CALENDERING OF AN ELASTIC-VISCOUS MATERIAL

by

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Calendering of an Elastic-viscous Material

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Submitted to the Department of Mechanical Engineering on December 17, 1954 in partial fulfillment of the requirements for the degree of Doctor of Science in Mechanical Engineering.

Abstract

Calendering is the term applied to the process of rolling bulk plastics into sheets. This process is widely used in industry for the production of plastic sheets and linoleum. In this process as well as any large-scale industrial process there is always the desire to have a given piece of equipment produce the largest amount of its product possible. For this reason an analytical approach to the problem seemed useful.

A study of the flow of an elastic-viscous material that is being calendered is undertaken. The solution of this problem is known for the case of a viscous material. Advantage is taken of the viscous solution to predict the behavior of an elastic-viscous material whose properties are "almost" viscous. The problem is broken into two parts, as follows:

1. A stress-strain rate equation is chosen that can give suitable predictions for simple types of flows while remaining simple enough to work with. This equation is given below.

$$(1+aJ_2)\dot{e}_{ij} = \frac{S_{ij}}{2\eta} + \frac{\dot{S}_{ij}}{2G}$$

J_2 = second invariant of the strain rate tensor

e_{ij} = strain

S_{ij} = reduced stress

η = viscosity

G = shear modulus

a = parameter selected to match material in question

2. An approximate solution of the calendering problem is obtained and calculated results are given to determine the effect of several parameters. (A formal perturbation solution is given in the Appendix.) This solution indicates the effect of elasticity on the calendering of a material which is very close to being viscous. This solution can be used as a guide in selecting suitable speeds of operation for calendering when the material to be calendered exhibits elastic as well as viscous behavior.

Thesis Supervisor: E. Orowan
Title: Professor of Mechanical Engineering

Cambridge, Massachusetts
December 17, 1954

Professor J. H. Keenan
Chairman, Departmental Committee on Graduate Students
Department of Mechanical Engineering
Cambridge 39, Massachusetts

Dear Professor Keenan:

In partial fulfillment of the requirements for the degree of Doctor of Science in Mechanical Engineering, I, herewith, submit my thesis entitled, "Calendering of an Elastic-viscous Material".

Very truly yours,

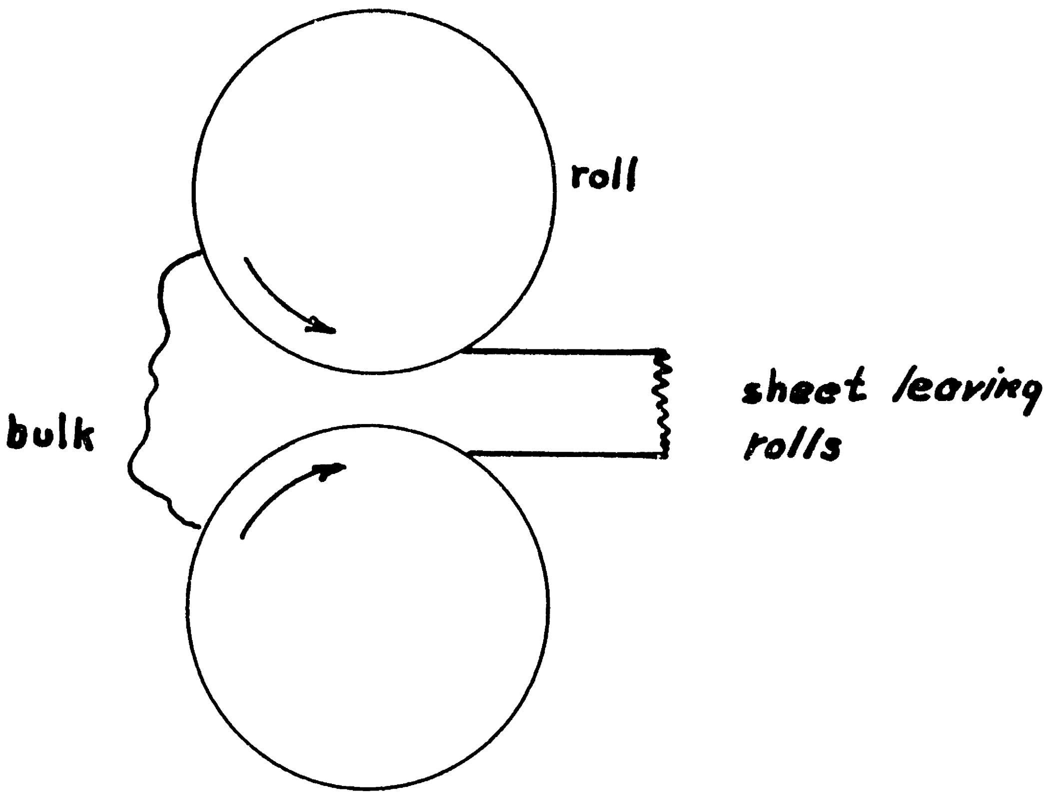
Paul R. Paslay

ACKNOWLEDGEMENT

The author wishes to express his thanks to Professor G. F. Carrier of Harvard University who suggested the subject of this thesis and offered continued help to the author throughout the entire investigation.

INTRODUCTION

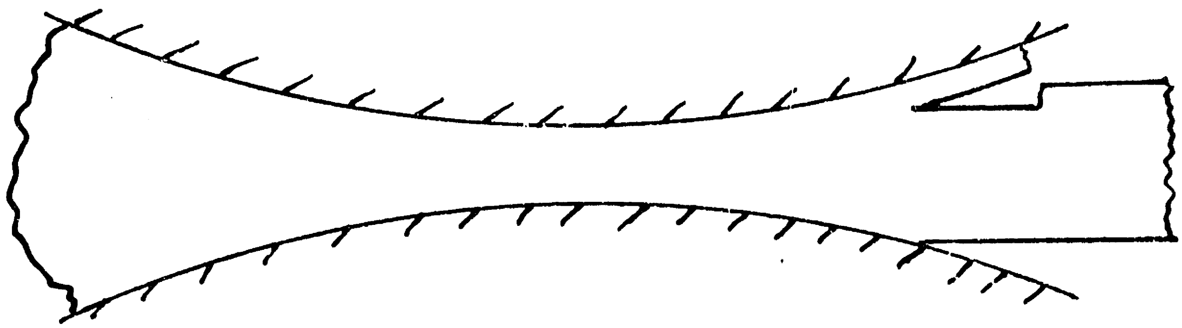
Calendering is the term applied to the process of rolling bulk plastic into sheets. The process finds wide application in such industries as the manufacture of linoleum and the production of plastic sheets. A diagram of the process is shown below.



The bulk material that will be considered in this thesis is a heated plastic. The properties of this heated plastic are those of an elastic-viscous material.

From a manufacturing point of view the speed of production from a set of rolls should be the maximum allowable for the production of a sufficiently high-quality sheet. One primary factor controlling the speed of manufacture is the force exerted by the material to force the rolls apart. The smaller the force is between the rolls, the lighter the rolling equipment may be. To decrease the force between the rolls the material may be heated (thereby reducing the viscosity of the material). In order that the material may retain its shape after rolling, the rolls are cooled.

If the material is above some temperature when it enters the rolls, one of two undesirable things may happen. The material may blister below the surface of the rolled sheet; this leaves an undesirable appearance and is not acceptable. The other thing that may happen is that part of the sheet may tear away from the rest and stick to the roll as shown below.



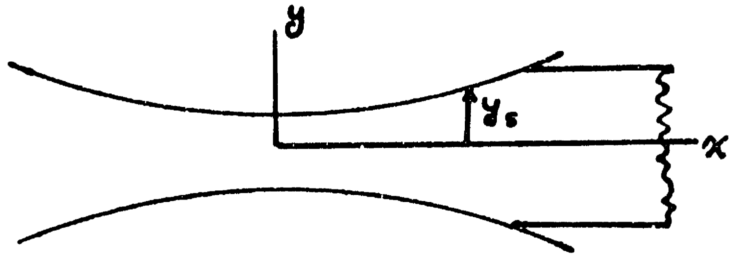
Both the blistering and tearing are the result of a common cause. This is the fact that the energy dissipated and the cooling accomplished in the rolling process produces a non-uniform temperature distribution across the sheet. Professor M. Finston* has shown that for a viscous material, taking into account the thermal conductivity, the position of the peak temperature of the sheet leaving the rolls is under the surface of the sheet. This explains both the blistering and peeling phenomena.

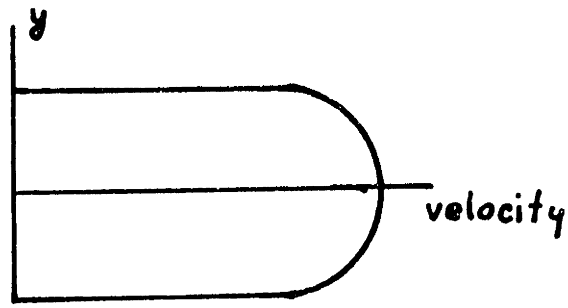
The groundwork for the calendering problem was laid by Dr. R. E. Gaskell** in 1950. Dr. Gaskell considered both Newtonian and non-Newtonian materials for the case of viscous flow neglecting momentum changes. The boundary conditions Dr. Gaskell used were that the velocity of the material in contact with the rolls be the same as the velocity of the surface of the rolls (i.e., no slipping between roll and material) and that the velocity profile of the material leaving the rolls be uniform. For the Newtonian material Dr. Gaskell's result was that the velocity profile was composed of a uniform velocity equal to the roll surface velocity plus a parabolic distribution that disappeared

* Reference 2   ** Reference 1

at the surface. Let V_o be the roll surface velocity, x the position along the passage and y the distance from the center of material to a general point of the material.

Then:





$$\text{velocity of } (x,y) = V_o + K(x)\left(1 - \frac{y^2}{y_s^2}\right)$$

$$\frac{\partial(\text{velocity at } (x,y))}{\partial y} = -2K(x)\frac{y}{y_s^2}$$

where y_s is the distance from the center of the material to the surface of the roll (y_s is a function of x) and $K(x)$ a function of x determined so that there is equal mass flow across each section. The shear strain rate (and shear stress as well) are linear functions of y for a prescribed value of x. The material is taken to be incompressible. Dr. Gaskell determined pressure distributions as well, considering normal stresses large as compared to shear stresses.

Professor M. Finston* extended Dr. Gaskell's solution by allowing the viscosity to be a function of temperature. Professor Finston carried out a perturbation solution around constant viscosity, that is, he allowed only

* Reference 2

small variations of viscosity. Let η be viscosity at temperature T and η_0 viscosity at temperature T_0 then Professor Finston's viscosity-temperature relation may be written as the following,

$$\eta = \eta_0 \left(1 - \delta\left(\frac{T - T_0}{T_0}\right)\right)$$

He then carried out a heat transfer solution of this problem. Professor Finston determined pressure, shear stress and velocity profiles as well as temperature distributions. The prediction of the point of maximum temperature in the crossection from Professor Finston's calculation agreed very well with the position of blisters formed in actual experiments. Professor Finston considered normal stress in each direction at a point to be equal, an incompressible material and he neglected momentum changes.

In 1953 Professor G. F. Carrier* discussed the determination of the temperature distribution of the calendering problem, after the velocity profile and pressure distribution are obtained. This work involves the manipulation of the conservation of energy equation to obtain a suitable approach to the heat transfer aspects of this problem.

The results obtained by Dr. Gaskell and Professor Finston compare favorably with experimental results, except in the case where the material exhibits elastic as well as viscous behavior. This thesis was undertaken as an attempt to shed light on what happens to an elastic viscous material when the material is calendered. A stress strain relation is chosen that gives suitable results for some of the simpler tests and an approximate solution is carried out for the calendering process.

*Reference 3

ANALYSIS

Determination of a Stress-Strain Rate Equation

In order to obtain a result suitable for engineering purposes for a problem of the sort treated here it is often necessary to make simplifying assumptions regarding the behavior of the material and the geometry of the problem. The first such simplification to be encountered here has to do with the stress-strain law to be used. To describe the behavior of an Hookean-elastic, Newtonian-viscous material one might heuristically proceed as follows for an incompressible material.

Let $$e_{ij} = \text{strain} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

where $\bar{u}_i =$ displacement in i^{th} direction

and $x_i = i^{th}$ cartesian coordinate.

Also let $S_{ij} =$ reduced stress $= \sigma_{ij} - \delta_{ij}\sigma$

where $\sigma_{ij} =$ stress

$$\delta_{ij} = \text{Kronecker delta} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and $$\sigma = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = -p.$$

Other notation used:

$t =$ time

$G =$ shear modulus

$\zeta =$ viscosity.

Then calculating elastic and viscous strains separately one obtains:

$$\left.\begin{array}{l} (e_{ij})_{elastic} = \dfrac{S_{ij}}{2G} \qquad (1) \\ e_{11} + e_{22} + e_{33} = 0 \qquad (2) \end{array}\right\} \text{Hooke's law}$$

and $$\left.\begin{aligned}(\dot{e}_{ij})\text{ viscous} &= \frac{S_{ij}}{2\eta} \\ e_{11}+e_{22}+e_{33} &= 0\end{aligned}\right\}$$ (3)

Newtonian Viscous Material.

(4)

Taking the time derivative of equation (1) yields:

$$(\dot{e}_{ij})\text{ elastic} = \frac{\dot{S}_{ij}}{2G}.$$

Now since the strain rates add linearly, one writes

$$(\dot{e}_{ij})\text{elastic} + (\dot{e}_{ij})\text{viscous} = (\dot{e}_{ij})\text{total} \tag{5}$$

and, therefore, $$\dot{e}_{ij} = \frac{S_{ij}}{2\eta} + \frac{\dot{S}_{ij}}{2G}. \tag{6}$$

A material which satisfies the stress-strain equation (6) is called a Maxwell material. Before discussing the physical interpretation of equation (6) two remarks are appropriate, first,

$$\dot{e}_{ij} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial \dot{\bar{u}}_i}{\partial x_j} + \frac{\partial \dot{\bar{u}}_j}{\partial x_i}\right)$$

where $\dot{\bar{u}}_i = u_i = i^{th}$ component of velocity

and, therefore, $$\dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{7}$$

The second observation is that the stress-strain law intrinsically relates the deformation (and its rates of change) to the stress acting on a given sample of the material, that is to say, it is the Lagrangian stress which must appear in the basic stress-strain law. However, in one flow problem, the stress one must discuss is the Eulerian stress. This implies that, although the $\dot{S}_{ij}$ of equation (6) is merely the partial derivative $\frac{\partial(S^*_{ij})}{\partial t}$ where S^*_{ij} is the Lagrangian stress, it can be anticipated that the corresponding operation on the Eulerian stress will be more involved. The determination of the correct form for $\dot{S}_{ij}$ has been discussed by Fromm (4,5) and the following equations are derived by him:

$$\overset{\circ}{S}_{ij} = \frac{\partial S_{ij}}{\partial t} + u_1 \frac{\partial S_{ij}}{\partial x_1} + u_2 \frac{\partial S_{ij}}{\partial x_2} + u_3 \frac{\partial S_{ij}}{\partial x_3}$$

$$+ \omega_{12}\Pi_{12} + \omega_{31}\Pi_{31} + \omega_{23}\Pi_{23}$$

where $\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$ = rotation at a point

$x_i = i^{th}$ coordinate

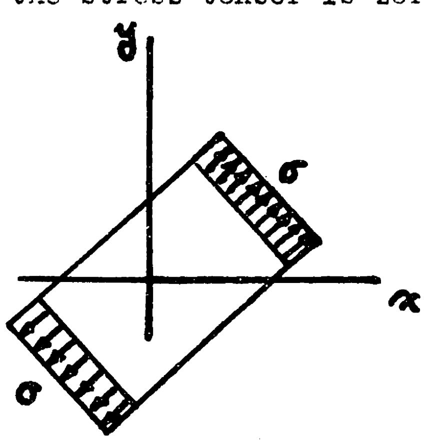
$$\Pi_{12} = \begin{Bmatrix} 2\sigma_{xy} & \sigma_{yy}-\sigma_{xx} & \sigma_{yz} \\ \sigma_{yy}-\sigma_{xx} & -2\sigma_{xy} & -\sigma_{xz} \\ \sigma_{yz} & -\sigma_{zx} & 0 \end{Bmatrix}$$

$$\Pi_{31} = \begin{Bmatrix} -2\sigma_{zx} & -\sigma_{yz} & \sigma_{xx}-\sigma_{zz} \\ -\sigma_{yz} & 0 & \sigma_{xy} \\ \sigma_{xx}-\sigma_{zz} & \sigma_{xy} & 2\sigma_{zx} \end{Bmatrix}$$

and

$$\Pi_{23} = \begin{Bmatrix} 0 & \sigma_{zx} & -\sigma_{xy} \\ \sigma_{xz} & 2\sigma_{yz} & \sigma_{zz}-\sigma_{yy} \\ -\sigma_{xy} & \sigma_{zz}-\sigma_{yy} & -2\sigma_{yz} \end{Bmatrix}.$$

A simple example will show that the rotation of the element must be taken into account. A bar subjected to a constant uniform tension in the x - y plane is rotated at constant angular velocity $\dot{\theta}_{xy} = \omega$ (see sketch below). The stress does not depend on time, therefore, the time derivative of the stress tensor is zero. This may be computed as follows:



$$\sigma_{xx} = \frac{\sigma}{2}(1 + \cos 2\omega t)$$
$$\sigma_{yy} = \frac{\sigma}{2}(1 - \cos 2\omega t)$$
$$\sigma_{xy} = \frac{\sigma}{2}(\sin 2\omega t)$$

$$\frac{d}{dt}\left\{\begin{matrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{matrix}\right\} = \frac{\partial}{\partial t}\left\{\begin{matrix} \frac{\sigma}{2}(1+\cos 2\omega t) & \frac{\sigma}{2}\sin 2\omega t & 0 \\ \frac{\sigma}{2}\sin 2\omega t & \frac{\sigma}{2}(1-\cos 2\omega t) & 0 \\ 0 & 0 & 0 \end{matrix}\right\}$$

$$+ \frac{\partial}{\partial s}\left\{\begin{matrix} \frac{\sigma}{2}(1+\cos 2\omega t) & \frac{\sigma}{2}\sin 2\omega t & 0 \\ \frac{\sigma}{2}\sin 2\omega t & \frac{\sigma}{2}(1-\cos 2\omega t) & 0 \\ 0 & 0 & 0 \end{matrix}\right\}$$

$$+ \omega\, W_{12}$$

$$\frac{d}{dt}\left\{\begin{matrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{matrix}\right\} = \left\{\begin{matrix} -\sigma\omega\sin 2\omega t & +\sigma\omega\cos 2\omega t & 0 \\ +\sigma\omega\cos 2\omega t & +\sigma\omega\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{matrix}\right\}$$

$$+\left\{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right\} + \left\{\begin{matrix} +\sigma\omega\sin 2\omega t & -\sigma\omega\cos 2\omega t & 0 \\ -\sigma\omega\cos 2\omega t & -\sigma\omega\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{matrix}\right\} = 0.$$

Thus, as expected, the rotation of an element of material must be taken into account in the time derivative.

One method of considering the physical interpretation of equation (6) is to determine what results are predicted by the equation for several simple problems. If the results predicted by equation (6) are in accord with what one expects for the simple cases, one may hope that reasonable results can be obtained for more complicated cases that are not so easily substantiated by experiment. Three cases are considered below.

<u>Case 1.</u> Consider a bar with a constant uniform uniaxial tension applied in the direction.

In this case:

$$e_{ij} = \begin{Bmatrix} e_{xx} & 0 & 0 \\ 0 & -\frac{1}{2}e_{xx} & 0 \\ 0 & 0 & -\frac{1}{2}e_{xx} \end{Bmatrix}$$

$$S_{ij} = \begin{Bmatrix} \frac{2}{3}\sigma_{xx} & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_{xx} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{xx} \end{Bmatrix}$$

$$\dot{S}_{ij} = 0$$

and, therefore, $\dot{e}_{xx} = \dfrac{\sigma_{xx}}{3\eta}$.

This case degenerates into the viscous case which is perfectly reasonable.

<u>Case 2.</u> Consider a bar on which is imposed a prescribed uniform uniaxial strain in the x direction at $t = 0$. The problem is to find the variation of stress with time.

For $t > 0$, $\dot{e}_{ij} = 0$ and σ_{xx} = function of time only. All other stress components are zero.

Now

$$\frac{1}{G}\begin{Bmatrix} \frac{2}{3}\dot{\sigma}_{xx} & 0 & 0 \\ 0 & -\frac{1}{3}\dot{\sigma}_{xx} & 0 \\ 0 & 0 & -\frac{1}{3}\dot{\sigma}_{xx} \end{Bmatrix} + \frac{1}{\eta}\begin{Bmatrix} \frac{2}{3}\sigma_{xx} & 0 & 0 \\ 0 & -\frac{1}{3}\sigma_{xx} & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_{xx} \end{Bmatrix} = 0$$

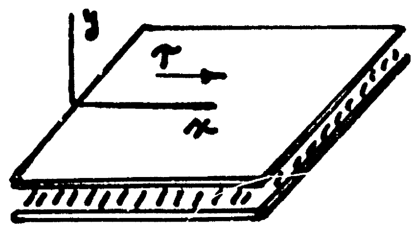
or $$\frac{d\sigma_{xx}}{dt} + \frac{G}{\eta}\sigma_{xx} = 0.$$

Solving the equation with the initial condition that $\sigma_{xx} = \sigma_{xx_0}$ at $t = 0^+$ yields:

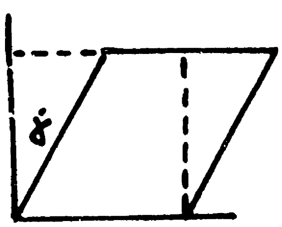
$$\sigma_{xx} = \sigma_{xx_0}\, e^{-\frac{t}{\eta/G}}. \tag{10}$$

This result shows that the stress decreases exponentially with time, a situation which one might expect.

Case 3. The following case is due to H. Fromm.* Determine the shear-stress versus rate-of-strain curve for the shearing of the material between two infinite parallel plates in a steady process.



Because of equilibrium the shear stress and the shear-strain rate are uniform throughout the material, therefore, we may define the following:



$$\dot e_{xx} = 0, \quad \dot e_{yy} = 0, \quad \dot e_{zz} = 0$$
$$\dot e_{yz} = 0, \quad \dot e_{xz} = 0$$
$$\dot e_{xy} = \tfrac{1}{2}\dot\gamma = \omega_{xy}.$$

Also

$$\frac{\partial S_{ij}}{\partial t} = 0, \quad \frac{\partial S_{ij}}{\partial x} = 0, \quad \frac{\partial S_{ij}}{\partial y} = 0, \quad \frac{\partial S_{ij}}{\partial z} = 0.$$

Now

$$\begin{Bmatrix} 0 & \frac{1}{2}\dot\gamma & 0 \\ \frac{1}{2}\dot\gamma & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} = \frac{\frac{1}{2}\dot\gamma}{2G}\begin{Bmatrix} 2\sigma_{xy} & \sigma_{yy}-\sigma_{xx} & 0 \\ \sigma_{yy}-\sigma_{xx} & -2\sigma_{xy} & 0 \\ 0 & 0 & 0 \end{Bmatrix} + \frac{1}{2\eta}\begin{Bmatrix} \frac{2}{3}\sigma_{xx}-\frac{1}{3}\sigma_{yy} & \sigma_{xy} & 0 \\ \sigma_{xy} & \frac{2}{3}\sigma_{yy}-\frac{1}{3}\sigma_{xx} & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

and by simplifying one obtains,

$$\frac{\sigma_{xy}}{G} = \frac{\frac{\eta}{G}\dot\gamma}{1+\left[\frac{\eta}{G}\dot\gamma\right]^2} \tag{11}$$

$$\frac{\sigma_{yy}}{G} = -\frac{\sigma_{xx}}{G} = \frac{\left[\frac{\eta}{G}\dot\gamma\right]^2}{1+\left[\frac{\eta}{G}\dot\gamma\right]^2}. \tag{12}$$

* Reference 5

A plot of $\frac{\sigma_{xy}}{G}$ versus $\frac{n}{G}\dot{\gamma}$ is shown in Figure 1. The questionable point of this result is that the shear stress reaches a maximum for a finite shear strain rate and that each shear stress below the maximum is associated with two strain rates. This result is not in accord with what one expects based on the available experimental data. To eliminate this difficulty one might assume variations in viscosity or shear modulus with strain rate or might alter the stress-strain equation. This result would not be too disturbing if one knew that the shear strain rates would be sufficiently low so that the calculated shear stress would not reach the peak indicated in Figure 1.

Unfortunately, for the calendering problem the primary interest is to be focused on shear stress and shear strain rate close to the surface of the material. The shear-stress and shear-strain rates are highest at the surface of the material as it is being calendered. If equation (6) could be modified to overcome the difficulty encountered in case three without unduly disturbing the results of cases one and two, then one might hope that the result obtained by the modified equation would be representative of the physical picture.

In modifying equation (6) care must be exercised not to alter the equation in such a way that if the equation were transformed to another Cartesian coordinate system x', y', z' that the meaning of the equation would be changed. In other words, these equations are for an isotropic material and should not change meaning when different Cartesian coordinates are used. The most obvious way to modify such an equation is by use of invariants of either strain or stress. For the problem to be considered in this thesis the following stress-strain rate relation was chosen:

$$\left(1+aJ_2\right)\dot{e}_{ij}=\frac{S_{ij}}{2\eta}+\frac{\dot{S}_{ij}}{2G} \tag{13}$$

where $J_2 =$ second invariant of the strain rate tensor

$$= \tfrac{1}{2}\left[(\dot{e}_{xx})^2+(\dot{e}_{yy})^2+(\dot{e}_{zz})^2\right]+(\dot{e}_{xy})^2+(\dot{e}_{zx})^2+(\dot{e}_{yz})^2$$

and $a =$ an arbitrary parameter.

In order to demonstrate the meaning of equation (13), the results of cases one, two and three are given below where equation (13) has replaced equation (6).

Case 1.

$$\frac{\sigma_{xx}}{3\eta}=\left(1-\frac{3a}{4}(\dot{e}_{xx})^2\right)\dot{e}_{xx}. \tag{14}$$

This equation is plotted in Figure 2 for various values of .

Case 2.

$$\sigma_{xx}=\sigma_{xx_0}e^{\frac{-t}{\eta/G}}. \qquad \text{(same result as with equation (6))}. \tag{15}$$

This equation is plotted in Figure 3. The equation is independent of a.

Case 3.

$$\frac{\sigma_{xy}}{G}=\frac{\left[1+\frac{a}{4}(\dot{\gamma})^2\right]\left[\frac{\eta}{G}\dot{\gamma}\right]}{1+\left[\frac{\eta}{G}\dot{\gamma}\right]^2} \tag{16}$$

$$-\frac{\sigma_{xx}}{G}=\frac{\sigma_{yy}}{G}=\frac{\left[1+\frac{a}{4}(\dot{\gamma})^2\right]\left[\frac{\eta}{G}\dot{\gamma}\right]^2}{1+\left[\frac{\eta}{G}\dot{\gamma}\right]^2} \tag{17}$$

Equations (16) and (17) are plotted in Figures 4 and 5 respectively for various ratios of $\frac{a}{\eta^2/G^2}$.

Figure 4 shows that by manipulating the parameter a in equation (13), the shear stress-strain rate curve can be adjusted to be in better accord with what one would expect. Reasonable values of $\frac{a}{\eta^2/G^2}$ would be $\frac{a}{\eta^2/G^2}>0$. a should be adjusted to agree with the material under consideration.

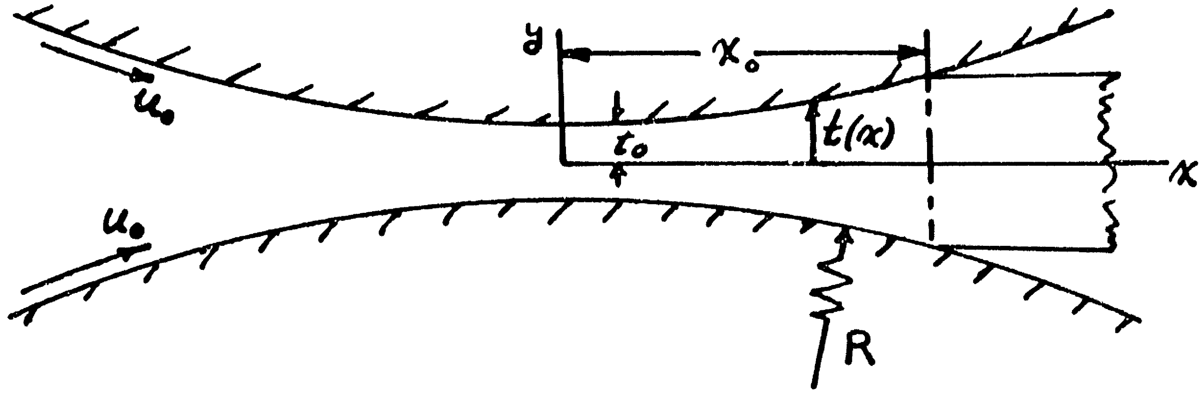
Experiments show that a could be adjusted to be in good agreement with many materials (see example reference 6 where curves are given for non-vulcanized rubber or reference 7 where results are shown for a 35% polystyrene solution). Before the stress-strain equation (equation 13) proposed here is used for a specific case results of several tests should verify the suitability of this equation. The experimental results of the three cases considered above should furnish sufficient data to judge the suitability of equation 13. Experimental investigations require great skill to avoid serious errors, for example, when concentric cylinders are rotated with respect to each other with the material to be tested between the cylinders there is a strong tendency for slip between the material and the cylinder. In the case of concentric cylinders, roughing of the cylinder to prevent slip often requires irregularities in the surface large enough to disrupt the flow pattern which invalidates the test.

If a material and geometrical configuration are chosen so that

$\frac{n}{G}(e_{ii}) << 1$ (where e_{ii} is a characteristic strain rate of the system) then by referring to Figure 2 or equation (14) one may see that the results are substantially unchanged. The cases to be considered for the calendering process fall into the class of materials and geometry where $\frac{n}{G} e_{ii}$ is small compared to one.

SOLUTION OF CALENDERING PROBLEM

Having chosen a suitable stress-strain rate equation, the calendering problem may now be considered. The following notation will be used:



u_o is the roll surface velocity.

x_o is the value of x where the material leaves the rolls.

t_o is one-half the minimum gap between the rolls.

$t(x)$ is one-half the thickness of the sheet at various values of x.

u is the component of velocity in the x direction.

v is the component of velocity in the y direction.

R is the radius of each of the rolls.

The subscript o refers to specified velocities or distances.

A complication that immediately presents itself is that the value of x_o must be arbitrarily prescribed. The real condition to be applied here is that the material leaves the roll when the stress pulling the material away from the roll equals the adhesion between the roll and the material. The determination of x_o must be left to experiment as the present status of the theoretical work does not allow a theoretical prediction of the position of the material leaving the rolls. Approximate values of x_o have been reported by Dr. R. E. Gaskell* as follows:

$$x_o = \sqrt{\frac{Rt_o}{2}}$$

This value of x_o makes it possible to determine solutions for comparative

* In reference 2 Professor M. Finston mentions values of x which were contained in a summary report to the Armstrong Cork Company by Dr. R. E. Gaskell.

purposes by a more precise value of $\underline{x}_0$ would be required for an actual specific problem.

This solution assumes as both Dr. Gaskell's and Professor Finston's solution do, that there is no slipping between the material and the rolls. One anticipates that this solution will be simular to Dr. Gaskell's and Professor Finston's solutions so that normal stresses will be large compared to shear stress.

This means that moderate values of the coefficient of friction (less than 1) will prevent slip between the roll and material.

Now it is possible to deduce certain important results from the geometry of the calendering process. The solution will be restricted to configurations that comply with the following restrictions (most actual calendering processes will be included in the solution).

1. The thickness variation $(2(t(x_0)-t_0))$ is of the order of magnitude of the minimum thickness $(2t_0)$. This means that $\dfrac{t(x_0)-t_0}{t_0}$ is of no higher order than one.
2. The passage length $(\approx 2x_0)$ is long compared to the thickness $2t_0$.
3. The axial length of the rolls is long compared with the length of the passage $(\approx 2x_0)$. This permits the approximation of plane strain. The plane strain approximation will be good except at the ends of the roll (edges of the sheet).

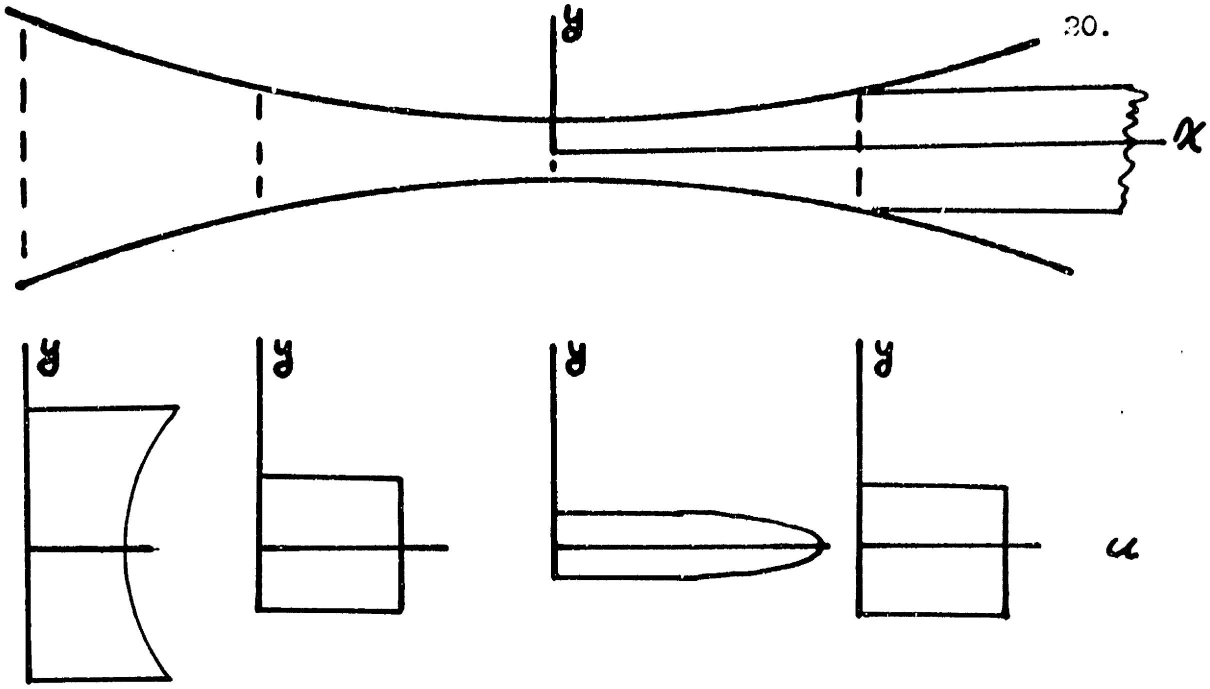
The implications of statement 3 are that:

$$e_{xz} = e_{yz} = e_{zz} = 0$$

$$S_{zz} = 0, \quad p = -\sigma_{zz} = -\frac{\sigma_{xx} + \sigma_{yy}}{2}$$

$$\tau_{xz} = \tau_{zy} = 0.$$

Since an incompressible material has been assumed from the outset, the velocity profiles at the various sections may be roughly pictured as follows:



Realizing the rough shape of the velocity profile and statement 1, one may say that $\frac{u_{max}-u_0}{u_0}$ is of order $\frac{t_{max}-t_0}{t_{max}}$. In the equations that follow $\rho = O(\xi)$ should be read "ρ is of the order of ξ".

Now $\frac{u_{max}-u_0}{u_0} = O(1)$

and from this the following statements may be obtained:

$$\frac{\partial u}{\partial x}\Big|_{max} = O\left(\frac{u_0}{x_0}\right) \qquad (u = u_{max} \text{ when } x=0,\ y=0\text{; and } u=u_0 \text{ at } x=x_0)$$

$$\frac{\partial u}{\partial y}\Big|_{max} = O\left(\frac{u_0}{t_0}\right) \qquad (u = u_{max} \text{ when } y=0,\ x=0\text{; and } u=u_0 \text{ at } y=t(x)).$$

Now referring to the conservation of mass equation (written for an incompressible material):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial y}\Big|_{max} = O\left(\frac{\partial u}{\partial x}\Big|_{max}\right) = O\left(\frac{u_0}{x_0}\right).$$

Then

$$v_{max} = O\left(\frac{u_0 t_0}{x_0}\right) \quad \text{so that} \quad \frac{\partial v}{\partial x}\Big|_{max} = O\left(\frac{u_0}{x_0}\frac{t_0}{x_0}\right).$$

From the above order of magnitude study and statement 2 $\left(\frac{t_0}{x_0} \ll 1\right)$ some consequences are:

$$u_{max} \gg v_{max} \qquad\qquad \frac{\partial u}{\partial y}\Big|_{max} \gg \frac{\partial u}{\partial x}\Big|_{max}$$

$$\frac{\partial u}{\partial y}\Big|_{max} \gg \frac{\partial v}{\partial y}\Big|_{max} \qquad\qquad \frac{\partial u}{\partial y}\Big|_{max} \gg \frac{\partial v}{\partial x}\Big|_{max}.$$

The stresses may be analyzed in a similar way. This solution is intended to be for a material and geometry that exhibit primarily viscous behavior so that for an order of magnitude study we may write:

$$\tau_{max} = O\left(\eta \frac{\partial u}{\partial y}\Big|_{max}\right) = O\left(\eta \frac{u_o}{t_o}\right)$$

and
$$\frac{\partial \tau}{\partial x}\Big|_{max} = O\left(\frac{\eta u_o}{t_o x_o}\right) \qquad (\tau = 0 \text{ when } x = x_o;\ \tau = \tau_{max} \text{ at } x = 0, y = 0.)$$

$$\frac{\partial \tau}{\partial y}\Big|_{max} = O\left(\eta \frac{u_o}{t_o^2}\right) \qquad (\tau = 0 \text{ when } y = 0;\ \tau = \tau_{max} \text{ at } x = 0, y = t_o).$$

The remaining stresses are $\sigma_x + p$ and $\sigma_y + p$ which reduce to a single variable in the following way:

$$\sigma_x + p = \sigma_x - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}$$

$$\sigma_y + p = \sigma_y - \frac{\sigma_x - \sigma_y}{2} = \frac{\sigma_y - \sigma_x}{2}$$

so that

$$\sigma_x + p = -(\sigma_y + p) = \frac{\sigma_x - \sigma_y}{2}.$$

Both of the stresses $\sigma_x + p$ and $\sigma_y + p$ are a result of shearing stresses action on a rotating element for this problem so that one would anticipate that:

$$(\sigma_x + p)_{max} = O(\tau_{max}) = O\left(\frac{\eta u_o}{t_o}\right)$$

$$\frac{\partial(\sigma_x + p)}{\partial x}\Big|_{max} = O\left(\frac{\partial \tau}{\partial x}\Big|_{max}\right) = O\left(\frac{\eta}{t_o}\frac{u_o}{x_o}\right)$$

and
$$\frac{\partial(\sigma_x + p)}{\partial y}\Big|_{max} = O\left(\frac{\partial \tau}{\partial y}\Big|_{max}\right) = O\left(\frac{\eta u_o}{t_o^2}\right).$$

The order of magnitude of $\sigma_x + p$ will have to be checked to see if this condition is satisfied in the solution.

The above equations make suitable simplifications so that an approximate solution is possible. Before considering the equations, however, there is a limitation on the stress-strain rate equations. This limitation is that

$\frac{\eta}{G} e_{ii} \ll 1$ (where e_{ii} is a characteristic strain rate of the system). For the calendering problem the following restriction is imposed:

$$\frac{\eta}{G}\frac{\partial u}{\partial y}\Big|_{max} = O(1) \quad \text{so that} \quad \frac{\eta}{G}\frac{\partial u}{\partial x} = O\left(\frac{y_0}{x_0}\right) \ll 1.$$

The stress-strain rate equations (with the implications of statement 3) are now written out for examination.

$$\left\{1 + \frac{a}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]\right\} \begin{Bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 0 \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

$$= \frac{1}{2\eta}\begin{Bmatrix} \sigma_x + p & \tau & 0 \\ \tau & -(\sigma_x+p) & 0 \\ 0 & 0 & 0 \end{Bmatrix} + \frac{1}{2G}\left[\!\left[u \begin{Bmatrix} \frac{\partial(\sigma_x+p)}{\partial x} & \frac{\partial \tau}{\partial x} & 0 \\ \frac{\partial \tau}{\partial x} & -\frac{\partial(\sigma_x+p)}{\partial x} & 0 \\ 0 & 0 & 0 \end{Bmatrix}\right.\right.$$

$$\left.\left. + v \begin{Bmatrix} \frac{\partial(\sigma_x+p)}{\partial y} & \frac{\partial \tau}{\partial y} & 0 \\ \frac{\partial \tau}{\partial y} & -\frac{\partial(\sigma_x+p)}{\partial y} & 0 \\ 0 & 0 & 0 \end{Bmatrix} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\begin{Bmatrix} 2\tau & \sigma_x - \sigma_y & 0 \\ \sigma_x - \sigma_y & 2\tau & 0 \\ 0 & 0 & 0 \end{Bmatrix}\right]\!\right]$$

The scheme to be followed is that the maximum order of each term in each stress-strain equation is to be written out and then only the highest order terms will be retained, thus, leading to simplified equations.

First taking the xx stress-strain rate equation:

$$\left\{1 + \frac{a}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]\right\}\frac{\partial u}{\partial x} = \frac{\sigma_x + p}{2\eta}$$
$$+ \frac{1}{2G}\left[u\frac{\partial(\sigma_x+p)}{\partial x} + v\frac{\partial(\sigma_x+p)}{\partial y} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)2\tau\right].$$

The orders of the respective terms are:

$$\left\{1 + a\left[\left(\frac{u_0}{x_0}\right)^2 + \left(\frac{u_0}{x_0}\right)^2 + \left(\frac{u_0}{t_0} + \frac{u_0}{x_0}\frac{t_0}{x_0}\right)^2\right]\right\}\frac{u_0}{x_0} = \frac{u_0}{t_0}$$
$$+ \left(\frac{\eta}{G}\frac{u_0}{t_0}\right)\left[\frac{u_0}{x_0} + \frac{u_0}{x_0} + \frac{u_0 t_0}{x_0 x_0} - \frac{u_0}{t_0}\right].$$

Recalling that $a = O\left(\frac{\eta^2}{G^2}\right)$, $\frac{t_0}{x_0} << 1$ ane $\left(\frac{\eta}{G}\frac{u_0}{t_0}\right) = O(1)$

the highest order terms turn out to be:

$$0 = \frac{\sigma_x + p}{2\eta} - \frac{\tau\frac{\partial u}{\partial y}}{2G} \quad \text{or} \quad \sigma_x + p = \frac{\eta}{G}\frac{\partial u}{\partial y}\tau. \tag{19}$$

$\sigma_x + p$ is of the order of τ, as anticipated.

Taking the xy stress-strain rate equation,

$$\left\{1 + \frac{a}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]\right\}\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{\tau}{2\eta}$$
$$+ \frac{1}{2G}\left[u\frac{\partial \tau}{\partial x} + v\frac{\partial \tau}{\partial y} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)(\sigma_y - \sigma_x)\right]$$

the orders of the respective terms are:

$$\left\{1 + a\left[\left(\frac{u_0}{x_0}\right)^2 + \left(\frac{u_0}{x_0}\right)^2 + \left(\frac{u_0}{t_0} - \frac{u_0}{x_0}\frac{t_0}{x_0}\right)^2\right]\right\}\left(\frac{u_0}{t_0} - \frac{u_0}{x_0}\frac{t_0}{x_0}\right) = \frac{u_0}{t_0}$$

$$+\left(\frac{\eta}{G}\frac{u_0}{t_0}\right)\left[\frac{u_0}{x_0} + \frac{u_0}{x_0} + \frac{u_0}{x_0}\frac{t_0}{x_0} - \frac{u_0}{t_0}\right].$$

The highest order terms of this equation turn out to be:

$$\frac{1}{2}\left(1 + \frac{a}{4}\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial u}{\partial y} = \frac{\tau}{2\eta} + \frac{\sigma_x + p}{2G}\left(\frac{\partial u}{\partial y}\right). \tag{20}$$

Substituting equation (19) into equation (20) yields:

$$\left(1 + \frac{a}{4}\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial u}{\partial y} = \frac{\tau}{\eta}\left(1 + \frac{\eta^2}{G^2}\left(\frac{\partial u}{\partial y}\right)^2\right). \tag{21}$$

The stress-strain rate equations have reduced to the following:

$$\left(1 + \frac{a}{4}\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial u}{\partial y} = \frac{\tau}{\eta}\left(1 + \frac{\eta^2}{G^2}\left(\frac{\partial u}{\partial y}\right)^2\right) \tag{21}$$

and

$$\sigma_x + p = -(\sigma_y + p) = \frac{\eta}{G}\frac{\partial u}{\partial y}\tau.$$

Next consider the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{22}$$

This equation states that when u has been determined then v may be determined.

The only equations that remain are the equilibrium equations. If kinetic energy terms are neglected these equations may be written as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \tag{23}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0. \tag{24}$$

The equilibrium equations may be simplified as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0 \tag{23}$$

$$\frac{\partial p}{\partial x} = \frac{\partial (\sigma_x + p)}{\partial x} + \frac{\partial \tau}{\partial y}.$$

This may be approximated by:

$$\frac{(\Delta p)^x}{2x_0} = \frac{\text{change in pressure along passage}}{\text{length of passage}} = O\left(\frac{\Delta(\sigma_x + p)}{2x_0} + \frac{\tau_{max}}{t_0}\right).$$

$(\Delta p)^x$ is of the order $\Delta(\sigma_x + p)|_{max} + \frac{u_0 \eta x_0}{t_0^2}$,

also $\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0$

$$\frac{\partial p}{\partial y} = \frac{\partial(\sigma_y + p)}{\partial y} + \frac{\partial \tau}{\partial x} = -\frac{\partial(\sigma_x + p)}{\partial y} + \frac{\partial \tau}{\partial x}. \tag{24}$$

This may be approximated by:

$$\frac{(\Delta p)^y}{t_0} = \frac{\text{change of } p \text{ across passage}}{\text{width of passage}} = O\left(-\frac{\Delta(\sigma_x + p)}{t_0} + \frac{\tau_{max}}{x_0}\right).$$

$(\Delta p)^y$ is of the order $\Delta(\sigma_z + p)|_{max} + \frac{u_0 \eta}{x_0}$.

Since $\Delta(\sigma_x + p)|_{max}$ is of the order τ_{max} one finds that $(\Delta p)^x \gg (\Delta p)^y$. The above illustrates that the variation of p with x is much larger than the variation of p with y. Now we make the approximation that

$$p = \text{function of } x \text{ only} = f(x). \tag{25}$$

By disregarding higher order terms in equation (23) one obtains

$$\frac{\partial \tau}{\partial y} = f'(x) \qquad \text{or} \qquad \tau = f'(x)y + \text{constant}.$$

$\tau = 0$ when $y = 0$ (from symmetry).

and, therefore,

$$\tau = f'(x)y = \eta F \frac{y}{t(x)} \quad (26). \text{ Where } \quad F = \frac{t(x) f'(x)}{\eta}. \qquad (27)$$

Substituting equation (26) into equation (21) there results the following:

$$\left(1 + \frac{a}{4}\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial u}{\partial y} = F\frac{y}{t(x)}\left[1 + \frac{\kappa^2}{G^2}\left(\frac{\partial u}{\partial y}\right)^2\right]. \qquad (28)$$

At this point the results of this approximate procedure are collected.

$$\left(1 + \frac{a}{4}\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial u}{\partial y} = F\frac{y}{t(x)}\left[1 + \frac{\kappa^2}{G^2}\left(\frac{\partial u}{\partial y}\right)^2\right] \qquad (28)$$

$$\tau = \eta F \frac{y}{t(x)} \quad (26) \qquad F = \frac{t(x) f'(x)}{\eta} \qquad (27)$$

$$p = f(x) \quad (25) \qquad \sigma_x + p = -(\sigma_y + p) = \frac{\kappa}{G}\tau\frac{\partial u}{\partial y}. \qquad (19)$$

Overall conservation of mass yields a further restriction:

$$\int_0^{t(x)} u\,dy = \text{constant} = u_0 t(x_0). \qquad (29).$$

A numerical procedure may be set up to solve these equations. The procedure is as follows:

1. The geometry, speed, $\frac{\kappa}{G}$ and a must be selected. $\frac{u_0}{t_0}\frac{\kappa}{G}$ must not be of order greater than one.

2. Arbitrarily select a value for F (equation 26). This may be done, to start with, as a value of F occuring in the viscous solution. Now with this value of F and a value of $\frac{\partial u}{\partial y}$ (chosen arbitrarily) equation 28 may be solved. For the selected value of F calculate several values of $\frac{y}{t(x)}$ (enough to give a sufficiently accurate curve of $\frac{du}{dy}$.

versus $\frac{y}{t(x)}$ in the region $0 < \frac{y}{t(x)} < 1$).

3. With the $\frac{\partial u}{\partial y}$ variation with $\frac{y}{t(x)}$ for a prescribed a, $\frac{n}{G}$ and F a numerical integration may be started with the condition that $u = u_0$ at $\frac{y}{t(x)} = 1$. This determines $\frac{u}{t(x)}$ at every point along the section. u is not known yet since $t(x)$ is unknown. Since $\frac{\partial u}{\partial y}$ versus $\frac{y}{t(x)}$ is nearly linear (the viscous case is linear), the trapezoidal rule furnishes a sufficiently accurate method of numerical integration.

4. $\frac{u}{t(x)}$ may now be plotted against $\frac{y}{t(x)}$. If the area under this curve is obtained (by Simpson's rule or some other appropriate means) then $t(x)$ may be determined (from equation 29). One method which is convenient and accurate is to plot $\frac{u}{t(x)} - \frac{u_0}{t(x)}$ versus $\frac{y}{t(x)}$. $\frac{u}{t(x)} - \frac{u_0}{t(x)}$ is the variation of the velocity profile from the uniform velocity profile. If the area under the $\frac{u}{t(x)} - \frac{u_0}{t(x)}$ versus $\frac{y}{t(x)}$ is determined and designated by A then from continuity (equation 29).

$$A\left(t(x)\right)^2 + 2u_0 t(x) - 2u_0 t(x_0) = 0$$

This quadratic may be solved for $t(x)$ and the value of x corresponding to this value of F may be determined as well.

5. Steps 1 through 4 are carried out for enough values of F to obtain a sufficiently accurate plot of F versus x. F is the maximum shear stress for the corresponding values of x so that τ_{max} versus x is known. Also $f'(x)$ in equation (25) may be found as a functions of x by using equation (27). A numerical integration may be used to find $f(x)$ for each value of x . The boundary condition to be used is that $f(x) = 0$ at $x = x_0$.

The scheme outlined above was used to find the solution of a problem with fixed geometry, viscosity and speed of rolling but with variable G .

The numerical values used for the geometry were:

Radius of rolls $= 14''$ *

$2t_o = .100''$

roll surface speed $= 100\ ''/sec$ and $x_o = .446''$.

Other parameters used were:

$$\frac{u_o}{t_o}\frac{n}{G} = \sqrt{.1},\ \sqrt{.5},\ 1.0 \qquad \text{(of order 1)} .$$

For each value of $\frac{u_o}{t_o}\frac{n}{G}$ the following values of a were used:

$$a = 0,\ a = n^2/G^2,\ a = 2.5\frac{n^2}{G^2},\ a = 4\frac{n^2}{G^2}.$$

The results obtained are tabulated in Tables I to X. Plots of all results are shown for $a=0$ and $\frac{u_o}{t_o}\frac{n}{G}=1$ in Figures 6 through 13. Plots of $\frac{\partial u}{\partial y}$ and $\frac{u}{t(x)} - \frac{u_o}{t(x)}$ versus $\frac{y}{t(x)}$ for $F=800$ are shown for the remaining cases in Figures 14 through 22. The dotted line on the $\frac{y}{t(x)}$ versus $\frac{\partial u}{\partial y}$ curve represents the viscous solution for the same value of F. Pressure distributions and τ max. distributions for the remaining cases are shown in Figures 23 through 38. Values given in the tables for the force pushing the rolls apart is the force exerted in the passage $(-x_o \leq x \leq x_o)$ only. For velocity distributions and positions in the passage for values of F below those tabulated the viscous solution is used.

Since $\frac{n}{G}\frac{u_o}{x_o}$, $a\frac{u_o}{x_o}$ and $\frac{t_o}{x_o}$ are small, a perturbation procedure seems in order. This is carried out in the Appendix. Unfortunately, the most conveniently chosen parameters restrict the solution far more than the approximate procedure given in this section.

* These values are reported by Finston in Reference 2. to be appropriate.

CONCLUSIONS

Figures 14 through 22 illustrate clearly the effect of replacing a viscous material by an elastic-viscous material of the type considered. The most important result shown here is the variation in $\frac{\partial u}{\partial y}$ from the viscous case at the surface of the material. The indication from these calculations is that $\frac{\partial u}{\partial y}$ varies across the section in a different manner (in most cases) from the viscous case. The percent difference between $\frac{\partial y}{\partial y}\Big|_{max}$ for the elastic viscous case and the viscous case was from 0 to 14% for equal values of maximum shearing stress. The percent difference depends on the value of a and the speed of operation. The percent difference goes up as the speed is increased and down as a in increased.

The difference in variation in $\frac{\partial u}{\partial y}$ across the section for an elastic-viscous material as compared to a viscous material explains the failure of the viscous solution in making suitable predictions regarding the behavior of an elastic viscous material. The calculations carried out are to indicate how the elastic-viscous solution varies from the viscous solution. The approximate method included here certainly indicates which direction the variables go from the viscous variables if a suitable value of a is known.

The results of the approximate solution indicate that the pressure and shear stress drop as G is lowered. This is because the viscosity is held constant and the lowering of G corresponds to a reduction of stiffness of the material. In other words, the material may accomodate a given load by elastic as well as viscous deformation so that the load necessary for a prescribed time dependent deformation will in general go down as G decreases. Therefore, these results show a slight decrease in total lateral force when G decreases (maximum decrease of 7.5% for these calculations).

The parameter $\frac{u_0}{t_0}\frac{\eta}{G}$ is seen to be an important parameter in this problem. This is not too surprising. $\left.\frac{\partial u}{\partial y}\right|_{max}$ and $\frac{\eta}{G}$ are certainly two major parameters and the solution would depend strongly on some combination of these since they are related through the stress-strain rate equation.

$\frac{u_0}{t_0}$ is the order of $\left.\frac{\partial u}{\partial y}\right|$ max.

The reader should note that this approximate solution does not allow the determination of the elastic recovery after the material has left the rolls.

This solution only includes the solution of the velocities and the stresses. The associated heat transfer problem was not undertaken but for values of $a < 4\frac{\eta^2}{G^2}$ one would anticipate an increase of peak temperature over the viscous case because the maximum shear stress is larger for the elastic viscous material.

This solution should be of use to manufactures who use the calendering process. The knowledge of how the variables of the problem change when the material has elastic as well as viscous properties may be used as a guide in selecting the speed of rolling, the temperature of the bulk to be rolled and the amount of cooling of the rolls necessary.

BIBLIOGRAPHY

1. R. E. Gaskell, "The Calendering of Plastic Materials," Journal of Applied Mechanics, Vol. 17, p. 334, 1950.

2. M. Finston, "Thermal Effects in Calendering Viscous Fluids," Journal of Applied Mechanics, Vol. 18, P. 12, 1951.

3. G. F. Carrier, "Boundary Layer Problems in Applied Mechanics," Advances in Applied Mechanics Volume III, R. von Mises and T. von Karman, P. 1, New York, Academic Press Inc., 1953.

4. H. Fromm, "Laminare Stromung Newtonscher und Maxwellscher Flussigkeiten," Zeitschrift fur Angewandte Mathematik und Mechanik, Band 25/27, Seite 147, 1947.

5. H. Fromm, "Laminare Stromung Newtonscher und Maxwellscher Flussigkeiten," Zeitschrift fur Augewandte Mathematik und Mechanik, Band 28, Seite 43-54, 1948.

6. G. G. H. Ham and A. Van Rossen, "Rheological Properties of Rubber in Non-vulcanized State," Proceedings International Congress on Rheology, 1948, p. II-210, New York, Interscience Publishers Inc., 1949.

7. F. Stainsby and A. G. Ward, "An Absolute Viscometer for Measuring the Viscoelastic Properties of Concentrated High Polymer Solutions," Proceedings International Congress on Rheology, 1948, P. II-108, New York Interscience Publishers Inc., 1949.

LIST OF TABLES

TABLE I

Numerical solution of equations 19, 25, 36, 27, 38 & 29

For $a=0$ $\qquad \frac{u_0}{c_0}\frac{h}{G}=\sqrt{.1}$

F	600	700	800
$t(x)$	.0518	.0509	.0503
x	.218	.162	.080
y/ε	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$
0	0	0	0
.1	60	70	80
.2	120	140	162
.3	180	209	242
.4	240	279	319
.5	300	349	400
.6	361	419	483
.7	421	494	565
.8	483	565	647
.9	544	636	728
1.0	607	708	812
y/ε	$\frac{u-u_0}{\varepsilon}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\varepsilon}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\varepsilon}\downarrow\left(\frac{1}{sec}\right)$
0	301	352	403
.1	298	349	399
.2	289	338	387
.3	274	324	267
.4	253	279	339
.5	226	265	303
.6	193	227	259
.7	154	180	206
.8	109	127	146
.9	58	67	77

x	$\frac{q}{x}\left(\frac{1}{sec}\right)$	$-\frac{q}{\varepsilon}\Big\vert_{lower}\left(\frac{1}{sec}\right)$	$\frac{q}{\varepsilon}\Big\vert_{lower}\left(\frac{1}{sec}\right)$
-.446	9190	9190	9190
-.40	9140	9130	9140
-.35	8960	8940	8970
-.30	8640	8610	8660
-.25	8190	8140	8230
-.20	7630	7560	7690
-.15	6970	6890	7050
-.10	6230	6130	6330
-.05	5430	5320	5540
0	4600	4480	4710
.05	3760	3650	3870
.10	2960	2870	3060
.15	2220	2140	2300
.20	1570	1500	1630
.25	1000	960	1050
.30	650	530	580
.35	240	220	250
.40	60	50	60
.446	0	0	0

$\frac{\text{Lateral force on rolls}}{\varepsilon} = 4180\,\frac{in^2}{sec}$

TABLE II

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a=0$ $\quad \frac{u_0 t}{\tau_0 G} = \sqrt{.5}$

F	300	400	500	600	700	800
$t(x)$	.0541	.0533	.0524	.0516	.0508	.0501
x	.340	.300	.258	.212	.150	.046
y/t	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$
0	0	0	0	0	0	0
.1	30	40	50	60	70	30
.2	60	80	100	120	140	160
.3	90	120	150	180	211	242
.4	120	160	202	242	283	324
.5	150	202	252	304	355	408
.6	180	242	304	365	430	494
.7	211	282	355	429	505	583
.8	242	324	405	495	583	677
.9	273	366	462	562	663	774
1.0	304	408	517	630	749	877
y/t	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$
0	151	202	254	307	363	418
.1	144	200	252	304	358	414
.2	145	194	244	295	348	402
.3	137	184	232	280	330	382
.4	127	170	214	259	305	354
.5	113	152	192	232	273	317
.6	97	130	164	198	234	272
.7	77	104	131	159	187	218
.8	55	73	92	112	133	155
.9	29	39	49	60	71	83

TABLE II (continued)

FOR $a=0$ $\frac{u_0}{\ell_0}\frac{\eta}{G}=\sqrt{.5}$

x	$\frac{p}{\eta}\left(\frac{1}{sec}\right)$	$-\frac{\sigma_x}{\eta}\Big\|_{max}\left(\frac{1}{sec}\right)$	$\frac{\sigma_y}{\eta}\Big\|_{max}\left(\frac{1}{sec}\right)$
-.446	8980	8980	8980
-.40	8920	8920	8930
-.35	8740	8710	8770
-.30	8430	8380	8480
-.25	8000	7800	8090
-.20	7440	7310	7580
-.15	6800	6620	6970
-.10	6080	5870	6280
-.05	5300	5080	5530
0	4490	4250	4730
.05	3680	3450	3900
.10	2900	2700	3110
.15	2180	2010	2360
.20	1540	1400	1670
.25	980	890	1080
.30	550	500	600
.35	240	210	270
.40	60	50	60
.446	0	0	0

$$\frac{\text{Lateral force on rolls}}{\eta} = 4140\,\frac{in^2}{sec}$$

TABLE III

Numerical solution of equations 14, 25, 26, 27, 28 & 29

For $a=0$ $\frac{u_0}{t_0}\frac{\eta}{G}=1.0$

x	$\frac{p}{\eta}\left(\frac{1}{sec}\right)$	$-\frac{\sigma_x}{\eta}\left(\frac{1}{sec}\right)$	$-\frac{\sigma_y}{\eta}\left(\frac{1}{sec}\right)$
-.446	8650	8650	8650
-.40	8590	8580	8600
-.35	8410	8370	8440
-.30	8100	8020	8180
-.25	7670	7550	7800
-.20	7140	6960	7320
-.15	6520	6300	6750
-.10	5830	5570	6100
-.05	5090	4800	5380
0	4320	4020	4630
.05	3550	3260	3840
.10	2810	2550	3080
.15	2120	1900	2350
.20	1500	1330	1680
.25	970	850	1100
.30	550	470	630
.35	240	200	280
.40	60	50	60
.446	0	0	0

$$\frac{\text{Lateral force on rolls}}{\eta} = 4030\ \frac{in^2}{sec}$$

TABLE IV

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a = \frac{\kappa^2}{\delta^2}$ $\quad \frac{u_0^2}{\epsilon_0}\frac{\kappa}{\delta} = \sqrt{.1}$

F	300	500	750	800
$t(x)$	.0541	.0525	.0506	.0503
x	.341	.264	.130	.082
$\frac{y}{\delta(x)}$	$\frac{du}{dy}\big\vert_\ell\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\big\vert_\ell\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\big\vert_\ell\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\big\vert_\ell\left(\frac{1}{sec}\right)$
0	0	0	0	0
.1	30	50	76	80
.2	60	100	150	160
.3	90	120	225	241
.4	126	200	300	321
.5	180	300	452	483
.7	210	350	528	564
.8	241	401	604	645
.9	270	451	681	726
1.0	301	502	758	809
$^{y}/_{\delta}$	$\frac{u-u_0}{\delta}\big\vert_\ell\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\big\vert_\ell\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\big\vert_\ell\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\big\vert_\ell\left(\frac{1}{sec}\right)$
0	150	250	378	403
.1	149	248	374	399
.2	144	240	362	387
.3	137	228	344	367
.4	126	210	318	338
.5	113	188	284	302
.6	96	160	242	258
.7	77	128	193	206
.8	54	90	137	145
.9	29	48	72	77
1.0	0	0	0	0

x	$\frac{Q}{\ell}\left(\frac{in^2}{sec}\right)$	$\frac{Q}{\ell}\big\vert_{max}\left(\frac{in^2}{sec}\right)$	$\frac{Q}{\ell}\big\vert_{min}\left(\frac{in^2}{sec}\right)$
−.446	9180	9180	9180
−.40	9120	9120	9130
−.35	8940	8930	8950
−.30	8620	8600	8650
−.25	8180	8140	8230
−.20	7620	7560	7690
−.25	6960	6880	7040
−.10	6220	6120	6320
0	4590	4480	4700
.05	3760	3650	3860
.10	2960	2860	3060
.15	2220	2140	2300
.20	1560	1490	1620
.25	1000	950	1040
.30	560	530	580
.35	240	230	250
.40	60	50	60
.446	0	0	0

$$\frac{\text{Lateral force on rolls}}{\ell} = 4180 \ \frac{in^2}{sec}$$

This case is very close to the viscous case

TABLE V

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a = \frac{K^2}{G^2}$ $\frac{u_0}{\epsilon_0}\frac{k}{G} = \sqrt{.5}$

F	300	400	500	600	700	800
$\delta(x)$	.0541	.0533	.0524	.0517	.0509	.0501
k	.341	.302	.262	.215	.157	.061
y/δ	$\frac{\partial u}{\partial y}\delta\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\delta\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\delta\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\delta\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\delta\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\delta\left(\frac{1}{sec}\right)$
0	0	0	0	0	0	0
.1	30	40	50	60	70	80
.2	60	80	100	121	140	160
.3	90	121	150	181	210	241
.4	121	160	200	241	282	323
.5	150	200	251	302	354	404
.6	180	241	302	364	427	491
.7	210	282	354	427	502	577
.8	241	323	406	491	577	666
.9	271	369	459	556	655	758
1.0	302	406	516	622	734	853
y/δ	$\frac{u-u_0}{\delta}\delta\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\delta\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\delta\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\delta\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\delta\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\delta}\delta\left(\frac{1}{sec}\right)$
0	150	202	253	305	358	413
.1	144	200	250	302	355	409
.2	144	193	243	293	344	397
.3	137	183	230	278	327	377
.4	126	169	213	257	302	348
.5	113	151	190	230	270	312
.6	96	129	163	197	232	267
.7	77	103	130	157	183	214
.8	54	73	92	111	131	152
.9	29	38	49	59	69	81
1.0	0	0	0	0	0	0

TABLE V (continued)

n	$\frac{p}{\eta}\left(\frac{1}{sec}\right)$	$\left.-\frac{\partial u}{\partial y}\right\rvert_{max}\left(\frac{1}{sec}\right)$	$\left.-\frac{\partial u}{\partial y}\right\rvert_{min}\left(\frac{1}{sec}\right)$
-.446	9070	9070	9070
-.40	9010	9010	9020
-.35	8630	8800	8860
-.30	8520	8460	8580
-.25	8080	7980	8180
-.20	7520	7380	7660
-.15	6870	6690	7050
-.10	6140	5930	6350
-.05	5350	5120	5580
0	4540	4300	4780
.05	3720	3490	3950
.10	2940	2720	3140
.15	2200	2030	2380
.20	1550	1410	1690
.25	990	900	1080
.30	560	500	610
.35	240	220	270
.40	60	50	70
.446	0	0	0

$$\frac{\text{Lateral force on rolls}}{\eta} = 4190\ \frac{in^2}{sec}$$

TABLE VI

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a = \frac{h^2}{G^2}$ [illegible] $= 1.0$

F	300	400	500	600	700	800
$f(x)$	.0541	.0533	.0524	.0515	.0507	.0500
x	.340	.302	.258	.204	.042	0
y/t	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$
0	0	0	0	0	0	0
.1	30	40	50	60	71	80
.2	60	80	100	120	140	160
.3	90	120	150	180	212	243
.4	120	160	201	243	289	326
.5	151	201	252	305	358	413
.6	181	243	305	369	435	502
.7	212	284	358	435	514	596
.8	243	326	412	502	596	696
.9	274	369	468	572	683	804
1.0	305	413	525	646	776	920
y/t	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$
0	151	203	256	315	368	428
.1	150	201	253	312	364	424
.2	145	195	246	303	354	412
.3	138	185	283	288	336	392
.4	127	171	216	267	312	364
.5	119	153	193	239	279	327
.6	97	131	165	206	240	281
.7	77	104	132	165	192	226
.8	55	74	94	119	137	161
.9	29	39	50	61	73	86
1.0	0	0	0	0	0	0

TABLE VI

α	$\frac{p}{\rho}\left(\frac{1}{sec}\right)$	$\frac{-\delta p}{\rho}\Big/_{max}\left(\frac{1}{sec}\right)$	$\frac{-\delta_1}{\rho}\Big/_{max}\left(\frac{1}{sec}\right)$
-.446	8830	8830	8830
-.40	8780	8770	8780
-.35	8600	8560	8630
-.30	8280	8200	8360
-.25	7850	7720	7980
-.20	7310	7120	7490
-.15	6670	6430	6910
-.10	5970	5690	6250
-.05	5200	4900	5510
0	4420	4100	4730
.05	3630	3320	3940
.10	2870	2590	3150
.15	2160	1920	2400
.20	1530	1340	1720
.25	930	850	1120
.30	550	470	630
.35	240	200	270
.40	60	50	60
.446	0	0	0

$$\frac{\text{Lateral force on rolls}}{\rho} = 4120 \, \frac{in^2}{sec}$$

TABLE VII

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a = 2.5\ \frac{in^2}{sec}$ $\quad \frac{u_0}{c_0}$ [illegible] $= \sqrt{.1}$

F	600	700	800
$\varepsilon(\alpha)$	.0517	.0510	.0503
α	.220	.168	.095
y/ε	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$	$\frac{\partial u}{\partial y}\downarrow\left(\frac{1}{sec}\right)$
0	0	0	0
.1	60	70	80
.2	120	140	160
.3	180	210	240
.4	240	280	321
.5	300	350	401
.6	361	421	481
.7	421	491	562
.8	491	562	642
.9	542	633	723
1.0	602	703	805
y/ε	$\frac{u-u_0}{\varepsilon}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\varepsilon}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{\varepsilon}\downarrow\left(\frac{1}{sec}\right)$
0	300	351	401
.1	329	347	397
.2	289	337	385
.3	274	319	365
.4	253	295	337
.5	226	263	301
.6	192	229	257
.7	153	179	205
.8	108	127	145
.9	57	67	76
1.0	0	0	0

x	$\frac{p}{\eta}\left(\frac{1}{sec}\right)$	[illegible] $\left(\frac{1}{sec}\right)$	[illegible] $\left(\frac{1}{sec}\right)$
−.446	9320	9320	9320
−.40	9270	9270	9270
−.35	9080	9070	9100
−.30	8760	8740	8790
−.25	8310	8260	8360
−.20	7740	6670	7810
−.15	7070	6980	7150
−.10	6310	6210	6410
−.05	5500	5390	5610
0	4670	4550	4780
.05	3820	3710	3930
.10	3010	2910	2110
.15	2260	2170	2340
.20	1580	1520	1650
.25	1010	970	1060
.30	560	530	590
.35	290	230	250
.40	60	50	60
.446	0	0	0

$$\frac{\text{Lateral force on rolls}}{\eta} = 4240\ \frac{in^2}{sec}$$

TABLE VIII

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a = 2.5 \frac{in^2}{G^2}$ $\frac{u_0}{t_0}\frac{K}{G} = \sqrt{.5}$

F	500	600	700	800
$t(a)$	.0525	.0517	.0509	.0502
κ	.263	.218	.161	.082
y/t	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$	$\frac{du}{dy}\downarrow\left(\frac{1}{sec}\right)$
0	0	0	0	0
.1	50	60	70	80
.2	100	120	140	160
.3	150	180	210	241
.4	200	241	281	322
.5	251	301	352	403
.6	300	362	424	485
.7	352	424	496	568
.8	403	485	568	652
.9	454	547	642	738
1.0	506	610	716	824
y/t	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t}\downarrow\left(\frac{1}{sec}\right)$
0	251	303	354	406
.1	249	300	351	402
.2	241	241	340	390
.3	229	276	323	370
.4	211	255	298	342
.5	189	227	266	306
.6	161	194	228	261
.7	129	155	182	209
.8	91	110	128	148
.9	48	58	69	78
1.0	0	0	0	0

x	$\frac{p}{\eta}\left(\frac{1}{sec}\right)$	$\frac{\sigma_x}{\eta}\left(\frac{1}{sec}\right)$	$\frac{\sigma_y}{\eta}\left(\frac{1}{sec}\right)$
-.446	9170	9170	9170
-.40	9120	9110	9120
-.35	8930	8910	8960
-.30	8620	8560	8680
-.25	8170	8070	8270
-.30	7610	7470	7750
-.15	6950	6770	7130
-.10	6210	6000	6430
-.05	5410	5480	5650
0	4580	4340	4830
.05	3760	3520	4000
.10	2860	2740	3180
.13	2220	2040	2400
.20	1560	1420	1700
.25	1000	900	1100
.30	550	490	610
.35	240	210	260
.40	60	50	60
.396	0	0	0

$\frac{\text{Lateral forces on rolls}}{\eta} = 4340 \frac{in^1}{sec}$

TABLE IX

Numerical solution of equations 19, 25, 26, 27, 28 & 29

For $a = 2.5\,\frac{h^2}{G^2}$ $\quad \frac{u_0}{t_0}\frac{h}{G} = 1.0$

F	400	500	600	700	800
$t(k)$	.0533	.0524	.0517	.0509	.0501
k	.304	.262	.215	.157	.064
y/t	$\frac{du}{dy} \downarrow \left(\frac{1}{sec}\right)$	$\frac{du}{dy} \downarrow \left(\frac{1}{sec}\right)$	$\frac{du}{dy} \downarrow \left(\frac{1}{sec}\right)$	$\frac{du}{dy} \downarrow \left(\frac{1}{sec}\right)$	$\frac{du}{dy} \downarrow \left(\frac{1}{sec}\right)$
0	0	0	0	0	0
.1	40	50	60	70	80
.2	80	100	120	140	160
.3	120	150	181	211	241
.4	160	201	241	282	323
.5	201	251	302	354	406
.6	241	302	364	427	490
.7	282	354	427	501	576
.8	323	406	490	576	666
.9	364	458	555	654	755
1.0	406	512	620	732	843
y/t	$\frac{u-u_0}{t} \downarrow \left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t} \downarrow \left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t} \downarrow \left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t} \downarrow \left(\frac{1}{sec}\right)$	$\frac{u-u_0}{t} \downarrow \left(\frac{1}{sec}\right)$
0	201	253	305	358	412
.1	199	250	302	355	408
.2	193	243	293	344	396
.3	183	230	278	327	376
.4	169	213	257	302	347
.5	151	190	230	270	311
.6	129	163	196	231	267
.7	103	130	157	185	213
.8	73	92	111	131	151
.9	39	49	59	69	80
1.0	0	0	0	0	0

TABLE IX (continued)

x	$\frac{p}{\eta}\left(\frac{1}{sec}\right)$	$\frac{-\sigma_x}{\eta_{max}}\left(\frac{1}{sec}\right)$	$\frac{-\sigma_y}{\eta_{max}}\left(\frac{1}{sec}\right)$
-.446	9030	9090	9090
-.40	9040	9030	9050
-.35	8860	8820	8890
-.30	8540	8460	8620
-.25	8100	7960	8230
-.20	7540	7340	7740
-.15	6880	6630	7190
-.10	6150	5850	6440
-.05	5360	5030	5690
0	4550	4210	4890
.05	3730	3410	4060
.10	2950	2650	3240
.15	2210	1960	2470
.20	1560	1360	1760
.25	1000	860	1190
.30	560	470	640
.35	240	200	270
.40	60	50	60
.446	0	0	0

$$\frac{\text{Lateral force on rolls}}{\eta} = 4240 \ \frac{in^2}{sec}$$

TABLE X

Numerical solution of equations 19, 25, 26, 27, 28 & 29

F	$t(\alpha)$	x_r
100	.0560	.410
200	.0550	.374
300	.0542	.342
400	.0533	.304
500	.0525	.265
566	.0520	.238
647	.0514	.202
703	.0510	.170
777	.0505	.118
821	.0502	.065
852	.0500	0

x	$\frac{p}{k}\left(\frac{1}{sec}\right)$
-.446	9310
-.40	9250
-.35	9070
-.30	8760
-.25	8310
-.20	7740
-.15	7060
-.10	6310
-.05	5500
0	4660
.05	3810
.10	3000
.15	2250
.20	1580
.25	1000
.30	560
.35	240
.40	60
.446	0

For $a = 4\frac{h^2}{G^2}$

all values of $\frac{u_0 \, h_1}{t_0 \, G}$

$$\frac{\partial u}{\partial y} = F\frac{y}{t(\alpha)}$$

$$u = u_0 + \left(1 - \frac{y^2}{y_1^2}\right) F \frac{t}{2}$$

when $\frac{k}{G}=0$ this is the viscous case

for other values of $\frac{k}{G}$

	$\frac{u_0}{c_0}\frac{k}{G}=\sqrt{.1}$		$\frac{u_0}{c_0}\frac{k}{G}=\sqrt{.5}$		$\frac{u_0}{c_0}\frac{k}{G}=1.0$	
x	$-\frac{\sigma_y}{q}/_{max}\left(\frac{1}{sec}\right)$	$-\frac{\sigma_y}{q}/_{max}\left(\frac{1}{sec}\right)$	$-\frac{\sigma_y}{q}/_{max}\left(\frac{1}{sec}\right)$	$-\frac{\sigma_y}{q}/_{max}\left(\frac{1}{sec}\right)$	$\frac{\sigma_y}{q}/_{max}\left(\frac{1}{sec}\right)$	$\frac{\sigma_y}{q}/_{max}\left(\frac{1}{sec}\right)$
-.446	9310	9320	9310	9310	9310	9310
-.40	9250	9560	9250	9260	9250	9260
-.35	9060	9080	9040	9100	9030	9110
-.30	8730	8780	8600	8830	8670	8840
-.25	8260	8350	8210	8410	8160	8450
-.20	7670	7800	7590	7890	7530	7950
-.15	6980	7150	6870	7250	6790	7330
-.10	6210	6410	6090	6530	5990	6620
-.05	5390	5610	5250	5740	5150	5850
0	4540	4770	4400	4910	4290	5020
.05	3700	3920	3570	4060	3470	4160
.10	2910	3100	2780	3230	2690	3320
.15	2170	2340	2060	2440	1980	2520
.20	1510	1640	1430	1720	1360	1790
.25	960	1050	900	1000	860	1150
.30	530	580	500	620	480	640
.35	230	250	220	270	200	280
.40	60	60	50	60	50	70
.446	0	0	0	0	0	0
	Lateral force on rolls = 4240#		Lateral force on rolls = 4310#		Lateral force on rolls = 4350#	

LIST OF GRAPHS

27. $\frac{p}{\eta}$ and $-\frac{\sigma_y}{\eta}\Big|_{max.}$ versus x for $a = 0$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.1}$

28. $a = 0$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.5}$

29. $a = 0$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = 1$

30. $a = \frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.1}$

31. $a = \frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.5}$

32. $a = \frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = 1$

33. $a = 2.5\frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.1}$

34. $a = 2.5\frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.5}$

35. $a = 2.5\frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = 1.0$

36. $a = 4\frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.1}$

37. $a = 4\frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = \sqrt{.5}$

38. $a = 4\frac{\mu^2}{\sigma^2}$ and $\frac{\omega_0}{\omega}\frac{\mu}{\sigma} = 1$

Figure 1

Plot of $(\frac{\tau}{G})$ Versus $(\frac{\eta}{G}\dot{\gamma})$ for Maxwell Fluid Between Parallel Plates

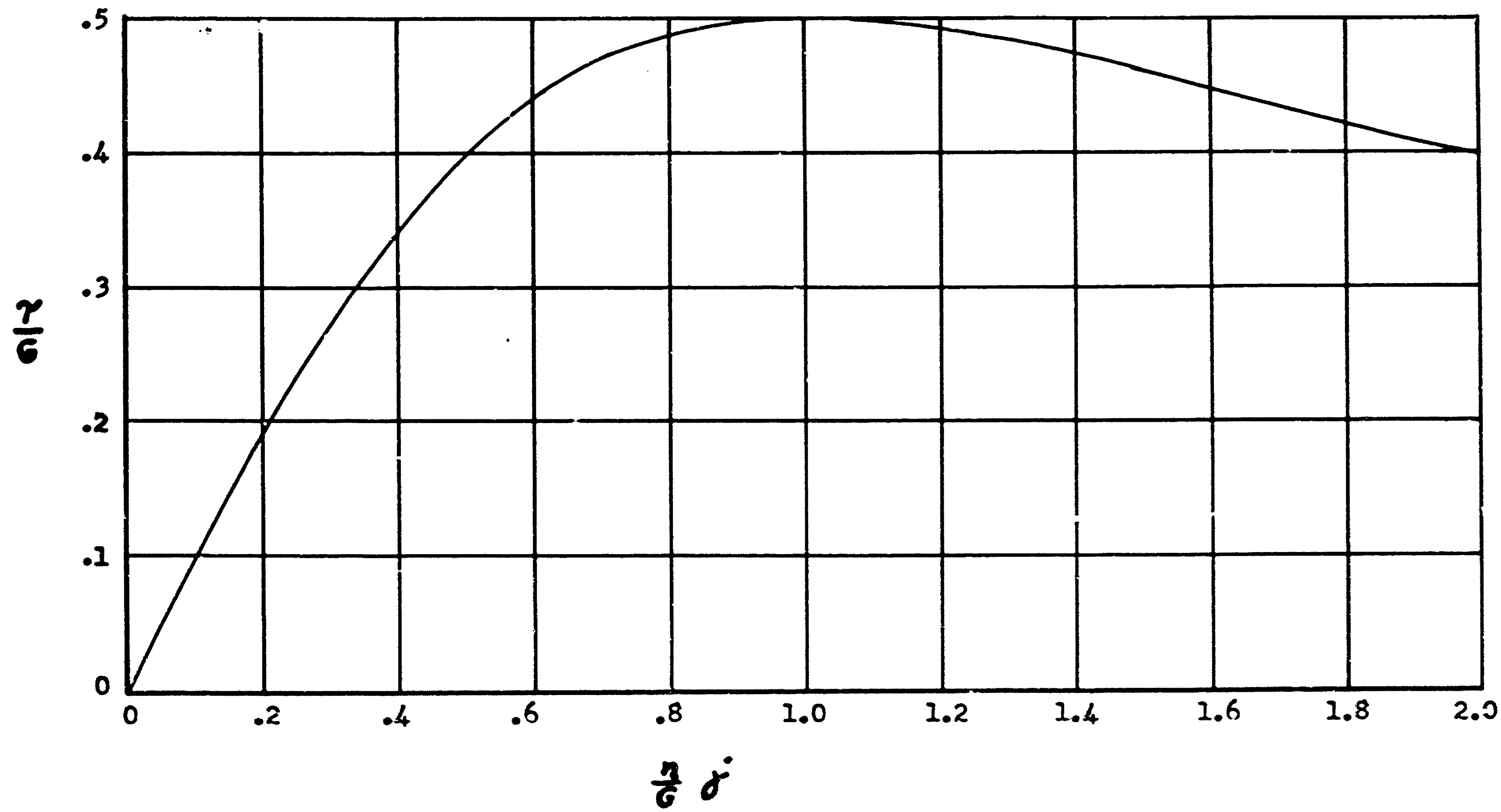


Figure 2

Tension Test with Constant Stress

$\frac{\sigma_{xx}}{3\eta}$ versus $\dot{e}_{xx}$

$$(1+\alpha J_2)\dot{e}_{ij} = \frac{\dot{S}_{ij}}{2G} + \frac{S_{ij}}{2\eta}$$

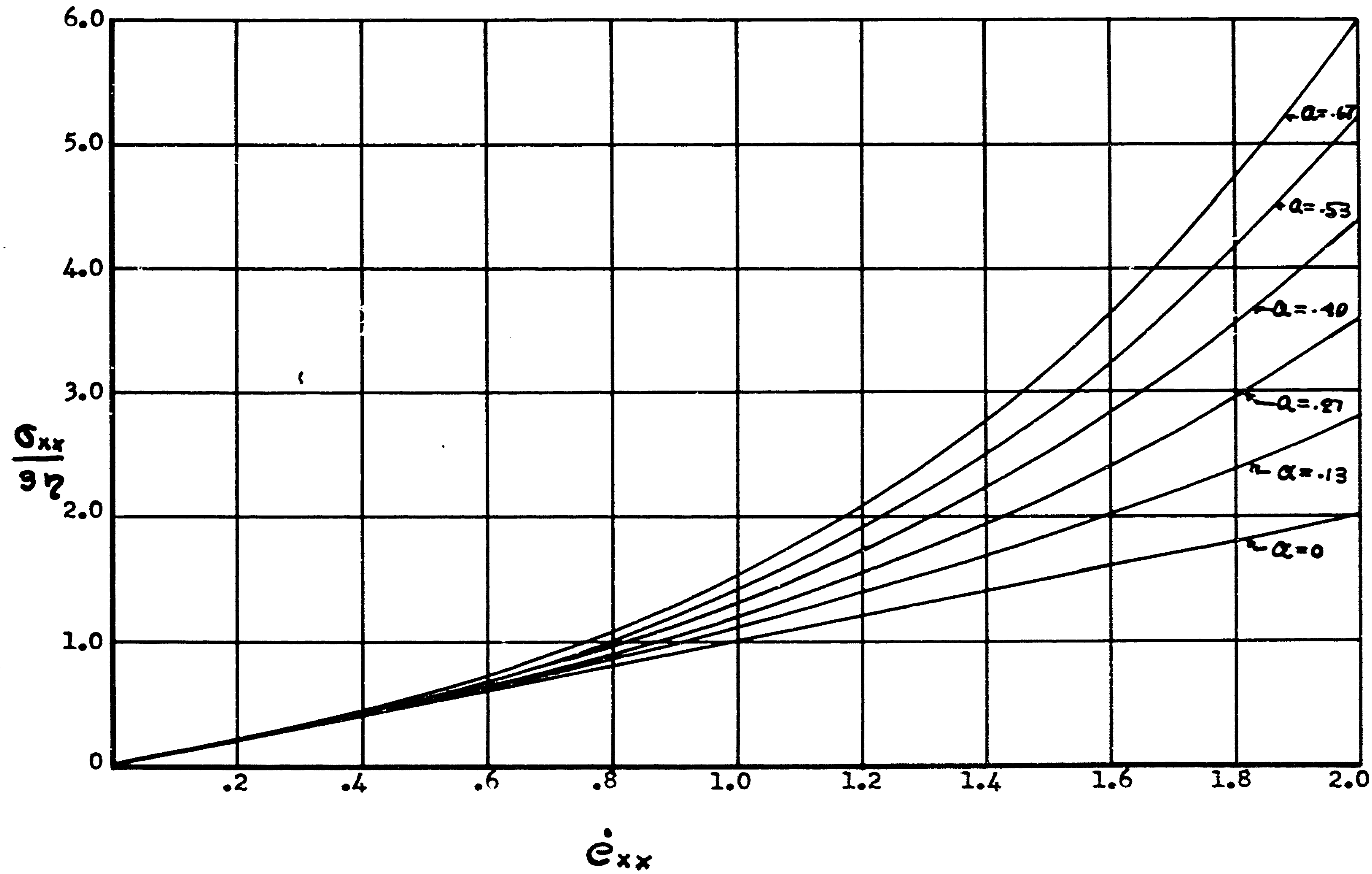


Figure 3

Relaxation Test in Tension with Constant Strain

$$\left(1+\alpha J_2\right)\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2G} + \frac{s_{ij}}{2\eta}$$

$\frac{\text{Normal Stress at Time} = t}{\text{Normal Stress at Time} = 0}$ versus $\frac{\text{Time}}{\eta/G}$

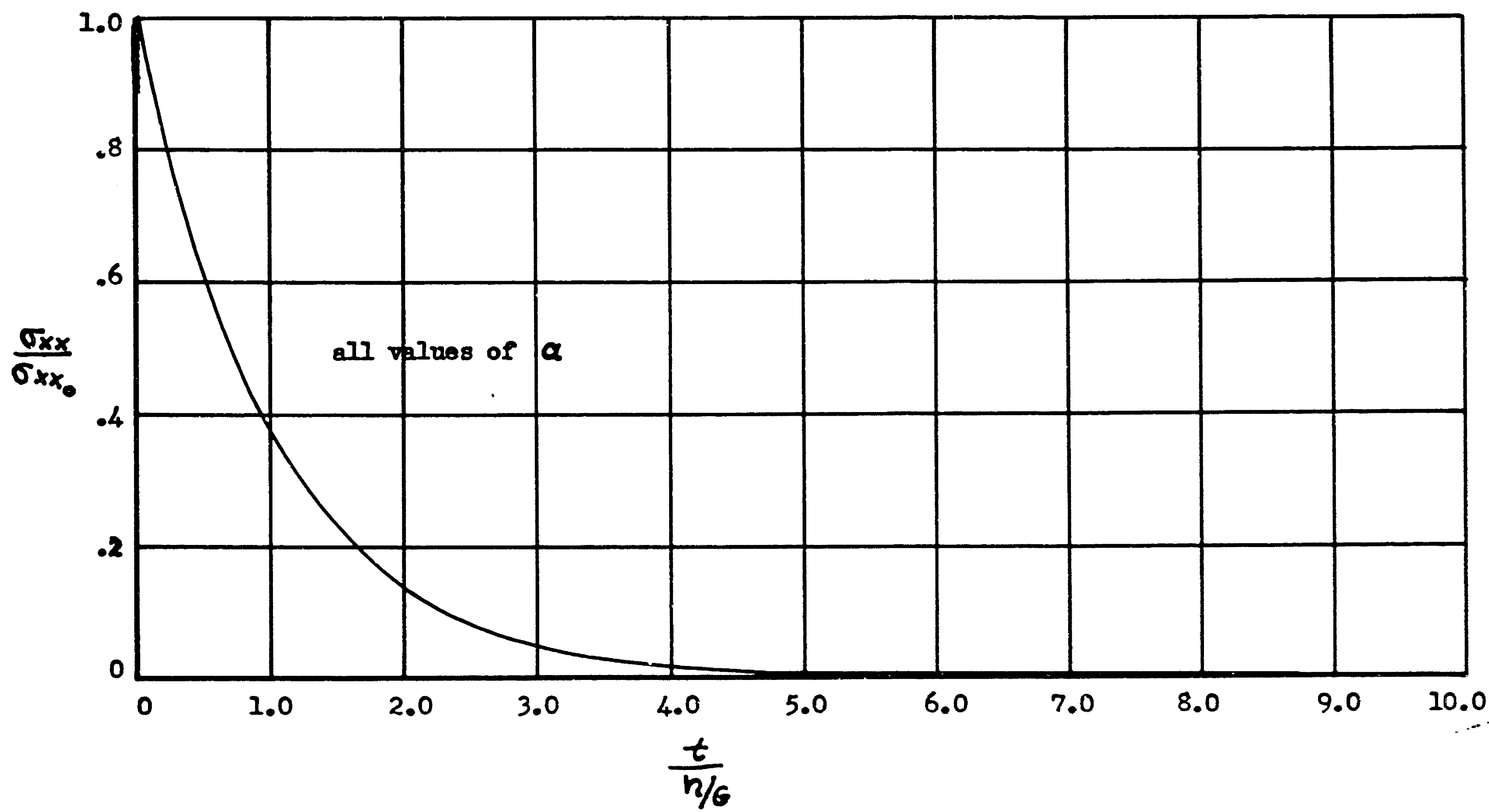


Figure 4
Fluid between Two Parallel Plates

$$(1+a J_2)\,\dot{e}_{ij} = \frac{\dot{S}_{ij}}{2G} + \frac{S_{ij}}{2\eta}$$

Shear Stress / G versus Shear Strain Rate / (G/η)

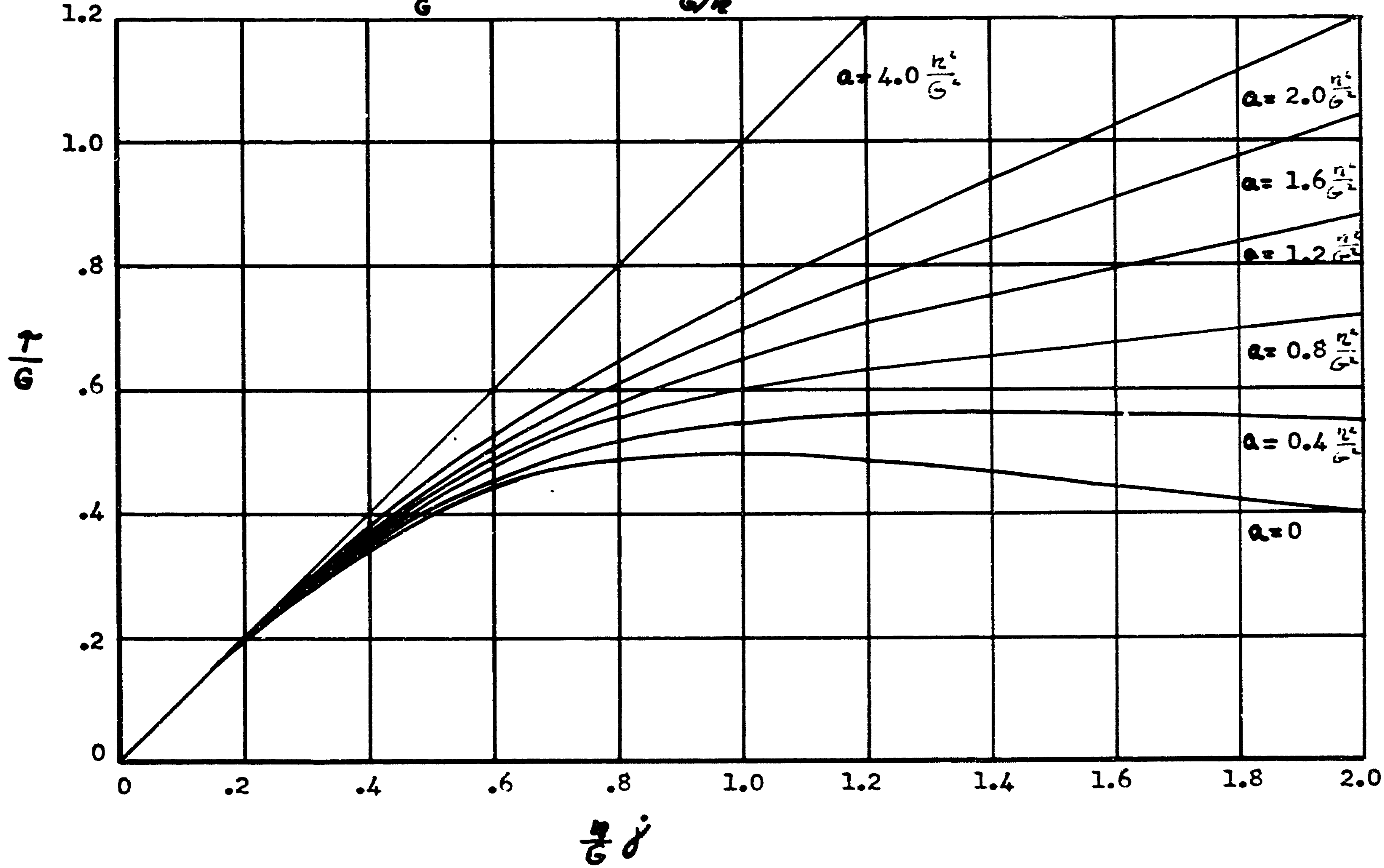


Figure 5

Fluid between Two Parallel Plates

$$(1 + a J_2)\dot{e}_{ij} = \frac{\dot{S}_{ij}}{2G} + \frac{S_{ij}}{2\eta}$$

$\frac{\text{Normal Stress}}{G}$ versus $\frac{\text{Shear Strain Rate}}{G/\eta}$

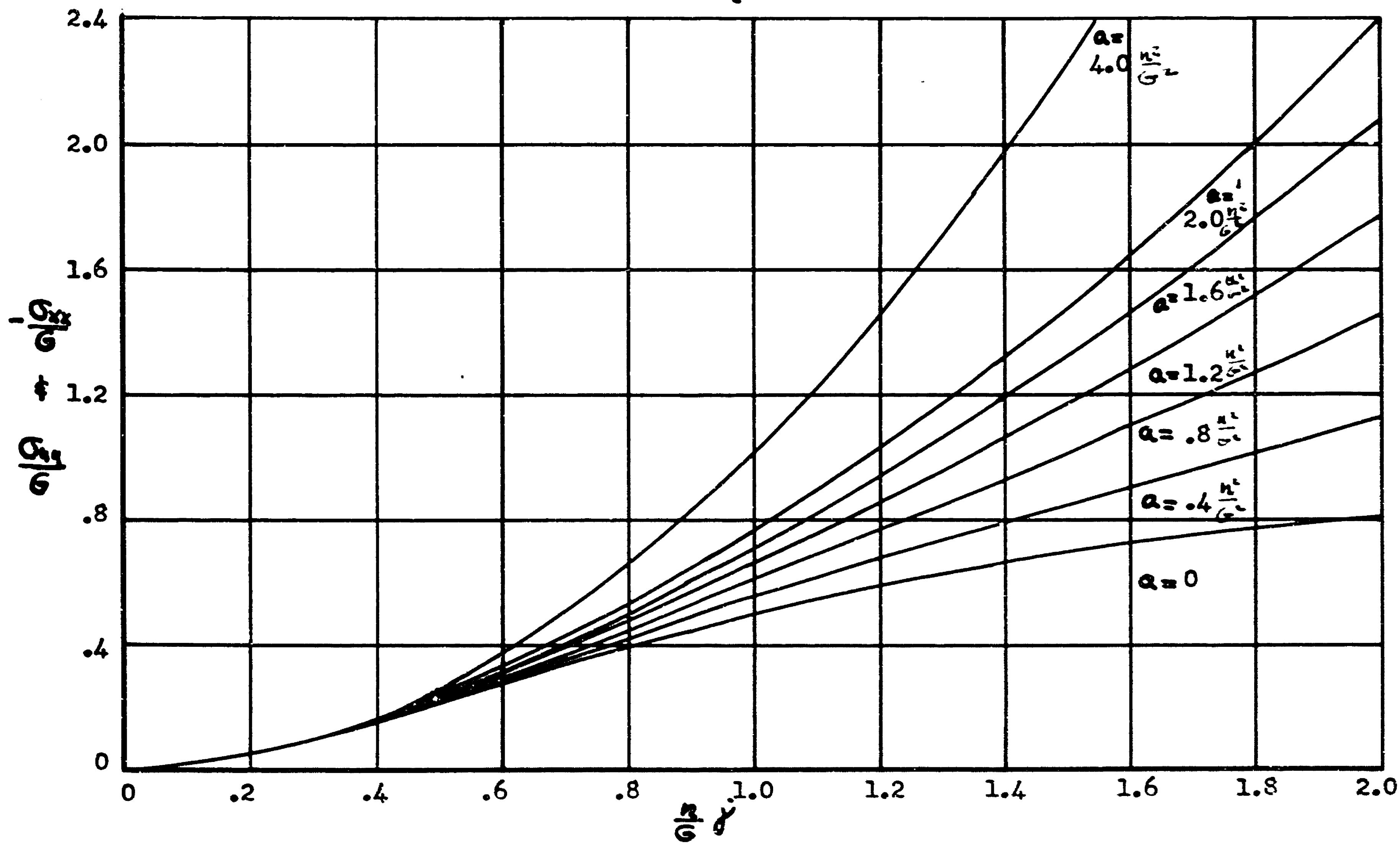
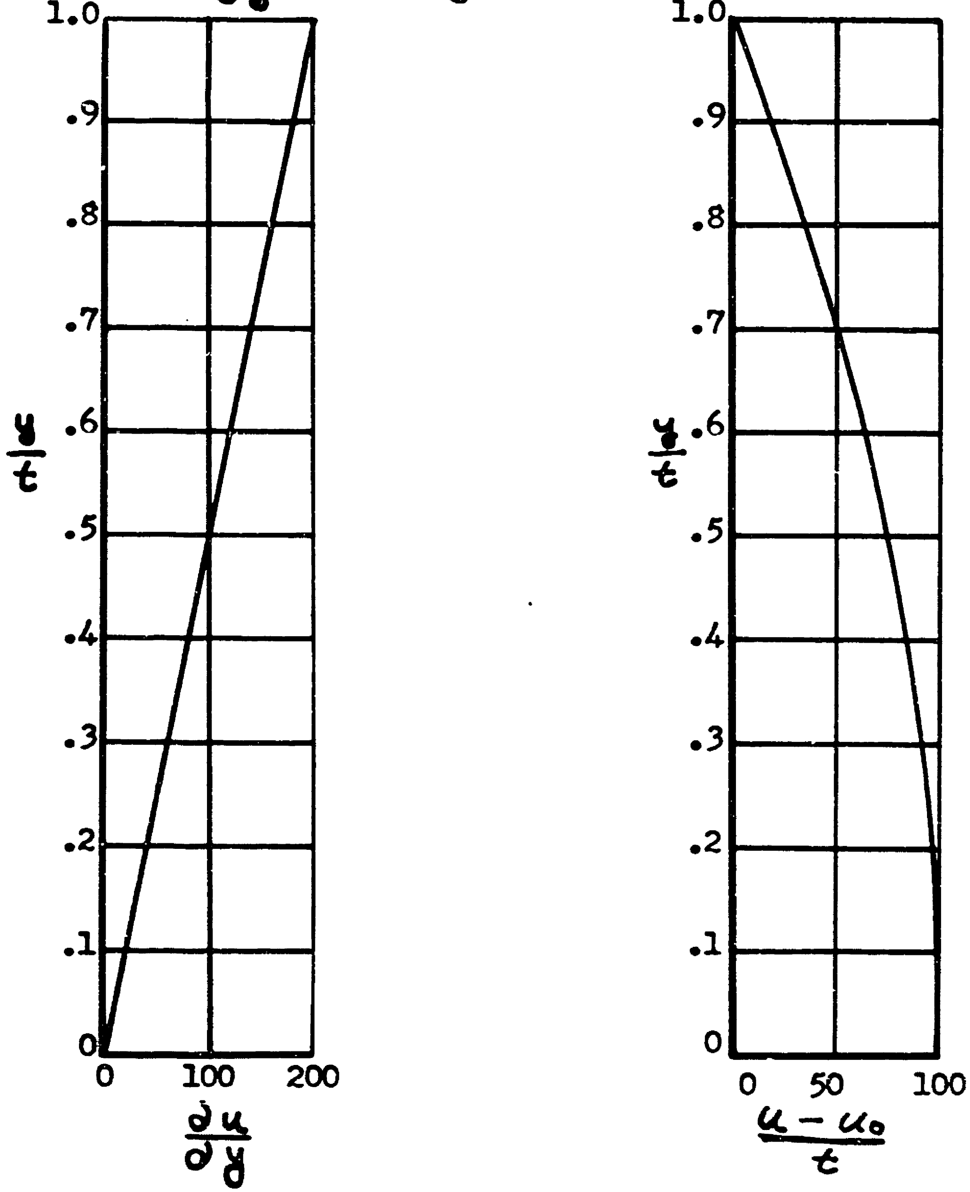


Figure 6

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a=0$ and $\frac{u_0}{\tau_0}\frac{\eta}{\sigma}=1$

$F = 200$

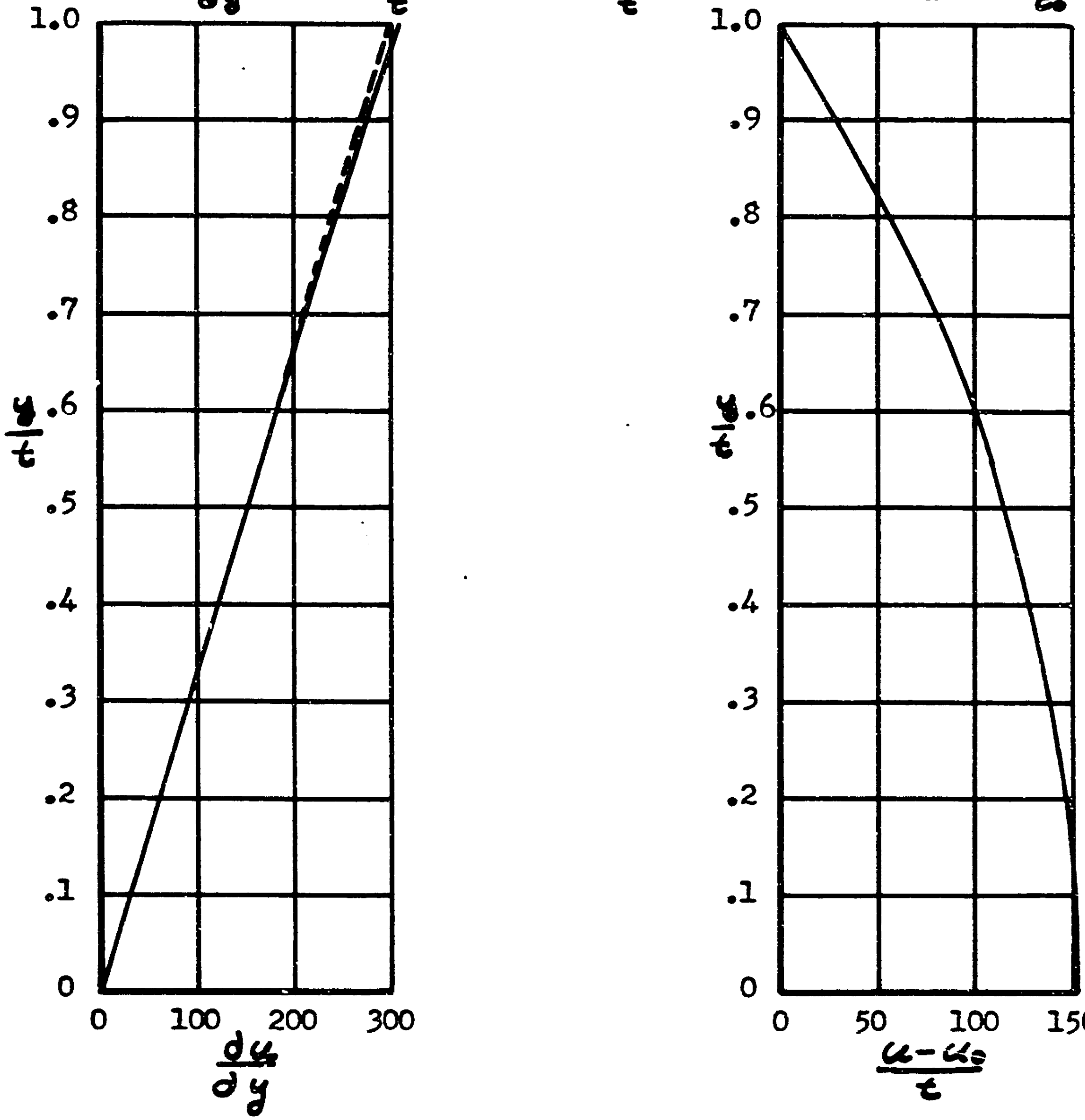
$x = .374$



Dotted line corresponds to viscous solution for same value of F.

Figure 7

$\frac{\partial u}{\partial y}$ and $\frac{u - u_0}{t}$ versus $\frac{y}{t}$ for $a = 0$ and $\frac{u_0}{t_0} \frac{\eta}{G} = 1$



F = 300

x = .342

Dotted line corresponds to viscous solution for same value of F.

Figure 8

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = 0$ and $\frac{u_0}{t_0}\frac{\eta}{\sigma} = 1$

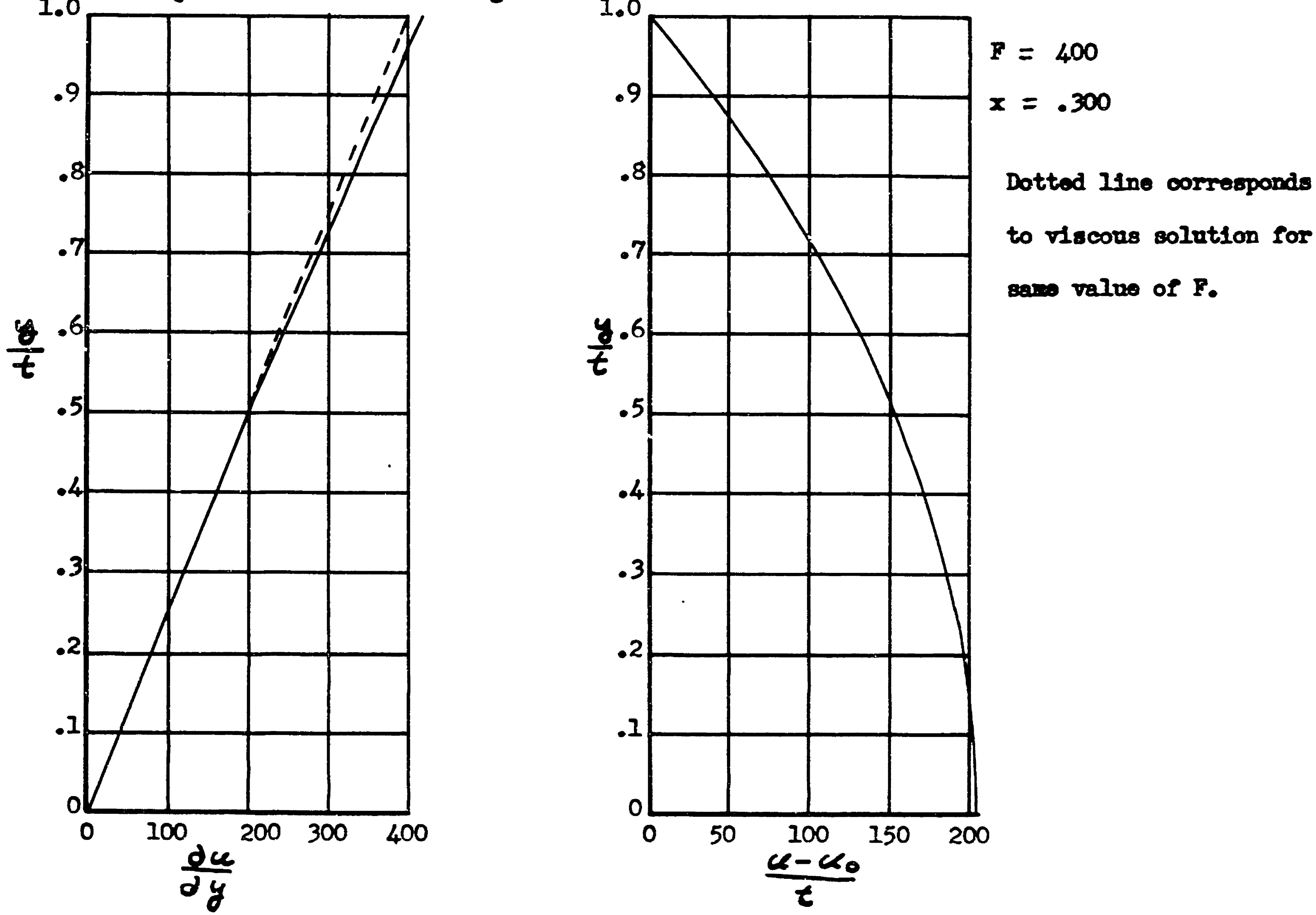
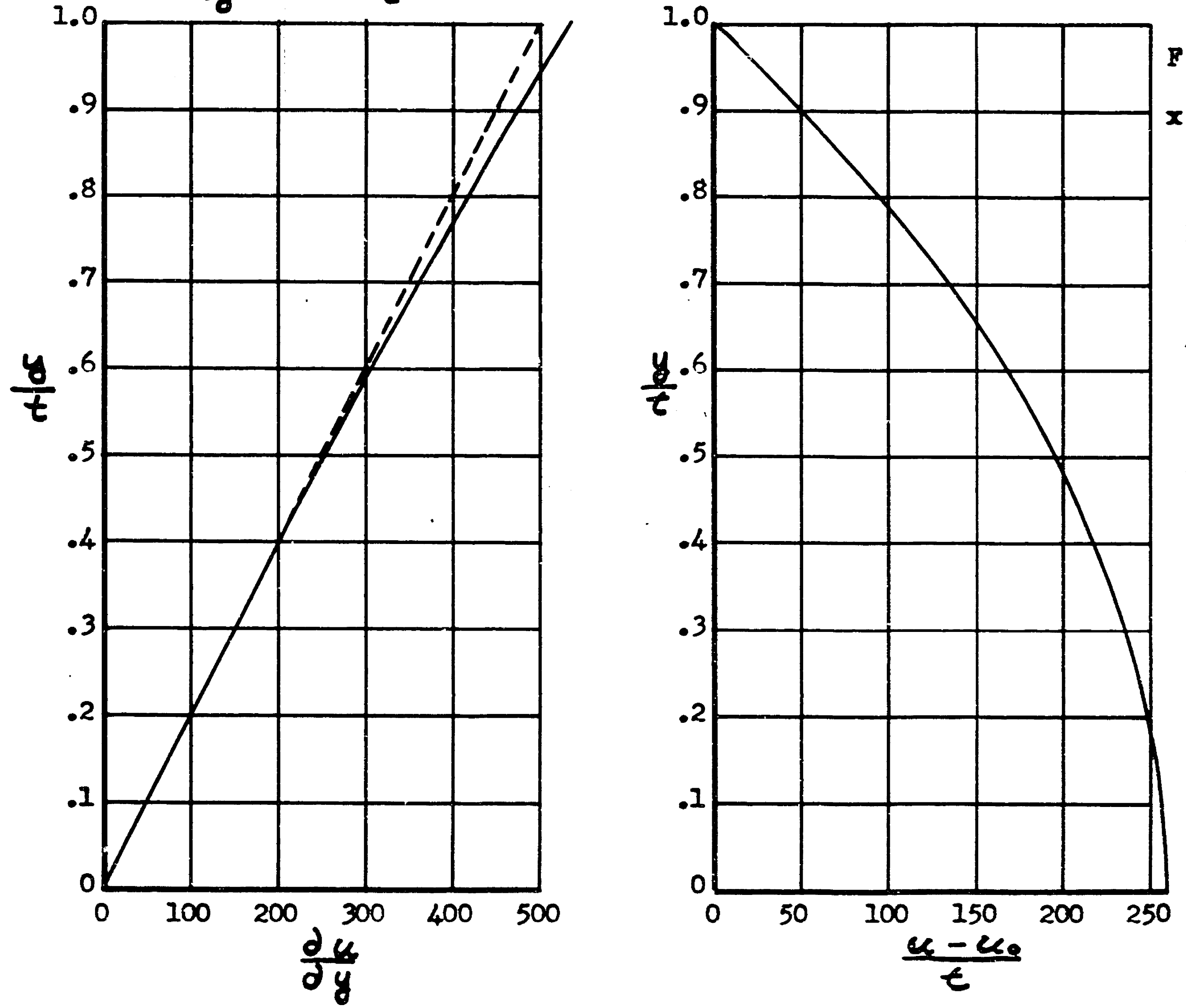


Figure 9

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a=0$ and $\frac{u_0}{\tau_0}\frac{\eta}{\sigma}=1.0$



F = 500

x = .252

Dotted line corresponds to viscous solution for same value of F.

Figure 10

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = 0$ and $\frac{u_0}{\tau_0}\frac{h}{\sigma} = 1$

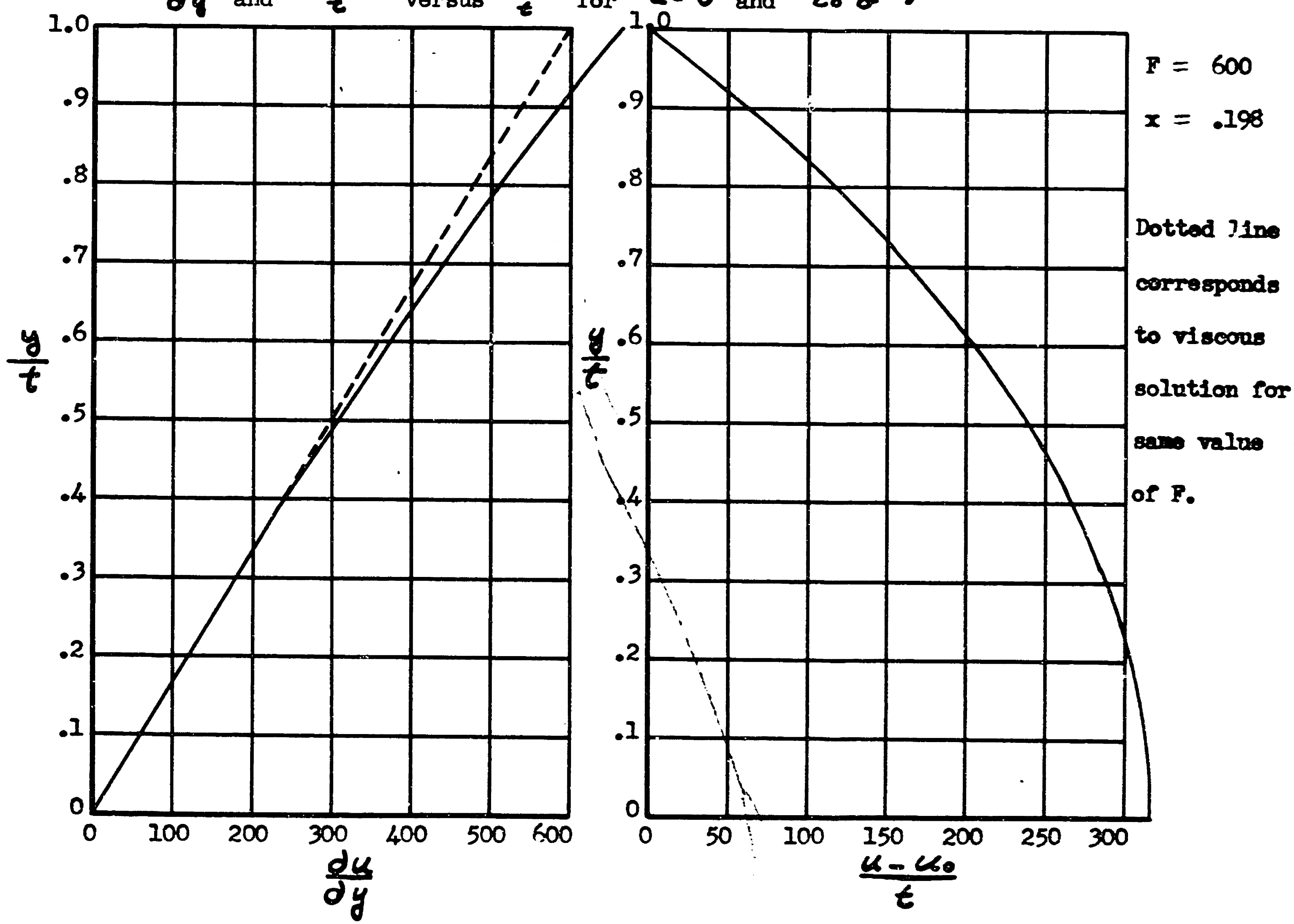
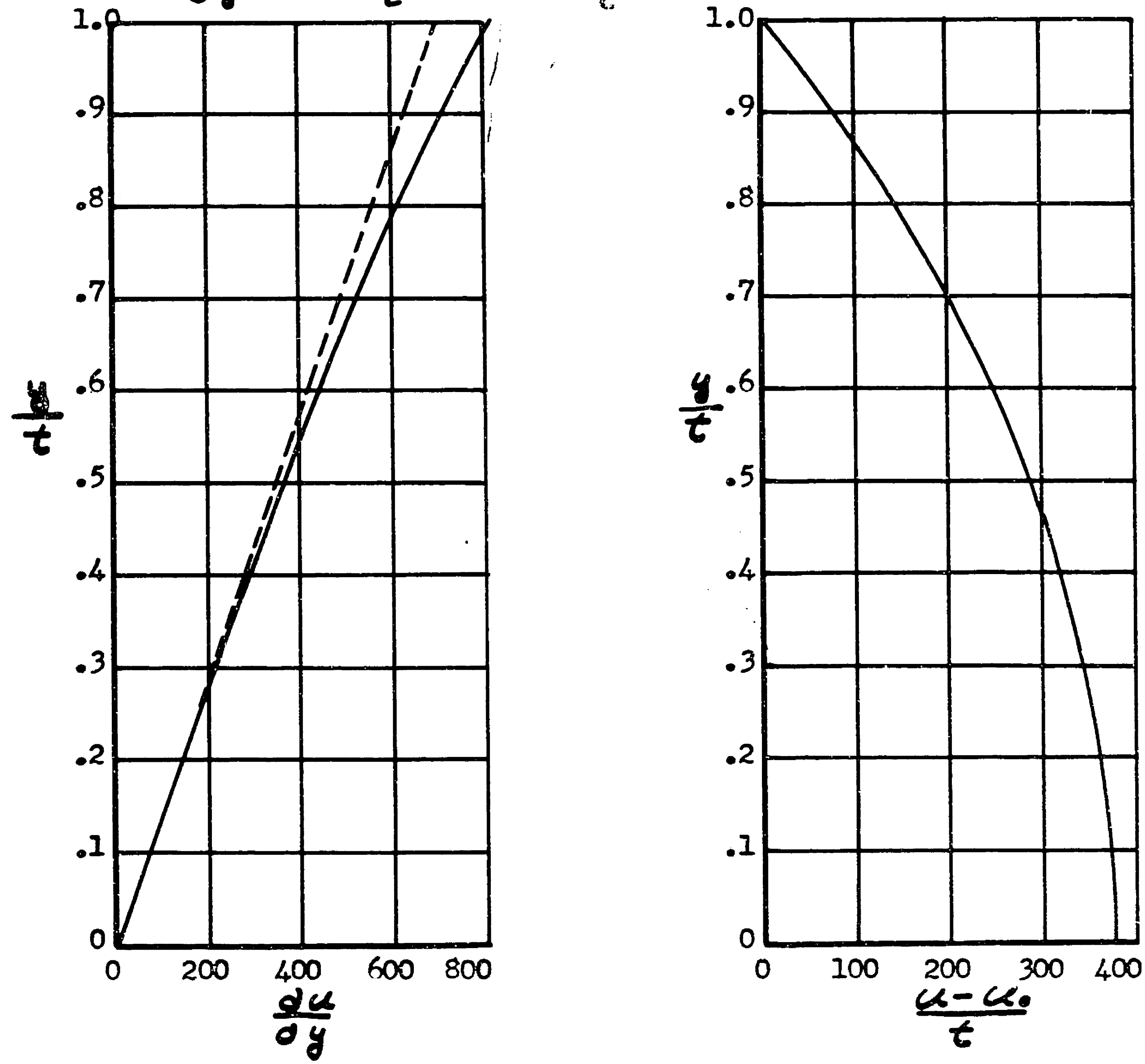


Figure 11

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $\alpha = 0$ and $\frac{u_0}{\tau_0}\,\frac{k}{G} = 1.0$



F = 700

x = .123

Dotted line corresponds to viscous solution for for same value of F.

Figure 12

$\frac{\tau_{max}}{\eta}$ versus x for $a=0$ and $\frac{u_0}{\tau_0}\frac{\eta}{\sigma}=1$

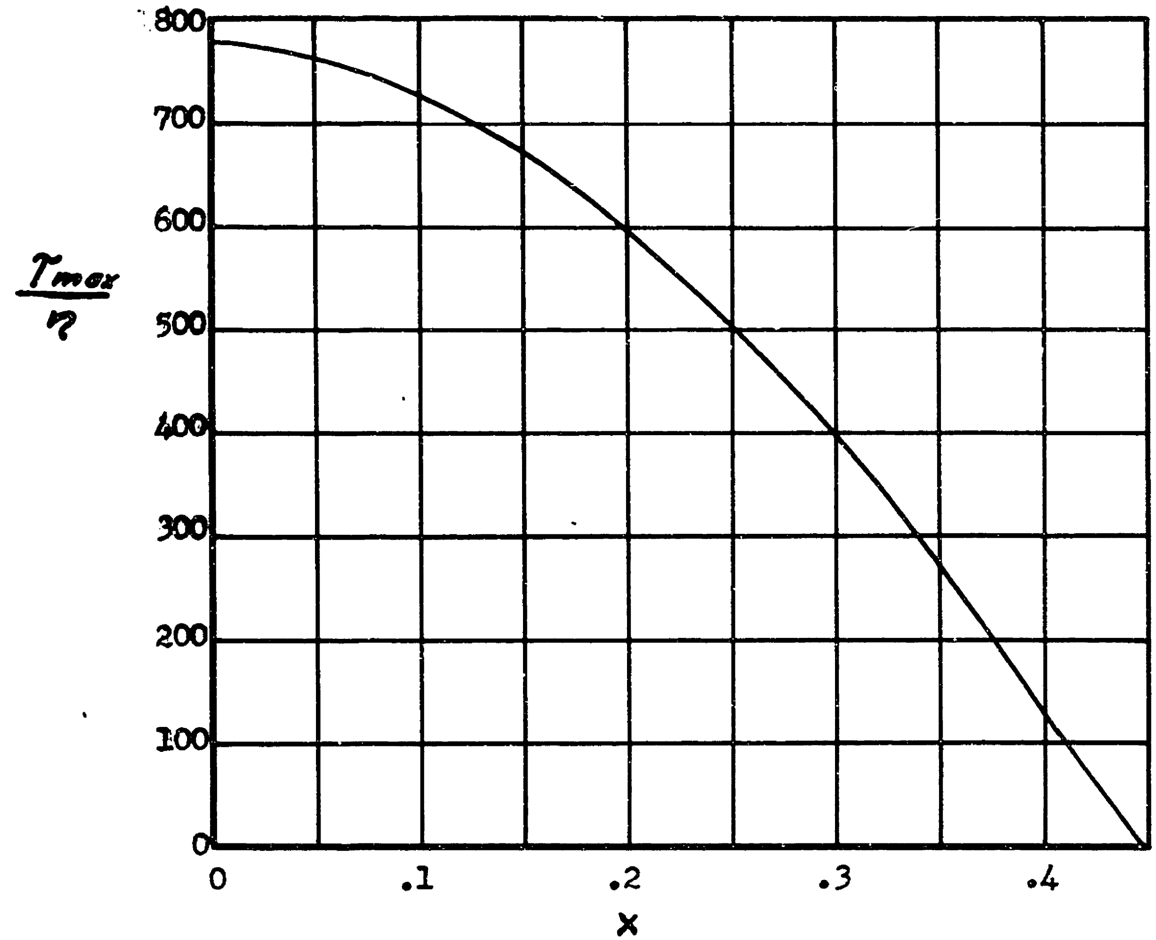


Figure 13

$\frac{p}{h}$ and $-\frac{\sigma_y}{h}\Big|_{max}$ versus x for $a = 0$ and $\frac{u_0}{t_0}\frac{h}{G} = 1$

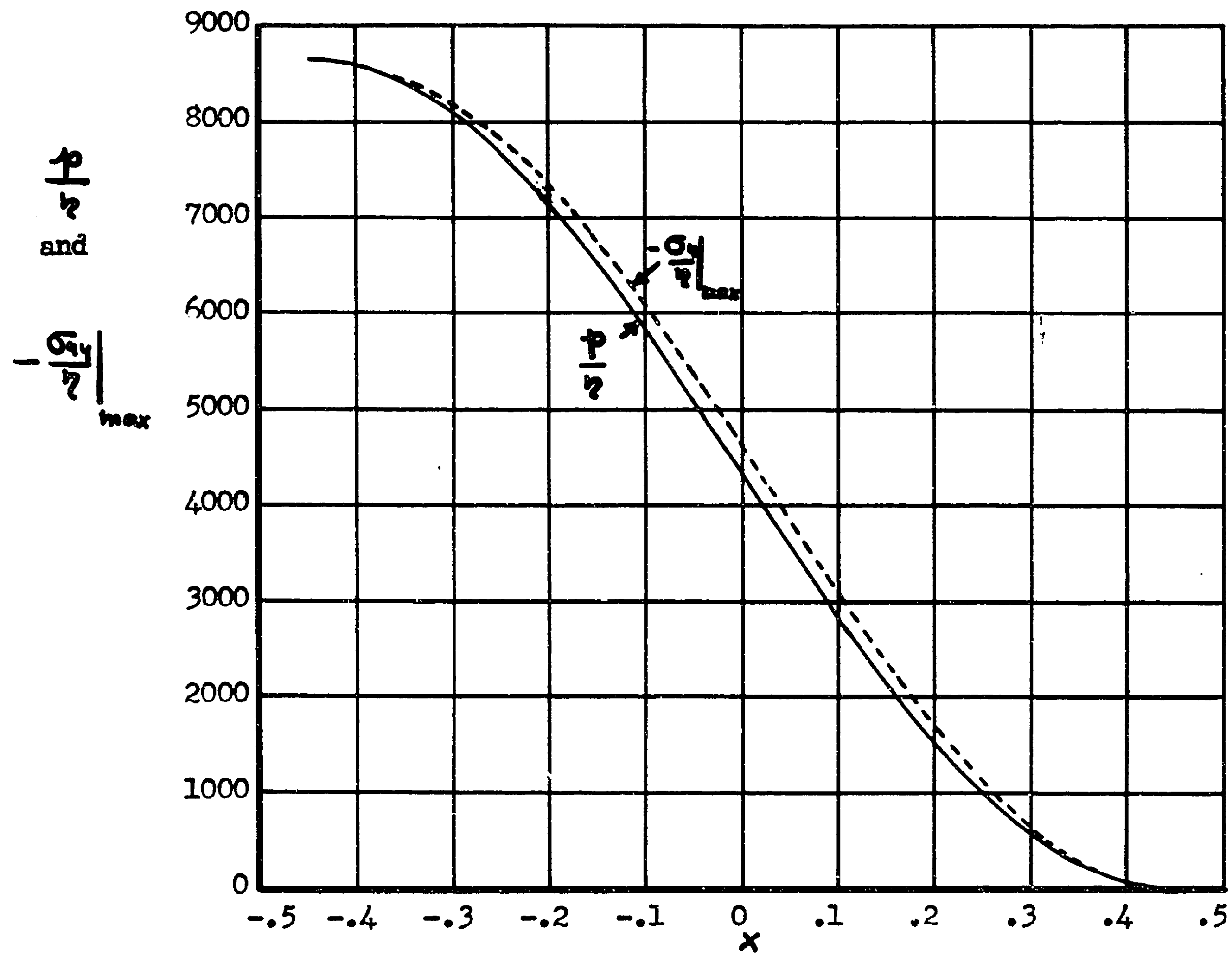
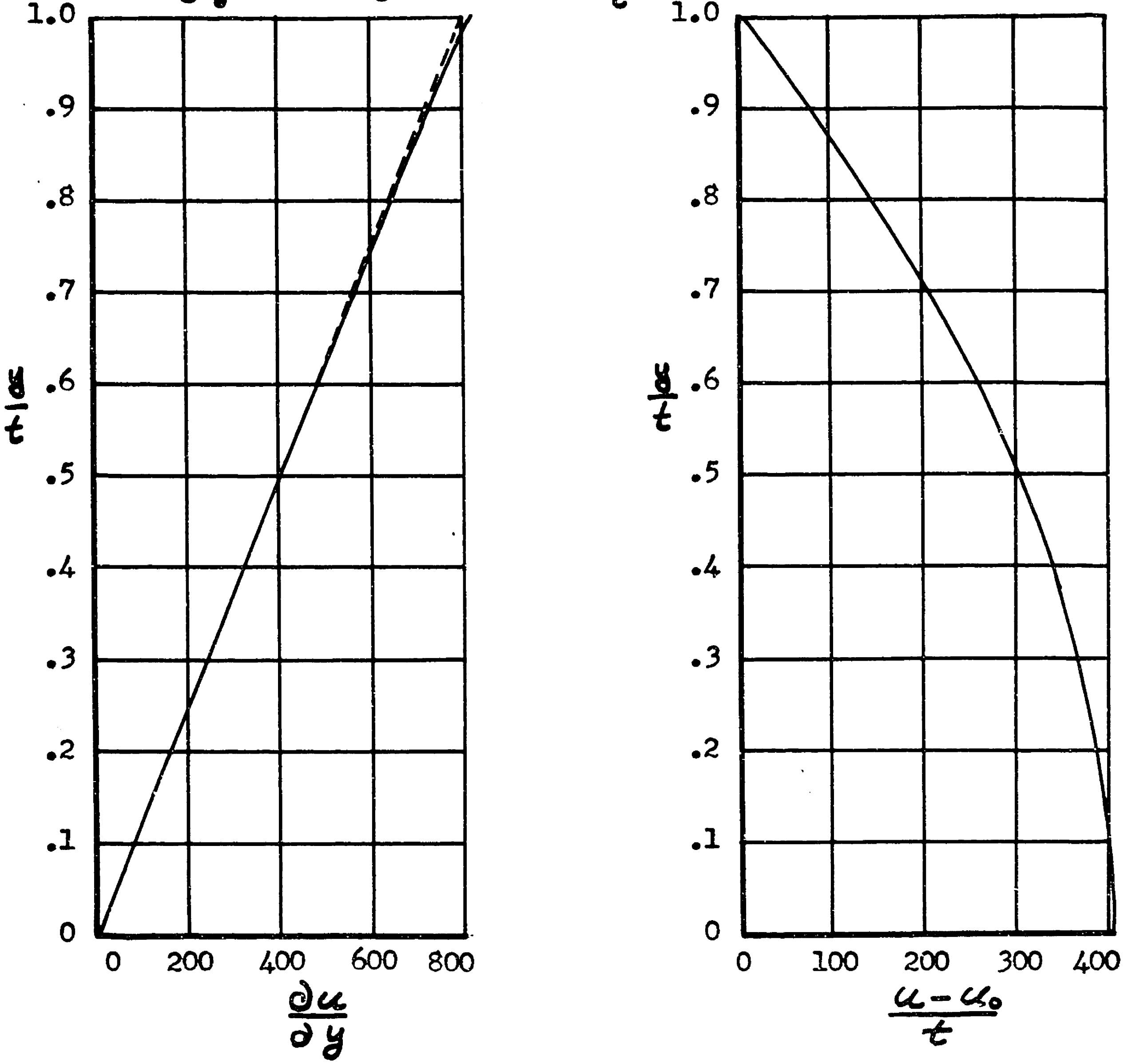


Figure 14

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a=0$ and $\frac{u_0}{t_0}\frac{\eta}{\sigma} = \sqrt{.1}$



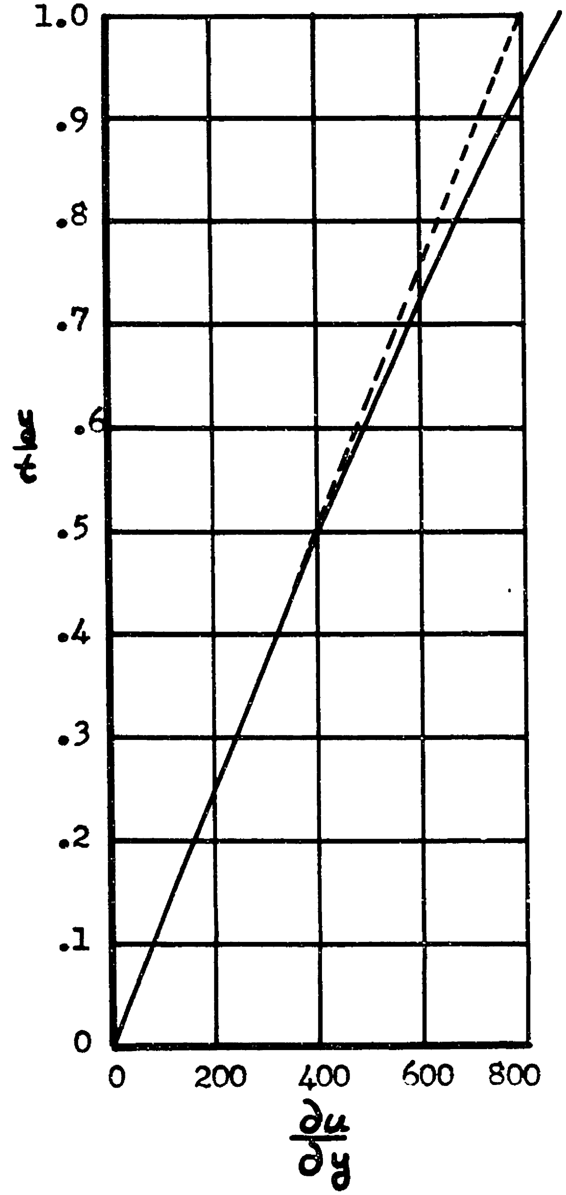
F 800

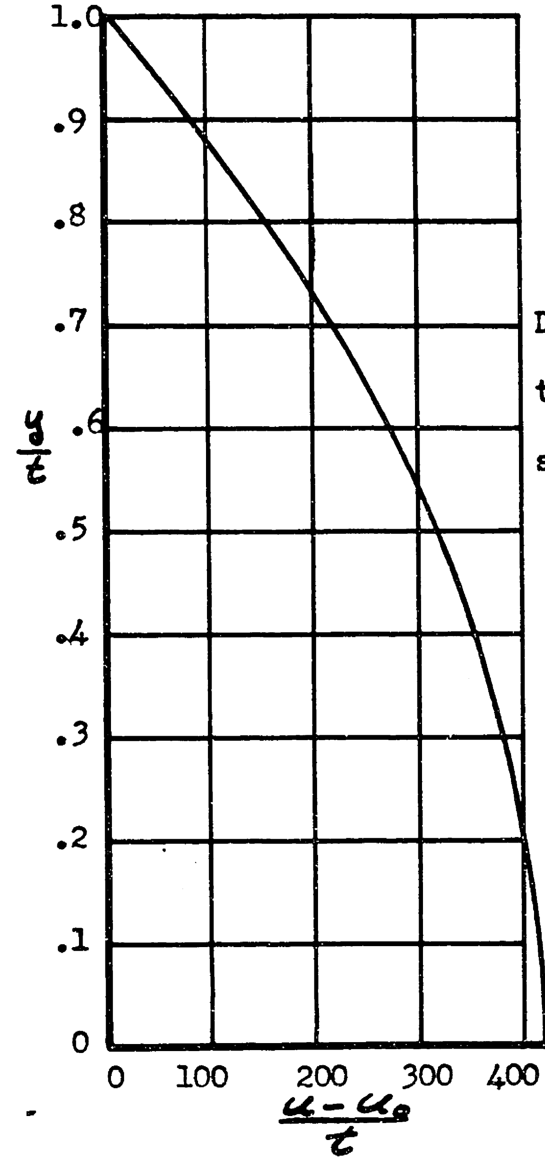
x .080

Dotted line corresponds to viscous solution for same value of F.

Figure 15

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a=0$ and $\frac{u_0}{\tau_0}\frac{\eta}{G}=\sqrt{.5}$





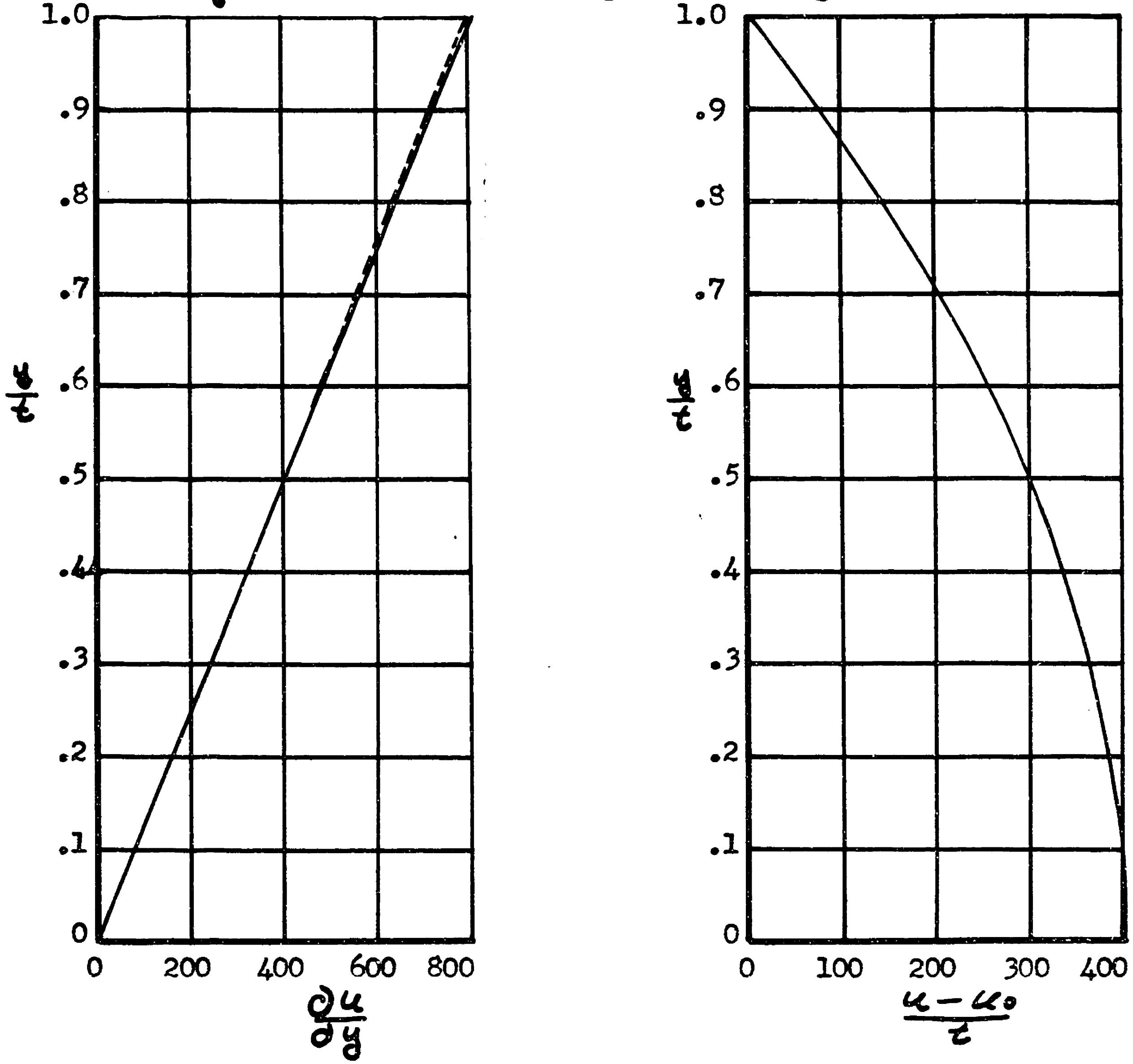
F 800

x .046

Dotted line corresponds to viscous solution for same value of F.

Figure 16

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = \frac{\eta^2}{G^2}$ and $\frac{u_0}{\tau_0}\frac{\eta}{G} = \sqrt{.1}$



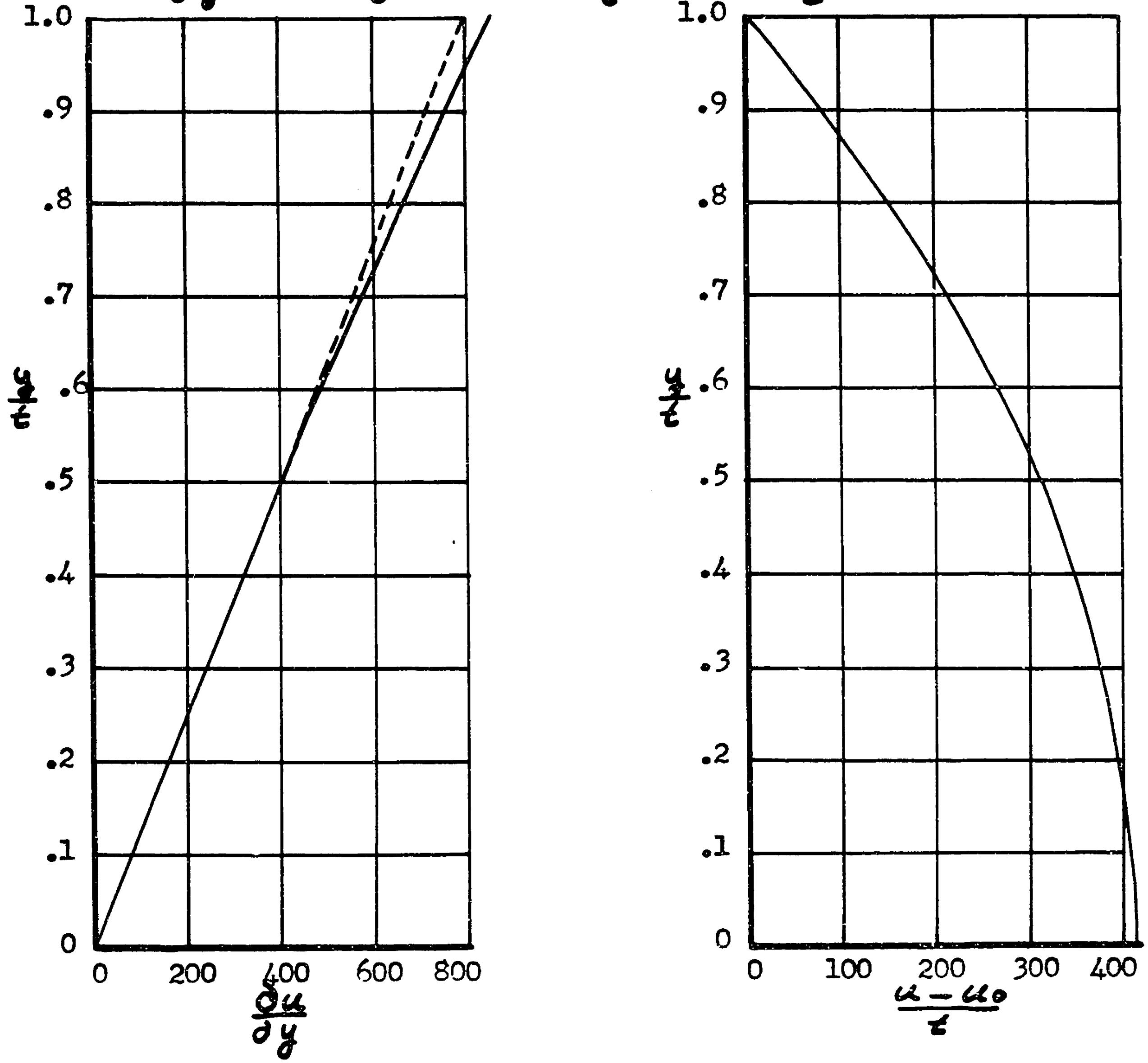
F 800

x .082

Dotted line corresponds to viscous solution for same value of F.

Figure 17

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a=\frac{n^2}{6^2}$ and $\frac{u_0}{t_0}\frac{h}{6}=\sqrt{.5}$



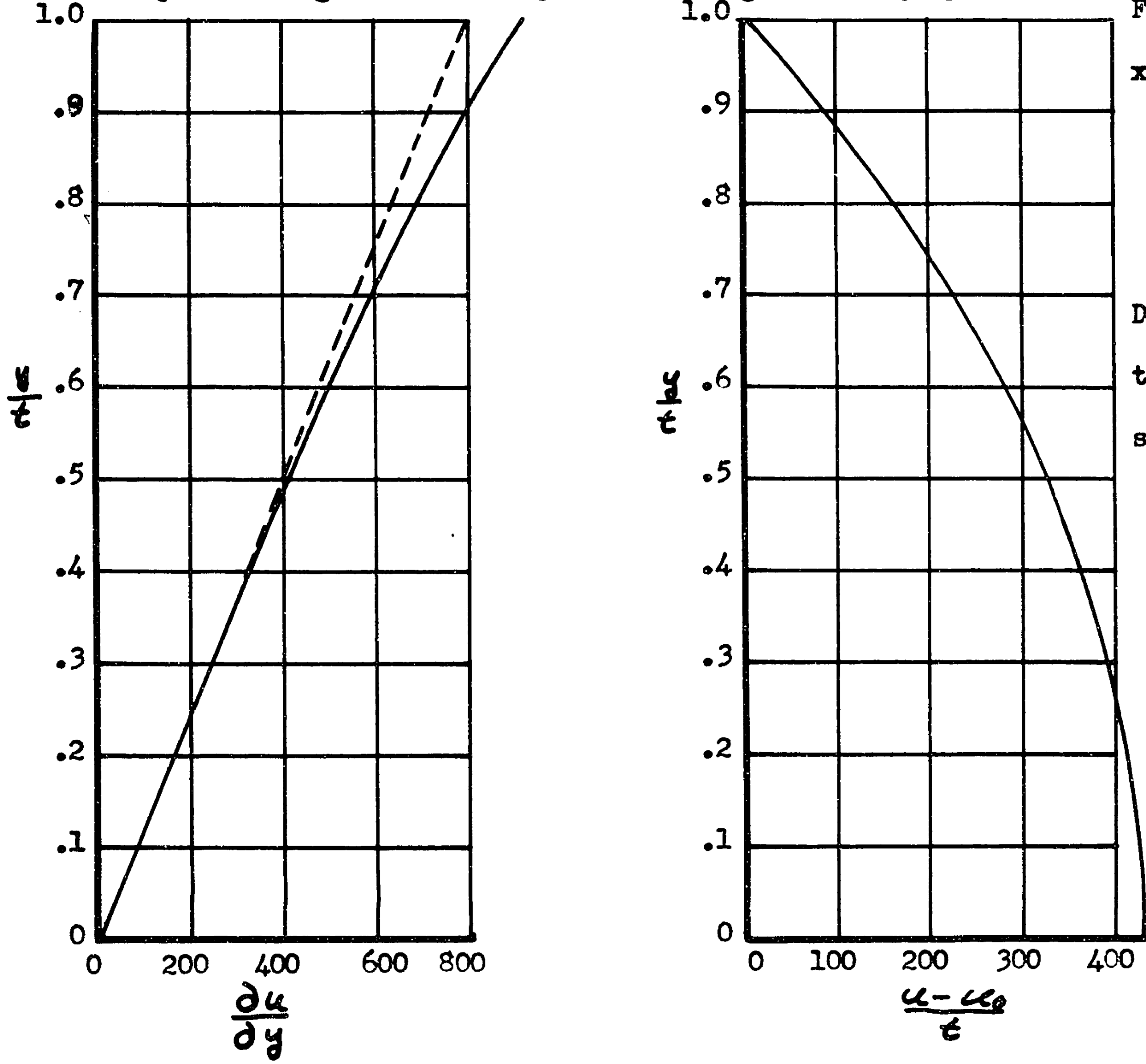
F = 800

x = .0611

Dotted line corresponds to viscous solution for same value of F.

Figure 18

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = \frac{\eta^2}{\sigma^2}$ and $\frac{u_0}{t_0}\frac{\eta}{\sigma} = 1$



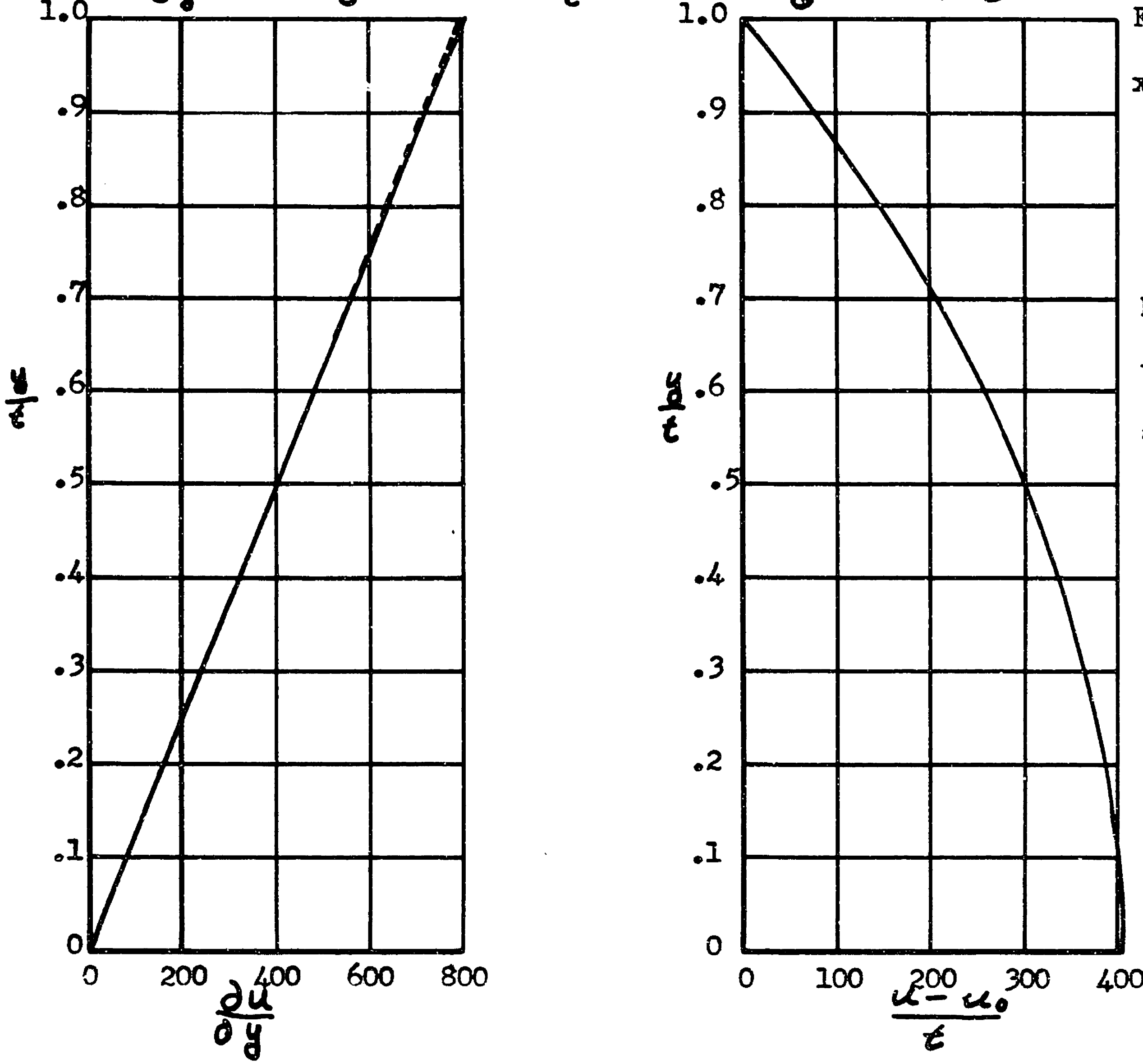
$F = 800$

$x \approx 0$

Dotted line corresponds to viscous solution for same value of F.

Figure 19

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = 2.5\frac{\eta^2}{G^2}$ and $\frac{u_0}{t_0}\frac{\eta}{G} = \sqrt{.1}$



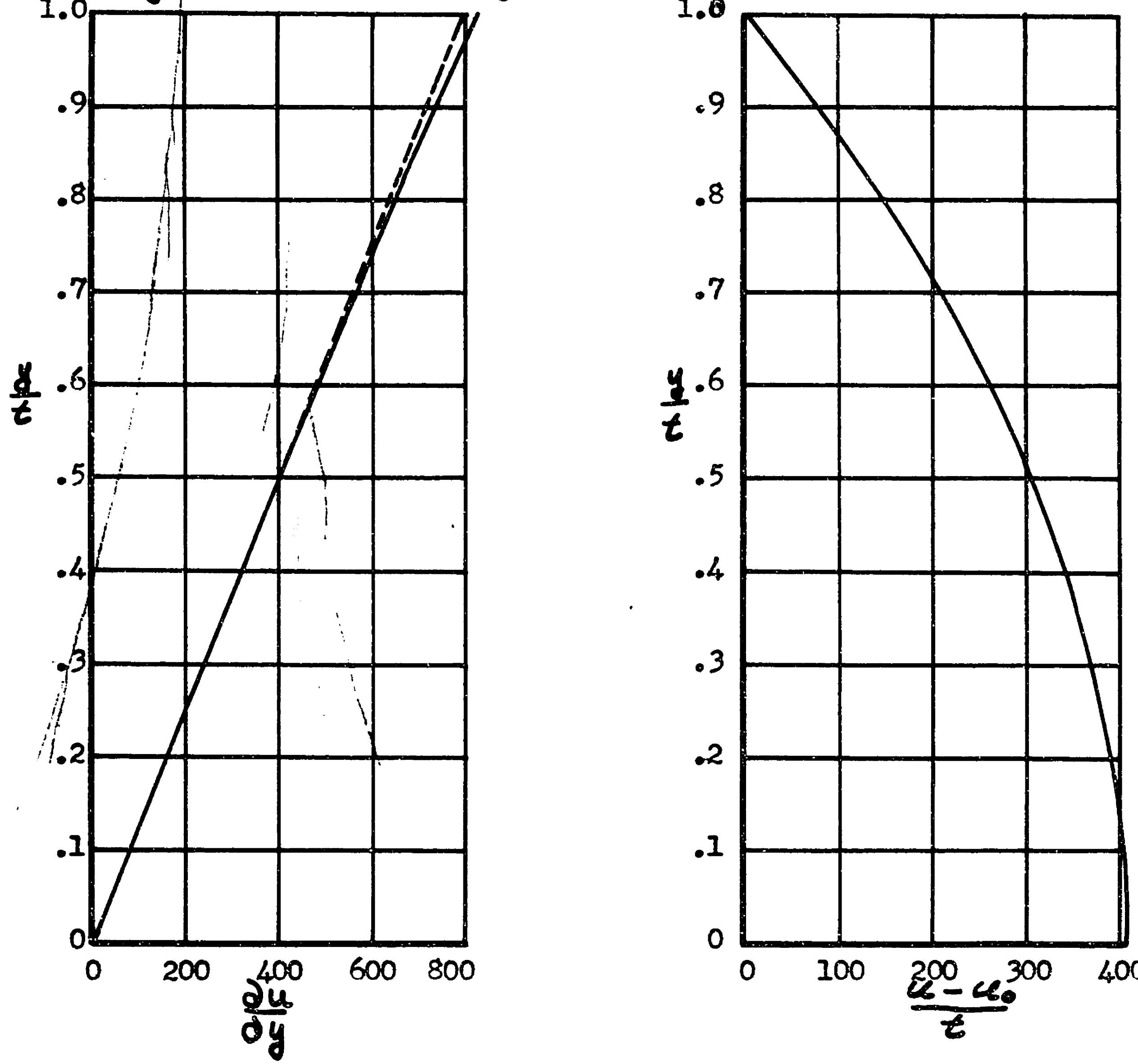
F = 800

x = .095

Dotted line corresponds to viscous solution for same value of F.

Figure 20

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = 2.5\,\frac{m^2}{c^2}$ and $\frac{u_0}{t_0}\,\frac{\eta}{c} = \sqrt{.5}$



F = 800

x = .082

Dotted line corresponds to viscous solution for same value of F.

Figure 21

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = 2.5\frac{h^2}{G^2}$ and $\frac{u_0}{t_0}\frac{h}{G} = 1$

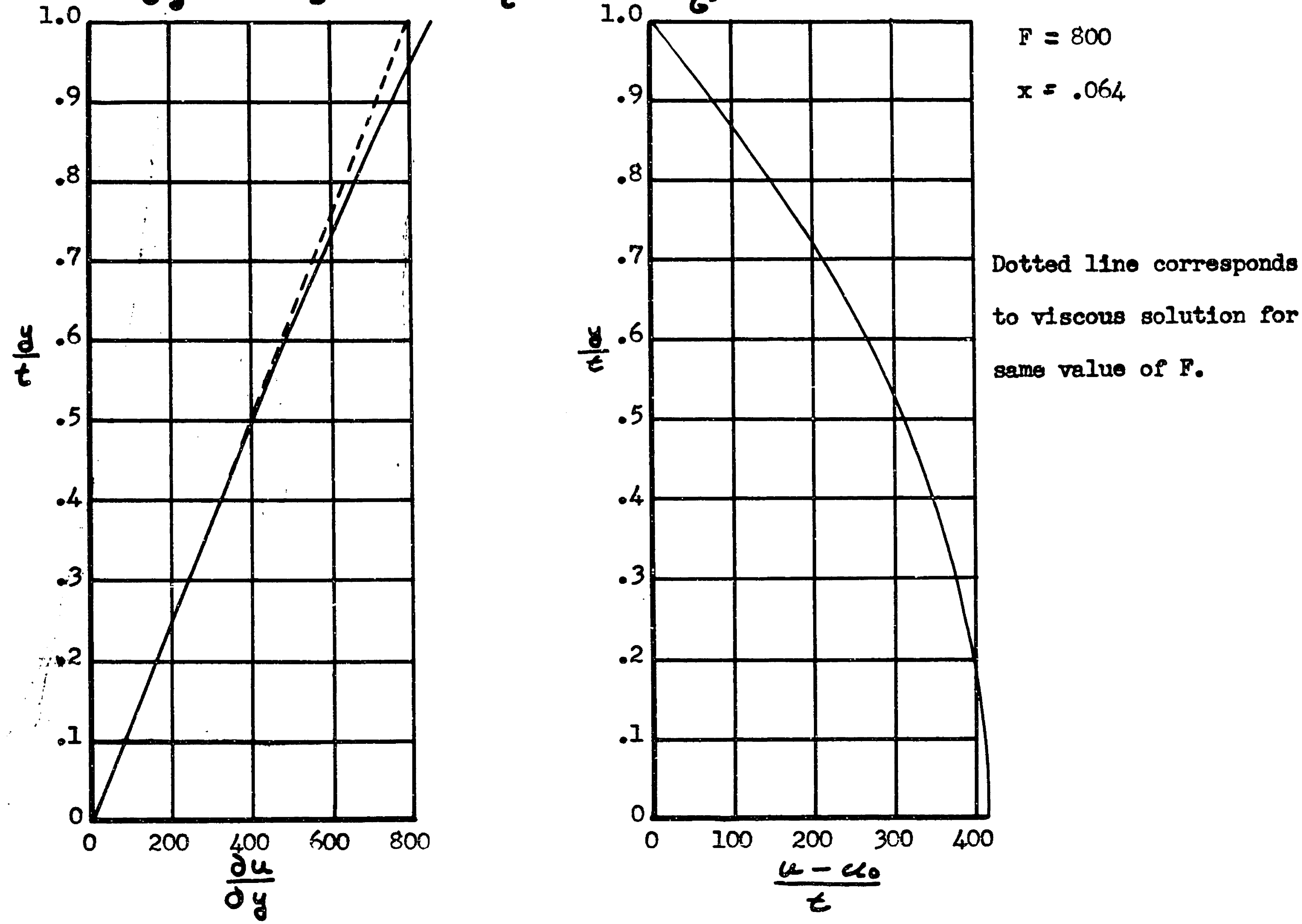
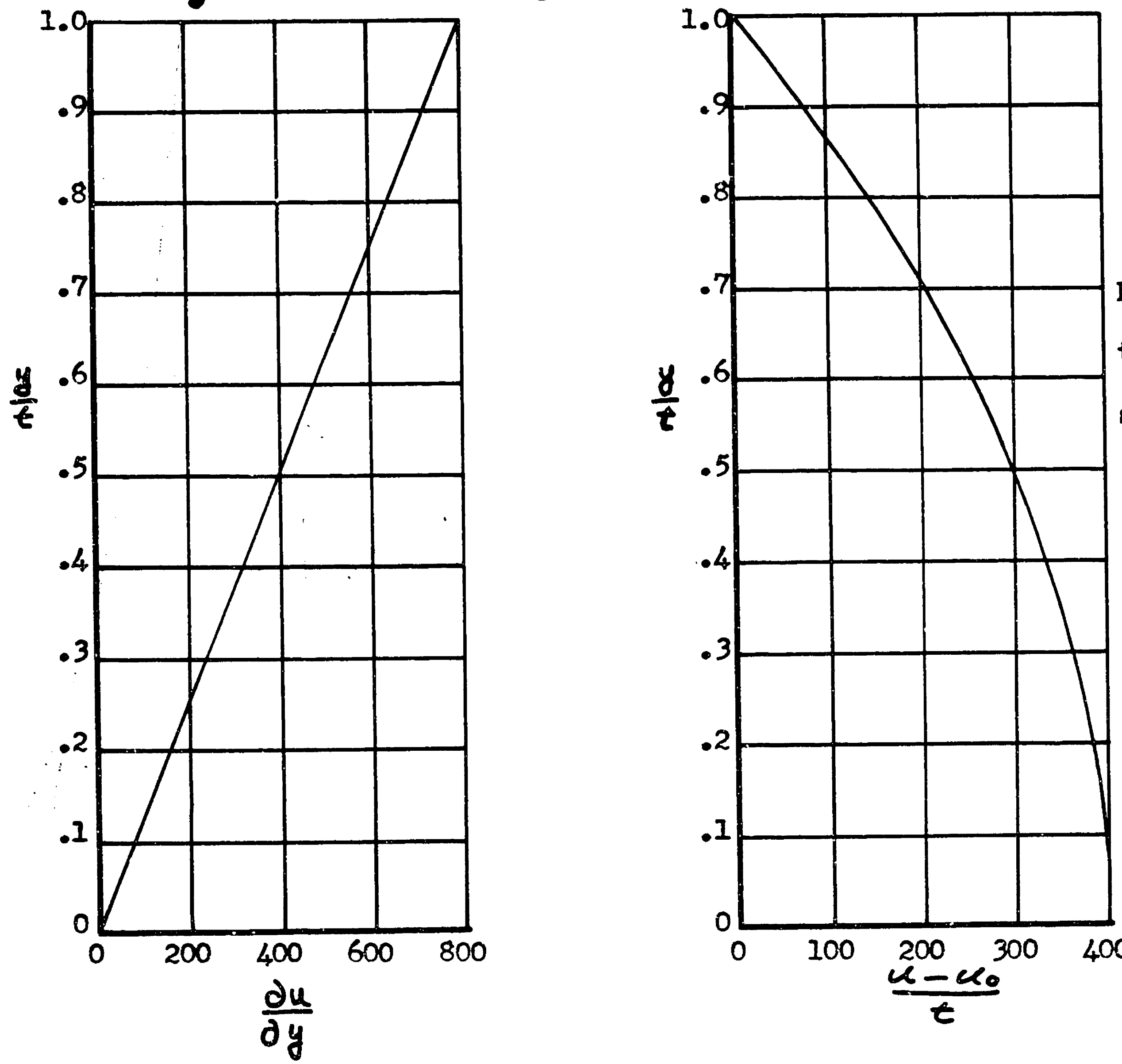


Figure 22

$\frac{\partial u}{\partial y}$ and $\frac{u-u_0}{t}$ versus $\frac{y}{t}$ for $a = 4\frac{k^2}{t^2}$ and all values of $\frac{u_0}{t_0}\frac{k}{G}$



F = 800

x = .095

Dotted line corresponds to viscous solution for same value of F.

Figure 23

$\frac{T_{max}}{h}$ versus x for $a = 0$ and various values of $\frac{u_0}{\tau_0}\frac{h}{G}$

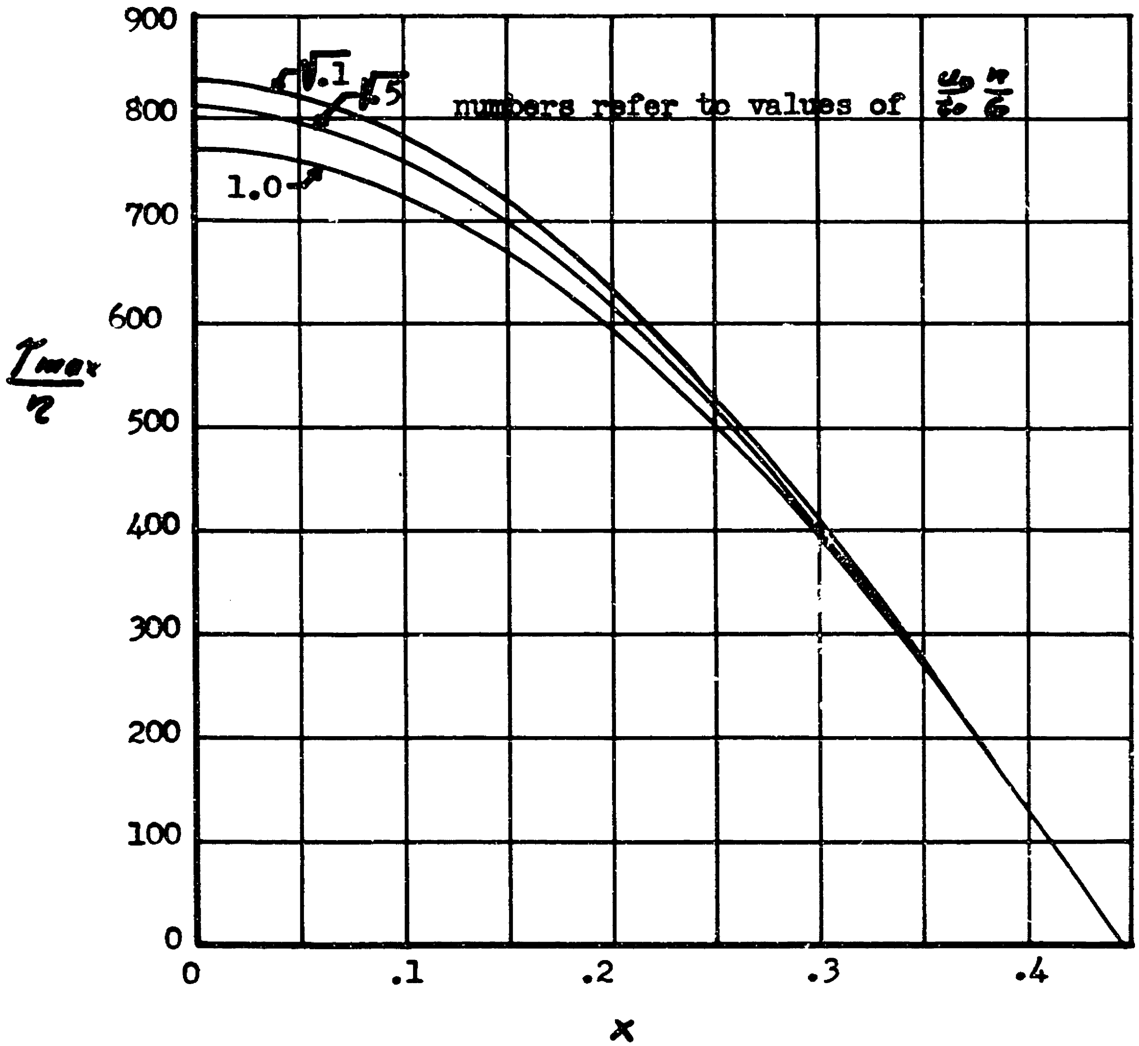


Figure 24

$\frac{T_{max}}{\eta}$ versus x for $a = \frac{k^2}{G^2}$ and various values of $\frac{u_0}{\varepsilon_0} \frac{k}{G}$

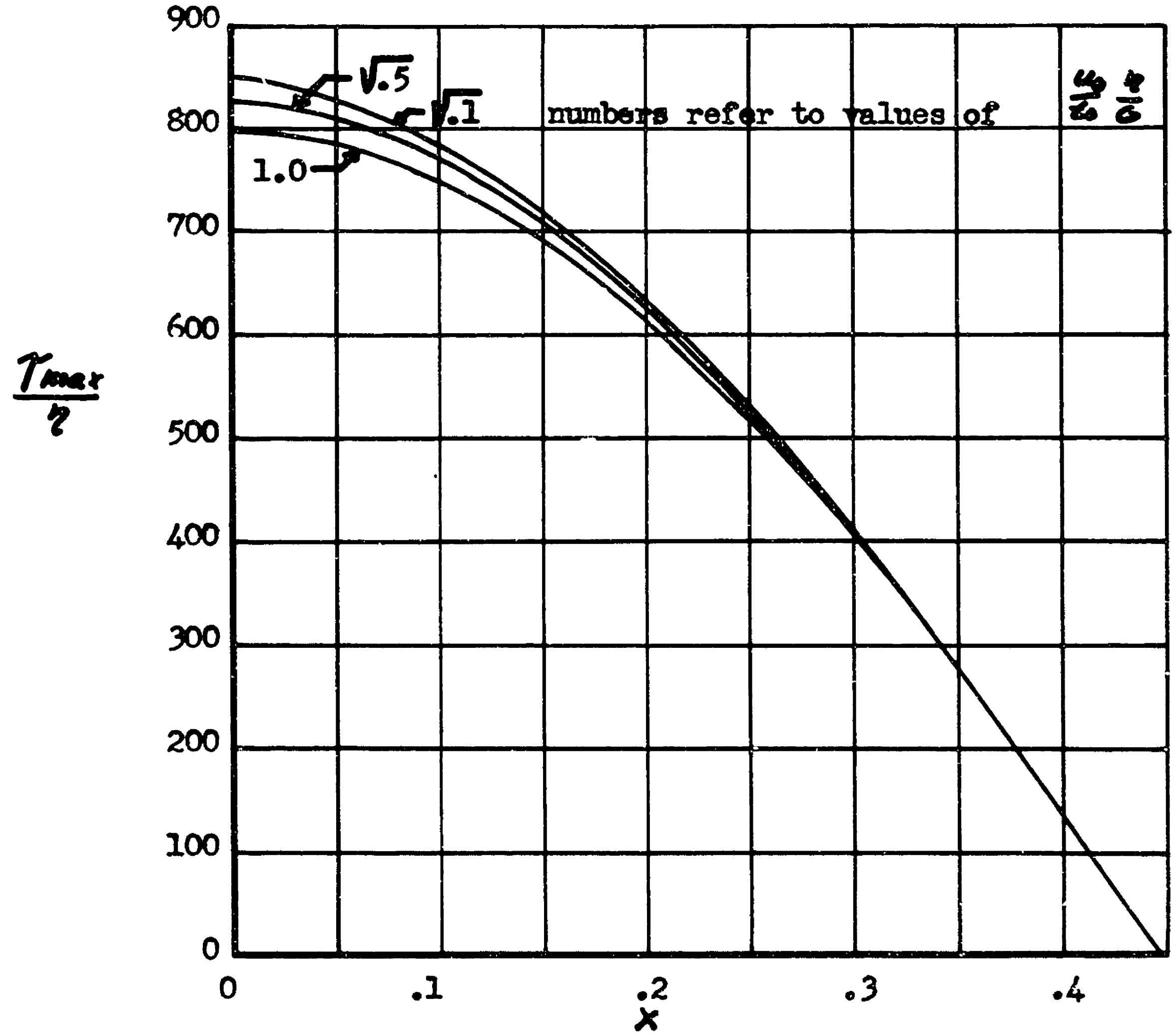


Figure 25

$\frac{\tau_{max}}{k}$ versus x for $\alpha = 2.5\,\frac{k^2}{G^2}$ and various values of $\frac{u_0}{\varepsilon_0}\frac{k}{G}$

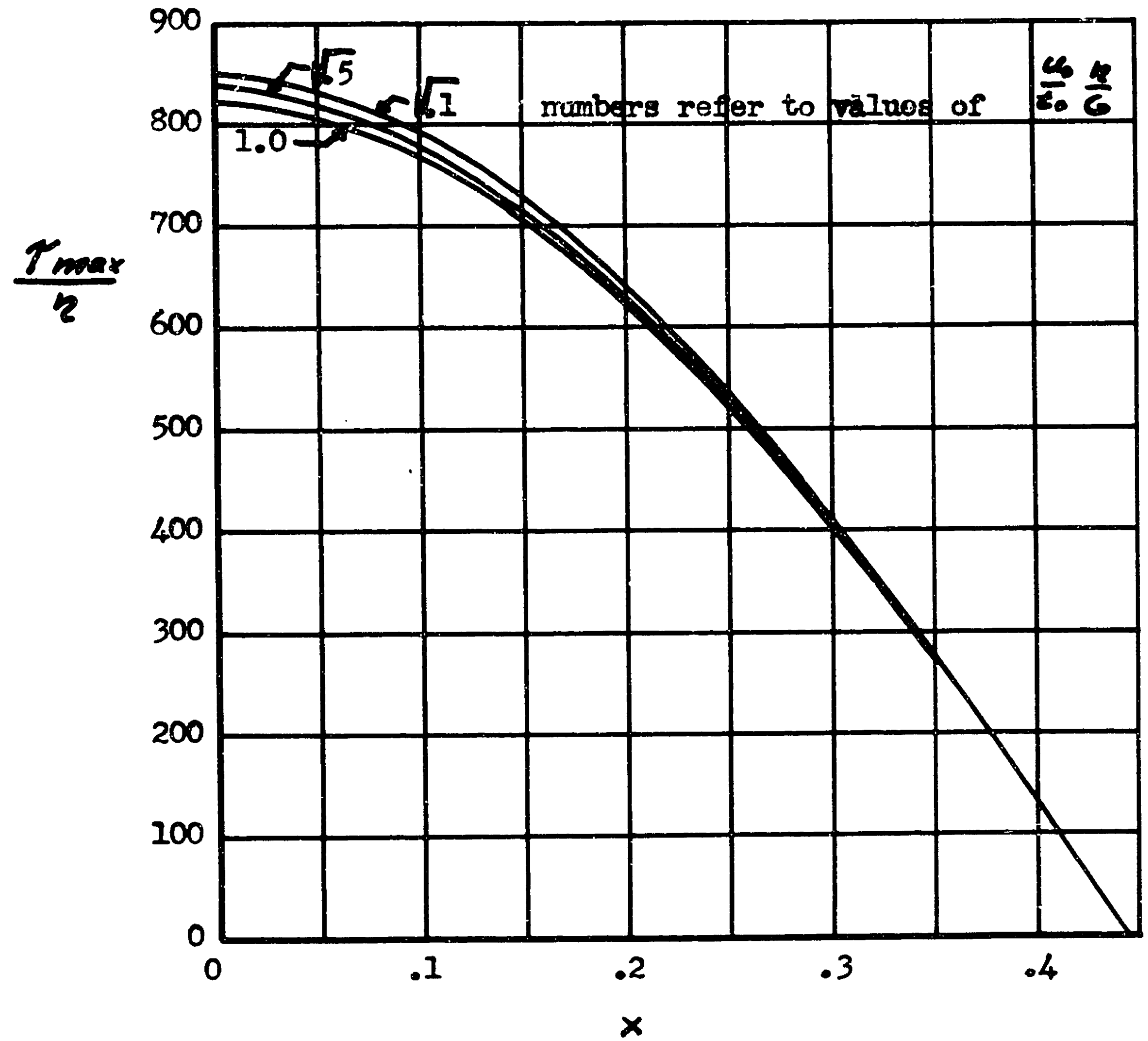


Figure 26

$\frac{T_{max}}{k}$ versus x for $a = 4\frac{k^2}{G^2}$ and all values of $\frac{u_0}{\varepsilon_0}\frac{k}{G}$

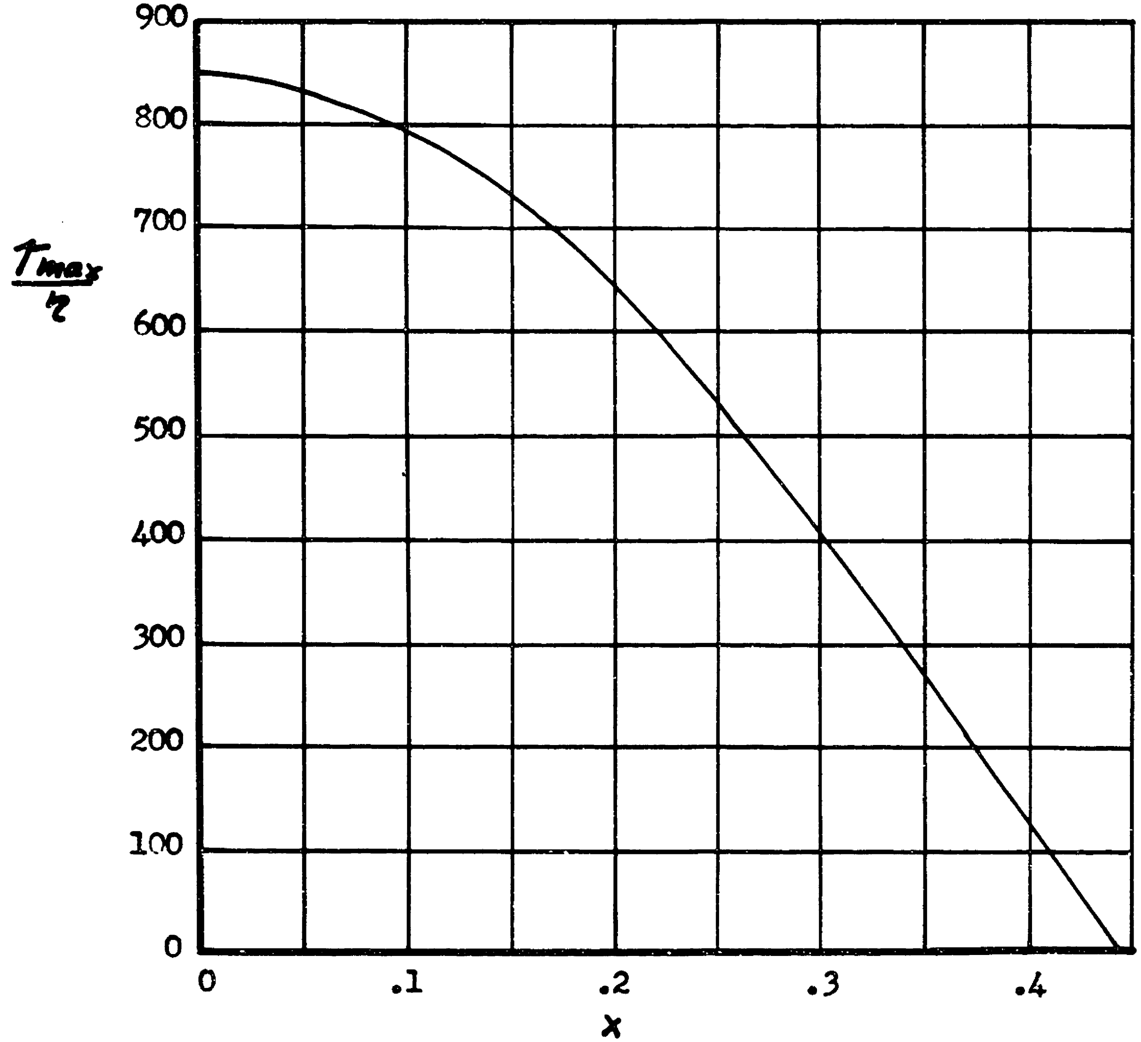


Figure 27

$\frac{p}{\eta}$ and $-\frac{\sigma_y}{\eta}\Big|_{max}$ versus x for $a = 0$ and $\frac{u_0}{\tau_0}\frac{\eta}{G} = \sqrt{.1}$

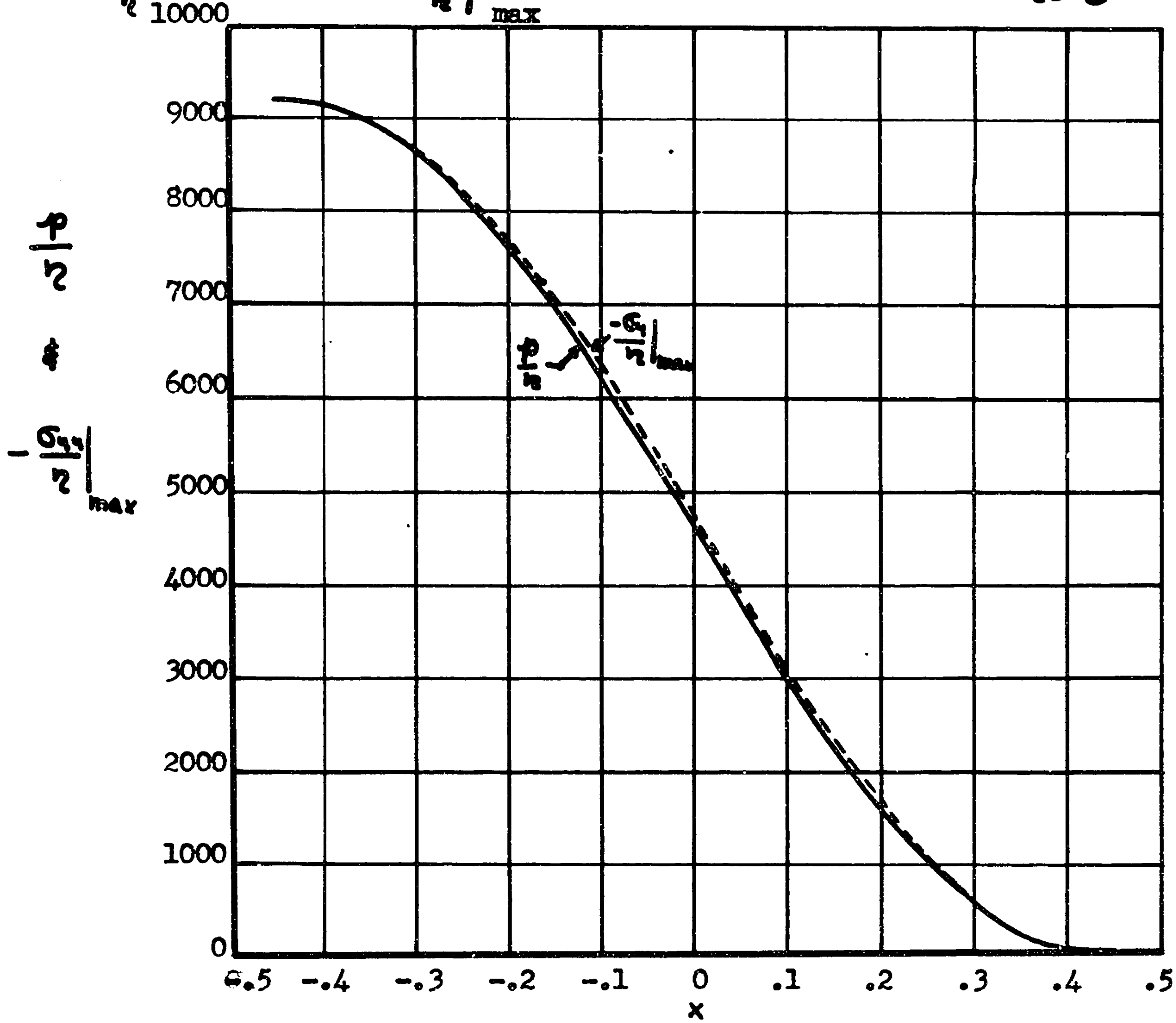


Figure 28

$\frac{p}{\eta}$ and $-\frac{\sigma_y}{\eta}\Big|_{max}$ versus x for $a = 0$ and $\frac{\alpha_0}{\varepsilon_0}\frac{\kappa}{G} = \sqrt{.5}$

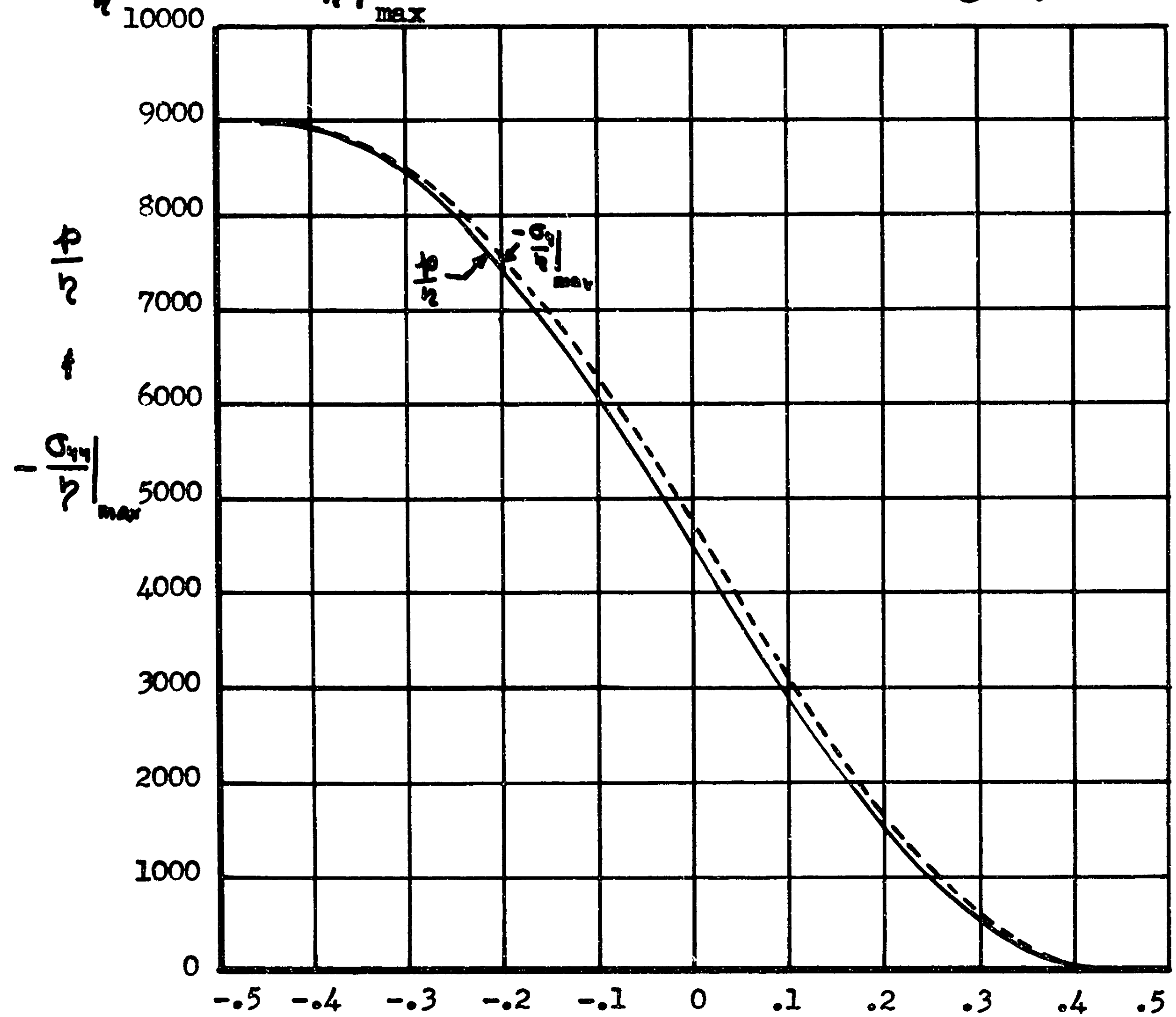


Figure 29

$\frac{p}{\eta}$ and $-\frac{\sigma_u}{\eta}\Big|_{max}$ versus x for $\alpha = 0$ and $\frac{\alpha_0}{c_0}\frac{k}{G} = 1$

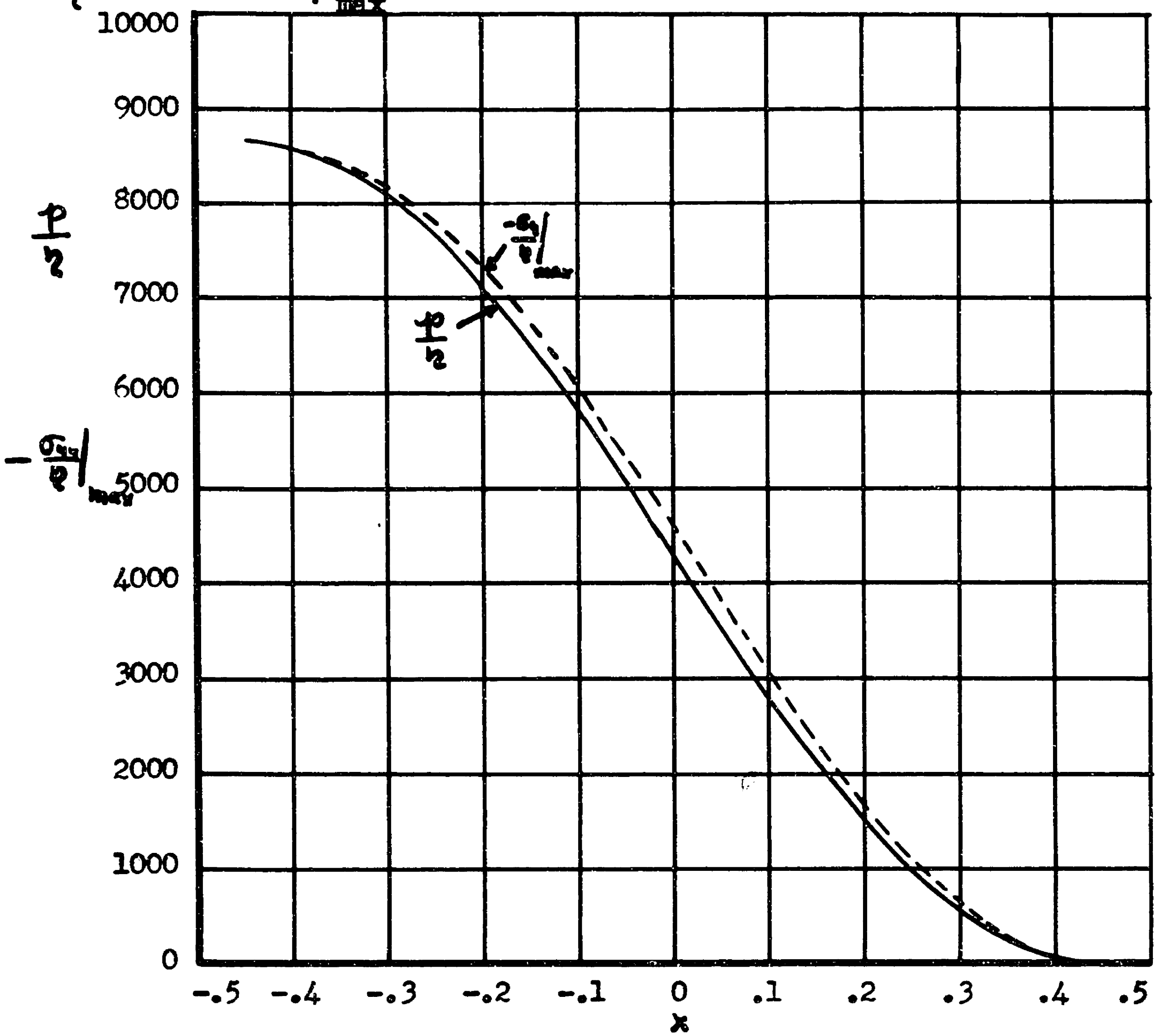


Figure 30

$\frac{p}{h}$ and $-\frac{\sigma_y}{h}\Big|_{max}$ versus x for $a = \frac{h^2}{G^2}$ and $\frac{\mu_0}{\epsilon_0}\,\frac{h}{G} = \sqrt{.1}$

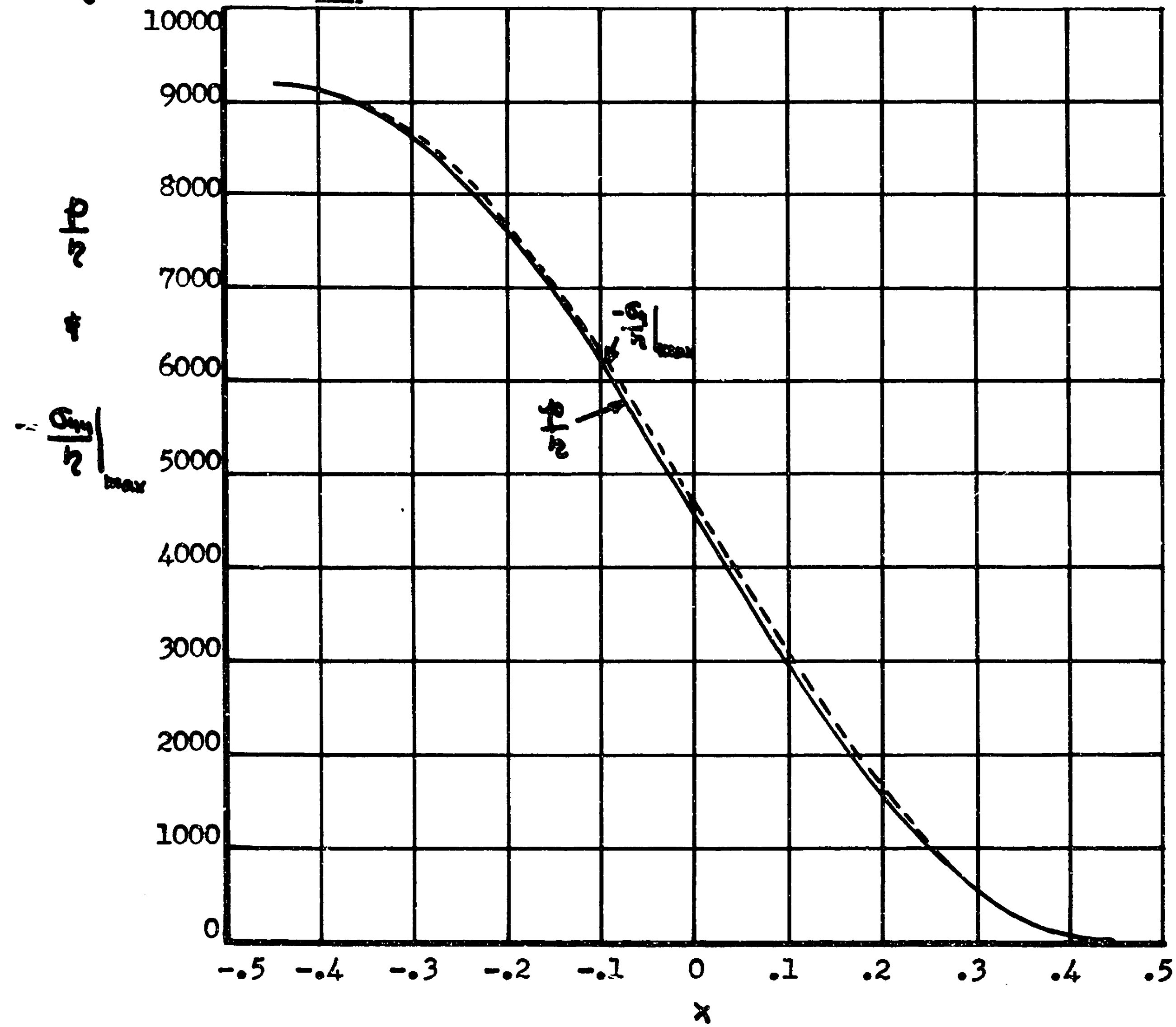


Figure 31

$\frac{p}{\eta}$ and $-\frac{\sigma_y}{\eta}\Big|_{max}$ versus x for $a = \frac{k^2}{G^2}$ and $\frac{u_0}{\varepsilon_0}\frac{k}{G} = \sqrt{.5}$

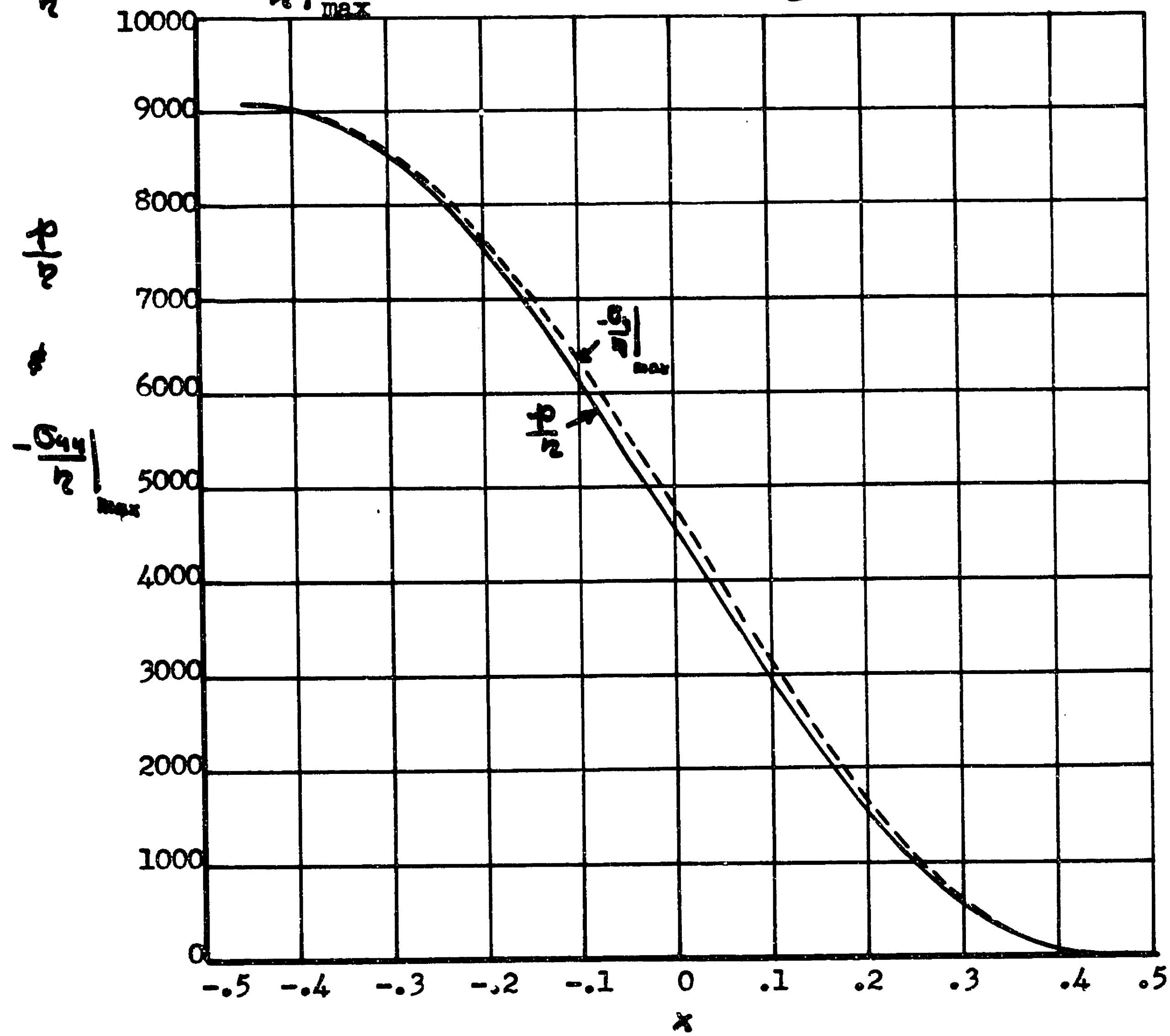


Figure 32

$\frac{p}{h}$ and $-\frac{\sigma_y}{h}\Big|_{max}$ versus x for $a = \frac{k^2}{G^2}$ and $\frac{\omega_0}{\tau_0}\frac{k}{G} = 1$

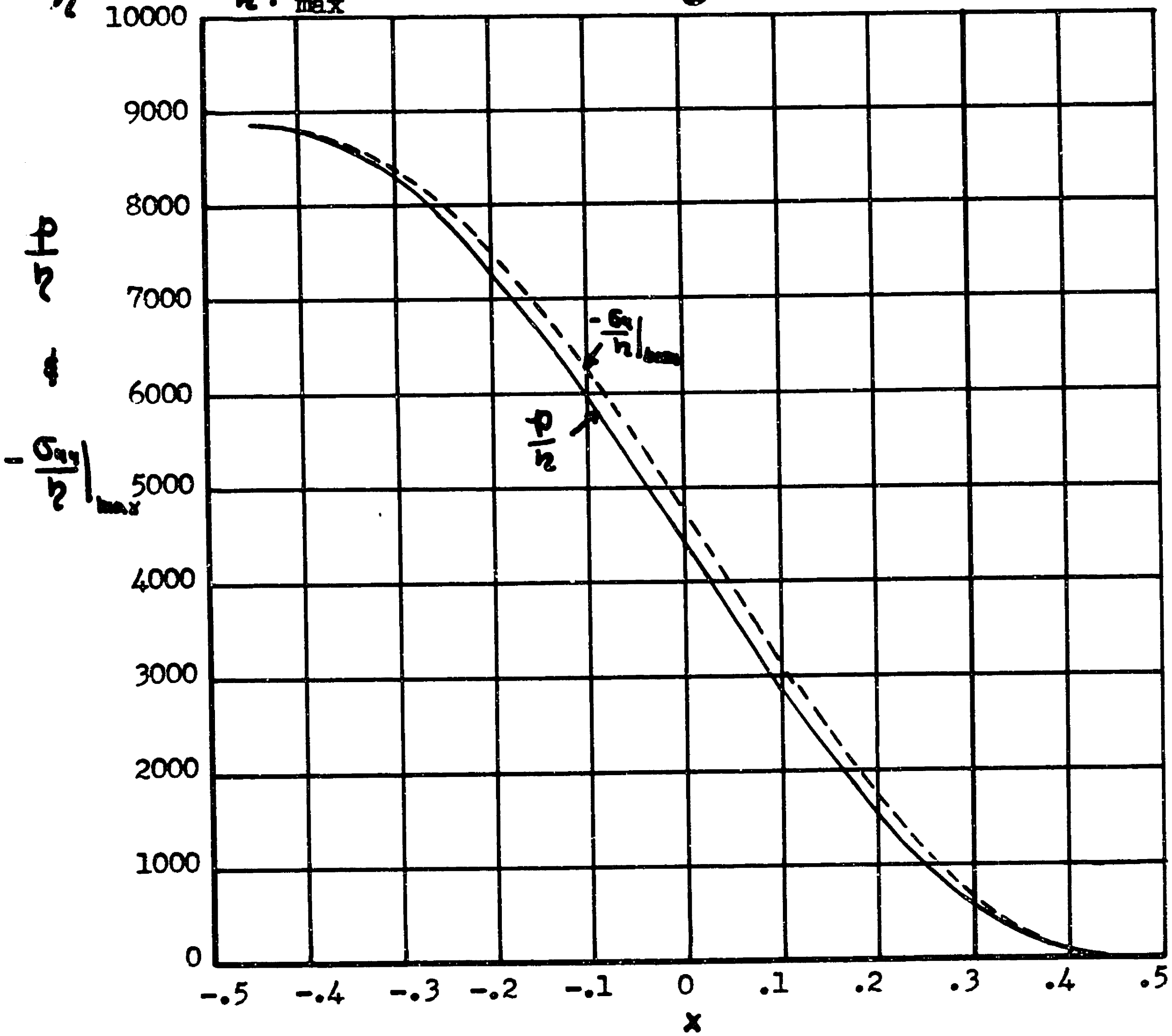


Figure 33

$\frac{p}{\eta}$ and $-\frac{\sigma_y}{\eta}\Big|_{max}$ versus x for $a = 2.5\frac{k^2}{G^2}$ and $\frac{\alpha_0}{\tau_0}\frac{k}{G} = \sqrt{.1}$

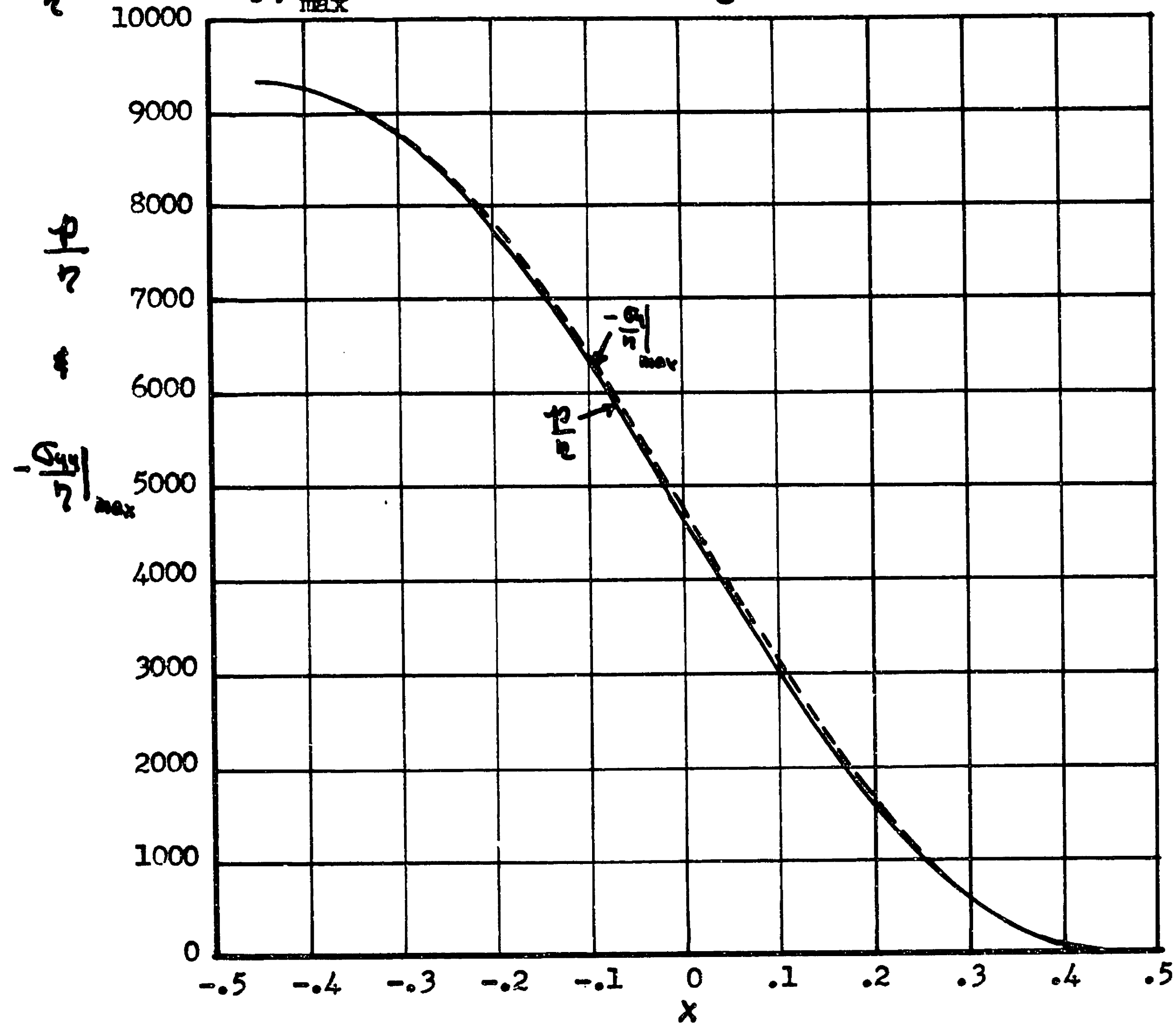


Figure 34

$\frac{p}{h}$ and $-\frac{\sigma_y}{h}\Big|_{max}$ versus x for $a = 2.5\,\frac{k^3}{G^2}$ and $\frac{a_0}{e_0}\,\frac{k}{G} = \sqrt{.5}$

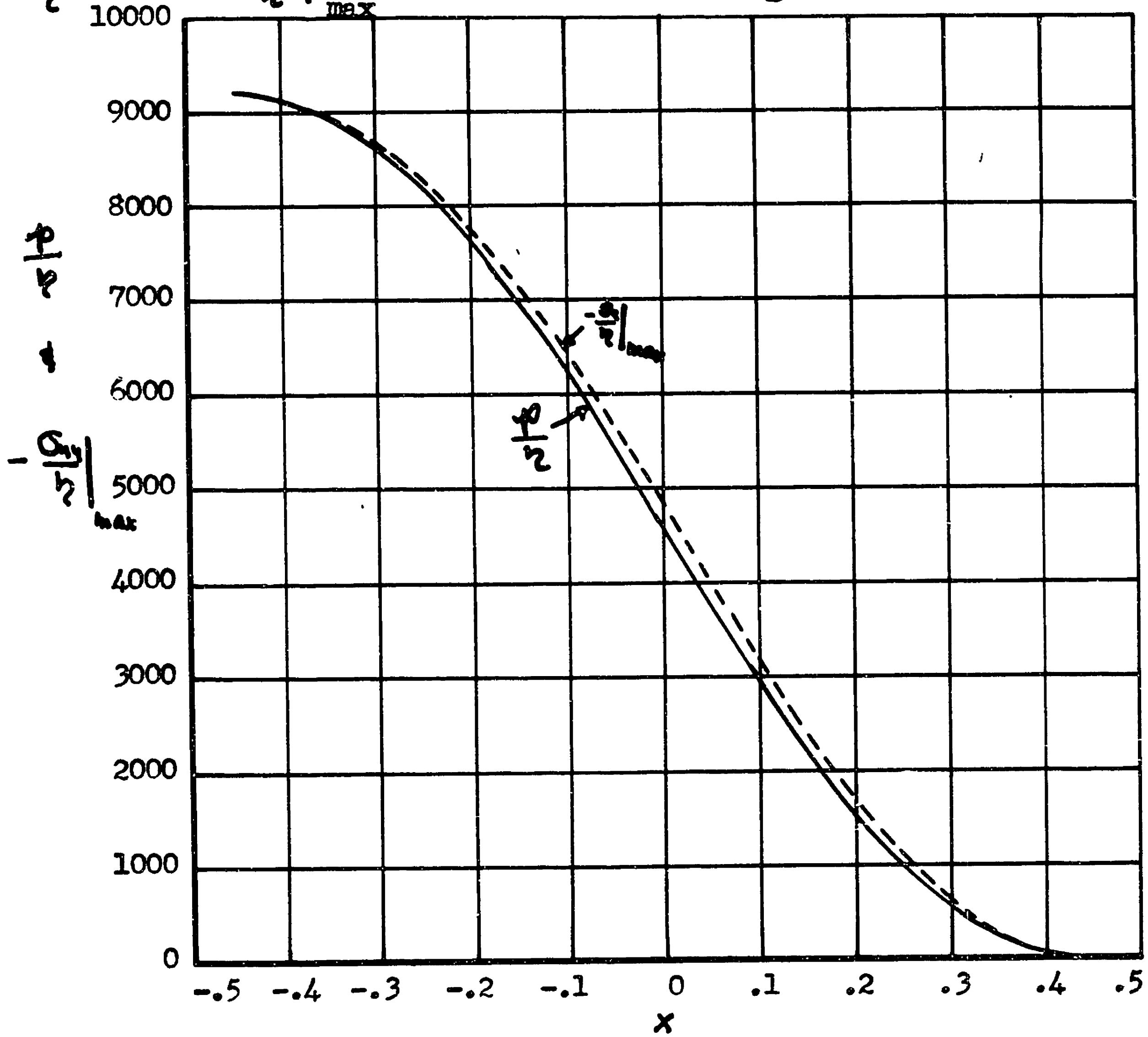


Figure 35

$\frac{p}{h}$ and $-\frac{\sigma_y}{h}\Big|_{max}$ versus x for $a = 8.5\frac{h^2}{G^2}$ and $\frac{u_0}{\tau_0}\frac{h}{G} = 1$

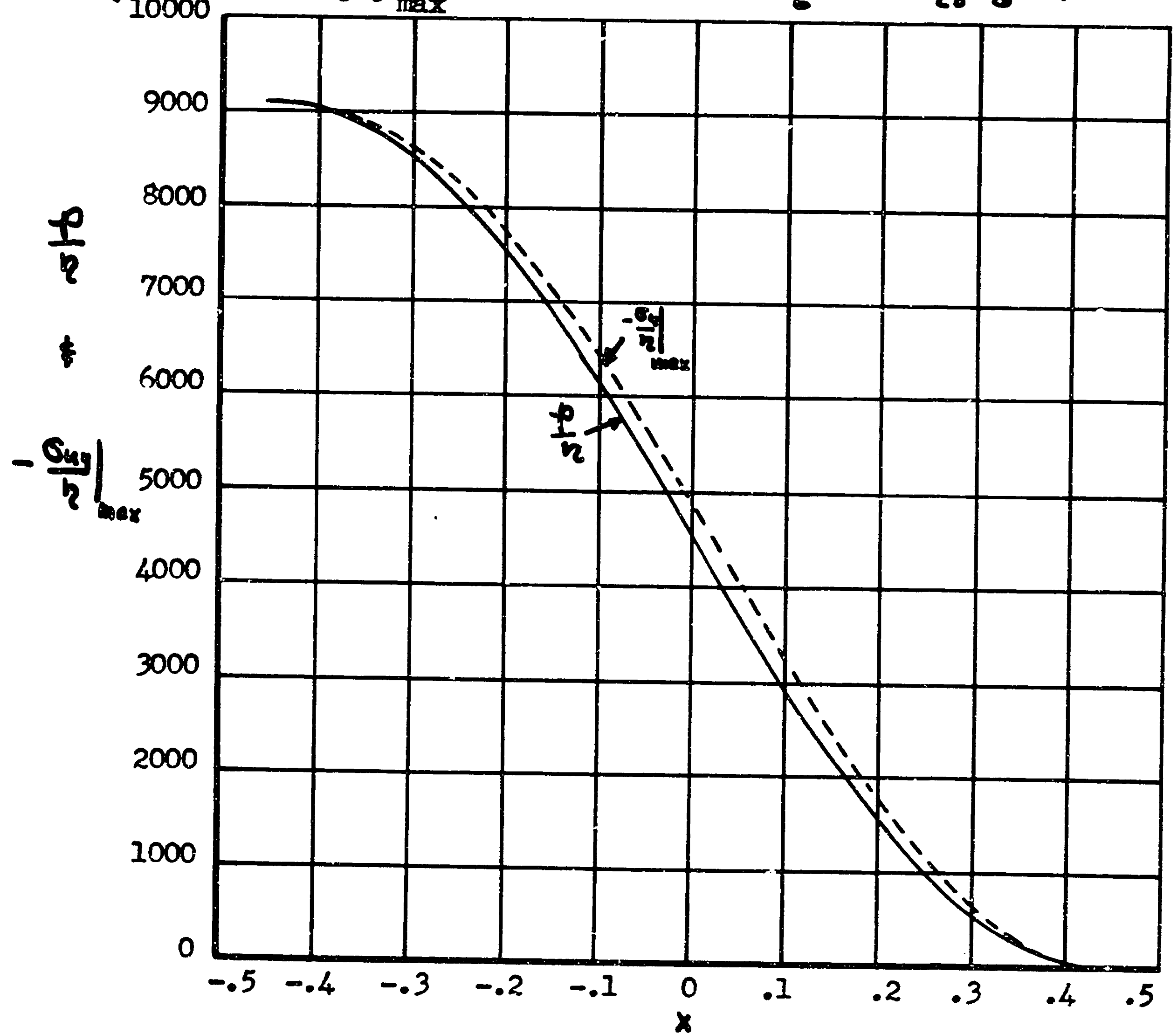


Figure 36

$\frac{p}{\eta}$ and $-\frac{\sigma_y}{\eta}\Big|_{max}$ versus x for $a = 4\frac{h^2}{G^2}$ and $\frac{u_0}{\tau_0}\frac{h}{G} = \sqrt{.1}$

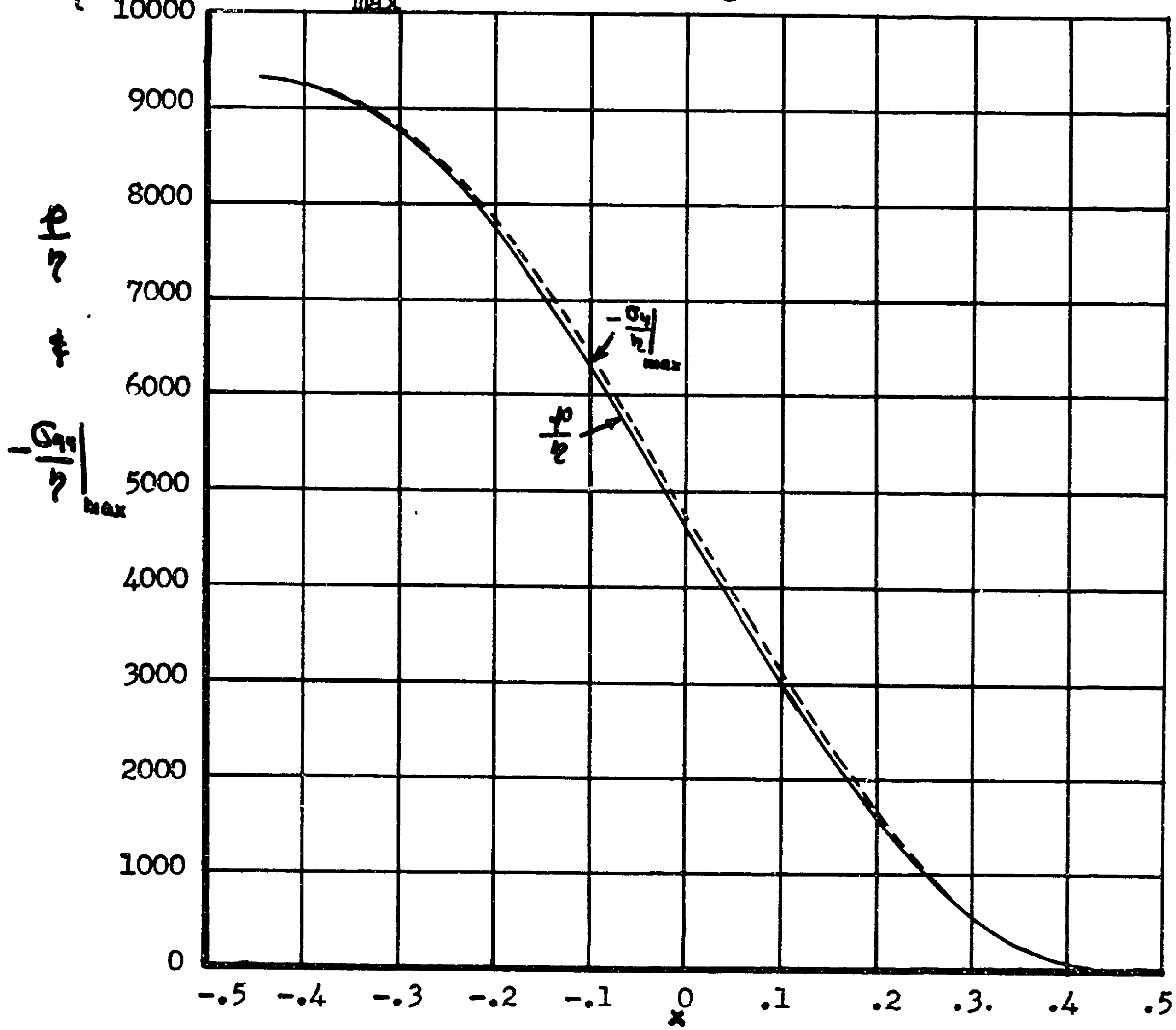


Figure 37

$\frac{p}{h}$ and $-\frac{\sigma_y}{h}\Big|_{max}$ versus x for $a = 4\frac{h^2}{G^2}$ and $\frac{u_0}{t_0}\frac{h}{G} = \sqrt{.5}$

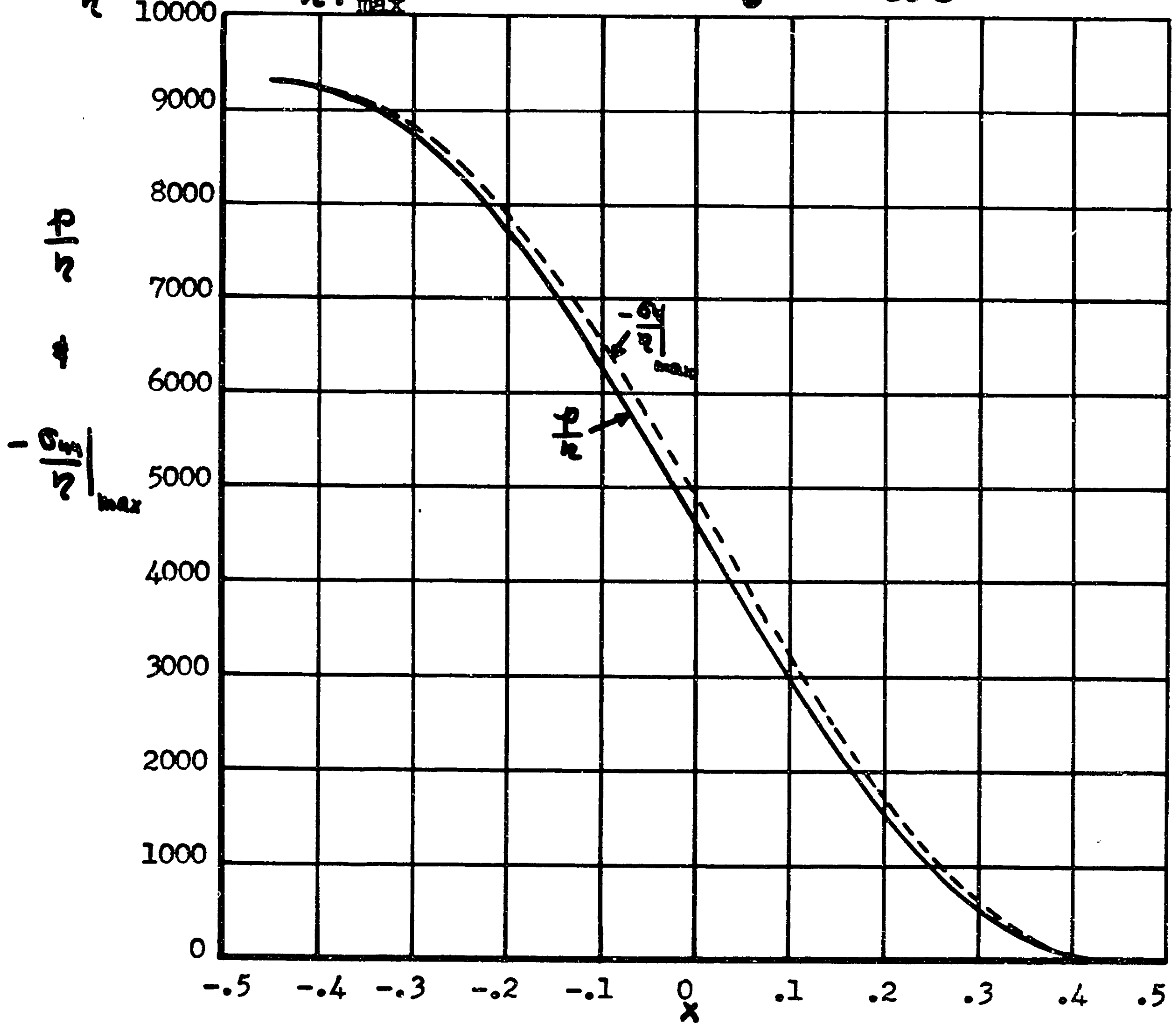
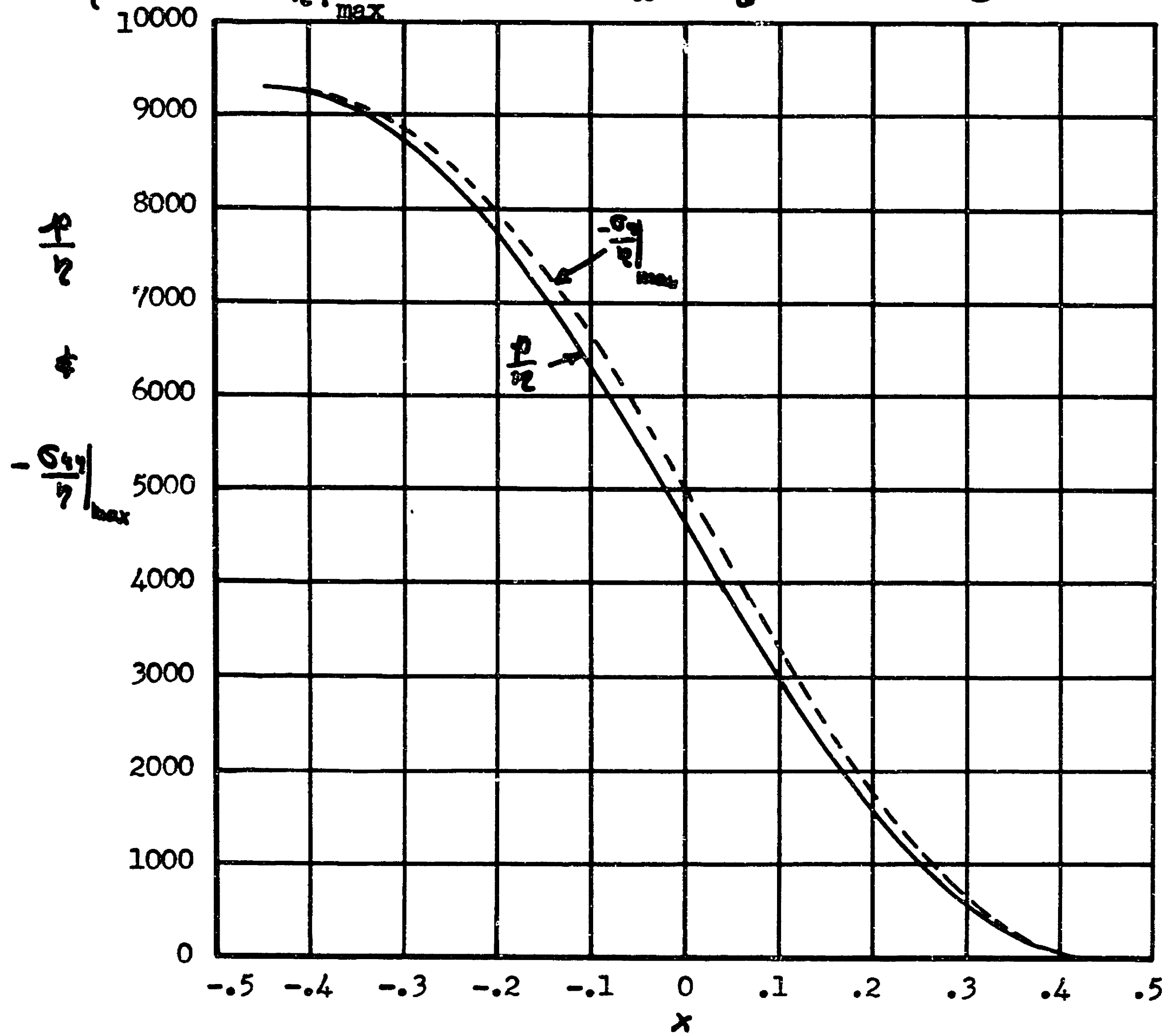


Figure 38

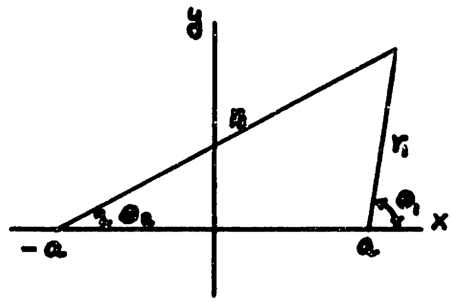
$\frac{p}{\eta}$ and $-\frac{\sigma_y}{\eta}\Big|_{max}$ versus x for $a = 4\,\frac{\kappa^2}{G^2}$ and $\frac{\mu_0}{\varepsilon_0}\,\frac{\kappa}{G} = 1$



APPENDIX

Perturbation Procedure

Before undertaking the formal perturbation, the boundary conditions may be simplified by converting the equations to bipolar coordinates.

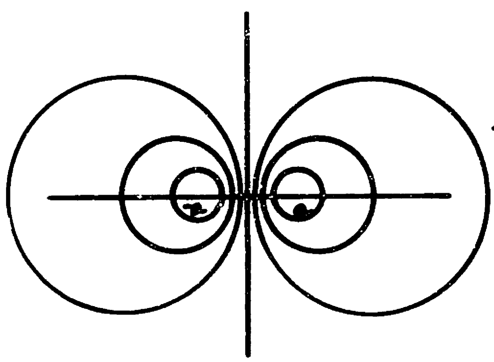


new coordinates

$$\eta = \log \frac{r_2}{r_1}\ , \quad \xi = \theta_1 - \theta_2$$

$$x = \frac{a \sinh \eta}{\cosh \eta - \cos \xi}\ , \quad y = \frac{a \sin \xi}{\cosh \eta - \cos \xi}$$

then lines of η = constant are shown below



η = constant lines

Now the two rolls may be presented by η = constant $\equiv \eta_0$ lines in the orthogonal coordinates. The equations to be transformed are:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \qquad \text{continuity}$$

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \end{aligned}\right\} \quad \text{equilibrium}$$

$$\left(1+\alpha J_2\right)\left(\frac{\partial u_x}{\partial x}\right) = \frac{\sigma_{xx}+p}{2\eta} + \frac{u_x\frac{\partial(\sigma_{xx}+p)}{\partial x} + u_y\frac{\partial(\sigma_{xx}+p)}{\partial y} + \left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)\tau_{xy}}{2G}$$

$$\frac{1}{2}\left(1+\alpha J_2\right)\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) = \frac{\tau_{xy}}{2\eta} + \frac{u_x\frac{\partial \tau_{xy}}{\partial x} + u_y\frac{\partial \tau_{xy}}{\partial y} + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)\left(\sigma_{yy}-\sigma_{xx}\right)}{2G}$$

Following conventional means the equation in $\eta\ \xi$ coordinates are:

$$\frac{\partial}{\partial \eta}\left(\frac{a u_\eta}{\cosh\eta-\cos\xi}\right) + \frac{\partial}{\partial \xi}\left(\frac{a v_\xi}{\cosh\eta-\cos\xi}\right) = 0 \qquad \text{continuity}$$

$$\frac{\partial \sigma_\eta}{\partial \eta} + \frac{\partial \tau_{\eta\xi}}{\partial \xi} + \frac{\sinh\eta\,(\sigma_\xi-\sigma_\eta)}{(\cosh\eta-\cos\xi)} - \frac{2\sin\xi\,(\tau_{\eta\xi})}{(\cosh\eta-\cos\xi)} = 0 \qquad \text{equilibrium}$$

$$\frac{\partial \sigma_\xi}{\partial \eta} + \frac{\partial \tau_{\eta\xi}}{\partial \eta} - \frac{\sin\xi\,(\sigma_\xi-\sigma_\eta)}{(\cosh\eta-\cos\xi)} - \frac{2\sinh\eta\,(\tau_{\eta\xi})}{(\cosh\eta-\cos\xi)} = 0 \qquad \text{equilibrium}$$

$$\begin{aligned} J_2 = {} & \frac{1}{2}\frac{(\sinh^2\eta+2\sin^2\xi)}{a^2}u_\xi^2 + \frac{\sin\xi\sinh\eta}{a^2}u_\xi u_\eta + \frac{1}{2}\frac{(2\sinh^2\eta+\sin^2\xi)}{a^2}u_\eta^2 \\ & + \frac{(\cosh\eta-\cos\xi)}{a^2}\left(u_\xi\sinh\eta+u_\eta\sin\xi\right)\left(\frac{\partial u_\eta}{\partial \xi}+\frac{\partial u_\xi}{\partial \eta}\right) \\ & - \frac{2(\cosh\eta-\cos\xi)}{a^2}\left(u_\eta\frac{\partial u_\xi}{\partial \xi}\sinh\eta + u_\xi\frac{\partial u_\eta}{\partial \eta}\sin\xi\right) \\ & + \frac{(\cosh\eta-\cos\xi)^2}{2a^2}\left[\left(\frac{\partial u_\eta}{\partial \xi}+\frac{\partial u_\xi}{\partial \eta}\right)^2 + 2\left(\frac{\partial u_\xi}{\partial \xi}\right)^2 + 2\left(\frac{\partial u_\eta}{\partial \eta}\right)^2\right] \end{aligned}$$

xx stress strain equation

$$\left(1+\alpha J_2\right)\left\{\frac{-\sin\xi\cosh\eta}{a(\cosh\eta-\cos\xi)}\left[u_\xi(1-\cos\xi\cosh\eta)+u_\eta\sin\xi\sinh\eta\right]\right.$$

$$+\frac{\sin\xi\sinh\eta}{a(\cosh\eta-\cos\xi)}\left(\frac{\partial u_\xi}{\partial\xi}\sin\xi\sinh\eta-\frac{\partial u_\eta}{\partial\xi}(1-\cos\xi\cosh\eta)\right)$$

$$\left.-\frac{(1-\cos\xi\cosh\eta)}{a(\cosh\eta-\cos\xi)}\left(\frac{\partial u_\xi}{\partial\eta}\sin\xi\sinh\eta-\frac{\partial u_\eta}{\partial\eta}(1-\cos\xi\cosh\eta)\right)\right\} =$$

$$\frac{(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)(\sigma_\xi-\sigma_\eta)}{4G(\cosh\eta-\cos\xi)^2}$$

$$+\frac{(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)}{4Ga(\cosh\eta-\cos\xi)}\left(u_\xi\frac{\partial(\sigma_\xi-\sigma_\eta)}{\partial\xi}+u_\eta\frac{\partial(\sigma_\xi-\sigma_\eta)}{\partial\eta}\right)$$

$$+\frac{\sin\xi\sinh\eta(1-\cos\xi\cosh\eta)(\sigma_\xi-\sigma_\eta)}{2Ga(\cosh\eta-\cos\xi)^2}\left(u_\eta\sin\xi-u_\xi\sinh\eta\right)$$

$$+\frac{(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)\tau_{\eta\xi}}{2Ga(\cosh\eta-\cos\xi)^2}\left(u_\xi\sinh\eta-u_\eta\sin\xi\right)$$

$$+\frac{(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)\tau_{\eta\xi}}{2Ga(\cosh\eta-\cos\xi)}\left(\frac{\partial u_\eta}{\partial\xi}-\frac{\partial u_\xi}{\partial\eta}\right)$$

$$+\frac{\sin\xi\sinh\eta(1-\cos\xi\cosh\eta)(\sigma_\xi-\sigma_\eta)}{2Ga(\cosh\eta-\cos\xi)}\left(\frac{\partial u_\eta}{\partial\xi}-\frac{\partial u_\xi}{\partial\eta}\right)$$

xy stress strain equation

$$\frac{(1+\alpha J_2)}{2a(\cosh\eta-\cos\xi)}\left\{u_\xi\left[\sin^2\xi\sinh\eta\cosh\eta+(1-\cos\xi\cosh\eta)\cos\xi\sinh\eta\right]\right.$$

$$+u_\eta\left[\sinh^2\eta\sin\xi\cos\xi-(1-\cos\xi\cosh\eta)\sin\xi\cosh\eta\right]$$

$$+\left(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2\right)\left(\frac{\partial u_\eta}{\partial\xi}+\frac{\partial u_\xi}{\partial\eta}\right)$$

$$\left.+2\sin\xi\sinh\eta(1-\cos\xi\cosh\eta)\left(\frac{\partial u_\xi}{\partial\xi}-\frac{\partial u_\eta}{\partial\eta}\right)\right\} =$$

$$\frac{(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)\tau_{\eta\xi}+\sin\xi\sinh\eta(1-\cos\xi\cosh\eta)(\sigma_\xi-\sigma_\eta)}{2G(\cosh\eta-\cos\xi)^2}$$

$$+\frac{\sin\xi\sinh\eta(1-\cos\xi\cosh\eta)}{2Ga(\cosh\eta-\cos\xi)}\left(u_\xi\frac{\partial(\sigma_\xi-\sigma_\eta)}{\partial\xi}+u_\eta\frac{\partial(\sigma_\xi-\sigma_\eta)}{\partial\eta}\right)$$

$$+\frac{(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)(\sigma_\xi-\sigma_\eta)}{4Ga(\cosh\eta-\cos\xi)^2}\left(u_\xi\sinh\eta-u_\eta\sin\xi\right)$$

$$+\frac{2\sin\xi\sinh\eta\,(1-\cos\xi\cosh\eta)\,\tau_{\eta\xi}}{2Ga\,(\cosh\eta-\cos\xi)^2}\,(u_\eta\sin\xi - u_\xi\sinh\eta)$$

$$+\frac{2\sin\xi\sinh\eta\,(1-\cos\xi\cosh\eta)\,\tau_{\eta\xi}}{2Ga\,(\cosh\eta-\cos\xi)}\left(\frac{\partial u_\eta}{\partial\xi}-\frac{\partial u_\xi}{\partial\eta}\right)$$

$$+\frac{(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)(\sigma_\xi-\sigma_\eta)}{4Ga\,(\cosh\eta-\cos\xi)}\left(\frac{\partial u_\xi}{\partial\eta}-\frac{\partial u_\eta}{\partial\xi}\right)$$

$$+\frac{(1-\cos\xi\cosh\eta)(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)}{2Ga\,(\cosh\eta-\cos\xi)^3}\left(\frac{\partial\tau_{\eta\xi}}{\partial\eta}\right)\Big((1-\cos\xi\cosh\eta)u_\eta-(\sin\xi\sinh\eta)u_\xi\Big)$$

$$+\frac{\sin\xi\sinh\eta\,(\sin^2\xi\sinh^2\eta-(1-\cos\xi\cosh\eta)^2)}{2Ga\,(\cosh\eta-\cos\xi)^3}\left(\frac{\partial\tau_{\eta\xi}}{\partial\xi}\right)\Big((\sin\xi\sinh\eta)u_\xi-(1-\cos\xi\cosh\eta)u_\eta\Big)$$

Before the perturbation can be carried out the equations must be written out in dimensionless form. The coordinates, velocities and stresses are collected in the following groups:

$$\frac{\eta}{\eta_0},\ \xi,\ \frac{u_\xi}{U},\ \frac{u_\eta}{U},\ \frac{\tau_{\eta\xi}\,t_0}{4U},\ \frac{\sigma_\xi\,t_0\,\eta_0}{4U},\ \frac{\sigma_\eta\,t_0\,\eta_0}{4U}$$

For the class of problem considered here $\frac{U^2}{t_0\eta}$ $\frac{\mu}{G}$, and η_0 are small. Because of this we hope the following expansions are permissable.

$$u_\xi(\eta,\xi,\eta_0,\delta,\epsilon) = u_{\xi_0}(\eta,\xi)+\eta_0 u_{\xi_1}(\eta,\xi)+\delta u_{\xi_2}(\eta,\xi)+\epsilon u_{\xi_3}(\eta,\xi)$$
$$+\eta_0^2 u_{\xi_4}(\eta,\xi)+\delta^2 u_{\xi_5}(\eta,\xi)+\epsilon^2 u_{\xi_6}(\eta,\xi)+\eta_0\delta u_{\xi_7}(\eta,\xi)$$
$$+\eta_0\epsilon u_{\xi_8}(\eta,\xi)+\delta\epsilon u_{\xi_9}(\eta,\xi)+\cdots$$

$$u_\eta(\eta,\xi,\eta_0,\delta,\epsilon) = \eta_0\big(u_{\eta_1}(\eta,\xi)+\eta_0 u_{\eta_4}(\eta,\xi)+\delta u_{\eta_7}(\eta,\xi)+\epsilon u_{\eta_8}(\eta,\xi)$$
$$+\eta_0^2 u_{\eta_{10}}(\eta,\xi)+\delta^2 u_{\eta_{11}}(\eta,\xi)+\epsilon^2 u_{\eta_{12}}(\eta,\xi)+\eta_0\delta u_{\eta_{13}}(\eta,\xi)$$
$$+\eta_0\epsilon u_{\eta_{14}}(\eta,\xi)+\delta\epsilon u_{\eta_{15}}(\eta,\xi)+\cdots\big)$$

$$\tau_{\eta\xi}(\eta,\xi,\eta_0,\delta,\epsilon) = \eta_0\big(\tau_{\eta\xi_1}(\eta,\xi)+\eta_0\tau_{\eta\xi_4}(\eta,\xi)+\delta\tau_{\eta\xi_7}(\eta,\xi)+\tau_{\eta\xi_8}(\eta,\xi)$$
$$+\eta_0^2\tau_{\eta\xi_{10}}(\eta,\xi)+\delta^2\tau_{\eta\xi_{11}}(\eta,\xi)+\epsilon^2\tau_{\eta\xi_{12}}(\eta,\xi)+\eta_0\delta\tau_{\eta\xi_{13}}(\eta,\xi)$$
$$+\eta_0\epsilon\tau_{\eta\xi_{14}}(\eta,\xi)+\delta\epsilon\tau_{\eta\xi_{15}}(\eta,\xi)+\cdots\big)$$

$$\sigma_\xi(\eta,\xi,\eta_0,\delta,\varepsilon) = \sigma_{\xi_0}(\eta,\xi) + \eta_0\sigma_{\xi_1}(\eta,\xi) + \delta\sigma_{\xi_2}(\eta,\xi) + \varepsilon\sigma_{\xi_3}(\eta,\xi)$$
$$+ \eta_0^2\sigma_{\xi_4}(\eta,\xi) + \delta^2\sigma_{\xi_5}(\eta,\xi) + \varepsilon^2\sigma_{\xi_6}(\eta,\xi) + \eta_0\delta\sigma_{\xi_7}(\eta,\xi)$$
$$+ \eta_0\varepsilon\sigma_{\xi_8}(\eta,\xi) + \delta\varepsilon\sigma_{\xi_9}(\eta,\xi) + \cdots$$
$$\sigma_\eta(\eta,\xi,\eta_0,\delta,\varepsilon) = \sigma_{\eta_0}(\eta,\xi) + \eta_0\sigma_{\eta_1}(\eta,\xi) + \delta\sigma_{\eta_2}(\eta,\xi) + \varepsilon\sigma_{\eta_3}(\eta,\xi)$$
$$+ \eta_0^2\sigma_{\eta_4}(\eta,\xi) + \delta^2\sigma_{\eta_5}(\eta,\xi) + \varepsilon^2\sigma_{\eta_6}(\eta,\xi) + \eta_0\delta\sigma_{\eta_7}(\eta,\xi)$$
$$+ \eta_0\varepsilon\sigma_{\eta_8}(\eta,\xi) + \delta\varepsilon\sigma_{\eta_9}(\eta,\xi) + \cdots$$

where η_0 = value of η on roll, $\varepsilon = \frac{4}{G}\frac{U_0}{\tau_0}$, $\delta = a\left(\frac{U_0}{\tau_0}\right)^2$

By substituting the expansions into the original equations and expanding the $\cosh\eta$ and $\sinh\eta$ in their respective series , a set of equations in powers of η , δ and ε are obtained. If equal powers are equated we obtain sets of equations which may be solved. The first of these sets of equations are coefficients which are not multiplied by η , δ or ε . These are:

$$\frac{\partial u_{\xi_0}}{\partial\xi} + \eta_0\frac{\partial u_{\eta_1}}{\partial\eta} = \frac{\sin\xi}{1-\cos\xi}u_{\xi_0} \qquad \text{continuity}$$

$$\frac{\partial\sigma_{\xi_0}}{\partial\xi} + \frac{\partial\tau_{\eta\xi}}{\partial\eta} = 0 \qquad \xi \text{ equilibrium}$$

$$\sigma_{\eta_0} = \frac{24U}{a\eta_0^2}f(\xi) \qquad \eta \text{ equation}$$

$$\sigma_{\eta_0} = \sigma_{\xi_0} \qquad xx \text{ stress strain}$$

$$\tau_{\eta\xi_1} = \frac{4}{a}(1-\cos\xi)\frac{\partial u_{\xi_0}}{\partial\eta} \qquad xy \text{ stress strain}$$

This set of equations may be solved if the following boundary conditions are imposed.

$$\tau_{\eta\xi_0} = 0 \quad \text{when} \quad \eta = 0$$

$$u_{\xi_0} = U \quad \text{when} \quad \eta = \pm\eta_0$$

$$\frac{2\alpha u_0 \eta_0}{1-\cos\xi_1} = 2\int_0^{\eta_0} \frac{\alpha u_{\xi_0}}{\cosh\eta - \cos\xi}\, d\eta \qquad \xi_1 = \text{entrance}$$

$$\int_0^{\eta_0} \frac{\alpha \sigma_{\xi_0}}{\cosh\eta - \cos\xi}\, d\eta \Big|_{\xi=\xi_E} = 0 \qquad \xi_E = \text{exit}$$

$$u_{\eta_1} = 0 \quad \text{when} \quad \eta = 0$$

The solution is:

$$u_{\xi_0} = \frac{3}{2} U \left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(1 - \frac{\eta^2}{\eta_0^2}\right) + U$$

$$u_{\eta_1} = \frac{U}{2} \frac{\sin\xi}{(1-\cos\xi)} \left(\frac{\eta^3}{\eta_0^3} - \frac{\eta}{\eta_0}\right)$$

$$\tau_{\eta\xi_1} = -3\frac{4U}{\alpha\eta_0}\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\frac{\eta}{\eta_0}$$

$$\sigma_{\xi_0} = \sigma_{\eta_0} = 3\frac{4U}{\alpha\eta_0^2}\left\{\frac{1}{(1-\cos\xi_1)}\left[\frac{3}{2}(\xi-\xi_E) - 2(\sin\xi - \sin\xi_E) + \frac{1}{4}(\sin 2\xi - \sin 2\xi_E) - (\xi-\xi_E) + (\sin\xi - \sin\xi_E)\right]\right\}$$

For the remainder of the solutions the boundary conditions are:

$$\tau_{\eta\xi_n} = 0 \quad \text{when} \quad \eta = 0$$

$$u_{\xi_n} = 0 \quad \text{when} \quad \eta = \pm\eta_0$$

$$\int_0^{\eta_0} \frac{u_{\xi_n}\, d\eta}{\cosh\eta - \cos\xi} = 0$$

$$\int_0^{\eta_0} \frac{\alpha\sigma_{\xi_n}\, d\eta}{\cosh\eta - \cos\xi}\Big|_{\xi=\xi_E} = 0$$

$$u_{\eta_n} = 0 \quad \text{when} \quad \eta = 0$$

The equations and resulting solutions are given on the following pages. In cases where the normal stresses are more easily obtained by a numerical integration rather than by substitution into the intergrated form, the equation has been left as an integral (for example see 6 order equation).

Zero Order Equations

$$\frac{\partial u_{\xi_0}}{\partial \xi} + \eta_0 \frac{\partial u_{\xi_1}}{\partial \eta} = \frac{\sin \xi}{1-\cos \xi} u_{\xi_0}$$ continuity

$$\frac{\partial \sigma_{\xi_0}}{\partial \xi} + \frac{\partial \tau_{\eta\xi_1}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{\eta_0} = \frac{24U}{a\,\eta_0^{3}} f(\xi)$$ equilibrium

$$\sigma_{\eta_0} = \sigma_{\xi_0}$$ stress strain

$$\tau_{\eta\xi_1} = \frac{4}{a}(1-\cos\xi)\frac{\partial u_{\xi_0}}{\partial \eta}$$ stress strain

Zero Order Solution

$$u_{\xi_0} = \frac{3U}{2}\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(1 - \frac{\eta^2}{\eta_0^2}\right) + U$$

$$u_{\eta_1} = \frac{U}{2}\left(\frac{\sin\xi}{1-\cos\xi_1}\right)\left(\frac{\eta^3}{\eta_0^3} - \frac{\eta}{\eta_0}\right)$$

$$\tau_{\eta\xi_1} = -\frac{3\mu U}{a\,\eta_0}\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)(1-\cos\xi)\frac{\eta}{\eta_0}$$

$$\begin{aligned}\sigma_{\xi_0} = \sigma_{\eta_0} = {} & \frac{3\mu U}{a\,\eta_0^2}\left(\frac{1}{(1-\cos\xi_1)}\left[\frac{3}{2}(\xi-\xi_E) - 2(\sin\xi - \sin\xi_E)\right]\right) \\ & + \frac{3\mu U}{a\,\eta_0^2}\left(\frac{1}{1-\cos\xi_1}\left(\frac{1}{4}\sin 2\xi - \frac{1}{4}\sin 2\xi_E\right) - (\xi - \xi_E)\right) \\ & + \frac{3\mu U}{a\,\eta_0^2}\left(\sin\xi - \sin\xi_E\right)\end{aligned}$$

η_0 Order Equations

$$\frac{\partial u_{\xi_1}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta_4}}{\partial \eta} = \frac{\sin \xi}{1 - \cos \xi} u_{\xi_1}$$ continuity

$$\frac{\partial \sigma_{\xi_1}}{\partial \xi} + \frac{\partial \tau_{\eta\xi_4}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{\eta_1} = \frac{2\psi U}{a \eta_0^2} f(\xi)$$ equilibrium

$$\sigma_{\eta_1} = \sigma_{\xi_1}$$ stress strain

$$\tau_{\eta\xi_4} = \frac{\psi}{a}(1 - \cos \xi) \frac{\partial u_{\xi_1}}{\partial \eta}$$ stress strain

η_0 Order Solution

trivial solution

ε Order Equations

$$\frac{\partial u_{\xi_3}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta_3}}{\partial \eta} = \frac{\sin\xi}{1-\cos\xi} u_{\xi_3}$$ continuity

$$\frac{\partial \sigma_{\xi_3}}{\partial \xi} + \frac{\partial \tau_{\eta\xi_3}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{\eta_3} = \sigma_{\xi_3} = \frac{24U}{a\,\eta_0^2} f(\xi)$$ equilibrium
stress strain

$$\tau_{\eta\xi_3} = \frac{4}{a}(1-\cos\xi)\frac{\partial u_{\xi_3}}{\partial \eta}$$ stress strain

ε Order Solution

trivial solution

δ Order Equations

$$\frac{\partial u_{\xi 2}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta 7}}{\partial \eta} = \frac{\sin \xi}{(1-\cos \xi)} u_{\xi 2}$$ continuity

$$\frac{\partial \sigma_{\xi 2}}{\partial \xi} + \frac{\partial \tau_{\eta \xi 7}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{\eta_2} = \frac{24U}{a\,\eta_0^2} f(\xi)$$ equilibrium

$$\sigma_{\eta_2} = \sigma_{\xi 2}$$ stress strain

$$\tau_{\eta \xi_7} = \frac{4}{a}(1-\cos \xi)\left(\frac{\partial u_{\xi 2}}{\partial \eta} + \eta_0^2 \frac{(1-\cos \xi)^2}{8U^2}\left(\frac{\partial u_{\xi 0}}{\partial \eta}\right)^3 \right)$$ stress strain

δ Order Solution

$$u_{\xi_2} = \frac{U}{160}(1-\cos\xi)^2\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^3\left(135\frac{\eta^4}{\eta_0^4}-162\frac{\eta^2}{\eta_0^2}+27\right)$$

$$u_{\eta_T} = -\frac{U}{160}\sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^2\left[4\frac{1-\cos\xi}{1-\cos\xi}-1\right]\left(27\frac{\eta^5}{\eta_0^5}-54\frac{\eta^3}{\eta_0^3}+27\frac{\eta}{\eta_0}\right)$$

$$\tau_{\varphi\xi_T} = -\frac{81\,\mu U}{40a\,\eta_0}(1-\cos\xi)^3\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^3\frac{\eta}{\eta_0}$$

$$\sigma_{\xi_2} = \sigma_{\eta_2} = +\frac{81\,\mu U}{40a\eta_0^2}\int_{\xi_E}^{\xi}(1-\cos\xi)^3\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^3 d\xi$$

η_0^2 Order Equations

$$\frac{\partial u_{\xi 4}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta 10}}{\partial \eta} = \frac{\sin\xi}{1-\cos\xi} u_{\xi 4} + \frac{\eta}{\eta_0(1-\cos\xi)} u_{\eta 1} - \frac{\eta^2}{2\eta_0^2(1-\cos\xi)}\left(\frac{\partial u_{\xi 0}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta 1}}{\partial \eta}\right)$$

continuity

$$\frac{\partial \sigma_{\xi 4}}{\partial \xi} + \frac{\partial \tau_{\eta\xi 10}}{\partial \eta} = \frac{\sin\xi}{1-\cos\xi}(\sigma_{\xi 4} - \sigma_{\eta 4}) + \frac{2\eta}{\eta_0^2(1-\cos\xi)} \tau_{\eta\xi 1} - \frac{\eta^2}{2\eta_0^2(1-\cos\xi)}\left(\frac{\partial \sigma_{\xi 0}}{\partial \xi} + \frac{\partial \tau_{\eta\xi 1}}{\partial \eta}\right)$$

equilibrium

$$\frac{\partial \sigma_{\eta 4}}{\partial \eta} = \frac{2\sin\xi}{\eta_0^2(1-\cos\xi)} \tau_{\eta\xi 1} - \frac{1}{\eta_0^2}\frac{\partial \tau_{\eta\xi 1}}{\partial \xi}$$

equilibrium

$$2\sin\xi\left(u_{\xi 0} + \eta\frac{\partial u_{\xi 0}}{\partial \eta}\right) - 2(1-\cos\xi)\eta_0 \frac{\partial u_{\eta 1}}{\partial \eta} = \frac{a\eta_0^2}{24}(\sigma_{\xi 4} - \sigma_{\eta 4})$$

stress strain

$$\frac{1}{2}(1-\cos\xi)\left[\frac{\eta}{\eta_0}\sin^2\xi + \frac{\eta}{\eta_0}\cos\xi(1-\cos\xi)\right] u_{\xi 0} - \sin\xi(1-\cos\xi)^2 u_{\eta 1}$$

$$-(1-\cos\xi)^3\left(\frac{\partial u_{\eta 1}}{\partial \xi} + \eta_0\frac{\partial u_{\xi 4}}{\partial \eta}\right) + 2\sin\xi(1-\cos\xi)^2\frac{\eta}{\eta_0}\left(\frac{\partial u_{\xi 0}}{\partial \xi} - \eta_0\frac{\partial u_{\eta 1}}{\partial \eta}\right)$$

$$+\frac{\eta_0}{24}(1-\cos\xi)^3\frac{\partial u_{\xi 0}}{\partial \eta} = -\frac{a\eta_0}{4}(1-\cos\xi)^2\tau_{\eta\xi 10}$$

$$+\frac{a\eta_0}{4}\left(\frac{\eta^2}{\eta_0^2}\sin^2\xi + \frac{\eta^2}{\eta_0^2}\cos\xi(1-\cos\xi) - \frac{1}{2}\frac{\eta^2}{\eta_0^2}(1-\cos\xi)\right)\tau_{\eta\xi 1}$$

$$+\frac{a\eta_0\eta}{4}\sin\xi(1-\cos\xi)(\sigma_{\xi 4} - \sigma_{\eta 4})$$

stress strain

η_0^2 Order Solution

$$u_{\xi_4} = \frac{U}{16(1-\cos\xi)}\left(\frac{\eta^4}{\eta_0^4} - \frac{6}{5}\frac{\eta^2}{\eta_0^2} + \frac{1}{5}\right)\left(-82\cos\xi - 39 + \frac{1-\cos\xi}{1-\cos\xi_1}(102\cos\xi + 55)\right)$$

$$u_{\eta_{10}} = \frac{U\sin\xi}{80(1-\cos\xi)^2}\left(\frac{\eta^5}{\eta_0^5} - 2\frac{\eta^3}{\eta_0^3} + \frac{\eta}{\eta_0}\right)\left(-82\cos\xi - 160 + \frac{1-\cos\xi}{1-\cos\xi_1}(157)\right)$$
$$+ \frac{U\sin\xi}{16(1-\cos\xi)^2}\left(\frac{\eta^5}{5\eta_0^5} - \frac{\eta^3}{3\eta_0^3}\right)\left(-4 + 12\frac{1-\cos\xi}{1-\cos\xi_1}\right) - \frac{U\sin\xi}{6(1-\cos\xi)^2}\frac{\eta^3}{\eta_0^3}$$

$$\tau_{\eta\xi_{10}} = \frac{4U}{40a\eta_0}\left(210 - 20\cos\xi + \frac{1-\cos\xi}{1-\cos\xi_1}(-80 + 240\cos\xi)\right)\frac{\eta^3}{\eta_0^3}$$
$$+ \frac{4U}{40a\eta_0}\left(199 + 157\cos\xi + \frac{1-\cos\xi}{1-\cos\xi_1}(-235 - 557\cos\xi)\right)\frac{\eta}{\eta_0}$$

$$\sigma_{\xi_4} = \frac{34U\sin\xi}{2a\eta_0^2}\left(\frac{\eta^2}{\eta_0^2}\right) - \frac{4U(297\sin\xi + 199\xi)}{40a\eta_0^2} + \frac{4U(153\xi + 164\sin\xi)}{80(1-\cos\xi_1)a\eta_0^2}$$
$$- \frac{4U(317\sin\xi\cos\xi)}{80(1-\cos\xi_1)a\eta_0^2} + \frac{24U}{a\eta_0^2}\sin\xi\left(3\frac{1-\cos\xi}{1-\cos\xi_1} - 2\right)\left(1 - 3\frac{\eta^2}{\eta_0^2}\right)$$
$$- \frac{4U\sin\xi_E}{2a\eta_0^2} + \frac{4U(297\sin\xi_E + 199\xi_E)}{40a\eta_0^2} - \frac{4U(153\xi_E + 164\sin\xi_E)}{80(1-\cos\xi_1)a\eta_0^2}$$
$$+ \frac{4U(317\sin\xi_E\cos\xi_E)}{80(1-\cos\xi_1)a\eta_0^2}$$

$$\sigma_{\xi_4} - \sigma_{\eta_4} = \frac{4U}{a\eta_0^2}\left(4\sin\xi + 2\sin\xi\left(3\frac{1-\cos\xi}{1-\cos\xi_1} - 2\right)\left(1 - 3\frac{\eta^2}{\eta_0^2}\right)\right)$$

ε^2 Order Equations

$$\frac{\partial u_{\xi 6}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta 12}}{\partial \eta} = \frac{\sin\xi}{1-\cos\xi} u_{\xi 6}$$ continuity

$$\frac{\partial \sigma_{\xi 6}}{\partial \xi} + \frac{\partial \tau_{\eta\xi 12}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{\eta 6} = \frac{24U}{a\eta_0^2} f(\xi)$$ equilibrium

$$\sigma_{\xi 6} = \sigma_{\eta 6}$$ stress strain

$$4(1-\cos\xi)\eta_0 \frac{\partial u_{\xi 6}}{\partial \eta} = a\eta_0 \tau_{\eta\xi 12} - \frac{a\eta_0^3}{4U}(1-\cos\xi)(\sigma_{\xi 8} - \sigma_{\eta 8})\frac{\partial u_{\xi 0}}{\partial \eta}$$
stress strain

also from $\eta_0 \varepsilon$ equations (since $\sigma_{\xi 8} - \sigma_{\eta 8}$ is unknown)

$$\sigma_{\xi 8} - \sigma_{\eta 8} = \tau_{\eta\xi 1} \frac{(1-\cos\xi)}{U} \frac{\partial u_{\xi 0}}{\partial \eta}$$ stress strain

ε^{12} Order Solution

$$u_{\xi_6} = \frac{27U}{16}(1-\cos\xi)^2\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^3\left(\frac{\eta^4}{\eta_0^4}-\frac{6}{5}\frac{\eta^2}{\eta_0^2}+\frac{1}{5}\right)$$

$$u_{\eta_{12}} = -\frac{27U}{16}\sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^2\left(4\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)\left(\frac{\eta^5}{5\eta_0^5}-\frac{2\eta^3}{5\eta_0^3}+\frac{\eta}{5\eta_0}\right)$$

$$\tau_{\eta\xi_{12}} = -\frac{81}{20}\frac{4U}{a\eta_0}(1-\cos\xi)^3\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^3\frac{\eta}{\eta_0}$$

$$\sigma_{\xi_6} = \sigma_{\eta_6} = \frac{81}{20}\frac{4U}{a\eta_0^2}\int_{\xi_{1i}}^{\xi}(1-\cos\xi)^3\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^3 d\xi$$

δ^2 Order Equations

$$\frac{\partial u_{\xi 5}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta 11}}{\partial \eta} = \frac{\sin \xi}{1-\cos \xi} u_{\xi 5}$$ continuity

$$\frac{\partial \sigma_{\xi 5}}{\partial \xi} + \frac{\partial \tau_{\eta \xi 11}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{\eta 5} = \frac{24U}{a \eta_0^2} f(\xi)$$ equilibrium

$$\sigma_{\xi 5} = \sigma_{\eta 5}$$ stress strain

$$\eta_0 (1-\cos \xi) \frac{\partial u_{\xi 5}}{\partial \eta} + \frac{3 \eta_0^3}{8 U^2} (1-\cos \xi)^3 \left(\frac{\partial u_{\xi 0}}{\partial \eta}\right)^2 \left(\frac{\partial u_{\xi 2}}{\partial \eta}\right) = \frac{a \eta_0}{4} \tau_{\eta \xi 11}$$ stress strain

δ^2 Order Solution

$$u_{\xi_5} = -U(1-\cos\xi)^4\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^5\left(\frac{2430}{1280}\frac{\eta^6}{\eta_0^6} - \frac{2187}{1280}\frac{\eta^4}{\eta_0^4} - \frac{2187}{8850}\frac{\eta^2}{\eta_0^2} + \frac{10{,}371}{18{,}880}\right)$$

$$u_{\eta_{11}} = U\sin\xi\,(1-\cos\xi)^3\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^4\left(8\frac{1-\cos\xi}{1-\cos\xi_1}-3\right)\left(\frac{2430}{8960}\frac{\eta^7}{\eta_0^7} - \frac{2187}{6400}\frac{\eta^5}{\eta_0^5} - \frac{729}{8850}\frac{\eta^3}{\eta_0^3} + \frac{10{,}371}{18{,}880}\frac{\eta}{\eta_0}\right)$$

$$\tau_{\eta\xi_{11}} = \frac{4374}{8850}\,\frac{4U}{a\eta_0}\,(1-\cos\xi)^5\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^5\frac{\eta}{\eta_0}$$

$$\sigma_{\xi_5} = \sigma_{\eta_5} = -\frac{4374}{8850}\,\frac{4U}{a\eta_0^2}\int_{\xi_1}^{\xi}(1-\cos\xi)^5\left(\frac{1-\cos\xi}{1-\cos\xi_1}-1\right)^5 d\xi$$

$\eta_0 \varepsilon$ Order Equations

$$\frac{\partial u_{\xi\varepsilon}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta\varepsilon 1}}{\partial \eta} = \frac{\sin\xi}{1-\cos\xi} u_{\xi\varepsilon}$$ continuity

$$\frac{\partial \sigma_{\xi\varepsilon}}{\partial \xi} + \frac{\partial \tau_{\eta\xi\varepsilon 1}}{\partial \eta} = \frac{\sin\xi}{1-\cos\xi}(\sigma_{\xi\varepsilon} - \sigma_{\eta\varepsilon})$$ equilibrium

$$\sigma_{\eta\varepsilon} = \frac{24U}{a\,\eta_0^2} f(\xi)$$ equilibrium

$$\sigma_{\xi\varepsilon} - \sigma_{\eta\varepsilon} = \frac{(1-\cos\xi)}{U} \tau_{\eta\xi 1} \frac{\partial u_{\xi 0}}{\partial \eta}$$ stress strain

$$\eta_0(1-\cos\xi)\frac{\partial u_{\xi\varepsilon}}{\partial \eta} = \frac{a\eta_0}{4}\tau_{\eta\xi\varepsilon 1} - \frac{a\eta\eta_0\sin\xi}{4(1-\cos\xi)}(\sigma_{\xi\varepsilon} - \sigma_{\eta\varepsilon})$$

$$+ \frac{a\eta\eta_0\sin\xi}{4U}\tau_{\eta\xi 1}\frac{\partial u_{\xi 0}}{\partial \eta} + \frac{a\eta_0^2}{44U}(1-\cos\xi)(\sigma_{\xi 1} - \sigma_{\eta 1})\frac{\partial u_{\xi 0}}{\partial \eta}$$

$$+ \frac{a\eta_0^2}{24U}\left(u_{\eta 1}(1-\cos\xi) - \frac{\eta}{\eta_0}\sin\xi\, u_{\xi 0}\right)\frac{\partial \tau_{\eta\xi 1}}{\partial \eta}$$

stress strain

$\eta_0 \epsilon$ Order Solution

$$u_{\xi_8} = U \sin\xi \left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(-\frac{9}{8} + \frac{9}{16}\frac{1-\cos\xi}{1-\cos\xi_1}\right)\left(\frac{\eta^4}{\eta_0^4} - \frac{6}{5}\frac{\eta^2}{\eta_0^2} + \frac{1}{5}\right)$$

$$\begin{aligned} u_{\eta_{14}} = {} & U(1+\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(-\frac{9}{8} + \frac{9}{16}\frac{1-\cos\xi}{1-\cos\xi_1}\right)\left(\frac{\eta^5}{5\eta_0^5} - \frac{2\eta^3}{5\eta_0^3} + \frac{\eta}{5\eta_0}\right) \\ & - \frac{9}{16} U(1+\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(\frac{1-\cos\xi}{1-\cos\xi_1}\right)\left(\frac{\eta^5}{5\eta_0^5} - \frac{2\eta^3}{5\eta_0^3} + \frac{\eta}{5\eta_0}\right) \\ & - U(1+\cos\xi)\left(-\frac{9}{8} + \frac{9}{16}\frac{1-\cos\xi}{1-\cos\xi_1}\right)\left(\frac{1-\cos\xi}{1-\cos\xi_1}\right)\left(\frac{\eta^5}{5\eta_0^5} - \frac{2\eta^3}{5\eta_0^3} + \frac{\eta}{5\eta_0}\right) \\ & - U\cos\xi\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(-\frac{9}{8} + \frac{9}{16}\frac{1-\cos\xi}{1-\cos\xi_1}\right)\left(\frac{\eta^5}{5\eta_0^5} - \frac{2\eta^3}{5\eta_0^3} + \frac{\eta}{5\eta_0}\right) \end{aligned}$$

$$\begin{aligned} \tau_{\eta\xi_{14}} = {} & \frac{34U}{a\eta_0}\sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(-3\frac{1-\cos\xi}{1-\cos\xi_1} + 1\right)\frac{\eta^3}{\eta_0^3} \\ & + \frac{4U}{a\eta_0}\sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(\frac{27}{10} + \frac{9}{10}\frac{1-\cos\xi}{1-\cos\xi_1}\right)\frac{\eta}{\eta_0} \end{aligned}$$

$$\sigma_{\xi_8} - \sigma_{\eta_8} = \frac{34U}{a\eta_0^2}(1-\cos\xi)^2\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)^2\frac{\eta^2}{\eta_0^2}$$

$$\begin{aligned} \sigma_{\xi_8} = {} & \frac{34U}{a\eta_0^2}\frac{\eta^2}{\eta_0^2}\left[\int\left\{(1-\cos\xi)^2\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)^2 - \sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(-3\frac{1-\cos\xi}{1-\cos\xi_1} + 1\right)\right\}d\xi\right] \\ & - \frac{4U}{a\eta_0^2}\left[\int\sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(\frac{27}{10} + \frac{9}{10}\frac{1-\cos\xi}{1-\cos\xi_1}\right)d\xi\right] \\ & - \frac{34U}{a\eta_0^2}\left[\int\left\{(1-\cos\xi)^2\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)^2 - \sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(-3\frac{1-\cos\xi}{1-\cos\xi_1} + 1\right)\right\}d\xi\right]_{\xi=\xi_E} \\ & + \frac{4U}{a\eta_0^2}\left[\int\sin\xi(1-\cos\xi)\left(\frac{1-\cos\xi}{1-\cos\xi_1} - 1\right)\left(\frac{27}{10} + \frac{9}{10}\frac{1-\cos\xi}{1-\cos\xi_1}\right)d\xi\right]_{\xi=\xi_E} \end{aligned}$$

$\eta_0 \delta$ Order Equations

$$\frac{\partial u_{\xi 7}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta 13}}{\partial \eta} = \frac{\sin \xi}{1-\cos \xi} u_{\xi 7}$$ continuity

$$\frac{\partial \sigma_{\xi 7}}{\partial \xi} + \frac{\partial \tau_{\eta \xi 13}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{\eta 7} = \frac{24U}{a \eta_0^2} f(\xi)$$ equilibrium

$$\sigma_{\xi 7} = \sigma_{\eta 7}$$ stress strain

$$\eta_0 (1-\cos \xi) \frac{\partial u_{\xi 7}}{\partial \eta} + \frac{3\eta_0^3}{8U^2}(1-\cos \xi)^3 \left(\frac{\partial u_{\xi 0}}{\partial \eta}\right)\left(\frac{\partial u_{\xi 1}}{\partial \eta}\right) = \frac{a \eta_0}{4} \tau_{\eta \xi 13}$$ stress strain

$\eta_0 \delta$ Order Solution

trivial solution

$\delta\varepsilon$ Order Equations

$$\frac{\partial u_{\xi 9}}{\partial \xi} + \eta_0 \frac{\partial u_{\eta, 15}}{\partial \eta} = \frac{\sin \xi}{1 - \cos \xi} u_{\xi 9}$$ continuity

$$\frac{\partial \sigma_{\xi 9}}{\partial \xi} + \frac{\partial \tau_{\eta \xi, 15}}{\partial \eta} = 0$$ equilibrium

$$\sigma_{r_9} = \frac{2\psi U}{a \eta_0^2} f(\xi)$$ equilibrium

$$\sigma_{\eta_9} = \sigma_{\xi_9}$$ stress strain

$$\eta_0 (1 - \cos \xi) \frac{\partial u_{\xi 9}}{\partial \eta} + \frac{\eta_0^3}{4U^2} (1 - \cos \xi)^6 \left(\frac{\partial u_{\xi 0}}{\partial \eta} \right)^2 \frac{\partial u_{\xi 3}}{\partial \eta} = \frac{a \eta_0}{4} \gamma_{\eta \xi, 15}$$

stress strain

$\delta\varepsilon$ Order Solution

trivial solution

BIOGRAPHY

The author was born on May 9, 1931. He received his elementary and high school training in New Orleans, Louisiana. In 1950 he received his Bachelor of Science degree in Mechanical Engineering from Louisiana State University, Baton Rouge, Louisiana and in 1952 his degree of Master of Science in Mechanical Engineering from The Rice Institute, Houston, Texas.

The author enrolled at M.I.T. in 1952 and became a Teaching Assistant in Mechanical Engineering. During the academic year 1953-1954 he was the Dupont Fellow in the Mechanical Engineering Department.

"Design of Shrink Fits" by Paul Paslay and Robert Plunkett, Transactions of the ASME, October 1953, is the author's only publication.