bigcap_{i=1}^{\infty} so(A_i).$$

Proof: In the case of the union, this result follows immediately from Lemma 2.7.

Let $x \in so(A_1 \cap A_2 \cap \ldots) - \{0\}$. Let $u = \alpha x/|x|$. Then $u \in A_1 \cap A_2 \cap \ldots$, and so $u \in A_i$ for all $i > 0$. But this holds if and only if $x \in so(A_i)$ for all $i > 0$. □

For all $A \subseteq \mathbf{S}^{n-1}$, the *indicator function* $1_A : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ is defined as follows. For all $u \in \mathbf{S}^{n-1}$, define $1_A(u) = 1$ if $u \in A$, and define $1_A(u) = 0$ otherwise. The indicator 1_A is a measurable function on $\mathbf{S}^{n-1}$, if and only if A is a Lebesgue measurable subset of $\mathbf{S}^{n-1}$.

Lemma 2.10 *Let $\alpha > 0$, and let $so(A)$ be the spherical cone with base $A \subseteq \alpha\mathbf{S}^{n-1}$. Let $A_1 = \frac{1}{\alpha}A = \{x/\alpha : x \in A\}$. Then $\rho_{so(A)} = \alpha 1_{A_1}$. It follows that $so(A) \in \mathcal{S}^n$.*

Note that A_1 is just the radial projection of $so(A) - \{0\}$ onto $\mathbf{S}^{n-1}$.

Proof: Let $u \in \mathbf{S}^{n-1}$. If $u \in A_1$, then $\alpha u \in A$, and so $\alpha u \in so(A)$. Since $so(A) \subseteq \alpha B$, it follows that $\rho_{so(A)}(u) = \alpha$. If $u \notin A_1$, then $\alpha u \in \alpha\mathbf{S}^{n-1} - A$. Hence, $\lambda u \in so(A)$ if and only if $\lambda = 0$, and $\rho_{so(A)}(u) = 0$.

Since A is a measurable subset of $\alpha\mathbf{S}^{n-1}$, its projection A_1 is a measurable subset of $\mathbf{S}^{n-1}$. It follows that $\rho_{so(A)} = 1_{A_1}$ is a measurable function and that $so(A) \in \mathcal{S}^n$. □

Let $A \subseteq \mathbf{S}^{n-1}$ be such that $so(A)$ is Lebesgue measurable in $\mathbf{R}^n$. The spherical volume S of A may be expressed as follows:

$$S(A) = \frac{1}{n}V(so(A)).$$

It follows from Lemma 2.7, and from the measure properties of V, that S is a countably additive rotation invariant measure on $\mathbf{S}^{n-1}$. These conditions determine S uniquely up to a constant factor [31]. It follows that S is equal to the spherical Lebesgue measure. Thus, if $so(A)$ is a Lebesgue measurable subset of $\mathbf{R}^n$, then A is a Lebesgue measurable subset of $\mathbf{S}^{n-1}$, and $so(A)$ is a spherical cone.

Definition 2.11 *A* polycone *P is defined to be a finite union of spherical cones.*

It follows from Lemma 2.10 that a polycone is also an $\mathcal{L}^n$-star.

Proposition 2.12 *Let P be a polycone. Then there exists a unique collection $\alpha_1, \ldots, \alpha_m > 0$ and a unique collection of disjoint measurable sets $A_1, \ldots, A_m \subseteq \mathbf{S}^{n-1}$ such that*

$$\rho_P = \sum_{j=1}^{m} \alpha_j 1_{A_j}.$$

Conversely, any linear combination of measurable indicator functions is the radial function of a polycone.

A finite linear combination of measurable indicator functions is also called a *simple measurable function.*

Proof: By the definition of polycone, there exist $\alpha_1, \ldots, \alpha_m > 0$ and spherical cones $C_1, \ldots, C_m$, with bases $\alpha_i D_i \subseteq \alpha_i \mathbf{S}^{n-1}$, such that $P = C_1 \cup \cdots \cup C_m$. It then follows from Lemma 2.10 that, for all $u \in \mathbf{S}^{n-1}$,

$$\rho_P(u) = \max_{i \in [m]} \{\rho_{C_i(u)}\} = \max_{i \in [m]} \{1_{D_i(u)}\}.$$

Hence ρ_P takes on values from the set $\{0, \alpha_1, \alpha_2, \ldots, \alpha_m\}$. Since each ρ_{C_i} is a measurable function, so is ρ_P. For each $i \in [m]$, let $A_i = \rho_P^{-1}(\alpha_i)$. Then $\rho_P = \alpha_1 1_{A_1} + \cdots + \alpha_m 1_{A_m}$, where the sets A_i are measurable and mutually disjoint.

Conversely, suppose that P is a star-shaped set with radial function $\rho_P = \beta_1 1_{D_1} + \cdots + \beta_k 1_{D_k}$, where $\beta_i > 0$ and $D_i \subseteq \mathbf{S}^{n-1}$ is measurable for each $i \in [k]$. Then ρ_P takes on a finite number of non-zero values; call them $\alpha_1, \ldots \alpha_m$. Let $A_i = \rho_P^{-1}(\alpha_i)$ for each $i \in [m]$. Then $\rho_P = \alpha_1 1_{A_1} + \cdots + \alpha_m 1_{A_m}$, where the sets A_i are mutually disjoint. It follows that $P = C_1 \cup \cdots \cup C_m$, a disjoint union, where C_i is the star-shaped set with radial function $\alpha_i 1_{A_i}$ for each $i \in [m]$. By Lemma 2.10, the sets C_i are spherical cones. Hence P is a polycone. □

The set of polycones will prove to be a useful tool for approximating arbitrary $\mathcal{L}^n$-stars. Specifically, we have the following proposition.

Proposition 2.13 *Let $K \in \mathcal{S}^n$. Then there exists an increasing sequence $P_1 \subseteq P_2 \subseteq \cdots$ of polycones such that*

$$\lim_{j \to \infty} P_j = K$$

in $\mathcal{S}^n$ and such that $\rho_{P_j} \longrightarrow \rho_K$ pointwise as well.

Proof: Since ρ_K is an $\mathcal{L}^n$ function on $\mathbf{S}^{n-1}$, there exists an increasing sequence of non-negative simple measurable functions ρ_j on $\mathbf{S}^{n-1}$ such that $\lim_{j\to\infty} \rho_j = \rho_K$, a pointwise limit of functions [35, p. 15]. By Proposition 2.12, each ρ_j is the radial function of a polycone P_j. Since the ρ_j are increasing, $P_i \subseteq P_j$ whenever $i < j$.

The decreasing sequence $|\rho_K - \rho_j|^n \longrightarrow 0$ pointwise on $\mathbf{S}^{n-1}$. By the monotone convergence theorem,

$$\lim_{j \to \infty} \|\rho_K - \rho_j\|_n = \lim_{j \to \infty} \left(\int_{\mathbf{S}^{n-1}} |\rho_K - \rho_j|^n \, dS \right)^{\frac{1}{n}} = 0.$$

Hence, $P_j \longrightarrow K$ in $\mathcal{S}^n$. □

Note that when K is a star body, the radial function ρ_K is bounded by some $\alpha > 0$ almost everywhere on $\mathbf{S}^{n-1}$. Hence an increasing (or decreasing) sequence ρ_j of simple measurable functions may be found that converges to ρ_K *uniformly* [35]. This is no longer true when K is a star-shaped set with an arbitrary $\mathcal{L}^p$ radial function.

An important tool in the study of $\mathcal{L}^n$-stars is the *polar coordinate formula* for volume in $\mathbf{R}^n$.

Theorem 2.14 *The formula*

$$V(K) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^n dS,$$

holds for all $K \in \mathcal{S}^n$.

Proof: Let $A \subseteq \mathbf{S}^{n-1}$. Recall that the radial function $\rho_{so(A)}$ of the cone $so(A)$ is just the indicator function 1_A, which takes the value 1 at points of A, and is zero elsewhere on $\mathbf{S}^{n-1}$. Hence,

$$\begin{aligned} V(so(A)) &= \frac{1}{n}S(A) \\ &= \frac{1}{n}\int_{\mathbf{S}^{n-1}} 1_A \, dS \\ &= \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho^n_{so(A)} \, dS. \end{aligned}$$

The exponent n in the last integral above appears (and is) unnecessary at this point, but it will be needed when we generalize this formula to cones with radii different from one.

Now let $A \subseteq \alpha\mathbf{S}^{n-1}$. Let $A_1 = \frac{1}{\alpha}A$, the radial projection of A onto $\mathbf{S}^{n-1}$. Note that $so(A)$ is the $\mathcal{L}^n$-star with radial function $\rho_{so(A)} = \alpha 1_{A_1}$.

Since the volume in $\mathbf{R}^n$ is homogeneous of degree n with respect to dilation, it follows that

$$\begin{aligned} V(so(A)) &= V(\alpha \cdot so(A_1)) \\ &= \alpha^n V(so(A_1)) \\ &= \frac{\alpha^n}{n}\int_{\mathbf{S}^{n-1}} 1_{A_1} \, dS \\ &= \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho^n_{so(A)} \, dS. \end{aligned}$$

Next let P be a polycone. By Proposition 2.12, the function ρ_P may be expressed

$$\rho_P = \sum_{j=1}^{m} \alpha_j 1_{A_j},$$

where the sets A_j are mutually disjoint. For each j, let $C_j = so(\alpha_j A_j)$. Then by Lemma 2.10,

$$\rho_P = \sum_{j=1}^{m} \rho_{C_j}.$$

Hence,

$$\rho_P^n = \left(\sum_{j=1}^{m} \rho_{C_j}\right)^n = \left(\sum_{j=1}^{m} \alpha_j 1_{A_j}\right)^n = \sum_{j=1}^{m} \alpha_j^n 1_{A_j} = \sum_{j=1}^{m} \rho^n_{C_j}.$$

The third equality in the above expression follows from the fact that the sets A_j are mutually disjoint. This fact also implies that $C_i \cap C_j = \{0\}$ whenever $i \neq j$. Using the formula for the volume of a spherical cone, we may conclude that

$$V(P) = \sum_{j=1}^{m} V(C_j) = \sum_{j=1}^{m} \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_{C_j}^n \, dS = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_P^n \, dS$$

Now suppose that $K \in \mathcal{S}^n$, and suppose that there exists $\alpha > 0$ such that $\rho_K \leq \alpha$ almost everywhere on $\mathbf{S}^{n-1}$. Let $\eta = \alpha - \rho_K$. Since $\eta \geq 0$ almost everywhere on $\mathbf{S}^{n-1}$, it follows that η is the radial function of an $\mathcal{L}^n$-star L. By Proposition 2.13, there exists an increasing sequence of polycones $\{Q_i\}_{i=1}^{\infty}$, with radial functions η_i, such that $\eta_i \longrightarrow \eta$ pointwise as $i \longrightarrow \infty$. For all $i > 0$, let $\rho_i = \alpha - \eta_i$. Note that each ρ_i is a simple measurable function on $\mathbf{S}^{n-1}$ such that

$$\rho_i = \alpha - \eta_i \geq \alpha - \eta = \rho_K$$

almost everywhere on $\mathbf{S}^{n-1}$. Moreover, the functions ρ_i form a *decreasing* sequence of simple measurable functions that converges to ρ_K pointwise almost everywhere on $\mathbf{S}^{n-1}$. For all $i > 0$, denote by P_i the polycone with radial function ρ_i. It then follows from Equation (2.1) that

$$K = \bigcap_{i=1}^{\infty} P_i.$$

In other words, K is a Lebesgue measurable subset of $\mathbf{R}^n$; i.e. the volume of K is defined. We now apply the monotone convergence theorem to conclude that

$$\begin{aligned} V(K) &= V\left(\bigcap_{i=1}^{\infty} P_i\right) \\ &= \inf_{i>0} V(P_i) \\ &= \inf_{i>0} \left(\frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_i^n \, dS\right) \\ &= \frac{1}{n} \int_{\mathbf{S}^{n-1}} \inf_{i>0} \rho_i^n \, dS \end{aligned}$$

$$= \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_K^n \, dS.$$

In other words,

$$V(K) = \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_K^n \, dS, \tag{2.2}$$

provided that there exists $\alpha > 0$ such that $\rho_K \leq \alpha$ almost everywhere on $\mathbf{S}^{n-1}$.

Finally, let $K \in \mathcal{S}^n$. For all $j \geq 0$, let $E_j = \{u \in \mathbf{S}^{n-1} : j \leq \rho_K(u) < j+1\}$. The sets E_j partition $\mathbf{S}^{n-1}$ into a countable collection of disjoint measurable subsets. Moreover, for all $j \geq 0$, the function $1_{E_j}\rho_K$ is an $\mathcal{L}^n$ function such that

$$j \leq 1_{E_j}\rho_K \leq j+1.$$

Let K_j be the $\mathcal{L}^n$-star with radial function $1_{E_j}\rho_K$. Since each K_j is a bounded subset of $\mathbf{R}^n$, Equation (2.2) holds for each K_j. It follows that each K_j is a Lebesgue measurable subset of $\mathbf{R}^n$. Moreover, note that $K = \bigcup_{j\geq 0} K_j$. Since the volume V is a countably additive measure on $\mathbf{R}^n$,

$$\begin{aligned} V(K) &= \sum_{j=0}^{\infty} V(K_j) \\ &= \sum_{j=0}^{\infty} \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_{K_j}^n \, dS \\ &= \sum_{j=0}^{\infty} \frac{1}{n}\int_{\mathbf{S}^{n-1}} 1_{E_j}\rho_K^n \, dS \\ &= \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_K^n \, dS, \end{aligned}$$

where the last equality follows from the monotone convergence theorem.

Hence, the equation

$$V(K) = \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_K^n \, dS.$$

is valid for all $\mathcal{L}^n$-stars K. □

We can define analogues to the Minkowski mixed volumes by using radial combinations instead of Minkowski sums [27]. For computing the volume of a radial combination, we have the following theorem.

Theorem 2.15 *If $K_1, K_2, \dots, K_m \in \mathcal{S}^n$, and if $\lambda_1, \lambda_2, \dots, \lambda_m > 0$ then*

$$V(\lambda_1 K_1 \tilde{+} \lambda_2 K_2 \tilde{+} \cdots \tilde{+} \lambda_m K_m) = \sum_{i_1, i_2, \dots, i_n \in [m]} \tilde{V}(K_{i_1}, K_{i_2}, \dots, K_{i_n}) \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n},$$

where each coefficient $\tilde{V}(K_{i_1}, K_{i_2}, \dots, K_{i_n})$ depends only on the bodies $K_{i_1}, K_{i_2}, \dots, K_{i_n}$.

Given $K_1, K_2, \dots, K_n \in \mathcal{S}^n$, the coefficient $\tilde{V}(K_1, K_2, \dots, K_n)$ given by Theorem 2.15 is called the *dual mixed volume* of $K_1, K_2, \dots, K_n$.

Proof: Let $\lambda_1 K_1 \tilde{+} \lambda_2 K_2 \tilde{+} \cdots \tilde{+} \lambda_m K_m$ be a radial combination of $\mathcal{L}^n$-stars. The polar coordinate formula for volume implies that

$$\begin{aligned} V(\lambda_1 K_1 \tilde{+} \lambda_2 K_2 \tilde{+} \cdots \tilde{+} \lambda_m K_m) &= \frac{1}{n} \int_{S^{n-1}} (\lambda_1 \rho_{K_1} + \lambda_2 \rho_{K_2} + \cdots + \lambda_m \rho_{K_m})^n dS \\ &= \frac{1}{n} \sum_{i_1, i_2, \dots, i_n \in [m]} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n} \int_{S^{n-1}} \rho_{K_{i_1}} \rho_{K_{i_2}} \cdots \rho_{K_{i_n}} \, dS. \end{aligned}$$

Note that this expression is a homogeneous polynomial in the real variables λ_i. The coefficient of the $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}$ term depends solely on the $\mathcal{L}^n$-stars $K_{i_1}, K_{i_2}, \dots, K_{i_n}$. Hence, we set

$$\tilde{V}(K_{i_1}, K_{i_2}, \dots, K_{i_n}) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_{i_1}} \rho_{K_{i_2}} \cdots \rho_{K_{i_n}} \, dS. \tag{2.3}$$

□

It follows from the integral representation (2.3) that $\tilde{V}$ is well-defined on $\mathcal{S}^n$, for the dual mixed volumes ignore sets of Lebesgue measure zero.

The following corollary is an immediate consequence of the integral representation (2.3) for dual mixed volumes.

Corollary 2.16 *The dual mixed volume $\tilde{V}(K_1, K_2, \ldots, K_n)$ is a non-negative function on $\mathcal{S}^n$ in* n *variables, and is monotonic with respect to the subset partial ordering on $\mathcal{S}^n$.*
□

In analogy to the quermassintegrals for convex bodies, the *dual quermassintegrals* $\widetilde{W}_i$ of an $\mathcal{L}^n$-star K are defined as follows. For $K \in \mathcal{S}^n$ the i-th dual quermassintegral $\widetilde{W}_i(K)$ is given by

$$\widetilde{W}_i(K) = \tilde{V}(K, \ldots, K, B, \ldots, B),$$

where the $\mathcal{L}^n$-star K appears $n - i$ times and the unit ball B appears i times.

The dual mixed volume $\tilde{V}$ is a real-valued function on $\mathcal{S}^n$ in n variables. The following lemma tells us that $\tilde{V}$ is a continuous function on the collection of $\mathcal{L}^n$-stars.

Lemma 2.17 *Let $K, K_i \in \mathcal{S}^n$ such that $K_i \longrightarrow K$. For all $Q_1, \ldots, Q_{n-1} \in \mathcal{S}^n$,*

$$\tilde{V}(Q_1, \ldots, Q_{n-1}, K_i) \longrightarrow \tilde{V}(Q_1, \ldots, Q_{n-1}, K),$$

as $i \to \infty$.

Proof: Let $\epsilon > 0$. Let $\delta = \frac{1}{n}\|\rho_{Q_1}\|_n \|\rho_{Q_2}\|_n \cdots \|\rho_{Q_{n-1}}\|_n$. Since $K_i \longrightarrow K$, there exists $m > 0$ such that $\|\rho_{K_i} - \rho_K\|_n < \frac{\epsilon}{\delta}$ whenever $i > m$.

Therefore, if $i > m$, then

$$\begin{aligned}
|\tilde{V}(Q_1, \ldots, Q_{n-1}, K_i) - \tilde{V}(Q_1, \ldots, Q_{n-1}, K)| &= \frac{1}{n}\left|\int_{S^{n-1}} \rho_{Q_1} \cdots \rho_{Q_{n-1}}(\rho_{K_i} - \rho_K)\, dS\right| \\
&\le \frac{1}{n}\int_{S^{n-1}} \rho_{Q_1} \cdots \rho_{Q_{n-1}}|\rho_{K_i} - \rho_K|\, dS \\
&\le \frac{1}{n}\|\rho_{Q_1}\|_n \cdots \|\rho_{Q_{n-1}}\|_n \|\rho_{K_i} - \rho_K\|_n \\
&\le \delta\frac{\epsilon}{\delta} \\
&= \epsilon,
\end{aligned}$$

where the second inequality follows from the Hölder inequality [35, p. 63]. It follows that $\tilde{V}$ is continuous in each variable. □

In particular, the dual quermassintegrals are star-continuous. This gives added evidence that the class of $\mathcal{L}^n$-stars is indeed the class of bodies we are interested in, as we pursue a study of the dual theory.

In analogy to the mean projection representation (1.1) for quermassintegrals, we have the following representation [27] for the dual quermassintegrals of a star body K:

$$\widetilde{W}_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\xi \in \mathrm{Gr}(n,i)} v_i(K \cap \xi)\, d\xi = \frac{\kappa_n}{i\kappa_i} \int_{\xi \in \mathrm{Gr}(n,i)} \int_{\xi \cap \mathbf{S}^{n-1}} \rho_K^i \, dS^{i-1}\, d\xi, \tag{2.4}$$

where dS^{i-1} denotes integration with respect to the Lebesgue measure on the $(i-1)$-dimensional unit sphere. In other words, $\widetilde{W}_{n-i}(K)$ is equal to the mean of the i-dimensional volumes v_i of the intersections of K with i-dimensional subspaces ξ of $\mathbf{R}^n$. This is one example of the way in which results in the Brunn-Minkowski theory translate into results in the theory of dual mixed volumes. In Chapter 5 this result is extended to the class of $\mathcal{L}^n$-stars (see Theorem 5.11).

Two $\mathcal{L}^p$-stars K and L are said to be *dilates* if there exists $c > 0$ such that $\rho_K = c\rho_L$ almost everywhere on $\mathbf{S}^{n-1}$; i.e. if $K = cL$ in $\mathcal{S}^n$. Just as there are radial analogues for the Brunn-Minkowski and Alexandrov-Fenchel inequalities for star-shaped sets [27], we have the following radial analogue for Theorem 1.6.

Theorem 2.18 *Let $K_1, K_2, \ldots, K_n \in \mathcal{S}^n$. Then*

$$\widetilde{V}(K_1, K_2, \ldots, K_n)^n \leq V(K_1)V(K_2)\cdots V(K_n),$$

with equality if and only if $K_1, K_2, \ldots, K_n$ are dilates. □

This radial analogue follows from the Hölder inequality for integrals [35, p. 63], and from the integral representation for the dual mixed volumes. Hence, this inequality holds for all $\mathcal{L}^n$-stars.

There is a natural action of the special linear group $SL(n)$ on the class of star-shaped sets. This action is especially nice when restricted to the special orthogonal group $SO(n)$. We begin with some preliminary results.

Proposition 2.19 *Let* $\zeta : \mathbf{S}^{n-1} \longrightarrow \mathbf{S}^{n-1}$ *be a diffeomorphism, and let* $E \subseteq \mathbf{S}^{n-1}$ *be a subset of spherical Lebesgue measure zero. Then* $S(\zeta(E)) = 0$ *as well.*

Proof: For all $x \in \mathbf{R}^n - \{0\}$, the point x may be expressed in polar coordinates, $x = (r, u)$, where $r = |x|$ and $u = x/|x| \in \mathbf{S}^{n-1}$. Define a map $\tilde{\zeta} : \mathbf{R}^n - \{0\} \longrightarrow \mathbf{R}^n - \{0\}$ by the equation

$$\tilde{\zeta}(r, u) = r\zeta(u).$$

Since ζ is a diffeomorphism of $\mathbf{S}^{n-1}$, $\tilde{\zeta}$ is a diffeomorphism of $\mathbf{R}^n - \{0\}$.

Suppose $E \subseteq \mathbf{S}^{n-1}$ has spherical Lebesgue measure $S(E) = 0$. This means that

$$V(so(E)) = S(E) = 0.$$

Since $\tilde{\zeta}$ is a diffeomorphism of the open set $\mathbf{R}^n - \{0\}$ in $\mathbf{R}^n$, it follows that $V(\tilde{\zeta}(so(E))) = 0$ (see [35, p. 153]). From the definitions of $\tilde{\zeta}$ and the measure S, it then follows that

$$S(\zeta(E)) = V(so(\zeta(E)) = V(\tilde{\zeta}(so(E))) = 0.$$

□

Proposition 2.20 *Let* $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ *be an* $\mathcal{L}^p$ *function, where* $p \geq 1$*, and let* $\zeta : \mathbf{S}^{n-1} \longrightarrow \mathbf{S}^{n-1}$ *be a diffeomorphism. Then the composed function* $f \circ \zeta : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ *is an* $\mathcal{L}^p$ *function.*

Proof: Suppose that $p = 1$. Define a set function ν on the Borel subsets of $\mathbf{S}^{n-1}$ as follows. For all $A \subseteq \mathbf{S}^{n-1}$, define

$$\nu(A) = S(\zeta^{-1}(A)).$$

Since ζ^{-1} is a diffeomorphism, ζ^{-1} maps open sets to open sets and closed sets to closed sets. Moreover, ζ^{-1} commutes with unions and intersections, for ζ^{-1} is a bijective function on $\mathbf{S}^{n-1}$. It follows that ζ^{-1} maps Borel sets to Borel sets, and that ν is a Borel measure on $\mathbf{S}^{n-1}$. If $S(A) = 0$, then Proposition 2.19 implies that $\nu(A) = 0$ as well. In other words, ν is a Borel measure that is absolutely continuous with respect to the invariant measure S on $\mathbf{S}^{n-1}$. By the Lebesgue-Radon-Nikodym theorem [35, p. 121], there exists an $\mathcal{L}^1$ function $g_\nu : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, such that

$$\nu(A) = \int_{\mathbf{S}^{n-1}} 1_A g_\nu \, dS$$

for all Borel sets $A \subseteq \mathbf{S}^{n-1}$. Since ν is a non-negative measure, $g_\nu \geq 0$.

Meanwhile, suppose that $h : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ is a *continuous* function. In this case,

$$\int_{\mathbf{S}^{n-1}} hg_\nu \, dS = \int_{\mathbf{S}^{n-1}} h \, d\nu = \int_{\mathbf{S}^{n-1}} h \circ \zeta \, dS = \int_{\mathbf{S}^{n-1}} hJ_\zeta \, dS,$$

where J_ζ is the Jacobian of ζ. In other words, $g_\nu = J_\zeta$. But J_ζ is a continuous function on $\mathbf{S}^{n-1}$. In particular, J_ζ is *bounded* on $\mathbf{S}^{n-1}$. Hence, there exists $M > 0$ such that $0 \leq g_\nu \leq M$.

Since $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ is an $\mathcal{L}^1$ function,

$$\int_{\mathbf{S}^{n-1}} f \circ \zeta \, dS = \int_{\mathbf{S}^{n-1}} f \, d\nu = \int_{\mathbf{S}^{n-1}} fg_\nu \, dS \leq M \int_{\mathbf{S}^{n-1}} f \, dS < \infty.$$

In other words, $f \circ \zeta$ is an $\mathcal{L}^1$ function.

Next suppose that f is an $\mathcal{L}^p$ function, where $p > 1$. Then f^p is an $\mathcal{L}^1$ function. It follows that $f^p \circ \zeta = (f \circ \zeta)^p$ is an $\mathcal{L}^1$ function, so that $f \circ \zeta$ is $\mathcal{L}^p$. □

For all $A \subseteq \mathbf{R}^n$ and all $\phi \in SL(n)$, define $\phi A = \{\phi(x) : x \in A\}$.

Proposition 2.21 *Let $\phi \in SL(n)$. For all star-shaped sets K, the set ϕK is also star-shaped. Moreover, for all $u \in \mathbf{S}^{n-1}$,*

$$\rho_{\phi K}(u) = \frac{1}{|\phi^{-1}(u)|} \rho_K \left(\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|} \right).$$

It follows that ϕK is an $\mathcal{L}^p$-star (or a star body) if and only if K is an $\mathcal{L}^p$-star (or a star body). If $\phi \in SO(n)$, then $\rho_{\phi K} = \rho_K \circ \phi^{-1}$.

Proof: Since ϕ is linear and bijective, $\phi(0) = 0$, and for all lines ℓ through the origin in $\mathbf{R}^n$, ϕ maps the closed line segment $K \cap \ell$ to the closed line segment $\phi K \cap \phi\ell$. It follows that ϕK is star-shaped.

For all $u \in \mathbf{S}^{n-1}$,

$$\begin{aligned}
\rho_{\phi K}(u) &= \max\{\lambda : \lambda u \in \phi K\} \\
&= \max\{\lambda : \lambda \phi^{-1}(u) \in K\} \\
&= \max\{\lambda : \lambda|\phi^{-1}(u)|\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|} \in K\} \\
&= \frac{1}{|\phi^{-1}(u)|}\max\{\lambda|\phi^{-1}(u)| : \lambda|\phi^{-1}(u)|\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|} \in K\} \\
&= \frac{1}{|\phi^{-1}(u)|}\rho_K\left(\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|}\right).
\end{aligned}$$

It follows that $\rho_{\phi K}$ is a continuous function if and only if ρ_K is continuous.

Suppose that $K \in \mathcal{S}^n$. Let $\zeta : \mathbf{S}^{n-1} \longrightarrow \mathbf{S}^{n-1}$ be given by

$$\zeta(u) = \frac{\phi^{-1}(u)}{|\phi^{-1}(u)|}.$$

Since ζ is a diffeomorphism on $\mathbf{S}^{n-1}$, we may apply Proposition 2.20 to conclude that $\rho_K \circ \zeta$ is $\mathcal{L}^n$. The function $1/|\phi^{-1}(u)|$ is continuous on $\mathbf{S}^{n-1}$ and is therefore bounded. It follows that the function

$$\rho_{\phi K}(u) = \frac{1}{|\phi^{-1}(u)|}\rho_K\left(\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|}\right) \tag{2.5}$$

is an $\mathcal{L}^n$ function, and that ϕK is an $\mathcal{L}^n$-star.

If $\phi \in SO(n)$, then ϕ preserves length, so that $|\phi^{-1}(u)| = |u| = 1$. It follows from (2.5) that $\rho_{\phi K}(u) = \rho_K(\phi^{-1}(u))$. □

Chapter 3

The Measure Structure of Dual Mixed Volumes

Recall that the quermassintegrals on $\mathbf{R}^n$ satisfy the properties of a measure, when considered as functions on the set of convex bodies [29, 37]. In this chapter we investigate the analogous measure properties of the dual mixed volumes.

First let us state a theorem that generalizes the claim above. Given $\mathcal{L}^n$-stars K and Q, recall that

$$\tilde{V}_i(K,Q) = \tilde{V}(K,\ldots,K,Q,\ldots,Q),$$

where K appears $n-i$ times and Q appears i times in the right-hand expression.

Proposition 3.1 *Let $Q \in \mathcal{S}^n$ be fixed. The function on $\mathcal{S}^n$*

$$K \longmapsto \tilde{V}_i(K,Q)$$

satisfies the properties of a star measure. Moreover, this measure is continuous on $\mathcal{S}^n$.

Proof: Recall that, for all $K, L \in \mathcal{S}^n$ and all $u \in \mathbf{S}^{n-1}$, we have $\rho_{K\cup L}(u) = \max\{\rho_K(u), \rho_L(u)\}$ and $\rho_{K\cap L}(u) = \min\{\rho_K(u), \rho_L(u)\}$. Hence, for all $u \in \mathbf{S}^{n-1}$ and all $k > 0$,

$$\rho^k_{K\cup L}(u) + \rho^k_{K\cap L}(u) \quad = \quad \max\{\rho_K(u), \rho_L(u)\}^k + \min\{\rho_K(u), \rho_L(u)\}^k$$

$$= \rho_K^k(u) + \rho_L^k(u).$$

It follows that

$$\begin{aligned}
\tilde{V}_i(K \cup L, Q) + \tilde{V}_i(K \cap L, Q) &= \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_{K\cup L}^{n-i}\rho_Q^i \, dS + \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_{K\cap L}^{n-i}\rho_Q^i \, dS \\
&= \frac{1}{n}\int_{\mathbf{S}^{n-1}} (\rho_{K\cup L}^{n-i} + \rho_{K\cap L}^{n-i})\rho_Q^i \, dS \\
&= \frac{1}{n}\int_{\mathbf{S}^{n-1}} (\rho_K^{n-i} + \rho_L^{n-i})\rho_Q^i \, dS \\
&= \tilde{V}_i(K, Q) + \tilde{V}_i(L, Q).
\end{aligned}$$

Hence, $\tilde{V}_i(K, Q)$ is a star measure in the variable K (see Definition 2.5). It follows from Lemma 2.17 and from the integral representation (2.3) for dual mixed volumes that this measure is star-continuous with respect to K (and, for that matter, with respect to Q as well). □

We will denote by $\tilde{V}_i(*, Q)$ the star measure on $\mathcal{S}^n$ given by the map

$$K \longmapsto \tilde{V}_i(K, Q).$$

The radial sum operation $\tilde{+}$ also satisfies inclusion-exclusion, as well as various distributive laws.

Proposition 3.2 *For all $K, L \in \mathcal{S}^n$,*

$$(K \cup L)\tilde{+}(K \cap L) = K\tilde{+}L.$$

Proof: For all $u \in \mathbf{S}^{n-1}$,

$$\begin{aligned}
\rho_{K\cup L}(u) + \rho_{K\cap L}(u) &= \max\{\rho_K(u), \rho_L(u)\} + \min\{\rho_K(u), \rho_L(u)\} \\
&= \rho_K(u) + \rho_L(u).
\end{aligned}$$

□

Proposition 3.3 *For all $K, L, Q \in \mathcal{S}^n$,*

$$(K \cup L)\tilde{+}Q = (K\tilde{+}Q) \cup (L\tilde{+}Q).$$

Proof: For all $u \in \mathbf{S}^{n-1}$,

$$\begin{aligned} \rho_{K \cup L}(u) + \rho_Q(u) &= \max\{\rho_K(u), \rho_L(u)\} + \rho_Q(u) \\ &= \max\{\rho_K(u) + \rho_Q(u), \rho_L(u) + \rho_Q(u)\} \\ &= \rho_{(K\tilde{+}Q)\cup(L\tilde{+}Q)}(u). \end{aligned}$$

□

Proposition 3.4 *For all $K, L, Q \in \mathcal{S}^n$,*

$$(K \cap L)\tilde{+}Q = (K\tilde{+}Q) \cap (L\tilde{+}Q).$$

Proof: Given $u \in \mathbf{S}^{n-1}$,

$$\begin{aligned} \rho_{K \cap L}(u) + \rho_Q(u) &= \min\{\rho_K(u), \rho_L(u)\} + \rho_Q(u) \\ &= \min\{\rho_K(u) + \rho_Q(u), \rho_L(u) + \rho_Q(u)\} \\ &= \rho_{(K\tilde{+}Q)\cap(L\tilde{+}Q)}(u). \end{aligned}$$

□

An important question now arises concerning the conditions one must place on the (fixed) body Q so that the star measure $\tilde{V}_i(*, Q)$ will be rotation invariant.

Theorem 3.5 *Let $i \in [n]$. Let $Q \in \mathcal{S}_c^n$. The star measure $\widetilde{V}_i(*, Q)$ is rotation invariant if and only if Q is a ball; that is, if and only if there exists $c \in \mathbf{R}$ such that $\widetilde{V}_i(*, Q) = c\widetilde{W}_i$.*

Proof: Let $\phi \in SO(n)$. From Theorem 2.18 it follows that

$$\begin{aligned} \widetilde{V}_i(\phi Q, Q)^n &= \widetilde{V}(\phi Q, \phi Q, \ldots, \phi Q, Q, \ldots, Q) \\ &\geq V(\phi Q)^{n-i} V(Q)^i \\ &= V(Q)^n. \end{aligned}$$

Meanwhile, rotation invariance implies that $\widetilde{V}_i(\phi Q, Q) = \widetilde{V}_i(Q, Q) = V(Q)$. The above inequality becomes an equality. It then follows from the equality conditions of Theorem 2.18 that ϕQ is a dilate of Q. Since ϕ preserves volume, $\phi Q = Q$. Hence $\rho_{\phi Q} = \rho_Q$ *almost everywhere.* Recall from Proposition 2.21 that $\rho_{\phi Q} = \rho_Q \circ \phi^{-1}$. It follows that, for each $\phi \in SO(n)$, $\rho_Q \circ \phi^{-1} = \rho_Q$ almost everywhere on $\mathbf{S}^{n-1}$.

Two continuous functions are equal almost everywhere if and only if they are identical. Hence,

$$\rho_Q \circ \phi^{-1} = \rho_Q$$

for all $\phi \in SO(n)$.

Let $u_0 \in S^{n-1}$. For any $u \in \mathbf{S}^{n-1}$, there exists a rotation $\phi_u \in SO(n)$ such that $\phi_u(u) = u_0$. It follows that $\rho_Q(u) = \rho_Q \circ \phi_u(u) = \rho_Q(u_0)$. Hence ρ_Q is a constant, and so Q must be a ball. □

This theorem is unsatisfactory, for one must assume that $Q \in \mathcal{S}_c^n$. In fact, the theorem holds in much greater generality, provided only that ρ_Q is an $\mathcal{L}^n$ function on $\mathbf{S}^{n-1}$ (see Theorem 4.5).

Chapter 4

Measures on the Unit Sphere

The dual mixed volumes supply us with a cornucopia of measures on the lattice of $\mathcal{L}^n$-stars. Our goal is to classify all such measures, or as many as we can. In the next chapter we will proceed with the classification of star measures that are homogeneous with respect to dilation of $\mathcal{L}^n$-stars. Such measures are closely related to measures on the unit sphere $\mathbf{S}^{n-1}$. These measures are the subject of the present chapter.

Of especial significance is the following theorem [31].

Theorem 4.1 *Let μ be a countably additive Borel measure on the unit sphere $\mathbf{S}^{n-1}$, such that μ is invariant under the action of the special orthogonal group $SO(n)$. Then there exists $k \in \mathbf{R}$ such that $\mu = kS$.*

Here S denotes the Lebesgue measure on $\mathbf{S}^{n-1}$. Before reviewing the proof of this theorem, let us briefly discuss the action of $SO(n)$ on $\mathbf{S}^{n-1}$. Let n_0 be a fixed point in $\mathbf{S}^{n-1}$, the north pole. Define a map $\Psi : SO(n) \longrightarrow \mathbf{S}^{n-1}$ by the equation

$$\Psi(\phi) = \phi(n_0).$$

We are mapping each rotation to the image of the north pole under that rotation. The stabilizer of the north pole n_0,

$$\mathrm{Stab}(n_0) = \{\phi \in SO(n) : \phi(n_0) = n_0\},$$

is isomorphic to $SO(n-1)$, that subgroup being the set of rotations of the equator $\mathbf{S}^{n-2} \subset \mathbf{S}^{n-1}$. Moreover, for any $\phi \in SO(n)$ and any $\eta \in \mathrm{Stab}(n_0)$,

$$\Psi(\phi\eta) = \phi \circ \eta(n_0) = \phi(n_0) = \Psi(\phi).$$

Since the action of $SO(n)$ on $\mathbf{S}^{n-1}$ is continuous, being the restriction of the action $SO(n)$ on $\mathbf{R}^n$, it follows that Ψ induces a homeomorphism

$$\tilde{\Psi} : \frac{SO(n)}{SO(n-1)} \longrightarrow \mathbf{S}^{n-1}.$$

Note that $\frac{SO(n)}{SO(n-1)}$ is not a group, but merely a quotient space. The result is a representation of the sphere $\mathbf{S}^{n-1}$ as a homogeneous space of the topological group $SO(n)$.

An *integral* μ on $\mathbf{S}^{n-1}$ is defined to be a linear functional on the vector space $C(\mathbf{S}^{n-1})$ of continuous real-valued functions on $\mathbf{S}^{n-1}$. A *positive integral* μ on $\mathbf{S}^{n-1}$ is an integral such that $\mu(f)$ is positive whenever f is a non-negative continuous function.

The following is a special case of a theorem of André Weil [31, p. 138], applied to the homogeneous space we have just described.

Theorem 4.2 *There exists a positive integral ν on the sphere $\mathbf{S}^{n-1}$ such that ν is invariant under the action of $SO(n)$. If η is another positive integral on $\mathbf{S}^{n-1}$ satisfying these conditions, then there exists $k > 0$ such that $\eta = k\nu$.*

In this case the existence part of the theorem is obvious, since the Lebesgue integral on $\mathbf{S}^{n-1}$ satisfies the conditions for ν. It is the uniqueness statement that will be of use to us.

Weil's theorem is actually more general. Given a topological group G and a homogeneous space E, he classifies all positive integrals on E that are invariant under the action of G. For a proof of his result, as well as a complete discussion of homogeneous spaces and Haar integrals, see [31].

Proposition 4.3 (The Jordan Decomposition) *Let μ be a countably additive Borel measure on the unit sphere $\mathbf{S}^{n-1}$. Then there exist non-negative countably additive Borel measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. The measure μ is rotation invariant if and only if the measures μ^+ and μ^- are rotation invariant.*

Proof: Recall that any real measure μ on $\mathbf{S}^{n-1}$ has a total variation norm $|\mu|$, defined as follows [35, p. 116]. Let $A \subseteq \mathbf{S}^{n-1}$ be measurable. The measure $|\mu|$ is defined by

$$|\mu|(A) = \sup\{\sum_i |\mu(A_i)|\}, \tag{4.1}$$

where the supremum is taken over all finite partitions of the set A into disjoint measurable subsets $\{A_1, \ldots, A_p\}$.

The total variation norm $|\mu|$ is also a measure. Recall also that $|\mu(A)| \leq |\mu|(A)$.

Take care to note that the ambiguous notation $\|\mu\|$ does *not* refer to the absolute value of the measure μ. Nor is this "norm" a norm in the sense of Banach spaces. The total variation norm of a measure is *a non-negative measure*, on the same measure space as the original measure, and related to the original measure by Equation (4.1).

Define non-negative measures μ^+ and μ^- on $\mathbf{S}^{n-1}$ by

$$\mu^+ = \frac{1}{2}(|\mu| + \mu) \quad \text{and} \quad \mu^- = \frac{1}{2}(|\mu| - \mu).$$

Then $\mu = \mu^+ - \mu^-$.

Suppose that μ is invariant under the action of $SO(n)$ on $\mathbf{S}^{n-1}$. Then it is clear from the definition of the total variation norm that $|\mu|$ is rotation invariant as well. It follows that μ^+ and μ^- are also rotation invariant measures.

Conversely, suppose that μ^+ and μ^- are rotation invariant measures. Since $\mu = \mu^+ - \mu^-$, the measure μ is also rotation invariant. □

This representation of μ is called the *Jordan decomposition* of μ. The measures μ^+ and μ^- are called the *Jordan components* of μ.

We will now prove Theorem 4.1.

Proof of Theorem 4.1: If $\mu = 0$ then the result is obvious, so let us assume that $\mu \neq 0$. Let μ be a non-negative measure on the unit sphere $\mathbf{S}^{n-1}$, such that μ is countably additive and invariant under the action of the orthogonal group $SO(n)$. Since all continuous functions on $\mathbf{S}^{n-1}$ are bounded and Borel measurable, integration with respect to μ restricts to a linear functional on $C(\mathbf{S}^{n-1})$ that satisfies the definition of a positive integral. By the uniqueness conclusion of Theorem 4.2, there exists $k > 0$ such that for all $f \in C(\mathbf{S}^{n-1})$,

$$\int_{\mathbf{S}^{n-1}} f\, d\mu = k \int_{\mathbf{S}^{n-1}} f\, dS.$$

Here dS denotes the Lebesgue integral on $\mathbf{S}^{n-1}$.

Now let $f \in \mathcal{L}^1(S)$. Then there exists a sequence of continuous functions $f_j : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ such that $\|f - f_j\|_1 \longrightarrow 0$ as $j \to \infty$. It follows that

$$\int_{\mathbf{S}^{n-1}} f\, d\mu = k \int_{\mathbf{S}^{n-1}} f\, dS,$$

for any Lebesgue integrable function f. In particular $\mu = kS$ on $\mathbf{S}^{n-1}$, where S denotes the Lebesgue measure on $\mathbf{S}^{n-1}$.

Now suppose that μ takes on both positive and negative values. By Proposition 4.3, there exists non-negative countably additive invariant measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. We may conclude from the previous argument that there exist $k_1, k_2 > 0$ such that $\mu^+ = k_1 S$ and $\mu^- = k_2 S$. It follows that $\mu = kS$, where $k = k_1 - k_2$. □

Corollary 4.4 *Let $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ be an $\mathcal{L}^1$ function. Suppose that, for each $\phi \in SO(n)$, we have $f \circ \phi = f$ almost everywhere. Then there exists $c \in \mathbf{R}$ such that $f = c$ almost everywhere on $\mathbf{S}^{n-1}$.*

Proof: Define a countably additive measure ν on $\mathbf{S}^{n-1}$ in the following way. For all

Lebesgue measurable $A \subseteq \mathbf{S}^{n-1}$, define

$$\nu(A) = \int_{\mathbf{S}^{n-1}} 1_A f \, dS.$$

Since the measure S is rotation invariant on $\mathbf{S}^{n-1}$, it follows that, for all $\mathcal{L}^1$ functions g on $\mathbf{S}^{n-1}$ and all $\phi \in SO(n)$,

$$\int_{\mathbf{S}^{n-1}} g \circ \phi \, dS = \int_{\mathbf{S}^{n-1}} g \, dS.$$

Meanwhile, for all $A \subseteq \mathbf{S}^{n-1}$ and all $u \in \mathbf{S}^{n-1}$, $1_{\phi A}(u) = 1$ if and only if $u \in \phi A$; that is, if and only if $\phi^{-1}(u) \in A$. Hence $1_{\phi A} = 1_A \circ \phi^{-1}$. Combining these two facts, we have

$$\begin{aligned}
\nu(\phi A) &= \int_{\mathbf{S}^{n-1}} 1_{\phi A} f \, dS \\
&= \int_{\mathbf{S}^{n-1}} (1_{\phi A} f) \circ \phi \, dS \\
&= \int_{\mathbf{S}^{n-1}} (1_{\phi A} \circ \phi)(f \circ \phi) \, dS \\
&= \int_{\mathbf{S}^{n-1}} 1_A (f \circ \phi) \, dS
\end{aligned}$$

for all $A \subseteq \mathbf{S}^{n-1}$ and all $\phi \in SO(n)$. Since $f \circ \phi = f$ almost everywhere with respect to S,

$$\nu(\phi A) = \int_{\mathbf{S}^{n-1}} 1_A (f \circ \phi) \, dS = \int_{\mathbf{S}^{n-1}} 1_A f \, dS = \nu(A).$$

Therefore ν satisfies the hypotheses of Theorem 4.1. It follows that there exists $c \in \mathbf{R}$ such that $d\nu = c \, dS$. Hence, for all Lebesgue measurable subsets $A \subseteq \mathbf{S}^{n-1}$,

$$\int_{\mathbf{S}^{n-1}} 1_A \, d\nu = \int_{\mathbf{S}^{n-1}} 1_A f \, dS = c \int_{\mathbf{S}^{n-1}} 1_A \, dS. \tag{4.2}$$

Let $D = \{u \in \mathbf{S}^{n-1} : f(u) \geq c\}$ and let $E = \{u \in \mathbf{S}^{n-1} : f(u) \leq c\}$. Then

$$\int_{\mathbf{S}^{n-1}} |f - c| \, dS = \int_{\mathbf{S}^{n-1}} 1_D (f - c) \, dS + \int_{\mathbf{S}^{n-1}} 1_E (c - f) \, dS - \int_{\mathbf{S}^{n-1}} 1_{D \cap E} \cdot 0 \, dS.$$

It follows from Equation (4.2) that the first two terms on the right are zero. The third term is also clearly zero. We may conclude that $|f - c| = 0$ almost everywhere. In other words, $f(u) = c$ for almost all $u \in \mathbf{S}^{n-1}$. □

Theorem 3.5 may now be extended to full generality.

Theorem 4.5 *Let $i \in [n]$. Let Q be an $\mathcal{L}^n$-star. The star measure $\widetilde{V}_i(*, Q)$ is rotation invariant if and only if Q is a ball; that is, if and only if there exists $c \in \mathbf{R}$ such that $\widetilde{V}_i(*, Q) = c\widetilde{W}_i$.*

Proof: Let $\phi \in SO(n)$. As in the proof of Theorem 3.5, Theorem 2.18 implies that $\rho_Q \circ \phi = \rho_Q$ *almost everywhere.* Hence $\rho_Q^i \circ \phi = \rho_Q^i$ almost everywhere. Since ρ_Q^i is a non-negative $\mathcal{L}^1$ function, Corollary 4.4 implies the existence of $c \in \mathbf{R}$ such that $\rho_Q^i = c$ almost everywhere. Since ρ_Q is non-negative, $c \geq 0$, and so $\rho_Q = c^{1/i}$ almost everywhere. Hence $Q = c^{1/i}B$. □

In analogy to Theorem 4.1, the following results give us a unique probability measure τ on Grassmannians such that τ is invariant under the action of $SO(n)$. Results concerning this measure will be necessary for the generalization of the mean intersection formula (Equation (2.4)) to all of $\mathcal{S}^n$.

Theorem 4.6 *There exists a unique countably additive Borel probability measure τ_i on the Grassmannian* $\mathrm{Gr}(n, i)$ *such that τ_i is rotation invariant.*

Proof: Let ξ_0 be a fixed element in $\mathrm{Gr}(n, i)$. Define a map $\Psi : SO(n) \longrightarrow \mathrm{Gr}(n, i)$, by $\Psi(\phi) = \phi(\xi_0)$. We are mapping each rotation to the image of the i-dimensional subspace ξ_0 under that rotation. Let $G_0 = \mathrm{Stab}(\xi_0) = \{\phi \in SO(n) : \phi(\xi_0) = \xi_0\}$ be the stabilizer group of ξ_0 under this action. Since the action of $SO(n)$ on $\mathrm{Gr}(n, i)$ is continuous, it follows that Ψ induces a homeomorphism

$$\tilde{\Psi} : \frac{SO(n)}{G_0} \longrightarrow \mathrm{Gr}(n, i).$$

The result is a representation of the Grassmannian $\mathrm{Gr}(n, i)$ as a homogeneous space of the topological group $SO(n)$.

It now follows from the general version of Weil's Theorem [31, p. 138] that there exists a positive integral τ_i on $\mathrm{Gr}(n,i)$ that is invariant under the action of $SO(n)$. Since $SO(n)$ is compact, we may assume without loss of generality that τ_i is a probability measure; i.e. that $\tau_i(\mathrm{Gr}(n,i)) = 1$. If η is another positive integral on $\mathrm{Gr}(n,i)$ that satisfies these conditions, then there exists $k > 0$ such that $\eta = k\tau_i$.

From here the proof follows that same argument as is given in the proof of Theorem 4.1, with $\mathrm{Gr}(n,i)$ and τ_i substituted for $\mathbf{S}^{n-1}$ and S, respectively. □

The following result is of particular importance.

Theorem 4.7 *Let $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ be a non-negative $\mathcal{L}^1$ function. Define a function $I_f : \mathrm{Gr}(n,i) \longrightarrow \mathbf{R}$ by the equation*

$$I_f(\xi) = \int_{\xi \cap \mathbf{S}^{n-1}} f \, dS^{i-1}. \tag{4.3}$$

The function I_f is defined almost everywhere on $\mathrm{Gr}(n,i)$ with respect to the measure τ_i. Moreover, I_f is integrable with respect to τ_i, with

$$\int_{\mathrm{Gr}(n,i)} I_f \, d\tau_i = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} f \, dS.$$

Here σ_j denotes the surface area of the j-dimensional unit sphere $\mathbf{S}^j$.

Because a careful proof Theorem 4.7 turns out to be somewhat long, we shall break it down into the following series of propositions.

Proposition 4.8 *Let $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ be a non-negative continuous function. Define a function $I_f : \mathrm{Gr}(n,i) \longrightarrow \mathbf{R}$ as in Equation (4.3). Then I_f is a continuous function on $\mathrm{Gr}(n,i)$. Moreover,*

$$\int_{\mathrm{Gr}(n,i)} I_f \, d\tau_i = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} f \, dS.$$

Proof: If a function $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ is continuous, then the function I_f is continuous on $\mathrm{Gr}(n,i)$. Notice also that I_f is linear with respect to the variable f. Therefore, we may use Equation (4.3) to define a positive integral on $\mathbf{S}^{n-1}$. That is, for all continuous functions $f \in C(\mathbf{S}^{n-1})$, define

$$T(f) = \int_{\mathrm{Gr}(n,i)} I_f \, d\tau_i.$$

Since the measures S^{i-1} and τ_i are invariant under the action of $SO(n)$, the linear functional T is rotation invariant on $C(\mathbf{S}^{n-1})$. It follows from Theorem 4.2 that there exists a constant $k \geq 0$ such that

$$T(f) = k \int_{\mathbf{S}^{n-1}} f \, dS$$

for all $f \in C(\mathbf{S}^{n-1})$. We then compute:

$$\begin{aligned} T(1) &= \int_{\mathrm{Gr}(n,i)} I_1 \, d\tau_i \\ &= \int_{\mathrm{Gr}(n,i)} \int_{\mathbf{S}^{i-1}} 1 \, dS^{i-1} \, d\tau_i \\ &= \sigma_{i-1} \int_{\mathrm{Gr}(n,i)} 1 \, d\tau_i \\ &= \sigma_{i-1}. \end{aligned}$$

Hence $k = \sigma_{i-1}/\sigma_{n-1} > 0$, and

$$T(f) = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} f \, dS \tag{4.4}$$

for all $f \in C(\mathbf{S}^{n-1})$. □

A *closed neighborhood* in a topological subspace X of $\mathbf{R}^n$ is a subset of X that can be expressed as the intersection of X with a closed n-dimensional ball in $\mathbf{R}^n$.

Proposition 4.9 *Let A be a countable union of closed neighborhoods in $\mathbf{S}^{n-1}$. Then I_{1_A}*

is a well-defined integrable function on $\mathrm{Gr}(n,i)$. *Moreover,*

$$\int_{\mathrm{Gr}(n,i)} I_{1_A}\, d\tau_i = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} 1_A\, dS.$$

Proof:

To begin, note that $\xi \cap A$ is a countable union of closed neighborhoods in $\xi \cap \mathbf{S}^{n-1}$ for all $\xi \in \mathrm{Gr}(n,i)$. It follows that $1_{\xi\cap A}$ is a measurable and integrable function on $\xi \cap \mathbf{S}^{n-1}$, so that I_{1_A} may be properly defined by Equation (4.3).

Now suppose that A is equal to the intersection of $\mathbf{S}^{n-1}$ with a *finite* union of closed balls in $\mathbf{R}^n$. There exists a decreasing sequence of continuous functions $g_0 \geq g_1 \geq g_2 \geq \cdots \geq 1_A$ such that $g_j \longrightarrow 1_A$ pointwise. It follows from Equation (4.3) and from the monotone convergence theorem that $I_{g_j} \longrightarrow I_{1_A}$ pointwise and monotonically, and that

$$\begin{aligned}
\int_{\mathrm{Gr}(n,i)} I_{1_A}\, d\tau_i &= \int_{\mathrm{Gr}(n,i)} \lim_{j\to\infty} I_{g_j}\, d\tau_i \\
&= \lim_{j\to\infty} \int_{\mathrm{Gr}(n,i)} I_{g_j}\, d\tau_i \\
&= \lim_{j\to\infty} \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} g_j\, dS \\
&= \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} 1_A\, dS.
\end{aligned}$$

Next, suppose that A is a countable union of closed neighborhoods in $\mathbf{R}^n$. In other words, A has the form

$$A = \bigcup_{j\geq 1} A_j,$$

where each A_j is the intersection of $\mathbf{S}^{n-1}$ with a closed ball in $\mathbf{R}^n$. For all $m > 0$, let $C_j = \bigcup_{j=1}^{m} A_j$, and let $s_j = 1_{C_j}$. Note that I_{s_j} is well-defined and integrable, by the previous argument. Moreover, the sequence s_j converges monotonically to 1_A. It follows from the monotone convergence theorem, and from Equation (4.5), that $I_{s_j} \longrightarrow I_{1_A}$

pointwise, and that

$$\begin{aligned}\int_{\mathrm{Gr}(n,i)} I_{1_A}\, d\tau_i &= \int_{\mathrm{Gr}(n,i)} \lim_{j\to\infty} I_{s_j}\, d\tau_i \\ &= \lim_{j\to\infty} \int_{\mathrm{Gr}(n,i)} I_{s_j}\, d\tau_i \\ &= \lim_{j\to\infty} \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} s_j\, dS. \\ &= \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} 1_A\, dS.\end{aligned}$$

□

Proposition 4.10 *Let $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ be measurable, and suppose that $f = 0$ almost everywhere on $\mathbf{S}^{n-1}$. Then I_f is a well-defined integrable function on $\mathrm{Gr}(n,i)$. Moreover, $I_f = 0$ almost everywhere on $\mathrm{Gr}(n,i)$, so that*

$$\int_{\mathrm{Gr}(n,i)} I_f\, d\tau_i = 0.$$

Proof:

Let $E = \{x \in \mathbf{S}^{n-1} : f(x) \neq 0\}$. Then $S(E) = 0$, and there exists a descending sequence of subsets $C_1 \supseteq C_2 \supseteq \cdots$ of $\mathbf{S}^{n-1}$ such that, for all $m > 0$:

- $E \subseteq C_m$.
- C_m is a countable union of closed neighborhoods in $\mathbf{S}^{n-1}$.
- $S(C_m) \leq \frac{1}{m}$.

Let $C = \bigcap_{m>0} C_m$. Then $E \subseteq C$, and $S(C) = 0$.

Note that $1_{C_m} \longrightarrow 1_C$ monotonically as $m \to \infty$. It then follows from Proposition 4.9 and from the monotone convergence theorem that I_{1_C} is defined, that $I_{1_{C_m}} \longrightarrow I_{1_C}$

monotonically, and that

$$\int_{\mathrm{Gr}(n,i)} I_{1_C}\, d\xi = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} 1_C \; dS = 0. \tag{4.5}$$

Since $I_{1_C} \geq 0$, Equation (4.5) implies that $I_{1_C}(\xi) = 0$ for almost all $\xi \in \mathrm{Gr}(n,i)$. Equation (4.3) then implies that, for almost all $\xi \in \mathrm{Gr}(n,i)$, the set $\xi \cap C$ has measure zero with respect to the measure S^{i-1} on $\mathbf{S}^{i-1} \cong \xi \cap \mathbf{S}^{n-1}$. Since $E \subseteq C$, it follows that $S^{i-1}(\xi \cap E) = 0$ in $\xi \cap \mathbf{S}^{n-1}$, for almost all $\xi \in \mathrm{Gr}(n,i)$. In other words, for almost all ξ, the restriction $f|_{\xi \cap \mathbf{S}^{n-1}} = 0$ almost everywhere with respect to the measure S^{i-1}. We may therefore use Equation (4.3) to define $I_f = 0$ for almost all $\xi \in \mathrm{Gr}(n,i)$, so that

$$\int_{\mathrm{Gr}(n,i)} I_f \; d\tau_i = 0.$$

□

Proposition 4.11 *Let $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ be a non-negative integrable function, and suppose that there exists $\alpha > 0$ such that $f \leq \alpha$ almost everywhere on $\mathbf{S}^{n-1}$. Define a function $I_f : \mathrm{Gr}(n,i) \longrightarrow \mathbf{R}$ by*

$$I_f(\xi) = \int_{\xi \cap \mathbf{S}^{n-1}} f \; dS^{i-1}. \tag{4.6}$$

Then I_f is a well-defined measurable function on $\mathrm{Gr}(n,i)$ and is bounded almost everywhere with respect to τ_i. Moreover,

$$\int_{\mathrm{Gr}(n,i)} I_f \; d\tau_i = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} f \; dS.$$

Proof: Recall from Lusin's Theorem (see [35, p. 55]) that there exists a sequence of continuous functions $g_j : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, with $0 \leq g_j \leq 2\alpha$, such that $g_j \longrightarrow f$ almost everywhere on $\mathbf{S}^{n-1}$. Let D be the set of all $x \in \mathbf{S}^{n-1}$ such that $g_j(x) \longrightarrow f(x)$ as $j \rightarrow \infty$, and let $E = \mathbf{S}^{n-1} - D$.

Since $S(E) = 0$, we have

$$0 \leq 1_E f(x) \leq 1_E \alpha = 0,$$

almost everywhere on $\mathbb{S}^{n-1}$. It follows from Proposition 4.10 that $I_{1_E f} = 0$ almost everywhere on $\mathrm{Gr}(n,i)$.

Meanwhile, for all $j > 0$, the function $g_j - 1_D g_j = 0$ almost everywhere on $\mathbb{S}^{n-1}$. Once again, Proposition 4.10 implies that $I_{g_j - 1_D g_j} = 0$ almost everywhere on $\mathrm{Gr}(n,i)$, so that $I_{1_D g_j}(\xi)$ is defined, with

$$I_{1_D g_j}(\xi) = I_{g_j}(\xi)$$

for almost all $\xi \in \mathrm{Gr}(n,i)$. Since the function $1_D f$ is the pointwise limit of the functions $1_D g_j$ (which are, in turn, dominated uniformly by the constant function 2α), the Lebesgue dominated convergence theorem enables us to define the function $1_D f$ by means of Equation (4.6), so that $I_{1_D f}$ is defined almost everywhere on $\mathrm{Gr}(n,i)$. That is,

$$1_D f = \lim_{j\to\infty} 1_D g_j.$$

Equation (4.6) and the Lebesgue dominated convergence theorem also imply that

$$\begin{aligned} \int_{Gr(n,i)} I_{1_D f}\, d\tau_i &= \int_{Gr(n,i)} \lim_{j\to\infty} I_{1_D g_j}\, d\tau_i \\ &= \lim_{j\to\infty} \int_{Gr(n,i)} I_{1_D g_j}\, d\tau_i \\ &= \lim_{j\to\infty} \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} 1_D g_j\, dS \\ &= \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} 1_D f\, dS. \end{aligned}$$

Since $f = 1_D f + 1_E f$, we may define

$$I_f = I_{1_D f} + I_{1_E f}.$$

Proposition 4.10 then implies that I_f is well-defined almost everywhere on $\mathrm{Gr}(n,i)$. Hence,

$$\begin{aligned}\int_{\mathrm{Gr}(n,i)} I_f \, d\tau_i &= \int_{\mathrm{Gr}(n,i)} I_{1_D f} + I_{1_E f} \, d\tau_i \\ &= \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} 1_D f \, dS + \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} 1_E f \, dS \\ &= \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} f \, dS\end{aligned}$$

□

There remains one small step to the proof of Theorem 4.7

Proof:

If the function f is bounded above, Proposition 4.11 applies. Suppose then that $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ is an arbitrary non-negative $\mathcal{L}^1$ function. There exists an increasing sequence of simple measurable functions $s_1 \leq s_2 \leq \cdots$ such that $s_j \longrightarrow f$ pointwise. For all $\xi \in \mathrm{Gr}(n,i)$, the restrictions $s_j|_{\xi \cap \mathbf{S}^{n-1}}$ converge to $f|_{\xi \cap \mathbf{S}^{n-1}}$ pointwise. For each j, the function $I_{s_j}(\xi)$ is defined almost everywhere on $\mathrm{Gr}(n,i)$, by Proposition 4.11. It follows from the monotone convergence theorem that the integral

$$I_f(\xi) = \int_{\xi \cap \mathbf{S}^{n-1}} f \, dS^{i-1} = \lim_{j \to \infty} \int_{\xi \cap \mathbf{S}^{n-1}} s_j \, dS^{i-1}$$

is defined for almost all $\xi \in \mathrm{Gr}(n,i)$, with the proviso that $I_f(\xi)$ may take the value $+\infty$. Recall, however, that

$$\int_{\mathrm{Gr}(n,i)} I_{s_j} \, d\tau_i = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} s_j \, dS$$

for all j. It follows from the monotone convergence theorem that

$$\int_{\mathrm{Gr}(n,i)} I_f \, d\tau_i = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} f \, dS$$

as well. Since the integral on the right is finite, so is the integral on the left. It follows that I_f takes finite values almost everywhere on $\mathrm{Gr}(n,i)$. In particular, the function I_f

is integrable on $\mathrm{Gr}(n, i)$. □

Chapter 5

Homogeneous Measures on $\mathcal{L}^n$-stars

We now present a classification theorem for star measures. Let $i > 0$. Recall that a star measure μ is said to be *homogeneous of degree* i if, for all $K \in \mathcal{S}^n$ and all $c > 0$, $\mu(cK) = c^i \mu(K)$. Let μ be a star measure that is continuous and homogeneous of degree i, where $i \in [n]$. Recall that two $\mathcal{L}^n$-stars are equal if their radial functions are equal almost everywhere. If an $\mathcal{L}^n$-star K has Lebesgue measure zero, then $\rho_K = 0$ almost everywhere, so that K is equal to the singleton $\{0\}$. Hence $\mu(K) = 0$ by the homogeneity of μ.

It will only be necessary to assume that the continuous star measure μ is *finitely* additive. We will use μ to define a measure $\tilde{\mu}$ on the sphere $\mathbf{S}^{n-1}$ that is absolutely continuous (see [35, p. 120]) with respect to the Lebesgue measure on $\mathbf{S}^{n-1}$. It will follow from the star continuity of μ that the induced measure $\tilde{\mu}$ on $\mathbf{S}^{n-1}$ is *countably* additive. The Lebesgue-Radon-Nikodym theorem will then lead us to a classification for the star measure μ.

Lemma 5.1 *Let $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ be a star measure that is continuous and homogeneous of degree k, where $k > 0$. Then μ induces a countably additive measure $\tilde{\mu}$ on $\mathbf{S}^{n-1}$ that is absolutely continuous with respect to spherical Lebesgue measure. Moreover, for all $A \in \mathcal{S}^n$,*

$$\mu(A) = \int_{\mathbf{S}^{n-1}} \rho_A^k \, d\tilde{\mu}.$$

Proof: Let $A \subseteq \mathbf{S}^{n-1}$ be a Lebesgue measurable set, and let $so(A)$ be the spherical cone with base A and apex 0, as defined in Chapter 2. Since A is measurable, the indicator function $1_A : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ is a measurable function. Therefore, $so(A)$ is an $\mathcal{L}^n$-star, since $\rho_{so(A)} = 1_A$. Note also that $\rho_{so(A)} = 0$ almost everywhere if and only if A has spherical Lebesgue measure zero. IN this case we have $A = \{0\}$ in $\mathcal{S}^n$.

Define a measure $\tilde{\mu}$ on $\mathbf{S}^{n-1}$ as follows. Given $A \subseteq \mathbf{S}^{n-1}$, set

$$\tilde{\mu}(A) = \mu(so(A)).$$

Recall from Lemma 2.9 that $so(A_1 \cup A_2) = so(A_1) \cup so(A_2)$ and $so(A_1 \cap A_2) = so(A_1) \cap so(A_2)$. It follows that $\tilde{\mu}$ is a real-valued measure on $\mathbf{S}^{n-1}$. Recall also that A has Lebesgue measure zero in $\mathbf{S}^{n-1}$ if and only if $so(A)$ has measure zero in $\mathbf{R}^n$. It follows from the homogeneity of μ that if A has Lebesgue measure zero in $\mathbf{S}^{n-1}$, then $\tilde{\mu}(A) = \mu(so(A)) = 0$. Thus $\tilde{\mu}$ is absolutely continuous with respect to Lebesgue measure.

The countable additivity of $\tilde{\mu}$ remains to be shown. Let $A_1, A_2, A_3, \ldots$ be a sequence of measurable subsets of $\mathbf{S}^{n-1}$ that are mutually disjoint. For each i, $\rho_{so(A_i)} = 1_{A_i}$. Because the sets A_i are disjoint,

$$1_{A_1 \cup A_2 \cup \cdots} = \sum_{i=1}^{\infty} 1_{A_i} = \lim_{m \to \infty} \sum_{i=1}^{m} 1_{A_i} = \lim_{m \to \infty} 1_{A_1 \cup \cdots \cup A_m},$$

where this limit is taken pointwise on $\mathbf{S}^{n-1}$. In other words,

$$\rho_{so(A_1 \cup A_2 \cup \cdots)} = \lim_{m \to \infty} \rho_{so(A_1 \cup \cdots \cup A_m)},$$

a pointwise limit of radial functions. Note also that the functions $\rho_{so(A_1 \cup \cdots \cup A_m)}$ form a monotonically increasing sequence. It follows from the monotone convergence theorem that

$$\|\rho_{so(A_1 \cup A_2 \cup \cdots)} - \rho_{so(A_1 \cup \cdots \cup A_m)}\|_n \longrightarrow 0$$

as $m \to \infty$. Since μ is star-continuous,

$$\tilde{\mu}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} so(A_i)\right) = \lim_{m\to\infty} \mu\left(\bigcup_{i=1}^{m} so(A_i)\right) =$$

$$= \lim_{m\to\infty} \tilde{\mu}\left(\bigcup_{i=1}^{m} A_i\right) = \lim_{m\to\infty} \sum_{i=1}^{m} \tilde{\mu}(A_i) = \sum_{i=1}^{\infty} \tilde{\mu}(A_i),$$

where the first and third equalities follow from Lemma 2.9.

Now let $D \subseteq \alpha\mathbf{S}^{n-1}$, where $\alpha\mathbf{S}^{n-1}$ is the sphere about the origin of radius α. Since μ is k-homogeneous, $\mu(so(D)) = \alpha^k \mu(\frac{1}{\alpha} so(D))$. Since $\frac{1}{\alpha} so(D)$ is a spherical cone with base $\frac{1}{\alpha} D \subseteq \mathbf{S}^{n-1}$,

$$\begin{aligned}
\mu(so(D)) = \alpha^k \mu\left(\frac{1}{\alpha} so(D)\right) &= \alpha^k \mu(so(\frac{1}{\alpha} D)) \\
&= \alpha^k \tilde{\mu}(\frac{1}{\alpha} D) \\
&= \alpha^k \int_{\mathbf{S}^{n-1}} 1_{\frac{1}{\alpha} D} \, d\tilde{\mu} \\
&= \alpha^k \int_{\mathbf{S}^{n-1}} \rho^k_{so(\frac{1}{\alpha} D)} \, d\tilde{\mu} \\
&= \int_{\mathbf{S}^{n-1}} (\alpha \rho_{so(\frac{1}{\alpha} D)})^k \, d\tilde{\mu} \\
&= \int_{\mathbf{S}^{n-1}} \rho^k_{so(D)} \, d\tilde{\mu}.
\end{aligned}$$

Let P be a polycone. By Proposition 2.12, $P = \bigcup_{j=1}^{m} C_j$, a disjoint union of spherical cones C_j. It follows that

$$\mu(P) = \sum_{j=1}^{m} \mu(C_j) = \sum_{j=1}^{m} \int_{\mathbf{S}^{n-1}} \rho^k_{C_j} \, d\tilde{\mu} = \int_{\mathbf{S}^{n-1}} \rho^k_P \, d\tilde{\mu}.$$

Finally, let $K \in \mathcal{S}^n$. By Proposition 2.13, there exists an increasing sequence P_j of polycones such that $P_j \longrightarrow K$, as $j \to \infty$. Since μ is star-continuous,

$$\mu(K) = \lim_{j\to\infty} \mu(P_j) = \lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} \rho^k_{P_j} \, d\tilde{\mu} = \int_{\mathbf{S}^{n-1}} \rho^k_K \, d\tilde{\mu},$$

where the final equality follows from the monotone convergence theorem. □

The situation becomes particularly interesting when the homogeneity degree of μ is an integer $i \in [n]$. We will require the following preliminary result from functional analysis (see also [35, p. 127]).

Lemma 5.2 *Let $p, q > 1$ such that $(1/p) + (1/q) = 1$. Let $f : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$. Suppose that for all $g \in \mathcal{L}^p(S)$*

$$\left| \int_{\mathbf{S}^{n-1}} gf \, dS \right| < \infty.$$

Then $f \in \mathcal{L}^q(S)$, and the linear functional defined by

$$g \longmapsto \int_{\mathbf{S}^{n-1}} gf \, dS$$

is a continuous map from $\mathcal{L}^p(S)$ into $\mathbf{R}$.

Proof: To begin, let us suppose that $f \geq 0$. Let $0 \leq s_1 \leq s_2 \leq \cdots$ be a sequence of simple measurable functions such that $f = \lim_{j \to \infty}\{s_j\}$. For all $j > 0$ and all $g \in \mathcal{L}^p(S)$, define

$$T_j(g) = \int_{\mathbf{S}^{n-1}} gs_j \, dS$$

For all $g \in \mathcal{L}^p(S)$,

$$|T_j(g)| \leq \int_{\mathbf{S}^{n-1}} |gs_j| \, dS = \int_{\mathbf{S}^{n-1}} |g|s_j \, dS \leq \int_{\mathbf{S}^{n-1}} |g|f \, dS = \left| \int_{\mathbf{S}^{n-1}} |g|f \, dS \right| < \infty, \quad (5.1)$$

where the final inequality follows by hypothesis.

For all $j > 0$, let $M_j = \|s_j\|_q$. Since $\mathbf{S}^{n-1}$ has finite spherical Lebesgue measure, it follows that $M_j < \infty$. The Hölder inequality then implies that

$$\sup_{\|g\|_p \leq 1} |T_j(g)| \leq \sup_{\|g\|_p \leq 1} \int_{\mathbf{S}^{n-1}} |gs_j| \, dS = \sup_{\|g\|_p \leq 1} \|gs_j\|_1 \leq \sup_{\|g\|_p \leq 1} \|g\|_p \, \|s_j\|_q \leq M_j < \infty,$$

for all $g \in \mathcal{L}^p(S)$ (see [35, p. 63]). In other words, the maps T_j are bounded linear functionals.

For all $g \in \mathcal{L}^p(S)$ define

$$T(g) = \int_{\mathbf{S}^{n-1}} gf \, dS.$$

Write $g = g^+ - g^-$, where $g^+(u) = \max\{g(u), 0\}$ and $g^-(u) = \max\{-g(u), 0\}$. Since T_j is linear,

$$\begin{aligned}
\lim_{j\to\infty} T_j(g) &= \lim_{j\to\infty} T_j(g^+ - g^-) \\
&= \lim_{j\to\infty} (T_j(g^+) - T_j(g^-)) \\
&= \lim_{j\to\infty} T_j(g^+) - \lim_{j\to\infty} T_j(g^-).
\end{aligned}$$

Since $g^+, g^- \geq 0$, it follows from the monotone convergence theorem that

$$\lim_{j\to\infty} T_j(g) = T(g^+) - T(g^-) = T(g). \tag{5.2}$$

By Equation (5.1), $|T_j(g)| \leq T(|g|) < \infty$ for all $g \in \mathcal{L}^p(S)$ and all $j > 0$. Hence,

$$\sup_j |T_j(g)| < \infty$$

for all $g \in \mathcal{L}^p(S)$. It follows from the Principle of Uniform Boundedness [35, p. 98] that there exists $M > 0$ such that $|T_j(g)| \leq M\|g\|_p$ for all $g \in \mathcal{L}^p(S)$. Equation (5.2) then implies that $|T(g)| \leq M\|g\|_p$ for all $g \in \mathcal{L}^p(S)$. In other words, T is a continuous linear functional on $\mathcal{L}^p(S)$.

For all $p > 1$, denote by $(\mathcal{L}^p(S))^*$ the space of real-valued linear functionals on $\mathcal{L}^p(S)$ that are continuous. Recall from [35, p. 126] that, if $p, q > 1$ such that $(1/p) + (1/q) = 1$, then the map $\Phi : \mathcal{L}^q \longrightarrow (\mathcal{L}^p)^*$ given by

$$(\Phi(h))(g) = \int_{\mathbf{S}^{n-1}} gh \, dS$$

is a linear isomorphism. Since the linear functional T is continuous on $\mathcal{L}^p(S)$, it follows that $T = \Phi(f_T)$ for some $f_T \in \mathcal{L}^q(S)$. This implies that $f = f_T$ almost everywhere on

$\mathbf{S}^{n-1}$, so that $f \in \mathcal{L}^q(S)$.

For the general case, let $E^+ = \{u \in \mathbf{S}^{n-1} : f(u) \geq 0\}$, and let $E^- = \{u \in \mathbf{S}^{n-1} : f(u) \leq 0\}$. Let $f^+ = 1_{E^+} f$, and let $f^- = -1_{E^-} f$, so that $f = f^+ - f^-$. For all $g \in \mathcal{L}^p(S)$, define

$$T^+(g) = \int_{\mathbf{S}^{n-1}} g f^+ \, dS,$$

and define T^- similarly. For all $g \in \mathcal{L}^p(S)$,

$$|T^+(g)| = \left| \int_{\mathbf{S}^{n-1}} g f^+ \, dS \right| = \left| \int_{\mathbf{S}^{n-1}} (g 1_{E^+}) f \, dS \right| < \infty,$$

and similarly $|T^-(g)| < \infty$. Since $f^+, f^- \geq 0$, it follows from the previous argument that T^+ and T^- are continuous, and that f^+ and f^- are $\mathcal{L}^q$ functions. Therefore, $T = T^+ - T^-$ is continuous, and $f \in \mathcal{L}^q(S)$. □

Theorem 5.3 *Let $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ be a star measure that is continuous and homogeneous of degree i, where $i \in [n]$. Then there exist $\mathcal{L}^n$-stars Q_1 and Q_2 such that $\mu(K) = \widetilde{V}_{n-i}(K, Q_1) - \widetilde{V}_{n-i}(K, Q_2)$ for all $K \in \mathcal{S}^n$.*

Proof: By Lemma 5.1, there exists a measure $\widetilde{\mu}$ on $\mathbf{S}^{n-1}$ that is absolutely continuous with respect to S, such that for all $K \in \mathcal{S}^n$,

$$\mu(K) = \int_{\mathbf{S}^{n-1}} \rho_K^i \, d\widetilde{\mu}.$$

It then follows from the Lebesgue-Radon-Nikodym theorem [35, p. 121] that $d\widetilde{\mu} = f_\mu dS$, where $f_\mu : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ is an $\mathcal{L}^1$ function on $\mathbf{S}^{n-1}$. Hence,

$$\mu(K) = \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu \, dS \tag{5.3}$$

for all $K \in \mathcal{S}^n$.

Since $|\mu(K)| < \infty$ for all $K \in \mathcal{S}^n$, the integral

$$\left| \int_{\mathbf{S}^{n-1}} g f_\mu \, dS \right| < \infty,$$

for all $\mathcal{L}^{\frac{n}{i}}$ functions $g : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$. It follows from Lemma 5.2 that f_μ is an $\mathcal{L}^{\frac{n}{n-i}}$ function on $\mathbf{S}^{n-1}$. Conversely, Equation (5.3) defines a continuous star measure for each function $f_\mu \in \mathcal{L}^{\frac{n}{n-i}}$.

Let

$$f_\mu^+(u) = \max\{f_\mu(u), 0\},$$

$$f_\mu^-(u) = -\min\{f_\mu(u), 0\},$$

for all $u \in \mathbf{S}^{n-1}$. Then $f_\mu = f_\mu^+ - f_\mu^-$, where f_μ^+ and f_μ^- are non-negative measurable functions. Since $\|f_\mu^+\|_{\frac{n}{n-i}} \leq \|f_\mu\|_{\frac{n}{n-i}}$ and $\|f_\mu^-\|_{\frac{n}{n-i}} \leq \|f_\mu\|_{\frac{n}{n-i}}$, both f_μ^+ and f_μ^- are $\mathcal{L}^{\frac{n}{n-i}}$ functions.

Let Q_1 and Q_2 be the star-shaped sets satisfying the conditions $\rho_{Q_1}^{n-i} = n f_\mu^+$ and $\rho_{Q_2}^{n-i} = n f_\mu^-$. Then $Q_1, Q_2 \in \mathcal{S}^n$. For all $K \in \mathcal{S}^n$,

$$\begin{aligned}
\mu(K) &= \int_{\mathbf{S}^{n-1}} \rho_K^i \, d\tilde{\mu} \\
&= \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu \, dS \\
&= \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu^+ \, dS - \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu^- \, dS \\
&= \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i \rho_{Q_1}^{n-i} \, dS - \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i \rho_{Q_2}^{n-i} \, dS \\
&= \tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2),
\end{aligned}$$

as desired. □

Corollary 5.4 *Let $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ be a star measure that is continuous and homogeneous of degree i, where $i \in [n]$. If μ is non-negative, then there exists a unique $\mathcal{L}^n$-star Q such that $\mu(K) = \tilde{V}_{n-i}(K, Q)$ for all $K \in \mathcal{S}^n$.*

Proof: In this case the function f_μ in the previous proof will be non-negative almost everywhere (and so we may choose it to be non-negative). Hence $f_\mu = f_\mu^+$, and $f_\mu^- = 0$. The Q_2 term from the previous result may be ignored, and we are left with $Q = Q_1$. The uniqueness of Q follows from the uniqueness of f_μ in the Lebesgue-Radon-Nikodym Theorem. □

The decomposition of a homogeneous star measure μ given in Theorem 5.3 is not unique, but it does satisfy the following minimality property.

Corollary 5.5 *Let $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ be a star measure that is continuous and homogeneous of degree i, where $i \in [n]$. Let $\widetilde{V}_{n-i}(*, Q_1) - \widetilde{V}_{n-i}(*, Q_2)$ be the decomposition of μ given in Theorem 5.3. If there exists another pair of star-shaped sets M_1 and M_2 such that $\mu = \widetilde{V}_{n-i}(*, M_1) - \widetilde{V}_{n-i}(*, M_2)$, then $Q_1 \subseteq M_1$ and $Q_2 \subseteq M_2$.*

Proof: For all $K \in \mathbf{S}^{n-1}$,

$$\mu(K) = \widetilde{V}_{n-i}(K, M_1) - \widetilde{V}_{n-i}(K, M_2) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i (\rho_{M_1}^{n-i} - \rho_{M_2}^{n-i}) \, dS.$$

The uniqueness of f_μ then implies that $f_\mu = (1/n)(\rho_{M_1}^{n-i} - \rho_{M_2}^{n-i})$. Let $u \in \mathbf{S}^{n-1}$. If $f_\mu(u) \geq 0$, then $f_\mu^+(u) = f_\mu(u) \leq (1/n)\rho_{M_1}^{n-i}(u)$. If $f_\mu(u) < 0$ then $f_\mu^+(u) = 0 \leq (1/n)\rho_{M_1}^{n-i}(u)$. Hence $(1/n)\rho_{Q_1}^{n-i}(u) = f_\mu^+(u) \leq (1/n)\rho_{M_1}^{n-i}(u)$ for all $u \in \mathbf{S}^{n-1}$ (except possibly on a set of spherical Lebesgue measure zero). It follows that $Q_1 \subseteq M_1$. A similar argument shows that $Q_2 \subseteq M_2$. □

There remains to discuss the case where the star measure μ is invariant under rotations of $\mathcal{L}^n$-stars. Here the result is particularly satisfying, for it mirrors Hadwiger's classification for measures on convex bodies (see Theorem 1.7).

Theorem 5.6 *Let $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ be a continuous rotation invariant star measure that is homogeneous of degree i, where $i \in [n]$. Then there exists $\alpha \in \mathbf{R}$ such that $\mu(K) = \alpha \widetilde{W}_{n-i}(K)$.*

Proof of Theorem 5.6: Given $\phi \in SO(n)$, the measure on $\mathbf{S}^{n-1}$ given by $d\widetilde{\mu}_\phi = f_\mu \circ \phi \, dS$ is equal to the measure $d\widetilde{\mu} = f_\mu \, dS$. By the uniqueness property of f_μ in the Lebesgue-

Radon-Nikodym theorem, it follows that $f_\mu \circ \phi = f_\mu$ almost everywhere. By Corollary 4.4, there exists $\alpha \in \mathbf{R}$ such that $f_\mu = \alpha$ almost everywhere. Thus for all $K \in \mathcal{S}^n$,

$$\mu(K) = \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu \, dS = \frac{\alpha}{n}\int_{\mathbf{S}^{n-1}} \rho_K^i \, dS = \alpha \widetilde{W}_{n-i}(K),$$

as desired. □

Examination of the arguments leading up to Theorem 5.6 reveals that this classification applies even if the homogeneity degree i is not an integer, a fact which motivates the following definition.

Definition 5.7 *Let $i > 0$. Let Q be a fixed $\mathcal{L}^n$-star. For all $K \in \mathcal{S}^n$, define $\widetilde{V}_{n-i}(K,Q)$ by the following expression:*

$$\widetilde{V}_{n-i}(K,Q) = \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_K^i \rho_Q^{n-i} \, dS.$$

Also, define $\widetilde{W}_{n-i}(K) = \widetilde{V}_{n-i}(K,B)$.

If $i > n$, it should be noted that $\rho_Q > 0$ almost everywhere. We would like to be certain that the expression $\widetilde{V}_{n-i}(*,Q)$ is still a star measure.

Lemma 5.8 *Let Q be a fixed $\mathcal{L}^n$-star, satisfying the conditions of Definition 5.7. Then $\widetilde{V}_{n-i}(*,Q)$ defines a star measure.*

Proof: The argument is almost identical to that given in the proof of Proposition 3.2. For all $K, L \in \mathcal{S}^n$ and all $u \in \mathbf{S}^{n-1}$, we have $\rho_{K\cup L}(u) = \max\{\rho_K(u), \rho_L(u)\}$, and $\rho_{K\cap L}(u) = \min\{\rho_K(u), \rho_L(u)\}$. Hence,

$$\begin{aligned}\rho_{K\cup L}^i(u) + \rho_{K\cap L}^i(u) &= (\max\{\rho_K(u), \rho_L(u)\})^i + (\min\{\rho_K(u), \rho_L(u)\})^i \\ &= \rho_K^i(u) + \rho_L^i(u).\end{aligned}$$

It follows that

$$\begin{aligned}\widetilde{V}_{n-i}(K \cup L, Q) + \widetilde{V}_{n-i}(K \cap L, Q) &= \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_{K\cup L}^{i}\rho_Q^{n-i}\, dS + \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_{K\cap L}^{i}\rho_Q^{n-i}\, dS \\ &= \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_{K}^{i}\rho_Q^{n-i}\, dS + \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_{L}^{i}\rho_Q^{n-i}\, dS \\ &= \widetilde{V}_{n-i}(K, Q) + \widetilde{V}_{n-i}(L, Q).\end{aligned}$$

□

Having at no point used the integer properties of the homogeneity degree i in the proofs of Theorem 5.3 and Theorem 5.6, we may immediately conclude the following.

Theorem 5.9 (Classification of Homogeneous Star Measures) *Let $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ be a star measure that is continuous and homogeneous of degree i, where $i > 0$.*

(1) There exists a unique $\mathcal{L}^1$ function $f_\mu : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ such that for all $K \in \mathcal{S}^n$,

$$\mu(K) = \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu \, dS.$$

Moreover, if $0 < i < n$, then there exist fixed $\mathcal{L}^n$-stars Q_1 and Q_2 such that

$$\mu(K) = \widetilde{V}_{n-i}(K, Q_1) - \widetilde{V}_{n-i}(K, Q_2)$$

for all $K \in \mathcal{S}^n$. The sets Q_1 and Q_2 satisfy the minimality property for μ defined in Corollary 5.5.

(2) If the star measure μ is also rotation invariant, then there exists $\alpha \in \mathbf{R}$ such that

$$\mu(K) = \alpha \int_{\mathbf{S}^{n-1}} \rho_K^i \, dS$$

for all $K \in \mathcal{S}^n$. If $0 \leq i \leq n$, then

$$\mu(K) = \alpha \widetilde{W}_{n-i}(K).$$

□

From Theorem 5.9 we may deduce the following star analogue to Theorems 1.8 and 1.9.

Theorem 5.10 *Let $\mu : \mathcal{S}^n \longrightarrow \mathbf{R}$ be a set function. Then μ is a continuous star measure, homogeneous of degree $i \in (0, n)$, if and only if there exist sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$ in $\mathcal{S}_c^n$ such that,*

$$\mu(K) = \lim_{j \to \infty}(\widetilde{V}_{n-i}(K, L_j) - \widetilde{V}_{n-i}(K, M_j))$$

Proof: Let μ be a continuous star measure that is homogeneous of degree i, where $0 < i < n$. By Theorem 5.9, there exist fixed $\mathcal{L}^n$-stars Q_1 and Q_2 such that

$$\mu(K) = \widetilde{V}_{n-i}(K, Q_1) - \widetilde{V}_{n-i}(K, Q_2)$$

for all $K \in \mathcal{S}^n$. Recall that the set $C(\mathbf{S}^{n-1})$ is dense in $\mathcal{L}^n(S)$ (see [35, p. 69]). In other words, there exist sequences of non-negative continuous functions $f_j, g_j : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ such that

$$\lim_{j \to \infty} \|f_j - \rho_{Q_1}\|_n = 0,$$

and

$$\lim_{j \to \infty} \|g_j - \rho_{Q_2}\|_n = 0.$$

For all $j > 0$, let L_j be the star body whose radial function satisfies the equation $\rho_{L_j} = f_j$, and let M_j be the star body whose radial function satisfies the equation $\rho_{M_j} = g_j$. Then $L_j \longrightarrow Q_1$ and $M_j \longrightarrow Q_2$ as $j \longrightarrow \infty$.

Let $K \in \mathcal{S}^n$. Recall from Proposition 3.1 that $\widetilde{V}_{n-i}(K, Q)$ is continuous in the variable Q. Hence,

$$\lim_{j \to \infty}(\widetilde{V}_{n-i}(K, L_j) - \widetilde{V}_{n-i}(K, M_j)) = (\widetilde{V}_{n-i}(K, Q_1) - \widetilde{V}_{n-i}(K, Q_2)) = \mu(K).$$

Conversely, suppose that there exist sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$ such that the following limit exists for all $K \in \mathcal{S}^n$:

$$\mu(K) = \lim_{j\to\infty} \widetilde{V}_{n-i}(K, L_j) - \widetilde{V}_{n-i}(K, M_j). \tag{5.4}$$

For all $j > 0$, let $f_j = \rho_{L_j}^{n-i} - \rho_{M_j}^{n-i}$. Note that f_j is continuous, and that

$$\widetilde{V}_{n-i}(K, L_j) - \widetilde{V}_{n-i}(K, M_j) = \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_K^i f_j \; dS, \tag{5.5}$$

for all $K \in \mathcal{S}^n$.

For all $j > 0$, let $\mu_j(K) = \widetilde{V}_i(K, L_j) - \widetilde{V}_{n-i}(K, M_j)$. It follows from Equation (5.5) and from the Hölder inequality that each μ_j determines a bounded (continuous) linear operator T_j on the space of all $\mathcal{L}^{\frac{n}{i}}$ functions on $\mathbf{S}^{n-1}$, given by

$$T_j(g) = \frac{1}{n}\int_{\mathbf{S}^{n-1}} g f_j \; dS,$$

for all $g \in \mathcal{L}^{\frac{n}{i}}(\mathbf{S}^{n-1})$. The limit μ also determines a linear operator T on $\mathcal{L}^{\frac{n}{i}}(\mathbf{S}^{n-1})$, given by

$$T(g) = \lim_{j\to\infty} T_j(g) = \lim_{j\to\infty} \frac{1}{n}\int_{\mathbf{S}^{n-1}} g f_j \; dS. \tag{5.6}$$

To show that the limit $T(g)$ exists, write $g = g^+ - g^-$ in the usual way, where $g^+, g^- \geq 0$. Let $\rho_P^i = g^+$ and $\rho_Q^i = g^-$. We then have

$$\begin{aligned} T(g) &= \lim_{j\to\infty} T_j(g) \\ &= \lim_{j\to\infty} (T_j(g^+) - T_j(g^-)) \\ &= \lim_{j\to\infty} \left(\frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_P^i f_j \; dS - \frac{1}{n}\int_{\mathbf{S}^{n-1}} \rho_Q^i f_j \; dS\right) \\ &= \lim_{j\to\infty} (\mu_j(P) - \mu_j(Q)) \\ &= \mu(P) - \mu(Q). \end{aligned}$$

Let $g \in \mathcal{L}^{\frac{n}{i}}(S)$, and let $\epsilon > 0$. It follows from Equation (5.6) that there exists $N > 0$ such that

$$|T_j(g)| - |T(g)| \leq |T_j(g) - T(g)| \leq \epsilon,$$

for all $j \geq N$. In other words,

$$|T_j(g)| \leq \epsilon + |T(g)|,$$

for all $j \geq N$. The Principle of Uniform Boundedness [35, p. 98] then implies that there exists $c > 0$ such that $|T_j(g)| \leq c\|g\|_{\frac{n}{i}}$ for all $g \in \mathcal{L}^{\frac{n}{i}}(S)$. Equation (5.6) then implies that $|T(g)| \leq c\|g\|_{\frac{n}{i}}$ for all $g \in \mathcal{L}^{\frac{n}{i}}(S)$. Therefore, T is a continuous linear functional on $\mathcal{L}^{\frac{n}{i}}(S)$, and there exists $f \in \mathcal{L}^{\frac{n}{n-i}}(S)$ such that

$$T(g) = \frac{1}{n}\int_{S^{n-1}} gf \, dS,$$

for all $g \in \mathcal{L}^{\frac{n}{i}}(S)$ (see [35, p. 126]). We can therefore write

$$\mu(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^i f \, dS,$$

for all $K \in \mathcal{S}^n$. Decompose the function f into the difference of non-negative functions $f = f^+ - f^-$ in the usual way. Since f is an $\mathcal{L}^{\frac{n}{n-i}}$ function, so are its components f^+ and f^-. Let $\rho_{Q_1}^{n-i} = f^+$, and let $\rho_{Q_2}^{n-i} = f^-$. Then $Q_1, Q_2 \in \mathcal{S}^n$, and

$$\begin{aligned}
\mu(K) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^i (f^+ - f^-) \, dS \\
&= \frac{1}{n}\int_{S^{n-1}} \rho_K^i \rho_{Q_1}^{n-i} \, dS - \frac{1}{n}\int_{S^{n-1}} \rho_K^i \rho_{Q_2}^{n-i} \, dS \\
&= \tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2)
\end{aligned}$$

for all $K \in \mathcal{S}^n$. It now follows from Theorem 5.9 that μ is a continuous star measure, and that μ is homogeneous of degree i. □

As another application of Theorem 5.9, we prove the following theorem relating the dual quermassintegrals to the *mean intersection integrals* of an $\mathcal{L}^n$-star. This result was originally obtained by E. Lutwak for star bodies [27].

Theorem 5.11 *Let $K \in \mathcal{S}^n$. For all $i \in [n]$,*

$$\widetilde{W}_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\xi \in \mathrm{Gr}(n,i)} v_i(K \cap \xi)\, d\xi = \frac{\kappa_n}{i\kappa_i} \int_{\xi \in \mathrm{Gr}(n,i)} \int_{\xi \cap \mathbf{S}^{n-1}} \rho_K^i \, dS^{i-1}\, d\xi.$$

Recall that $\mathrm{Gr}(n,i)$ denotes the space of all i-dimensional subspaces of $\mathbf{R}^n$ (also known as the *Grassmannian*). The integral over $\mathrm{Gr}(n,i)$ is taken with respect to the rotation invariant probability measure τ_i on $\mathrm{Gr}(n,i)$ (see Theorem 4.6).

Proof: Let $K \in \mathcal{S}^n$. By Proposition 4.7 there exists a function $I_K = I_{\rho_K^i} : \mathrm{Gr}(n,i) \longrightarrow \mathbf{R}$ which is integrable with respect to the measure τ_i. Recall that I_K is given by the equation

$$I_K(\xi) = \int_{\xi \cap \mathbf{S}^{n-1}} \rho_K^i \, dS^{i-1}. \tag{5.7}$$

For all $K \in \mathcal{S}^n$ define a set function ν as follows:

$$\nu(K) = \frac{\kappa_n}{i\kappa_i} \int_{\xi \in \mathrm{Gr}(n,i)} \int_{\xi \cap \mathbf{S}^{n-1}} \rho_K^i \, dS^{i-1}\, d\xi. \tag{5.8}$$

Let $K, L \in \mathcal{S}^n$. Recall that, for all $u \in \mathbf{S}^{n-1}$, we have

$$\rho_{K\cup L}^i(u) + \rho_{K\cap L}^i(u) = \rho_K^i(u) + \rho_L^i(u).$$

It then follows from Equation (5.8) that $\nu(K \cup L) + \nu(K \cap L) = \nu(K) + \nu(L)$. Since the integrals in (5.8) are taken with respect to rotation invariant measures, we may conclude that ν is a continuous rotation invariant measure on $\mathcal{S}^n$. It is also clear from (5.8) that ν is homogeneous of degree i. By Theorem 5.9, there exists $c > 0$ such that $\nu = c\widetilde{W}_{n-i}$.

To compute the constant, note that

$$\begin{aligned}
\nu(B) &= \frac{\kappa_n}{i\kappa_i}\int_{\xi\in \mathrm{Gr}(n,i)}\int_{\xi\cap S^{n-1}} 1\ dS^{i-1}\ d\xi \\
&= \frac{\kappa_n}{i\kappa_i} i\kappa_i \int_{\xi\in \mathrm{Gr}(n,i)} 1\ d\xi \\
&= \kappa_n \\
&= V(B) \\
&= \widetilde{W}_{n-i}(B).
\end{aligned}$$

Therefore, $c = 1$ and $\nu = \widetilde{W}_{n-i}$. □

This concludes our discussion of homogeneous star measures.

Chapter 6

Rotation Invariant Measures on $\mathcal{L}^n$-stars

We now have a classification for star measures that satisfy the condition of homogeneity. This condition may be done away with, providing we work with star measures that are rotation invariant.

Definition 6.1 *Let $K \in \mathcal{S}^n$. The $\mathcal{L}^n$-star K is* bounded *if there exists $\alpha > 0$ such that $\rho_K < \alpha$ almost everywhere on $\mathbf{S}^{n-1}$. Denote by $\mathcal{S}_b^n$ the set of all bounded $\mathcal{L}^n$-stars.*

Note that if K is a bounded $\mathcal{L}^n$-star then K is a bounded $\mathcal{L}^p$-star for all $p \geq 1$.

Theorem 6.2 *Let μ be a continuous star measure that is invariant under rotations. Let $\mu(\{0\}) = 0$. Then there exists a unique continuous function $g : [0, \infty) \longrightarrow \mathbf{R}$ such that for all $K \in \mathcal{S}^n$,*

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS.$$

The measure μ will be homogeneous of degree i, if and only if $g(x) = cx^i$.

Proof: Let us re-examine the argument behind Lemma 5.1. Beginning with a homogeneous star measure μ, a new measure was constructed on the unit sphere, by evaluating

μ on spherical cones with apex at the origin and with subsets of $\mathbf{S}^{n-1}$ as bases. For this part of the proof of Lemma 5.1, the homogeneity of μ was unnecessary. Indeed, any continuous star measure will induce a countably additive measure on each sphere centered at the origin. Such induced measures will in each case be absolutely continuous with respect to spherical Lebesgue measure, provided that $\mu(\{0\}) = 0$. Nothing goes wrong until we look at more than one sphere at a time.

Now consider a continuous star measure μ that is rotation invariant (but not homogeneous). Let $\tilde{\mu}$ be the measure on $\mathbf{S}^{n-1}$ induced by μ. By Theorem 4.1, there exists $g_1 \in \mathbf{R}$ such that $\tilde{\mu} = g_1 S$, where S denotes the Lebesgue measure on the unit sphere. This constant is denoted g_1 for a reason that shall become clear below.

The argument just given may be applied to each sphere centered at the origin. For all $\alpha > 0$ denote by S_α the Lebesgue measure on $\alpha\mathbf{S}^{n-1}$. The star measure μ induces an invariant measure $\tilde{\mu}_\alpha$ on $\alpha\mathbf{S}^{n-1}$. Once again, $\tilde{\mu}_\alpha = g_\alpha S_\alpha$ on $\alpha\mathbf{S}^{n-1}$, where g_α is a real constant.

It follows that, given a spherical cone C with base $A \subseteq \alpha\mathbf{S}^{n-1}$ and apex at the origin,

$$\mu(C) = \tilde{\mu}_\alpha(A) = g_\alpha S_\alpha(A) = g_\alpha \int_{\mathbf{S}^{n-1}} \rho_C^{n-1}\, dS = \int_{\mathbf{S}^{n-1}} g_{\rho_C} \rho_C^{n-1}\, dS.$$

Let P be a polycone. By Proposition 2.12, there exist disjoint spherical cones $C_1, \ldots, C_m$ such that $P = \bigcup_{j=1}^m C_j$, and $\rho_P = \sum_{i=1}^m \rho_{C_i}$. From the argument above, and from the linearity of the integral, it follows that

$$\mu(P) = \sum_{i=1}^m \mu(C_i) = \sum_{i=1}^m \int_{\mathbf{S}^{n-1}} g_{\rho_{C_i}} \rho_{C_i}^{n-1}\, dS = \int_{\mathbf{S}^{n-1}} g_{\rho_P} \rho_P^{n-1}\, dS,$$

where the last equality follows from the fact that the cones C_i are disjoint.

Let $g : [0, \infty) \longrightarrow \mathbf{R}$ be defined by $g(x) = g_x x^{n-1}$. The expression above now becomes

$$\mu(P) = \int_{\mathbf{S}^{n-1}} g(\rho_P)\, dS,$$

for any polycone P.

The function g is determined uniquely by the action of the star measure μ on balls centered at the origin. This follows from the fact that

$$\mu(\alpha B) = \int_{\mathbf{S}^{n-1}} g(\rho_{\alpha B})\, dS = \int_{\mathbf{S}^{n-1}} g(\alpha)\, dS = g(\alpha)\sigma_{n-1},$$

where αB is the ball of radius α, and where σ_{n-1} is the surface area of the sphere $\mathbf{S}^{n-1}$.

Since the measure μ is star-continuous, the expression $\mu(\alpha B)$ defines a function on the positive reals that is continuous in the variable α. It follows that g is a continuous function. Since $\mu(\{0\}) = 0$, it follows that $g(0) = 0$.

Next, let $K \in \mathcal{S}_b^n$. By Proposition 2.13, there exists an increasing sequence of polycones P_i such that $P_i \longrightarrow K$ and such that $\rho_{P_i} \longrightarrow \rho_K$ pointwise as $i \longrightarrow \infty$. Since g is continuous, it follows that $g \circ \rho_{P_i} \longrightarrow g \circ \rho_K$ pointwise. Meanwhile there exists $\alpha > 0$ such that $\rho_{P_i} \leq \rho_K \leq \alpha$ almost everywhere on $\mathbf{S}^{n-1}$. Let $\beta = \max\{g(x) : x \in [0, \alpha]\}$. Since $g \circ \rho_{P_i}, g \circ \rho_K \leq \beta$ almost everywhere, it follows that $g \circ \rho_{P_i}$ and $g \circ \rho_K$ are measurable functions that are bounded almost everywhere; i.e. these functions are $\mathcal{L}^p$ for all $p \geq 1$.

Since μ is star-continuous, $\mu(P_i) \longrightarrow \mu(K)$ as $i \longrightarrow \infty$. It then follows from the Lebesgue-dominated convergence theorem that

$$\mu(K) = \lim_{i \to \infty} \mu(P_i) = \lim_{i \to \infty} \int_{\mathbf{S}^{n-1}} g(\rho_{P_i})\, dS = \int_{\mathbf{S}^{n-1}} \lim_{i \to \infty} g(\rho_{P_i})\, dS = \int_{\mathbf{S}^{n-1}} g(\rho_K)\, dS. \quad (6.1)$$

Finally, let $K \in \mathcal{S}^n$. For all $j \geq 0$ let $E_j = \{u \in \mathbf{S}^{n-1} : 0 \leq \rho_K(u) \leq j\}$. The sets E_j form an increasing sequence of measurable subsets of $\mathbf{S}^{n-1}$, such that

$$\bigcup_j E_j = \mathbf{S}^{n-1}. \quad (6.2)$$

For all $j \geq 0$, let $\rho_j = 1_{E_j}\rho_K$. Each ρ_j is an $\mathcal{L}^n$ function such that

$$0 \leq \rho_j \leq j.$$

Since the sets E_j form an increasing sequence with respect to inclusion, the functions ρ_j form an increasing sequence of functions, bounded above uniformly by ρ_K. Meanwhile, Equation (6.2) implies that

$$\lim_{j\to\infty} \rho_j = \lim_{j\to\infty} 1_{E_j}\rho_K = \rho_K \lim_{j\to\infty} 1_{E_j} = \rho_K. \tag{6.3}$$

Hence, the non-negative functions $\{\rho_K - \rho_j\}$ form a *decreasing* sequence that converges to zero pointwise. By the monotone convergence theorem,

$$\lim_{j\to\infty} \|\rho_K - \rho_j\|_n^n = \lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} |\rho_K - \rho_j|^n \, dS = 0. \tag{6.4}$$

Let K_j be the $\mathcal{L}^n$-star with radial function ρ_j. Equation (6.4) implies that $K_j \longrightarrow K$ in $\mathcal{S}^n$, as $j \longrightarrow \infty$. The star-continuity of μ then implies that

$$\lim_{j\to\infty} \mu(K_j) = \mu(K).$$

Since g is a continuous function,

$$g \circ \rho_j \longrightarrow g \circ \rho_K$$

pointwise, as $j \longrightarrow \infty$. Since $g(0) = 0$, we have

$$g(\rho_j(u)) = g(1_{E_j}(u)\rho_K(u)) = g(\rho_K(u)),$$

if $u \in E_j$, and

$$g(\rho_j(u)) = g(1_{E_j}(u)\rho_K(u)) = g(0) = 0,$$

if $u \notin E_j$. In other words,

$$g \circ \rho_j = g(1_{E_j}\rho_K) = (1_{E_j})(g \circ \rho_K) = (1_{[0,j]} \circ \rho_K)(g \circ \rho_K).$$

For all $u \in \mathbf{S}^{n-1}$, let $g^+(u) = \max\{g(u), 0\}$, and let $g^-(u) = \max\{-g(u), 0\}$. Then $g = g^+ - g^-$, where both g^+ and g^- are non-negative continuous functions. The composed functions $g^+ \circ \rho_j$ form an increasing sequence of functions on $\mathbf{S}^{n-1}$ that converges pointwise to $g^+ \circ \rho_K$. Similarly, the composed functions $g^- \circ \rho_j$ form a decreasing sequence of functions on $\mathbf{S}^{n-1}$ that converges pointwise to $g^- \circ \rho_K$.

Since each K_j is a *bounded* subset of $\mathbf{R}^n$, Equation (6.1) implies that

$$\mu(K_j) = \int_{\mathbf{S}^{n-1}} g(\rho_j)\, dS,$$

for all $j > 0$. It follows that

$$\begin{aligned}
\mu(K) &= \lim_{j\to\infty} \mu(K_j) \\
&= \lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} g(\rho_j)\, dS \\
&= \lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} g^+(\rho_j) - g^-(\rho_j)\, dS \\
&= \lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} g^+(\rho_j)\, dS - \lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} g^-(\rho_j)\, dS \\
&= \int_{\mathbf{S}^{n-1}} \lim_{j\to\infty} g^+(\rho_j)\, dS - \int_{\mathbf{S}^{n-1}} \lim_{j\to\infty} g^-(\rho_j)\, dS \\
&= \int_{\mathbf{S}^{n-1}} g^+(\rho_K)\, dS - \int_{\mathbf{S}^{n-1}} g^-(\rho_K)\, dS \\
&= \int_{\mathbf{S}^{n-1}} g(\rho_K)\, dS,
\end{aligned}$$

where the fifth equality follows from the monotone convergence theorem. □

The following propositions will be useful for the classification of the rotation invariant star measures.

Proposition 6.3 *Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a non-negative continuous function. The equation*

$$\int_{\mathbf{S}^{n-1}} g \circ \rho\, dS < \infty \tag{6.5}$$

holds for all non-negative $\mathcal{L}^n$ functions $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, if and only if there exists $k > 0$ such that $g(x) \leq kx^n$ for all $x \geq 1$.

Proof: To begin, suppose that there exists $k > 0$ such that $g(x) \leq kx^n$ for all $x \geq 1$.

Let $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ be a non-negative $\mathcal{L}^n$ function. Let $D = \{u \in \mathbf{S}^{n-1} : \rho(u) < 1\}$, and let $E = \{u \in \mathbf{S}^{n-1} : \rho(u) \geq 1\}$. Since ρ is bounded on D, the continuity of g implies that $g \circ \rho$ is also a bounded function on D, so that

$$\int_D g \circ \rho \, dS < \infty.$$

Meanwhile,

$$\begin{aligned} \int_E g \circ \rho \, dS &\leq \int_E k\rho^n \, dS \\ &\leq k \int_{\mathbf{S}^{n-1}} \rho^n \, dS \\ &< \infty, \end{aligned}$$

since ρ is a non-negative $\mathcal{L}^n$ function. Hence,

$$\int_{\mathbf{S}^{n-1}} g \circ \rho \, dS = \int_D g \circ \rho \, dS + \int_E g \circ \rho \, dS < \infty.$$

Conversely, suppose that Equation (6.5) holds for all $\rho \in \mathcal{L}^n(\mathbf{S}^{n-1})$, and suppose also that there does *not* exist $k > 0$ such that $g(x) \leq kx^n$ for all $x \geq 1$. Then for all $k > 0$ there exists $\alpha_k > 1$ such that $g(\alpha_k) > k\alpha_k^n$. Since $k > 0$, this statement is equivalent to the following claim:

For all $k > 0$ there exists $\alpha_k > 1$ such that $g(\alpha_k) > 2^k\alpha_k^n$.

Let $U_1, U_2, \ldots$ be a sequence of disjoint open subsets of $\mathbf{S}^{n-1}$ such that $S(U_k) = \frac{1}{2^k\alpha_k^n}$, for all $k > 0$. Let $Z = \mathbf{S}^{n-1} - \cup_{k>0} U_k$.

Define a function $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ as follows. For all $u \in \mathbf{S}^{n-1}$, set $\rho(u) = \alpha_k$ if $u \in U_k$. If $u \in Z$ then set $\rho(u) = 0$. It then follows that

$$\int_{\mathbf{S}^{n-1}} \rho^n \, dS = \sum_{k>0} \int_{U_k} \rho^n \, dS$$

$$
\begin{aligned}
&= \sum_{k>0} \alpha_k^n S(U_k) \\
&= \sum_{k>0} \alpha_k^n \frac{1}{2^k \alpha_k^n} \\
&= \sum_{k>0} \frac{1}{2^k} \\
&= 2.
\end{aligned}
$$

In other words, the function ρ is a non-negative $\mathcal{L}^n$ function on $\mathbf{S}^{n-1}$. Meanwhile,

$$
\begin{aligned}
\int_{\mathbf{S}^{n-1}} g \circ \rho \, dS &= \int_Z g \circ \rho \, dS + \sum_{k>0} \int_{U_k} g \circ \rho \, dS \\
&= \int_Z g(0) \, dS + \sum_{k>0} \int_{U_k} g(\alpha_k) \, dS \\
&= g(0)S(Z) + \sum_{k>0} g(\alpha_k) S(U_k) \\
&= g(0)S(Z) + \sum_{k>0} g(\alpha_k) \frac{1}{2^k \alpha_k^n} \\
&> g(0)S(Z) + \sum_{k>0} 2^k \alpha_k^n \frac{1}{2^k \alpha_k^n} \\
&= g(0)S(Z) + \sum_{k>0} 1 \\
&= \infty.
\end{aligned}
$$

In other words, $g \circ \rho$ does not satisfy Equation (6.5), contradicting our assumption. Therefore, there must exist $k > 0$ such that $g(x) \leq kx^n$ for all $x \geq 1$. □

Proposition 6.4 *Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function. The equation*

$$
\left| \int_{\mathbf{S}^{n-1}} g \circ \rho \, dS \right| < \infty \tag{6.6}
$$

holds for all non-negative $\mathcal{L}^n$ functions $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, if and only if there exists $k > 0$ such that $|g(x)| \leq kx^n$ for all $x \geq 1$.

Proof: To begin, suppose that $g(0) = 0$.

For all $u \in \mathbf{S}^{n-1}$, let $g^+(u) = \max\{g(u), 0\}$, and let $g^-(u) = \max\{-g(u), 0\}$. Then $g = g^+ - g^-$, where both g^+ and g^- are non-negative continuous functions. Let $D = \{x \in \mathbf{R} : g(x) \geq 0\}$, and let $E = \{x \in \mathbf{R} : g(x) < 0\}$. Since g is a continuous function, D is a closed set, and E is an open set in $\mathbf{R}$.

For all $x \in \mathbf{R}$, we have $g^+(x) = 1_D(x)g(x)$. In other words, $g^+(x) = g(x)$ if $x \in D$; otherwise $g^+(x) = 0$. Since $g(0) = 0$, it then follows that $g^+(x) = g(1_D x)$ for all $x \in \mathbf{R}$.

Similarly, for all $x \in \mathbf{R}$, we have $g^-(x) = -1_E(x)g(x)$. In other words, $g^-(x) = -g(x)$ if $x \in E$; otherwise $g^-(x) = 0$. Since $g(0) = 0$, it then follows that $g^-(x) = -g(1_E x)$ for all $x \in \mathbf{R}$.

Suppose that

$$\left|\int_{\mathbf{S}^{n-1}} g \circ \rho \, dS\right| < \infty,$$

for all non-negative $\mathcal{L}^n$ functions $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$. It then follows that

$$\left|\int_{\mathbf{S}^{n-1}} g^+ \circ \rho \, dS\right| = \left|\int_{\mathbf{S}^{n-1}} g \circ (1_D \rho) \, dS\right| < \infty,$$

since $1_D\rho$ is also a non-negative $\mathcal{L}^n$ function.

Similarly,

$$\left|\int_{\mathbf{S}^{n-1}} g^- \circ \rho \, dS\right| = \left|\int_{\mathbf{S}^{n-1}} -g \circ (1_E \rho) \, dS\right| = \left|\int_{\mathbf{S}^{n-1}} g \circ (1_E \rho) \, dS\right| < \infty,$$

since $1_E\rho$ is also a non-negative $\mathcal{L}^n$ function.

Since $g^+, g^- \geq 0$, Proposition 6.3 implies that there exists $k > 0$ such that

$$|g(x)| = g^+(x) + g^-(x) \leq kx^n,$$

for all $x \geq 1$.

Now suppose that $g(0) \neq 0$. Let $h = g - g(0)$. If

$$\left|\int_{\mathbf{S}^{n-1}} g \circ \rho \, dS\right| < \infty,$$

for all non-negative $\mathcal{L}^n$ functions $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, then

$$\begin{aligned}\left|\int_{\mathbf{S}^{n-1}} h \circ \rho \, dS\right| &= \left|\int_{\mathbf{S}^{n-1}} g \circ \rho - g(0) \, dS\right| \\ &\leq \left|\int_{\mathbf{S}^{n-1}} g \circ \rho \, dS\right| + \left|\int_{\mathbf{S}^{n-1}} g(0) \, dS\right| \\ &< \infty.\end{aligned}$$

It then follows from the previous argument that there exists $k > 0$ such that $|h(x)| \leq kx^n$, for all $x \geq 1$. Let $m > k + |g(0)|$. We then have $(m-k)x^n > |g(0)|$ for all $x \geq 1$, so that

$$|g(x)| = |h(x) + g(0)| \leq kx^n + |g(0)| < kx^n + (m-k)x^n = mx^n,$$

for all $x \geq 1$.

This completes half of the proof. To prove the converse, let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function, and suppose that there exists $k > 0$ such that $|g(x)| \leq kx^n$ for all $x \geq 1$.

Let $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$ be a non-negative $\mathcal{L}^n$ function. Let $D = \{u \in \mathbf{S}^{n-1} : \rho(u) < 1\}$, and let $E = \{u \in \mathbf{S}^{n-1} : \rho(u) \geq 1\}$. Since ρ is bounded on D, the continuity of g implies that $|g \circ \rho|$ is also a bounded function on D, so that

$$\int_D |g \circ \rho| \, dS < \infty.$$

Meanwhile,

$$\begin{aligned}\int_E |g \circ \rho| \, dS &\leq \int_E k\rho^n \, dS \\ &\leq k \int_{\mathbf{S}^{n-1}} \rho^n \, dS \\ &< \infty,\end{aligned}$$

since ρ is a non-negative $\mathcal{L}^n$ function. Hence,

$$\left|\int_{\mathbf{S}^{n-1}} g \circ \rho \, dS\right| \leq \int_{\mathbf{S}^{n-1}} |g \circ \rho| \, dS = \int_D |g \circ \rho| \, dS + \int_E |g \circ \rho| \, dS < \infty.$$

This concludes the proof. □

Proposition 6.5 *Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function. The equation*

$$\left|\int_{\mathbf{S}^{n-1}} g \circ \rho \, dS\right| < \infty \tag{6.7}$$

holds for all non-negative $\mathcal{L}^n$ functions $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, if and only if there exist $a, b \geq 0$ such that $|g(x)| \leq ax^n + b$ for all $x \geq 0$.

Proof: Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function. By Proposition 6.4, Equation (6.7) holds for all non-negative $\mathcal{L}^n$ functions $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$, if and only if there exists $k > 0$ such that $|g(x)| \leq kx^n$ for all $x \geq 1$.

Let $a = k$. Since $|g|$ is continuous on the compact interval $[0, 1]$, the function $|g|$ must attain a maximum value b on this interval. If $x \in [0, 1]$, then

$$|g(x)| \leq b \leq ax^n + b.$$

If $x \geq 1$, then

$$|g(x)| \leq kx^n = ax^n \leq ax^n + b.$$

In other words, $|g(x)| \leq ax^n + b$ for all $x \geq 0$.

Conversely, suppose that, for all $x \geq 0$, we have $|g(x)| \leq ax^n + b$, where $a, b \geq 0$. Let $k > 0$ such that $k \geq (a + b)$. For all $x \geq 1$,

$$|g(x)| \leq ax^n + b \leq ax^n + bx^n \leq kx^n.$$

It now follows from Proposition 6.4 that Equation (6.7) holds for all non-negative $\mathcal{L}^n$ functions $\rho : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$. □

Theorem 6.2 can now be improved as follows.

Theorem 6.6 *Let μ be a continuous star measure that is invariant under rotations. Let $\mu(\{0\}) = 0$. Then there exists a unique continuous function $g : [0, \infty) \longrightarrow \mathbf{R}$ such that for all $K \in \mathcal{S}^n$,*

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS.$$

Moreover, there exist $a, b > 0$ such that $|g(x)| \leq ax^n + b$ for all $x \geq 0$.

Proof: This theorem is just a restatement of Theorem 6.2, with an added restriction on the continuous function g associated to the star measure μ.

Since $\mu(K)$ takes on finite values for all $K \in \mathcal{S}^n$, we have

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g(\rho_K) \, dS < \infty,$$

for all non-negative $\mathcal{L}^n$ functions $\rho_K : \mathbf{S}^{n-1} \longrightarrow \mathbf{R}$. It then follows from Proposition 6.5 that there exist $a, b > 0$ such that $|g(x)| \leq ax^n + b$ for all $x \geq 0$. □

So far this classification is one-sided. To each continuous rotation invariant star measure μ (such that $\mu(\{0\}) = 0$) there is associated a unique continuous function $g : [0, \infty) \longrightarrow \mathbf{R}$ (such that $g(0) = 0$), satisfying the inequality conditions of Theorem 6.6. This injective mapping from the rotation invariant star measures to the "sub n-th degree" continuous functions on the nonnegative reals is in fact a *bijective* mapping. In order to see this, one final lemma is required.

Lemma 6.7 *Let f, g be non-negative $\mathcal{L}^1$ functions on $\mathbf{S}^n$. Let f_i be a sequence of non-negative $\mathcal{L}^1$ functions such that $f_i \longrightarrow f$ pointwise as $i \to \infty$, and such that*

$$\lim_{i \to \infty} \int_{\mathbf{S}^{n-1}} f_i \, dS = \int_{\mathbf{S}^{n-1}} f \, dS.$$

Let g_i be a sequence of non-negative $\mathcal{L}^1$ functions such that $g_i \longrightarrow g$ pointwise as $i \to \infty$, and such that

$$g_i \leq f_i$$

for all i. Then there exists a subsequence $\{g_{i_j}\} \subseteq \{g_i\}$ such that

$$\lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} g_{i_j} \; dS = \int_{\mathbf{S}^{n-1}} g \; dS.$$

Proof: To begin, note that

$$\int_{\mathbf{S}^{n-1}} \liminf(f_i - g_i) \; dS = \int_{\mathbf{S}^{n-1}} (f - g) \; dS = \int_{\mathbf{S}^{n-1}} f \; dS - \int_{\mathbf{S}^{n-1}} g \; dS. \tag{6.8}$$

Meanwhile, since $f_i - g_i \geq 0$ for all i,

$$\int_{\mathbf{S}^{n-1}} \liminf(f_i - g_i) \; dS \leq \liminf \int_{\mathbf{S}^{n-1}} (f_i - g_i) \; dS,$$

by Fatou's Lemma [35, p. 23]. Hence,

$$\begin{aligned}
\int_{\mathbf{S}^{n-1}} \liminf(f_i - g_i) \; dS &\leq \liminf \int_{\mathbf{S}^{n-1}} (f_i - g_i) \; dS \\
&\leq \limsup \int_{\mathbf{S}^{n-1}} (f_i - g_i) \; dS \\
&\leq \limsup \int_{\mathbf{S}^{n-1}} f_i \; dS + \limsup \int_{\mathbf{S}^{n-1}} -g_i \; dS \\
&= \limsup \int_{\mathbf{S}^{n-1}} f_i \; dS - \liminf \int_{\mathbf{S}^{n-1}} g_i \; dS \\
&= \int_{\mathbf{S}^{n-1}} f \; dS - \liminf \int_{\mathbf{S}^{n-1}} g_i \; dS
\end{aligned}$$

It then follows from Equation (6.8) that

$$\begin{aligned}
\int_{\mathbf{S}^{n-1}} f \; dS - \int_{\mathbf{S}^{n-1}} g \; dS &= \int_{\mathbf{S}^{n-1}} \liminf(f_i - g_i) \; dS \\
&\leq \int_{\mathbf{S}^{n-1}} f \; dS - \liminf \int_{\mathbf{S}^{n-1}} g_i \; dS,
\end{aligned}$$

so that

$$\liminf \int_{\mathbf{S}^{n-1}} g_i \; dS \leq \int_{\mathbf{S}^{n-1}} g \; dS.$$

Applying Fatou's Lemma once again, we have,

$$\liminf \int_{\mathbf{S}^{n-1}} g_i \, dS \leq \int_{\mathbf{S}^{n-1}} g \, dS = \int_{\mathbf{S}^{n-1}} \liminf g_i \, dS \leq \liminf \int_{\mathbf{S}^{n-1}} g_i \, dS.$$

In other words,

$$\int_{\mathbf{S}^{n-1}} g \, dS = \liminf \int_{\mathbf{S}^{n-1}} g_i \, dS.$$

Therefore, there exists a subsequence $\{g_{i_j}\}$ such that

$$\int_{\mathbf{S}^{n-1}} g \, dS = \lim_{j \to \infty} \int_{\mathbf{S}^{n-1}} g_{i_j} \, dS.$$

□

Proposition 6.8 *Let $g : [0, \infty) \longrightarrow \mathbf{R}$ be a continuous function such that $g(0) = 0$, and suppose that there exist $a, b > 0$ such that $|g(x)| \leq ax^n + b$ for all $x \geq 0$. Define a set function μ on $\mathcal{S}^n$ by the equation*

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS,$$

for all $K \in \mathcal{S}^n$. Then μ is a continuous rotation-invariant measure on $\mathcal{S}^n$. Moreover, $\mu(\{0\}) = 0$.

Proof:

First we must show that μ is defined on $\mathcal{S}^n$. Let $K \in \mathcal{S}^n$. Since ρ_K is a non-negative $\mathcal{L}^n$ function on $\mathbf{S}^{n-1}$, it follows from Proposition 6.5 that

$$\left| \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS \right| < \infty.$$

Next, let $K, L \in \mathcal{S}^n$. For all $u \in \mathbf{S}^{n-1}$,

$$\begin{aligned} g \circ \rho_{K \cup L}(u) + g \circ \rho_{K \cap L}(u) &= g(\max\{\rho_K(u), \rho_L(u)\}) + g(\min\{\rho_K(u), \rho_L(u)\}) \\ &= g(\rho_K(u)) + g(\rho_L(u)). \end{aligned}$$

It follows that

$$\begin{aligned}\mu(K \cup L) + \mu(K \cap L) &= \int_{\mathbf{S}^{n-1}} g \circ \rho_{K\cup L}\, dS + \int_{\mathbf{S}^{n-1}} g \circ \rho_{K\cap L}\, dS \\ &= \int_{\mathbf{S}^{n-1}} g \circ \rho_K\, dS + \int_{\mathbf{S}^{n-1}} g \circ \rho_L\, dS \\ &= \mu(K) + \mu(L).\end{aligned}$$

In other words, μ is a measure on $\mathcal{S}^n$.

To show that μ is rotation invariant, let $K \in \mathcal{S}^n$, and let $\phi \in SO(n)$. Proposition 2.21 implies that

$$\mu(\phi K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_{\phi K}\, dS = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \circ \phi^{-1}\, dS.$$

Since the measure S on $\mathbf{S}^{n-1}$ is invariant under the action of $SO(n)$, it follows that

$$\mu(\phi K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \circ \phi^{-1}\, dS = \int_{\mathbf{S}^{n-1}} g \circ \rho_K\, dS = \mu(K).$$

In other words, μ is a rotation invariant measure on $\mathcal{S}^n$. It remains to show that μ is star-continuous.

The proof of star-continuity shall rely on the following principle.

> Let a_i be a sequence of real numbers, and let $a \in \mathbf{R}$. Then $\lim_{i\to\infty} a_i = a$ if and only if, for every subsequence a_{i_j}, there exists a *sub*-subsequence $a_{i_{j_k}}$ such that $\lim_{k\to\infty} a_{i_{j_k}} = a$.

Assume first that $g \geq 0$. Let $K_m, K \in \mathcal{S}^n$ such that $K_m \longrightarrow K$ as $m \to \infty$, and let ρ_m be the radial function of K_m for all m. Let $\{K_i\}$ be a *subsequence* of $\{K_m\}$. Then as $i \to \infty$, we have $K_i \longrightarrow K$ as well. By choosing a (sub)subsequence if necessary, we may assume without loss of generality that $\rho_i \longrightarrow \rho_K$ pointwise [35, p. 68].

For all $i > 0$ let $g_i = g \circ \rho_i$, and let $f_i = a\rho_i^n + b$. Similarly, let $g_K = g \circ \rho_K$, and let $f_K = a\rho_K^n + b$. Then,

$$\mu(K_i) = \int_{\mathbf{S}^{n-1}} g(\rho_i)\, dS = \int_{\mathbf{S}^{n-1}} g_i\, dS,$$

and

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g(\rho_K)\, dS = \int_{\mathbf{S}^{n-1}} g_K\, dS.$$

Since g is a continuous function, $g_i \longrightarrow g_K$ pointwise. Similarly, $f_i \longrightarrow f_K$ pointwise. Since $K_i \longrightarrow K$, we also have

$$\lim_{i\to\infty} \int_{\mathbf{S}^{n-1}} \rho_i^n\, dS = \int_{\mathbf{S}^{n-1}} \rho_K^n\, dS,$$

so that

$$\lim_{i\to\infty} \int_{\mathbf{S}^{n-1}} f_i\, dS = \int_{\mathbf{S}^{n-1}} f_K\, dS.$$

By our original assumption,

$$0 \leq g_i \leq f_i, \quad \text{and} \quad 0 \leq g_K \leq f_K.$$

By Lemma 6.7, there exists a subsequence g_{i_j} such that

$$\lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} g_{i_j}\, dS = \int_{\mathbf{S}^{n-1}} g_K\, dS.$$

In other words, for every subsequence $\{K_i\}$ of the original sequence $\{K_m\}$, there exists a sub-subsequence $\{K_{i_j}\}$ such that

$$\lim_{j\to\infty} \mu(K_{i_j}) = \lim_{j\to\infty} \int_{\mathbf{S}^{n-1}} g_{i_j}\, dS = \int_{\mathbf{S}^{n-1}} g_K\, dS = \mu(K).$$

It then follows that

$$\lim_{m\to\infty} \mu(K_m) = \mu(K),$$

so that μ is star-continuous.

The proof is nearly finished. Suppose finally that g is an arbitrary continuous function on $\mathbf{R}$ such that $g(0) = 0$ and such that $|g(x)| \leq ax^n + b$ for all $x \geq 0$. In the usual way we express g as the difference $g = g^+ - g^-$, where g^+ and g^- are nonnegative continuous functions. Note that $g^+(0) = g^-(0) = g(0) = 0$. For all $K \in \mathcal{S}^n$, define

$$\mu^+(K) = \int_{\mathbf{S}^{n-1}} g^+ \circ \rho_K \, dS,$$

and

$$\mu^-(K) = \int_{\mathbf{S}^{n-1}} g^- \circ \rho_K \, dS.$$

Since $g^+, g^- \geq 0$, the previous argument implies that μ^+ and μ^- are both continuous star measures. Since $\mu = \mu^+ - \mu^-$, the star measure μ is also continuous. □

In other words, the injective correspondence given by Theorem 6.6 is, in fact, a bijection. These results are summarized in the following theorem.

Theorem 6.9 *Let μ be a continuous star measure that is invariant under rotations. Let $\mu(\{0\}) = 0$. Then there exists a unique continuous function $g : [0, \infty) \longrightarrow \mathbf{R}$ such that for all $K \in \mathcal{S}^n$,*

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS. \tag{6.9}$$

The function g satisfies the following two conditions:

- *$g(0) = 0$.*
- *There exist $a, b > 0$ such that $|g(x)| \leq ax^n + b$ for all $x \geq 0$.*

Conversely, Equation (6.9) defines a continuous rotation invariant star measure μ, for all continuous functions $g : [0, \infty) \longrightarrow \mathbf{R}$ satisfying the above conditions. □

In order for the preceding arguments to flow gracefully, it was necessary to assume that $\mu(\{0\}) = 0$. This assumption implied, among other things, the absolute continuity of the induced measures $\tilde{\mu}_\alpha$ on the spheres $\alpha\mathbf{S}^{n-1}$, which in turn enabled us to apply the Lebesgue-Radon-Nikodym theorem. But there are many rotation invariant star-continuous measures for which $\mu(\{0\}) \neq 0$. An important example of such a measure is the following.

Definition 6.10 *Define the star measure* $\chi : \mathcal{S}^n \longrightarrow \mathbf{R}$*, as follows. For each* $\mathcal{L}^n$*-star* K*, define* $\chi(K) = 1$.

This constant set function is obviously continuous and rotation invariant. That χ is a measure is also clear.

Now let μ be any continuous rotation invariant star measure. Let $c = \mu(\{0\})$. Since the measure $\nu = \mu - c\chi$ satisfies the conditions of Theorem 6.9, there exists a unique continuous function $g : [0, \infty) \longrightarrow \mathbf{R}$ such that $g(0) = 0$, and such that

$$\begin{aligned} \mu(K) &= \nu(K) - c \\ &= \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS - c \\ &= \int_{\mathbf{S}^{n-1}} g \circ \rho_K - \frac{c}{\sigma_{n-1}} \, dS \end{aligned}$$

for all $K \in \mathcal{S}^n$.

Hence, there is a unique continuous function $G = g - (c/\sigma_{n-1})$ such that

$$\mu(K) = \int_{\mathbf{S}^{n-1}} G \circ \rho_K \, dS$$

for all $K \in \mathcal{S}^n$.

Note once again that the function G is determined uniquely by the action of the star measure μ on balls centered at the origin. As in the proof of Theorem 6.2,

$$\mu(\alpha B) = \int_{\mathbf{S}^{n-1}} G(\rho_{\alpha B}) \, dS = \int_{\mathbf{S}^{n-1}} G(\alpha) \, dS = G(\alpha)\sigma_{n-1},$$

where αB is the ball of radius α, and where σ_{n-1} is the surface area of the sphere $\mathbf{S}^{n-1}$.

Since the functions G and g differ only by a constant, the condition that $|g|$ is bounded above by a polynomial of the form $ax^n + b$ (where $a, b \geq 0$) is equivalent to the same condition on the function $|G|$.

This result is summarized in the following theorem.

Theorem 6.11 (Classification of Rotation Invariant Star Measures) *There is a bijective correspondence between continuous star measures μ that are invariant under rotations and continuous functions $G : [0, \infty) \longrightarrow \mathbf{R}$ such that $|G(x)| \leq ax^n + b$ for some $a, b \geq 0$.*

This correspondence is given by the following equations:

$$\mu(K) = \int_{\mathbf{S}^{n-1}} G \circ \rho_K \; dS$$

for all $K \in \mathcal{S}^n$, and

$$G(\alpha) = \frac{1}{\sigma_{n-1}} \mu(\alpha B)$$

for all $\alpha \geq 0$. □

Many important properties of μ translate into analogous properties of the associated function G.

Corollary 6.12 *Let μ be a continuous star measure that is invariant under rotations. Let $G : [0, \infty) \longrightarrow \mathbf{R}$ be the continuous function associated to μ in Theorem 6.11.*

- *The measure μ is non-negative if and only if the function G is non-negative on $[0, \infty)$.*
- *The measure μ is monotonic on $\mathcal{S}^n$ if and only if G is a monotonically increasing function on $[0, \infty)$.*

Proof: Given $\alpha \geq 0$,

$$\mu(\alpha B) = \int_{\mathbf{S}^{n-1}} G(\alpha) \; dS = G(\alpha)\sigma_{n-1}. \tag{6.10}$$

Hence μ is non-negative if and only if the function G is non-negative on $[0, \infty)$.

If μ is a monotonic measure, then $\mu(\alpha B)$ increases monotonically with respect to α in Equation (6.10), so that G must also be a monotonically increasing function. Conversely, suppose that G is a monotonically increasing function. If $K, L \in \mathcal{S}^n$ and $K \subseteq L$, then $\rho_K(u) \leq \rho_L(u)$ for all $u \in \mathbf{S}^{n-1}$. Therefore $G(\rho_K(u)) \leq G(\rho_L(u))$. It follows that

$$\mu(K) = \int_{\mathbf{S}^{n-1}} G(\rho_K)\, dS \leq \int_{\mathbf{S}^{n-1}} G(\rho_L)\, dS = \mu(L),$$

so that μ is a monotonic measure. □

This concludes our investigation of rotation invariant star measures.

Chapter 7

$SL(n)$ Invariant Measures on $\mathcal{L}^n$-stars

Recall from Proposition 2.21 that the special linear group $SL(n)$ acts on $\mathcal{S}^n$. This fact motivates an investigation of $SL(n)$-invariant measures.

To begin, note that such a measure μ is rotation invariant. Theorem 6.11 then implies the existence of an associated continuous function $G : [0, \infty) \longrightarrow \mathbf{R}$ such that for all $K \in \mathcal{S}^n$,

$$\mu(K) = \int_{\mathbf{S}^{n-1}} G \circ \rho_K \, dS.$$

Lemma 7.1 *Let μ be an $SL(n)$-invariant continuous star measure. Suppose that $\mu(\{0\}) = 0$. Then μ is either a non-negative measure or a non-positive measure.*

Proof: Let G be the continuous function associated to μ, as discussed above. If $\mu = 0$, then we are done. If not, assume without loss of generality that $G(a) > 0$, for some $a > 0$. Suppose there exists $y \in (0, a)$ such that $G(y) < 0$. Since G is continuous, G attains a minimum $G(r_0) = m < 0$ on $[0, a]$.

Let B_0 be the ball of radius r_0, centered at the origin. Let $\lambda = \frac{a}{r_0}$. Let $T \in SL(n)$ be the map represented by a diagonal matrix, with n diagonal entries $\lambda, \frac{1}{\lambda}, 1, \ldots, 1$. Finally, let $E = T(B_0)$. In other words, E is the image of the ball B_0 under the linear map T, an ellipsoid. Since $r_0 < a$, we know that $\frac{1}{\lambda} < 1 < \lambda$.

Let ∂E denote the boundary of E. For all $y \in \partial E$, we have $y = Tx$ for some

$x \in \partial B_0 = r_0\mathbf{S}^{n-1}$. Hence,

$$|y| = |Tx| = \left(\lambda^2 x_1^2 + \frac{1}{\lambda^2}x_2^2 + x_3^2 + \cdots + x_n^2\right)^{\frac{1}{2}} \leq \lambda|x| = \lambda r_0 = a.$$

It follows that the image $Im(\rho_E) \subseteq [0, a]$. For $x = (r_0, 0, \ldots, 0)$, $|Tx| = |(a, 0, \ldots, 0)| = a$, so that $a = \rho_E(x/|x|) \in Im(\rho_E)$.

Since G is minimized at r_0 on $[0, a]$,

$$\mu(E) = \int_{\mathbf{S}^{n-1}} G \circ \rho_E \, dS \geq \int_{\mathbf{S}^{n-1}} G(r_0) \, dS = m\sigma_{n-1}.$$

Meanwhile, the $SL(n)$-invariance of μ implies that $\mu(E) = \mu(B_0) = m\sigma_{n-1}$. The inequality is an equality. This can only be so if $G(x) = m$ almost everywhere on the closed interval $Im(\rho_E)$. Since G is continuous, it follows that $G(x) = m$ on $Im(\rho_E)$. In particular, $G(a) = m < 0$. This is a contradiction.

Thus G must be non-negative on [0,a]. Now suppose that $G(b) < 0$ for some $b > a$. The same argument, applied to the measure $-\mu$ on the interval $[0, b]$, implies that this is also impossible. It follows that G is a non-negative function. By Corollary 6.12, μ is a non-negative measure. □

The trick of deforming a ball of local minimum or maximum measure into a convenient ellipse will be used again in the proof of this next lemma.

Lemma 7.2 *Let μ be an $SL(n)$-invariant continuous star measure. Suppose that $\mu(\{0\}) = 0$. Then μ is monotonic.*

Proof: If $\mu = 0$, we are done. Suppose that $\mu \neq 0$. By Lemma 7.1, one may assume without loss of generality that μ is a non-negative measure. Let G be the non-negative continuous function associated to μ. Let $a > 0$. Since G is continuous, G attains a maximum $G(r_1) = M$ on $[0, a]$.

Let B_1 be the ball of radius r_1, centered at the origin. Let $\lambda = \frac{a}{r_1}$. Let $T \in SL(n)$ be the map represented by a diagonal matrix, with n diagonal entries $\lambda, \frac{1}{\lambda}, 1, \ldots, 1$. Finally, let $E = T(B_1)$. In other words, E is the image of the ball B_1 under the linear map T, an ellipsoid. In this case $r_1 \leq a$, so that $\frac{1}{\lambda} \leq 1 \leq \lambda$.

We follow an argument almost identical to that given in the proof of Lemma 7.1.

As before, for all $y \in \partial E$, $y = Tx$ for some $x \in \partial B_1 = r_1 \mathbf{S}^{n-1}$. Hence,

$$|y| = |Tx| = \left(\lambda^2 x_1^2 + \frac{1}{\lambda^2} x_2^2 + x_3^2 + \cdots + x_n^2\right)^{\frac{1}{2}} \leq \lambda |x| = \lambda r_1 = a.$$

It follows that the image $Im(\rho_E) \subseteq [0, a]$. For $x = (r_1, 0, \ldots, 0)$, $|Tx| = |(a, 0, \ldots, 0)| = a$, so that $a = \rho_E(x/|x|) \in Im(\rho_E)$.

Since G is maximized at r_1 on $[0, a]$,

$$\mu(E) = \int_{\mathbf{S}^{n-1}} G \circ \rho_E \, dS \leq \int_{\mathbf{S}^{n-1}} G(r_1) \, dS = M\sigma_{n-1}.$$

Meanwhile, the $SL(n)$-invariance of μ implies that $\mu(E) = \mu(B_1) = M\sigma_{n-1}$. The inequality is an equality. This can only be so if $G(x) = M$ almost everywhere on the closed interval $Im(\rho_E)$. Since G is continuous, we have $G(x) = M$ on $Im(\rho_E)$. In particular, $G(a) = M$.

Thus, on any closed interval $[0, a]$, where $a > 0$, the function G attains its maximum at a. It follows that if $0 \leq a \leq b$, then $G(a) \leq G(b)$. In other words, G is a monotonic function. By Corollary 6.12, μ is a monotonic measure. □

Theorem 7.3 *Let μ be an $SL(n)$-invariant continuous star measure. Suppose that $\mu(\{0\}) = c_0$. Then $\mu = c_0\chi + c_1 V$, where $c_0, c_1 \in \mathbf{R}$ are constants.*

Here V denotes volume in $\mathbf{R}^n$, and χ denotes the constant unit measure on $\mathcal{L}^n$-stars, defined in the previous chapter.

Proof: Suppose that $\mu(\{0\}) = 0$. Let G be the continuous function associated to μ. Lemma 7.2 and Corollary 6.12 imply that G is a monotonic function on $[0, \infty)$. Let

$\alpha = \mu(B)$, where B is the unit ball, centered at the origin. For all $K \in \mathcal{S}^n$, define

$$\nu(K) = \mu(K) - \frac{\alpha}{V(B)} V(K).$$

Since the measure ν satisfies the conditions of Lemma 7.2, ν must be a monotonic measure. But $\nu(\{0\}) = 0$, and $\nu(B) = 0$. Hence $\nu(K) = 0$ for all $\mathcal{L}^n$-stars $K \subseteq B$. In other words, $\mu(K) = \frac{\alpha}{V(B)} V(K)$ for all star bodies $K \subseteq B$.

This argument may be repeated, using larger and larger balls centered at the origin, instead of the unit ball B. Since each larger ball will contain the last, the constant $\frac{\alpha}{V(B)}$ must never change. Since every polycone is contained in some ball centered at the origin,

$$\mu(P) = \frac{\alpha}{V(B)} V(P)$$

for all polycones P. It then follows from Proposition 2.12 and from the star-continuity of μ and V that

$$\mu(K) = \frac{\alpha}{V(B)} V(K)$$

for all $K \in \mathcal{S}^n$. Let $c_0 = 0$ and $c_1 = \frac{\alpha}{V(B)}$.

Next suppose that $\mu(\{0\}) = c_0 \neq 0$. Repeat the preceding argument, using the measure $\mu - c_0\chi$. It follows that there exists $c_1 \in \mathbf{R}$ such that $\mu - c_0\chi = c_1 V$. □

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